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# INTERMITTENCY AT THE ONSET OF STOCHASTICITY IN NONLINEAR RESONANT COUPLING PROCESSES

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A simple model for the nonlinear saturation of a plasma instability is studied via numerical solution of the resonant three-wave coupling equations. When parameters are varied the attractors of motion undergo bifurcations of several types. Intermittency is shown to occur in a transition from a limit cycle to a strange attractor. Such a transition might be indicative of an intermittent onset of turbulence in certain plasma experiments.

## 1. Introduction

The occurrence of stochasticity in low-dimensional dynamical systems is now common knowledge [1, 2]. However, most models up to now studied in fluid mechanics [2–5] are derived from arbitrary truncations of partial differential equations. In this way one is easily led to finite order differential systems.

Such a procedure may be grounded on reasons of mathematical simplicity, but the models thus obtained can hardly describe any realistic situation of fluid physics. On the other hand, certain laboratory plasma experiments effectively involve but a few modes [6–8]. The peculiar nature of plasma instabilities brings a natural explanation to this situation. The plasma endures as long as the instability saturation level remains low enough. Such a low level implies a small growth rate of instabilities and a structure of those perturbations which does not greatly differ from the structure of linear modes. Besides most waves are so strongly damped in these situations that they take no part in the system's evolution. Then, only a finite number of modes remains and those are endowed with small growth or damping rates.

Those considerations have led to investigate an elementary model which pictures the quadratic interaction of two modes [9, 10]. In previous works on this system [11–13] chaotic behaviours have been obtained for particular values of the parameters. Subharmonic bifurcations cascades have been found together with a tangent bifurcation [12, 13] leading to a three loops cycle. In our work, we first thoroughly investigate a large domain of parameter values, and we determine the nature of the attractors existing in different regions, thus completing the previous works. More importantly we find that transitions with intermittency effectively occurs at certain tangent bifurcations. Such a phenomenon occurs in fluids experiments, where turbulence often appears in the guise of random bursts [17–19]. Though such a behaviour has already been found for the Lorenz system [22], it is of interest to show that the same feature can be observed for the simple real dynamical system we consider. It must be also stressed that our system has important differences with the Lorenz system. Namely, it does not have either symmetry or Lyapounov functions ensuring the boundedness of solutions.

## 2. The model

We consider the non-linear decay of a linearly unstable high-frequency wave into a linearly damped low-frequency one.  $\gamma_0$  and  $\gamma_1$  represent the growth and damping rates, respectively, of those two modes. The real parameter  $V$  is responsible for the quadratic coupling. We also introduce a frequency mismatch  $\Delta\omega = \omega_0 - 2\omega_1$ . Then the equations for complex mode amplitudes  $A_0 = |A_0| e^{i\omega_0 t}$  and  $A_1 = |A_1| e^{i\omega_1 t}$  write

$$i \left( \frac{dA_0}{dt} - \gamma_0 A_0 \right) = VA_1^2 \exp(-i\Delta\omega t),$$

$$i \left( \frac{dA_1}{dt} - \gamma_1 A_1 \right) = VA_0 A_1^* \exp(+i\Delta\omega t).$$

This system calls into play three independent variables; namely  $|A_0|$ ,  $|A_1|$  and  $\theta = \varphi_0 - 2\varphi_1 + \Delta\omega t$ , a relative phase.

Following ref. [9] we set

$$x = \frac{V}{\gamma_0} |A_0| \sin \theta, \quad y = \frac{V}{\gamma_0} |A_0| \cos \theta,$$

$$z = \frac{V^2}{\gamma_0^2} |A_1|^2, \quad \tau = \gamma_0 t,$$

and reduce it to the dimensionless form:

$$\frac{dx}{d\tau} = x + \delta y - z + 2xy^2,$$

$$\frac{dy}{d\tau} = y - \delta x - 2xy,$$

$$\frac{dz}{d\tau} = -2\gamma z + 2xz.$$

These equations depend on the two parameters:

$$\gamma = -\frac{\gamma_1}{\gamma_0}, \quad \delta = \frac{\Delta\omega}{\gamma_0}.$$

The inequality  $\gamma > 1$  ensures volume contraction in phase space  $(x, y, z)$ . Hence we shall limit our study to the half-plane  $\gamma > 1$  in parameter space  $(\gamma, \delta)$ .

## 3. The attracting sets

### 3.1 The fixed points

The system has two stationary solutions. One is located at the origin and always remains unstable. Its unstable and stable manifolds are the  $x$ - $y$  plane and 1-manifold tangent to the  $z$  axis, respectively.

The second fixed point C has coordinates

$$\left( \gamma, \frac{\gamma\delta}{1-2\gamma}, \gamma \left[ 1 + \left( \frac{\delta}{1-2\gamma} \right)^2 \right] \right).$$

Provided that  $\gamma > (1 + \sqrt{3})/2$ , this point remains linearly stable as long as  $|\delta| > \delta_c$  where  $\delta_c^2 = (1-2\gamma)^2 (2\gamma^2 - 2\gamma + 1) / (2\gamma^2 - 2\gamma - 1)$ .

When the system's representative point in parameter space crosses the critical curve

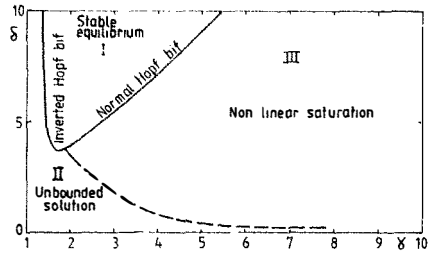


Fig. 1. The three regions in  $(\gamma, \delta)$  space.

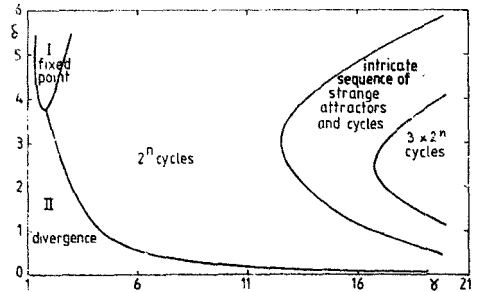


Fig. 2. Observed attractors in region III. Curve (a) is the location of accumulation points for the subharmonic bifurcation cascade originating from C; Curve (b) is the location of tangent transition to 3 loops cycle.

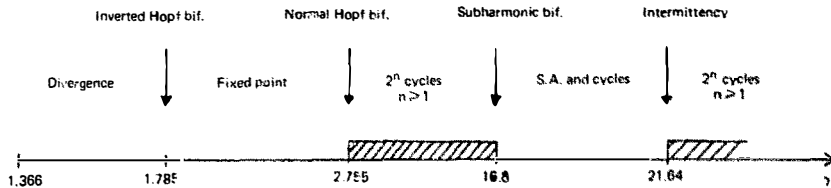


Fig. 3. Successive bifurcations for  $\delta = 5$ .

$\delta = \delta_c(\gamma)$ , C loses stability by either subcritical or supercritical Hopf bifurcation [11].

We thus are led to distinguish three regions in the  $(\gamma, \delta)$  plane which we have extensively studied by numerical methods (for  $0 < \delta < 10$ ,  $(1 + \sqrt{3})/2 < \gamma < 50$  (see fig. 1).

-region I: C is locally stable, that is to say the unstable high frequency wave saturates at a constant amplitude.

### 3.2 Region II. the adiabatic regime

In this region, C has lost stability by subcritical bifurcation. No bounded solutions exist; namely the variables  $x(\tau)$ ,  $y(\tau)$ ,  $z(\tau)$  oscillate with exponentially growing amplitude and frequencies. The decay process is not efficient for saturating the linear instability.

### 3.3 Region III. periodic and aperiodic regimes

In this region, C has lost its stability by supercritical bifurcation. The system displays an intricate bifurcation sequence with alternation of periodic (multilooped cycles) and non-periodic attractors (see fig. 2). Those attracting sets never achieve global stability, their attraction basins are large, however.

The transitions between limit cycles and strange attractors follow two different schemes depending on parameter values:

-In the first one, the cycles undergo a subharmonic bifurcation cascade. Akin bifurcation sequences may be obtained in iterations of single-hump functions [14-16].

-In the second, a simple periodic attractor abruptly takes an aperiodic one's place. In the

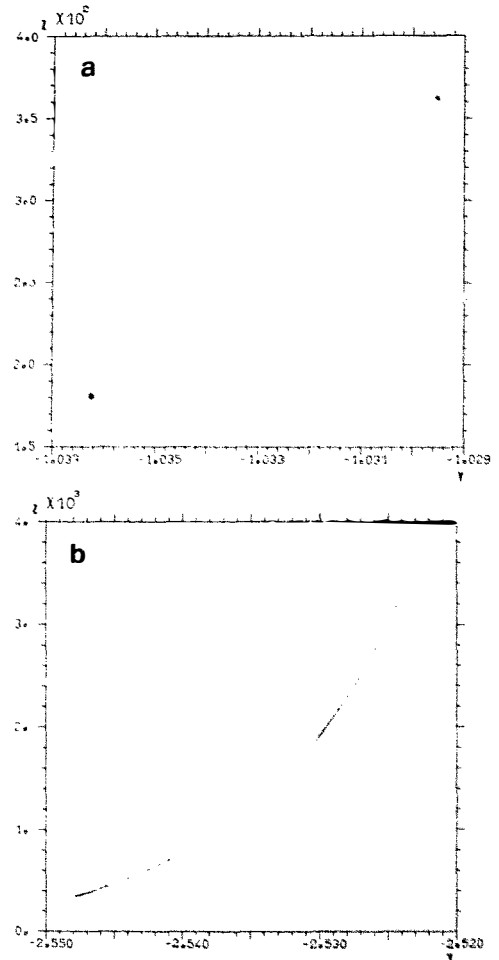


Fig. 4. Sections of the attractor. (a) for  $\gamma = 10$ ,  $\delta = 2$ ; (b) for  $\gamma = 21.6$ ,  $\delta = 5$ .

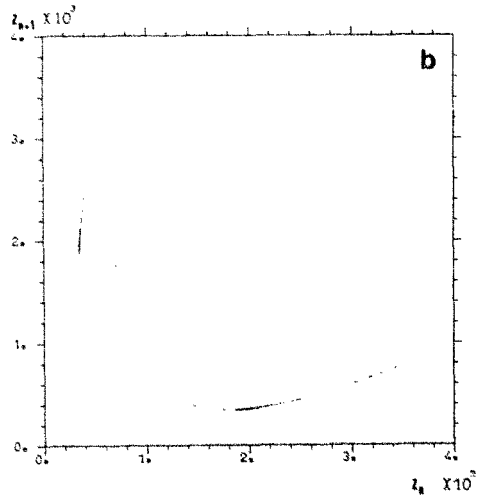
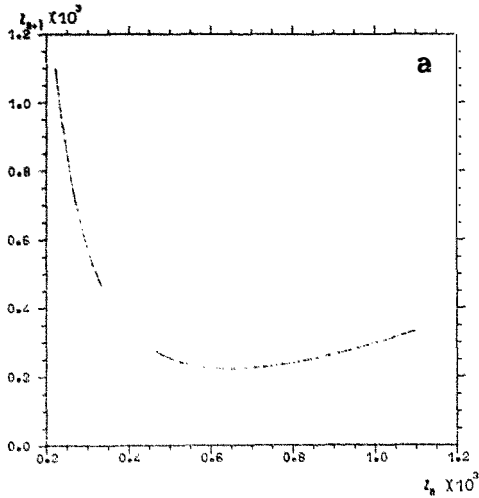


Fig. 5. Poincaré maps. (a) for  $\gamma = 13.18$ ,  $\delta = 3$ ; (b) for  $\gamma = 21.6$ ,  $\delta = 5$ .

case of period 2 and period 3 cycles, intermittency occurs at that transition. It appears to be the most striking result of this work. That latter behavior may be understood when considering an appropriate first return map of the flow.

We choose for convenience the plane  $x = +\gamma$  as the section surface, together with the condition  $dx/d\tau > 0$ . The section of the attractor by that plane consists of a finite set of points (limit cycles) or of arcs of curves (strange attractors) (see fig. 4).

Actually those curves do have a transversal structure owing to the uniqueness of solutions for such a differential system. However, that fine structure does not appear at the scale used in figures, owing to the strong contraction rate and we shall take no account of it. Hence, we can parametrize the position of a point by its  $z$  coordinate, since the curves present no singularity. We shall accordingly consider in the sequel the reduced Poincaré map  $f_z$ . Let us notice that  $z$  takes its largest values in the plane  $x = \gamma$ . Hence, our reduced Poincaré map  $f_z$  yields nothing else than the recurrence scheme for  $z$  maxima, i.e. the maximal values of the low-frequency wave energy.

The representative curve  $f_z$ , smoothly depends on parameters values (see fig. 5). For low  $\gamma$  values, we obtain a smooth parabolic

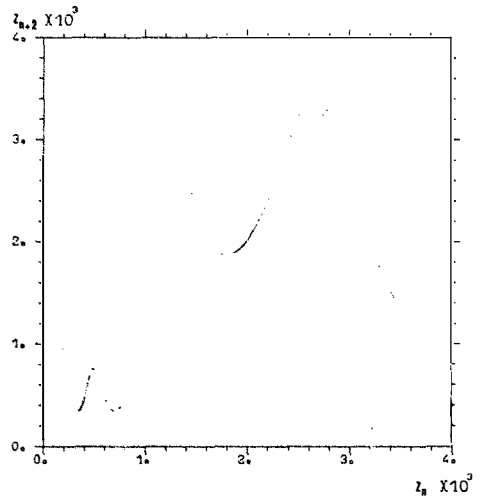


Fig. 6. Second iterate of the Poincaré map at  $\gamma = 21.6$ ,  $\delta = 5$ .

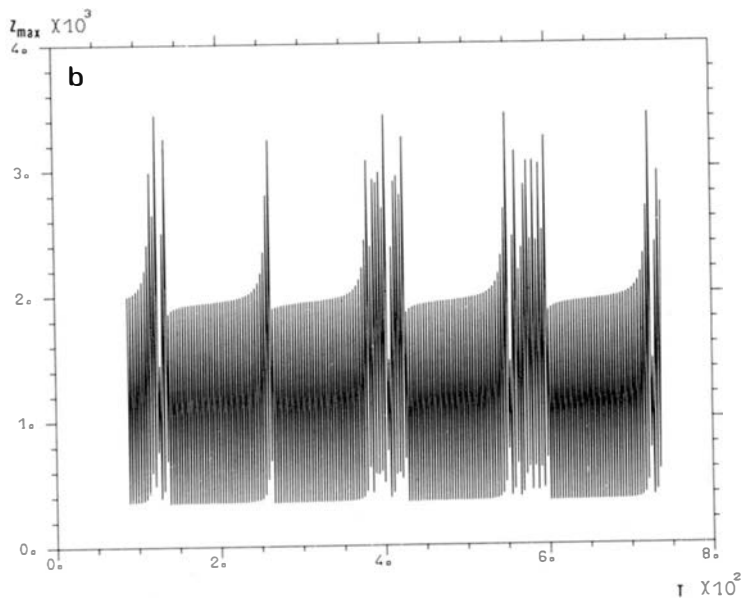
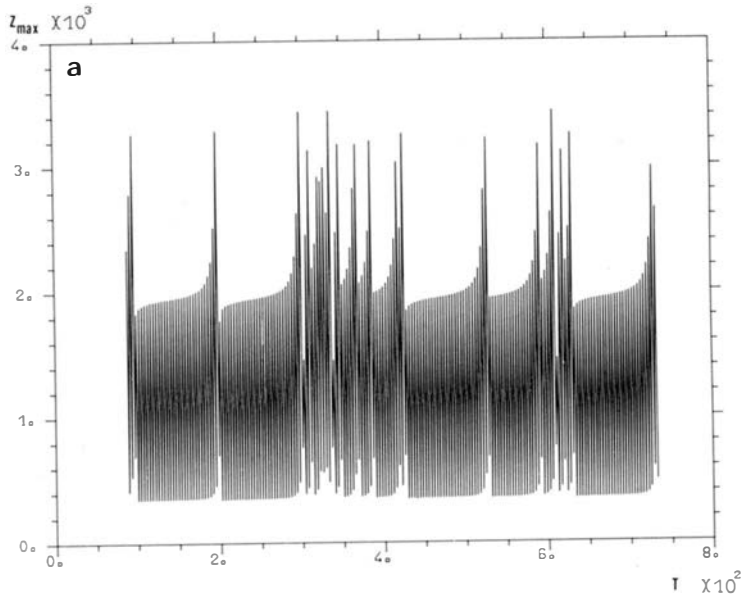
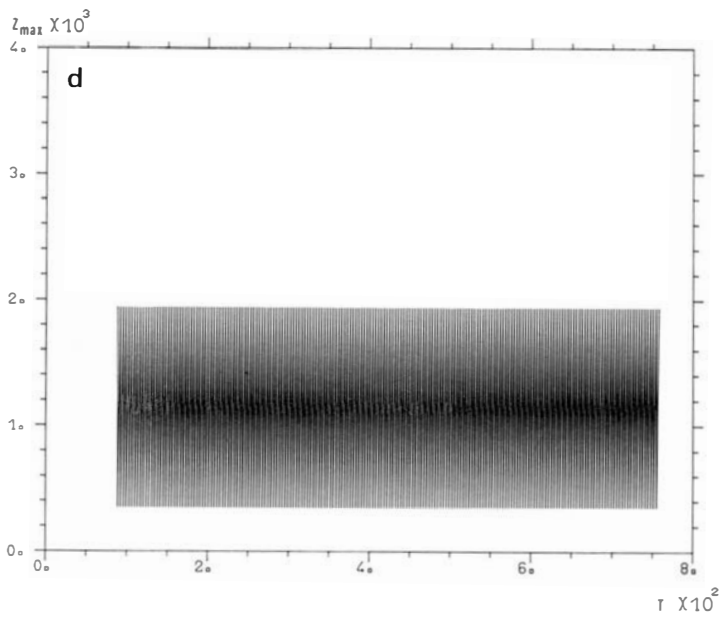
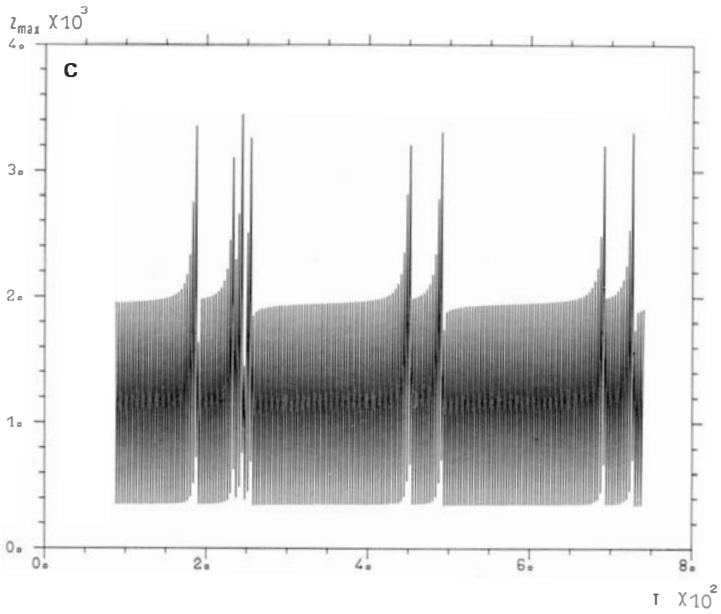


Fig. 7. Time plot of  $z_{\max}(t)$  for several values of  $\gamma$  at  $\delta = 5$ . (a)  $\gamma = 21.61$ , (b)  $\gamma = 21.62$ , (c)  $\gamma = 21.63$ , (d)  $\gamma = 21.64$ .



**Fig. 7. (Continued).**

curve (see fig. 5a). For higher  $\gamma$  values, the curve presents a hump (see fig. 5b). In that parameter range the system displays the transition from a strange attractor to a double looped cycle. Near the bifurcation threshold we observe an intermittent behavior.

#### 4. Intermittency

For  $\delta = 5$ , the transition occurs at  $\gamma \approx 21.64$ , see fig. 3. Since the limit cycle has two loops, we consider the second-return map i.e. the second iterate of the Poincaré map  $f_2$ .

For  $\gamma > 21.6$ , the representative curve intersects four times the first bisectrix, giving rise to two pairs of fixed points. To each pair belong a stable fixed point (section of one loop of the cycle) and an unstable one. Each pair collapses to a single point at  $\gamma_c = 21.64$ . Then the curve is tangent to the bisectrix in two points. For  $\gamma$  slightly smaller than 21.64, the curve no longer crosses the bisectrix but remains close to it (see fig. 6). Locally, the change in the trend of the curve reduces to the translation of a parabolic curve, together with a change of its second derivative.

Such bifurcations have been studied by Pomeau and Manneville [20, 21]. Following those authors, we shall refer to that phenomenon as type I intermittency.

We have plotted successive values of  $z$  as a function of time and observed the expected behavior (see fig. 7). For  $\gamma$  slightly smaller than  $\gamma_c$ , laminar periods (no loss of correlations) are interrupted by random turbulent bursts (loss of correlations). The duration of laminar periods is upper bounded.

The upper bound depends on the distance to the bifurcation threshold. Consequently, we have plotted that maximal duration as a function of  $\epsilon = (\gamma_c - \gamma)/\gamma_c$ . As predicted by Pomeau and Manneville, we have then obtained a  $\epsilon^{-1/2}$  law with a good accuracy (see fig. 8) (6%). Similar features are observed at the intermittent onset of turbulence following a period 3 cycle.

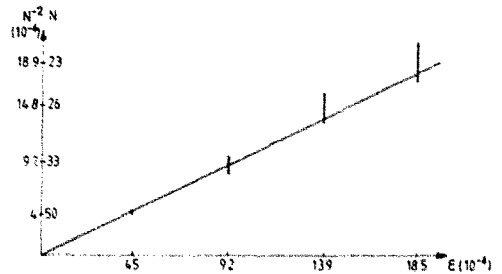


Fig. 8. The square of  $N$ , the largest number of cycles during a laminar period is proportional to the distance from the threshold  $(\gamma_c - \gamma)/\gamma_c$ .  $N$  is given within 1 cycle to account for the uncertainty in the definition of a laminar phase.

#### 5. Conclusion

That simple system displays a rich variety of behavior, yielding both subharmonic and intermittent bifurcations to stochasticity. That latter feature is observed in fluid experiments [17, 18]. The results we obtain suggest that plasma turbulence might also set in under such a guise.

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#### References

- [1] D. Ruelle and F. Takens, On the nature of turbulence, *Commun. Math. Phys.* 20, (1971) 167-192.
- [2] E.N. Lorenz, Deterministic non-periodic flows, *J. Atmospheric Sci.* 20 (1963) 130-141.
- [3] C. Boldrighini and V. Franceschini, A five-dimensional truncation of the plane incompressible Navier-Stokes equations, *Commun. Math. Phys.* 64 (1979) 159-170.
- [4] J.H. Curry, A generalized Lorenz system, *Commun. Math. Phys.* 60 (1978) 3.

- [5] J.B. MacLaughlin, Successive bifurcations leading to stochastic behavior, *J. Stat. Phys.* 15 (1976) 307-326.
- [6] J. Weiland and H. Wilhelmsson, *Coherent Nonlinear Interaction of Waves in Plasmas* (Pergamon, London, New York, 1977).
- [7] B.I. Cohen, J.A. Krommes, W.N. Tang and M.N. Rosenbluth, Nonlinear saturation of the dissipative trapped-ion mode by mode coupling, *Nucl. Fusion* 16 (1976) 6.
- [8] N.R. Sauthoff, S. Galeev and W. Stodiek, *Nucl. Fusion* 18 (1978) 1445.
- [9] G. Laval and R. Pellat, in: *Plasma Physics, Les Houches, 1972* (Gordon and Breach, New York, 1972).
- [10] S.Ya. Vyshkind, Occurrence of stochasticity in the case of nondegenerate wave interaction in media with amplification, *Izv. Vysshikh Uchebn. Zavedenii, Radiofiz.* 21 6 (1978) 850-856.
- [11] J.C. Adam, M.N. Bussac and G. Laval, Stabilization of a linearly unstable mode by resonant nonlinear coupling to damped modes, and J.M. Wersinger, J.M. Finn and E. Ott, Bifurcations and strange behaviour in coupled three waves system, in: *Intrinsic Stochasticity in Plasmas* (Les Editions de Physique, Orsay, France, 1979).
- [12] J.M. Wersinger, J.M. Finn and E. Ott, *Phys. Rev. Lett.* 44 (1980) 453.
- [13] J.M. Wersinger, J.M. Finn and E. Ott, *Phys. Fluids* 23 (1980) 1142.
- [14] M.J. Feigenbaum, *J. Stat. Phys.* 19 (178) 25.
- [15] R.M. May, Simple mathematical models with very complicated dynamics, *Nature* 261 (10 June 1976).
- [16] P. Coullet and C. Tresser, Iterations d'endomorphismes et groupe de renormalisation, *J. de Phys. Colloq.* C5 (1978) 5-25.
- [17] A. Libchaber and J. Maurer, *J. de Phys. Lett.*, to appear.
- [18] P. Bergé and M. Dubois, private communication.
- [19] D.J. Tritton, *Physical Fluid Dynamics* (Van Nostrand-Reinhold, New York, 1977).
- [20] Y. Pomeau and P. Manneville, Intermittent transition to turbulence in dissipative dynamical systems, *Commun. Math. Phys.* 74 (1980) 189-197.
- [21] P. Manneville and Y. Pomeau, Intermittency and the Lorenz model, *Phys. Lett.* 75A (1, 2) (1979).