



**HAL**  
open science

# PROJECTIVE STRUCTURE ON RIEMANN SURFACE AND NATURAL DIFFERENTIAL OPERATORS

Indranil Biswas, Sorin Dumitrescu

► **To cite this version:**

Indranil Biswas, Sorin Dumitrescu. PROJECTIVE STRUCTURE ON RIEMANN SURFACE AND NATURAL DIFFERENTIAL OPERATORS. *Differential Geometry and its Applications*, 2020, 72. hal-02387407

**HAL Id: hal-02387407**

**<https://hal.science/hal-02387407>**

Submitted on 29 Nov 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# PROJECTIVE STRUCTURE ON RIEMANN SURFACE AND NATURAL DIFFERENTIAL OPERATORS

INDRANIL BISWAS AND SORIN DUMITRESCU

ABSTRACT. We investigate the holomorphic differential operators on a Riemann surface  $M$ . This is done by endowing  $M$  with a projective structure. Let  $\mathcal{L}$  be a theta characteristic on  $M$ . We explicitly describe the jet bundle  $J^k(E \otimes \mathcal{L}^{\otimes n})$ , where  $E$  is a holomorphic vector bundle on  $M$  equipped with a holomorphic connection, for all  $k$  and  $n$ . This provides a description of holomorphic differential operators from  $E \otimes \mathcal{L}^{\otimes n}$  to another holomorphic vector bundle  $F$  using the natural isomorphism  $\text{Diff}^k(E \otimes \mathcal{L}^{\otimes n}, F) = F \otimes (J^k(E \otimes \mathcal{L}^{\otimes n}))^*$ .

## 1. INTRODUCTION

The study of natural differential operators between natural bundles is a major topic in global analysis on manifolds (see, for instance, [Ni], [Te] and [KMS]).

Our aim here is to study natural holomorphic differential operators between natural holomorphic vector bundles on a Riemann surface  $M$ . To be able to formulate their description, we make use of the existence of compatible projective structures on  $M$  (see [Gu]).

We recall that a projective structure on  $M$  is given by a holomorphic coordinate atlas such that all the transition functions are restrictions of Möbius transformations. This approach to holomorphic differential operators on  $M$  entails a detailed understanding of the action of  $\text{SL}(2, \mathbb{C})$  on the jet bundles on  $\mathbb{C}\mathbb{P}^1$ .

Let  $\mathbb{V}$  be a complex vector space of dimension two. For notational convenience, denote by  $X$  the space  $\mathbb{P}(\mathbb{V})$  of complex lines in  $\mathbb{V}$ . Let  $L$  be the tautological line bundle of degree one on  $X$ . The trivial holomorphic vector bundle  $X \times \mathbb{V}$  on  $X$  with fiber  $\mathbb{V}$  will be denoted by  $\mathbb{V}_1$ , while the symmetric product  $\text{Sym}^j(\mathbb{V}_1)$  will be denoted by  $\mathbb{V}_j$ . The holomorphically trivial line bundle  $X \times \mathbb{C}$  on  $X$  will be denoted by  $\mathbb{V}_0$ .

We prove the following (see Theorem 3.5):

**Theorem 1.1.** *If  $n < 0$  or  $n \geq k$ , then*

$$J^k(L^n) = L^{n-k} \otimes \mathbb{V}_k.$$

*If  $k > n \geq 0$ , then*

$$J^k(L^n) = \mathbb{V}_n \oplus (L^{-(k+1)} \otimes \mathbb{V}_{k-n-1}).$$

*Both the isomorphisms are  $\text{SL}(\mathbb{V})$ -equivariant.*

---

2010 *Mathematics Subject Classification.* 14F10, 14H40.

*Key words and phrases.* Differential operator, projective structure, jet bundle, theta characteristic.

The natural forgetful homomorphism  $J^k(L^n) \longrightarrow J^{k-1}(L^n)$  of holomorphic vector bundles with action of  $\mathrm{SL}(\mathbb{V})$  is described in Proposition 3.1, Proposition 3.2 and Proposition 3.4.

Since all the above mentioned isomorphisms and homomorphisms are  $\mathrm{SL}(\mathbb{V})$ -equivariant, we are able to translate them into the set-up of a Riemann surface equipped with a projective structure (see Section 4 to see that any projective structure admits an underlying atlas for which the transition maps are restrictions of elements in  $\mathrm{SL}(\mathbb{V})$ ).

Let  $M$  be a compact Riemann surface equipped with a projective structure  $P$ . Using  $P$ , the above tautological line bundle  $L$  on  $X$  produces a holomorphic line bundle  $\mathbf{L}$  on  $M$  with the property that  $\mathbf{L}^{\otimes 2} = TM$ . Moreover, using  $P$ , the trivial vector bundle  $\mathbb{V}_1$  produces a holomorphic vector bundle  $\mathcal{V}_1$  of rank two over  $M$  that fits in the following short exact sequence of holomorphic vector bundles on  $M$ :

$$0 \longrightarrow \mathbf{L}^* \longrightarrow \mathcal{V}_1 \longrightarrow \mathbf{L} \longrightarrow 0.$$

This involves fixing a nonzero element of  $\bigwedge^2 \mathbb{V}$ . The symmetric power  $\mathrm{Sym}^j(\mathcal{V}_1)$  will be denoted by  $\mathcal{V}_j$ , while the trivial line bundle  $\mathcal{O}_M$  will be denoted by  $\mathcal{V}_0$ .

Let us now describe our results about natural differential holomorphic operators on a Riemann surface endowed with a projective structure.

The following is deduced using Theorem 1.1 (see Corollary 4.2):

**Corollary 1.2.** *If  $n < 0$  or  $n \geq k$ , then there is a canonical isomorphism*

$$J^k(\mathbf{L}^n) = \mathbf{L}^{n-k} \otimes \mathcal{V}_k.$$

*If  $k > n \geq 0$ , then there is a canonical isomorphism*

$$J^k(\mathbf{L}^n) = \mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1}).$$

This enables us to deduce the following result (see Lemma 4.4):

**Lemma 1.3.** *Let  $M$  be a compact connected Riemann surface endowed with a projective structure  $P$ . Consider  $E$  a holomorphic vector bundle on  $M$  equipped with a holomorphic connection.*

*If  $n < 0$  or  $n \geq k$ , then there is a canonical isomorphism*

$$J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k.$$

*If  $k > n \geq 0$ , then there is a canonical isomorphism*

$$J^k(E \otimes \mathbf{L}^n) = E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1})).$$

Let  $E$  be a holomorphic vector bundle on  $M$  equipped with a holomorphic connection, as in the previous Lemma. Given a holomorphic vector bundle  $F$  on  $M$ , denote by  $\mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F)$  the sheaf of holomorphic differential operators of order  $k$  from  $E \otimes \mathbf{L}^n$  to  $F$ .

The associated symbol homomorphism

$$\mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F) \longrightarrow \mathrm{Hom}(E \otimes \mathbf{L}^n, F) \otimes (TM)^{\otimes k}$$

is describe in Corollary 4.7 and Corollary 4.11.

Fix integers  $k$ ,  $n$  and  $l$  such that at least one of the following two conditions is valid:

- (1)  $n < 0$ , or
- (2)  $n, l \geq k$ .

Also fix a section

$$\theta_0 \in H^0(M, \text{End}(E) \otimes \mathbf{L}^l).$$

Using  $\theta_0$ , and a projective structure  $P$  on  $M$ , we construct a holomorphic differential operator

$$\mathcal{T}_\theta \in H^0(M, \text{Diff}^k(E \otimes \mathbf{L}^n, E \otimes \mathbf{L}^{l+n-2k})).$$

The following theorem shows that  $\mathcal{T}_\theta$  is a lift of the symbol  $\theta_0$  to a holomorphic differential operator (see Theorem 5.1):

**Theorem 1.4.** *The symbol of the above holomorphic differential operator  $\mathcal{T}_\theta$  is the section  $\theta_0$ .*

## 2. PRELIMINARIES

**2.1. Jet bundles and differential operators.** Let  $X$  be a connected Riemann surface. The holomorphic cotangent bundle of  $X$  will be denoted by  $K_X$ . Consider the Cartesian product  $X \times X$ . For  $i = 1, 2$ , let

$$p_i : X \times X \longrightarrow X$$

be the projection to the  $i$ -th factor. Let

$$\Delta := \{(x, x) \mid x \in X\} \subset X \times X \tag{2.1}$$

be the diagonal divisor. At times we shall identify  $\Delta$  with  $X$  using the mapping  $x \mapsto (x, x)$ . For any nonnegative integer  $k$ , consider the natural exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow \mathcal{O}_{X \times X}(-k\Delta) \\ &\longrightarrow \mathcal{O}_{X \times X}(-k\Delta)/\mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow 0. \end{aligned} \tag{2.2}$$

Using the Poincaré adjunction formula, we have

$$\mathcal{O}_{X \times X}(-k\Delta)/\mathcal{O}_{X \times X}(-(k+1)\Delta) = K_X^{\otimes k}; \tag{2.3}$$

here  $X$  is identified with  $\Delta$ . Let  $z$  be a holomorphic coordinate function on an open subset  $U \subset X$ . The isomorphism in (2.3) over  $(U \times U) \cap \Delta$  sends the section  $(z \circ p_2 - z \circ p_1)^k$  of  $\mathcal{O}_{X \times X}(-k\Delta)/\mathcal{O}_{X \times X}(-(k+1)\Delta)$  over  $(U \times U) \cap \Delta$  to the section  $(dz)^{\otimes k}$  of  $K_X^{\otimes k}|_U$  (after identifying  $U$  with  $(U \times U) \cap \Delta$ ).

Let  $W$  be a holomorphic vector bundle on  $X$  of rank  $r_W$ . For any nonnegative integer  $k$ , consider the subsheaf

$$(p_2^*W) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \subset p_2^*W.$$

The  $k$ -th order jet bundle  $J^k(W)$  is defined to be the direct image

$$J^k(W) := p_{1*}((p_2^*W)/((p_2^*W) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta))) \longrightarrow X, \tag{2.4}$$

which is a holomorphic vector bundle on  $X$  of rank  $(k+1)r_W$ .

For any  $k \geq 1$ , using the natural inclusions

$$(p_2^*W) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \subset (p_2^*W) \otimes \mathcal{O}_{X \times X}(-k\Delta) \subset p_2^*W$$

together with (2.3), we get a short exact sequence of holomorphic vector bundles on  $X$

$$0 \longrightarrow W \otimes K_X^{\otimes k} \xrightarrow{\iota} J^k(W) \xrightarrow{\phi_k} J^{k-1}(W) \longrightarrow 0. \quad (2.5)$$

Let  $V$  be a holomorphic vector bundle on  $X$ . Let  $\text{Diff}^k(W, V)$  denote the sheaf of holomorphic differential operators of order  $k$  from  $W$  to  $V$ . It is the sheaf of holomorphic sections of

$$\text{Diff}^k(W, V) := \text{Hom}(J^k(W), V) = V \otimes J^k(W)^*. \quad (2.6)$$

Consider the homomorphism

$$\text{Diff}^k(W, V) = V \otimes J^k(W)^* \xrightarrow{\text{Id}_V \otimes \iota^*} V \otimes (W \otimes K_X^{\otimes k})^* = (TX)^{\otimes k} \otimes \text{Hom}(W, V), \quad (2.7)$$

where  $\iota$  is the homomorphism in (2.5). This homomorphism in (2.7) is known as the *symbol map*.

**2.2. Line bundles on the projective line.** Let  $\mathbb{V}$  be a complex vector space of dimension two. Now set

$$X := \mathbb{P}(\mathbb{V})$$

to be the compact Riemann surface of genus zero parametrizing all the complex lines in  $\mathbb{V}$ . Let

$$L := \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}(\mathbb{V})}(1) \longrightarrow X \quad (2.8)$$

be the tautological holomorphic line bundle of degree one whose fiber over the point corresponding to a line  $\xi \subset \mathbb{V}$  is the quotient line  $\mathbb{V}/\xi$ . For any integer  $n$ , by  $L^n$  we shall denote the holomorphic line bundle

- $L^{\otimes n}$ , if  $n > 0$ ,
- $(L^{\otimes -n})^*$ , if  $n < 0$ , and
- $\mathcal{O}_X$  (the trivial line bundle), if  $n = 0$ .

A theorem of Grothendieck asserts that any holomorphic vector bundle  $F$  on  $X$  of rank  $r$  is of the form

$$F = \bigoplus_{i=1}^r L^{a_i},$$

where  $a_i \in \mathbb{Z}$  [Gr, p. 122, Théorème 1.1]; moreover, the above integers  $\{a_i\}$  are uniquely determined, up to a permutation, by  $F$  [Gr, p. 122, Théorème 1.1], [At1, p. 315, Theorem 2]. The holomorphic cotangent bundle  $K_X$  is holomorphically isomorphic to  $L^{-2}$ .

Let

$$\mathbb{L} \longrightarrow X \quad (2.9)$$

be the tautological holomorphic line bundle of degree  $-1$  whose fiber over the point corresponding to a line  $\xi \subset \mathbb{V}$  is  $\xi$  itself. This  $\mathbb{L}$  is holomorphically isomorphic to  $L^{-1}$ . We note that there is no natural isomorphism between  $L^{-1}$  and  $\mathbb{L}$ .

Let  $\text{SL}(\mathbb{V})$  be the complex Lie group consisting of all linear automorphisms of  $\mathbb{V}$  that act trivially on the determinant line  $\bigwedge^2 \mathbb{V}$ . This group  $\text{SL}(\mathbb{V})$  acts on  $\mathbb{P}(\mathbb{V})$  as follows: the action of any  $A \in \text{SL}(\mathbb{V})$  sends a line  $\xi \subset \mathbb{V}$  to the line  $A(\xi)$ . The corresponding action homomorphism

$$\text{SL}(\mathbb{V}) \longrightarrow \text{Aut}(X)$$

is holomorphic, and its kernel is  $\pm \text{Id}_{\mathbb{V}}$ . The action of  $\text{SL}(\mathbb{V})$  has a tautological lift to an action of  $\text{SL}(\mathbb{V})$  on  $K_X^{\otimes m}$  for every integer  $m$ .

The above action of  $\mathrm{SL}(\mathbb{V})$  on  $\mathbb{P}(\mathbb{V})$  clearly lifts to the line bundle  $L$ . Indeed, the action of any  $A \in \mathrm{SL}(\mathbb{V})$  on  $\mathbb{V}$  produces an isomorphism

$$\mathbb{V}/\xi \xrightarrow{\sim} \mathbb{V}/A(\xi)$$

for every line  $\xi \subset \mathbb{V}$ . The resulting action of  $\mathrm{SL}(\mathbb{V})$  on  $L$  produces an action of  $\mathrm{SL}(\mathbb{V})$  on  $L^n$  for every  $n$ .

The diagonal action of  $\mathrm{SL}(\mathbb{V})$  on  $X \times X$  preserves the divisor  $\Delta$  in (2.1). This action of  $\mathrm{SL}(\mathbb{V})$  on  $X \times X$ , and the action of  $\mathrm{SL}(\mathbb{V})$  on  $L^n$ , together produce an action of  $\mathrm{SL}(\mathbb{V})$  on  $J^k(L^n)$  defined in (2.4). The homomorphism  $\phi_k$  in (2.5) is in fact  $\mathrm{SL}(\mathbb{V})$ -equivariant.

### 3. DESCRIPTION OF JET BUNDLES ON THE PROJECTIVE LINE

Consider the short exact sequence of coherent sheaves on  $X = \mathbb{P}(\mathbb{V})$

$$0 \longrightarrow (p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow p_2^*L^n \longrightarrow (p_2^*L^n)|_{(k+1)\Delta} \longrightarrow 0. \quad (3.1)$$

From (2.4) we know that  $J^k(L^n)$  is the direct image of  $(p_2^*L^n)|_{(k+1)\Delta}$  to  $X$  under the projection  $p_1$ . We would investigate  $J^k(L^n)$  using the long exact sequence of direct images for the short exact sequence of sheaves in (3.1).

Let

$$\begin{aligned} 0 &\longrightarrow R^0p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow R^0p_{1*}p_2^*L^n \longrightarrow R^0p_{1*}((p_2^*L^n)|_{(k+1)\Delta}) \\ &=: J^k(L^n) \longrightarrow R^1p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow R^1p_{1*}p_2^*L^n \longrightarrow 0 \end{aligned} \quad (3.2)$$

be the long exact sequence of direct images, under  $p_1$ , associated to the short exact sequence in (3.1). Note that

$$R^1p_{1*}((p_2^*L^n)|_{(k+1)\Delta}) = 0,$$

because the support of  $(p_2^*L^n)|_{(k+1)\Delta}$  is finite over  $X$  (for the projection  $p_1$ ).

Since the diagonal action of  $\mathrm{SL}(\mathbb{V})$  on  $X \times X$  preserves the divisor  $\Delta$ , each sheaf in (3.2) is equipped with an action of  $\mathrm{SL}(\mathbb{V})$  that lifts the action of  $\mathrm{SL}(\mathbb{V})$  on  $X$ . Moreover, all the homomorphisms in the exact sequence (3.2) are  $\mathrm{SL}(\mathbb{V})$ -equivariant.

In (3.2), the direct image  $R^i p_{1*} p_2^* L^n$  is the trivial holomorphic vector bundle

$$R^i p_{1*} p_2^* L^n = X \times H^i(X, L^n) \quad (3.3)$$

over  $X$  with fiber  $H^i(X, L^n)$  for  $i = 0, 1$ .

Next, to understand  $R^i p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta))$ , we first note that the holomorphic line bundle  $\mathcal{O}_{X \times X}(\Delta)$  is holomorphically isomorphic to  $(p_1^*L) \otimes (p_2^*L)$ ; however, there is no natural isomorphism between these two isomorphic line bundles. To construct an explicit isomorphism between them, fix a nonzero element

$$0 \neq \omega \in \bigwedge^2 \mathbb{V}. \quad (3.4)$$

Note that  $\omega$  defines a symplectic structure on  $\mathbb{V}$ , in particular,  $\omega$  produces an isomorphism

$$\mathbb{V}^* \xrightarrow{\sim} \mathbb{V}. \quad (3.5)$$

The action of  $\mathrm{SL}(\mathbb{V})$  on  $\bigwedge^2 \mathbb{V}$ , given by the action of  $\mathrm{SL}(\mathbb{V})$  on  $\mathbb{V}$ , is the trivial action, and hence this action fixes the element  $\omega$ . We note that  $\omega$  produces a holomorphic isomorphism

$$\Psi : L^{-1} \longrightarrow \mathbb{L}, \quad (3.6)$$

where  $\mathbb{L}$  was constructed in (2.9). Indeed, from the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathbb{L} \longrightarrow X \times \mathbb{V} \longrightarrow L \longrightarrow 0 \quad (3.7)$$

we conclude that  $\mathbb{L} \otimes L = X \times \bigwedge^2 \mathbb{V}$ . Therefore, the element  $\omega \in \bigwedge^2 \mathbb{V}$  gives a nowhere vanishing holomorphic section of  $\mathbb{L} \otimes L$ . This section produces the isomorphism  $\Psi$  in (3.6). Note that the diagonal action of  $\mathrm{SL}(\mathbb{V})$  on  $X \times \mathbb{V}$  preserves the subbundle  $\mathbb{L}$  in (3.7); more precisely, the homomorphisms in (3.7) are  $\mathrm{SL}(\mathbb{V})$ -equivariant. The isomorphism  $\Psi$  in (3.6) is evidently  $\mathrm{SL}(\mathbb{V})$ -equivariant. It should be mentioned that  $\Psi$  depends on  $\omega$ .

We have  $H^0(X, L) = \mathbb{V}$ . So

$$H^0(X \times X, (p_1^* L) \otimes (p_2^* L)) = H^0(X, L) \otimes H^0(X, L) = \mathbb{V} \otimes \mathbb{V},$$

and hence  $\omega \in \bigwedge^2 \mathbb{V} \subset \mathbb{V} \otimes \mathbb{V}$  in (3.4) defines a holomorphic section

$$\widehat{\omega} \in H^0(X \times X, (p_1^* L) \otimes (p_2^* L)). \quad (3.8)$$

This section  $\widehat{\omega}$  vanishes exactly on the diagonal  $\Delta \subset X \times X$ . Consequently, there is a unique holomorphic isomorphism

$$\Phi : (p_1^* L) \otimes (p_2^* L) \xrightarrow{\sim} \mathcal{O}_{X \times X}(\Delta)$$

such that the section  $\Phi(\widehat{\omega})$ , where  $\widehat{\omega}$  is constructed in (3.8), coincides with the holomorphic section of  $\mathcal{O}_{X \times X}(\Delta)$  given by the constant function 1 on  $X \times X$ . Therefore, combining  $\Phi$  with  $\Psi$  in (3.6), we have

$$(p_1^* \mathbb{L}) \otimes (p_2^* \mathbb{L}) = (p_1^* L^{-1}) \otimes (p_2^* L^{-1}) = (p_1^* \mathbb{L}) \otimes (p_2^* L^{-1}) = \mathcal{O}_{X \times X}(-\Delta). \quad (3.9)$$

Using the identification of  $X$  with  $\Delta$ , the Poincaré adjunction formula gives that  $\mathcal{O}_{X \times X}(\Delta)|_{\Delta} = TX$ , where  $TX$  is the holomorphic tangent bundle of  $X$ ; it is the dual of the isomorphism in (2.3) for  $k = 1$ . Therefore, restricting the isomorphisms in (3.9) to  $\Delta \subset X \times X$ , we have a holomorphic isomorphism

$$\Phi_0 : L^2 = (\mathbb{L}^*)^{\otimes 2} \xrightarrow{\sim} TX. \quad (3.10)$$

Using the isomorphisms in (3.9), we have

$$(p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) = (p_1^* L^{-k-1}) \otimes (p_2^* L^{n-k-1}) = (p_1^* \mathbb{L}^{k+1}) \otimes (p_2^* L^{n-k-1}).$$

Hence by the projection formula, [Ha, p. 426, A4],

$$\begin{aligned} R^i p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) &= R^i p_{1*}((p_1^* \mathbb{L}^{k+1}) \otimes (p_2^* L^{n-k-1})) \\ &= \mathbb{L}^{k+1} \otimes R^i p_{1*}(p_2^* L^{n-k-1}) \end{aligned} \quad (3.11)$$

for  $i = 0, 1$ .

We shall break  $(n, k)$  into three cases.

**3.1. The case of  $n < 0$ .** In (3.2), we have  $R^0 p_{1*} p_2^* L^n = 0$ , because  $n < 0$ . Consequently, the short exact sequence in (3.2) becomes

$$0 \longrightarrow J^k(L^n) \longrightarrow R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow R^1 p_{1*} p_2^* L^n \longrightarrow 0. \quad (3.12)$$

Now using (3.11), the exact sequence in (3.12) becomes the short exact sequence

$$0 \longrightarrow J^k(L^n) \longrightarrow \mathbb{L}^{k+1} \otimes R^1 p_{1*}(p_2^* L^{n-k-1}) \xrightarrow{\gamma_1} R^1 p_{1*} p_2^* L^n \longrightarrow 0. \quad (3.13)$$

Invoking Serre duality and using the isomorphism  $\Phi_0$  in (3.10),

$$H^1(X, L^n) = H^0(X, L^{-n-2})^* = \text{Sym}^{-n-2}(\mathbb{V})^*,$$

where  $\text{Sym}^{-n-2}(\mathbb{V})$  denotes:

- the  $(-n-2)$ -th symmetric power of  $\mathbb{V}$ , if  $-n-2 > 0$ ,
- $\mathbb{C}$ , if  $-n = 2$ , and
- 0, if  $-n = 1$ .

Therefore,  $R^1 p_{1*} p_2^* L^n$  in (3.13) is the trivial holomorphic vector bundle

$$X \times \text{Sym}^{-n-2}(\mathbb{V})^* \longrightarrow X$$

(see (3.3)).

For notational convenience, the trivial holomorphic vector bundle  $X \times \text{Sym}^j(\mathbb{V})$  over  $X$  with fiber  $\text{Sym}^j(\mathbb{V})$  will be denoted by  $\mathbb{V}_j$ . The trivial line bundle  $\mathcal{O}_X$  will be denoted by  $\mathbb{V}_0$ . So we have

$$R^1 p_{1*} p_2^* L^n = (\mathbb{V}_{-n-2})^* = \mathbb{V}_{-n-2}^*. \quad (3.14)$$

Similarly, we have

$$R^1 p_{1*}(p_2^* L^{n-k-1}) = X \times \text{Sym}^{k-n-1}(\mathbb{V})^* = \mathbb{V}_{k-n-1}^* \longrightarrow X. \quad (3.15)$$

Hence  $\mathbb{L}^{k+1} \otimes R^1 p_{1*}(p_2^* L^{n-k-1})$  in (3.13) has the following description:

$$\mathbb{L}^{k+1} \otimes R^1 p_{1*}(p_2^* L^{n-k-1}) = \mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}^*. \quad (3.16)$$

Using (3.14) and (3.16), the homomorphism  $\gamma_1$  in (3.13) is a homomorphism

$$\gamma_1 : \mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}^* \longrightarrow \mathbb{V}_{-n-2}^*. \quad (3.17)$$

The homomorphism  $\gamma_1$  in (3.17) is simply the contraction of elements of  $\mathbb{V}$  by  $\mathbb{V}^*$  (recall that  $\mathbb{L} \subset X \times \mathbb{V}$ ). More precisely, consider the contraction homomorphism

$$\mathbb{V} \otimes (\mathbb{V}^*)^{\otimes j} \longrightarrow (\mathbb{V}^*)^{\otimes(j-1)}.$$

Since  $\mathbb{L} \subset \mathbb{V}_1$  (see (3.7)), this contraction produces a holomorphic homomorphism of vector bundles

$$\mathbb{L} \otimes \mathbb{V}_j^* \longrightarrow \mathbb{V}_{j-1}^*.$$

The homomorphism  $\gamma_1$  in (3.17) is  $(k+1)$ -times iteration of this homomorphism starting with  $j = k - n - 1$ .

We shall compute the kernel of the homomorphism  $\gamma_1$  in (3.17).

Taking dual of the short exact sequence in (3.7), we have

$$L^* \subset X \times \mathbb{V}^* = \mathbb{V}_1^*.$$



Hence, using the isomorphism  $\Psi$  in (3.6), we have

$$\mathbb{L} = L^* \subset \mathbb{V}_1^*. \quad (3.18)$$

For the natural duality pairing  $\mathbb{V}_1 \otimes \mathbb{V}_1^* \rightarrow \mathcal{O}_X$ , the annihilator of the subbundle  $\mathbb{L} \subset \mathbb{V}_1$  in (3.7) is the subbundle  $\mathbb{L} \subset \mathbb{V}_1^*$  in (3.18). Therefore, from the above description of the homomorphism  $\gamma_1$  in (3.17) we conclude that the subbundle

$$\mathbb{L}^{k+1} \otimes (\mathbb{V}_{k-n-1}^*) \supset \mathbb{L}^{k+1} \otimes (\mathbb{L}^{-n-1} \otimes \mathbb{V}_k^*) = \mathbb{L}^{k-n} \otimes \mathbb{V}_k^*$$

is contained in the kernel of the surjective homomorphism  $\gamma_1$ ; note that  $\mathbb{L}^{-n-1} \otimes \mathbb{V}_k^*$  is realized as a subbundle of  $\mathbb{V}_{k-n-1}^*$  using the inclusion  $\mathbb{L} \subset \mathbb{V}_1^*$  in (3.18). From (3.13) we know that  $\gamma_1$  is surjective. Since

$$\text{rank}(\mathbb{L}^{k-n} \otimes \mathbb{V}_k^*) = k+1 = (k-n) - (-n-1) = \text{rank}(\mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}^*) - \text{rank}(\mathbb{V}_{-n-2}^*),$$

we now conclude that

$$\mathbb{L}^{k-n} \otimes \mathbb{V}_k^* = \text{kernel}(\gamma_1).$$

Therefore, from (3.13) it follows that

$$J^k(L^n) = \text{kernel}(\gamma_1) = \mathbb{L}^{k-n} \otimes \mathbb{V}_k^*. \quad (3.19)$$

Now using the isomorphism between  $\mathbb{V}_k$  and  $\mathbb{V}_k^*$  given by the isomorphism in (3.5), from (3.19) we deduce that

$$J^k(L^n) = \mathbb{L}^{k-n} \otimes \mathbb{V}_k.$$

Moreover, using the isomorphism  $\Psi$  in (3.6) we have

$$J^k(L^n) = L^{n-k} \otimes \mathbb{V}_k. \quad (3.20)$$

Recall that all the homomorphisms in (3.2) are  $\text{SL}(\mathbb{V})$ -equivariant. From the construction of the isomorphism in (3.19) between  $J^k(L^n)$  and  $\mathbb{L}^{k-n} \otimes \mathbb{V}_k^*$  it is evident that it is  $\text{SL}(\mathbb{V})$ -equivariant. Therefore, the isomorphism in (3.20) is  $\text{SL}(\mathbb{V})$ -equivariant.

Fix  $k \geq 1$  an integer. Using (3.20), the homomorphism  $\phi_k$  in (2.5) is a projection

$$\phi_k : L^{n-k} \otimes \mathbb{V}_k = J^k(L^n) \rightarrow J^{k-1}(L^n) = L^{n-k+1} \otimes \mathbb{V}_{k-1}. \quad (3.21)$$

We shall describe the homomorphism in (3.21).

Recall that  $\mathbb{V}_j = \text{Sym}^j(\mathbb{V}_1) = \text{Sym}^j(X \times \mathbb{V})$ , and the line bundle  $L$  is a quotient of  $\mathbb{V}_1$  (see (3.7)). Therefore, we have a natural projection

$$\varpi_k^0 : \mathbb{V}_k \rightarrow L \otimes \mathbb{V}_{k-1}. \quad (3.22)$$

To describe this  $\varpi_k^0$  explicitly, let  $q_L : \mathbb{V}_1 \rightarrow L$  be the quotient map in (3.7). Consider the homomorphism

$$q_L \otimes \text{Id}_{\mathbb{V}_1^{\otimes(k-1)}} : \mathbb{V}_1^{\otimes k} \rightarrow L \otimes \mathbb{V}_1^{\otimes(k-1)}.$$

Note that  $q_L \otimes \text{Id}_{\mathbb{V}_1^{\otimes(k-1)}}$  sends the subbundle

$$\text{Sym}^k(\mathbb{V}_1) \subset \mathbb{V}_1^{\otimes k}$$

to the subbundle  $L \otimes \mathbb{V}_{k-1} \subset L \otimes \mathbb{V}_1^{\otimes(k-1)}$ . The homomorphism  $\varpi_k^0$  in (3.22) is this restriction of  $q_L \otimes \text{Id}_{\mathbb{V}_1^{\otimes(k-1)}}$ .

**Proposition 3.1.** *The homomorphism  $\phi_k$  in (3.21) coincides with  $\text{Id}_{L^{n-k}} \otimes \varpi_k^0$ , where  $\varpi_k^0$  is the homomorphism in (3.22).*

*Proof.* Consider the short exact sequence in (3.12). We have the commutative diagram of homomorphisms

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & L^n \otimes K_X^{\otimes k} & = & L^n \otimes K_X^{\otimes k} & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^k(L^n) & \longrightarrow & R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) & \longrightarrow & R^1 p_{1*} p_2^* L^n \longrightarrow 0 \\
& & \downarrow & & \downarrow \mu & & \parallel \\
0 & \longrightarrow & J^{k-1}(L^n) & \longrightarrow & R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta)) & \longrightarrow & R^1 p_{1*} p_2^* L^n \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & \\
& & & & & & (3.23)
\end{array}$$

where the bottom exact sequence is the one in (3.12) with  $k$  substituted by  $k-1$ , the top exact sequence is the one in (3.12), the left-vertical exact sequence is the one in (2.5) for  $W = L^n$ , and the other vertical exact sequence is the long exact sequence of direct images, under  $p_1$ , for the short exact sequence

$$0 \longrightarrow (p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow (p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta) \longrightarrow (L^n \otimes K_X^{\otimes k})|_{\Delta} \longrightarrow 0$$

of sheaves on  $X \times X$  obtained by tensoring with  $p_2^* L^n$  the short exact sequence given by (2.2).

Using (3.11) and (3.16) we have

$$R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) = \mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}^*.$$

Replacing  $k$  by  $k-1$ , we have

$$R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta)) = \mathbb{L}^k \otimes \mathbb{V}_{k-n-2}^*.$$

So the homomorphism  $\mu$  in (3.23) becomes a homomorphism

$$\mu : \mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}^* \longrightarrow \mathbb{L}^k \otimes \mathbb{V}_{k-n-2}^*.$$

This homomorphism is simply the contraction of elements of  $\mathbb{V}^*$  by elements of  $\mathbb{L} \subset \mathbb{V}$ . The proposition follows from this and the commutative diagram in (3.23).  $\square$

**3.2. The case of  $n \geq k$ .** Note that  $n \geq 0$ , because  $n \geq k$ . Since now we have  $n - k - 1 \geq -1$ , it follows that

$$H^1(X, L^{n-k-1}) = 0,$$

and hence using (3.11) and (3.3) it is deduced that

$$R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) = 0.$$

Consequently, the exact sequence in (3.2) becomes the short exact sequence

$$0 \longrightarrow R^0 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow R^0 p_{1*} p_2^* L^n \longrightarrow J^k(L^n) \longrightarrow 0. \quad (3.24)$$

Therefore, using (3.11) and (3.3), the exact sequence in (3.24) becomes the short exact sequence

$$0 \longrightarrow \mathbb{L}^{k+1} \otimes R^0 p_{1*}(p_2^* L^{n-k-1}) = \mathbb{L}^{k+1} \otimes \mathbb{V}_{n-k-1} \xrightarrow{\gamma^2} X \times \text{Sym}^n(\mathbb{V}) \quad (3.25)$$

$$=: \mathbb{V}_n \longrightarrow J^k(L^n) \longrightarrow 0.$$

Consider the homomorphism given by the composition of the natural homomorphisms

$$\mathbb{V} \otimes \mathbb{V}_j \longrightarrow X \times \mathbb{V}^{\otimes(j+1)} \longrightarrow X \times \text{Sym}^{j+1}(\mathbb{V}) = \mathbb{V}_{j+1}.$$

Since  $\mathbb{L} \subset \mathbb{V}$  (see (3.7)), this restricts to a homomorphism

$$\mathbb{L} \otimes \mathbb{V}_j \longrightarrow \mathbb{V}_{j+1}.$$

The homomorphism  $\gamma_2$  in (3.25) is  $(k+1)$ -fold iteration of this homomorphism starting with  $j = n - k - 1$ .

To compute the cokernel of this homomorphism  $\gamma_2$ , consider the surjective homomorphism  $\mathbb{V}_1 \longrightarrow L$  in (3.7). It produces a surjective homomorphism

$$\theta_2 : \mathbb{V}_n \longrightarrow L^{n-k} \otimes \mathbb{V}_k.$$

It is straight-forward to check that  $\theta_2 \circ \gamma_2 = 0$ . Consequently,  $\theta_2$  produces a surjective homomorphism from  $\text{cokernel}(\gamma_2)$  to  $L^{n-k} \otimes \mathbb{V}_k$ . On the other hand,

$$\text{rank}(\mathbb{V}_n) - \text{rank}(\mathbb{L}^{k+1} \otimes \mathbb{V}_{n-k-1}) = k+1 = \text{rank}(L^{n-k} \otimes \mathbb{V}_k).$$

Hence we conclude that the above surjection from  $\text{cokernel}(\gamma_2)$  to  $L^{n-k} \otimes \mathbb{V}_k$  is also injective.

Therefore, from (3.25) it follows that

$$J^k(L^n) = L^{n-k} \otimes \mathbb{V}_k. \quad (3.26)$$

From the construction of the isomorphism in (3.26) it is evident that this isomorphism is in fact  $\text{SL}(\mathbb{V})$ -equivariant.

Consider  $k \geq 1$  an integer. Using (3.26), the homomorphism  $\phi_k$  in (2.5) is a projection

$$\phi_k : L^{n-k} \otimes \mathbb{V}_k = J^k(L^n) \longrightarrow J^{k-1}(L^n) = L^{n-k+1} \otimes \mathbb{V}_{k-1}. \quad (3.27)$$

**Proposition 3.2.** *The homomorphism  $\phi_k$  in (3.27) coincides with  $\text{Id}_{L^{n-k}} \otimes \varpi_k^0$ , where  $\varpi_k^0$  is the homomorphism in (3.22).*

*Proof.* Consider the short exact sequence of sheaves in (3.24). We have the commutative diagram of homomorphism

$$\begin{array}{ccccccc}
& & & & & & 0 \\
& & & & & & \downarrow \\
& & & & & & L^n \otimes K_X^{\otimes k} \\
& & & & & & \downarrow \\
0 & \longrightarrow & R^0 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) & \longrightarrow & R^0 p_{1*} p_2^* L^n & \longrightarrow & J^k(L^n) \longrightarrow 0 \\
& & & & & & \parallel \\
0 & \longrightarrow & R^0 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta)) & \longrightarrow & R^0 p_{1*} p_2^* L^n & \longrightarrow & J^{k-1}(L^n) \longrightarrow 0 \\
& & & & & & \downarrow \\
& & & & & & 0 \\
& & & & & & \downarrow \\
& & & & & & 0 \\
& & & & & & \downarrow \\
& & & & & & 0 \\
& & & & & & \downarrow \\
& & & & & & 0
\end{array}$$

(3.28)

where the bottom exact row is (3.24) with  $k - 1$  substituted in place of  $k$ , the top exact row is the one in (3.24), the right vertical exact sequence is the one in (2.5) and the left vertical exact sequence is a part of the long exact sequence of direct images, under  $p_1$ , for the short exact sequence

$$0 \longrightarrow (p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow (p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta) \longrightarrow (L^n \otimes K_X^{\otimes k})|_\Delta \longrightarrow 0$$

of sheaves on  $X \times X$  obtained by tensoring (2.3) with  $p_2^* L^n$ . Using (3.11), in (3.25) we saw that

$$R^0 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) = \mathbb{L}^{k+1} \otimes \mathbb{V}_{n-k-1},$$

So replacing  $k$  by  $k - 1$ , we have

$$R^0 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta)) = \mathbb{L}^k \otimes \mathbb{V}_{n-k}.$$

Using these isomorphisms, the injective homomorphism

$$\mu : \mathbb{L}^{k+1} \otimes \mathbb{V}_{n-k-1} = \mathbb{L}^{k+1} \otimes \text{Sym}^{n-k-1}(\mathbb{V}_1) \longrightarrow \mathbb{L}^k \otimes \text{Sym}^{n-k}(\mathbb{V}_1)$$

in (3.28) is the natural homomorphism induced by the inclusion of  $\mathbb{L}$  in  $\mathbb{V}_1$ . Now the proposition follows from (3.28).  $\square$

The remaining case will now be considered.

**3.3. The case of  $k > n \geq 0$ .** Since  $n \geq 0$ , it follows that  $H^1(X, L^n) = 0$ . Hence in (3.2) we have

$$R^1 p_{1*} p_2^* L^n = 0;$$

see (3.3). Since  $n - k - 1 < 0$ , we have  $H^0(X, L^{n-k-1}) = 0$ . Therefore, from (3.11) and (3.3) we conclude that

$$R^0 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) = 0.$$

Consequently, the exact sequence in (3.2) becomes the short exact sequence

$$0 \longrightarrow R^0 p_{1*} p_2^* L^n \longrightarrow J^k(L^n) \longrightarrow R^1 p_{1*}((p_2^* L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow 0. \quad (3.29)$$

Now using (3.11) and (3.3) together with the projection formula, the exact sequence in (3.29) becomes the exact sequence

$$0 \longrightarrow \mathbb{V}_n \xrightarrow{\gamma_3} J^k(L^n) \xrightarrow{\theta_3} \mathbb{L}^{k+1} \otimes R^1 p_{1*}(p_2^* L^{n-k-1}) \longrightarrow 0. \quad (3.30)$$

The homomorphisms  $\gamma_3$  and  $\theta_3$  in (3.30) are clearly  $\text{SL}(\mathbb{V})$ -equivariant.

**Lemma 3.3.** *The short exact sequence in (3.30) has a canonical  $\text{SL}(\mathbb{V})$ -equivariant holomorphic splitting.*

*Proof.* Since  $k > n$ , there is a natural projection

$$J^k(L^n) \longrightarrow J^n(L^n)$$

(see (2.5)). Now  $J^n(L^n) = \mathbb{V}_n$  by (3.26), so this projection defines a projection

$$\beta : J^k(L^n) \longrightarrow \mathbb{V}_n. \quad (3.31)$$

It is straight-forward to check that  $\beta \circ \gamma_3 = \text{Id}_{\mathbb{V}_n}$ , where  $\gamma_3$  and  $\beta$  are constructed in (3.30) and (3.31) respectively. This produces an  $\text{SL}(\mathbb{V})$ -equivariant holomorphic splitting

$$J^k(L^n) = \text{image}(\gamma_3) \oplus \text{kernel}(\beta)$$

of the short exact sequence in (3.30).  $\square$

Lemma 3.3 gives a holomorphic decomposition

$$J^k(L^n) = \mathbb{V}_n \oplus (\mathbb{L}^{k+1} \otimes R^1 p_{1*}(p_2^* L^{n-k-1})) \quad (3.32)$$

which is compatible with the actions of  $\mathrm{SL}(\mathbb{V})$ . Now,  $R^1 p_{1*}(p_2^* L^{n-k-1}) = \mathbb{V}_{k-n-1}^*$  (see (3.15)), and  $\mathbb{V}_{k-n-1}^* = \mathbb{V}_{k-n-1}$  which is constructed using the isomorphism in (3.5). Therefore, the decomposition in (3.32) can be reformulated as

$$J^k(L^n) = \mathbb{V}_n \oplus (\mathbb{L}^{k+1} \otimes \mathbb{V}_{k-n-1}) = \mathbb{V}_n \oplus (L^{-(k+1)} \otimes \mathbb{V}_{k-n-1}) \quad (3.33)$$

using  $\Psi$  in (3.6). As before, the isomorphisms in (3.33) are compatible with the actions of  $\mathrm{SL}(\mathbb{V})$ .

If  $k-1 > n \geq 0$ , then from (3.33) we know that

$$J^{k-1}(L^n) = \mathbb{V}_n \oplus (\mathbb{L}^k \otimes \mathbb{V}_{k-n-2}) = \mathbb{V}_n \oplus (L^{-k} \otimes \mathbb{V}_{k-n-2}),$$

and if  $k-1 = n \geq 0$ , then from (3.26) we have

$$J^{k-1}(L^n) = \mathbb{V}_{k-1}.$$

The proof of the following proposition is similar to the proofs of Proposition 3.1 and Proposition 3.2.

**Proposition 3.4.** *If  $k-1 > n \geq 0$ , then the projection  $J^k(L^n) \rightarrow J^{k-1}(L^n)$  in (2.5) is the homomorphism*

$$\mathbb{V}_n \oplus (L^{-(k+1)} \otimes \mathbb{V}_{k-n-1}) \rightarrow \mathbb{V}_n \oplus (L^{-k} \otimes \mathbb{V}_{k-n-2})$$

which is the direct sum of the identity map of  $\mathbb{V}_n$  and the natural contraction map  $L^{-(k+1)} \otimes \mathbb{V}_{k-n-1} \rightarrow L^{-k} \otimes \mathbb{V}_{k-n-2}$ .

If  $k-1 = n \geq 0$ , then the projection  $J^k(L^n) \rightarrow J^{k-1}(L^n)$  in (2.5) is the homomorphism

$$\mathbb{V}_n \oplus (L^{-(k+1)} \otimes \mathbb{V}_{k-n-1}) \rightarrow \mathbb{V}_n$$

which is the direct sum of the identity map of  $\mathbb{V}_n$  and the zero homomorphism of  $L^{-(k+1)} \otimes \mathbb{V}_{k-n-1} = L^{-(k+1)}$  (recall that  $\mathbb{V}_0 = \mathcal{O}_X$ ).

**3.4. The jet bundle.** Combining (3.20), (3.26) and (3.33), we have the following:

**Theorem 3.5.** *If  $n < 0$  or  $n \geq k$ , then*

$$J^k(L^n) = L^{n-k} \otimes \mathbb{V}_k.$$

If  $k > n \geq 0$ , then

$$J^k(L^n) = \mathbb{V}_n \oplus (L^{-(k+1)} \otimes \mathbb{V}_{k-n-1}).$$

Both the isomorphisms are  $\mathrm{SL}(\mathbb{V})$ -equivariant.

#### 4. HOLOMORPHIC CONNECTIONS ON VECTOR BUNDLES

In this section we construct and study natural differential holomorphic operators between natural holomorphic vector bundles on a Riemann surface equipped with a projective structure.

**4.1. Jets of Riemann surfaces with a projective structure.** Let  $M$  be a compact connected Riemann surface of genus  $g$ . A holomorphic coordinate chart on  $M$  is a pair of the form  $(U, \varphi)$ , where  $U \subset M$  is an open subset and  $\varphi : U \rightarrow \mathbb{P}(\mathbb{V}) = X$  is a holomorphic embedding. A holomorphic coordinate atlas on  $M$  is a collection of coordinate charts  $\{(U_i, \varphi_i)\}_{i \in I}$  such that  $M = \bigcup_{i \in I} U_i$ . A projective structure on  $M$  is given by a holomorphic coordinate atlas  $\{(U_i, \varphi_i)\}_{i \in I}$  such that for all pairs  $(i, j) \in I \times I$  for which  $U_i \cap U_j \neq \emptyset$ , there exists an element  $\tau_{j,i} \in \text{Aut}(\mathbb{P}(\mathbb{V})) = \text{PGL}(\mathbb{V})$  such that  $\varphi_j \circ \varphi_i^{-1}$  is the restriction of  $\tau_{j,i}$  to  $\varphi_i(U_i \cap U_j)$ . A holomorphic coordinate atlas satisfying this condition is called a projective atlas. Two such projective atlases  $\{(U_i, \varphi_i)\}_{i \in I}$  and  $\{(U_i, \varphi_i)\}_{i \in J}$  are called equivalent if their union  $\{(U_i, \varphi_i)\}_{i \in I \cup J}$  is again a projective atlas. A *projective structure* on  $M$  is an equivalence class of projective atlases (see [Gu]).

Note that the collection  $\{\tau_{j,i}\}$  in the definition of a projective atlas forms a 1-cocycle with values in  $\text{PGL}(\mathbb{V})$ . The corresponding cohomology class in  $H^1(M, \text{PGL}(\mathbb{V}))$  depends only on the projective structure and can be lifted to  $H^1(M, \text{SL}(\mathbb{V}))$ . Moreover the space of such lifts is in bijection with the theta characteristics of  $M$  (see [Gu]). We recall that the theta characteristic on  $M$  is a holomorphic line bundle  $K_M^{1/2}$  on  $M$  together with a holomorphic isomorphism of  $K_M^{1/2} \otimes K_M^{1/2}$  with the holomorphic cotangent bundle  $K_M$ .

We fix a theta characteristic on  $M$ . So a projective structure on  $M$  is given by a coordinate atlas  $\{(U_i, \varphi_i)\}_{i \in I}$  together with an element

$$\tau_{j,i} \in \text{SL}(\mathbb{V})$$

for every ordered pair  $(i, j) \in I \times I$  such that  $U_i \cap U_j \neq \emptyset$  such that the following four conditions hold:

- (1)  $\varphi_j \circ \varphi_i^{-1}$  is the restriction to  $\varphi_i(U_i \cap U_j)$  of the automorphism of  $X$  given by  $\tau_{j,i}$ ,
- (2)  $\tau_{i,k} \tau_{k,j} \tau_{j,i} = \text{Id}_{\mathbb{V}}$  for all  $i, j, k \in I$  such that  $U_i \cap U_j \cap U_k \neq \emptyset$ ,
- (3)  $\tau_{i,i} = \text{Id}_{\mathbb{V}}$  for all  $i \in I$ , and
- (4) the theta characteristic corresponding to the data is the given one.

Two such collections of triples  $\{ \{(U_i, \varphi_i)\}_{i \in I}, \{\tau_{i,j}\} \}$  and  $\{ \{(U_i, \varphi_i)\}_{i \in J}, \{\tau_{i,j}\} \}$  are called equivalent if their union  $\{ \{(U_i, \varphi_i)\}_{i \in I \cup J}, \{\tau_{i,j}\} \}$  is again a part of a collection of triples satisfying the above conditions. A  $\text{SL}(\mathbb{V})$ -*projective structure* on  $M$  is an equivalence class of collection of triples satisfying the above conditions.

Let  $P = \{ \{(U_i, \varphi_i)\}_{i \in I}, \{\tau_{i,j}\} \}$  be a  $\text{SL}(\mathbb{V})$ -projective structure on  $M$ . Recall that the collection  $\{\tau_{i,j}\}$  forms a 1-cocycle on  $M$  with values in the group  $\text{SL}(\mathbb{V})$ . Hence  $\{\tau_{i,j}\}$  produce a holomorphic rank two bundle  $\mathcal{V}$  on  $M$  equipped with a holomorphic connection  $D$  (see [At2] for holomorphic connections). To construct the pair  $(\mathcal{V}, D)$  explicitly, for each  $i \in I$ , consider the trivial holomorphic vector bundle  $U_i \times \mathbb{V} \rightarrow U_i$  equipped with the trivial connection. For any ordered pair  $(i, j) \in I \times I$  with  $U_i \cap U_j \neq \emptyset$ , glue  $U_i \times \mathbb{V}$  and  $U_j \times \mathbb{V}$  over  $U_i \cap U_j$  using the automorphism of  $(U_i \cap U_j) \times \mathbb{V}$  that sends any

$$(y, v) \in (U_i \cap U_j) \times \mathbb{V} \subset U_j \times \mathbb{V}$$

to  $(y, \tau_{i,j}(v)) \in (U_i \cap U_j) \times \mathbb{V} \subset U_i \times \mathbb{V}$ . This automorphism of  $(U_i \cap U_j) \times \mathbb{V}$  is holomorphic, and it evidently preserves the trivial connection. Therefore, this gluing operation produces a holomorphic vector bundle on  $M$ , which we shall denote by  $\mathcal{V}$ , and a holomorphic connection on  $\mathcal{V}$ , which we shall denote by  $D$ .

The diagonal action of  $\mathrm{SL}(\mathbb{V})$  on  $X \times \mathbb{V}$  preserves the subbundle  $\mathbb{L}$  in (3.7). Therefore, the subbundles  $\varphi_i(U_i) \times \mathbb{L} \subset \varphi(U_i) \times \mathbb{V}$  patch together compatibly to produce a holomorphic line subbundle of the vector bundle  $\mathcal{V}$  on  $M$  constructed above. This line subbundle of  $\mathcal{V}$  will be denoted by  $\mathcal{L}$ . Let

$$\mathbf{L} := \mathcal{V}/\mathcal{L} \quad (4.1)$$

be the quotient bundle.

Recall that the isomorphism  $\Psi$  in (3.6) and the isomorphism  $\Phi_0$  in (3.10) are both  $\mathrm{SL}(\mathbb{V})$ -equivariant. However, it should be clarified that a nonzero element  $\omega \in \bigwedge^2 \mathbb{V}$  (see (3.4)) was used in their construction. Since  $\Psi$  and  $\Phi_0$  are  $\mathrm{SL}(\mathbb{V})$ -equivariant, they produce isomorphisms

$$\mathbf{L} = \mathcal{L}^* \quad \text{and} \quad \mathbf{L}^{\otimes 2} = (\mathcal{L}^*)^{\otimes 2} = TM \quad (4.2)$$

respectively. Recall that we fixed a theta characteristic on  $M$ . The fourth condition in the definition of a projective structure says that the theta characteristic  $\mathcal{L}$  coincides with the chosen one.

For any  $j \geq 1$ , the vector bundle  $\mathrm{Sym}^j(\mathcal{V})$  will be denoted by  $\mathcal{V}_j$ . Also,  $\mathcal{V}_0$  will denote the trivial line bundle  $\mathcal{O}_M$ . The holomorphic connection  $D$  on  $\mathcal{V}$  induces a holomorphic connection on  $\mathcal{V}_j$  for every  $j \geq 1$ ; this connection on  $\mathcal{V}_j$  will be denoted by  $D_j$ . The trivial connection on  $\mathcal{V}_0 = \mathcal{O}_M$  will be denoted by  $D_0$ . The connections  $D_m$  and  $D_n$  on  $\mathcal{V}_m$  and  $\mathcal{V}_n$  respectively together produce a holomorphic connection on  $\mathcal{V}_m \otimes \mathcal{V}_n$ .

**Proposition 4.1.** *Let  $M$  be a compact connected Riemann surface equipped with a projective structure  $P$ . Take integers  $m > n \geq 0$ . Then  $\mathcal{V}_m \otimes \mathcal{V}_n$  has a canonical decomposition*

$$\mathcal{V}_m \otimes \mathcal{V}_n = \bigoplus_{i=0}^n \mathcal{V}_{m+n-2i}.$$

The above isomorphism takes the connection on  $\mathcal{V}_m \otimes \mathcal{V}_n$  induced by  $D_m$  and  $D_n$  to the connection  $\bigoplus_{i=0}^n D_{m+n-2i}$  on  $\bigoplus_{i=0}^n \mathcal{V}_{m+n-2i}$ .

*Proof.* First consider the case of  $X = \mathbb{P}(\mathbb{V})$  investigated in Section 2. The  $\mathrm{SL}(\mathbb{V})$ -module  $\mathrm{Sym}^j(\mathbb{V})$  is irreducible for every  $j \geq 0$  [FH, p. 150], and furthermore, for  $m > n \geq 0$ , the  $\mathrm{SL}(\mathbb{V})$ -module  $\mathrm{Sym}^m(\mathbb{V}) \otimes \mathrm{Sym}^n(\mathbb{V})$  decomposes as

$$\mathrm{Sym}^m(\mathbb{V}) \otimes \mathrm{Sym}^n(\mathbb{V}) = \bigoplus_{i=0}^n \mathrm{Sym}^{m+n-2i}(\mathbb{V}) \quad (4.3)$$

[FH, p. 151, Ex. 11.11].

Choose data  $\{\{(U_i, \varphi_i)\}_{i \in I}, \{\tau_{i,j}\}\}$  representing a  $\mathrm{SL}(\mathbb{V})$ -projective structure associated to  $P$  and to a fixed theta characteristic on  $M$ . On each  $U_i$  we have the holomorphic vector bundles  $\varphi_i^* \mathbb{V}_j = U_i \times \mathrm{Sym}^j(\mathbb{V})$  equipped with the trivial connection. Patching these together using  $\{\tau_{i,j}\}$  as the transition functions we get  $\mathcal{V}_j$  equipped with the connection  $D_j$  for every  $j \geq 0$ . Now it can be shown that the isomorphism in (4.3) over each  $U_i$  patch together compatibly to give a global isomorphism over  $M$ . Indeed, this is an immediate consequence of the fact that the isomorphism in (4.3) intertwines the actions of  $\mathrm{SL}(\mathbb{V})$ . This completes the proof.  $\square$

Henceforth we assume that  $\mathrm{genus}(M) = g \geq 1$ . Since all the isomorphisms in Theorem 3.5 are  $\mathrm{SL}(\mathbb{V})$ -equivariant, the following corollary is deduced from Theorem 3.5.

**Corollary 4.2.** *Let  $M$  be a connected Riemann surface equipped with a projective structure  $P$ . If  $n < 0$  or  $n \geq k$ , then there is a canonical isomorphism*

$$J^k(\mathbf{L}^n) = \mathbf{L}^{n-k} \otimes \mathcal{V}_k.$$

*If  $k > n \geq 0$ , then there is a canonical isomorphism*

$$J^k(\mathbf{L}^n) = \mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1}).$$

*Remark 4.3.* Note that Proposition 4.1, Corollary 4.2 and (4.2) together give a description of any jet bundles of the form  $J^k(\mathbf{L}^a) \otimes J^\ell(\mathbf{L}^b)$  as a direct sum of vector bundles of the form  $\mathcal{V}^p \otimes \mathbf{L}^q$ .

**4.2. Jets of vector bundles with a holomorphic connection.** Let  $M$  be a compact connected Riemann surface. Let  $E$  be a holomorphic vector bundle on  $M$  equipped with a holomorphic connection  $D_E$ . Take a holomorphic vector bundle  $W$  on  $M$ . We will construct a holomorphic homomorphism from  $J^k(E \otimes W)$  to  $E \otimes J^k(W)$  for all  $k \geq 0$ . To construct this homomorphism, let  $f_1, f_2 : M \times M \rightarrow M$  be the projections to the first and second factors respectively. The diagonal divisor

$$\{(x, x) \mid x \in M\} \subset M \times M$$

will be denoted by  $\Delta_M$ . Recall from (2.4) that

$$\begin{aligned} J^k(E \otimes W) &:= f_{1*}((f_2^*(E \otimes W))/((f_2^*(E \otimes W)) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))) \\ &= f_{1*}(((f_2^*E) \otimes (f_2^*W))/((f_2^*E) \otimes (f_2^*W) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))) \rightarrow M. \end{aligned} \quad (4.4)$$

Consider the pulled back connection  $f_2^*D_E$  on  $f_2^*E$ . Since the sheaf of holomorphic two-forms on  $M$  is the zero sheaf, the connection  $D_E$  is flat (the curvature vanishes identically). Hence the connection  $f_2^*D_E$  is also flat. Therefore, on any analytic neighborhood  $U$  of  $\Delta_M \subset M \times M$  that admits a deformation retraction to  $\Delta_M$ , the two vector bundles  $(p_1^*E)|_U$  and  $(p_2^*E)|_U$  are identified using parallel translation. Invoking this isomorphism between  $(p_1^*E)|_U$  and  $(p_2^*E)|_U$ , from (4.4) we have

$$J^k(E \otimes W) = f_{1*}(((f_1^*E) \otimes (f_2^*W))/((f_1^*E) \otimes (f_2^*W) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))).$$

Hence the projection formula gives that

$$J^k(E \otimes W) = E \otimes f_{1*}((f_2^*W)/((f_2^*W) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))) = E \otimes J^k(W). \quad (4.5)$$

**Lemma 4.4.** *Let  $M$  be a connected Riemann surface equipped with a projective structure  $P$ . Let  $E$  be a holomorphic vector bundle on  $M$  equipped with a holomorphic connection  $D_E$ . If  $n < 0$  or  $n \geq k$ , then there is a canonical isomorphism*

$$J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k.$$

*If  $k > n \geq 0$ , then there is a canonical isomorphism*

$$J^k(E \otimes \mathbf{L}^n) = E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1})).$$

*Proof.* This follows from (4.5) and Corollary 4.2. We omit the details.  $\square$



**4.3. The symbol map.** As before,  $M$  is a compact connected Riemann surface equipped with a projective structure  $P$ , and  $E$  is a holomorphic vector bundle over  $M$  equipped with a holomorphic connection  $D_E$ . Consider the projection

$$\phi_k : J^k(E \otimes \mathbf{L}^n) \longrightarrow J^{k-1}(E \otimes \mathbf{L}^n) \quad (4.6)$$

in (2.5). We will describe  $\phi_k$  in terms of the isomorphisms in Lemma 4.4.

4.3.1. *Case where either  $n < 0$  or  $n \geq k$ .* Assume that at least one of the following two conditions is valid:

- (1)  $n < 0$ , or
- (2)  $n \geq k$ .

In view of Lemma 4.4, this implies that

$$J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k \quad \text{and} \quad J^{k-1}(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k+1} \otimes \mathcal{V}_{k-1} \quad (4.7)$$

when  $k \geq 1$  (note that in (4.6) we have  $k \geq 1$ ). Using (4.7), the projection  $\phi_k$  in (4.6) is a surjective homomorphism

$$\phi_k : E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k \longrightarrow E \otimes \mathbf{L}^{n-k+1} \otimes \mathcal{V}_{k-1}. \quad (4.8)$$

Recall that  $\mathcal{V}_j = \text{Sym}^j(\mathcal{V})$ , and the line bundle  $\mathbf{L}$  is a quotient of  $\mathcal{V}$  (see (4.1)). Therefore, we have a natural projection

$$\varpi_k : \mathcal{V}_k \longrightarrow \mathbf{L} \otimes \mathcal{V}_{k-1}. \quad (4.9)$$

To describe  $\varpi_k$  explicitly, let  $q_{\mathbf{L}} : \mathcal{V} \longrightarrow \mathbf{L}$  be the quotient map. Consider the homomorphism

$$q_{\mathbf{L}} \otimes \text{Id}_{\mathcal{V}^{\otimes(k-1)}} : \mathcal{V}^{\otimes k} \longrightarrow \mathbf{L} \otimes \mathcal{V}^{\otimes(k-1)}.$$

Note that  $q_{\mathbf{L}} \otimes \text{Id}_{\mathcal{V}^{\otimes(k-1)}}$  sends the subbundle  $\text{Sym}^k(\mathcal{V}) \subset \mathcal{V}^{\otimes k}$  to the subbundle  $\mathbf{L} \otimes \mathcal{V}_{k-1} \subset \mathbf{L} \otimes \mathcal{V}^{\otimes(k-1)}$ . The homomorphism  $\varpi_k$  in (4.9) is this restriction of the homomorphism  $q_{\mathbf{L}} \otimes \text{Id}_{\mathcal{V}^{\otimes(k-1)}}$ .

**Proposition 4.5.** *The homomorphism  $\phi_k$  in (4.8) coincides with  $\text{Id} \otimes \varpi_k$ , where  $\text{Id}$  denotes the identity map of  $E \otimes \mathbf{L}^{n-k}$ , and  $\varpi_k$  is the homomorphism in (4.9).*

*Proof.* This follows from Proposition 3.1 and Proposition 3.2. □

Consider the inclusion homomorphism  $\iota_{\mathcal{L}} : \mathcal{L} \hookrightarrow \mathcal{V}$  (see (4.1)). We have the subbundle

$$\iota' := \text{Id}_E \otimes \iota_{\mathcal{L}}^{\otimes k} : E \otimes \mathcal{L}^{\otimes k} \hookrightarrow E \otimes \text{Sym}^k(\mathcal{V}) = E \otimes \mathcal{V}_k.$$

Using the isomorphism  $\mathbf{L} = \mathcal{L}^*$  in (4.2), the above homomorphism  $\iota'$  produces a homomorphism

$$\tilde{\iota} := \iota' \otimes \text{Id}_{\mathbf{L}^{n-k}} : E \otimes \mathbf{L}^{n-2k} = E \otimes \mathcal{L}^{\otimes k} \otimes \mathbf{L}^{n-k} \hookrightarrow E \otimes \mathcal{V}_k \otimes \mathbf{L}^{n-k}. \quad (4.10)$$

The following is an immediate consequence of Proposition 4.5.

**Corollary 4.6.** *The kernel of the surjective homomorphism  $\phi_k$  in (4.8) coincides with the subbundle  $E \otimes \mathbf{L}^{n-2k} \hookrightarrow E \otimes \mathcal{V}_k \otimes \mathbf{L}^{n-k}$  given by the image of the homomorphism  $\tilde{\iota}$  in (4.10).*

Let  $F$  be any holomorphic vector bundle on  $M$ . Using (2.6) and (4.7), we have

$$\mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F) = F \otimes (E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k)^* = F \otimes \mathbf{L}^{k-n} \otimes (E \otimes \mathcal{V}_k)^*. \quad (4.11)$$

Using the homomorphism  $\tilde{\iota}$  in (4.10) construct the homomorphism

$$\mathrm{Id}_F \otimes \tilde{\iota}^* : F \otimes (E \otimes \mathcal{V}_k \otimes \mathbf{L}^{n-k})^* \longrightarrow F \otimes (E \otimes \mathbf{L}^{n-2k})^* = \mathrm{Hom}(E, F) \otimes \mathbf{L}^{2k-n}. \quad (4.12)$$

Combining (4.11) and (4.12), we have a surjective homomorphism

$$\sigma_k : \mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F) \longrightarrow \mathrm{Hom}(E, F) \otimes \mathbf{L}^{2k-n}.$$

The following is an immediate consequence of Corollary 4.6.

**Corollary 4.7.** *The symbol homomorphism*

$$\mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F) \longrightarrow \mathrm{Hom}(E \otimes \mathbf{L}^n, F) \otimes (TM)^{\otimes k}$$

in (2.7) coincides with the above homomorphism  $\sigma_k$  after  $TM$  is identified with  $\mathbf{L}^{\otimes 2}$  as in (4.2).

4.3.2. *Case where  $k > n \geq 0$ .* We first assume that  $k - 1 > n \geq 0$ . Therefore, from Lemma 4.4 we have  $J^k(E \otimes \mathbf{L}^n) = E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1}))$  and

$$J^{k-1}(E \otimes \mathbf{L}^n) = E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-k} \otimes \mathcal{V}_{k-n-2})).$$

Let

$$\begin{aligned} \phi_k : E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1})) &= J^k(E \otimes \mathbf{L}^n) \\ \longrightarrow J^{k-1}(E \otimes \mathbf{L}^n) &= E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-k} \otimes \mathcal{V}_{k-n-2})) \end{aligned} \quad (4.13)$$

be the projection in (2.5). Just as the homomorphism  $\mathrm{Id} \otimes \varpi_k$  in Proposition 4.5, we have a homomorphism

$$\widehat{\varpi}_k : E \otimes \mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1} \longrightarrow E \otimes \mathbf{L}^{-k} \otimes \mathcal{V}_{k-n-2}.$$

**Lemma 4.8.** *The homomorphism  $\phi_k$  in (4.13) coincides with the homomorphism*

$$\mathrm{Id}_{E \otimes \mathcal{V}_n} \oplus \widehat{\varpi}_k : E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1})) \longrightarrow E \otimes (\mathcal{V}_n \oplus (\mathbf{L}^{-k} \otimes \mathcal{V}_{k-n-2})).$$

*Proof.* This follows from Proposition 3.4. □

Note that using (4.2),

$$\mathrm{kernel}(\widehat{\varpi}_k) = E \otimes \mathbf{L}^{-(k+1)} \otimes \mathcal{L}^{\otimes (k-n-1)} = E \otimes K_M^{\otimes k} \mathbf{L}^n.$$

The following is an immediate consequence of Lemma 4.8.

**Corollary 4.9.** *Let  $F$  be any holomorphic vector bundle on  $M$ . The symbol homomorphism*

$$\mathrm{Diff}_M^k(E \otimes \mathbf{L}^n, F) \longrightarrow \mathrm{Hom}(E \otimes \mathbf{L}^n, F) \otimes (TM)^{\otimes k}$$

coincides with  $\mathrm{Id}_F \otimes \iota^*$ , where  $\iota : \mathrm{kernel}(\widehat{\varpi}_k) \hookrightarrow E \otimes \mathbf{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1}$  is the inclusion map.

We now assume that  $k - 1 = n \geq 0$ .

Under this assumption, from Lemma 4.4 we have

$$J^k(E \otimes \mathbf{L}^n) = E \otimes (\mathcal{V}_n \oplus \mathbf{L}^{-(k+1)}) \quad \text{and} \quad J^{k-1}(E \otimes \mathbf{L}^n) = E \otimes \mathcal{V}_n.$$

Let

$$\phi_k : E \otimes (\mathcal{V}_n \oplus \mathbf{L}^{-(k+1)}) = J^k(E \otimes \mathbf{L}^n) \longrightarrow J^{k-1}(E \otimes \mathbf{L}^n) = E \otimes \mathcal{V}_n \quad (4.14)$$

be the projection in (2.5).

**Lemma 4.10.** *The homomorphism  $\phi_k$  in (4.14) coincides with the projection*

$$E \otimes (\mathcal{V}_n \oplus \mathbf{L}^{-(k+1)}) \longrightarrow E \otimes \mathcal{V}_n$$

*to the first factor.*

*Proof.* This follows from Proposition 3.4. □

The following is an immediate consequence of Lemma 4.10.

**Corollary 4.11.** *Let  $F$  be any holomorphic vector bundle on  $M$ . The symbol homomorphism*

$$\text{Diff}_M^k(E \otimes \mathbf{L}^n, F) \longrightarrow \text{Hom}(E \otimes \mathbf{L}^n, F) \otimes (TM)^{\otimes k}$$

*coincides with  $\text{Id}_F \otimes p'$ , where*

$$p' : E^* \otimes (\mathcal{V}_n \oplus \mathbf{L}^{-(k+1)})^* \longrightarrow E^* \otimes (\mathbf{L}^{-(k+1)})^*$$

*is the natural projection.*

## 5. LIFTING OF SYMBOL

As before,  $M$  is a connected Riemann surface equipped with a projective structure  $P$ , and  $E$  is a holomorphic vector bundle on  $M$  equipped with a holomorphic connection  $D_E$ . The connection  $D_E$  on  $E$  induces a holomorphic connection on  $\text{End}(E)$ . This induced connection on  $\text{End}(E)$  will be denoted by  $\mathcal{D}_E$ .

Take integers  $k, n$  and  $l$  such that at least one of the following two conditions is valid:

- (1)  $n < 0$ , or
- (2)  $n, l \geq k$ .

Let

$$\theta_0 \in H^0(M, \text{End}(E) \otimes \mathbf{L}^l) \quad (5.1)$$

be a holomorphic section. Note that  $\theta_0$  defines a section

$$\theta \in H^0(M, J^k(\text{End}(E) \otimes \mathbf{L}^l)).$$

Recall that  $D_E$  induces a holomorphic connection  $\mathcal{D}_E$  on  $\text{End}(E)$ . In view of this connection, Lemma 4.4 says that  $J^k(\text{End}(E) \otimes \mathbf{L}^l) = \text{End}(E) \otimes \mathbf{L}^{l-k} \otimes \mathcal{V}_k$ . So, we have

$$\theta \in H^0(M, \text{End}(E) \otimes \mathbf{L}^{l-k} \otimes \mathcal{V}_k). \quad (5.2)$$

Recall from Lemma 4.4 that  $J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k$ . Let

$$\mathcal{T}_\theta^0 : J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k \longrightarrow E \otimes \mathbf{L}^{l+n-2k} \otimes \mathcal{V}_k \otimes \mathcal{V}_k$$

be the homomorphism defined by  $a \otimes b \mapsto \sum_i (a'_i(a) \otimes b \otimes b'_i)$ , where  $a$  and  $b$  are local sections of  $E$  and  $\mathbf{L}^{n-k} \otimes \mathcal{V}_k$  respectively, while  $\theta$  in (5.2) is locally expressed as  $\sum_i a'_i \otimes b'_i$  with  $a'_i$  and  $b'_i$  being local sections of  $\text{End}(E)$  and  $\mathbf{L}^{l-k} \otimes \mathcal{V}_k$  respectively. Let

$$p_0 : \mathcal{V}_k \otimes \mathcal{V}_k \longrightarrow \mathcal{V}_0 = \mathcal{O}_M$$

be the projection constructed using the decomposition in Proposition 4.1. Now define the homomorphism

$$\mathcal{T}_\theta := (\text{Id} \otimes p_0) \circ \mathcal{T}_\theta^0 : J^k(E \otimes \mathbf{L}^n) = E \otimes \mathbf{L}^{n-k} \otimes \mathcal{V}_k \longrightarrow E \otimes \mathbf{L}^{l+n-2k}, \quad (5.3)$$

where  $\text{Id}$  denotes the identity map of  $E \otimes \mathbf{L}^{l+n-2k}$ . Therefore, we have

$$\mathcal{T}_\theta \in H^0(M, \text{Diff}^k(E \otimes \mathbf{L}^n, E \otimes \mathbf{L}^{l+n-2k})) \quad (5.4)$$

(see (2.6)). Let

$$\hat{\sigma} : \text{Diff}^k(E \otimes \mathbf{L}^n, E \otimes \mathbf{L}^{l+n-2k}) \longrightarrow \text{End}(E) \otimes \mathbf{L}^{l-2k} \otimes (TM)^{\otimes k} = \text{End}(E) \otimes \mathbf{L}^l$$

be the symbol homomorphism in (2.7); in (4.2) it was shown that  $TM = \mathbf{L}^{\otimes 2}$ .

**Theorem 5.1.** *The symbol  $\hat{\sigma}(\mathcal{T}_\theta)$  of the differential operator  $\mathcal{T}_\theta$  in (5.4) is the section  $\theta_0$  in (5.1).*

*Proof.* This theorem can be derived from Corollary 4.7 that describes the symbol. The following is needed to derive the theorem from Corollary 4.7.

Consider the decomposition of the  $\text{SL}(\mathbb{V})$ -module

$$\text{Sym}^m(\mathbb{V}) \otimes \text{Sym}^m(\mathbb{V}) = \bigoplus_{i=0}^m \text{Sym}^{2(m-i)}(\mathbb{V})$$

in (4.3). Let

$$f : \text{Sym}^m(\mathbb{V}) \otimes \text{Sym}^m(\mathbb{V}) \longrightarrow \text{Sym}^0(\mathbb{V}) = \mathbb{C}$$

be the projection constructed using this decomposition (by setting  $i = m$  in the decomposition). Then for any line  $L_0 \subset \mathbb{V}$ , we have

$$f((L_0^{\otimes m}) \otimes (L_0 \otimes \text{Sym}^{m-1}(\mathbb{V}))) = 0,$$

so  $f$  produces a homomorphism

$$\hat{f} : (L_0^{\otimes m}) \otimes ((\text{Sym}^m(\mathbb{V}))/((L_0 \otimes \text{Sym}^{m-1}(\mathbb{V})))) = (L_0^{\otimes m}) \otimes (\mathbb{V}/L_0)^{\otimes m} \longrightarrow \mathbb{C}.$$

This homomorphism  $\hat{f}$  sends  $v^{\otimes m} \otimes u^{\otimes m}$ , where  $v \in L_0$  and  $u \in \mathbb{V}/L_0$ , to  $\lambda^m \in \mathbb{C}$  that satisfies the equation

$$v \wedge u = \lambda \cdot \omega,$$

where  $\omega$  is the fixed element in (3.4); note that  $v \wedge u$  is a well-defined element of  $\bigwedge^2 \mathbb{V}$ . The theorem follows from this observation and Corollary 4.7.  $\square$

*Remark 5.2.* It may be mentioned that symbols do not always lift to a holomorphic differential operator. For such an example, set the symbol to be the constant function 1. It does not lift to a first order holomorphic differential operator from  $\mathcal{L}^n$  to  $\mathcal{L}^n \otimes K_M$  if  $g \geq 2$  and  $n \neq 0$ . Indeed, a first order holomorphic differential operator from  $\mathcal{L}^n$  to  $\mathcal{L}^n \otimes K_M$  with symbol 1 is a holomorphic connection on  $\mathcal{L}^n$ . But  $\mathcal{L}^n$  does not admit a holomorphic connection because its degree is nonzero.

## ACKNOWLEDGEMENTS

This work has been supported by the French government through the UCAJEDI Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR2152IDEX201. The first author is partially supported by a J. C. Bose Fellowship.

## REFERENCES

- [At1] M. F. Atiyah, On the Krull-Schmidt theorem with application to sheaves, *Bull. Soc. Math. Fr.* **84** (1956), 307–317.
- [At2] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.
- [FH] W. Fulton and J. Harris, *Representation theory. A first course*, Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York,
- [Gr] A. Grothendieck, Sur la classification des fibrés holomorphes sur la sphère de Riemann, *Amer. Jour. Math.* **79** (1957), 121–138.
- [Gu] R. C. Gunning, *On uniformization of complex manifolds: the role of connections*, Princeton Univ. Press, 1978.
- [Ha] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.
- [KMS] I. Kolar, P. Michor and J. Slovák, *Natural operations in differential geometry*, Springer-Verlag, Berlin-Heidelberg, 1993.
- [Ni] A. Nijenhuis, Natural bundles and their general properties, in *Differential Geometry*, in honour of K. Yano (Kinokuniya, Tokyo, 1972), 317–334.
- [Te] C. Terng, Natural vector bundles and natural differential operators, *Amer. Jour. Math.* **100** (1978), 775–828.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI 400005, INDIA

*Email address:* `indranil@math.tifr.res.in`

UNIVERSITÉ CÔTE D’AZUR, CNRS, LJAD, FRANCE

*Email address:* `dumitres@unice.fr`