Well-Posedness of nonconvex degenerate sweeping process via unconstrained evolution problems

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Abstract

The main concern of this paper is the study of degenerate sweeping process involving uniform prox-regular sets via an unconstrained differential inclusion by showing that the sets of solutions of the two problems coincide. This principle of reduction to unconstrained evolution problem can be seen as a penalization of the subdifferential of the distance function. Using this reduction technique, an existence and uniqueness result of a Lipschitz perturbed version of the degenerate sweeping process is proved in the finite dimensional setting. An application is given to quasistatic unilateral dynamics in nonsmooth mechanics where the moving set is described by a finite number of inequalities. We provide sufficient verifiable conditions ensuring both the prox-regularity and the Lipschitz continuity with respect to the Hausdorff distance of the moving set.

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1. Introduction

Moreau's sweeping process appears as a model in several contexts [3, 8, 14, 17]. Its importance comes from the study of elastoplasticity in nonsmooth

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mechanics [18], modeling and simulation of switched electrical circuits [2], crowd motion modeling [15], etc. The main subject of investigation continues to be the existence of solutions, under increasing degrees of generality.

In this paper, we consider the evolution differential inclusion

$$\begin{cases} -\dot{u}(t) \in \mathrm{N}_{C(t)}\big(Au(t)\big) \ a.e.\ t \in [0,T] \\ u(0) = u_0, Au(0) \in C(0) \end{cases} \tag{1}$$

where $A : H \to H$ is a possibly nonlinear Lipschitz and strongly monotone operator in a real Hilbert space H, and $C : [0,T] \rightrightarrows H,\ t \longmapsto C(t)$ is a set-valued mapping, the precise assumptions will be described below. Moreover, for any subset S in H the set $\mathrm{N}_S(v)$ denotes the Clarke normal cone to S at $v \in S$. Inclusion of type (1) arise in plastic flow problems, in quasistatic elastoplasticity [18]. It has been introduced and studied by M. Kunze and M.D.P. Monteiro Marques [13] for the convex case i.e. when the set-valued map $C(\cdot)$ has a nonempty closed and convex values. This kind of dynamic (1) could be considered as a hybrid system since when $Au(t)$ lies in the interior of $C(t)$, the velocity $\dot{u}(t) = 0$ and when $Au(t)$ touches the boundary of $C(t)$, a discrete event occurs which corresponds to choosing a selection in the normal cone (a Lagrange multiplier) so that $Au(t)$ remains in $C(t)$. The presence of the operator A in (1) is more complicated than the classical case where $A = \mathrm{Id}$ (see [12, 13] for more details). The authors in [13] proved the existence of solutions of (1) where $C(t)$ is a closed and convex subset of the Hilbert space H moving in a Lipschitz continuous way and A is a maximal strongly monotone operator. The method in [13] used a discretization technique based on a surjectivity of the sum of two maximal monotone operators (one of these operators being the normal cone $\mathrm{N}_{C(t)}(.)$. In the nonconvex setting, the existence of solutions for (1) when $C(t)$ is an uniformly prox-regular set of the Hilbert space H moving in an absolutely continuous way is still an open problem. The method used in [13] can't be applied in the nonconvex setting since the normal cone $\mathrm{N}_{C(t)}(.)$ is not a maximal monotone operator.

In the present paper, using ideas from [20] for the classical sweeping process

($A = \mathrm{Id}$), we study the relationship between the constrained evolution problem (1) , and the following unconstrained differential inclusion

$$\begin{cases} -\dot{u}(t) \in \frac{|\dot{\zeta}(t)|}{\beta} \partial d_{C(t)}\big(Au(t)\big) \ a.e.\ t \in [0,T] \\ u(0) = u_0, Au(0) \in C(0) \end{cases} \tag{2}$$

where $\partial d_S(v)$ denotes the Clarke subdifferential of the distance function d_S by assuming that

$$\big|d_{C(t)}(x) - d_{C(s)}(x)\big| \leq |\zeta(t) - \zeta(s)| \tag{3}$$

for all $x \in H$ and $t, s \in [0,T]$ with ζ is an absolutely continuous nonnegative function and A is $\beta-$coercive operator. We show in Theorem 3.1 the coincidence between the solution sets of (1) and (2) under the r-prox-regularity of the moving set $C(\cdot)$. The coerciveness of the operator A will play an important role in our analysis. The principle of reduction to unconstrained problem can be seen as a penalization of the subdifferential of the distance function with an appropriate rate depending on the velocity of the absolute continuous function ζ and the coerciveness coefficient β of the operator A. Thanks to this reduction technique, an existence and uniqueness result of the Lipschitz perturbed version of problem (1) is showed in Theorem 4.1 in the finite dimensional setting. An application is given to quasistatic unilateral dynamics in nonsmooth mechanics where the moving set is described by a finite number of inequalities. We provide new sufficient verifiable conditions ensuring both the prox-regularity and the Lipschitz continuity with respect to the Hausdorff distance of the moving set (see Proposition 5.1).

This work owes its originality to overcome the difficulty of the presence of the operator A inside the normal cone and to the extension to the prox-regularity of the moving set with the reduction to unconstrained differential inclusion. This unifies and extends some known results in the literature [12, 13, 20] which paves the way for new research and applications such as the study the Bolza problem in optimal control with dynamic constraint governed by degenerate sweeping process [9].

The paper is organized as follows. Section 2 is devoted to some definitions

and important results needed later. Section 3 provides the reduction result. It is applied in Section 4 to prove an existence and uniqueness result for (1) and its Lipschitz perturbation form (see (16) after) in the finite dimensional framework when the sets $C(t)$ are uniformly prox-regular. Section 5 is dedicated to an application of the degenerate sweeping process to quasistatic unilateral dynamics involving a moving set described by a finite number of inequalities.

2. Notations and preliminaries

Throughout the paper, H is a real Hilbert space whose inner product is denoted by $\langle .,. \rangle$ and the associated norm by $\|.\|$. For any $x \in H$ and $r > 0$ the closed ball (respectively, open ball) centered at x with radius r will be denoted by $\mathbb{B}[x, r]$ (respectively, $\mathbb{B}(x, r)$). For $x = 0$ and $r = 1$ we set $\mathbb{B}$ the closed unit ball in H. Let C be a nonempty closed subset of H. We denote by $d_C(x)$ the usual distance function associated with C, i.e.,

$$d_C(x) := \inf \{\|y - x\|, y \in C\}$$

and the metric projection of x onto C is the set of points in C which realize the infimum. When this set is a singleton, we denote this point by $\mathrm{Proj}_C(x)$.

We will reduce the differential inclusion related to the normal cone to a differential inclusion involving the subdifferential of the distance function. For a function $f : H \to \mathbb{R} \cup \{+\infty\}$, a vector $\zeta \in H$ is a *proximal subgradient* of f at a point x with $f(x) < +\infty$ (see, e.g., [16, 8]) if there exist some reals $\sigma \geq 0$ and $\delta > 0$ such that

$$\langle \zeta, y - x \rangle \leq f(y) - f(x) + \sigma \|y - x\|^2 \quad \text{for all } y \in \mathbb{B}(x, \delta).$$

The set $\partial^P f(x)$ of all proximal subgradients of f at x is the *proximal subdifferential* of f at x. Of course, $\partial^P f(x) = \emptyset$ if $f(x) = +\infty$.

Taking the proximal subdifferential $\partial^P \psi_C(x)$ of the indicator function ψ_C (with $\psi(y) = 0$ if $y \in C$ and $\psi(y) = +\infty$ if $y \notin C$) we obtain the *proximal normal cone* $\mathrm{N}_C^P(x)$ of C et the point $x \in C$. Equivalently, a vector $\zeta \in H$ is a

proximal normal vector of C at $x \in C$ if and only if there are reals $\sigma \geq 0$ and $\delta > 0$ such that

$$\langle \zeta, y - x \rangle \leq \sigma \|y - x\|^2 \quad \text{for all } y \in C \cap \mathbb{B}(x, \delta). \tag{4}$$

The proximal normal cone can be described in the following geometrical way (see, e.g., [8])

$$\mathrm{N}_C^P(x) = \{\zeta \in H : \ \exists \rho > 0 \text{ such that } x \in \mathrm{Proj}_C(x + \rho\zeta)\},$$

where

$$\mathrm{Proj}_C(u) := \{y \in S : \ d_C(u) = \|u - y\|\}.$$

The proximal normal cone is also connected with the distance function to C through the equalities (see e.g. [8, 16, 19])

$$\partial^P d_C(x) = \mathrm{N}_S^P(x) \cap \mathbb{B}_H \text{ and } \mathrm{N}_C^P(x) = \mathbb{R}_+ \partial^P d_C(x),$$

where $\mathbb{R}_+ := [0, +\infty[$.

Besides the concept of proximal subdifferential, we need to recall the concept of the Fréchet subdifferential. A vector $\zeta \in H$ is a *Fréchet subgradient* of f at x (where f is finite) provided, for each real $\varepsilon > 0$, there exists a neighborhood U of x such that

$$\langle \zeta, y - x \rangle \leq f(y) - f(x) + \varepsilon \|y - x\|, \text{ for all } y \in U.$$

The set of all Fréchet subgradients of f at x is the *Fréchet subdifferential* $\partial^F f(x)$ of f at x and one set $\partial^F f(x) = \emptyset$ whenever $f(x)$ is not finite. The cone $\mathrm{N}_S^F(x) = \partial^F \psi_S(x)$ is the *Fréchet normal cone* of S at x. For $x \in S$, it is known that (see [16, 8, 19])

$$\partial^F d_S(x) = \mathrm{N}_S^F(x) \cap \mathbb{B}_H \, , \, \mathrm{N}_S^F(x) = \mathbb{R}_+ \partial^F d_S(x). \tag{5}$$

Prox-regular sets will play an important role in the sequel. The definition was first given by Federer, under the name of sets which positive reach, and later studied by several authors (see for instance [8])

Definition 2.1. *The set $C \subset H$ is said to be r-prox-regular (or uniformly prox-regular with constant $r > 0$) whenever, for all $x \in C$, for all $\zeta \in \mathrm{N}_C^P(x) \cap \mathbb{B}_H$ and for all $\alpha \in]0, r[$, one has $x \in \mathrm{Proj}_C(x + \alpha\zeta)$.*

The following proposition summarizes some important consequences of prox-regularity needed in the paper. For the proof of these results, we refer the reader to [8].

Proposition 2.1. *Let C be a nonempty closed subset in H and let $r \in]0, +\infty]$. Then the following are equivalent:*

(i) *The subset C is (uniformly) r-prox-regular;*

(ii) *For any point x' in the open enlargement $U_r(C) = \{x \in H : d_C(x) < r\}$, the set $\mathrm{Proj}_C(x')$ is a singleton and the mapping $\mathrm{Proj}_C(.)$ is continuous on $U_r(C)$;*

(iii) *The equality $\partial^P d_C(x) = \partial d_C(x)$ for all $x \in H$ satisfying $d_C(x) < r$;*

(iv) *The distance function $d_C(.)$ is continuously Fréchet differentiable on the open tube $\{x \in H : 0 < d_C(x) < r\}$;*

(v) *The normal cone $\mathrm{N}_C^P(.)$ satisfies the hypomonotonicity condition*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq - \|x_1 - x_2\|^2$$

for all $x_i \in C$ and all $y_i \in \mathrm{N}_C^P(x_i)$ with $\|y_i\| \leq r$ where $i \in \{1, 2\}$.

At any $x \in H$ with $0 < d_C(x) < r$, by the Fréchet differentiability of $d_C(.)$ we have

$$\|\nabla d_C(x)\| = 1 \tag{6}$$

Remark 2.1. (i) As a consequence of (iii) in Proposition 2.1, for uniformly prox-regular sets C, the Fréchet and the proximal normal cones coincide, i.e. $\mathrm{N}_C^P(x) = \mathrm{N}_C^F(x)$.

(ii) The closed r-prox-regular set C is convex if and only if $r = +\infty$. Some familiar examples of prox-regular sets can be found in [8].

(iii) The finite union of disjoint intervals is nonconvex but r-prox-regular where r depends on the distances between the intervals (see e.g. [8]).

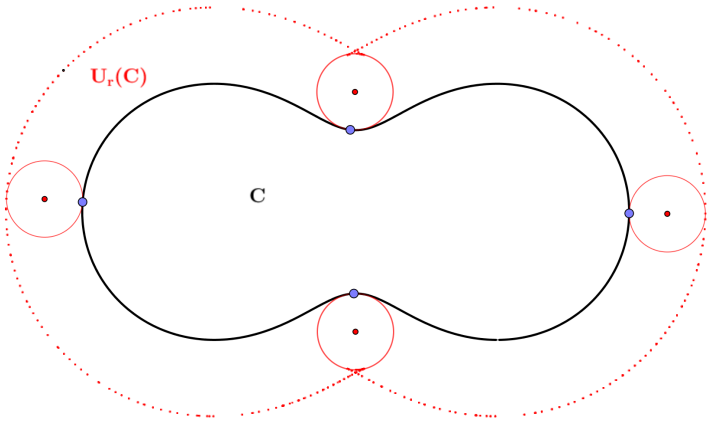

Figure 1: A prox-regular set with its enlargement $U_r(C)$ in $\mathbb{R}^2$.

(iv) More generally, any finite union of disjoint convex subsets in H is nonconvex but r-prox-regular where r depends on the distances between the sets (see e.g. [8]).

3. Reduction result

In this section, we prove the coincidence between the solution sets of (1) and (2) under the r-prox-regularity assumption. First let us introduce the set-valued mapping $C : [0, T] \rightrightarrows H$ and the mapping $A : H \to H$ with the following assumptions:

($\mathcal{H}_1$) For all $t \in [0, T]$, $C(t)$ is a nonempty closed and uniformly r-prox-regular subset of H;

($\mathcal{H}_2$) There exists an absolutely continuous nonnegative function $\zeta : [0, T] \longrightarrow \mathbb{R}_+$ (with $\zeta(0) = 0$) such that

$$\left|d_{C(t)}(x) - d_{C(s)}(x)\right| \leq |\zeta(t) - \zeta(s)|, \quad \text{for all } s,\ t \in [0, T] \text{ and all } x \in H;$$

$(\mathcal{H}_3)$ $A : H \longrightarrow H$ is a (possibly) nonlinear, Lipschitz and strongly monotone operator, i.e. satisfying

$$\|Ax - Ay\| \leq L\|x - y\|, \text{ for all } x, y \in H, \text{ for some constant } L > 0.$$

and

$$\langle Ax - Ay, x - y\rangle \geq \beta \|x - y\|^2, \text{ for all } x, y \in H, \text{ for some constant } \beta > 0.$$

Remark 3.1. *Note that, if there is an absolutely continuous function $\gamma : [0, T] \to \mathbb{R}_+$ satisfing $(\mathcal{H}_2)$, then by setting $\zeta(t) = \int_0^t (|\dot{\gamma}(z) + \varepsilon)dz$, the function $\zeta(\cdot)$ fulffils the same inequality i.e. $(\mathcal{H}_2)$ as well as the condition $\zeta(0) = 0$ and $\dot{\zeta}(\cdot) \geq \varepsilon > 0$.*

Using some ideas from [20] for the classical sweeping process and adapted in our context to the degenerate sweeping process, the following theorem states the coincidence between the solution sets of (1) and (2) under the r-prox-regularity assumption.

Theorem 3.1. *Assume that $(\mathcal{H}_1), (\mathcal{H}_2)$ and $(\mathcal{H}_3)$ hold. A mapping $u(.)$ is a solution of the constrained evolution inclusion* (1) *if and only if it is a solution of the unconstrained differential inclusion* (2)*.*

Proof. Put $v(t) = Au(t)$ for all $t \in [0, T]$. Note first that v is differentiable a.e., since A is Lipschitz continuous and a mapping $u(\cdot) : [0, T] \to H$ is absolutely continuous. Let $h > 0$ and $t \in [0, T]$ such that $\dot{u}(t)$ and $\dot{v}(t)$ exist. It follows from the strong monotonicity of A by taking $x = u(t + h)$ and $y = u(t)$ that a.e.

$$\langle v(t + h) - v(t), u(t + h) - u(t)\rangle \geq \beta \|u(t + h) - u(t)\|^2$$

or

$$\left\langle \frac{v(t + h) - v(t)}{h}, \frac{u(t + h) - u(t)}{h} \right\rangle \geq \beta \left\| \frac{u(t + h) - u(t)}{h} \right\|^2$$

and let $h \to 0^+$

$$\langle \dot{v}(t), \dot{u}(t)\rangle \geq \beta \|\dot{u}(t)\|^2. \tag{7}$$

• (1) $\Longrightarrow$ (2). Let $u(.)$ be any solution of the constrained differential inclusion (1) . Let $t \in]0,T[$ be fix such that $u(.)$, $v(.)$ and $\zeta(.)$ are derivable at t with $\dot{u}(t) \neq 0$ and $-\dot{u}(t) \in \mathrm{N}_{C(t)}(v(t))$. Using Remark 2.1 (i), we have: $\mathrm{N}^F_{C(t)}\big(v(t)\big) = \mathrm{N}_{C(t)}\big(v(t)\big)$. Therefore,

$$-\frac{\dot{u}(t)}{\|\dot{u}(t)\|} \in \mathrm{N}^F_{C(t)}\big(v(t)\big)$$

and so by (5), we obtain

$$-\frac{\dot{u}(t)}{\|\dot{u}(t)\|} \in \partial^F d_{C(t)}\big(v(t)\big). \tag{8}$$

Also, the absolute continuity of $u(.)$ and the Lipschitz continuity of A give

$$\|v(t) - v(s)\| \leq L \int_s^t \|\dot{u}(\tau)\|\, d\tau.$$

From the definition of Fréchet subdifferential, we have for any $\varepsilon > 0$ and for $s < t$ sufficiently close to t,

$$\begin{aligned}
\left\langle -\frac{\dot{u}(t)}{\|\dot{u}(t)\|}, v(s) - v(t) \right\rangle &\leq d_{C(t)}\left(v(s)\right) + \varepsilon \|v(t) - v(s)\| \\
&= d_{C(t)}\left(v(s)\right) - d_{C(s)}\left(v(s)\right) + \varepsilon \|v(t) - v(s)\| \\
&\leq |\zeta(t) - \zeta(s)| + \varepsilon \|v(t) - v(s)\|,
\end{aligned}$$

and hence

$$\left\langle -\frac{\dot{u}(t)}{\|\dot{u}(t)\|}, \frac{v(s) - v(t)}{t - s} \right\rangle \leq \left| \frac{\zeta(t) - \zeta(s)}{t - s} \right| + \varepsilon \left\| \frac{v(t) - v(s)}{t - s} \right\|.$$

By passing to the limit as $s \uparrow t$, we get for any $\varepsilon > 0$,

$$\left\langle \frac{\dot{u}(t)}{\|\dot{u}(t)\|}, \dot{v}(t) \right\rangle \leq \left| \dot{\zeta}(t) \right| + \varepsilon \|\dot{v}(t)\|,$$

and using (7), we get

$$\|\dot{u}(t)\| \leq \frac{\left|\dot{\zeta}(t)\right|}{\beta}. \tag{9}$$

So, the inequality (9) holds when $\dot{u}(t) = 0$, we get $\|\dot{u}(t)\| \leq \frac{|\dot{\zeta}(t)|}{\beta}$ for almost all $t \in [0,T]$ and hence by the definition of the Fréchet subdifferential

$$\dot{u}(t) \in -\|\dot{u}(t)\|\, \partial^F d_{C(t)}\big(Au(t)\big) \subset -\frac{\left|\dot{\zeta}(t)\right|}{\beta} \partial^F d_{C(t)}\big(Au(t)\big).$$

Consequently,

$$-\dot{u}(t) \in \frac{\left|\dot{\zeta}(t)\right|}{\beta}\partial d_{C(t)}\big(Au(t)\big),$$

and hence any solution of (1) is a solution of (2).

• (2) $\Longrightarrow$ (1). Let $u(.)$ be any solution of the unconstrained differential inclusion (2) . We know that $\partial d_C(x) \subset \mathrm{N}_C(x)$ for all $x \in C$, so it is enough to show that $v(t) \in C(t)$ for all $t \in [0,T]$. As $\dot{\zeta}(.) \in L^1([0,T],\mathbb{R})$, then

$$\lim_{\lambda(S)\longrightarrow 0}\int_S \left|\dot{\zeta}(t)\right| dt = 0,$$

where λ is the Lebesgue measure in $\mathbb{R}$ and S is a measurable set of $\mathbb{R}$.

We can fixe some real $\alpha > 0$ such that

$$[1+\frac{L}{\beta}]\int_S \left|\dot{\zeta}(t)\right| dt < r, \tag{10}$$

for all Lebesgue measurable sets $S \subset [0,T]$ satisfying $\lambda(S) < \alpha$.

Let us denote, for $r \in \,]0,+\infty]$, by $\mathcal{U}_r(C)$, the open $r-$tube around the set C involved in (ii) of Proposition 2.1, that is

$$\mathcal{U}_r(C) = U_r(C) \setminus C.$$

The proof is divided into two steps.

1. First, we suppose that $T < \alpha$. We prove that $u(.)$ is a solution of (1), that is, $v(t) \in C(t)$, for all $t \in [0,T]$.

Let $\psi : [0,T] \longrightarrow \mathbb{R}$ be the function defined by

$$\psi(t) := d_{C(t)}\big(v(t)\big).$$

The function ψ is absolutely continuous since, by $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, we have

$$|\psi(t) - \psi(s)| \leq L\,\|u(t) - u(s)\| + |\zeta(t) - \zeta(s)|$$

for all $s,t \in [0,T]$. So the set

$$\Omega := \{t \in [0,T] : v(t) \notin C(t)\} = \psi^{-1}\left(]0,+\infty]\right)$$

is open in $[0,T]$. We claim that this set Ω is in fact empty. Indeed, let us suppose that $\Omega \neq \emptyset$. As $0 \notin \Omega$, there exists a nonempty open interval $]a,b[$ such that $\psi(a)=0$. Indeed, we can write $\Omega = \cup_{i\geq 1}]a_i,b_i[$ with $]a_i,b_i[\cap]a_j,b_j[=\emptyset, \forall i \neq j$ and $a_1 < a_2 < a_2 < ...$ and $b_1 < b_2 < b_3 < ....$ As $\Omega \subset]0,T[$, if $a_1 = 0$, then we take $a := 0$ and $b := b_1$. If $a_1 \neq 0$, i.e., $a_1 > 0$, then $[0,a_1] \not\subset \Omega$. So $\psi(a_1) = 0$ and we take $a := a_1$ and $b := b_1$.

Let s be any point of $]a,b[$ where $\dot{\psi}(s), \dot{u}(s), \dot{v}(s), \dot{\zeta}(s)$ exist, with $\dot{u}(s)$ satisfies the inclusion (2). Then for some mapping $\varepsilon(\delta)\to 0$ as $\delta \downarrow 0$, we can write

$$v(s+\delta) = v(s) + \delta\dot{v}(s) + \delta\varepsilon(\delta).$$

So for some $\delta > 0$ sufficiently small, by $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ we obtain

$$\begin{aligned}
\delta^{-1}\left[\psi(s+\delta)-\psi(s)\right] &= \delta^{-1}\left[d_{C(s+\delta)}\left(v(s+\delta)\right) - d_{C(s)}\left(v(s)\right)\right] \\
&= \delta^{-1}\left[d_{C(s+\delta)}\left(v(s)+\delta\dot{v}(s)+\delta\varepsilon(s)\right) - d_{C(s+\delta)}\left(v(s)+\delta\dot{v}(s)\right)\right] + \\
&+ \delta^{-1}\left[d_{C(s+\delta)}\left(v(s)+\delta\dot{v}(s)\right) - d_{C(s)}\left(v(s)+\delta\dot{v}(s)\right)\right] + \\
&+ \delta^{-1}\left[d_{C(s)}\left(v(s)+\delta\dot{v}(s)\right) - d_{C(s)}\left(v(s)\right)\right] \\
&\leq \|\varepsilon(\delta)\| + \delta^{-1}\left|\zeta(s+\delta)-\zeta(s)\right| + \\
&+ \delta^{-1}\left[d_{C(s)}\left(v(s)+\delta\dot{v}(s)\right) - d_{C(s)}\left(v(s)\right)\right]. \qquad (11)
\end{aligned}$$

Since $\dot{\zeta}(s) > 0$ (see Remark 3.1), then $\dot{u}(s)$ satisfies (2) and $v(s) \notin C(s)$. We get by $(\mathcal{H}_2), (\mathcal{H}_3)$ and $v(0) \in C(0)$,

$$\begin{aligned}
d_{C(s)}\left(v(s)\right) &= d_{C(s)}\left(v(s)\right) - d_{C(0)}\left(v(0)\right) \\
&\leq L\left\|u(s)-u_0\right\| + \left|\zeta(s)-\zeta(0)\right| \\
&= L\left\|\int_0^s \dot{u}(\tau)d\tau\right\| + \left|\int_0^s \dot{\zeta}(\tau)d\tau\right|.
\end{aligned}$$

Using the fact that

$$\|\dot{u}(\tau)\| \leq \frac{\left|\dot{\zeta}(\tau)\right|}{\beta}, \text{ for a.e. } \tau \in [0,T],$$

we have

$$\begin{aligned} d_{C(s)}\left(Au(s)\right) &\leq L\int_0^s \|\dot{u}(\tau)\|\, d\tau + \int_0^s \left|\dot{\zeta}(\tau)\right| d\tau \\ &\leq \frac{L}{\beta}.\int_0^s \left|\dot{\zeta}(\tau)\right| d\tau + \int_0^s \left|\dot{\zeta}(\tau)\right| d\tau \\ &\leq \left(\frac{L}{\beta}+1\right).\int_0^s \left|\dot{\zeta}(\tau)\right| d\tau. \end{aligned} \tag{12}$$

Taking into account (10), (12) and the inequality $T < \alpha$, we have

$$0 < d_{C(s)}\left(v(s)\right) < r, i.e.\ v(s) \in \mathcal{U}_r(C(s)). \tag{13}$$

By $(\mathcal{H}_1)$ and (iv) of Proposition (2.1), the function $d_{C(s)}(.)$ is Fréchet differentiable at $v(s)$. Then the inclusion (2) for $\dot{u}(s)$ means that $\left(-\beta.\frac{\dot{u}(s)}{\dot{\zeta}(s)}\right)$ is the Fréchet derivative at $v(s)$ of the function $d_{C(s)}(.)$, which combined with the fact that $v(s) \notin C(s)$, entails that

$$\beta\,\|\dot{u}(s)\| = \left|\dot{\zeta}(s)\right|. \tag{14}$$

As $\dot{\zeta}(s) > 0$ (see Remark 3.1), then $\dot{v}(s) \neq 0$. Indeed, suppose the contrary, i.e., $\dot{v}(s) = 0$ then (7) implies that $\dot{u}(s) = 0$ and hence we get from $\beta > 0$ and (14) that $\dot{\zeta}(s) = 0$. Then we have $\dot{v}(s) \neq 0$ and

$$\lim_{x \to v(s)} \frac{1}{\|x - v(s)\|}\left[d_{C(s)}(x) - d_{C(s)}(v(s)) + \beta\langle\frac{\dot{u}(s)}{\dot{\zeta}(s)}, x - v(s)\rangle\right] = 0.$$

In particular, for $x = v(s) + \delta\dot{v}(s)$ and for some function $\eta(\delta) \longrightarrow 0$ we get

$$\begin{aligned} \delta^{-1}\left[d_{C(s)}\left(v(s)+\delta\dot{v}(s)\right) - d_{C(s)}\left(v(s)\right)\right] &= -\beta\left\langle \frac{\dot{u}(s)}{\left|\dot{\zeta}(s)\right|}, \dot{v}(s)\right\rangle + \eta(\delta) \\ &= \frac{-\beta}{\left|\dot{\zeta}(s)\right|}\left\langle \dot{u}(s), \dot{v}(s)\right\rangle + \eta(\delta). \end{aligned}$$

The differential inequality in (7) permits us to write

$$\delta^{-1}\left[d_{C(s)}\left(v(s)+\delta\dot{v}(s)\right)-d_{C(s)}\left(v(s)\right)\right] \leq -\frac{\beta}{\left|\dot{\zeta}(s)\right|}\beta\left\|\dot{u}(s)\right\|^2+\eta(\delta)$$
$$= -\left|\dot{\zeta}(s)\right|+\eta(\delta). \tag{15}$$

Inequalities (11) and (15) permit us to write

$$\dot{\psi}(s) \leq \left|\dot{\zeta}(s)\right| - \left|\dot{\zeta}(s)\right| = 0.$$

Therefore, for almost every $s \in]a,b[$, we have $\dot{\psi}(s) \leq 0$ and so for every $t \in]a,b[$, we get

$$\psi(t) = \psi(a) + \int_a^t \dot{\psi}(s)ds = \int_a^t \dot{\psi}(s)ds \leq 0,$$

which constitutes a contradiction with $]a,b[\subset \Omega$. Consequently, $\Omega = \emptyset$ and the proof of the first step is completed.

2. Consider now the case where no restriction on the length of the interval $[0,T]$ is assumed.

Taking an integer N such that $\frac{T}{N} < \alpha$ and consider the subdivision of $[0,T]$ given by $0 = T_0, T_1, ..., T_N = T$ with

$$T_k = T_0 + k\frac{T}{N}, \text{ for } k = 0,1,...,N.$$

For each $k = 1,...,N$ denote by $u^k(.)$ the restriction of $u(.)$ to $[T_{k-1}, T_k]$, that is

$$u^k := u\mid_{[T_{k-1},T_k]}$$

It suffices to apply step 1 to the restriction $u\mid_{[T_{k-1},T_k]}$ on $[T_{k-1},T_k]$ of the solution of (2) to conclude.

4. Existence and Uniqueness Result with Lipschitz perturbation

In this section, and under the assumptions $(\mathcal{H}_1), (\mathcal{H}_2)$ and $(\mathcal{H}_3)$, we derive from the above section the existence and uniqueness of solution to the evolution

inclusion

$$\begin{cases} -\dot{u}(t) \in \mathrm{N}_{C(t)}\big(Au(t)\big) + f(t,u(t)) \text{ a.e. } t \in [0,T] \\ u(0) = u_0, Au(0) \in C(0) \end{cases} \tag{16}$$

with the Lipschitz perturbation $f : [0,T] \times H \to H$.

Theorem 4.1. *Assume that $(\mathcal{H}_1), (\mathcal{H}_2)$ and $(\mathcal{H}_3)$ hold. Let $f : [0,T] \times H \to H$ be a single-valued mapping such that the mapping $t \mapsto f(t, v(t))$ is Lebesgue measurable for each Lebesgue measurable mapping $v(\cdot)$ from $[0,T]$ into H and Lipschitz continuous with respect to the second variable i.e. there exists $\kappa(.) \in L^1([0,T], \mathbb{R}^+)$ such that*

$$\|f(t,x) - f(t,y)\| \leq \kappa(t)\|x - y\|, \quad \text{for all } x,\, y \in H \text{ and } t \in [0,T]. \tag{17}$$

(a) If the mapping f satisfies, for some nonnegative Lebesgue integrable function $\gamma(\cdot)$ the boundedness condition $\|f(t,x)\| \leq \gamma(t)$ for all $t \in [0,T]$ and all $x \in H$, then any solution of the unconstrained evolution inclusion

$$\begin{cases} -\dot{u}(t) \in \frac{|\dot{\zeta}(t)| + L\gamma(t)}{\beta} \partial d_{C(t)}\big(Au(t)\big) + f(t,u(t)) \text{ a.e. } t \in [0,T] \\ u(0) = u_0, Au(0) \in C(0) \end{cases} \tag{18}$$

is a solution of (16).

(b) Let $\gamma(\cdot)$ and $\sigma(\cdot)$ be non negative Lebesgue integrable functions for which the mapping f fulfills the linear growth condition

$$\|f(t,x)\| \leq \gamma(t) + \sigma(t)\|x\|$$

for all $(t,x) \in [0,T] \times H$. Then any solution of the unconstrained evolution inclusion

$$\begin{cases} -\dot{u}(t) \in \frac{|\dot{\zeta}(t)| + L\dot{m}(t)}{\beta} \partial d_{C(t)}\big(Au(t)\big) + f(t,u(t)) \text{ a.e. } t \in [0,T] \\ u(0) = u_0, Au(0) \in C(0) \end{cases} \tag{19}$$

is a solution of (16)*, where $m(\cdot) : [0,T] \longrightarrow \mathbb{R}$ is the absolutely continuous solution of the ordinary differential equation*

$$\begin{cases} \dot{m}(t) &= \big[\gamma(t) + \sigma(t)(\|u_0\| + \exp(\int_0^t \gamma(s)\,ds) \int_0^t (\frac{|\dot{\zeta}(s)|}{\beta} + \gamma(s))ds\big] + \\ & \quad + \frac{L}{\beta}\sigma(t) \exp(\int_0^t \gamma(s)\,ds) m(t), \\ m(0) &= 0. \end{cases} \tag{20}$$

If H is finite dimensional, then (16) *has at least one solution $u(.)$ which satisfies $\|\dot{u}(t) + f(t, u(t))\| \leq \left[\frac{|\dot{\zeta}(t)|+L\dot{m}(t)}{\beta}\right]$ for almost all $t \in [0, T]$. If the operator $A : H \to H$ is linear and symmetric , then the solution is unique.*

Proof. (a) We suppose that $u(.)$ is a solution of (18). For all $t \in [0, T]$, let us define
$z(t) := \int_0^t f(s, u(s))ds$, $D(t) := C(t) + Az(t)$ and $v(t) := u(t) + z(t)$.
Clearly, $D(t)$ is $r-$ prox-regular. Further, for $x \in H$ and $t, s \in [0, T]$ with $s \leq t$ by one has

$$\begin{aligned} |d(x, \mathrm{D}(t)) - d(x, \mathrm{D}(s))| &\leq |d(x - Az(t), C(t)) - d(x - Az(s), C(s))| \\ &\leq L\|z(t) - z(s)\| + |\zeta(t) - \zeta(s)| \\ &\leq |\xi(t) - \xi(s)|, \end{aligned}$$

where

$$\xi(t) := \int_0^t \big[L\gamma(s) + |\dot{\zeta}(s)|\big]ds.$$

Hence $D(\cdot)$ satisfies also $(\mathcal{H}_2)$ with the absolutely continuous function $\xi(\cdot)$. As $Au_0 \in C(0) = D(0)$, we can rewrite the evolution inclusion (18) as

$$\begin{cases} -\dot{v}(t) \in \frac{\dot{\xi}(t)}{\beta}\partial d_{D(t)}\big(Av(t)\big) \text{ a.e. } t \in [0, T] \\ v(0) = u_0, Av(0) \in D(0), . \end{cases}$$

According to Theorem (3.1), the map $v(\cdot)$ satisfies also the evolution inclusion

$$\begin{cases} -\dot{v}(t) \in \mathrm{N}_{D(t)}\big(Av(t)\big) \text{ a.e. } t \in [0, T] \\ v(0) = u_0, Av(0) \in D(0), \end{cases}$$

and so

$$\begin{cases} -\dot{u}(t) \in \mathrm{N}_{C(t)}\big(Au(t)\big) + f(t, u(t)) \text{ a.e. } t \in [0, T] \\ u(0) = u_0, Au(0) \in C(0), \end{cases}$$

that is, $u(\cdot)$ is a solution of (16).

(b) Let $u(\cdot)$ be a solution of (19) with $m(\cdot)$ as given in (20) . According to the Linear growth condition that

$$\|\dot{u}(t)\| \leq \frac{|\dot{\zeta}(t)| + L\dot{m}(t)}{\beta} + \gamma(t) + \sigma(t)\|u(t)\|. \tag{21}$$

Putting $\theta(t) := \|u_0\| + \int_0^t \|\dot{u}(s)\|\,ds$ and $\delta(t) := \dfrac{|\dot{\zeta}(t)| + L\dot{m}(t)}{\beta} + \gamma(t)$. We note that by the triangular inequality we have $\|u(t)\| \le \theta(t)$. The inequality (21) ensures us that $\dot{\theta}(t) \le \delta(t) + \sigma(t)\theta(t)$ for a.e. $t \in [0,T]$. The Gronwall lemma yields for every $t \in [0,T]$ that

$$\theta(t) \le \theta(0) + \exp(\int_0^t \sigma(s)\,ds) \int_0^t \delta(s)\,ds,$$

which implies

$$\|u(t)\| \le \|u_0\| + \exp(\int_0^t \sigma(s)\,ds) \int_0^t \delta(s)\,ds.$$

Taking into account this inequality and the linear growth condition we obtain

$$\begin{aligned}\|f(t,u(t))\| &\le \gamma(t) + \sigma(t)\Big((\|u_0\| + \exp(\int_0^t \gamma(s)\,ds) \int_0^t (\frac{|\dot{\zeta}(s)| + L\dot{m}(s)}{\beta} + \gamma(s))ds\Big) \\ &= \Big(\gamma(t) + \sigma(t)\Big(\|u_0\| + \exp(\int_0^t \gamma(s)\,ds) \int_0^t (\frac{|\dot{\zeta}(s)|}{\beta} + \gamma(s))ds\Big) + \\ &+ \frac{L}{\beta}\sigma(t)\exp(\int_0^t \gamma(s)\,ds)m(t),\end{aligned}$$

which gives by (20)

$$\|f(t,u(t))\| \le \dot{m}(t).$$

Since $t \mapsto \dot{m}(t)$ is integrable as an absolutely continuous function and nonnegative on $[0,T]$, we then apply (a) to obtain that $u(\cdot)$ is a solution of (16), which completes the proof.

Now we prove in the finite dimensional setting, the existence of solutions for (16) . So, from above, it is enough to prove that the unconstrained evolution inclusion (19) has a solution. Putting for all $(t,x) \in [0,T] \times H$

$$(t,x) \mapsto G(t,x) := -\Big[\frac{|\dot{\zeta}(t)| + L\dot{m}(t)}{\beta}\Big]\partial d_{C(t)}\big(Ax\big) + f(t,x).$$

Then, the evolution inclusion (19) appears as a classical evolution inclusion

$$\dot{u}(t) \in G(t,u(t)) \text{ with } u(0) = u_0.$$

Indeed, the right hand side is given by a set-valued mapping with compact convex values that is measurable and upper semicontinuous with respect to x.

Also we have $G(t,x) \subset \left[\frac{|\dot{\zeta}(t)|+L\dot{m}(t)}{\beta} + \dot{m}(t)\right]\mathbb{B}_H$. Then, it follows from Theorem VI-13 in [10] that the evolution inclusion (19) has at least one solution.

Suppose now that $A : H \to H$ is linear, symmetric and that $u_i, i = 1, 2$ are two solutions of (16) such that

$$-\dot{u}_i(t) \in \mathrm{N}_{C(t)}(Au_i(t)) + f(t, u_i(t)), \quad i = 1, 2.$$

Let $\dot{\xi}(\cdot) := L\dot{m}(\cdot) + |\dot{\zeta}(\cdot)| \in L^1(0,T)$ such that $\|\dot{u}_i(t) + f(t, u_i(t))]\| \leq \frac{\dot{\xi}(t)}{\beta}$ we have

$$\|\frac{r\beta}{\dot{\xi}(t)}[-\dot{u}_i(t) - f(t, u_i(t))]\| \leq r, \quad i = 1, 2.$$

Therefore using the hypomonotonicity of the normal cone for r-prox-regular sets, we have

$$\langle \dot{u}_1(t) - \dot{u}_2(t) + f(t, u_1(t)) - f(t, u_2(t)), Au_1(t) - Au_2(t)\rangle \leq \frac{\dot{\xi}(t)}{r\beta}\|Au_1(t) - Au_2(t)\|^2.$$

On the other hand, using the Lipschitz assumption (17) on f , we obtain

$$\begin{aligned}\langle \dot{u}_1(t) - \dot{u}_2(t), Au_1(t) - Au_2(t)\rangle &\leq \frac{\dot{\xi}(t)L^2}{r\beta}\|u_1(t) - u_2(t)\|^2 \\ &\quad - \langle f(t, u_1(t)) - f(t, u_2(t)), Au_1(t) - Au_2(t)\rangle \\ &\leq [\frac{\dot{\xi}(t)L^2}{r\beta} + \kappa(t)L]\|u_1(t) - u_2(t)\|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle u_1(t) - u_2(t), Au_1(t) - Au_2(t)\rangle &= \langle \dot{u}_1(t) - \dot{u}_2(t), Au_1(t) - Au_2(t)\rangle \\ &\leq \alpha(t)\|u_1(t) - u_2(t)\|^2,\end{aligned}$$

where $\alpha(t) := [\frac{\dot{\xi}(t)L^2}{r\beta} + \kappa(t)L]$. By integrating, using the initial conditions $u_1(0) = u_2(0) = u_0$ and the strong monotonicity of A, we get

$$\begin{aligned}\beta\|u_1(t) - u_2(t)\|^2 &\leq \langle u_1(t) - u_2(t), Au_1(t) - Au_2(t)\rangle \\ &\leq \int_0^t 2\alpha(s)\|u_1(s) - u_2(s)\|^2 ds.\end{aligned}$$

Gronwall's Lemma permits us to obtain that $\|u_1(t) - u_2(t)\|^2 = 0$. Therefore $u_1 = u_2$. Which completes the proof.

Remark 4.1. (i) We note that for the perturbed problem (16) the penalization parameter in both (18) and (19) depends not only on the derivative $\dot{\zeta}(t)$ and the coefficient β but also on the Lipschitz constant L of the operator A and the coefficients of the linear growth of $f(t,x)$.
(ii) The result obtained by L. Thibault in [20] can be derived as a particular case by setting $A = \mathrm{Id}$ (and so $\beta = 1$) in Theorem 4.1(a).

5. Application to nonsmooth mechanical systems

Many mechanical systems can be formulated by Lagrangian equations. The Lagrangian function is defined by

$$\mathcal{L}(q,\dot{q}) = \mathcal{T}(q,\dot{q}) - \mathcal{V}(q),$$

where $\mathcal{T}(q,\dot{q}) = \frac{1}{2}\langle M(q)\dot{q},\dot{q}\rangle$ is the kinetic energy and $\mathcal{V}(q)$ the potential energy. The matrix $M(q) \in \mathbb{R}^{n\times n}$ is the usual inertia matrix, which is symmetric and supposed to be positive definite and analytic with respect to q. With the generalized coordinates $q \in \mathbb{R}^n$, an external force $t \mapsto f_0(t) \in \mathbb{R}^n$ and a perturbation $F(\cdot,q,\dot{q})$, the Lagrange equations have the following form

$$\frac{d}{dt}\big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\big) - \frac{\partial \mathcal{L}}{\partial q} + F(\cdot,q,\dot{q}) = f_0. \tag{22}$$

Using classical arguments, we can rewrite (22) in the form

$$M(q)\ddot{q} + N(q,\dot{q}) + \nabla\mathcal{V}(q) + F(\cdot,q,\dot{q}) = f_0, \tag{23}$$

where $N(q,\dot{q})$ is the nonlinear inertial terms called the gyroscopic accelerations. In the setting of unilateral constraints described by a set of m inequalities, we defined the feasible set by

$$C(t) = \{q \in \mathbb{R}^n \ : \ g_i(t,q) \le 0, \ \ i = 1,2,\ldots,m\}, \tag{24}$$

where $g_i : [0,T]\times\mathbb{R}^n \to \mathbb{R}$, $(t,q) \mapsto g_i(t,q)$ are given functions (assumed to be continuously differentiable and not necessarily convex). The unilateral

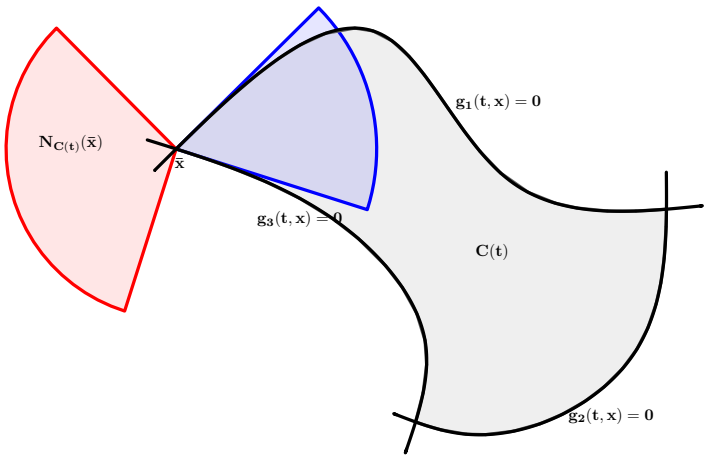

Figure 2: A moving feasible set in $\mathbb{R}^2$ and its normal cone at a point.

constraints will generate a generalized reaction force R such that the equation of motion is given by

$$M(q)\ddot{q} + N(q,\dot{q}) + \nabla\mathcal{V}(q) + F(\cdot,q,\dot{q}) = f_0 + R \tag{25}$$

$$-R \in \mathrm{N}_{C(t)}(q(t)). \tag{26}$$

Assume that the Mangasarian-Fromovitz qualification condition is satisfied at a point $\bar{x}$, that is: there is a vector $\bar{v} \in \mathbb{R}^n$ such that: $\langle \nabla g_k(\cdot,\bar{x}), \bar{v}\rangle < 0 \quad$ for all $k \in I(\bar{x})$, where $I(\bar{x}) := \{k \in \{1,\ldots,m\} : g_k(\cdot,\bar{x}) = 0\}$. Then the Clarke and the Fréchet normal cones of $C(\cdot)$ at $\bar{x}$ coincide and

$$\mathrm{N}_{C(t)}(\bar{x}) = \Big\{\sum_{k=1}^{m} \lambda_k(t)\nabla g_k(t,\bar{x}) : \lambda_k(t) \geq 0,\ \lambda_k(t)g_k(t,\bar{x}) = 0,\ g_k(t,\bar{x}) \leq 0,\ k = 1,2,\ldots,m\Big\}.$$

At this stage two main questions arise:

Question 1: under which conditions on the data g_k, the set of constraints $C(\cdot)$ defined in (24) is prox-regular? and how to estimate the parameter of prox-regularity $r > 0$?

Question 2: under which conditions on the data g_k, the set of constraints $C(\cdot)$

defined in (24) is Lipschitz-continuous with respect to the Hausdorff distance? The main concern is to give for each case some sufficient conditions verifiable in practice.

The answer to the first question is given in the following Theorem (see [5]).

Theorem 5.1. *Let $C(t)$ defined as in (24) and assume that, there exists an extended real $\rho \in]0, +\infty]$ such that:*

(i) *for all $t \in [0,T]$, for all $k \in \{1,\dots,m\}$, $g_k(t,\cdot)$ is continuously differentiable on $U_\rho(C(t)) = \{q \in \mathbb{R}^n \ : \ d_{C(t)}(q) < \rho\}$;*

(ii) *there exists a real $\gamma \geq 0$ such that, for all $t \in [0,T]$, for all $k \in \{1,\dots,m\}$, for all $x, y \in U_\rho(C(t))$,*

$$\langle \nabla g_k(t,\cdot)(x) - \nabla g_k(t,\cdot)(y), x - y\rangle \geq -\gamma \left\|x-y\right\|^2, \tag{27}$$

that is, $\nabla g_k(t,\cdot)$ is γ-hypomonotone on $U_\rho(C(t))$.

(iii) *there is a real $\delta > 0$ such that, for all $(t,x) \in [0,T] \times \mathbb{R}^n$ with $x \in \mathrm{bdry}(C(t))$, there exists $\overline{v} \in \mathbb{B}$ satisfying, for all $k \in \{1,\dots,m\}$,*

$$\langle \nabla g_k(t,\cdot)(x), \overline{v}\rangle \leq -\delta. \tag{28}$$

Then, for all $t \in [0,T]$, the set $C(t)$ is r-prox-regular with $r = \min\left\{\rho, \frac{\delta}{\gamma}\right\}$.

Proof. For the proof of this result see Theorem 4.1 in [5].

Let specify here some assumptions under which the set-valued map $C : [0,T] \rightrightarrows \mathbb{R}^n$ is Lipschitz continuous with respect to the Hausdorff distance and answer the second question.

Proposition 5.1. *Let $C(t)$ defined as in (24). Assume that there exist reals $\gamma \geq 0$, $\delta > 0$ and $v \in H$ with $\|v\| = 1$ such that for each $i = 1, \cdots, m$*

$$g_i(t,u) \leq g_i(s,u) + \gamma|t-s| \quad \textit{for all } u \in U_r\big(C(s)\big); \tag{29}$$

$$\langle \nabla g_i(t,.)(u), v\rangle \leq -\delta \quad \textit{for all } t \in I, u \in U_r\big(C(t)\big), \tag{30}$$

where r denotes the prox-regularity constant of all sets $C(t)$.

Then $C(\cdot)$ is λ-Lipschitzian on $[0,T]$, with $\lambda \geq \frac{\gamma}{\delta}$.

PROOF. Let $I := [0,T]$ and fix a real $\lambda \geq \delta^{-1}\gamma$ and choose a subdivision $0 < T_1 < \cdots < T_p = T$ such that $T_k - T_{k-1} < \lambda r$. Fix any $k = 1, \cdots, p$ and $s,t \in I_k := [T_{k-1}, T_k]$. Take any $i = 1, \cdots, m$. Putting $\mu(s,t) := \lambda|t-s|$, for any $x \in C(s)$ we can write by (29)

$$\begin{aligned} g_i(t, x+\mu(s,t)v) &= \big(g_i(t, x+\mu(s,t)v) - g_i(s, x+\mu(s,t)v)\big) + g_i(s, x+\mu(s,t)v) \\ &\leq \gamma|t-s| + g_i(s, x+\mu(s,t)v) \\ &= \gamma|t-s| + g_i(s,x) + \mu(s,t)\int_0^1 \nabla_2 g(s, x+\theta\mu(s,t)v)\cdot v\, d\theta. \end{aligned}$$

According to (30) and to the inclusion $x \in C(s)$ it ensues that

$$g_i(t, x+\mu(s,t)v) \leq \gamma|t-s| - \mu(s,t)\delta = (\gamma - \lambda\delta)|t-s| \leq 0,$$

where the last inequality is due to the choice choise of λ. This being true for every $i = 1, \cdots, m$, it follows that $x + \mu(s,t)v = x + \lambda|t-s|v$ belongs to $C(t)$, otherwise stated, $x \in C(t) + \lambda|t-s|(-v)$. It results that

$$C(s) \subset C(t) + \lambda|t-s|\mathbb{B}_H,$$

Since the variables s and t play symmetric roles, the set-valued mapping $C(\cdot)$ is λ-Lipschitzian on $[T_{k-1}, T_k]$. From this we clearly derive that $C(\cdot)$ is λ-Lipschitzian on $I = [0,T]$.

In this paper, we concentrate on quasistatic problems which can be formulated as a degenerate sweeping process of the form (1) or its equivalent Lipschitz perturbation form (16). More precisely, we assume that the applied forces vary slowly with a relatively slow system response such that the inertial terms in the equation of motion (25) can be neglected. In this case problem (25)-(26) reduces to

$$F(t, q(t), \dot{q}(t)) = f_0(t) - \nabla\mathcal{V}(q) + R \text{ a.e. } t \in [0,T] \tag{31}$$

$$-R \in \mathrm{N}_{C(t)}(q(t)). \tag{32}$$

The linear time invariant case is given when

$$F(t, q, \dot{q}) = D\dot{q} + Kq,$$

where $D \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ are the viscosity (damping) and the stiffness matrices respectively. We note that usually in mechanical systems these two matrices contain viscous and stiffness coefficients respectively and are symmetric and positive definite matrices. Therefore, in this case problem (25)-(26) takes the form

$$D\dot{q}(t) + Kq(t) + \nabla V(q(t)) + \mathrm{N}_{C(t)}(q(t)) \ni f_0(t) \text{ a.e. } t \in [0,T]. \tag{33}$$

We note that problem (33) could be reduced to a classical sweeping process if we assume that the matrix D is positive definite.
The nonlinear time-invariant case is given when

$$F(t,q,\dot{q}) = \nabla D(q)\dot{q},$$

where $D : \mathbb{R}^n \to \mathbb{R}^n,\ q \mapsto D(q)$ is a mapping of class C^1. In this case problem (25)-(26) takes the form

$$\nabla D(q(t))\dot{q}(t) + \nabla V(q(t)) + \mathrm{N}_{C(t)}(q(t)) \ni f_0(t) \text{ a.e. } t \in [0,T]. \tag{34}$$

We have the following existence result for quasistatic nonsmooth systems with unilateral constraints.

Definition 5.1. *A mapping $D : \mathbb{R}^n \to \mathbb{R}^n$ is said to be*

(i) α-expansive if there exists a constant $\alpha > 0$ such that

$$\|D(q_1) - D(q_2)\| \geq \alpha\|q_1 - q_2\|,\ \forall q_1,\ q_2 \in \mathbb{R}^n.$$

(ii) β-cocoercive if there exists $\beta > 0$ such that

$$\langle D(q_1) - D(q_2), q_1 - q_2\rangle \geq \beta\|D(q_1) - D(q_2)\|^2,\ \forall q_1,\ q_2 \in \mathbb{R}^n.$$

Remark 5.1. *We note that if $D : \mathbb{R}^n \to \mathbb{R}^n$ is β-cocoercive, then it is both monotone and $\frac{1}{\beta}$-Lipschitz continuous. However, a β-cocoercive map is not necessarily strongly monotone. On the other hand, a strongly monotone and Lipschitz continuous map is cocoercive.*

Proposition 5.2. *Assume that for each $k = 1, \dots, m$, the functions $g_k(t, \cdot)$ satisfies the assumption (i)-(ii) and (iii) in Theorem 5.1 and that the assumptions* (29) *and* (30) *in Proposition 5.1 are satisfied. Moreover, suppose that the potential $q \in \mathbb{R}^n \mapsto V(q)$ is a continuously differentiable function with an L-Lipschitz continuous gradient, $t \mapsto f_0(t) \in L^1([0,T];\mathbb{R}^n)$ and that $D : \mathbb{R}^n \to \mathbb{R}^n$ is bijective, α-expansive and β-cocoercive, then the quasistatic problem* (34) *with unilateral constraints has at least one solution.*

Proof. By setting $f(t,q) = \nabla V(q) - f_0(t)$, $u = D(q)$ and $A = D^{-1}$, it is easy to see that (34) is equivalent to

$$-\dot{u}(t) \in f(t, Au(t)) + \mathrm{N}_{C(t)}(Au(t)),$$

which is exactly of the form defined in (16).
Since D is β-cocoercive and α-expansive, then A is β-strongly monotone and $\frac{1}{\alpha}$-Lipschitz continuous. By Theorem 5.1 and Proposition 5.1, $C(t)$ is prox-regular and Lipschitz continuous and hence absolutely continuous. To conclude, we apply Theorem 4.1(b).

Example 5.1. Let us consider for $t \in [0, 2\pi]$, the following set

$$C(t) = \{x = (x_1, x_2) \in \mathbb{R}^2 \ : \ g_1(t,x) \leq 0, \ g_2(t,x) \leq 0\},$$

where

$$\begin{cases} g_1(t,x) = (x_1\cos(t) - x_2\sin(t))^2 + 3(x_1\sin(t) + x_2\cos(t))^2 - 5 \\ g_2(t,x) = -25(x_1\cos(t) - x_2\sin(t))^2 - 49(x_1\sin(t) + x_2\cos(t))^2 + 80x_1 + 2x_2 + 25 \end{cases}$$

The moving set $C(t)$ is given by the blue shaded part in Figure 3. The condition (iii) in Theorem 5.1 is important for proving the uniform prox-regularity of the set $C(\cdot)$. The red shad part in Figure 3 gives the set of vectors v satisfying (28).

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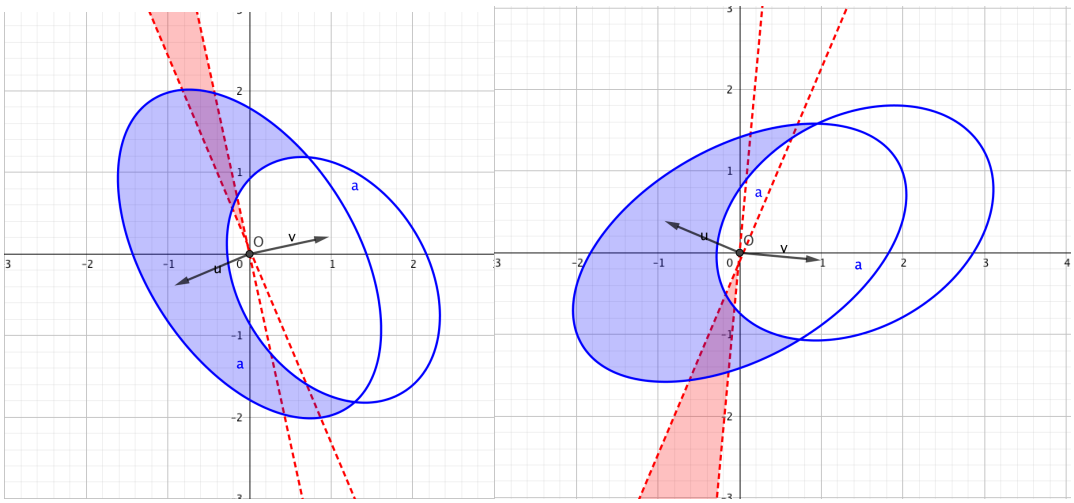

Figure 3: A moving set with the set of v satisfying (28).

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