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Asymptotic expansions for some integrals of quotients with degenerated divisors

Sergei Kuksin*

September 10, 2017

Abstract

We study asymptotic expansion as $\nu \to 0$ for integrals over $\mathbb{R}^{2d} = \{(x,y)\}$ of quotients $F(x,y)/\big((x\cdot y)^2 + (\nu\Gamma(x,y))^2\big)$, where Γ is strictly positive and F decays at infinity sufficiently fast. Integrals of this kind appear in description of the four–waves interactions.

1 Introduction

s2

Our concern is the integrals

$$I_\nu = \int_{\mathbb{R}^d\times\mathbb{R}^d} dx\,dy\ \frac{F(x,y)}{(x\cdot y)^2 + (\nu\Gamma(x,y))^2}\,, \quad d\ge 2\,,\ 0<\nu\ll 1. \tag{1.1}$$ I_s

Such integrals and their singular limits $\nu \to 0$ appear in physical works on the four-waves interaction, where the latter is suggested as a mechanism, dictating the long-time behaviour of solutions for nonlinear Hamiltonian PDEs with cubic nonlinearities and large values of the space-period. Usually the integrals I_ν appear there in an implicit form, and become visible as a result of rigorous mathematical analysis of the objects and constructions, involved in the heuristic physical argument (see below in this section).

We denote $(x,y) = z \in \mathbb{R}^{2d}$, $\omega(z) = x\cdot y$, and assume that F and Γ are C^2–smooth real functions, satisfying [1]

$$|\partial_z^\alpha F(z)| \le K\langle z\rangle^{-M-|\alpha|} \quad \forall\, z,\ \forall\,|\alpha|\le 2\,; \tag{1.2}$$ F_1

$$|\Gamma(z)| \ge K^{-1}\langle z\rangle^{r_*}\ \ \forall\, z\,, \qquad |\partial_z^\alpha\Gamma(z)| \le K\langle z\rangle^{r_*-|\alpha|}\ \ \forall\, z,\ \forall\,|\alpha|\le 2\,. \tag{1.3}$$ Ga_1

Here r_*, M, K are any real constants such that

$$M + r_* > 2d-2, \qquad M > 2d-4, \qquad K>1\,. \tag{1.4}$$ hr

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[1] For example, $\Gamma = \langle z\rangle^{2m}$, $m\in\mathbb{R}$, and F is a Schwartz function.

As usual we denote $\langle z\rangle = \sqrt{|z|^2+1}$.

The main difficulty in the study of I_ν comes from the vicinity of the quadric $\Sigma = \{\omega(z) = 0\}$. The latter has a locus at $0 \in \mathbb{R}^{2d}$ and is smooth outside it. Firstly we will study I_ν near 0, next – near the smooth part of the quadric, $\Sigma \setminus \{0\} =: \Sigma_*$, and finally will combine the results obtained to get the main result of this work:

t_singint **Theorem 1.1.** *As $\nu \to 0$, the integral I_ν has the following asymptotic:*

$$I_\nu = \pi\nu^{-1} \int_{\Sigma_*} \frac{F(z)}{|z|\,\Gamma(z)}\, d_{\Sigma_*} z + I^\Delta. \tag{1.5}$$ p22

Here $d_{\Sigma_*} z$ is the volume element on Σ_* [2] *and $|I^\Delta| \le C\,\chi_d(\nu)$, where*

$$\chi_d(\nu) = \begin{cases} 1, & d \ge 3\,, \\ \max(1, \ln(\nu^{-1})), & d = 2\,. \end{cases} \tag{1.6}$$ chi_d

The integral in (1.5) *converges absolutely, and the constant C depends on d, K, M and r_*.*

The integral in (1.5) may be regarded as integrating of the function F/Γ against a measure in $\mathbb{R}^{2d}$, supported by Σ, which we will denote $|z|^{-1}\delta_{\Sigma_*}$ (here δ_{Σ_*} is the delta–function of the hypersurface Σ_*). For any real number m let $C_m(\mathbb{R}^{2d})$ be the space of continuous functions on $\mathbb{R}^{2d}$ with the finite norm $|f|_m = \sup_z |f(z)|\langle z\rangle^m$. By (1.2) and (1.3), $F/\Gamma \in C_{M+r_*}(\mathbb{R}^{2d})$, where $M + r_* > 2d-2$.

prop **Proposition 1.2.** *The measure $|z|^{-1}\delta_{\Sigma_*}$ is an atomlesss σ-finite Borel measure on $\mathbb{R}^{2d}$. The integrating over it defines a continuous linear functional on the space $C_m(\mathbb{R}^{2d})$ if $m > 2d-2$.*

Since any function $F \in C_0^\infty(\mathbb{R}^{2d})$ satisfies (1.2) for every M, then we have

Corollary 1.3. *Let a C^2–function Γ meets* (1.3) *with some r_*. Then the function $\nu/\big((x\cdot y)^2 + (\nu\Gamma(x,y))^2\big)$ converges to the measure $|z|^{-1}\delta_{\Sigma_*}$ as $\nu \to 0$, in the space of distributions.*

The theorem and the proposition are proved below in Sections 2–4.

In the mentioned above works from the non-linear physics, to describe the long-time behaviour of solutions for nonlinear Hamiltonian PDEs with cubic nonlinearities, physicists derived nonlinear kinetic equations, called the (four-) wave kinetic equations. The k-th component of the kinetic kernel K ($k \in \mathbb{R}^d$) for such equation is given by an integral of the following form:

$$K_k = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F_k(k_1,k_2,k_3)\delta^{kk_3}_{k_1k_2}\delta(\omega^{kk_3}_{k_1k_2})\, dk_1 dk_2 dk_3\,. \tag{1.7}$$ heur

Here $\delta^{kk_3}_{k_1k_2}$ is the delta-function $\{k + k_3 = k_1 + k_2\}$ and $\delta(\omega^{kk_3}_{k_1k_2})$ is the delta-function $\{\omega_k + \omega_{k_3} = \omega_{k_1} + \omega_{k_2}\}$, where $\{\omega_k\}$ is the spectrum of oscillations for

[2] corresponding to the Riemann structure on Σ_*, obtained by the restricting to Σ_* the standard Riemann stricture on $\mathbb{R}^{2d}$.

the linearised at zero equation. If the corresponding nonlinear PDE is the cubic NLS equation, then $\omega_k = |k|^2$. In this case the two delta-functions define the following algebraic set:

$$\{(k_1, k_2, k_3) \in \mathbb{R}^{3d} : k + k_3 = k_1 + k_2, \ \ |k|^2 + |k_3|^2 = |k_1|^2 + |k_2|^2\},$$

see [5], p.91, and [3]. Excluding s_3 using the first relation we write the second as $-2(k_1 - k) \cdot (k_2 - k) = 0$. Or $-2x \cdot y = 0$, if we denote $x = k_1 - k$, $y = k_2 - k$. That is, K_k is given by an integral over the set $\Sigma_* \subset \mathbb{R}^{2d}$ as in (1.5). In a work in progress (see [4]) we make an attempt to derive rigorously a wave kinetic equation for NLS with added small dissipation and small random force (see [3, 4] for a discussion of this model). On this way nonlinearities of the form (1.7) appear naturally as limits for $\nu \to 0$ of certain integrals of the form (1.1), where, again, $x = k_1 - k$, $y = k_2 - k$. [3] We strongly believe that more asymptotical expansions of integrals, similar to (1.1), will appear when more works on rigorous justification of physical methods to treat nonlinear waves will come out.

Proof of Theotem 1.1, given below in Sections 2–4, is rather general and applies to other integrals with singular divisors. Some of these applications are discussed in Section 5.

Notation. By χ_A we denote the characteristic function of a set A. For an integral $I = \int_{\mathbb{R}^{2d}} f(z)\, dz$ and a submanifold $M \subset \mathbb{R}^{2d}$, $\dim M = m \leq 2d$, compact or not (if $m = 2d$, then M is an open domain in $\mathbb{R}^{2d}$) we write

$$\langle I, M \rangle = \int_M f(z)\, d_M(z),$$

where $d_M(z)$ is the volume–element on M, induced from $\mathbb{R}^{2d}$.

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2 Integral over the vicinity of 0.

s_71

For $0 < \delta \leq 1$ consider the domain

$$K_\delta = \{|x| \leq \delta, |y| \leq \delta\} \subset \mathbb{R}^d \times \mathbb{R}^d,$$

and the integral

$$\int_{K_\delta} \frac{|F(x,y)|\, dx\, dy}{(x \cdot y)^2 + (\nu \Gamma(x,y))^2}. \tag{2.1}$$

int_delta

[3] So the integrand F_k depends on the parameter $k \in \mathbb{R}^d$. This dependence should be controlled, which can be done with some extra efforts.

Obviously, everywhere in K_δ, $|F(x,y)| \le C_1$ and $\Gamma(x,y) \ge C$. So the integral is bounded by $C_1 \bar{I}_\nu(\delta)$, where

$$\bar{I}_\nu(\delta) = \int_{|x|\le\delta} \int_{|y|\le\delta} \frac{dx\, dy}{(x\cdot y)^2 + (C\nu)^2}\,.$$

We write $\bar{I}_\nu(\delta)$ as

$$\bar{I}_\nu(\delta) = \int_{|x|\le\delta} J_x\, dx\,, \qquad J_x = \int_{|y|\le\delta} \frac{dy}{(x\cdot y)^2 + (C\nu)^2}\,.$$

Let us introduce in the y–space a coordinate system $(y_1, \dots, y_d)$ with the first basis vector $e_1 = x/r$, where $r = |x|$. Since the volume of the layer, lying in the ball $\{|y| \le \delta\}$ above an infinitesimal segment $[y_1, y_1 + dy_1]$ is $\le C_d \delta^{d-1} dy_1$ and since $(x\cdot y) = r y_1$, then

$$J_x \le C_d r^{-2} \int_0^\delta dy_1\, \frac{\delta^{d-1}}{{y_1}^2 + (C\nu/r)^2} = C_d \delta^{d-1} \frac{\tan^{-1}(r\delta/C\nu)}{Cr\nu} \le \frac{\pi}{2} C_d\, \frac{\delta^{d-1}}{Cr\nu}\,.$$

So

$$\bar{I}_\nu(\delta) = \int_{|x|\le\delta} J_x dx \le C_d \frac{\delta^{d-1}}{C\nu} \int_0^\delta r^{d-2}\, dr \le C'_d \delta^{2d-2} \nu^{-1}\,.$$

Thus we have proved

l_nearsing **Lemma 2.1.** *The integral* (2.1) *is bounded by* $C\nu^{-1}\delta^{2d-2}$.

Now we pass to the global study of the integral (1.1) and begin with studying the geometry of the manifold Σ_* and its vicinity in $\mathbb{R}^{2d}$.

3 The manifold Σ_* and its vicinity.

s_3

The set $\Sigma_* = \Sigma \setminus (0,0)$ is a smooth submanifold of $\mathbb{R}^{2d}$ of dimension $2d-1$. Let $\xi \in \mathbb{R}^{2d-1}$ be a local coordinate on Σ_* with the coordinate mapping $\xi \mapsto (x_\xi, y_\xi) = z_\xi \in \Sigma_*$. Abusing notation we write $|\xi| = |(x_\xi, y_\xi)|$. The vector $N(\xi) = (y_\xi, x_\xi)$ is a normal to Σ_* at ξ of length $|\xi|$, and

$$N(\xi)\cdot(x_\xi, y_\xi) = 2\, x_\xi \cdot y_\xi = 0\,. \tag{3.1}$$ orth

For any $0 \le R_1 < R_2$ we denote

$$\begin{aligned} S^{R_1} &= \{z \in \mathbb{R}^{2d} : |z| = R_1\}\,, \quad \Sigma^{R_1} = \Sigma \cap S^{R_1}\,, \\ S^{R_2}_{R_1} &= \{z : R_1 < |z| < R_2\}\,, \quad \Sigma^{R_2}_{R_1} = \Sigma \cap S^{R_2}_{R_1}\,, \end{aligned} \tag{3.2}$$ nota

and for $t > 0$ denote by D_t the dilation operator

$$D_t : \mathbb{R}^{2d} \to \mathbb{R}^{2d}, \quad z \mapsto tz\,.$$

It preserves Σ_*, and for any $\xi \in \Sigma_*$ we denote by $t\xi$ the point $D_t(x_\xi, y_\xi)$.

l_p1 **Lemma 3.1.** *1) There exists $\theta_0^* \in (0,1]$ such that for any $0 < \theta_0 \le \theta_0^*$ a suitable neighbourhood $\Sigma^{nbh} = \Sigma^{nbh}(\theta_0)$ of Σ_* in $\mathbb{R}^{2d} \setminus \{0\}$ may be uniquely parametrised as*

$$\Sigma^{nbh} = \{\pi(\xi,\theta) : \xi \in \Sigma_*,\ |\theta| < \theta_0\}\,, \tag{3.3}$$ par

where $\pi(\xi,\theta) = (x_\xi, y_\xi) + \theta N_\xi = (x_\xi, y_\xi) + \theta(y_\xi, x_\xi)$.
2) For any vector $\pi = \pi(\xi,\theta) \in \Sigma^{nbh}$ its length equals

$$|\pi| = |\xi|\sqrt{1+\theta^2}. \tag{3.4}$$ length

The distance from π to Σ equals $|\xi||\theta|$, and the shortest path from π to Σ is the segment $[\xi,\pi] = \{\pi(\xi,t\theta) : 0 \le t \le 1\} =: S$.
3) If $z = (x,y) \in S^R$ is such that $\operatorname{dist}(z,\Sigma) \le \frac{1}{2}R\theta_0$, then $z = \pi(\xi,\theta) \in \Sigma^{nbh}$, where $|\theta| < \theta_0$ and $|\xi| \le R \le |\xi|\sqrt{1+\theta_0^2}$.
4) If $\pi(\xi,\theta) \in \Sigma^{nbh}$, then

$$\omega\big(\pi(\xi,\theta)\big) = |\xi|^2\theta. \tag{3.5}$$ p0

5) If $(x,y) \in S^R \cap (\Sigma^{nbh})^c$, then $|x \cdot y| \ge cR^2$ for some $c = c(\theta_0) > 0$.

The coordinates (3.3) are known as the normal coordinates, and their existence follows easily from the implicit function theorem. The assertion 1) is a bit more precise than the general result since it specifies the size of the neighbourhood Σ^{nbh}.

Proof. 1) Fix any positive $\kappa < 1$. Then for θ_0^* small enough it is well known that the points $\pi(\xi,\theta)$ with $\xi \in \Sigma_{1-\kappa}^{1+\kappa}$ and $|\theta| < \theta_0 \le \theta_0^*$ form a neighbourhood of $\Sigma_{1-\kappa}^{1+\kappa}$ in $\mathbb{R}^{2d}$ and parametrise it in a unique and smooth way. Besides, any point $\pi' \in \mathbb{R}^{2d}$ such that $\operatorname{dist}(\pi',\Sigma^1) \le \frac{1}{2}\theta_0$, may be represented as

$$\pi' = \pi(\xi',\theta'), \quad \xi' \in \Sigma_{1-\kappa}^{1+\kappa},\ \ |\theta'| < \theta_0\,, \tag{3.6}$$ p3

and

$$\pi(\xi_1,\theta_1) = \pi(\xi_2,\theta_2),\ \xi_1,\xi_2 \in \Sigma_{1-\kappa}^{1+\kappa} \ \Rightarrow\ |\theta_1|,|\theta_2| \ge 2\theta_0^*\,. \tag{3.7}$$ p39

We may assume that $\theta_0^* < \frac{1}{2}\kappa$. The mapping D_t sends $\Sigma_{1-\kappa}^{1+\kappa}$ to $\Sigma_{t-t\kappa}^{t+t\kappa}$ and sends $\pi\big(\Sigma_{1-\kappa}^{1+\kappa} \times (-\theta_0,\theta_0)\big)$ to $\pi\big(\Sigma_{t-t\kappa}^{t+t\kappa} \times (-\theta_0,\theta_0)\big)$. This implies that the set Σ^{nbh}, defined as a collection of all points $\pi(\xi,\theta)$ as in (3.3), makes a neighbourhood of Σ_*. To prove that the parametrisation is unique assume that it is not. Then there exist $t_1 > t_2 > 0$, $\theta_1,\theta_2 \in (-\theta_0,\theta_0)$ and $\xi_1 \in \Sigma^{t_1}, \xi_2 \in \Sigma^{t_2}$ such that $\pi(\xi_1,\theta_1) = \pi(\xi_2,\theta_2)$. So $\pi_1 = \pi_2$, where

$$\pi_1 = \pi(t_1^{-1}\xi_1,\theta_1), \quad \pi_2 = \pi(t_1^{-1}\xi_2,\theta_2),$$

and $1 = |t_1^{-1}\xi_1| > |t_1^{-1}\xi_2|$. Let us write $|t_1^{-1}\xi_2|$ as $1-\kappa', \kappa' > 0$. If $\kappa' < \kappa$, then $(\xi_1,\theta_1) = (\xi_2,\theta_2)$ by what was said above. If $\kappa' > \kappa$, then $|t_1^{-1}\xi_1 - t_1^{-1}\xi_2| \ge |t_1^{-1}\xi_1| - |t_1^{-1}\xi_2| \ge \kappa$. Since

$$|\pi_1 - t_1^{-1}\xi_1| = \theta_1, \qquad |\pi_2 - t_2^{-1}\xi_2| = |\theta_2 N_{t_1^{-1}\xi_2}| \le \theta_2\,,$$

then $|\pi_1 - \pi_2| \geq \kappa - \theta_1 - \theta_2 \geq \kappa - 2\theta_0$. Decreasing θ_0^* if needed, we achieve that $\kappa > 2\theta_0$, so $|\pi_1 - \pi_2| > 0$. Contradiction.

2) The first assertion holds since by (3.1) the vector N_ξ is orthogonal to (x_ξ, y_ξ) and since its norm equals $|\xi|$. The second assertion holds since the segment S is a geodesic from π to Σ_*, orthogonal to Σ_*. Any other geodesic from π to Σ_* must be a segment $S' = [\pi, \xi']$, $\xi' \in \Sigma_*$, orthogonal to Σ_*. It is longer than $|\theta||\xi|$. To prove this, by scaling (i.e. by applying a delation operator), we reduce the problem to the case $|\xi| = 1$. Now, if $\xi' \in \Sigma_{1-\kappa}^{1+\kappa}$, then $\pi(\xi', \theta') = \pi = \pi(\xi, \theta)$ for some real number θ'. So by (3.7), $|\theta| \geq 2|\theta_0^*|$, which is a contradiction. While if $\xi' \notin \Sigma_{1-\kappa}^{1+\kappa}$, then the distance from π to ξ' is bigger than $\kappa - \theta_0 > \theta_0$. Indeed, if $|\xi'| \geq 1 + \kappa$, then the distance is bigger than $1 + \kappa - |\pi| \geq 1 + \kappa - 1 - \theta_0 = \kappa - \theta_0$. The case $|\xi'| \leq 1 - \kappa$ is similar.

3) If $R = 1$, then the assertion follows from (3.6) and (3.4). If $R \neq 1$, we apply the operator $D_{R^{-1}}$ and use the result with $R = 1$.

4) Follows immediately from (3.1).

5) If $R = 1$, then the assertion with some $c > 0$ follows from the compactness of $S^1 \cap (\Sigma^{nbh})^c$. If $R \neq 1$, then again we apply $D_{R^{-1}}$ and use the result with $R = 1$. □

Let as fix any $0 < \theta_0 \leq \theta_0^*$, and consider the manifold $\Sigma^{nbh} = \Sigma^{nbh}(\theta_0)$. Below we provide it with some additional structures and during the corresponding constructions decrease θ_0^*, if needed. Consider the set Σ^1. It equals

$$\Sigma^1 = \{(x, y) : x \cdot y = 0, x^2 + y^2 = 1\}.$$

Since the differentials of the two relations, defining Σ^1, are independent on Σ^1, then this set is a smooth compact submanifold of $\mathbb{R}^{2d}$ of codimension 2. Let us cover it by some finite system of charts $\mathcal{N}_1, \ldots, \mathcal{N}_{\tilde{n}}, \mathcal{N}_j = \{\eta^j = (\eta_1^j, \ldots, \eta_{2d-2}^j)\}$. Denote by $m(d\eta)$ the volume element on Σ^1, induced from $\mathbb{R}^{2d}$, and denote the coordinate maps as $\mathcal{N}_j \ni \eta^j \to (x_{\eta^j}, y_{\eta^j}) \in \Sigma^1$. We will write points of Σ^1 both as η and (x_η, y_η).

The mapping

$$\Sigma^1 \times \mathbb{R}^+ \to \Sigma_*, \quad ((x_\eta, y_\eta), t) \to D_t(x_\eta, y_\eta),$$

is 1-1 and is a local diffeomorphism; so this is a global diffeomorphism. Accordingly, we can cover Σ_* by the $\tilde{n}$ charts $\mathcal{N}_j \times \mathbb{R}_+$, with the coordinate maps

$$(\eta^j, t) \mapsto D_t(x_{\eta^j}, y_{\eta^j}), \quad \eta^j \in \mathcal{N}_j, \ \ t > 0,$$

and can apply Lemma 3.1, taking (η, t) for the coordinates ξ. In these coordinates the volume element on Σ^t is $t^{2d-2} m(d\eta)$. Since $\partial/\partial t \in T_{\eta,t}\Sigma_*$ is a vector of unit length, perpendicular to Σ^t,[4] then the volume element on Σ_* is

$$d_{\Sigma_*} = t^{2d-2} m(d\eta)\, dt. \tag{3.8}$$

vol_on_*

[4] as $\partial/\partial t \perp S^t$ and $S^t \supset \Sigma^t$.

The coordinates (η, t, θ) with $\eta \in \mathcal{N}_j, t > 0, |\theta| < \theta_0$, where $1 \le j \le \tilde{n}$, make coordinate systems on the open set Σ^{nbh}. Since the vectors $\partial/\partial t$ and $t^{-1}\partial/\partial\theta$ form an orthonormal base of the orthogonal complement in $\mathbb{R}^{2d}$ to $T_{(\eta,t,0)}\Sigma^t$,[5] then in Σ^{nbh} the volume element $dx\,dy$ may be written as

$$dx\,dy = t^{2d-1}\mu(\eta, t, \theta)m(d\eta)dt\,d\theta\,, \quad \text{where } \mu(\eta, t, 0) = 1\,. \tag{3.9}$$ p4

For $r > 0$ the transformation D_r multiplies the form in the l.h.s. by r^{2d}, preserves $d\eta$ and $d\theta$, and multiplies dt by r. Hence, μ does not depend on t, and we have got

l_p2 **Lemma 3.2.** *The coordinates*

$$(\eta^j, t, \theta), \quad \textit{where } \eta^j \in \mathcal{N}_j,\ t > 0,\ |\theta| < \theta_0\,, \tag{3.10}$$ chj

and $1 \le j \le \tilde{n}$, define on Σ^{nbh} coordinate systems, jointly covering Σ^{nbh}. In these coordinates the dilations D_r, $r > 0$, reed as

$$D_r : (\eta, t, \theta) \mapsto (\eta, rt, \theta)\,,$$

and the volume element has the form (3.9)*, where μ does not depend on t.*

Besides, since at a point $z = (x, y) = \pi(\xi, \theta) \in \Sigma^{nbh}$ we have $(\partial/\partial\theta) = \nabla_z \cdot (y, x)$, then in view of (1.2), (1.3)

$$\Big|\frac{\partial^k}{\partial\theta^k}F(\eta, t, \theta)\Big| \le K'(1+t)^{-M}, \quad \Big|\frac{\partial^k}{\partial\theta^k}\Gamma(\eta, t, \theta)\Big| \le K'(1+t)^{r_*} \tag{3.11}$$ new_est

for $0 \le k \le 2$ and for all (η, t, θ).

For $0 \le R_1 < R_2$ we denote

$$\big(\Sigma^{nbh}\big)_{R_1}^{R_2} = \pi\big(\Sigma_{R_1}^{R_2} \times (-\theta_0, \theta_0)\big)\,.$$

In a chart (3.10) this domain is $\{(\eta^j, t, \theta) : \eta^j \in \mathcal{N}_j,\ R_1 < t < R_2,\ |\theta| < \theta_0\}$.

4 Global study of the integral (1.1)

s4

To study I_ν and the integrals $\langle I_\nu, \big(\Sigma^{nbh}\big)_{R_1}^{R_2}\rangle$, $0 \le R_1 < R_2$, we pass to the coordinates (η, t, θ) and write the former using (3.9). First we will examine in details the integrals over $\mu(\eta, \theta)\,d\theta$, and next integrate them over $t^{2d-1}m(d\eta)\,dt$.

[5] Since the vector $\partial/\partial t$ is perpendicular to Σ^t and lies in $T_{(\eta,t,0)}\Sigma_*$, and $\partial/\partial\theta$ is proportional to the vector $N_{(\eta,t,0)}$, normal to Σ_* at $(\eta, t, 0)$.

4.1 Desintegration of I_ν and integrating over $\mu\, d\theta$.

Using (3.9) we write the integral $\langle I_\nu, \big(\Sigma^{nbh}\big)_{R_1}^{R_2}\rangle$ as

$$\int_{\Sigma^1} m(d\eta) \int_{R_1}^{R_2} dt\, t^{2d-1} \int_{-\theta_0}^{\theta_0} d\theta \frac{F(\eta,t,\theta)\mu(\eta,\theta)}{(x\cdot y)^2 + (\nu\Gamma(\eta,t,\theta))^2} \\ = \int_{\Sigma^1} m(d\eta) \int_{R_1}^{R_2} dt\, t^{2d-1} J_\nu(\eta,t)\,, \tag{4.1}$$

3.9

where by (3.5)

$$J_\nu(\eta,t) = \int_{-\theta_0}^{\theta_0} d\theta \frac{F(\eta,t,\theta)\mu(\eta,\theta)}{t^4\theta^2 + (\nu\Gamma(\eta,t,\theta))^2}\,.$$

To study $J_\nu(\eta,t) =: J_\nu$ we write Γ as

$$\Gamma(\eta,t,\theta) = h_{\eta,t}(\theta)\Gamma(\eta,t,0)\,.$$

The function $h(\theta) := h_{\eta,t}(\theta)$ is C^2-smooth, and in view of (3.11) and (1.3) it satisfies

$$|h(\theta)| \ge C_0^{-1}, \quad \Big|\frac{\partial^k}{\partial\theta^k} h(\theta)\Big| \le C_k \quad \forall\, \eta,t,\theta,\ 0\le k\le 2\,. \tag{4.2}$$

p11

Denoting

$$\varepsilon = \nu t^{-2}\Gamma(\eta,t,0),$$

we write J_ν as

$$J_\nu = t^{-4}\int_{-\theta_0}^{\theta_0} \frac{F(\eta,t,\theta)\mu(\eta,\theta)h^{-2}(\theta)\, d\theta}{\theta^2 h^{-2}(\theta) + \varepsilon^2}\,.$$

Since $h(0) = 1$, then in view of (4.2) the mapping

$$f = f_{\eta,t} : [-\theta_0,\theta_0] \ni \theta \mapsto \bar\theta = \theta/h(\theta)$$

is a C^2–diffeomorphism on its image such that $f(0) = 0, f'(0) = 1$ and the C^2-norms of f and f^{-1} are bounded by a constant, independent from η, t (to achieve that, if needed, we decrease θ_0^*). Denote

$$\theta_0^+ = f(\theta_0), \quad \theta_0^- = -f(-\theta_0), \quad \bar\theta_0 = \min(\theta_0^+,\theta_0^-)\,.$$

Then $2^{-1}\theta_0 \le \theta_0^\pm \le 2\theta_0$ if θ_0^* is small, and

$$J_\nu = t^{-4}\int_{\theta_0^-}^{\theta_0^+} \frac{F(\eta,t,\theta)\mu(\eta,\theta)h^{-2}(\theta)(f^{-1}(\theta))'\, d\bar\theta}{\bar\theta^2 + \varepsilon^2}\,.$$

Denote the nominator of the integrand as $\Phi(\eta,t,\bar\theta)$. This is a C^2–smooth function, and by (3.11) and (4.2) it satisfies

$$|\frac{\partial^k}{\partial\theta^k}\Phi| \le C(1+t)^{-M} \quad \text{for} \quad 0\le k\le 2\,.$$

Moreover, since $h(0) = 1$ and $(f^{-1}(0))' = f'(0) = 1$, then in view of (3.9) we have that

$$\Phi(\eta, t, 0) = F(\eta, t, 0)\,. \tag{4.3}$$ p13

Consider the interval $\Delta_{\eta,t} = f^{-1}_{\eta,t}(-\bar\theta_0, \bar\theta_0)$. Then

$$(-\theta_0/2, \theta_0/2) \subset \Delta_{\eta,t} \subset (-\theta_0, \theta_0)$$

for all η and t. Now we modify the neighbourhood $\Sigma^{nbh}(\theta_0)$ to

$$\Sigma^{nbh_m} = \Sigma^{nbh_m}(\theta_0) = \{\pi(\eta, t, \theta) : \eta \in \Sigma^1,\ t > 0,\ \theta \in \Delta_{\eta,t}\}\,.$$

Then

$$\Sigma^{nbh}(\tfrac{1}{2}\theta_0) \subset \Sigma^{nbh_m}(\theta_0) \subset \Sigma^{nbh}(\theta_0)\,. \tag{4.4}$$ Smod

The modified analogy J^m_ν of the integral J_ν has the same form as J_ν, but the domain of integrating becomes not $(-\theta_0, \theta_0)$, but $\Delta_{\eta,t}$. Then

$$J^m_\nu = t^{-4} \int_{-\bar\theta_0}^{\bar\theta_0} \frac{\Phi(\eta, t, \theta)\, d\bar\theta}{\bar\theta^2 + \varepsilon^2}\,.$$

To estimate J^m_ν, consider first the integral J^{0m}_ν, obtained from J^m_ν by frozening Φ at $\bar\theta = 0$:

$$J^{0m}_\nu = t^{-4} \int_{-\bar\theta_0}^{\bar\theta_0} \frac{\Phi(\eta, t, 0)\, d\,\bar\theta}{\bar\theta^2 + \varepsilon^2} = 2t^{-4} F(\eta, t, 0)\varepsilon^{-1} \tan^{-1} \frac{\bar\theta_0}{\varepsilon}$$

(we use (4.3)). From here

$$|J^{0m}_\nu| \le \pi\varepsilon^{-1} t^{-4} |F(\eta, t, 0)|\,. \tag{4.5}$$ p_triv

As $0 < \frac{\pi}{2} - \tan^{-1}\frac{1}{\bar\varepsilon} < \bar\varepsilon$ for $0 < \bar\varepsilon \le \frac{1}{2}$, then also

$$0 < \pi\nu^{-1} t^{-2} (F/\Gamma)\,|_{\theta=0} - J^{0m}_\nu < \frac{2}{\bar\theta_0} t^{-4} F(\eta, t, 0)\,, \tag{4.6}$$ p17

if

$$\nu t^{-2} \Gamma(\eta, t, 0) \le \tfrac{1}{2}\bar\theta_0. \tag{4.7}$$ p18

Now we estimate the difference between J^m_ν and J^{0m}_ν. We have:

$$J^m_\nu - J^{0m}_\nu = t^{-4} \int_{-\bar\theta_0}^{\bar\theta_0} \frac{\Phi(\eta, t, \bar\theta) - \Phi(\eta, t, 0)}{\bar\theta^2 + \varepsilon^2} d\bar\theta\,.$$

Since each C^k–norm of Φ , $k \le 2$, is bounded by $C(1 + t)^{-M}$, then

$$\Phi(\eta, t, \bar\theta) - \Phi(\eta, t, 0) = A(\eta, t)\bar\theta + B(\eta, t, \bar\theta)\bar\theta^2\,,$$

where $|A|, |B| \le C(1 + t)^{-M}$. From here

$$|J^m_\nu - J^{0m}_\nu| \le C_1(1 + t)^{-M} t^{-4} \int_0^{\bar\theta_0} \frac{\bar\theta^2\, d\bar\theta}{\bar\theta^2 + \varepsilon^2} \le C_1(1 + t)^{-M} t^{-4} \bar\theta_0\,.$$

Denote

$$\mathcal{J}_\nu(\eta, t) = \pi t^{-2} \big(F\Gamma^{-1}\big)(\eta, t, 0)\,. \tag{4.8}$$ p20

Then, jointly with (4.6), the last estimate tell us that

$$|J^m_\nu - \nu^{-1}\mathcal{J}_\nu(\eta, t)| \le C(1 + t)^{-M} t^{-4} \bar\theta_0^{-1} \quad \text{if (4.7) holds} \tag{4.9}$$ p19 p19

s_76

4.2 Proof of Theorem 1.1

To estimate I_ν we write it as $I_\nu = \langle I_\nu, (\Sigma^{nbh_m})_0^{C^{-1}\sqrt{\nu}}\rangle + \langle I_\nu, (\Sigma^{nbh_m})_{C^{-1}\sqrt{\nu}}^{\infty}\rangle$. We will show that the first term is small by Lemma 2.1, and that the second may be approximately calculated using the representation (4.1) and the estimate (4.9). Doing that we have to distinguish the cases $r_* \le 2$ and $r_* > 2$.

Let $r_* \le 2$**.** Then by (1.3) assumption (4.7) holds if $t \ge C^{-1}\sqrt{\nu}$ (where C depends on $\bar\theta_0$). Integrating $J_\nu^m(\eta,t)$ and $\nu^{-1}\mathcal{J}_\nu(\eta,t)$ with respect to the measure $t^{2d-1}m(d\eta)\,dt$ and using (4.9), (4.1) and (1.4) we get that

$$\begin{aligned}&|\langle I_\nu, (\Sigma^{nbh_m})_{C^{-1}\sqrt{\nu}}^{\infty}\rangle - \nu^{-1}\int_{\Sigma^1} m(d\eta)\int_{C^{-1}\sqrt{\nu}}^{\infty} dt\, t^{2d-1}\mathcal{J}_\nu(\eta,t)|\\ &\le C\int_{\Sigma^1} m(d\eta)\int_{C^{-1}\sqrt{\nu}}^{\infty} dt\, t^{2d-1}t^{-4}(1+t)^{-M} \le C\chi_d(\nu)\end{aligned} \tag{4.10}$$

p41

(for the quantity $\chi_d(\nu)$ see (1.6)). In view of (4.8) and Lemma 2.1 with $\delta = 2C^{-1}\sqrt{\nu}$,

$$\begin{aligned}&|\langle I_\nu, (\Sigma^{nbh_m})_0^{C^{-1}\sqrt{\nu}}\rangle - \nu^{-1}\int_{\Sigma^1} m(d\eta)\int_0^{C^{-1}\sqrt{\nu}} dt\, t^{2d-1}\mathcal{J}_\nu(\eta,t)|\\ &\le C\nu^{-1}\nu^{d-1} + C\nu^{-1}\int_0^{C^{-1}\sqrt{\nu}} t^{2d-1}t^{-2}\,dt \le C_1\end{aligned} \tag{4.11}$$

p42

as $d \ge 2$. Next, by (4.8) and (1.4)

$$\int_{\Sigma^1} m(d\eta)\int_0^{\infty} dt\, t^{2d-1}|\mathcal{J}_\nu(\eta,t)| \le C\int_0^{\infty} t^{2d-1-2}(1+t)^{-M-r_*} \le C_1\,, \tag{4.12}$$

p43

and by (3.8)

$$\begin{aligned}\int_{\Sigma^1} m(d\eta)\int_0^{\infty} dt\, t^{2d-1}\mathcal{J}_\nu =&\pi\int_{\Sigma^1} m(d\eta)\int_0^{\infty} dt\, t^{2d-3}(F/\Gamma)\,|_{\theta=0}\\ =&\pi\int_{\Sigma_*} |z|^{-1}(F/\Gamma)(z)d_{\Sigma_*}z\,.\end{aligned} \tag{4.13}$$

p44

This gives us asymptotic description as $\nu \to 0$ of the integral (1.1), calculated over the vicinity Σ^{nbh_m} of Σ_*. It remains to estimate the integral over the complement to Σ^{nbh_m}. But this is easy: by (4.4),

$$\begin{aligned}|\langle I_\nu, \mathbb{R}^{2d}\setminus\Sigma^{nbh_m}\rangle| \le& |\langle I_\nu, \{|(x,y)|\le 2\nu\}\rangle|\\ &+ C_d\Big|\int_\nu^\infty dr\, r^{2d-1}\int_{S^r\setminus\Sigma^{nbh}(\theta_0/2)}\frac{F(x,y)\,d_{S^r}}{(x\cdot y)^2+(\nu\Gamma((x,y))^2}\Big|\,.\end{aligned}$$

By item 5) of Lemma 3.1 the divisor of the integrand is $\ge C^{-2}r^4$. Due to this and (1.2), the second term in the r.h.s. is bounded by

$$C\int_\nu^\infty (1+r)^{-M}r^{2d-5}\,dr \le C_1\chi_d(\nu)\,.$$

This estimate and Lemma 2.1 with $\delta = 2\nu$ imply that

$$|\langle I_\nu, \mathbb{R}^{2d} \setminus \Sigma^{nbh_m}\rangle| \le C\chi_d(\nu)\,. \tag{4.14}$$ p45

Now relations (4.10), (4.11), (4.13), (4.14) imply (1.5), while (3.8) and (4.12) imply that the integral in (1.5) converges absolutely.

Let $r_* > 2$. Then condition (4.6) holds if

$$C_\beta \nu^{-\beta} \ge t \ge C^{-1}\sqrt{\nu}\,, \qquad \beta = \frac{1}{r_* - 2}\,.$$

Accordingly, the term in the l.h.s. of (4.10) should be split in two. The first corresponds to the integrating from $C^{-1}\sqrt{\nu}$ to $C_\beta\nu^{-\beta}$ and estimates exactly as before. The second is

$$|\langle I_\nu, (\Sigma^{nbh_m})^\infty_{C_\beta\nu^{-\beta}}\rangle - \nu^{-1}\int_{\Sigma^1} m(d\eta)\int_{C_\beta\nu^{-\beta}}^\infty dt\, t^{2d-1}\mathcal{J}_\nu(\eta,t)|\,. \tag{4.15}$$ p02

To bound it we estimate the norm of the difference of the two integrals via the sum of their norms. In view of (4.5) and (4.8) both of them are bounded by

$$C\nu^{-1}\int_{\Sigma_1} m(d\eta)\int_{C_\beta\nu^{-\beta}}^\infty dt\, t^{2d-3}(F/\Gamma)(\eta,t,0)\,.$$

So

$$(4.15) \;\le\; C_\beta\nu^{-1+\beta(M+r_*+2-2d)} \le C_\beta$$

since $M > 2d-4$.

Adding this relation to (4.10), applied to the integrating from $C^{-1}\sqrt{\nu}$ to $C_\beta\nu^{-\beta}$, and – as before – using this jointly with (4.11), (4.13), (4.14), we again get (1.5) (while the absolute convergence of the integral still follows from (4.12)). □

4.3 Proof of Proposition 1.2

For any $R > r > 0$ let us denote by B^R and B^R_r the ball $\{|z| \le R\}$ and the spherical layer $\{r \le |z| \le R\}$, and consider the measure $\mu^R_r = \chi_{B^R_r}(z)|z|^{-1}\delta_{\Sigma_*}$. This is a well defined Borel measure on the manifold Σ_* and on the space $\mathbb{R}^{2d}$, supported by B^R. By the Riesz theorem the family of measures $\{\mu^R_\varepsilon, 0 < \varepsilon \le 1\}$ is weakly compact in the space of measures on B^R, and it is easy to see that for any bounded continuos function $f(z)$ the curve $\varepsilon \to \int f(z)\,d\mu^R_\varepsilon(dz)$ satisfies the Cauchy condition as $\varepsilon \to 0$. So the measures μ^R_ε weakly converge to a limit as $\varepsilon \to 0$. This is the restriction of the measure $|z|^{-1}\delta_{\Sigma_*}$ to B^R, and its further restriction to B^R_r equals μ^R_r. So $|z|^{-1}\delta_{\Sigma_*}$ is a σ–additive Borel measure on $\mathbb{R}^{2d}$, and it has no atoms outside the origin. Let us abbreviate $|z|^{-1}\delta_{\Sigma_*} = \mu$. By (3.8), for any $0 < r \le R$ and $\varepsilon \le r$,

$$\int_{B^R_r} d\,\mu = \int_{B^R_r} d\,\mu^R_\varepsilon = \int_r^R t^{2d-2}t^{-1}\,dt = \frac{1}{2d-2}(R^{2d-2} - r^{2d-2}).$$

From here and the weak convergence of the measures μ_ε^R to $\mu\,|_{B^R}$ we get that $\mu\{|z|<\rho\}\le\rho^{2d-2}/(2d-2)$ if $\rho<R$, so μ has no atom in the origin and is atomless. Next, for any function $f\in C_m(\mathbb{R}^{2d})$ we have

$$\begin{aligned}\int f(z)\,\mu(dz) &\le |f|_m\sum_{R=0}^{\infty}\int_{B_R^{R+1}}\langle z\rangle^{-m}\,\mu(dz)\\ &\le C_1|f|_m\sum_{R=0}^{\infty}\langle R\rangle^{-m}\,\frac{(R+1)^{2d-2}-R^{2d-2}}{2d-2}\\ &\le C_2|f|_m\sum_{R=0}^{\infty}\frac{(R+1)^{2d-1}}{\langle R\rangle^{-m}}=C_3|f|_m,\end{aligned}$$

if $m>2d-2$. This proves the proposition.

5 Other integrals

s_other_integrals

The geometrical approach to treat integrals (1.1), developed above, applies to various modifications of these integrals. Below we briefly discuss three more examples.

5.1 Integrals (1.1) with $d=1$

s_d1

The restriction $d\ge 2$ was imposed in the previous sections since in the one-dimensional case some integrals, involved in the construction, strongly diverge at the locus of the quadric Σ. This problem disappears if the function F vanishes near the locus. Indeed, consider

$$I'_\nu=\int_{\mathbb{R}^2}\frac{F(x,y)}{x^2y^2+(\nu\Gamma(x,y))^2}\,dxdy\,,$$

where $F\in C_0^2(\mathbb{R}^2)$ vanishes near the origin. The quadric $\Sigma'=\{xy=0\}\subset\mathbb{R}^2$ is one dimensional, has a singularity at the origin, and its smooth part $\Sigma'^*=\Sigma'\setminus 0$ has four connected components. Consider one of them: $\mathcal{C}_1=\{(x,y):y=0,x>0\}$. Now the coordinate ξ is a point in $\mathbb{R}_+$ with $(x_\xi,y_\xi)=(\xi,0)$ and with the normal $N(\xi)=(0,\xi)$, the set $\Sigma_1\cap\mathcal{C}_1$ is the single point $(1,0)$ and the coordinate (η,t,θ) in the vicinity of $\mathcal{C}_1$ degenerates to (t,θ), $t>0$, $|\theta|<\theta_0$, with the coordinate-map $(t,\theta)\mapsto(t,t\theta)$. The relations (3.8) and (3.9) are now obvious, and the integral (2.1) vanishes if $\delta>0$ is sufficiently small. Interpreting $z=(x,y)$ as a complex number, we write the assertion of Theorem 1.1 as

$$\big|J'_\nu-\pi\nu^{-1}\int_{\Sigma'}\frac{F(z)}{|z|\,\Gamma(z)}\,dz\big|\le C\,,$$

where the integral is a contour integral in the complex plane.

5.2 Integrals of quotients with divisors, linear in ω.

Consider

$$I'_\nu = \int_{\mathbb{R}^{2d}} \frac{F(x,y)\,dxdy}{x\cdot y + i\nu\Gamma(x,y)}\,. \tag{5.1}$$

another_integral

Now there is no need to separate the integral over the vicinity of the origin, and we just split I'_ν to an integral over Σ^{nbh_m} and over its complement.

To calculate $\langle I'_\nu, \Sigma^{nbh_m}\rangle$ we observe that an analogy of J^{0m}_ν is the integral

$$\int_{-\bar\theta_0}^{\bar\theta_0} \frac{F(\eta,t,0)\,d\bar\theta}{t^2\bar\theta + i\nu\Gamma(\eta,t,0)} = t^{-2}F(\eta,t,0)\ln\frac{\theta_0^+ + i\nu t^{-2}\Gamma(\eta,t,0)}{-\bar\theta_0 + i\nu t^{-2}\Gamma(\eta,t,0)},$$

which equals $\pi t^{-2}F(\eta,t,0) + O(\nu)$. So $\langle I'_\nu, \Sigma^{nbh_m}\rangle = \pi\int_{\Sigma_*}\frac{F(z)}{|z|}d_{\Sigma_*}z + O(\nu)$. The integral over the complement to Σ^{nbh_m} is

$$\int_{\mathbb{R}^{2d}\setminus\Sigma^{nbh_m}} \frac{F(x,y)\,dxdy}{x\cdot y + i\Gamma(x,y)} = \int_{\mathbb{R}^{2d}\setminus\Sigma^{nbh_m}} \frac{F(x,y)\,dxdy}{x\cdot y} + o(1)$$

as $\nu\to 0$ (the integral in the r.h.s. is regular). In difference with (1.1) the last integral is of the same order as the integral over Σ^{nbh_m}. So we have that

$$I'_\nu = \pi\int_{\Sigma_*}\frac{F(z)}{|z|}d_{\Sigma_*}z + \int_{\mathbb{R}^{2d}\setminus\Sigma^{nbh}} \frac{F(x,y)\,dxdy}{x\cdot y} + o(1)\,,$$

in agreement with the estimate (6.4), applied to (5.1).

5.3 Integrals, coming from the three-waves interaction

The three-waves interacting systems lead to integrals, similar to (1.7), where $\mathbb{R}^{3d}$ is replaced by $\mathbb{R}^{2d}$ and the δ-factor is replaced by $\delta^k_{k_1k_2}\delta(\omega^k_{k_1k_2})$, which gives rise to the algebraic set

$$\{(k_1,k_2): k_1+k_2=k,\ \ |k_1|^2+|k_2|^2=|k|^2\}$$

(see [5], Section 6). I.e., $k_2 = k - k_1$, where $k_1\in\{r\in\mathbb{R}^d : |r-\frac12 k|^2 = \frac14|k|^2\}$. Accordingly, in the variable $z = r - \frac12 k$ some constructions from the study of the three-waves interaction lead to the integrals

$$I'_\nu = \int_{\mathbb{R}^d} dz\,\frac{F(z)}{\omega(z)^2 + (\nu\Gamma(z))^2}\,, \qquad d\ge 2,$$

with $\omega(z) = |z|^2 - \frac14|k|^2$. Now the quadric $\Sigma = \{\omega = 0\}$ is a sphere, i.e. a smooth compact manifold. Denoting by η a local coordinate on Σ with a coordinate mapping $\eta\mapsto z(\eta)\in\Sigma$ and the volume form $m(d\eta)$ we see that, similar to Section 3, the local coordinate in the vicinity Σ^{nbh} of Σ is (η,θ), $|\theta|<\theta_0$, with the coordinate mapping $(\eta,\theta)\mapsto z(\eta)(1+\theta)$ and the volume form $(\frac12|k|)^{-1}\mu(\eta,\theta)m(d\eta)d\theta$, $\mu(\eta,0)\equiv 0$. The proof in Sections 2–4 simplifies and leads to the asymptotic

$$\big|I'_\nu - \pi\nu^{-1}\int_\Sigma \big(F(z)/\Gamma(z)\big)dz\big| \le \text{Const},$$

valid for C^2-functions F and Γ, satisfying some mild restriction.

6 Appendix

Let $\varphi(x)$ and $g(x)$ be smooth functions on $\mathbb{R}^n$ and φ has a compact support. Consider the integral

$$I(\lambda) = \int_{\mathbb{R}^n} \varphi(x) e^{i\lambda g(x)}\, dx, \quad \lambda \geq 1.$$

Assume that $g(x)$ has a unique critical point x_0, which is non-degenerate. Then, by the stationary phase method,

$$I(\lambda) = \big(\frac{2\pi}{\lambda}\big)^{n/2} |\det g_{xx}(x_0)|^{-1/2} \varphi(x_0) e^{i\lambda g(x_0) + (i\pi/4)\,\mathrm{sgn}\, g_{xx}(x_0)} + R\lambda^{-n/2-1} \tag{6.1}$$

stph_thm

for $\lambda \geq 1$, where R depends on $\|\varphi\|_{C^2}$, $\|g\|_{C^3}$, the measure of the support of φ and on $\sup_{x\in \mathrm{supp}\,\varphi}(|x-x_0|/|\nabla g(x)|) =: C^{\#}(g)$. See Section 7.7 of [2] and Section 5 of [1].

If the functions φ and g are not C^∞–smooth, but $\varphi \in C_0^2(\mathbb{R}^n)$ and $g \in C^3(\mathbb{R}^n)$, then, approximating φ and g by smooth functions and applying the result above we get from (6.1) that

$$|I(\lambda)| \leq C' \lambda^{-n/2} \qquad \forall\, \lambda \geq 1\,, \tag{6.2}$$

stph_est

with C' depending on $\|\varphi\|_{C^2}$, $\|g\|_{C^3}$ and $C^{\#}(g)$.

Now let $f(x) \in C_0^2(\mathbb{R}^d)$ and $g(x) \in C^3(\mathbb{R}^d)$ be such that

$$|f| \leq C, \;\; \mathrm{meas}(\,\mathrm{supp} f) \leq C.$$

Let an x_0 be the unique critical point of $g(x)$ and

$$C^{-1} \leq |\det \mathrm{Hess}\, g(x_0)| \leq C. \tag{6.3}$$

hess

Consider the integral

$$\mathcal{I}(\nu) = \int_{\mathbb{R}^d} \frac{f(x)}{\Gamma + i\nu^{-1} g(x)}\, dx = \nu \int_{\mathbb{R}^d} \frac{f(x)}{\nu\Gamma + i g(x)}\, dx\,, \;\; 0 < \nu \leq 1\,,$$

where Γ is a positive constant. Let us write it as

$$\mathcal{I}(\nu) = \int_{\mathbb{R}^d} \int_{-\infty}^0 f(x) e^{t(\Gamma + i\nu^{-1} g(x))}\, dt\, dx =: \int_{\mathbb{R}^d} \int_{-\infty}^0 F_\nu(t,x)\, dt\, dx = \mathcal{I}_1 + \mathcal{I}_2\,,$$

where

$$\mathcal{I}_1 = \int_{\mathbb{R}^d} \int_{-\nu}^0 F_\nu(t,x)\, dt\, dx\,, \quad \mathcal{I}_2 = \int_{\mathbb{R}^d} \int_{-\infty}^{-\nu} F_\nu(t,x)\, dt\, dx\,.$$

Clearly, $|\mathcal{I}_1| \leq C^2 \nu$. To estimate $\mathcal{I}_2$ consider the internal integral

$$J(t) = e^{t\Gamma} \int_{\mathbb{R}^d} f(x) e^{-i\nu^{-1}|t| g(x)}\, dx\,,$$

and apply to it the stationary phase method with $\lambda = \nu^{-1}|t| \geq 1$. By (6.2) and (6.3), $|J(t)|$ is bounded by $e^{t\Gamma} K_1(f,g)(\nu^{-1}|t|)^{-d/2}$. So

$$|\mathcal{I}_2| = \int_{-\infty}^{\nu} J(t)\, dt \leq K_1(f,g)\nu^{d/2} \int_{-\infty}^{\nu} |t|^{-d/2} e^{t\Gamma}\, dt \leq K_2(f,g)\nu^{d/2}\nu^{-d/2+1}\chi_d(\nu)$$

since $\Gamma > 0$, where $\chi_d(\nu)$ is defined in (1.6).

Thus

$$|\mathcal{I}(\nu)| \leq |\mathcal{I}_1| + |\mathcal{I}_2| \leq K(f,g)\, \nu\, \chi_d(\nu)\,. \tag{6.4}$$

Aa3

The constant $K(f,g)$ depends on C, $\|f\|_{C^2}$, $\|g\|_{C^3}$ and $C^{\#}(g)$.

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