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Indranil Biswas, Sorin Dumitrescu

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# PROJECTIVE STRUCTURE ON RIEMANN SURFACE AND NATURAL DIFFERENTIAL OPERATORS

INDRANIL BISWAS AND SORIN DUMITRESCU

ABSTRACT. We investigate the holomorphic differential operators on a Riemann surface  $M$ . This is done by endowing  $M$  with a projective structure. Let  $\mathcal{L}$  be a theta characteristic on  $M$ . We explicitly describe the jet bundle  $J^k(E \otimes \mathcal{L}^{\otimes n})$ , where  $E$  is a holomorphic vector bundle on  $M$  equipped with a holomorphic connection, for all  $k$  and  $n$ . This provides a description of holomorphic differential operators from  $E \otimes \mathcal{L}^{\otimes n}$  to another holomorphic vector bundle  $F$  using the natural isomorphism  $\text{Diff}^k(E \otimes \mathcal{L}^{\otimes n}, F) = F \otimes (J^k(E \otimes \mathcal{L}^{\otimes n}))^*$ .

## 1. INTRODUCTION

The study of natural differential operators between natural bundles is a major topic in global analysis on manifolds (see, for instance, [Ni], [Te] and [KMS]).

Our aim here is to study natural holomorphic differential operators between natural holomorphic vector bundles on a Riemann surface  $M$ . To be able to formulate their description, we make use of the existence of compatible projective structures on  $M$  (see [Gu]).

We recall that a projective structure on  $M$  is given by a holomorphic coordinate atlas such that all the transition functions are restrictions of Möbius transformations. This approach to holomorphic differential operators on  $M$  entails a detailed understanding of the action of  $\text{SL}(2, \mathbb{C})$  on the jet bundles on  $\mathbb{C}\mathbb{P}^1$ .

Let  $\mathbb{V}$  be a complex vector space of dimension two. For notational convenience, denote by  $X$  the space  $\mathbb{P}(\mathbb{V})$  of complex lines in  $\mathbb{V}$ . Let  $L$  be the tautological line bundle of degree one on  $X$ . The trivial holomorphic vector bundle  $X \times \mathbb{V}$  on  $X$  with fiber  $\mathbb{V}$  will be denoted by  $\mathbb{V}_1$ , while the symmetric product  $\text{Sym}^j(\mathbb{V}_1)$  will be denoted by  $\mathbb{V}_j$ . The holomorphically trivial line bundle  $X \times \mathbb{C}$  on  $X$  will be denoted by  $\mathbb{V}_0$ .

We prove the following (see Theorem 3.5):

**Theorem 1.1.** *If  $n < 0$  or  $n \geq k$ , then*

$$J^k(L^n) = L^{n-k} \otimes \mathbb{V}_k.$$

*If  $k > n \geq 0$ , then*

$$J^k(L^n) = \mathbb{V}_n \oplus (L^{-(k+1)} \otimes \mathbb{V}_{k-n-1}).$$

*Both the isomorphisms are  $\text{SL}(\mathbb{V})$ -equivariant.*

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The natural forgetful homomorphism  $J^k(L^n) \longrightarrow J^{k-1}(L^n)$  of holomorphic vector bundles with action of  $\mathrm{SL}(\mathbb{V})$  is described in Proposition 3.1, Proposition 3.2 and Proposition 3.4.

Since all the above mentioned isomorphisms and homomorphisms are  $\mathrm{SL}(\mathbb{V})$ -equivariant, we are able to translate them into the set-up of a Riemann surface equipped with a projective structure (see Section 4 to see that any projective structure admits an underlying atlas for which the transition maps are restrictions of elements in  $\mathrm{SL}(\mathbb{V})$ ).

Let  $M$  be a compact Riemann surface equipped with a projective structure  $P$ . Using  $P$ , the above tautological line bundle  $L$  on  $X$  produces a holomorphic line bundle  $\mathbf{L}$  on  $M$  with the property that  $\mathbf{L}^{\otimes 2} = TM$ . Moreover, using  $P$ , the trivial vector bundle  $\mathbb{V}_1$  produces a holomorphic vector bundle  $\mathcal{V}_1$  of rank two over  $M$  that fits in the following short exact sequence of holomorphic vector bundles on  $M$ :

$$0 \longrightarrow \mathbf{L}^* \longrightarrow \mathcal{V}_1 \longrightarrow \mathbf{L} \longrightarrow 0.$$

This involves fixing a nonzero element of  $\bigwedge^2 \mathbb{V}$ . The symmetric power  $\mathrm{Sym}^j(\mathcal{V}_1)$  will be denoted by  $\mathcal{V}_j$ , while the trivial line bundle  $\mathcal{O}_M$  will be denoted by  $\mathcal{V}_0$ .

Let us now describe our results about natural differential holomorphic operators on a Riemann surface endowed with a projective structure.

The following is deduced using Theorem 1.1 (see Corollary 4.2):

**Corollary 1.2.** *If  $n < 0$  or  $n \geq k$ , then there is a canonical isomorphism*

$$J^k(\mathbf{L}^n) = \mathbf{L}^{n-k} \otimes \mathcal{V}_k.$$

*If  $k > n \geq 0$ , then there is a canonical isomorphism*

$$J^k(\mathbf{L}^n) = \mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1}).$$

This enables us to deduce the following result (see Lemma 4.4):

**Lemma 1.3.** *Let  $M$  be a compact connected Riemann surface endowed with a projective structure  $P$ . Consider  $E$  a holomorphic vector bundle on  $M$  equipped with a holomorphic connection.*

*If  $n < 0$  or  $n \geq k$ , then there is a canonical isomorphism*

$$J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k.$$

*If  $k > n \geq 0$ , then there is a canonical isomorphism*

$$J^k(E \otimes \mathbf{L}^n) = E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1})).$$

Let  $E$  be a holomorphic vector bundle on  $M$  equipped with a holomorphic connection, as in the previous Lemma. Given a holomorphic vector bundle  $F$  on  $M$ , denote by  $\mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F)$  the sheaf of holomorphic differential operators of order  $k$  from  $E \otimes \mathbf{L}^n$  to  $F$ .

The associated symbol homomorphism

$$\mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F) \longrightarrow \mathrm{Hom}(E \otimes \mathbf{L}^n, F) \otimes (TM)^{\otimes k}$$

is describe in Corollary 4.7 and Corollary 4.11.

Fix integers  $k$ ,  $n$  and  $l$  such that at least one of the following two conditions is valid:

- (1)  $n < 0$ , or
- (2)  $n, l \geq k$ .

Also fix a section

$$\theta_0 \in H^0(M, \text{End}(E) \otimes \mathbf{L}^l).$$

Using  $\theta_0$ , and a projective structure  $P$  on  $M$ , we construct a holomorphic differential operator

$$\mathcal{T}_\theta \in H^0(M, \text{Diff}^k(E \otimes \mathbf{L}^n, E \otimes \mathbf{L}^{l+n-2k})).$$

The following theorem shows that  $\mathcal{T}_\theta$  is a lift of the symbol  $\theta_0$  to a holomorphic differential operator (see Theorem 5.1):

**Theorem 1.4.** *The symbol of the above holomorphic differential operator  $\mathcal{T}_\theta$  is the section  $\theta_0$ .*

## 2. PRELIMINARIES

**2.1. Jet bundles and differential operators.** Let  $X$  be a connected Riemann surface. The holomorphic cotangent bundle of  $X$  will be denoted by  $K_X$ . Consider the Cartesian product  $X \times X$ . For  $i = 1, 2$ , let

$$p_i : X \times X \longrightarrow X$$

be the projection to the  $i$ -th factor. Let

$$\Delta := \{(x, x) \mid x \in X\} \subset X \times X \tag{2.1}$$

be the diagonal divisor. At times we shall identify  $\Delta$  with  $X$  using the mapping  $x \mapsto (x, x)$ . For any nonnegative integer  $k$ , consider the natural exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow \mathcal{O}_{X \times X}(-k\Delta) \\ &\longrightarrow \mathcal{O}_{X \times X}(-k\Delta)/\mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow 0. \end{aligned} \tag{2.2}$$

Using the Poincaré adjunction formula, we have

$$\mathcal{O}_{X \times X}(-k\Delta)/\mathcal{O}_{X \times X}(-(k+1)\Delta) = K_X^{\otimes k}; \tag{2.3}$$

here  $X$  is identified with  $\Delta$ . Let  $z$  be a holomorphic coordinate function on an open subset  $U \subset X$ . The isomorphism in (2.3) over  $(U \times U) \cap \Delta$  sends the section  $(z \circ p_2 - z \circ p_1)^k$  of  $\mathcal{O}_{X \times X}(-k\Delta)/\mathcal{O}_{X \times X}(-(k+1)\Delta)$  over  $(U \times U) \cap \Delta$  to the section  $(dz)^{\otimes k}$  of  $K_X^{\otimes k}|_U$  (after identifying  $U$  with  $(U \times U) \cap \Delta$ ).

Let  $W$  be a holomorphic vector bundle on  $X$  of rank  $r_W$ . For any nonnegative integer  $k$ , consider the subsheaf

$$(p_2^*W) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \subset p_2^*W.$$

The  $k$ -th order jet bundle  $J^k(W)$  is defined to be the direct image

$$J^k(W) := p_{1*}((p_2^*W)/((p_2^*W) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta))) \longrightarrow X, \tag{2.4}$$

which is a holomorphic vector bundle on  $X$  of rank  $(k+1)r_W$ .

For any  $k \geq 1$ , using the natural inclusions

$$(p_2^*W) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \subset (p_2^*W) \otimes \mathcal{O}_{X \times X}(-k\Delta) \subset p_2^*W$$

together with (2.3), we get a short exact sequence of holomorphic vector bundles on  $X$

$$0 \longrightarrow W \otimes K_X^{\otimes k} \xrightarrow{\iota} J^k(W) \xrightarrow{\phi_k} J^{k-1}(W) \longrightarrow 0. \quad (2.5)$$

Let  $V$  be a holomorphic vector bundle on  $X$ . Let  $\text{Diff}^k(W, V)$  denote the sheaf of holomorphic differential operators of order  $k$  from  $W$  to  $V$ . It is the sheaf of holomorphic sections of

$$\text{Diff}^k(W, V) := \text{Hom}(J^k(W), V) = V \otimes J^k(W)^*. \quad (2.6)$$

Consider the homomorphism

$$\text{Diff}^k(W, V) = V \otimes J^k(W)^* \xrightarrow{\text{Id}_V \otimes \iota^*} V \otimes (W \otimes K_X^{\otimes k})^* = (TX)^{\otimes k} \otimes \text{Hom}(W, V), \quad (2.7)$$

where  $\iota$  is the homomorphism in (2.5). This homomorphism in (2.7) is known as the *symbol map*.

**2.2. Line bundles on the projective line.** Let  $\mathbb{V}$  be a complex vector space of dimension two. Now set

$$X := \mathbb{P}(\mathbb{V})$$

to be the compact Riemann surface of genus zero parametrizing all the complex lines in  $\mathbb{V}$ . Let

$$L := \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}(\mathbb{V})}(1) \longrightarrow X \quad (2.8)$$

be the tautological holomorphic line bundle of degree one whose fiber over the point corresponding to a line  $\xi \subset \mathbb{V}$  is the quotient line  $\mathbb{V}/\xi$ . For any integer  $n$ , by  $L^n$  we shall denote the holomorphic line bundle

- $L^{\otimes n}$ , if  $n > 0$ ,
- $(L^{\otimes -n})^*$ , if  $n < 0$ , and
- $\mathcal{O}_X$  (the trivial line bundle), if  $n = 0$ .

A theorem of Grothendieck asserts that any holomorphic vector bundle  $F$  on  $X$  of rank  $r$  is of the form

$$F = \bigoplus_{i=1}^r L^{a_i},$$

where  $a_i \in \mathbb{Z}$  [Gr, p. 122, Théorème 1.1]; moreover, the above integers  $\{a_i\}$  are uniquely determined, up to a permutation, by  $F$  [Gr, p. 122, Théorème 1.1], [At1, p. 315, Theorem 2]. The holomorphic cotangent bundle  $K_X$  is holomorphically isomorphic to  $L^{-2}$ .

Let

$$\mathbb{L} \longrightarrow X \quad (2.9)$$

be the tautological holomorphic line bundle of degree  $-1$  whose fiber over the point corresponding to a line  $\xi \subset \mathbb{V}$  is  $\xi$  itself. This  $\mathbb{L}$  is holomorphically isomorphic to  $L^{-1}$ . We note that there is no natural isomorphism between  $L^{-1}$  and  $\mathbb{L}$ .

Let  $\text{SL}(\mathbb{V})$  be the complex Lie group consisting of all linear automorphisms of  $\mathbb{V}$  that act trivially on the determinant line  $\bigwedge^2 \mathbb{V}$ . This group  $\text{SL}(\mathbb{V})$  acts on  $\mathbb{P}(\mathbb{V})$  as follows: the action of any  $A \in \text{SL}(\mathbb{V})$  sends a line  $\xi \subset \mathbb{V}$  to the line  $A(\xi)$ . The corresponding action homomorphism

$$\text{SL}(\mathbb{V}) \longrightarrow \text{Aut}(X)$$

is holomorphic, and its kernel is  $\pm \text{Id}_{\mathbb{V}}$ . The action of  $\text{SL}(\mathbb{V})$  has a tautological lift to an action of  $\text{SL}(\mathbb{V})$  on  $K_X^{\otimes m}$  for every integer  $m$ .

The above action of  $\mathrm{SL}(\mathbb{V})$  on  $\mathbb{P}(\mathbb{V})$  clearly lifts to the line bundle  $L$ . Indeed, the action of any  $A \in \mathrm{SL}(\mathbb{V})$  on  $\mathbb{V}$  produces an isomorphism

$$\mathbb{V}/\xi \xrightarrow{\sim} \mathbb{V}/A(\xi)$$

for every line  $\xi \subset \mathbb{V}$ . The resulting action of  $\mathrm{SL}(\mathbb{V})$  on  $L$  produces an action of  $\mathrm{SL}(\mathbb{V})$  on  $L^n$  for every  $n$ .

The diagonal action of  $\mathrm{SL}(\mathbb{V})$  on  $X \times X$  preserves the divisor  $\Delta$  in (2.1). This action of  $\mathrm{SL}(\mathbb{V})$  on  $X \times X$ , and the action of  $\mathrm{SL}(\mathbb{V})$  on  $L^n$ , together produce an action of  $\mathrm{SL}(\mathbb{V})$  on  $J^k(L^n)$  defined in (2.4). The homomorphism  $\phi_k$  in (2.5) is in fact  $\mathrm{SL}(\mathbb{V})$ -equivariant.

### 3. DESCRIPTION OF JET BUNDLES ON THE PROJECTIVE LINE

Consider the short exact sequence of coherent sheaves on  $X = \mathbb{P}(\mathbb{V})$

$$0 \longrightarrow (p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow p_2^*L^n \longrightarrow (p_2^*L^n)|_{(k+1)\Delta} \longrightarrow 0. \quad (3.1)$$

From (2.4) we know that  $J^k(L^n)$  is the direct image of  $(p_2^*L^n)|_{(k+1)\Delta}$  to  $X$  under the projection  $p_1$ . We would investigate  $J^k(L^n)$  using the long exact sequence of direct images for the short exact sequence of sheaves in (3.1).

Let

$$\begin{aligned} 0 &\longrightarrow R^0p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow R^0p_{1*}p_2^*L^n \longrightarrow R^0p_{1*}((p_2^*L^n)|_{(k+1)\Delta}) \\ &=: J^k(L^n) \longrightarrow R^1p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow R^1p_{1*}p_2^*L^n \longrightarrow 0 \end{aligned} \quad (3.2)$$

be the long exact sequence of direct images, under  $p_1$ , associated to the short exact sequence in (3.1). Note that

$$R^1p_{1*}((p_2^*L^n)|_{(k+1)\Delta}) = 0,$$

because the support of  $(p_2^*L^n)|_{(k+1)\Delta}$  is finite over  $X$  (for the projection  $p_1$ ).

Since the diagonal action of  $\mathrm{SL}(\mathbb{V})$  on  $X \times X$  preserves the divisor  $\Delta$ , each sheaf in (3.2) is equipped with an action of  $\mathrm{SL}(\mathbb{V})$  that lifts the action of  $\mathrm{SL}(\mathbb{V})$  on  $X$ . Moreover, all the homomorphisms in the exact sequence (3.2) are  $\mathrm{SL}(\mathbb{V})$ -equivariant.

In (3.2), the direct image  $R^i p_{1*} p_2^* L^n$  is the trivial holomorphic vector bundle

$$R^i p_{1*} p_2^* L^n = X \times H^i(X, L^n) \quad (3.3)$$

over  $X$  with fiber  $H^i(X, L^n)$  for  $i = 0, 1$ .

Next, to understand  $R^i p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta))$ , we first note that the holomorphic line bundle  $\mathcal{O}_{X \times X}(\Delta)$  is holomorphically isomorphic to  $(p_1^*L) \otimes (p_2^*L)$ ; however, there is no natural isomorphism between these two isomorphic line bundles. To construct an explicit isomorphism between them, fix a nonzero element

$$0 \neq \omega \in \bigwedge^2 \mathbb{V}. \quad (3.4)$$

Note that  $\omega$  defines a symplectic structure on  $\mathbb{V}$ , in particular,  $\omega$  produces an isomorphism

$$\mathbb{V}^* \xrightarrow{\sim} \mathbb{V}. \quad (3.5)$$

The action of  $\mathrm{SL}(\mathbb{V})$  on  $\bigwedge^2 \mathbb{V}$ , given by the action of  $\mathrm{SL}(\mathbb{V})$  on  $\mathbb{V}$ , is the trivial action, and hence this action fixes the element  $\omega$ . We note that  $\omega$  produces a holomorphic isomorphism

$$\Psi : L^{-1} \longrightarrow \mathbb{L}, \quad (3.6)$$

where  $\mathbb{L}$  was constructed in (2.9). Indeed, from the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathbb{L} \longrightarrow X \times \mathbb{V} \longrightarrow L \longrightarrow 0 \quad (3.7)$$

we conclude that  $\mathbb{L} \otimes L = X \times \bigwedge^2 \mathbb{V}$ . Therefore, the element  $\omega \in \bigwedge^2 \mathbb{V}$  gives a nowhere vanishing holomorphic section of  $\mathbb{L} \otimes L$ . This section produces the isomorphism  $\Psi$  in (3.6). Note that the diagonal action of  $\mathrm{SL}(\mathbb{V})$  on  $X \times \mathbb{V}$  preserves the subbundle  $\mathbb{L}$  in (3.7); more precisely, the homomorphisms in (3.7) are  $\mathrm{SL}(\mathbb{V})$ -equivariant. The isomorphism  $\Psi$  in (3.6) is evidently  $\mathrm{SL}(\mathbb{V})$ -equivariant. It should be mentioned that  $\Psi$  depends on  $\omega$ .

We have  $H^0(X, L) = \mathbb{V}$ . So

$$H^0(X \times X, (p_1^* L) \otimes (p_2^* L)) = H^0(X, L) \otimes H^0(X, L) = \mathbb{V} \otimes \mathbb{V},$$

and hence  $\omega \in \bigwedge^2 \mathbb{V} \subset \mathbb{V} \otimes \mathbb{V}$  in (3.4) defines a holomorphic section

$$\widehat{\omega} \in H^0(X \times X, (p_1^* L) \otimes (p_2^* L)). \quad (3.8)$$

This section  $\widehat{\omega}$  vanishes exactly on the diagonal  $\Delta \subset X \times X$ . Consequently, there is a unique holomorphic isomorphism

$$\Phi : (p_1^* L) \otimes (p_2^* L) \xrightarrow{\sim} \mathcal{O}_{X \times X}(\Delta)$$

such that the section  $\Phi(\widehat{\omega})$ , where  $\widehat{\omega}$  is constructed in (3.8), coincides with the holomorphic section of  $\mathcal{O}_{X \times X}(\Delta)$  given by the constant function 1 on  $X \times X$ . Therefore, combining  $\Phi$  with  $\Psi$  in (3.6), we have

$$(p_1^* \mathbb{L}) \otimes (p_2^* \mathbb{L}) = (p_1^* L^{-1}) \otimes (p_2^* L^{-1}) = (p_1^* \mathbb{L}) \otimes (p_2^* L^{-1}) = \mathcal{O}_{X \times X}(-\Delta). \quad (3.9)$$

Using the identification of  $X$  with  $\Delta$ , the Poincaré adjunction formula gives that  $\mathcal{O}_{X \times X}(\Delta)|_{\Delta} = TX$ , where  $TX$  is the holomorphic tangent bundle of  $X$ ; it is the dual of the isomorphism in (2.3) for  $k = 1$ . Therefore, restricting the isomorphisms in (3.9) to  $\Delta \subset X \times X$ , we have a holomorphic isomorphism

$$\Phi_0 : L^2 = (\mathbb{L}^*)^{\otimes 2} \xrightarrow{\sim} TX. \quad (3.10)$$

Using the isomorphisms in (3.9), we have

$$(p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) = (p_1^* L^{-k-1}) \otimes (p_2^* L^{n-k-1}) = (p_1^* \mathbb{L}^{k+1}) \otimes (p_2^* L^{n-k-1}).$$

Hence by the projection formula, [Ha, p. 426, A4],

$$\begin{aligned} R^i p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) &= R^i p_{1*}((p_1^* \mathbb{L}^{k+1}) \otimes (p_2^* L^{n-k-1})) \\ &= \mathbb{L}^{k+1} \otimes R^i p_{1*}(p_2^* L^{n-k-1}) \end{aligned} \quad (3.11)$$

for  $i = 0, 1$ .

We shall break  $(n, k)$  into three cases.

**3.1. The case of  $n < 0$ .** In (3.2), we have  $R^0 p_{1*} p_2^* L^n = 0$ , because  $n < 0$ . Consequently, the short exact sequence in (3.2) becomes

$$0 \longrightarrow J^k(L^n) \longrightarrow R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow R^1 p_{1*} p_2^* L^n \longrightarrow 0. \quad (3.12)$$

Now using (3.11), the exact sequence in (3.12) becomes the short exact sequence

$$0 \longrightarrow J^k(L^n) \longrightarrow \mathbb{L}^{k+1} \otimes R^1 p_{1*}(p_2^* L^{n-k-1}) \xrightarrow{\gamma_1} R^1 p_{1*} p_2^* L^n \longrightarrow 0. \quad (3.13)$$

Invoking Serre duality and using the isomorphism  $\Phi_0$  in (3.10),

$$H^1(X, L^n) = H^0(X, L^{-n-2})^* = \text{Sym}^{-n-2}(\mathbb{V})^*,$$

where  $\text{Sym}^{-n-2}(\mathbb{V})$  denotes:

- the  $(-n-2)$ -th symmetric power of  $\mathbb{V}$ , if  $-n-2 > 0$ ,
- $\mathbb{C}$ , if  $-n = 2$ , and
- 0, if  $-n = 1$ .

Therefore,  $R^1 p_{1*} p_2^* L^n$  in (3.13) is the trivial holomorphic vector bundle

$$X \times \text{Sym}^{-n-2}(\mathbb{V})^* \longrightarrow X$$

(see (3.3)).

For notational convenience, the trivial holomorphic vector bundle  $X \times \text{Sym}^j(\mathbb{V})$  over  $X$  with fiber  $\text{Sym}^j(\mathbb{V})$  will be denoted by  $\mathbb{V}_j$ . The trivial line bundle  $\mathcal{O}_X$  will be denoted by  $\mathbb{V}_0$ . So we have

$$R^1 p_{1*} p_2^* L^n = (\mathbb{V}_{-n-2})^* = \mathbb{V}_{-n-2}^*. \quad (3.14)$$

Similarly, we have

$$R^1 p_{1*}(p_2^* L^{n-k-1}) = X \times \text{Sym}^{k-n-1}(\mathbb{V})^* = \mathbb{V}_{k-n-1}^* \longrightarrow X. \quad (3.15)$$

Hence  $\mathbb{L}^{k+1} \otimes R^1 p_{1*}(p_2^* L^{n-k-1})$  in (3.13) has the following description:

$$\mathbb{L}^{k+1} \otimes R^1 p_{1*}(p_2^* L^{n-k-1}) = \mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}^*. \quad (3.16)$$

Using (3.14) and (3.16), the homomorphism  $\gamma_1$  in (3.13) is a homomorphism

$$\gamma_1 : \mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}^* \longrightarrow \mathbb{V}_{-n-2}^*. \quad (3.17)$$

The homomorphism  $\gamma_1$  in (3.17) is simply the contraction of elements of  $\mathbb{V}$  by  $\mathbb{V}^*$  (recall that  $\mathbb{L} \subset X \times \mathbb{V}$ ). More precisely, consider the contraction homomorphism

$$\mathbb{V} \otimes (\mathbb{V}^*)^{\otimes j} \longrightarrow (\mathbb{V}^*)^{\otimes(j-1)}.$$

Since  $\mathbb{L} \subset \mathbb{V}_1$  (see (3.7)), this contraction produces a holomorphic homomorphism of vector bundles

$$\mathbb{L} \otimes \mathbb{V}_j^* \longrightarrow \mathbb{V}_{j-1}^*.$$

The homomorphism  $\gamma_1$  in (3.17) is  $(k+1)$ -times iteration of this homomorphism starting with  $j = k - n - 1$ .

We shall compute the kernel of the homomorphism  $\gamma_1$  in (3.17).

Taking dual of the short exact sequence in (3.7), we have

$$L^* \subset X \times \mathbb{V}^* = \mathbb{V}_1^*.$$

Hence, using the isomorphism  $\Psi$  in (3.6), we have

$$\mathbb{L} = L^* \subset \mathbb{V}_1^*. \quad (3.18)$$

For the natural duality paring  $\mathbb{V}_1 \otimes \mathbb{V}_1^* \rightarrow \mathcal{O}_X$ , the annihilator of the subbundle  $\mathbb{L} \subset \mathbb{V}_1$  in (3.7) is the subbundle  $\mathbb{L} \subset \mathbb{V}_1^*$  in (3.18). Therefore, from the above description of the homomorphism  $\gamma_1$  in (3.17) we conclude that the subbundle

$$\mathbb{L}^{k+1} \otimes (\mathbb{V}_{k-n-1}^*) \supset \mathbb{L}^{k+1} \otimes (\mathbb{L}^{-n-1} \otimes \mathbb{V}_k^*) = \mathbb{L}^{k-n} \otimes \mathbb{V}_k^*$$

is contained in the kernel of the surjective homomorphism  $\gamma_1$ ; note that  $\mathbb{L}^{-n-1} \otimes \mathbb{V}_k^*$  is realized as a subbundle of  $\mathbb{V}_{k-n-1}^*$  using the inclusion  $\mathbb{L} \subset \mathbb{V}_1^*$  in (3.18). From (3.13) we know that  $\gamma_1$  is surjective. Since

$$\text{rank}(\mathbb{L}^{k-n} \otimes \mathbb{V}_k^*) = k+1 = (k-n) - (-n-1) = \text{rank}(\mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}^*) - \text{rank}(\mathbb{V}_{-n-2}^*),$$

we now conclude that

$$\mathbb{L}^{k-n} \otimes \mathbb{V}_k^* = \text{kernel}(\gamma_1).$$

Therefore, from (3.13) it follows that

$$J^k(L^n) = \text{kernel}(\gamma_1) = \mathbb{L}^{k-n} \otimes \mathbb{V}_k^*. \quad (3.19)$$

Now using the isomorphism between  $\mathbb{V}_k$  and  $\mathbb{V}_k^*$  given by the isomorphism in (3.5), from (3.19) we deduce that

$$J^k(L^n) = \mathbb{L}^{k-n} \otimes \mathbb{V}_k.$$

Moreover, using the isomorphism  $\Psi$  in (3.6) we have

$$J^k(L^n) = L^{n-k} \otimes \mathbb{V}_k. \quad (3.20)$$

Recall that all the homomorphisms in (3.2) are  $\text{SL}(\mathbb{V})$ -equivariant. From the construction of the isomorphism in (3.19) between  $J^k(L^n)$  and  $\mathbb{L}^{k-n} \otimes \mathbb{V}_k^*$  it is evident that it is  $\text{SL}(\mathbb{V})$ -equivariant. Therefore, the isomorphism in (3.20) is  $\text{SL}(\mathbb{V})$ -equivariant.

Fix  $k \geq 1$  an integer. Using (3.20), the homomorphism  $\phi_k$  in (2.5) is a projection

$$\phi_k : L^{n-k} \otimes \mathbb{V}_k = J^k(L^n) \rightarrow J^{k-1}(L^n) = L^{n-k+1} \otimes \mathbb{V}_{k-1}. \quad (3.21)$$

We shall describe the homomorphism in (3.21).

Recall that  $\mathbb{V}_j = \text{Sym}^j(\mathbb{V}_1) = \text{Sym}^j(X \times \mathbb{V})$ , and the line bundle  $L$  is a quotient of  $\mathbb{V}_1$  (see (3.7)). Therefore, we have a natural projection

$$\varpi_k^0 : \mathbb{V}_k \rightarrow L \otimes \mathbb{V}_{k-1}. \quad (3.22)$$

To describe this  $\varpi_k^0$  explicitly, let  $q_L : \mathbb{V}_1 \rightarrow L$  be the quotient map in (3.7). Consider the homomorphism

$$q_L \otimes \text{Id}_{\mathbb{V}_1^{\otimes(k-1)}} : \mathbb{V}_1^{\otimes k} \rightarrow L \otimes \mathbb{V}_1^{\otimes(k-1)}.$$

Note that  $q_L \otimes \text{Id}_{\mathbb{V}_1^{\otimes(k-1)}}$  sends the subbundle

$$\text{Sym}^k(\mathbb{V}_1) \subset \mathbb{V}_1^{\otimes k}$$

to the subbundle  $L \otimes \mathbb{V}_{k-1} \subset L \otimes \mathbb{V}_1^{\otimes(k-1)}$ . The homomorphism  $\varpi_k^0$  in (3.22) is this restriction of  $q_L \otimes \text{Id}_{\mathbb{V}_1^{\otimes(k-1)}}$ .

**Proposition 3.1.** *The homomorphism  $\phi_k$  in (3.21) coincides with  $\text{Id}_{L^{n-k}} \otimes \varpi_k^0$ , where  $\varpi_k^0$  is the homomorphism in (3.22).*

*Proof.* Consider the short exact sequence in (3.12). We have the commutative diagram of homomorphisms

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & L^n \otimes K_X^{\otimes k} & = & L^n \otimes K_X^{\otimes k} & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^k(L^n) & \longrightarrow & R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) & \longrightarrow & R^1 p_{1*} p_2^* L^n \longrightarrow 0 \\
& & \downarrow & & \downarrow \mu & & \parallel \\
0 & \longrightarrow & J^{k-1}(L^n) & \longrightarrow & R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta)) & \longrightarrow & R^1 p_{1*} p_2^* L^n \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & \\
& & & & & & (3.23)
\end{array}$$

where the bottom exact sequence is the one in (3.12) with  $k$  substituted by  $k-1$ , the top exact sequence is the one in (3.12), the left-vertical exact sequence is the one in (2.5) for  $W = L^n$ , and the other vertical exact sequence is the long exact sequence of direct images, under  $p_1$ , for the short exact sequence

$$0 \longrightarrow (p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow (p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta) \longrightarrow (L^n \otimes K_X^{\otimes k})|_{\Delta} \longrightarrow 0$$

of sheaves on  $X \times X$  obtained by tensoring with  $p_2^* L^n$  the short exact sequence given by (2.2).

Using (3.11) and (3.16) we have

$$R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) = \mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}^*.$$

Replacing  $k$  by  $k-1$ , we have

$$R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta)) = \mathbb{L}^k \otimes \mathbb{V}_{k-n-2}^*.$$

So the homomorphism  $\mu$  in (3.23) becomes a homomorphism

$$\mu : \mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}^* \longrightarrow \mathbb{L}^k \otimes \mathbb{V}_{k-n-2}^*.$$

This homomorphism is simply the contraction of elements of  $\mathbb{V}^*$  by elements of  $\mathbb{L} \subset \mathbb{V}$ . The proposition follows from this and the commutative diagram in (3.23).  $\square$

**3.2. The case of  $n \geq k$ .** Note that  $n \geq 0$ , because  $n \geq k$ . Since now we have  $n - k - 1 \geq -1$ , it follows that

$$H^1(X, L^{n-k-1}) = 0,$$

and hence using (3.11) and (3.3) it is deduced that

$$R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) = 0.$$

Consequently, the exact sequence in (3.2) becomes the short exact sequence

$$0 \longrightarrow R^0 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow R^0 p_{1*} p_2^* L^n \longrightarrow J^k(L^n) \longrightarrow 0. \quad (3.24)$$

Therefore, using (3.11) and (3.3), the exact sequence in (3.24) becomes the short exact sequence

$$0 \longrightarrow \mathbb{L}^{k+1} \otimes R^0 p_{1*}(p_2^* L^{n-k-1}) = \mathbb{L}^{k+1} \otimes \mathbb{V}_{n-k-1} \xrightarrow{\gamma^2} X \times \text{Sym}^n(\mathbb{V}) \quad (3.25)$$



where the bottom exact row is (3.24) with  $k - 1$  substituted in place of  $k$ , the top exact row is the one in (3.24), the right vertical exact sequence is the one in (2.5) and the left vertical exact sequence is a part of the long exact sequence of direct images, under  $p_1$ , for the short exact sequence

$$0 \longrightarrow (p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow (p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta) \longrightarrow (L^n \otimes K_X^{\otimes k})|_\Delta \longrightarrow 0$$

of sheaves on  $X \times X$  obtained by tensoring (2.3) with  $p_2^* L^n$ . Using (3.11), in (3.25) we saw that

$$R^0 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) = \mathbb{L}^{k+1} \otimes \mathbb{V}_{n-k-1},$$

So replacing  $k$  by  $k - 1$ , we have

$$R^0 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta)) = \mathbb{L}^k \otimes \mathbb{V}_{n-k}.$$

Using these isomorphisms, the injective homomorphism

$$\mu : \mathbb{L}^{k+1} \otimes \mathbb{V}_{n-k-1} = \mathbb{L}^{k+1} \otimes \text{Sym}^{n-k-1}(\mathbb{V}_1) \longrightarrow \mathbb{L}^k \otimes \text{Sym}^{n-k}(\mathbb{V}_1)$$

in (3.28) is the natural homomorphism induced by the inclusion of  $\mathbb{L}$  in  $\mathbb{V}_1$ . Now the proposition follows from (3.28).  $\square$

The remaining case will now be considered.

**3.3. The case of  $k > n \geq 0$ .** Since  $n \geq 0$ , it follows that  $H^1(X, L^n) = 0$ . Hence in (3.2) we have

$$R^1 p_{1*} p_2^* L^n = 0;$$

see (3.3). Since  $n - k - 1 < 0$ , we have  $H^0(X, L^{n-k-1}) = 0$ . Therefore, from (3.11) and (3.3) we conclude that

$$R^0 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) = 0.$$

Consequently, the exact sequence in (3.2) becomes the short exact sequence

$$0 \longrightarrow R^0 p_{1*} p_2^* L^n \longrightarrow J^k(L^n) \longrightarrow R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow 0. \quad (3.29)$$

Now using (3.11) and (3.3) together with the projection formula, the exact sequence in (3.29) becomes the exact sequence

$$0 \longrightarrow \mathbb{V}_n \xrightarrow{\gamma_3} J^k(L^n) \xrightarrow{\theta_3} \mathbb{L}^{k+1} \otimes R^1 p_{1*}(p_2^* L^{n-k-1}) \longrightarrow 0. \quad (3.30)$$

The homomorphisms  $\gamma_3$  and  $\theta_3$  in (3.30) are clearly  $\text{SL}(\mathbb{V})$ -equivariant.

**Lemma 3.3.** *The short exact sequence in (3.30) has a canonical  $\text{SL}(\mathbb{V})$ -equivariant holomorphic splitting.*

*Proof.* Since  $k > n$ , there is a natural projection

$$J^k(L^n) \longrightarrow J^n(L^n)$$

(see (2.5)). Now  $J^n(L^n) = \mathbb{V}_n$  by (3.26), so this projection defines a projection

$$\beta : J^k(L^n) \longrightarrow \mathbb{V}_n. \quad (3.31)$$

It is straight-forward to check that  $\beta \circ \gamma_3 = \text{Id}_{\mathbb{V}_n}$ , where  $\gamma_3$  and  $\beta$  are constructed in (3.30) and (3.31) respectively. This produces an  $\text{SL}(\mathbb{V})$ -equivariant holomorphic splitting

$$J^k(L^n) = \text{image}(\gamma_3) \oplus \text{kernel}(\beta)$$

of the short exact sequence in (3.30).  $\square$

Lemma 3.3 gives a holomorphic decomposition

$$J^k(L^n) = \mathbb{V}_n \oplus (\mathbb{L}^{k+1} \otimes R^1 p_{1*}(p_2^* L^{n-k-1})) \quad (3.32)$$

which is compatible with the actions of  $\mathrm{SL}(\mathbb{V})$ . Now,  $R^1 p_{1*}(p_2^* L^{n-k-1}) = \mathbb{V}_{k-n-1}^*$  (see (3.15)), and  $\mathbb{V}_{k-n-1}^* = \mathbb{V}_{k-n-1}$  which is constructed using the isomorphism in (3.5). Therefore, the decomposition in (3.32) can be reformulated as

$$J^k(L^n) = \mathbb{V}_n \oplus (\mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}) = \mathbb{V}_n \oplus (L^{-(k+1)} \otimes \mathbb{V}_{k-n-1}) \quad (3.33)$$

using  $\Psi$  in (3.6). As before, the isomorphisms in (3.33) are compatible with the actions of  $\mathrm{SL}(\mathbb{V})$ .

If  $k-1 > n \geq 0$ , then from (3.33) we know that

$$J^{k-1}(L^n) = \mathbb{V}_n \oplus (\mathbb{L}^k \otimes \mathbb{V}_{k-n-2}) = \mathbb{V}_n \oplus (L^{-k} \otimes \mathbb{V}_{k-n-2}),$$

and if  $k-1 = n \geq 0$ , then from (3.26) we have

$$J^{k-1}(L^n) = \mathbb{V}_{k-1}.$$

The proof of the following proposition is similar to the proofs of Proposition 3.1 and Proposition 3.2.

**Proposition 3.4.** *If  $k-1 > n \geq 0$ , then the projection  $J^k(L^n) \rightarrow J^{k-1}(L^n)$  in (2.5) is the homomorphism*

$$\mathbb{V}_n \oplus (L^{-(k+1)} \otimes \mathbb{V}_{k-n-1}) \rightarrow \mathbb{V}_n \oplus (L^{-k} \otimes \mathbb{V}_{k-n-2})$$

which is the direct sum of the identity map of  $\mathbb{V}_n$  and the natural contraction map  $L^{-(k+1)} \otimes \mathbb{V}_{k-n-1} \rightarrow L^{-k} \otimes \mathbb{V}_{k-n-2}$ .

If  $k-1 = n \geq 0$ , then the projection  $J^k(L^n) \rightarrow J^{k-1}(L^n)$  in (2.5) is the homomorphism

$$\mathbb{V}_n \oplus (L^{-(k+1)} \otimes \mathbb{V}_{k-n-1}) \rightarrow \mathbb{V}_n$$

which is the direct sum of the identity map of  $\mathbb{V}_n$  and the zero homomorphism of  $L^{-(k+1)} \otimes \mathbb{V}_{k-n-1} = L^{-(k+1)}$  (recall that  $\mathbb{V}_0 = \mathcal{O}_X$ ).

**3.4. The jet bundle.** Combining (3.20), (3.26) and (3.33), we have the following:

**Theorem 3.5.** *If  $n < 0$  or  $n \geq k$ , then*

$$J^k(L^n) = L^{n-k} \otimes \mathbb{V}_k.$$

If  $k > n \geq 0$ , then

$$J^k(L^n) = \mathbb{V}_n \oplus (L^{-(k+1)} \otimes \mathbb{V}_{k-n-1}).$$

Both the isomorphisms are  $\mathrm{SL}(\mathbb{V})$ -equivariant.

#### 4. HOLOMORPHIC CONNECTIONS ON VECTOR BUNDLES

In this section we construct and study natural differential holomorphic operators between natural holomorphic vector bundles on a Riemann surface equipped with a projective structure.

**4.1. Jets of Riemann surfaces with a projective structure.** Let  $M$  be a compact connected Riemann surface of genus  $g$ . A holomorphic coordinate chart on  $M$  is a pair of the form  $(U, \varphi)$ , where  $U \subset M$  is an open subset and  $\varphi : U \rightarrow \mathbb{P}(\mathbb{V}) = X$  is a holomorphic embedding. A holomorphic coordinate atlas on  $M$  is a collection of coordinate charts  $\{(U_i, \varphi_i)\}_{i \in I}$  such that  $M = \bigcup_{i \in I} U_i$ . A projective structure on  $M$  is given by a holomorphic coordinate atlas  $\{(U_i, \varphi_i)\}_{i \in I}$  such that for all pairs  $(i, j) \in I \times I$  for which  $U_i \cap U_j \neq \emptyset$ , there exists an element  $\tau_{j,i} \in \text{Aut}(\mathbb{P}(\mathbb{V})) = \text{PGL}(\mathbb{V})$  such that  $\varphi_j \circ \varphi_i^{-1}$  is the restriction of  $\tau_{j,i}$  to  $\varphi_i(U_i \cap U_j)$ . A holomorphic coordinate atlas satisfying this condition is called a projective atlas. Two such projective atlases  $\{(U_i, \varphi_i)\}_{i \in I}$  and  $\{(U_i, \varphi_i)\}_{i \in J}$  are called equivalent if their union  $\{(U_i, \varphi_i)\}_{i \in I \cup J}$  is again a projective atlas. A *projective structure* on  $M$  is an equivalence class of projective atlases (see [Gu]).

Note that the collection  $\{\tau_{j,i}\}$  in the definition of a projective atlas forms a 1-cocycle with values in  $\text{PGL}(\mathbb{V})$ . The corresponding cohomology class in  $H^1(M, \text{PGL}(\mathbb{V}))$  depends only on the projective structure and can be lifted to  $H^1(M, \text{SL}(\mathbb{V}))$ . Moreover the space of such lifts is in bijection with the theta characteristics of  $M$  (see [Gu]). We recall that the theta characteristic on  $M$  is a holomorphic line bundle  $K_M^{1/2}$  on  $M$  together with a holomorphic isomorphism of  $K_M^{1/2} \otimes K_M^{1/2}$  with the holomorphic cotangent bundle  $K_M$ .

We fix a theta characteristic on  $M$ . So a projective structure on  $M$  is given by a coordinate atlas  $\{(U_i, \varphi_i)\}_{i \in I}$  together with an element

$$\tau_{j,i} \in \text{SL}(\mathbb{V})$$

for every ordered pair  $(i, j) \in I \times I$  such that  $U_i \cap U_j \neq \emptyset$  such that the following four conditions hold:

- (1)  $\varphi_j \circ \varphi_i^{-1}$  is the restriction to  $\varphi_i(U_i \cap U_j)$  of the automorphism of  $X$  given by  $\tau_{j,i}$ ,
- (2)  $\tau_{i,k} \tau_{k,j} \tau_{j,i} = \text{Id}_{\mathbb{V}}$  for all  $i, j, k \in I$  such that  $U_i \cap U_j \cap U_k \neq \emptyset$ ,
- (3)  $\tau_{i,i} = \text{Id}_{\mathbb{V}}$  for all  $i \in I$ , and
- (4) the theta characteristic corresponding to the data is the given one.

Two such collections of triples  $\{ \{(U_i, \varphi_i)\}_{i \in I}, \{\tau_{i,j}\} \}$  and  $\{ \{(U_i, \varphi_i)\}_{i \in J}, \{\tau_{i,j}\} \}$  are called equivalent if their union  $\{ \{(U_i, \varphi_i)\}_{i \in I \cup J}, \{\tau_{i,j}\} \}$  is again a part of a collection of triples satisfying the above conditions. A  $\text{SL}(\mathbb{V})$ -*projective structure* on  $M$  is an equivalence class of collection of triples satisfying the above conditions.

Let  $P = \{ \{(U_i, \varphi_i)\}_{i \in I}, \{\tau_{i,j}\} \}$  be a  $\text{SL}(\mathbb{V})$ -projective structure on  $M$ . Recall that the collection  $\{\tau_{i,j}\}$  forms a 1-cocycle on  $M$  with values in the group  $\text{SL}(\mathbb{V})$ . Hence  $\{\tau_{i,j}\}$  produce a holomorphic rank two bundle  $\mathcal{V}$  on  $M$  equipped with a holomorphic connection  $D$  (see [At2] for holomorphic connections). To construct the pair  $(\mathcal{V}, D)$  explicitly, for each  $i \in I$ , consider the trivial holomorphic vector bundle  $U_i \times \mathbb{V} \rightarrow U_i$  equipped with the trivial connection. For any ordered pair  $(i, j) \in I \times I$  with  $U_i \cap U_j \neq \emptyset$ , glue  $U_i \times \mathbb{V}$  and  $U_j \times \mathbb{V}$  over  $U_i \cap U_j$  using the automorphism of  $(U_i \cap U_j) \times \mathbb{V}$  that sends any

$$(y, v) \in (U_i \cap U_j) \times \mathbb{V} \subset U_j \times \mathbb{V}$$

to  $(y, \tau_{i,j}(v)) \in (U_i \cap U_j) \times \mathbb{V} \subset U_i \times \mathbb{V}$ . This automorphism of  $(U_i \cap U_j) \times \mathbb{V}$  is holomorphic, and it evidently preserves the trivial connection. Therefore, this gluing operation produces a holomorphic vector bundle on  $M$ , which we shall denote by  $\mathcal{V}$ , and a holomorphic connection on  $\mathcal{V}$ , which we shall denote by  $D$ .

The diagonal action of  $\mathrm{SL}(\mathbb{V})$  on  $X \times \mathbb{V}$  preserves the subbundle  $\mathbb{L}$  in (3.7). Therefore, the subbundles  $\varphi_i(U_i) \times \mathbb{L} \subset \varphi(U_i) \times \mathbb{V}$  patch together compatibly to produce a holomorphic line subbundle of the vector bundle  $\mathcal{V}$  on  $M$  constructed above. This line subbundle of  $\mathcal{V}$  will be denoted by  $\mathcal{L}$ . Let

$$\mathbf{L} := \mathcal{V}/\mathcal{L} \quad (4.1)$$

be the quotient bundle.

Recall that the isomorphism  $\Psi$  in (3.6) and the isomorphism  $\Phi_0$  in (3.10) are both  $\mathrm{SL}(\mathbb{V})$ -equivariant. However, it should be clarified that a nonzero element  $\omega \in \bigwedge^2 \mathbb{V}$  (see (3.4)) was used in their construction. Since  $\Psi$  and  $\Phi_0$  are  $\mathrm{SL}(\mathbb{V})$ -equivariant, they produce isomorphisms

$$\mathbf{L} = \mathcal{L}^* \quad \text{and} \quad \mathbf{L}^{\otimes 2} = (\mathcal{L}^*)^{\otimes 2} = TM \quad (4.2)$$

respectively. Recall that we fixed a theta characteristic on  $M$ . The fourth condition in the definition of a projective structure says that the theta characteristic  $\mathcal{L}$  coincides with the chosen one.

For any  $j \geq 1$ , the vector bundle  $\mathrm{Sym}^j(\mathcal{V})$  will be denoted by  $\mathcal{V}_j$ . Also,  $\mathcal{V}_0$  will denote the trivial line bundle  $\mathcal{O}_M$ . The holomorphic connection  $D$  on  $\mathcal{V}$  induces a holomorphic connection on  $\mathcal{V}_j$  for every  $j \geq 1$ ; this connection on  $\mathcal{V}_j$  will be denoted by  $D_j$ . The trivial connection on  $\mathcal{V}_0 = \mathcal{O}_M$  will be denoted by  $D_0$ . The connections  $D_m$  and  $D_n$  on  $\mathcal{V}_m$  and  $\mathcal{V}_n$  respectively together produce a holomorphic connection on  $\mathcal{V}_m \otimes \mathcal{V}_n$ .

**Proposition 4.1.** *Let  $M$  be a compact connected Riemann surface equipped with a projective structure  $P$ . Take integers  $m > n \geq 0$ . Then  $\mathcal{V}_m \otimes \mathcal{V}_n$  has a canonical decomposition*

$$\mathcal{V}_m \otimes \mathcal{V}_n = \bigoplus_{i=0}^n \mathcal{V}_{m+n-2i}.$$

The above isomorphism takes the connection on  $\mathcal{V}_m \otimes \mathcal{V}_n$  induced by  $D_m$  and  $D_n$  to the connection  $\bigoplus_{i=0}^n D_{m+n-2i}$  on  $\bigoplus_{i=0}^n \mathcal{V}_{m+n-2i}$ .

*Proof.* First consider the case of  $X = \mathbb{P}(\mathbb{V})$  investigated in Section 2. The  $\mathrm{SL}(\mathbb{V})$ -module  $\mathrm{Sym}^j(\mathbb{V})$  is irreducible for every  $j \geq 0$  [FH, p. 150], and furthermore, for  $m > n \geq 0$ , the  $\mathrm{SL}(\mathbb{V})$ -module  $\mathrm{Sym}^m(\mathbb{V}) \otimes \mathrm{Sym}^n(\mathbb{V})$  decomposes as

$$\mathrm{Sym}^m(\mathbb{V}) \otimes \mathrm{Sym}^n(\mathbb{V}) = \bigoplus_{i=0}^n \mathrm{Sym}^{m+n-2i}(\mathbb{V}) \quad (4.3)$$

[FH, p. 151, Ex. 11.11].

Choose data  $\{\{(U_i, \varphi_i)\}_{i \in I}, \{\tau_{i,j}\}\}$  representing a  $\mathrm{SL}(\mathbb{V})$ -projective structure associated to  $P$  and to a fixed theta characteristic on  $M$ . On each  $U_i$  we have the holomorphic vector bundles  $\varphi_i^* \mathbb{V}_j = U_i \times \mathrm{Sym}^j(\mathbb{V})$  equipped with the trivial connection. Patching these together using  $\{\tau_{i,j}\}$  as the transition functions we get  $\mathcal{V}_j$  equipped with the connection  $D_j$  for every  $j \geq 0$ . Now it can be shown that the isomorphism in (4.3) over each  $U_i$  patch together compatibly to give a global isomorphism over  $M$ . Indeed, this is an immediate consequence of the fact that the isomorphism in (4.3) intertwines the actions of  $\mathrm{SL}(\mathbb{V})$ . This completes the proof.  $\square$

Henceforth we assume that  $\mathrm{genus}(M) = g \geq 1$ . Since all the isomorphisms in Theorem 3.5 are  $\mathrm{SL}(\mathbb{V})$ -equivariant, the following corollary is deduced from Theorem 3.5.

**Corollary 4.2.** *Let  $M$  be a connected Riemann surface equipped with a projective structure  $P$ . If  $n < 0$  or  $n \geq k$ , then there is a canonical isomorphism*

$$J^k(\mathbf{L}^n) = \mathbf{L}^{n-k} \otimes \mathcal{V}_k.$$

*If  $k > n \geq 0$ , then there is a canonical isomorphism*

$$J^k(\mathbf{L}^n) = \mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1}).$$

*Remark 4.3.* Note that Proposition 4.1, Corollary 4.2 and (4.2) together give a description of any jet bundles of the form  $J^k(\mathbf{L}^a) \otimes J^\ell(\mathbf{L}^b)$  as a direct sum of vector bundles of the form  $\mathcal{V}^p \otimes \mathbf{L}^q$ .

**4.2. Jets of vector bundles with a holomorphic connection.** Let  $M$  be a compact connected Riemann surface. Let  $E$  be a holomorphic vector bundle on  $M$  equipped with a holomorphic connection  $D_E$ . Take a holomorphic vector bundle  $W$  on  $M$ . We will construct a holomorphic homomorphism from  $J^k(E \otimes W)$  to  $E \otimes J^k(W)$  for all  $k \geq 0$ . To construct this homomorphism, let  $f_1, f_2 : M \times M \rightarrow M$  be the projections to the first and second factors respectively. The diagonal divisor

$$\{(x, x) \mid x \in M\} \subset M \times M$$

will be denoted by  $\Delta_M$ . Recall from (2.4) that

$$\begin{aligned} J^k(E \otimes W) &:= f_{1*}((f_2^*(E \otimes W))/((f_2^*(E \otimes W)) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))) \\ &= f_{1*}(((f_2^*E) \otimes (f_2^*W))/((f_2^*E) \otimes (f_2^*W) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))) \rightarrow M. \end{aligned} \quad (4.4)$$

Consider the pulled back connection  $f_2^*D_E$  on  $f_2^*E$ . Since the sheaf of holomorphic two-forms on  $M$  is the zero sheaf, the connection  $D_E$  is flat (the curvature vanishes identically). Hence the connection  $f_2^*D_E$  is also flat. Therefore, on any analytic neighborhood  $U$  of  $\Delta_M \subset M \times M$  that admits a deformation retraction to  $\Delta_M$ , the two vector bundles  $(p_1^*E)|_U$  and  $(p_2^*E)|_U$  are identified using parallel translation. Invoking this isomorphism between  $(p_1^*E)|_U$  and  $(p_2^*E)|_U$ , from (4.4) we have

$$J^k(E \otimes W) = f_{1*}(((f_1^*E) \otimes (f_2^*W))/((f_1^*E) \otimes (f_2^*W) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))).$$

Hence the projection formula gives that

$$J^k(E \otimes W) = E \otimes f_{1*}((f_2^*W)/((f_2^*W) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))) = E \otimes J^k(W). \quad (4.5)$$

**Lemma 4.4.** *Let  $M$  be a connected Riemann surface equipped with a projective structure  $P$ . Let  $E$  be a holomorphic vector bundle on  $M$  equipped with a holomorphic connection  $D_E$ . If  $n < 0$  or  $n \geq k$ , then there is a canonical isomorphism*

$$J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k.$$

*If  $k > n \geq 0$ , then there is a canonical isomorphism*

$$J^k(E \otimes \mathbf{L}^n) = E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1})).$$

*Proof.* This follows from (4.5) and Corollary 4.2. We omit the details.  $\square$

**4.3. The symbol map.** As before,  $M$  is a compact connected Riemann surface equipped with a projective structure  $P$ , and  $E$  is a holomorphic vector bundle over  $M$  equipped with a holomorphic connection  $D_E$ . Consider the projection

$$\phi_k : J^k(E \otimes \mathbf{L}^n) \longrightarrow J^{k-1}(E \otimes \mathbf{L}^n) \quad (4.6)$$

in (2.5). We will describe  $\phi_k$  in terms of the isomorphisms in Lemma 4.4.

**4.3.1. Case where either  $n < 0$  or  $n \geq k$ .** Assume that at least one of the following two conditions is valid:

- (1)  $n < 0$ , or
- (2)  $n \geq k$ .

In view of Lemma 4.4, this implies that

$$J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k \quad \text{and} \quad J^{k-1}(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k+1} \otimes \mathcal{V}_{k-1} \quad (4.7)$$

when  $k \geq 1$  (note that in (4.6) we have  $k \geq 1$ ). Using (4.7), the projection  $\phi_k$  in (4.6) is a surjective homomorphism

$$\phi_k : E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k \longrightarrow E \otimes \mathbf{L}^{n-k+1} \otimes \mathcal{V}_{k-1}. \quad (4.8)$$

Recall that  $\mathcal{V}_j = \text{Sym}^j(\mathcal{V})$ , and the line bundle  $\mathbf{L}$  is a quotient of  $\mathcal{V}$  (see (4.1)). Therefore, we have a natural projection

$$\varpi_k : \mathcal{V}_k \longrightarrow \mathbf{L} \otimes \mathcal{V}_{k-1}. \quad (4.9)$$

To describe  $\varpi_k$  explicitly, let  $q_{\mathbf{L}} : \mathcal{V} \longrightarrow \mathbf{L}$  be the quotient map. Consider the homomorphism

$$q_{\mathbf{L}} \otimes \text{Id}_{\mathcal{V}^{\otimes(k-1)}} : \mathcal{V}^{\otimes k} \longrightarrow \mathbf{L} \otimes \mathcal{V}^{\otimes(k-1)}.$$

Note that  $q_{\mathbf{L}} \otimes \text{Id}_{\mathcal{V}^{\otimes(k-1)}}$  sends the subbundle  $\text{Sym}^k(\mathcal{V}) \subset \mathcal{V}^{\otimes k}$  to the subbundle  $\mathbf{L} \otimes \mathcal{V}_{k-1} \subset \mathbf{L} \otimes \mathcal{V}^{\otimes(k-1)}$ . The homomorphism  $\varpi_k$  in (4.9) is this restriction of the homomorphism  $q_{\mathbf{L}} \otimes \text{Id}_{\mathcal{V}^{\otimes(k-1)}}$ .

**Proposition 4.5.** *The homomorphism  $\phi_k$  in (4.8) coincides with  $\text{Id} \otimes \varpi_k$ , where  $\text{Id}$  denotes the identity map of  $E \otimes \mathbf{L}^{n-k}$ , and  $\varpi_k$  is the homomorphism in (4.9).*

*Proof.* This follows from Proposition 3.1 and Proposition 3.2. □

Consider the inclusion homomorphism  $\iota_{\mathcal{L}} : \mathcal{L} \hookrightarrow \mathcal{V}$  (see (4.1)). We have the subbundle

$$\iota' := \text{Id}_E \otimes \iota_{\mathcal{L}}^{\otimes k} : E \otimes \mathcal{L}^{\otimes k} \hookrightarrow E \otimes \text{Sym}^k(\mathcal{V}) = E \otimes \mathcal{V}_k.$$

Using the isomorphism  $\mathbf{L} = \mathcal{L}^*$  in (4.2), the above homomorphism  $\iota'$  produces a homomorphism

$$\tilde{\iota} := \iota' \otimes \text{Id}_{\mathbf{L}^{n-k}} : E \otimes \mathbf{L}^{n-2k} = E \otimes \mathcal{L}^{\otimes k} \otimes \mathbf{L}^{n-k} \hookrightarrow E \otimes \mathcal{V}_k \otimes \mathbf{L}^{n-k}. \quad (4.10)$$

The following is an immediate consequence of Proposition 4.5.

**Corollary 4.6.** *The kernel of the surjective homomorphism  $\phi_k$  in (4.8) coincides with the subbundle  $E \otimes \mathbf{L}^{n-2k} \hookrightarrow E \otimes \mathcal{V}_k \otimes \mathbf{L}^{n-k}$  given by the image of the homomorphism  $\tilde{\iota}$  in (4.10).*

Let  $F$  be any holomorphic vector bundle on  $M$ . Using (2.6) and (4.7), we have

$$\mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F) = F \otimes (E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k)^* = F \otimes \mathbf{L}^{k-n} \otimes (E \otimes \mathcal{V}_k)^*. \quad (4.11)$$

Using the homomorphism  $\tilde{\iota}$  in (4.10) construct the homomorphism

$$\mathrm{Id}_F \otimes \tilde{\iota}^* : F \otimes (E \otimes \mathcal{V}_k \otimes \mathbf{L}^{n-k})^* \longrightarrow F \otimes (E \otimes \mathbf{L}^{n-2k})^* = \mathrm{Hom}(E, F) \otimes \mathbf{L}^{2k-n}. \quad (4.12)$$

Combining (4.11) and (4.12), we have a surjective homomorphism

$$\sigma_k : \mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F) \longrightarrow \mathrm{Hom}(E, F) \otimes \mathbf{L}^{2k-n}.$$

The following is an immediate consequence of Corollary 4.6.

**Corollary 4.7.** *The symbol homomorphism*

$$\mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F) \longrightarrow \mathrm{Hom}(E \otimes \mathbf{L}^n, F) \otimes (TM)^{\otimes k}$$

in (2.7) coincides with the above homomorphism  $\sigma_k$  after  $TM$  is identified with  $\mathbf{L}^{\otimes 2}$  as in (4.2).

4.3.2. *Case where  $k > n \geq 0$ .* We first assume that  $k - 1 > n \geq 0$ . Therefore, from Lemma 4.4 we have  $J^k(E \otimes \mathbf{L}^n) = E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1}))$  and

$$J^{k-1}(E \otimes \mathbf{L}^n) = E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-k} \otimes \mathcal{V}_{k-n-2})).$$

Let

$$\begin{aligned} \phi_k : E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1})) &= J^k(E \otimes \mathbf{L}^n) \\ \longrightarrow J^{k-1}(E \otimes \mathbf{L}^n) &= E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-k} \otimes \mathcal{V}_{k-n-2})) \end{aligned} \quad (4.13)$$

be the projection in (2.5). Just as the homomorphism  $\mathrm{Id} \otimes \varpi_k$  in Proposition 4.5, we have a homomorphism

$$\widehat{\varpi}_k : E \otimes \mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1} \longrightarrow E \otimes \mathbf{L}^{-k} \otimes \mathcal{V}_{k-n-2}.$$

**Lemma 4.8.** *The homomorphism  $\phi_k$  in (4.13) coincides with the homomorphism*

$$\mathrm{Id}_{E \otimes \mathcal{V}_n} \oplus \widehat{\varpi}_k : E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1})) \longrightarrow E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-k} \otimes \mathcal{V}_{k-n-2})).$$

*Proof.* This follows from Proposition 3.4. □

Note that using (4.2),

$$\mathrm{kernel}(\widehat{\varpi}_k) = E \otimes \mathbf{L}^{-(k+1)} \otimes \mathcal{L}^{\otimes(k-n-1)} = E \otimes K_M^{\otimes k} \mathbf{L}^n.$$

The following is an immediate consequence of Lemma 4.8.

**Corollary 4.9.** *Let  $F$  be any holomorphic vector bundle on  $M$ . The symbol homomorphism*

$$\mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F) \longrightarrow \mathrm{Hom}(E \otimes \mathbf{L}^n, F) \otimes (TM)^{\otimes k}$$

coincides with  $\mathrm{Id}_F \otimes \iota^*$ , where  $\iota : \mathrm{kernel}(\widehat{\varpi}_k) \hookrightarrow E \otimes \mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1}$  is the inclusion map.

We now assume that  $k - 1 = n \geq 0$ .

Under this assumption, from Lemma 4.4 we have

$$J^k(E \otimes \mathbf{L}^n) = E \otimes (\mathcal{V}_n \oplus \mathbf{L}^{-(k+1)}) \quad \text{and} \quad J^{k-1}(E \otimes \mathbf{L}^n) = E \otimes \mathcal{V}_n.$$

Let

$$\phi_k : E \otimes (\mathcal{V}_n \oplus \mathbf{L}^{-(k+1)}) = J^k(E \otimes \mathbf{L}^n) \longrightarrow J^{k-1}(E \otimes \mathbf{L}^n) = E \otimes \mathcal{V}_n \quad (4.14)$$

be the projection in (2.5).

**Lemma 4.10.** *The homomorphism  $\phi_k$  in (4.14) coincides with the projection*

$$E \otimes (\mathcal{V}_n \oplus \mathbf{L}^{-(k+1)}) \longrightarrow E \otimes \mathcal{V}_n$$

*to the first factor.*

*Proof.* This follows from Proposition 3.4. □

The following is an immediate consequence of Lemma 4.10.

**Corollary 4.11.** *Let  $F$  be any holomorphic vector bundle on  $M$ . The symbol homomorphism*

$$\text{Diff}_M^k(E \otimes \mathbf{L}^n, F) \longrightarrow \text{Hom}(E \otimes \mathbf{L}^n, F) \otimes (TM)^{\otimes k}$$

*coincides with  $\text{Id}_F \otimes p'$ , where*

$$p' : E^* \otimes (\mathcal{V}_n \oplus \mathbf{L}^{-(k+1)})^* \longrightarrow E^* \otimes (\mathbf{L}^{-(k+1)})^*$$

*is the natural projection.*

## 5. LIFTING OF SYMBOL

As before,  $M$  is a connected Riemann surface equipped with a projective structure  $P$ , and  $E$  is a holomorphic vector bundle on  $M$  equipped with a holomorphic connection  $D_E$ . The connection  $D_E$  on  $E$  induces a holomorphic connection on  $\text{End}(E)$ . This induced connection on  $\text{End}(E)$  will be denoted by  $\mathcal{D}_E$ .

Take integers  $k, n$  and  $l$  such that at least one of the following two conditions is valid:

- (1)  $n < 0$ , or
- (2)  $n, l \geq k$ .

Let

$$\theta_0 \in H^0(M, \text{End}(E) \otimes \mathbf{L}^l) \quad (5.1)$$

be a holomorphic section. Note that  $\theta_0$  defines a section

$$\theta \in H^0(M, J^k(\text{End}(E) \otimes \mathbf{L}^l)).$$

Recall that  $D_E$  induces a holomorphic connection  $\mathcal{D}_E$  on  $\text{End}(E)$ . In view of this connection, Lemma 4.4 says that  $J^k(\text{End}(E) \otimes \mathbf{L}^l) = \text{End}(E) \otimes \mathbf{L}^{l-k} \otimes \mathcal{V}_k$ . So, we have

$$\theta \in H^0(M, \text{End}(E) \otimes \mathbf{L}^{l-k} \otimes \mathcal{V}_k). \quad (5.2)$$

Recall from Lemma 4.4 that  $J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k$ . Let

$$\mathcal{T}_\theta^0 : J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k \longrightarrow E \otimes \mathbf{L}^{l+n-2k} \otimes \mathcal{V}_k \otimes \mathcal{V}_k$$

be the homomorphism defined by  $a \otimes b \mapsto \sum_i (a'_i(a) \otimes b \otimes b'_i)$ , where  $a$  and  $b$  are local sections of  $E$  and  $\mathbf{L}^{n-k} \otimes \mathcal{V}_k$  respectively, while  $\theta$  in (5.2) is locally expressed as  $\sum_i a'_i \otimes b'_i$  with  $a'_i$  and  $b'_i$  being local sections of  $\text{End}(E)$  and  $\mathbf{L}^{l-k} \otimes \mathcal{V}_k$  respectively. Let

$$p_0 : \mathcal{V}_k \otimes \mathcal{V}_k \longrightarrow \mathcal{V}_0 = \mathcal{O}_M$$

be the projection constructed using the decomposition in Proposition 4.1. Now define the homomorphism

$$\mathcal{T}_\theta := (\text{Id} \otimes p_0) \circ \mathcal{T}_\theta^0 : J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k \longrightarrow E \otimes \mathbf{L}^{l+n-2k}, \quad (5.3)$$

where  $\text{Id}$  denotes the identity map of  $E \otimes \mathbf{L}^{l+n-2k}$ . Therefore, we have

$$\mathcal{T}_\theta \in H^0(M, \text{Diff}^k(E \otimes \mathbf{L}^n, E \otimes \mathbf{L}^{l+n-2k})) \quad (5.4)$$

(see (2.6)). Let

$$\hat{\sigma} : \text{Diff}^k(E \otimes \mathbf{L}^n, E \otimes \mathbf{L}^{l+n-2k}) \longrightarrow \text{End}(E) \otimes \mathbf{L}^{l-2k} \otimes (TM)^{\otimes k} = \text{End}(E) \otimes \mathbf{L}^l$$

be the symbol homomorphism in (2.7); in (4.2) it was shown that  $TM = \mathbf{L}^{\otimes 2}$ .

**Theorem 5.1.** *The symbol  $\hat{\sigma}(\mathcal{T}_\theta)$  of the differential operator  $\mathcal{T}_\theta$  in (5.4) is the section  $\theta_0$  in (5.1).*

*Proof.* This theorem can be derived from Corollary 4.7 that describes the symbol. The following is needed to derive the theorem from Corollary 4.7.

Consider the decomposition of the  $\text{SL}(\mathbb{V})$ -module

$$\text{Sym}^m(\mathbb{V}) \otimes \text{Sym}^m(\mathbb{V}) = \bigoplus_{i=0}^m \text{Sym}^{2(m-i)}(\mathbb{V})$$

in (4.3). Let

$$f : \text{Sym}^m(\mathbb{V}) \otimes \text{Sym}^m(\mathbb{V}) \longrightarrow \text{Sym}^0(\mathbb{V}) = \mathbb{C}$$

be the projection constructed using this decomposition (by setting  $i = m$  in the decomposition). Then for any line  $L_0 \subset \mathbb{V}$ , we have

$$f((L_0^{\otimes m}) \otimes (L_0 \otimes \text{Sym}^{m-1}(\mathbb{V}))) = 0,$$

so  $f$  produces a homomorphism

$$\hat{f} : (L_0^{\otimes m}) \otimes ((\text{Sym}^m(\mathbb{V}))/((L_0 \otimes \text{Sym}^{m-1}(\mathbb{V})))) = (L_0^{\otimes m}) \otimes (\mathbb{V}/L_0)^{\otimes m} \longrightarrow \mathbb{C}.$$

This homomorphism  $\hat{f}$  sends  $v^{\otimes m} \otimes u^{\otimes m}$ , where  $v \in L_0$  and  $u \in \mathbb{V}/L_0$ , to  $\lambda^m \in \mathbb{C}$  that satisfies the equation

$$v \wedge u = \lambda \cdot \omega,$$

where  $\omega$  is the fixed element in (3.4); note that  $v \wedge u$  is a well-defined element of  $\bigwedge^2 \mathbb{V}$ . The theorem follows from this observation and Corollary 4.7.  $\square$

*Remark 5.2.* It may be mentioned that symbols do not always lift to a holomorphic differential operator. For such an example, set the symbol to be the constant function 1. It does not lift to a first order holomorphic differential operator from  $\mathcal{L}^n$  to  $\mathcal{L}^n \otimes K_M$  if  $g \geq 2$  and  $n \neq 0$ . Indeed, a first order holomorphic differential operator from  $\mathcal{L}^n$  to  $\mathcal{L}^n \otimes K_M$  with symbol 1 is a holomorphic connection on  $\mathcal{L}^n$ . But  $\mathcal{L}^n$  does not admit a holomorphic connection because its degree is nonzero.

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SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI 400005, INDIA

*Email address:* `indranil@math.tifr.res.in`

UNIVERSITÉ CÔTE D’AZUR, CNRS, LJAD, FRANCE

*Email address:* `dumitres@unice.fr`