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Resonant averaging for small-amplitude solutions of stochastic nonlinear Schrödinger equations

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We consider the free linear Schrödinger equation on a torus $\mathbb{T}^d$, perturbed by a Hamiltonian nonlinearity, driven by a random force and subject to a linear damping:

$$u_t - \mathrm{i}\Delta u + \mathrm{i}\nu\rho|u|^{2q_*}u = -\nu f(-\Delta)u + \sqrt{\nu}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\boldsymbol{k}\in\mathbb{Z}^d} b_{\boldsymbol{k}}\boldsymbol{\beta}^{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x}.$$

Here $u = u(t,x)$, $x \in \mathbb{T}^d$, $0 < \nu \ll 1$, $q_* \in \mathbb{N}$, f is a positive continuous function, ρ is a positive parameter and $\boldsymbol{\beta}^{\boldsymbol{k}}(t)$ are standard independent complex Wiener processes. We are interested in limiting, as $\nu \to 0$, behaviour of distributions of solutions for this equation and of its stationary measure. Writing the equation in the slow time $\tau = \nu t$, we prove that the limiting behaviour of them both is described by the *effective equation*

$$u_\tau + f(-\Delta)u = -\mathrm{i}F(u) + \frac{\mathrm{d}}{\mathrm{d}\tau}\sum b_{\boldsymbol{k}}\boldsymbol{\beta}^{\boldsymbol{k}}(\tau)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x},$$

where the nonlinearity $F(u)$ is made out of the resonant terms of the monomial $|u|^{2q_*}u$.

Keywords: stochastic PDEs; NLS equation; weak turbulence; averaging theorem; effective equation

2010 *Mathematics subject classification:* Primary 35R60

1. Introduction

1.1. Equations

The nonlinear Schrödinger equation on the torus with small nonlinearity

$$u_t(t,x) - \mathrm{i}\Delta u(t,x) = -\mathrm{i}\varepsilon^{2q_*}|u|^{2q_*}u, \quad u = u(t,x),\ x \in \mathbb{T}^d_L = \mathbb{R}^d/(2\pi L\mathbb{Z}^d), \tag{1.1}$$

where $q_* \in \mathbb{N}$ and $0 < \varepsilon \leqslant 1$, is a popular model in various branches of science. The nonlinearity in (1.3) is Hamiltonian and may be written as

$$-\mathrm{i}\varepsilon^{2q_*}|u|^{2q_*}u = \varepsilon^{2q_*}\mathrm{i}\nabla\mathcal{H}(u), \qquad \mathcal{H}(u) = \mathcal{H}^{2q_*+2}(u) = -\frac{1}{2q_*+2}\int |u(x)|^{2q_*+2}\,\mathrm{d}x, \tag{1.2}$$

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so the equation describes a conservative system. To describe systems interacting with the 'environment', at some stage physicists often add (to this and to other similar equations) terms that describe the pumping of energy into the system and its dissipation (see, for example, [28, § 2.2.3]). A way to describe the pumping of energy is to add to the equation a small random force, which is usually Gaussian, smooth in x, and often white in time t, while to describe the dissipation a suitable function of the Laplacian is usually used. In this way we arrive at (see [27, (5)] and [3, (1.2), § 1.2])

$$u_t - \mathrm{i}\Delta u = -\mathrm{i}\varepsilon^{2q_*}|u|^{2q_*}u - \nu f(-\Delta)u + \sqrt{\nu}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\boldsymbol{k}\in\mathbb{Z}_L^d} b_{\boldsymbol{k}}\boldsymbol{\beta}^{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x}, \quad x\in\mathbb{T}_L^d. \tag{1.3}$$

Here $0<\nu\leqslant 1$ and $\mathbb{Z}_L^d$ denotes the set of vectors of the form $\boldsymbol{k}=\boldsymbol{l}/L$ with $\boldsymbol{l}\in\mathbb{Z}^d$. The damping $-f(-\Delta)$ is the self-adjoint linear operator in $L_2(\mathbb{T}_L^d)$, which acts on the exponentials $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x}$, $\boldsymbol{k}\in\mathbb{Z}_L^d$, according to

$$f(-\Delta)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x} = \gamma_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x}, \quad \gamma_{\boldsymbol{k}} = f(\lambda_{\boldsymbol{k}}), \text{ where } \lambda_{\boldsymbol{k}} = |\boldsymbol{k}|^2. \tag{1.4}$$

The real-valued smooth function $f(t)$, $t\geqslant 0$, is positive and $f'>0$. To avoid technicalities not relevant for this work, we assume that $f(t)\geqslant C_1|t|+C_2$ for all t, for suitable positive constants C_1, C_2 (for example, $f(-\Delta)u=-\Delta u+u$). The processes $\boldsymbol{\beta}^{\boldsymbol{k}}$, $\boldsymbol{k}\in\mathbb{Z}_L^d$, are standard independent complex Wiener processes, i.e. $\boldsymbol{\beta}^{\boldsymbol{k}}(t)=\beta_+^{\boldsymbol{k}}(t)+\mathrm{i}\beta_-^{\boldsymbol{k}}(t)$, where $\beta_\pm^{\boldsymbol{k}}(t)$ are standard independent real Wiener processes. The real numbers $b_{\boldsymbol{k}}$ are all non-zero and decay fast when $|\boldsymbol{k}|\to\infty$. The factor in front of the random force is chosen to be $\sqrt{\nu}$ to guarantee that solutions of (1.3) stay of order 1 when $t\gg 1$ and $0<\nu\ll 1$.

We *assume* that (1.3) with sufficiently smooth initial data $u(0,x)=u_0(x)$ is well posed. It is well known that this assumption holds (at least) under some restriction on d, q_* and the growth of $f(t)$ at infinity; see § 2.1

The parameters ν and ε measure, respectively, the inverse time-scale of the forced oscillations and their amplitude. Physicists consider different regimes, where the two parameters are tied in various ways.[1] To do this they assume some relations between ε and ν, explicitly or implicitly. In our work we choose

$$\varepsilon^{2q_*} = \rho\nu,$$

where $\rho>0$ is a constant. This assumption is within the usually imposed bounds (see [21]). Passing to the slow time $\tau=\nu t$, we get the rescaled equation

$$\dot u + \mathrm{i}\nu^{-1}(-\Delta u) = -f(-\Delta)u - \mathrm{i}\rho|u|^{2q_*}u + \sum b_{\boldsymbol{k}}\dot{\boldsymbol{\beta}}^{\boldsymbol{k}}(\tau)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x}, \tag{1.5}$$

where $u=u(\tau,x)$, $x\in\mathbb{T}_L^d$ and the upper dot stands for $\mathrm{d}/\mathrm{d}\tau$. If we write $u(\tau,x)$ as a Fourier series, $u(\tau,x)=\sum_{\boldsymbol{k}} v_{\boldsymbol{k}}(\tau)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x}$, then in view of (1.2), (1.5) may be written as the system

$$\dot v_{\boldsymbol{k}} + \mathrm{i}\nu^{-1}\lambda_{\boldsymbol{k}}v_{\boldsymbol{k}} = -\gamma_{\boldsymbol{k}}v_{\boldsymbol{k}} + 2\rho\mathrm{i}\frac{\partial\mathcal{H}(v)}{\partial\bar v_{\boldsymbol{k}}} + b_{\boldsymbol{k}}\dot\beta^{\boldsymbol{k}}(\tau), \quad \boldsymbol{k}\in\mathbb{Z}_L^d. \tag{1.6}$$

[1] When forcing and dissipation are not present, a parameter T is introduced that measures the time-scale on which averaging is performed (see [21]): to heuristically compare with the present case, we put $\nu=1/T$.

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Here $\mathcal{H}(v)$ is the Hamiltonian $\mathcal{H}$ expressed in terms of the Fourier coefficients $v = (v_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{Z}_L^d)$,

$$\mathcal{H}(v) = -\frac{1}{2q_* + 2} \sum_{\boldsymbol{k}_1,\dots,\boldsymbol{k}_{2q_*+2} \in \mathbb{Z}_L^d} v_{\boldsymbol{k}_1} \cdots v_{\boldsymbol{k}_{q_*+1}} \bar{v}_{\boldsymbol{k}_{q_*+2}} \cdots \bar{v}_{\boldsymbol{k}_{2q_*+2}} \delta^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+2}, \quad (1.7)$$

and we use the standard notation (see [21])

$$\delta^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+2} = \begin{cases} 1 & \text{if } \boldsymbol{k}_1 + \cdots + \boldsymbol{k}_{q_*+1} - \boldsymbol{k}_{q_*+2} - \cdots - \boldsymbol{k}_{2q_*+2} = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (1.8)$$

As before, we are interested in the limit $\nu \to 0$.[2]

We note that our method also applies to (1.6) with Hamiltonians $\mathcal{H}$ of the form (1.2), where the density of the Hamiltonian is a real-valued polynomial of u and $\bar{u}$ (not necessarily a polynomial of $|u|^2$). For instance, we could work with the cubic Hamiltonians $\mathcal{H}^3 = \int |u|^2(u + \bar{u})\,\mathrm{d}x$ or $\mathcal{H}^3 = \int (u^3 + \bar{u}^3)\,\mathrm{d}x$.

1.2. Discrete turbulence

In physics, (1.1) and (1.5) with $\nu, \varepsilon \ll 1$ are treated by the theory of weak turbulence (WT); see, for instance, [21, 28]. That theory either deals with (1.5), where $L = \infty$ by formally replacing Fourier series for L-periodic functions with Fourier integrals and making them bold transformations, or considers the limit $\nu, \varepsilon \to 0$ simultaneously with the limit $L \to \infty$. That is, WT theory considers the iterated limit

$$L \to \infty, \qquad \varepsilon, \nu \to 0, \quad (1.9)$$

and treats it in an equally bold way. Concerning this limit, the WT theory makes a number of remarkable predictions based on tools and ideas developed in the community that can be traced back to the work [23]. The relation between the parameters in (1.9) is not quite clear, and it may be better to talk about the WT limits (rather than about a single case).

In order to understand the double limit above, it is natural to first study the limit $\nu \to 0$ (with L fixed). Its deterministic version recently received attention in the physical literature as the 'discrete turbulence (DT) limit' (see [10] and [21, § 10]). Similar limits were considered by mathematicians interested in related problems (see [7]), and were used by them for intermediate arguments (see, for example, [5]).

Our work is dedicated to rigorous justification of the DT limit for the damped–driven equation $(1.3)_{\nu\sim\varepsilon^{2q_*}}$=(1.5). Namely, we show that *when $\nu \to 0$, statistical characteristics of actions of solution u^ν for (1.5) have limits of order 1, described by actions of solutions for a certain* effective equation *that is a nonlinear stochastic equation with coefficients of order 1 and with a Hamiltonian nonlinearity that is made out of the resonant terms of the nonlinearity $|u|^{2q_*}u$.*

The effective equation above is a natural stochastic version of similar equations from the deterministic (physical) DT (see [10,21]). So, in a sense, our results justify the physical DT in the stochastic setting. But in the stochastic case we do more

[2] See [16] for a theory of (1.5) for the case in which $f(t) = t + 1$ and $\nu = \infty$.

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than that since we also treat the stationary regime for (1.5) and show that it converges to that for the effective equation. So solutions of the latter approximate (in distribution) solutions of the former as $t \to \infty$ and $\nu \to 0$. Remarkably, in the stationary regime the effective equation approximates not only the actions of solutions with $\nu \ll 1$, but also their angles (see § 1.4).

As the title of the paper suggests, our argument is a form of averaging. The latter is a tool that is used by the WT community on a regular basis, either explicitly (see, for example, [21]), or implicitly.

1.3. Inviscid limits for damped–driven Hamiltonian PDE, effective equations and interaction representation

Equation (1.3) is the linear Hamiltonian partial differential equation (HPDE) $(1.1)_{\varepsilon=0}$, driven by a random force, subject to the linear damping $-\nu f(-\Delta u)$ and perturbed by the Hamiltonian nonlinearity $-\varepsilon^{2q_*}\mathrm{i}\rho|u|^{2q_*}u$. Damped–driven HPDEs and the inviscid limits in these equations when the random force and the damping go to zero are very important for physics. In particular, since the d-dimensional Navier–Stokes equation (NSE) with a random force can be regarded as a damped–driven Euler equation (which is an HPDE), the inviscid limit for the NSE describes the d-dimensional turbulence. The NSE with a random force, especially when $d = 2$, has been intensively studied in the past years, but the corresponding inviscid limit turned out to be very complicated even for $d = 2$ (see [19]). The problem of this limit becomes feasible when the underlying HPDE is integrable or linear. The most famous integrable PDE is the Korteweg–de Vries (KdV) equation. Its damped–driven perturbations and the corresponding inviscid limits were studied in [12, 17]. In [13] the method of those works was applied to the situation when the unperturbed HPDE is the Schrödinger equation

$$u_t + \mathrm{i}(-\Delta u + V(x)u) = 0, \quad x \in \mathbb{T}^d_L, \tag{1.10}$$

where the potential $V(x)$ is in general a function of position. Crucial for the previously mentioned works is that the unperturbed equation is free from strong resonances: for [12, 17] it means that all solutions of the KdV equation are almost-periodic functions of time such that for a typical solution the corresponding frequency vector is free from resonances; for [13] it means that, for the typical potentials $V(x)$ considered there, the spectrum of the linear operator in (1.10) is non-resonant.

In contrast, now the linear operator in the unperturbed equation $(1.1)_{\varepsilon=0}$ has the eigenvalues $\lambda_{\boldsymbol{k}} \in \boldsymbol{k}^{-2}\mathbb{Z}$, $\boldsymbol{k} \in \mathbb{Z}^d_L$ (see (1.4)), which are highly resonant (accordingly, all solutions for $(1.1)_{\varepsilon=0}$ are periodic with the same period $2\pi L^{-2}$). This gives rise to an additional difficulty. To explain it, we rewrite (1.5)=(1.6) as a fast–slow system, defining $I_{\boldsymbol{k}} = \frac{1}{2}|v_{\boldsymbol{k}}|^2$, $\varphi_{\boldsymbol{k}} = \operatorname{Arg} v_{\boldsymbol{k}}$ (these are the action angles for the linear Hamiltonian system $(1.1)_{\varepsilon=0}$). In the new variables (1.5) reads

$$\dot{I}_{\boldsymbol{k}}(\tau) = v_{\boldsymbol{k}} \cdot P_{\boldsymbol{k}}(v) + b^2_{\boldsymbol{k}} + b_{\boldsymbol{k}}(v_{\boldsymbol{k}} \cdot \dot{\boldsymbol{\beta}}^{\boldsymbol{k}}), \tag{1.11}$$

$$\dot{\varphi}_{\boldsymbol{k}}(\tau) = -\nu^{-1}\lambda_{\boldsymbol{k}} + I^{-1}_{\boldsymbol{k}} \cdots, \tag{1.12}$$

where $\boldsymbol{k} \in \mathbb{Z}^d_L$, $\cdot$ indicates the real scalar product in $\mathbb{C} \simeq \mathbb{R}^2$, $P(v)$ is the vector field on the right-hand side of the v-equation (1.6) and $\cdots$ abbreviates a factor of

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order 1 (as $\nu \to 0$). If the frequencies $\{\lambda_{\boldsymbol{k}}\}$ are resonant, then the equations for some linear combinations of the phases $\varphi_{\boldsymbol{k}}$ are slow, which makes it more difficult to analyse the system. The method of resonant averaging treats this problem in finite dimension; see [1] and § 2.2. In the situation at hand, we have an additional problem: the φ-equations (1.12) have singularities at the locus

$$\eth = \{I \colon I_{\boldsymbol{k}} = 0 \text{ for some } \boldsymbol{k}\}, \tag{1.13}$$

which is dense in the space of sequences $(I_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{Z}_L^d)$, and the averaged I-equations

$$\dot{I}_{\boldsymbol{k}}(\tau) = \langle v_{\boldsymbol{k}} \cdot P_{\boldsymbol{k}} \rangle(I) + b_{\boldsymbol{k}}^2 + b_{\boldsymbol{k}} \sqrt{2I_{\boldsymbol{k}}} \dot{\beta}^{\boldsymbol{k}}(\tau), \quad \boldsymbol{k} \in \mathbb{Z}_L^d, \tag{1.14}$$

where $\langle \cdot \rangle$ signifies the average in $\varphi \in \mathbb{T}^\infty$, have weak singularities there. A way to overcome these difficulties is to find for (1.11), (1.12) an *effective equation*, which is a system of regular equations

$$\dot{v}_{\boldsymbol{k}} = R_{\boldsymbol{k}}(v) + b_{\boldsymbol{k}} \dot{\boldsymbol{\beta}}^{\boldsymbol{k}}(\tau), \quad \boldsymbol{k} \in \mathbb{Z}_L^d, \tag{1.15}$$

such that under the natural projection

$$v_{\boldsymbol{k}} \mapsto I_{\boldsymbol{k}} = \tfrac{1}{2}|v_{\boldsymbol{k}}|^2, \quad \boldsymbol{k} \in \mathbb{Z}_L^d,$$

solutions of (1.15) transform to solutions of (1.14). In [12] this approach was used to study the perturbed KdV equation when written as a fast–slow system, similar to (1.11), (1.12). That system has strongly nonlinear behaviour, and in [12] the effective equation was constructed as a kind of averaging of the corresponding I-equations. In [13] an effective equation for the damped–driven non-resonant equation (1.10) was derived in a similar way. If the introduced damping is linear and the nonlinearity is Hamiltonian, as in (1.3), then the effective equation in [13] is linear.

When the unperturbed Hamiltonian system is linear, an alternative way to find an effective equation is to use the *interaction representation*, i.e. to pass from the complex variables $v_{\boldsymbol{k}}(\tau)$ (which diagonalize the linear system) to the fast rotating variables

$$a_{\boldsymbol{k}}(\tau) = \mathrm{e}^{\mathrm{i}\nu^{-1}\lambda_{\boldsymbol{k}}\tau} v_{\boldsymbol{k}}(\tau), \quad \boldsymbol{k} \in \mathbb{Z}_L^d. \tag{1.16}$$

Since $|a_{\boldsymbol{k}}| = |v_{\boldsymbol{k}}|$, the limiting dynamics of the a-variables controls the limiting behaviour of the actions $I_{\boldsymbol{k}}$. So a regular system of equations, describing the limiting a-dynamics, is the effective equation. Bogolyubov used this approach for finite-dimensional deterministic averaging, calling it *averaging in quasilinear systems* (see [1]). The interaction representation is systematically used in WT theory.

Now consider the fast–slow equations (1.11), (1.12), which come from (1.6), where the fast motion (1.12) is highly resonant. Repeating the construction of the effective equation from [13], but replacing there the usual averaging by the resonant averaging, we find an effective equation corresponding to (1.6). It turns out to be another damped–driven Hamiltonian system with a Hamiltonian $\mathcal{H}^{\mathrm{res}}$ obtained via the resonant averaging of $\mathcal{H}(v)$; see § 3.2. As we said above, an alternative way to derive the effective equation is through the interaction representation, i.e. by transition from the v-variables to the a-variables (1.16). In view of (1.6), the a-variables

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satisfy the system of equations

$$\begin{aligned}\dot a_{\boldsymbol k} = -\gamma_{\boldsymbol k} a_{\boldsymbol k} + \mathrm{e}^{\mathrm{i}\nu^{-1}\lambda_{\boldsymbol k}\tau} b_{\boldsymbol k}\dot\beta^{\boldsymbol k}(\tau)\\ - \rho\mathrm{i}\sum_{\boldsymbol k_1,\dots,\boldsymbol k_{2q_*+1}\in\mathbb Z^d_L} a_{\boldsymbol k_1}\cdots a_{\boldsymbol k_{q_*+1}}\bar a_{\boldsymbol k_{q_*+2}}\cdots\bar a_{\boldsymbol k_{2q_*+1}}\delta^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol k}\\ \times\exp(-\mathrm{i}\nu^{-1}\tau(\lambda_{\boldsymbol k_1}+\cdots+\lambda_{\boldsymbol k_{q_*+1}}-\lambda_{\boldsymbol k_{q_*+2}}\\ -\cdots-\lambda_{\boldsymbol k_{2q_*+1}}-\lambda_{\boldsymbol k})),\end{aligned}\tag{1.17}$$

for $\boldsymbol k\in\mathbb Z^d_L$. The terms constituting the nonlinearity oscillate rapidly as ν goes to zero unless the sum of the eigenvalues in the exponent in the final two lines vanishes. The processes $\{\mathrm{e}^{\mathrm{i}\nu^{-1}\lambda_{\boldsymbol k}\tau}\dot\beta^{\boldsymbol k}(\tau),\ \boldsymbol k\in\mathbb Z^d_L\}$ make another set of standard independent complex white noises. This leads to the right guess that only the terms for which this sum equals zero (i.e. the resonant terms) contribute to the limiting dynamics, and that the effective equation is the following damped–driven Hamiltonian system:

$$\dot v_{\boldsymbol k} = -\gamma_{\boldsymbol k} v_{\boldsymbol k} + 2\rho\mathrm{i}\frac{\partial\mathcal H^{\mathrm{res}}(v)}{\partial\bar v_{\boldsymbol k}} + b_{\boldsymbol k}\dot\beta^{\boldsymbol k}(\tau),\quad \boldsymbol k\in\mathbb Z^d_L.\tag{1.18}$$

Here the Hamiltonian $\mathcal H^{\mathrm{res}}(v)$ is given by the sum

$$-\frac{1}{2q_*+2}\sum_{\boldsymbol k_1,\dots,\boldsymbol k_{2q_*+2}\in\mathbb Z^d_L} v_{\boldsymbol k_1}\cdots v_{\boldsymbol k_{q_*+1}}\bar v_{\boldsymbol k_{q_*+2}}\cdots\bar v_{\boldsymbol k_{2q_*+2}}\delta^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+2}\delta(\lambda^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+2}),\tag{1.19}$$

so that $2\rho\mathrm{i}(\partial\mathcal H^{\mathrm{res}}/\partial\bar v_{\boldsymbol k})(v)$ is

$$-\rho\mathrm{i}\sum_{\boldsymbol k_1,\dots,\boldsymbol k_{2q_*+2}\in\mathbb Z^d_L} v_{\boldsymbol k_1}\cdots v_{\boldsymbol k_{q_*+1}}\bar v_{\boldsymbol k_{q_*+2}}\cdots\bar v_{\boldsymbol k_{2q_*+1}}\delta^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol k}\delta(\lambda^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol k}),\tag{1.20}$$

where we use another abbreviation commonly used in physics

$$\delta(\lambda^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+2}) = \begin{cases}1 & \text{if } \lambda_{\boldsymbol k_1}+\cdots+\lambda_{\boldsymbol k_{q_*+1}}-\lambda_{\boldsymbol k_{q_*+2}}-\cdots-\lambda_{\boldsymbol k_{2q_*+2}}=0,\\ 0 & \text{otherwise.}\end{cases}\tag{1.21}$$

This representation for $\mathcal H^{\mathrm{res}}$ is different from that given by the resonant averaging. Its advantage is the natural relation with the a-variables, with which it is convenient to study the limit $\nu\to 0$. The representation for $\mathcal H^{\mathrm{res}}$ by means of the resonant averaging turns out to be more useful in the study of properties of $\mathcal H^{\mathrm{res}}$ and of the corresponding Hamiltonian vector field.

We saw that the effective equation can be obtained from system (1.6) by a simple procedure: drop the fast rotations and replace the Hamiltonian $\mathcal H$ by its resonant average $\mathcal H^{\mathrm{res}}$. In contrast with the non-resonant case, this is a nonlinear system. The corresponding Hamiltonian system

$$\dot v_{\boldsymbol k} = 2\rho\mathrm{i}\frac{\partial\mathcal H^{\mathrm{res}}(v)}{\partial\bar v_{\boldsymbol k}},\quad \boldsymbol k\in\mathbb Z^d_L,\tag{1.22}$$

has a vector field, locally Lipschitz in sufficiently smooth spaces, so (1.18) is well posed locally in time. In fact, it is globally well posed. We get this result in § 5.1 as a simple consequence of our main theorems.

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The Hamiltonian $\mathcal{H}^{\rm res}$ has two convex quadratic integrals

$$H_0(v) = \frac{1}{2}\sum |v_{\boldsymbol{k}}|^2, \qquad H_1 = \frac{1}{2}\sum \lambda_{\boldsymbol{k}}|v_{\boldsymbol{k}}|^2,$$

which are similar to the energy and the enstrophy integrals for the two-dimensional Euler equation on $\mathbb{T}^2$ (see (3.33)), and the vector integral of moments $M(u) = \frac{1}{2}\sum \boldsymbol{k}|u_{\boldsymbol{k}}|^2 \in \mathbb{R}^d$, which can be compared with the extra integrals of the two-dimensional Euler equation. In addition, the vector field (1.20) is nonlinear homogeneous and Hamiltonian, as is that of the Euler equation. This makes the effective equation (1.18) similar to the two-dimensional Navier–Stokes system on $\mathbb{T}^2$. Fortunately the former is significantly simpler than the latter.

1.4. Results

The main results of our work are stated and proved in § 5, and are based on properties of the effective equation, established earlier. They imply that the long-time behaviour of solutions for (1.5), when $\nu \to 0$, is controlled in distribution by solutions for the effective equation. We start with the results on the Cauchy problem. So, let $v^\nu(\tau)$ be a solution of (1.6) such that $v^\nu(0) = v_0$, where $v_0 = (v_{0\boldsymbol{k}},\ \boldsymbol{k} \in \mathbb{Z}^d_L)$ corresponds to a sufficiently smooth function $u_0(x)$. Let us fix any $T > 0$.

Consider the vector of actions $I(v^\nu(\tau)) = \{I_{\boldsymbol{k}}(v^\nu(\tau)),\ \boldsymbol{k} \in \mathbb{Z}^d_L\}$.

Theorem A. *When $\nu \to 0$, we have the weak convergence of measures*

$$\mathcal{D}(I(v^\nu(\tau))) \rightharpoonup \mathcal{D}(I(v^0(\tau))), \tag{1.23}$$

where $v^0(\tau)$, $0 \leqslant \tau \leqslant T$, is a unique solution of (1.18) such that $v^0(0) = v_0$.

For any $\xi \in \mathbb{Z}^\infty_0$, where $\mathbb{Z}^\infty_0$ is the set of integer vectors $(\xi_{\boldsymbol{k}},\ \boldsymbol{k} \in \mathbb{Z}^d_L)$ of finite length, we define

$$\Phi^\xi(v^\nu(\tau)) := \sum \xi_{\boldsymbol{k}}\varphi_{\boldsymbol{k}}(v^\nu(\tau)) \in S^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

Then, in addition to (1.23), for a resonant vector $\xi \in \mathbb{Z}^\infty_0$ (i.e. for a vector ξ such that $\sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}}\lambda_{\boldsymbol{k}} = 0$) the distribution of $\Phi^\xi(v^\nu(\tau))$, mollified in τ, converges as $\nu \to 0$ to that of $\Phi^\xi(v^0(\tau))$. Conversely, if ξ is non-resonant, then the measure $\mathcal{D}(\Phi^\xi(v^\nu(\tau)))$, mollified in τ, converges to the Lebesgue measure on S^1. All of this is proved in § 5.1 using the interaction representation (1.17) for (1.3).

The limiting behaviour of solutions $v^\nu(\tau)$ can be described without evoking the effective equation (see proposition 5.5).

Now consider a stationary measure μ^ν for (1.5) (it always exists). We have the following theorem.

Theorem B. *Every sequence $\nu'_j \to 0$ has a subsequence $\nu_j \to 0$ such that*

$$I \circ \mu^{\nu_j} \rightharpoonup I \circ m^0, \qquad \Phi^{(\xi)} \circ \mu^{\nu_j} \rightharpoonup \Phi^{(\xi)} \circ m^0$$

for any resonant vector $\xi \in \mathbb{Z}^\infty_0$, where m^0 is a stationary measure for (1.18). If a vector ξ is non-resonant, then the measure $\Phi^\xi \circ \mu^\nu$ converges, as $\nu \to 0$, to the Lebesgue measure on S^1.

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If the effective equation has a unique stationary measure m^0, then the limits in theorem 1.4 do not depend on the sequence $\nu_j \to 0$, so the convergences hold as $\nu \to 0$. Remarkably, in this case the measure m^0 controls not only the slow, but also the fast components of the measures μ^ν.

THEOREM C. *If the effective equation has a unique stationary measure m^0, then $\mu^\nu \rightharpoonup m^0$ as $\nu \to 0$.*

In particular, if the effective equation has a unique stationary measure m^0 and (1.3) is mixing,[3] then m^0 describes asymptotic behaviour of distributions of solutions $u(t)$ for (1.3) as $t \to \infty$ and $\nu \to 0$:

$$\lim_{\nu\to 0}\lim_{t\to\infty} \mathcal{D}(u(t)) = m^0.$$

In view of the last theorem, it is important to understand when the effective equation has a unique stationary measure and is mixing. This is discussed in § 5.3. In particular, the mixing holds if $q_* = 1$, $f(t) = t + 1$ and $d \leqslant 3$.

Other equations. Our approach applies to other equations usually considered in WT theory. In particular, in [15] we applied it to the two-dimensional quasi-geostrophic equation on the β-plane with random force:

$$(-\Delta + K)\psi - \rho J(\psi, \Delta\psi) - \beta\psi_x = \langle \text{random force}\rangle - \kappa\Delta^2\psi + \Delta\psi. \tag{1.24}$$

Here ψ is the stream function, $\psi = \psi(t, x, y)$, where $x \in \mathbb{R}/L\mathbb{Z}$ and $y \in \mathbb{R}/\mathbb{Z}$, $\Delta\psi$ is the Ekman damping, $-\kappa\Delta^2\psi$ is the kinematic viscosity and the random force is similar to that in (1.3) and (1.5). The equation has the same structure as (1.5), and our approach is to prove that for typical values of the horizontal period L (when the structure of resonances is relatively simple) the limiting, as $\beta \to \infty$, behaviour of solutions for (1.24) exists, is uniform in $\kappa \in (0, 1]$, and is described by an effective equation that is an infinite system of stochastic equations. This system splits into invariant systems of complex dimension less than or equal to 3; each of them is an integrable Hamiltonian system, coupled with a Langevin thermostat. Under the iterated limits $\lim_{L=\rho\to\infty}\lim_{\beta\to\infty}$ and $\lim_{\kappa\to 0}\lim_{\beta\to\infty}$ we get similar systems. In particular, none of the three limiting systems exhibits the energy cascade to high frequencies.

1.5. Weak turbulence

The most famous prediction of WT theory (see § 1.2) deals with the distribution of the energy of solutions for (1.1) and (1.3) between the frequencies. To describe the corresponding claims, consider the quantity $\mathbb{E}|v_{\boldsymbol{k}}(\tau)|^2$, average it in time[4] τ and in wave vectors $\boldsymbol{k} \in \mathbb{Z}_L^d$ such that $|\boldsymbol{k}| \approx r > 0$; next, properly scale this and denote the result E_r. The function $r \to E_r$ is called the *energy spectrum*. It is predicted by WT theory that, in a certain *inertial range* $r \in [r_1, r_2]$, which is contained in the

[3] Both these conditions hold, for example, if $q_* = 1$ and $f(\lambda) = c_1 + \lambda^{c_d}$, where c_d is sufficiently big in terms of d.

[4] Certainly this is not needed if we consider stationary solutions of the equation.

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spectral zone where the random force is negligible (i.e. where $|b_{\boldsymbol{k}}| \lll (\mathbb{E}|v_{\boldsymbol{k}}|^2)^{1/2}$), the energy spectrum has an algebraic behaviour

$$E_r \sim r^{-\alpha} \quad \text{for } r \in [r_1, r_2], \tag{1.25}$$

for a suitable $\alpha > 0$. If the WT is stated in terms of the iterated limits (1.9), then only the limits that lead to the algebraic energy spectra (1.25) are relevant.

In our companion work [14] we study the effective equation for (1.5) under the limit $L \to \infty$, evoking the heuristic tools from WT theory as presented in [28] and mimicking the logic of that book. There we show (heuristically) that a suitable choice of the function $\rho(L)$ leads, in the limit of $L \to \infty$, to a wave kinetic equation for the averaged actions $n_{\boldsymbol{k}}(t) = \frac{1}{2}\mathbb{E}|v_{\boldsymbol{k}}(t)|^2$. That equation is different from the heuristic kinetic equations obtained by WT methods (see [3, § 1.2] and [28, § 2.2.3]), but is closely related to them so that the Zakharov ansatz applies and allows one to obtain stationary solutions of the equation, algebraic in $r = |\boldsymbol{k}|$ and corresponding to energy spectra of the desired form (1.25).[5]

The rigorous and heuristic results obtained in this work and in [14] encourage us to pursue our program to study the WT in the model, given by (1.3), which brings to WT theory the advantage of a rigorous foundation based on the recent results of stochastic calculus. We believe that some predictions of WT theory (including the fact that the limiting behaviour of the averaged actions $n_{\boldsymbol{k}}(t)$ is described by a certain wave kinetic equation that admits stationary solutions of the algebraical form (1.25)) may be obtained if not under the iterated limit 'first $\nu \to 0$, next $L \to \infty$', then under its suitable modification (for example, '$\nu \to 0$ and $L \to \infty$ in such a way that $\nu L \to 0$').

Notation and agreement. The *stochastic terminology* that we use agrees with [9]. All filtered probability spaces we work with satisfy the *usual condition* (see [9]).[6] Sometimes we neglect to mention that a certain relation holds almost surely (a.s.)

Spaces of integer vectors. We denote by $\mathbb{Z}_0^\infty$ the set of vectors in $\mathbb{Z}^\infty$ of finite length, and define $\mathbb{Z}_{+0}^\infty = \{s \in \mathbb{Z}_0^\infty : s_k \geqslant 0 \ \forall k\}$. See also (2.15) and (3.9).

Infinite vectors. For an infinite vector $\xi = (\xi_1, \xi_2, \dots)$ (integer, real or complex) and $N \in \mathbb{N}$ we denote by ξ^N the vector $(\xi_1, \dots, \xi_N)$ or the vector $(\xi_1, \dots, \xi_N, 0, \dots)$, depending on the context. For a complex vector ξ and $s \in \mathbb{Z}_{+0}^\infty$ we define $\xi^s = \prod_j \xi_j^{s_j}$.

Norms. We use $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^d$ and in $\mathbb{C} \simeq \mathbb{R}^2$, as well as the ℓ_1-norm in $\mathbb{Z}_0^\infty$. For the norms $|\cdot|_{h^m}$ and $|\cdot|_{h_I^m}$ see (2.13) onwards.

[5] The work [3] mainly treats systems in which the sources and sinks of energy are well separated, so that there is a large spectral region where the forcing and the dissipation are practically absent and where the kinetic equation does not depend explicitly on the form of the forcing and dissipation. For this reason, the stationary spectra, found in [3] for (1.3), are equal to those obtained by WT methods for the HPDE (1.1). This explains the difference between the kinetic equation in [3] and that in [14] (which deals with the case in which the dissipation is present on the whole spectral range). Contrary to [3], the method of [27] applies, at least formally, to all kinds of forcing and damping, so the kinetic equation of that work may depend on the forcing and dissipation.

[6] That is, the corresponding filtrations $\{\mathcal{F}_t\}$ are continuous from the right, and each $\mathcal{F}_t$ contains all negligible sets.

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Scalar products. The notation '$\cdot$' stands for the scalar product in $\mathbb{Z}_0^\infty$, the pairing of $\mathbb{Z}_0^\infty$ with $\mathbb{Z}^\infty$, the Euclidean scalar product in $\mathbb{R}^d$ and in $\mathbb{C}$. The latter means that if $u, v \in \mathbb{C}$, then $u \cdot v = \mathrm{Re}(\bar{u}v)$. The L_2-product is denoted by $\langle \cdot, \cdot \rangle$, and we also denote by $\langle f, \mu \rangle = \langle \mu, f \rangle$ the integral of a function f against a measure μ.

Maximum and minimum. We define $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

2. Preliminaries

Since in this work we are not interested in the dependence of the results on L, from now on it will be kept fixed and equal to 1, apart from in § 4. There we make explicit calculations, controlling how the results depend on L.

2.1. *A priori* estimates

In this section we discuss preliminary properties of solutions for (1.5). We found it convenient to parametrize the vectors from the trigonometric basis $\{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x}\}$ with natural numbers and to normalize them. That is, to use the basis $\{e^j(x),\ j \geqslant 1\}$, where

$$e^j(x) = (2\pi)^{-d/2}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x}, \quad \boldsymbol{k} = \boldsymbol{k}(j). \tag{2.1}$$

The functions $e^j(x)$ are eigenfunctions of the Laplacian, $-\Delta e^j = \lambda_j e^j$, so ordered that $0 = \lambda_1 < \lambda_2 \leqslant \cdots$. Accordingly, (1.5) reads

$$\dot{u} + \mathrm{i}\nu^{-1}(-\Delta u) = -f(-\Delta)u - \mathrm{i}\rho|u|^{2q_*}u + \frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{j=1}^{\infty} b_j \boldsymbol{\beta}^j(\tau)e^j(x), \tag{2.2}$$

$u = u(\tau, x)$, where $f(-\Delta)e^j = \gamma_j e^j$ with $\gamma_j = f(\lambda_j)$. The processes $\boldsymbol{\beta}^j = \beta^j + \mathrm{i}\beta^{-j}$, $j \geqslant 1$, are standard independent complex Wiener processes. The real numbers b_j are such that for a suitable sufficiently large even integer r (defined in (3.13)) we have

$$B_r := 2\sum_{j=1}^{\infty} \lambda_j^r b_j^2 < \infty.$$

By $\mathcal{H}^p$, $p \in \mathbb{R}$, we denote the Sobolev space $\mathcal{H}^p = H^p(\mathbb{T}^d, \mathbb{C})$, regarded as a real Hilbert space, and by $\langle \cdot, \cdot \rangle$ we denote the real L^2-scalar product on $\mathbb{T}^d$. We provide $\mathcal{H}^p$ with the norm $\|\cdot\|_p$,

$$\|u\|_p^2 = \sum_{j=1}^{\infty} |u_j|^2(\lambda_j \vee 1)^p \quad \text{for } u(x) = \sum_{j=1}^{\infty} u_j e^j(x).$$

Let $u(t, x)$ be a solution of (2.2) such that $u(0, x) = u_0$; it satisfies standard *a priori* estimates that we now discuss, following [13]. Firstly, for a suitable $\varepsilon_0 > 0$, uniformly in $\nu > 0$ one has

$$\mathbb{E}\mathrm{e}^{\varepsilon_0\|u(\tau)\|_0^2} \leqslant C(B_0, \|u_0\|_0) \quad \forall \tau \geqslant 0. \tag{2.3}$$

Assume that

$$q_* < \begin{cases} \infty & \text{if } d = 1, 2, \\ \dfrac{2}{d-2} & \text{if } d \geqslant 3. \end{cases} \tag{2.4}$$

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Then the following bounds on the Sobolev norms of the solution hold for each $2m \leqslant r$ and every n:

$$\mathbb{E}\bigg(\sup_{0\leqslant\tau\leqslant T}\|u(\tau)\|_{2m}^{2n}+\int_0^T\|u(s)\|_{2m+1}^2\|u(s)\|_{2m}^{2n-2}\,\mathrm{d}s\bigg)$$
$$\leqslant \|u_0\|_{2m}^{2n}+C(m,n,T)(1+\|u_0\|_0^{c_{m,n}}), \tag{2.5}$$

$$\mathbb{E}\|u(\tau)\|_{2m}^{2n}\leqslant C(m,n)\quad\forall\tau\geqslant 0, \tag{2.6}$$

where $C(m,n,T)$ and $C(m,n)$ also depend on B_{2m}.

Estimates (2.5), (2.6) are assumed everywhere in our work. As we have explained, they are fulfilled under assumption (2.4), but if the function $f(t)$ grows superlinearly, then restriction (2.4) may be weakened.

Relations (2.5) imply in the usual way (see [8, 18, 22, 24]) that (2.2) is *regular in the space* $\mathcal{H}^r$ in the sense that for any $u_0\in\mathcal{H}^r$ it has a unique strong solution $u(t,x)$, equal to u_0 at $t=0$, and satisfying estimates (2.3), (2.5) for any n. By the Bogolyubov–Krylov argument applied to a solution of (2.2), starting from the origin at $t=0$, this equation has a stationary measure μ^ν, supported by the space $\mathcal{H}^r$, and a corresponding stationary solution $u^\nu(\tau)$, $\mathcal{D}u^\nu(\tau)\equiv\mu^\nu$, also satisfies (2.3) and (2.6).

2.2. Resonant averaging

Let $W\in\mathbb{Z}^n$, $n\geqslant 1$, be a non-zero integer vector such that its components are relatively prime (so if $W=mV$, where $m\in\mathbb{Z}$ and $V\in\mathbb{Z}^n$, then $m=\pm1$). We call the set

$$\mathcal{A}=\mathcal{A}(W):=\{s\in\mathbb{Z}^n\colon W\cdot s=0\} \tag{2.7}$$

the *set of resonances for* W. This is a $\mathbb{Z}$-module. Denote its rank by r. Here and everywhere below the finite-dimensional vectors are regarded as column vectors.

LEMMA 2.1. *The rank r equals $n-1$. There exists a system $\zeta^1,\dots,\zeta^n$ of integer vectors in $\mathbb{Z}^n$ such that* $\operatorname{span}_{\mathbb{Z}}\{\zeta^1,\dots,\zeta^{n-1}\}=\mathcal{A}$, *and the $n\times n$ matrix $R=(\zeta^1\zeta^2\cdots\zeta^n)$ is unimodular* (*i.e.* $\det R=\pm1$).

That is to say, the vectors $(\zeta^1,\dots,\zeta^{n-1})$ make an integer basis of the hyperspace $W^\perp\subset\mathbb{R}^n$.

Proof. We restrict ourselves to the case in which some component of the vector W equals 1 since this is the result we need below. For the general case and for a more general statement, see, for example, [2, § 7].

Without loss of generality we assume that $W_n=1$. Consider the matrix such that its nth column is W and for $j<n$ the jth column is the vector $e^j=(e_1^j,\dots,e_n^j)^{\mathrm{T}}$, where $e_l^j=\delta_{j,l}$. It is unimodular and transforms the basis vector e^n to W. Its inverse is a unimodular matrix B such that $BW=e^n$. Let s be any vector in $\mathcal{A}$. Since

$$W\cdot s=0\iff BW\cdot(B^{\mathrm{T}})^{-1}s=0\iff e^n\cdot(B^{\mathrm{T}})^{-1}s=0,$$

we have

$$(B^{\mathrm{T}})^{-1}s=\sum_{j=1}^{n-1}m_je^j,$$

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where the m_js are some integers. This proves the lemma if we choose $\zeta^j = B^{\mathrm{T}} e^j$, $j = 1, \dots, n$. Note that the matrix R equals B^{T}. □

Since $R^{\mathrm{T}} W = BW = e^n$, the automorphism of the torus $\mathbb{T}^n \to \mathbb{T}^n$, $\varphi \to y = R^{\mathrm{T}}\varphi$, 'resolves the resonances' in the differential equation $\dot{\varphi} = W$ in the sense that it transforms it into the equation

$$\dot{y} = R^{\mathrm{T}} W = (0, \dots, 0, 1)^{\mathrm{T}}. \tag{2.8}$$

Let us consider a mapping $L = L_{\mathcal{A}} \colon \mathbb{T}^n \to \mathbb{T}^{n-1}$ that is 'dual to the module $\mathcal{A}$':

$$L \colon \mathbb{T}^n \ni \varphi \to (\varphi \cdot Re^1, \dots, \varphi \cdot Re^{n-1})^{\mathrm{T}} \in \mathbb{T}^{n-1}. \tag{2.9}$$

The basis $\{\eta^j = (R^{\mathrm{T}})^{-1} e^j,\ 1 \leqslant j \leqslant n\}$ is dual to the basis $\{\zeta^j = Re^j,\ 1 \leqslant j \leqslant n\}$, since

$$\eta^j \cdot \zeta^l = (R^{\mathrm{T}})^{-1} e^j \cdot Re^l = \delta_{j,l}.$$

Therefore, if we decompose $\varphi \in \mathbb{T}^n$ in the η-basis, $\varphi = \sum_k y_k \eta^k = (R^{\mathrm{T}})^{-1} y$, then $L\varphi = (y_1, \dots, y_{n-1})^{\mathrm{T}}$. That is,

$$L \circ (R^{\mathrm{T}})^{-1} (y_1, \dots, y_n)^{\mathrm{T}} = (y_1, \dots, y_{n-1})^{\mathrm{T}}. \tag{2.10}$$

In particular, the fibres of the mapping L are the circles $R(\{\boldsymbol{y}\} \times S^1)$, where $\boldsymbol{y} = (y_1, \dots, y_{n-1})^{\mathrm{T}} \in \mathbb{T}^{n-1}$.

For a continuous function f on $\mathbb{T}^n$ we define its *resonant average with respect to the integer vector* W as the function

$$\langle f \rangle_W(\varphi) := \int_0^{2\pi} f(\varphi + tW)\, đt, \tag{2.11}$$

where we have set $đt := \mathrm{d}t/2\pi$.

Lemma 2.2. *Let f be a C^∞-function on $\mathbb{T}^n$, $f(\varphi) = \sum f_s \mathrm{e}^{\mathrm{i}s\cdot\varphi}$. Then*

$$\langle f \rangle_W(\varphi) = \sum f_s \delta_{0, s\cdot W} \mathrm{e}^{\mathrm{i}s\cdot\varphi} = \sum_{s \in \mathcal{A}(W)} f_s \mathrm{e}^{\mathrm{i}s\cdot\varphi}. \tag{2.12}$$

Proof. It is immediate that (2.12) holds for a trigonometric polynomial. Since for C^∞-functions the series in (2.12) converges well, by continuity the result holds for smooth functions f. □

2.3. Resonant averaging in a Hilbert space

Consider the Fourier transform for complex functions on $\mathbb{T}^d$, which we write as the mapping

$$\mathcal{F} \colon \mathcal{H} \ni u(x) \mapsto v = (v_1, v_2, \dots) \in \mathbb{C}^\infty,$$

defined by the relation $u(x) = \sum v_k e^k(x)$. In the space of complex sequences we introduce the norms

$$|v|^2_{h^p} = \sum_{k \geqslant 1} |v_k|^2 (\lambda_k \vee 1)^p, \quad p \in \mathbb{R}, \tag{2.13}$$

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and set $h^p = \{v \mid |v|_{h^p} < \infty\}$. Then

$$|\mathcal{F}u|_{h^p} = \|u\|_p \quad \forall p.$$

For $k \geqslant 1$ let us define $I_k = I(v_k) = \frac{1}{2}|v_k|^2$ and $\varphi_k = \varphi(v_k)$, where, for $v \in \mathbb{C}$, $\varphi(v) = \operatorname{Arg} v \in S^1$ if $v \neq 0$, and $\varphi(0) = 0 \in S^1$. For any $r \geqslant 0$ consider the mappings

$$\Pi_I \colon h^r \ni v \mapsto I = (I_1, I_2, \dots) \in h^r_{I+}, \qquad \Pi_\varphi \colon h^r \ni v \mapsto \varphi = (\varphi_1, \varphi_2, \dots) \in \mathbb{T}^\infty. \tag{2.14}$$

Here h^r_{I+} is the positive octant $\{I \colon I_k \geqslant 0 \ \forall k\}$ in the space h^r_I, where

$$h^r_I = \left\{ I \;\middle|\; |I|_{h^r_I} = 2\sum_k (\lambda_k \vee 1)^r |I_k| < \infty \right\}.$$

Abusing notation slightly, we will write $\Pi_I(\mathcal{F}(u)) = I(u)$, $\Pi_\varphi(\mathcal{F}(u)) = \varphi(u)$. The mapping $I \colon \mathcal{H}^r \to h^r_I$ is 2-homogeneous continuous, while the mapping $\varphi \colon \mathcal{H}^r \to \mathbb{T}^\infty$ is Borel measurable (the torus $\mathbb{T}^\infty$ is given the Tikhonov topology and the corresponding Borel σ-algebra).

For infinite integer vectors $s = (s_1, s_2, \dots)$ (and only for them) we will write the l_1-norm of s as $|s|$,

$$|s| = \sum_j |s_j|.$$

We define $\mathbb{Z}^\infty_0 = \{s \in \mathbb{Z}^\infty \colon |s| < \infty\}$, and for a vector $s = (s_1, s_2, \dots) \in \mathbb{Z}^\infty_0$ write

$$\Lambda \cdot s = \sum_k \lambda_k s_k, \qquad \operatorname{supp} s = \{k \colon s_k \neq 0\}, \qquad \lceil s \rceil = \max\{k \colon s_k \neq 0\}. \tag{2.15}$$

Similarly, for $\varphi \in \mathbb{T}^\infty$ and $s \in \mathbb{Z}^\infty_0$ we write $\varphi \cdot s = s \cdot \varphi = \sum_k \varphi_k s_k \in S^1$.

Let us fix some $m \in \mathbb{N} \cup \infty$ and define the set of resonances of order m for the (integer) frequency vector $\Lambda = (\lambda_1, \lambda_2, \dots)$ as

$$\mathcal{A}(\Lambda, m) = \{s \in \mathbb{Z}^\infty_0 \colon |s| \leqslant m, \ \Lambda \cdot s = 0\}. \tag{2.16}$$

We will abbreviate $\mathcal{A}(\Lambda) = \mathcal{A}(\Lambda, \infty) = \{s \in \mathbb{Z}^\infty_0 \colon \Lambda \cdot s = 0\}$.

Let us define $\mathbb{Z}^\infty_{+0} = \{s \in \mathbb{Z}^\infty_0 \colon s_k \geqslant 0 \ \forall k\}$, and consider a series on some space h^r, $r \geqslant 0$,

$$F(v) = \sum_{p,q,l \in \mathbb{Z}^\infty_{+0}} C_{pql} (2I)^p v^q \bar{v}^l, \tag{2.17}$$

where $I = I(v)$, $C_{pql} = 0$ if $\operatorname{supp} q \cap \operatorname{supp} l \neq \emptyset$, and for $v \in h^r$, $q \in \mathbb{Z}^\infty_{+0}$ we write $v^q = \prod v_j^{q_j}$. We assume that the series converges normally in h^r in the sense that for each $R > 0$ we have

$$\sum_{p,q,l \in \mathbb{Z}^\infty_{+0}} |C_{pql}| \sup_{|v|_{h^r}, |w|_{h^r} \leqslant R} |v^p w^p v^q w^l| < \infty. \tag{2.18}$$

Clearly, $F(v) = \boldsymbol{F}(v, \bar{v})$, where $\boldsymbol{F}$ is a (complex) analytic function on $h^r \times h^r$. With some abuse of language and following a tradition in physics we will say that F *is analytic in* v *and* $\bar{v}$. In particular, $F(v)$ is a real-analytic (so continuous) function of v, and the series (2.17) converges absolutely.

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The resonant averaging of F can be conveniently defined by introducing, for any $\theta \in \mathbb{T}^\infty$, the rotation operator Ψ_θ, which is a linear operator in h^0:

$$\Psi_\theta(v) = v', \quad v'_k = \mathrm{e}^{\mathrm{i}\theta_k} v_k.$$

Clearly, this is a unitary isomorphism of every space h^r. Note that $(I \times \varphi)(\Psi_\theta v) \equiv (I(v), \varphi(v) + \theta)$. Using that Λ is an integer vector and based on definition (2.11), we give the following.

Definition. If a function $F \in C(h^r)$ is given by a normally converging series (2.17), then its resonant average with respect to Λ is the function

$$\langle F \rangle_\Lambda(v) := \int_0^{2\pi} F(\Psi_{t\Lambda}(v))\,đt, \quad đt = \mathrm{d}t/2\pi. \tag{2.19}$$

Defining a function $\tilde{F}(I, \varphi)$ by the relation $F(v) = \tilde{F}(I(v), \varphi(v))$, we see that $\langle F \rangle_\Lambda(v) = \int_0^{2\pi} \tilde{F}(I, \varphi + t\Lambda)\,đt$. So this definition agrees well with (2.11).

Consider a monomial $F = (2I)^p v^q \bar{v}^l$. By lemma 2.2 we have

$$\langle (2I)^p v^q \bar{v}^l \rangle_\Lambda = (2I)^p v^q \bar{v}^l \delta_{0,(q-l)\cdot\Lambda}.$$

Now assume that F is given by a normally convergent series (2.17) and has degree $\leqslant m \leqslant \infty$ in the sense that $C_{pql} = 0$ unless $|q| + |l| \leqslant m$. Then

$$\langle F \rangle_\Lambda(v) = \sum_{q-l \in \mathcal{A}(\Lambda, m)} C_{pql}(2I)^p v^q \bar{v}^l = \sum_{(q-l)\cdot\Lambda = 0} C_{pql}(2I)^p v^q \bar{v}^l. \tag{2.20}$$

If the series (2.17) converges normally, then the series on the right-hand side also does. It defines an analytic-in-$(v, \bar{v})$ function. Note that, in view of (2.20),

$$\langle F \rangle_\Lambda \text{ is a function of } I_1,\, I_2,\, \ldots \text{ and the variables } \{s \cdot \varphi,\ s \in \mathcal{A}(\Lambda, m)\}. \tag{2.21}$$

3. Averaging for (2.2)

Everywhere below T is a fixed positive number.

3.1. Equation (2.2) in v-variables, resonant monomials and combinations of phases

Let us pass in (2.2), with $u \in \mathcal{H}^r$, $r > d/2$, to the v-variables, $v = \mathcal{F}(u) \in h^r$:

$$\mathrm{d}v_k + \mathrm{i}\nu^{-1}\lambda_k v_k\,\mathrm{d}\tau = P_k(v)\,\mathrm{d}\tau + b_k\,\mathrm{d}\boldsymbol{\beta}^k(\tau), \quad k \geqslant 1,\ v(0) = \mathcal{F}(u_0) =: v_0. \tag{3.1}$$

Here

$$P_k = P_k^1 + P_k^0, \tag{3.2}$$

where P^1 and P^0 are, correspondingly, the linear and nonlinear Hamiltonian parts of the perturbation. So P_k^1 is the Fourier image of $-f(-\Delta)$, i.e. $P_k^1 = \mathrm{diag}\{-\gamma_k,\ k \geqslant 1\}$, while the operator P^0 is the mapping $u \mapsto -\mathrm{i}\rho|u|^{2q_*}u$, written in the v-variables. That is,

$$P^0(v) = -\mathrm{i}\rho\mathcal{F}(|u|^{2q_*}u), \quad u = \mathcal{F}^{-1}(v).$$

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Every component P_k^0 of it is a sum of monomials

$$P_k^0(v) = \sum_{p,q,l\in\mathbb{Z}_{+0}^\infty} C_k^{pql}(2I)^p v^q \bar{v}^l = \sum_{p,q,l\in\mathbb{Z}_{+0}^\infty} P_k^{0pql}(v), \quad k \geqslant 1, \tag{3.3}$$

where $C_k^{pql} = 0$ unless $2|p|+|q|+|l| = 2q_*+1$ and $|q| = |l|+1$. It is straightforward that $P_k^0(I,\varphi)$ (see (2.14)) is a function of $\varphi = (\varphi_j,\ j \geqslant 1)$ of order $2q_*+1$, and that the mapping P^0 is analytic and of polynomial growth.

Lemma 3.1. *The nonlinearity P^0 defines a real-analytic transformation of h^r if $r > d/2$. The mapping $P^0(v)$ and its differential $\mathrm{d}P^0(v)$ both have polynomial growth in $|v|_{h^r}$.*

We will refer to (3.1) as the *v-equations*.

For any $s \in \mathbb{Z}_0^\infty$ consider the linear combination of phases

$$\Phi^s\colon h^0 \to S^1, \quad v \mapsto s\cdot\varphi(v).$$

We fix

$$m = 2q_*+2$$

and find the corresponding set $\mathcal{A} = \mathcal{A}(\Lambda, m)$ of resonances of order m (see (2.16)). We order vectors in the set $\mathcal{A}$, that is, we write it as $\mathcal{A} = \{s^{(1)}, s^{(2)}, \dots\}$ in such a way that $\lceil s^{(j_1)}\rceil \leqslant \lceil s^{(j_2)}\rceil$ if $j_1 \leqslant j_2$, and for $N \geqslant 1$ define

$$J(N) = \max\{j\colon \lceil s^{(j)}\rceil \leqslant N\}. \tag{3.4}$$

For any $s^{(j)} \in \mathcal{A}$ consider the corresponding resonant combination of phases $\varphi(v)$, $\Phi_j(v) = \Phi^{s_j}(v)$, and introduce the Borel measurable mappings

$$\begin{aligned} &h^r \ni v \mapsto \Phi = (\Phi_1, \Phi_2, \dots) \in S^1\times S^1\times\cdots =: \mathcal{T}^\infty,\\ &h^r \ni v \mapsto (I\times\Phi) \in h^r_{I+}\times\mathcal{T}^\infty. \end{aligned}$$

Note that the system Φ of resonant combinations is highly overdetermined: there are many linear relations between its components Φ_j.

Let us pass in (3.1) from the complex variables v_k to the action-angle variables I, φ

$$\mathrm{d}I_k(\tau) = (v_k\cdot P_k)(v)\,\mathrm{d}\tau + b_k^2\,\mathrm{d}\tau + b_k(v_k\cdot\mathrm{d}\boldsymbol{\beta}^k) \tag{3.5}$$

(here $\cdot$ indicates the real scalar product in $\mathbb{C}\simeq\mathbb{R}^2$), and

$$\mathrm{d}\varphi_k(\tau) = (-\nu^{-1}\lambda_k + |v_k|^{-2}(\mathrm{i}v_k\cdot P_k(v)))\,\mathrm{d}\tau + |v_k|^{-2}b_k(\mathrm{i}v_k\cdot\mathrm{d}\boldsymbol{\beta}^k). \tag{3.6}$$

The equations for the actions are slow, while equations for the angles are fast since $\mathrm{d}\varphi_k \sim \nu^{-1}$. But the resonant combinations Φ_j of angles satisfy slow equations:

$$\mathrm{d}\Phi_j(\tau) = \sum_{k\geqslant 1} s_k^{(j)}(|v_k|^{-2}(\mathrm{i}v_k\cdot P_k)\,\mathrm{d}\tau + |v_k|^{-2}b_k(\mathrm{i}v_k\cdot\mathrm{d}\boldsymbol{\beta}^k)), \quad j\geqslant 1. \tag{3.7}$$

Repeating for (3.1) and (3.5) the argument of [17, § 7] (see also [12, § 6.2]), we get lower bounds for the norms of the components $v_k(\tau)$ of $v(\tau)$.

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LEMMA 3.2. *Let $v^\nu(\tau)$ be a solution of (3.1) and let $I^\nu(\tau) = I(v^\nu(\tau))$. Then for any $k \geqslant 1$ the convergence*

$$\int_0^T \mathbb{P}\{I_k^\nu(\tau) \leqslant \delta\}\,\mathrm{d}\tau \to 0 \quad \text{as } \delta \to 0 \tag{3.8}$$

holds uniformly in $\nu > 0$ (the rate of the convergence depends on k).

Now we define and study corresponding resonant monomials of v. For any $s \in \mathbb{Z}_0^\infty$, vectors $s^+, s^- \in \mathbb{Z}_{+0}^\infty$ such that $s = s^+ - s^-$ and $\operatorname{supp} s = \operatorname{supp} s^+ \cup \operatorname{supp} s^-$, $\operatorname{supp} s^+ \cap \operatorname{supp} s^- = \emptyset$ are uniquely defined. Denote by V^s the monomial

$$V^s(v) = v^{s^+}\bar{v}^{s^-} = \prod_l v_l^{s_l^+} \prod_l \bar{v}_l^{s_l^-}. \tag{3.9}$$

This is a real-analytic function on every space h^l, and $\varphi(V^s(v)) = \Phi^s(v)$. *Resonant monomials* are the functions[7]

$$V_j(v) = V^{s^{(j)}}(v), \quad j = 1, 2, \dots.$$

Clearly, they satisfy

$$I(V_j(v)) = (2I)^{|s^{(j)}|/2} := \prod_l (2I_l)^{|s_l^{(j)}|/2}, \quad \varphi(V_j(v)) = \Phi_j(v). \tag{3.10}$$

Now consider the mapping

$$V\colon h^l \ni v \mapsto (V_1, V_2, \dots) \in \mathbb{C}^\infty, \tag{3.11}$$

where $\mathbb{C}^\infty$ is given the Tikhonov topology. It is continuous for any l. For $N \geqslant 1$ define

$$V^{(N)}(v) = (V_1, \dots, V_J(v)) \in \mathbb{C}^J,$$

where $J = J(N)$ (see (3.4)).

For any $s \in \mathbb{Z}_0^\infty$, applying the Itô formula to the process $V^s(v(\tau))$, we get that

$$\begin{aligned}\mathrm{d}V^s = V^s\Bigg(&-\mathrm{i}\nu^{-1}(\Lambda\cdot s)\,\mathrm{d}\tau + \sum_{j\in\operatorname{supp} s^+} s_j^+ v_j^{-1}(P_j(v)\,\mathrm{d}\tau + b_j\,\mathrm{d}\boldsymbol{\beta}_j)\\ &+ \sum_{j\in\operatorname{supp} s^-} s_j^- \bar{v}_j^{-1}(\bar{P}_j(v)\,\mathrm{d}\tau + b_j\,\mathrm{d}\bar{\boldsymbol{\beta}}_j)\Bigg).\end{aligned} \tag{3.12}$$

If $s = \tilde{s} \in \mathbb{Z}_0^\infty$ is perpendicular to Λ, then the first term on the right-hand side vanishes. So $V^{\tilde{s}}(\tau)$ is a slow process, $\mathrm{d}V^{\tilde{s}} \sim 1$. In particular, the processes $\mathrm{d}V_j$, $j \geqslant 1$, are slow.

Estimates (2.5) and equation (3.12) readily imply the following lemma.

LEMMA 3.3. *For any $j \geqslant 1$ we have $\mathbb{E}|V_j(v(\cdot))|_{C^{1/3}[0,T]} \leqslant C_j(T) < \infty$, uniformly in $0 < \nu \leqslant 1$.*

[7] It may be better to call $V_j(v)$ a minimal resonant monomial since for any $l \in \mathbb{Z}_{+0}^\infty$ the monomial $I^l V_j(v)$ is also resonant and corresponds to the same resonance.

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Let us provide the space $C([0,T];\mathbb{C}^\infty)$ with the Tikhonov topology, identifying it with the space $C([0,T];\mathbb{C})^\infty$. This topology is metrizable by way of the Tikhonov distance. From now on we fix an even integer r,

$$r \geqslant \frac{d}{2}+1, \tag{3.13}$$

and write

$$h^r = h, \qquad h_I^r = h_I, \qquad C([0,T],h_{I+}) \times C([0,T],\mathbb{C}^\infty) =: \mathcal{H}_{I,V}.$$

We provide $\mathcal{H}_{I,V}$ with Tikhonov's distance, the corresponding Borel σ-algebra and the natural filtration of the σ-algebras $\{\mathcal{F}_t,\ 0 \leqslant t \leqslant T\}$.

Let us consider a solution $u^\nu(\tau)$ of (2.2) satisfying $u(0)=u_0$. Define $v^\nu(\tau) = \mathcal{F}(u^\nu(\tau))$ and write

$$I(v^\nu(\tau)) = I^\nu(\tau), \qquad V(v^\nu(\tau)) = V^\nu(\tau) \in \mathbb{C}^\infty.$$

Lemma 3.4.

(1) *Assume that $u_0 \in \mathcal{H}^r$. Then the set of laws $\mathcal{D}(I^\nu(\cdot),V^\nu(\cdot))$, $0<\nu\leqslant 1$, is tight in $\mathcal{H}_{I,V}$.*

(2) *Any limiting measure $\mathcal{Q}$ for the set of laws in (1) satisfies*

$$\mathbb{E}^{\mathcal{Q}}|I|^n_{C([0,T],h_I^r)} \leqslant C_n \quad \forall n\in\mathbb{N}, \qquad \mathbb{E}^{\mathcal{Q}}\int_0^T |I(\tau)|_{h_I^{r+1}}\,\mathrm{d}\tau \leqslant C', \tag{3.14}$$

$$\mathbb{E}^{\mathcal{Q}}\mathrm{e}^{\varepsilon_0|I(\tau)|_{h_I^0}} \leqslant C'' \quad \forall \tau\in[0,T]. \tag{3.15}$$

Proof. (1) Due to lemma 3.3 and the Arzelà theorem, the laws of processes $V_j(v^\nu(\cdot))$, $0<\nu\leqslant 1$, are tight in $C([0,T],\mathbb{C})$ for all j. Due to (2.5) with $n=1$, and since the actions I_k^ν satisfy the slow equations (3.5), the laws of processes $I^\nu(\tau)$ are tight in $C([0,T],h_{I+})$ (see, for example, [25]). Therefore, for every N, any sequence $\nu_\ell \to 0$ contains a subsequence such that the laws $\mathcal{D}(I^\nu(\cdot),V^{(N)}(v^\nu(\cdot)))$ converge along it to a limit. Applying the diagonal process we get another subsequence ν'_ℓ such that the convergence holds for each N. The corresponding limit is a measure m^N on the space $C([0,T],h_{I+})\times C([0,T],\mathbb{C})^{J(N)}$. Different measures m^N agree, so by Kolmogorov's theorem they correspond to some measure m on the σ-algebra, generated by cylindric subsets of the space $C([0,T],h_{I+})\times C([0,T],\mathbb{C})^\infty$, which coincides with the Borel σ-algebra for that space. It is not hard to check that $\mathcal{D}(I^\nu(\cdot),V^\nu(\cdot)) \rightharpoonup m$ as $\nu=\nu'_\ell\to 0$. This proves the first assertion.

(2) Estimates (3.14) follow from (2.3), (2.5), the weak convergence to $\mathcal{Q}$ and the Fatou lemma; see [19, lemma 1.2.17]. □

3.2. Averaged equations, effective equation, interaction representation

Fix $u_0\in\mathcal{H}^r$ and consider any limiting measure $\mathcal{Q}^0$, which exists by lemma 3.4, for the laws

$$\mathcal{D}(I^{\nu_\ell}(\cdot),V^{\nu_\ell}(\cdot)) \rightharpoonup \mathcal{Q}^0 \quad \text{as } \nu_\ell\to 0. \tag{3.16}$$

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Our goal is to show that the limit $\mathcal{Q}^0$ does not depend on the sequence $\nu_\ell \to 0$ and to develop tools for its study. We begin with writing down averaged equations for the slow components I and Φ of the process $v(\tau)$, using the rules of stochastic calculus (see [6, 11]), and formally replacing there the usual averaging in φ by the resonant averaging $\langle\cdot\rangle_\Lambda$. Let us first consider the I-equations (3.5). The drift in the kth equation is

$$b_k^2 + v_k \cdot P_k = b_k^2 + v_k \cdot P_k^1 + v_k \cdot P_k^0,$$

where

$$v_k \cdot P_k^1 = -2\gamma_k I_k \quad \text{and} \quad v_k \cdot P_k^0(v) = \sum_{p,q,l \in \mathbb{Z}^\infty_{+0}} v_k \cdot P_k^{0pql}(v)$$

(see (3.3)). From § 4, the sum converges normally, so the resonant averaging of the drift is well defined. The dispersion matrix for (3.5) with respect to the real Wiener processes $(\beta^1, \beta^{-1}, \beta^2, \dots)$ is $\operatorname{diag}\{b_k(\operatorname{Re} v_k \operatorname{Im} v_k),\ k \geqslant 1\}$ (it is formed by (1×2)-blocks). The diffusion matrix equals the dispersion matrix times its conjugate and equals $\operatorname{diag}\{b_k^2 |v_k|^2,\ k \geqslant 1\}$. It is independent from the angles, so the averaging does not change it. For its square root we take $\operatorname{diag}\{b_k\sqrt{2I_k}\}$, and accordingly write the Λ-averaged I-equations as

$$\mathrm{d}I_k(\tau) = \langle v_k \cdot P_k\rangle_\Lambda(I, V)\,\mathrm{d}\tau + b_k^2\,\mathrm{d}\tau + b_k\sqrt{2I_k}\,\mathrm{d}\beta^k(\tau), \quad k \geqslant 1 \tag{3.17}$$

(see (2.21)).

Now consider (3.7) for resonant combinations Φ_j of the angles. The corresponding dispersion matrix $D = (D_{jk})$ is formed by (1×2)-blocks

$$D_{jk} = -s_k^{(j)} b_k (2I_k)^{-1}(\operatorname{Im} v_k - \operatorname{Re} v_k).$$

Again the diffusion matrix does not depend on the angles and equals $M = (M_{j_1 j_2})$,

$$M_{j_1 j_2} = \sum_k s_k^{(j_1)} s_k^{(j_2)} b_k^2 (2I_k)^{-1}.$$

The matrix D^{new} with the entries

$$D_{jk}^{\mathrm{new}} = s_k^{(j)} b_k (2I_k)^{-1/2}$$

satisfies $|D^{\mathrm{new}}|^2 = M$, and we write the averaged equations for the Φ_js as

$$\mathrm{d}\Phi_j(\tau) = \sum_{k \geqslant 1} s_k^{(j)} \left(\frac{\langle \mathrm{i} v_k \cdot P_k\rangle_\Lambda(I, V)}{2I_k}\,\mathrm{d}\tau + \frac{b_k}{\sqrt{2I_k}}\,\mathrm{d}\beta^{-k}(\tau) \right), \quad j \geqslant 1 \tag{3.18}$$

(we use here Wiener processes that are independent from those in (3.17) since the differentials $v_k \cdot \mathrm{d}\boldsymbol{\beta}^k$ and $\mathrm{i}v_k \cdot \mathrm{d}\boldsymbol{\beta}^k$, corresponding to the noises in (3.5) and (3.6), are independent).

Equations (3.17) and (3.18) constitute a system of stochastic differential equations for the process $(I, V)(\tau)$ since each Φ_j is a function of I and V_j. It is overdetermined since there are linear relations between various Φ_js. In addition, (3.17) has a weak singularity at the locus $\eth(h) = \bigcup_k \{v \in h\colon v_k = 0\}$, while (3.18) has a strong singularity there.

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Consider a component $\langle v_k \cdot P_k^0 \rangle_\Lambda(v)$ of the averaged drift in the equation for I_k. It may be written as

$$\langle v_k \cdot P_k^0 \rangle_\Lambda(v) = \int_0^{2\pi} v_k \cdot (\mathrm{e}^{-\mathrm{i}t\lambda_k} P_k^0(\Psi_{t\Lambda}(v)))\,\text{đ}t = v_k \cdot R_k^0(v), \tag{3.19}$$

where we set $R_k^0(v) = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}t\lambda_k} P_k^0(\Psi_{t\Lambda}(v))\,\text{đ}t$. That is,

$$R^0(v) = \int_0^{2\pi} \Psi_{-t\Lambda} P^0(\Psi_{t\Lambda} v)\,\text{đ}t. \tag{3.20}$$

Repeating the derivation of (2.20) and using that $|q| + |l| \leqslant m - 1$, we see that

$$R_k^0(v) = \sum_{\substack{p,q,l \in \mathbb{Z}_{+0}^\infty, \\ q-l \in \mathcal{A}(\Lambda,m)+e^k, \\ |q|+|l|+1 \leqslant m}} C_k^{pql} (2I)^p v^q \bar{v}^l. \tag{3.21}$$

Relation (3.21) interprets $R^0(v)$ as a sum of resonant terms of the mapping $P^0(v)$, very much in the spirit of WT theory, while (3.20) interprets it as a result of the resonant averaging of P^0.

The vector field R^0 defines locally Lipschitz operators in the spaces h^p, $p > d/2$:

$$|R^0(v) - R^0(w)|_{h^p} \leqslant C_p (|v|_{h^p} \vee |w|_{h^p})^{2q_*} |v - w|_{h^p}. \tag{3.22}$$

Indeed, in view of (3.20), for any v, w such that $|v|_{h^p}, |w|_{h^p} \leqslant R$ we have

$$|(R^0(v) - R^0(w))|_{h^p} \leqslant \int_0^{2\pi} |\Psi_{-t\Lambda}(P^0(\Psi_{t\Lambda} v) - P^0(\Psi_{t\Lambda} w))|_{h^p}\,\text{đ}t. \tag{3.23}$$

Since $P^0(v) = -\mathrm{i}\rho\mathcal{F}(|\hat{v}|^{2q_*}\hat{v})$, where $\hat{v} = \mathcal{F}^{-1}v$, defining $\Psi_{t\Lambda} v = v_t$, defining w_t similarly, and using that the operators Ψ_θ define isometries of h^p, we bound the right-hand side of (3.23) by

$$\begin{aligned}
\int_0^{2\pi} |P^0(v_t) - P^0(w_t)|_{h^p}\,\text{đ}t &= \rho \int_0^{2\pi} \||\widehat{v_t}|^{2q_*}\widehat{v_t} - |\widehat{w_t}|^{2q_*}\widehat{w_t}\|_p\,\text{đ}t \\
&\leqslant \rho C_p R^{2q_*} \int_0^{2\pi} \|\widehat{v_t} - \widehat{w_t}\|_p\,\text{đ}t \\
&\leqslant \rho C_p R^{2q_*} |v - w|_{h^p}.
\end{aligned}$$

Finally, we set

$$R = R^0 + R^1, \quad \text{where } R_k^1(v) = P_k^1(v) = -\gamma_k v_k.$$

Since $\langle v_k \cdot P_k^1 \rangle_\Lambda = \langle -\sum 2\gamma_k I_k \rangle_\Lambda = v_k \cdot P_k^1 = v_k \cdot R_k^1$, in view of (3.19) we have

$$\langle v_k \cdot P_k \rangle_\Lambda(v) = v_k \cdot R_k(v). \tag{3.24}$$

For later use, by the same argument we note that $\langle \mathrm{i}v_k \cdot P_k^0 \rangle_\Lambda = \mathrm{i}v_k \cdot R_k^0$ and $\langle \mathrm{i}v_k \cdot P_k^1 \rangle_\Lambda = 0 = \mathrm{i}v_k \cdot R_k^1$. So also

$$\langle \mathrm{i}v_k \cdot P_k \rangle_\Lambda(v) = \mathrm{i}v_k \cdot R_k(v). \tag{3.25}$$

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Motivated by the averaging theory for equations without resonances in [12, 13], we now consider the following *effective equation* for the slow dynamics in (3.5):

$$\mathrm{d}v_k = R_k(v)\,\mathrm{d}\tau + b_k\,\mathrm{d}\boldsymbol{\beta}^k, \quad k \geqslant 1. \tag{3.26}$$

In contrast to the averaged equations (3.17) and (3.18), the effective equation is regular, i.e. it does not have singularities at the locus $\ni(h)$. Since $R^0\colon h \to h$ is locally Lipschitz, strong solutions for (3.26) exist locally in time and are unique.

Lemma 3.5. *A strong solution of (3.26) with a specified initial datum $v(0) = v_0 \in h$ is unique a.s.*

The relevance of the effective equation for the study of the long-time dynamics in (2.2)=(3.1) is clear from the next lemma.

Lemma 3.6. *Let a continuous process $v(\tau) \in h$ be a weak solution of (3.26) such that all moments of the random variable $\max_{0\leqslant\tau\leqslant T}|v(\tau)|_h$ are finite. Then $I(v(\tau))$ is a weak solution of (3.17). Let stopping times $0 \leqslant \tau_1 < \tau_2 \leqslant T$ and numbers $\delta_* > 0$, $N \in \mathbb{N}$ be such that*

$$I_k(v(\tau)) \geqslant \delta_* \quad \text{for } \tau_1 \leqslant \tau \leqslant \tau_2 \text{ and } k \leqslant N. \tag{3.27}$$

Then the process $(I(v(\tau)),\ \Phi_j(v(\tau)),\ j \leqslant J(N))$ is a weak solution of the system of averaged equations (3.17), $(3.18)_{j\leqslant J}$.[8]

Proof. Let $v(\tau)$ satisfy (3.26). Applying Itô's formula to $I_k(v(\tau))$ and $\Phi_j(v(\tau))$, $j \leqslant J$, we get that

$$\mathrm{d}I_k = v_k \cdot R_k\,\mathrm{d}\tau + b_k^2\,\mathrm{d}\tau + b_k v_k \cdot \mathrm{d}\boldsymbol{\beta}^k \tag{3.28}$$

and

$$\mathrm{d}\Phi_j = \sum_{k\in\operatorname{supp} s^{(j)}} s_k^{(j)}\left(\frac{\mathrm{i}v_k\cdot R_k}{|v_k|^2}\,\mathrm{d}\tau + \frac{b_k}{|v_k|^2}\mathrm{i}v_k\cdot\mathrm{d}\boldsymbol{\beta}^k\right).$$

Using (3.24) and (3.25) we see that (3.28) has the same drift and diffusion as (3.17). So $I(v(\tau))$ is a weak solution of (3.17) (see [20, 26]). Similarly, for $\tau \in [\tau_1, \tau_2]$, in view of (3.25), the process $(I,\ \Phi_j,\ j \leqslant J)$, is a weak solution of the system (3.17), $(3.18)_{j\leqslant J}$. □

Now we show that the effective equation describes the limiting (as $\nu \to 0$) dynamics for the equations of motions, written in the a-variables of the interaction representation (1.16). Indeed, let $u^\nu(\tau)$ be a solution of (2.2) satisfying $u(0) = u_0$. Define $v^\nu(\tau) = \mathcal{F}(u^\nu(\tau))$ and consider the vector of a-variables $a^\nu(\tau) = (a_k^\nu(\tau) = \mathrm{e}^{\mathrm{i}\nu^{-1}\lambda_k\tau}v_k^\nu(\tau),\ k \geqslant 1)$ (cf. (1.16)). Notice that we obviously have

$$|v^\nu(\tau)|_{h^m} \equiv |a^\nu(\tau)|_{h^m} \quad \forall m, \qquad I(v^\nu(\tau)) \equiv I(a^\nu(\tau)), \qquad V(v^\nu(\tau)) \equiv V(a^\nu(\tau)) \tag{3.29}$$

(see (3.11)). From (3.1), for the vector $a^\nu(\tau)$ we obtain the system of equations

$$\mathrm{d}a_k^\nu = (R_k(a^\nu) + \mathcal{R}_k(a^\nu, \nu^{-1}\tau))\,\mathrm{d}\tau + b_k\mathrm{e}^{\mathrm{i}\nu^{-1}\lambda_k\tau}\,\mathrm{d}\boldsymbol{\beta}^k(\tau), \quad k \geqslant 1,$$

[8] This system is heavily underdetermined.

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where we have defined

$$\mathcal{R}_k(a,\nu^{-1}\tau) = \sum_{\substack{p,q,l\in\mathbb{Z}^\infty_{+0},\\ q-l-e^k\notin\mathcal{A}(\Lambda,m),\\ |q|+|l|+1\leqslant m}} P_k^{0pql}(a)\exp(-\mathrm{i}\nu^{-1}\tau(\Lambda\cdot(q-l-e^k))). \tag{3.30}$$

This is the non-resonant, fast oscillating part of the nonlinearity (because $|\Lambda\cdot(q-l-e^k)|\geqslant 1$). Since $\{\bar{\boldsymbol{\beta}}^k(\tau) := \int \mathrm{e}^{\mathrm{i}\nu^{-1}\lambda_k\tau}\,\mathrm{d}\boldsymbol{\beta}^k(\tau),\ k\geqslant 1\}$ is another set of standard independent complex Wiener processes, the process $a^\nu(\tau)$ is a weak solution of the system of equations

$$\mathrm{d}a_k^\nu = (R_k(a^\nu) + \mathcal{R}_k(a^\nu,\nu^{-1}\tau))\,\mathrm{d}\tau + b_k\,\mathrm{d}\boldsymbol{\beta}^k(\tau), \quad k\geqslant 1. \tag{3.31}$$

We will refer to (3.31) as the *a-equations*. It is crucial that they are identical to the effective equation (3.26), apart from terms that oscillate fast as $\nu\to 0$.

3.3. Properties of resonant Hamiltonian $\mathcal{H}^{\mathrm{res}}$ and the effective equation

Lemma 3.7. *The vector field R^0,*

$$R^0 = \mathrm{i}\rho\nabla\mathcal{H}^{\mathrm{res}}(v) \quad \forall v\in h^p,\ p>d/2, \tag{3.32}$$

is Hamiltonian, where $\mathcal{H}^{\mathrm{res}}(v) = \langle\mathcal{H}\rangle_\Lambda(v)$ and $\mathcal{H}$ is the Hamiltonian (1.2).

Proof. Indeed, since $P^0(v) = \mathrm{i}\rho\nabla\mathcal{H}(v)$,

$$R^0(v) = \int_0^{2\pi}\Psi_{-t\Lambda}(\mathrm{i}\rho\nabla\mathcal{H}(\Psi_{t\Lambda}(v)))\,đt = \mathrm{i}\rho\nabla_v\int_0^{2\pi}\mathcal{H}(\Psi_{t\Lambda}(v))\,đt = \mathrm{i}\rho\nabla_v\mathcal{H}^{\mathrm{res}}(v),$$

as $\Psi_\theta^* \equiv \Psi_{-\theta}$, and where we have used (3.20). □

Clearly, $\mathcal{H}^{\mathrm{res}}(0) = 0$. Since $\mathcal{H}(u)\leqslant -C\|u\|_0^{2q_*+2}$ by the Hölder inequality and since the transformations $\Psi_{t\Lambda}$ preserve $\|u\|_0$, we have that

$$\mathcal{H}^{\mathrm{res}}(u)\leqslant -C\|u\|_0^{2q_*+2} \quad \forall u.$$

The resonant Hamiltonian $\mathcal{H}^{\mathrm{res}}$ has symmetries, given by some rotations Ψ_m, $m\in\mathbb{R}^\infty$.

Lemma 3.8. *The following hold:*

(i) *letting $\mathbf{1} = (1,1,\dots)$, we have $\mathcal{H}^{\mathrm{res}}(\Psi_{t\mathbf{1}}v) = \mathrm{const.}$ (i.e. it does not depend on t);*

(ii) *letting $\mathcal{M}^l$ be the lth component of the sequence $(\boldsymbol{k}(1),\boldsymbol{k}(2),\dots)$, $l=1,\dots,d$ (see (2.1)), we have $\mathcal{H}^{\mathrm{res}}(\Psi_{t\mathcal{M}^l}v) = \mathrm{const.}$ for each l;*

(iii) $\mathcal{H}^{\mathrm{res}}(\Psi_{t\Lambda}v) = \mathrm{const.}$

Proof. (i) By (2.19) we have

$$\mathcal{H}^{\mathrm{res}}(\Psi_{t\mathbf{1}}v) = \int_0^{2\pi}\mathcal{H}(\Psi_{t'\Lambda}(\Psi_{t\mathbf{1}}v))\,đt' = \int_0^{2\pi}\mathcal{H}(\Psi_{t\mathbf{1}}(\Psi_{t'\Lambda}v))\,đt'.$$

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Let us define $\Psi_{t\mathbf{1}}(\Psi_{t'\Lambda}v) = v(t;t')$. Then $(\mathrm{d}/\mathrm{d}t)v(t;t') = \mathrm{i}v$. The flow of this Hamiltonian equation commutes with that of the equation with the Hamiltonian $\mathcal{H}$.[9] So $\mathcal{H}(v(t;t'))$ is independent from t for each t', and (i) follows since $\mathcal{H}^{\mathrm{res}}(\Psi_{t\mathbf{1}}v) = \int \mathcal{H}(v(t;t'))\,\mathrm{d}t'$.

(ii) The proof is the same since the transformations $\Psi_{t\mathcal{M}^l}$, $t \in \mathbb{R}$, are the flow of the momentum Hamiltonian $M^l(u) = \frac{1}{2}\sum_{j=1}^{\infty} \boldsymbol{k}^l(j)|u_j|^2$, which commutes with $\mathcal{H}$.

(iii) This is a straightforward consequence of the definition of the resonant averaging (2.19). □

Since the transformations $\Psi_{t\mathbf{1}}$ form the flow of the Hamiltonian

$$H_0(v) = \frac{1}{2}\sum |v_j|^2 = \tfrac{1}{2}|v|^2_{h^0},$$

the transformations $\Psi_{t\Lambda}$ form the flow of $H_1(v) = \frac{1}{2}\sum \lambda_j |v_j|^2$, and the transformations $\Psi_{t\mathcal{M}^l}$, $t \in \mathbb{R}$, form the flow of the momentum Hamiltonian, we may recast the assertions of the last lemma as

$$\{\mathcal{H}^{\mathrm{res}}, H_0\} = 0, \qquad \{\mathcal{H}^{\mathrm{res}}, H_1\} = 0, \qquad \{\mathcal{H}^{\mathrm{res}}, \mathcal{M}^l\} = 0 \quad \forall l. \tag{3.33}$$

Here $\{\cdot,\cdot\}$ signifies the Poisson bracket. As the transformations Ψ_m, $m \in \mathbb{R}^\infty$, are symplectic, the symmetries in the lemma above preserve the Hamiltonian vector field R^0 and commute with it. In particular, since $\Psi_{t\Lambda} = \mathrm{e}^{-\mathrm{i}t\Delta}$, the spectral spaces E_λ of the operator $-\Delta$,

$$E_\lambda = \mathrm{span}\{e^j : \lambda_j = \lambda\},$$

are invariant for the flow maps of R^0.

Since the transformations Ψ_m, $m \in \mathbb{R}^\infty$, obviously preserve the vector field R^1 as well as the random force law in (3.26) (see the proof of lemma 3.9), those Ψ_m that are symmetries of R^0 (equivalently, are symmetries of the Hamiltonian $\mathcal{H}^{\mathrm{res}}$) preserve weak solutions of (3.26). So we have the following lemma.

Lemma 3.9. *If $v(\tau)$ is a solution of (3.26) and $m \in \mathbb{R}^\infty$ is either a vector $m = t\mathbf{1}$, $t \in \mathbb{R}$, or a vector $m = t\Lambda$, or $m = t\mathcal{M}^l$, $l = 1,\dots,d$, then also $\Psi_m v(\tau)$ is a weak solution.*

Proof. Define $\Psi_m v(\tau) = v'(\tau)$. Applying Ψ_m to (3.26), using lemma 3.8 and exploiting the invariance of the operator R^1 with respect to Ψ_m, we get

$$\mathrm{d}v'_k = (\Psi_m R(v(\tau)))_k\,\mathrm{d}\tau + \mathrm{e}^{\mathrm{i}m_k} b_k\,\mathrm{d}\boldsymbol{\beta}^k = (R(v'(\tau)))_k + b_k(\mathrm{e}^{\mathrm{i}m_k}\,\mathrm{d}\boldsymbol{\beta}^k).$$

Since $\{\mathrm{e}^{\mathrm{i}m_k}\boldsymbol{\beta}^k(\tau),\ k \geqslant 1\}$ is another set of standard independent Wiener processes, $v'(\tau)$ is a weak solution of (3.26). □

Corollary 3.10. *If μ is a stationary measure for (3.26) and a vector m is as in lemma 3.9, then the measure $\Psi_m \circ \mu$ is also stationary.*

[9] This follows from the fact that the functional $\frac{1}{2}|v|^2_{h^0}$ is an integral of motion for the Hamiltonian $\mathcal{H}$, which becomes obvious if we note that in the u-representation $\mathcal{H}$ has the form (1.2) and $\frac{1}{2}|v|^2_{h^0}$ is $\frac{1}{2}\int |u|^2(x)\,\mathrm{d}x$.

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The next lemma characterizes the increments of $R^0(v)$ in the space h^0. It will be needed below to study the ergodic properties of the effective equation.

LEMMA 3.11. *Let $p > d/2$. Then for any $v, w \in h^p$ we have*

$$|R^0(v) - R^0(w)|_{h^0} \leqslant C(|v|_{h^p} + |w|_{h^p})^{2q_*}|v - w|_{h^0}.$$

Proof. Repeating the proof of the Lipschitz property of R^0 in the space h (see (3.22)) and using the notation of that proof, i.e. defining $\Psi_{t\Lambda}v = v_t$, $\hat{v} = \mathcal{F}^{-1}v$, and similarly for the vector w, we get that

$$\begin{aligned}
|R^0(v) - R^0(w)|_{h^0} &\leqslant \int_0^{2\pi} |\Psi_{-t\Lambda}(P^0(\Psi_{t\Lambda}v) - P^0(\Psi_{t\Lambda}w))|_{h^0}\,đt \\
&= \int_0^{2\pi} |P^0(v_t) - P^0(w_t)|_{h^0}\,đt \\
&= \int_0^{2\pi} \||\widehat{v_t}|^{2q_*}\widehat{v_t} - |\widehat{w_t}|^{2q_*}\widehat{w_t}\|_0\,đt \\
&\leqslant C\int_0^{2\pi} (|\widehat{v_t}|_{L^\infty} + |\widehat{w_t}|_{L^\infty})^{2q^*}\|\widehat{v_t} - \widehat{w_t}\|_0\,đt \\
&\leqslant C_1(|v|_{h^p} + |w|_{h^p})^{2q^*}|v - w|_{h^0}.
\end{aligned}$$

□

4. Explicit calculation

Here we intend to explicitly calculate the effective equation (3.26), keeping track of the dependence on the size L of the torus. To do that, it is convenient to use the natural parametrization of the exponential basis by vectors $\boldsymbol{k} \in \mathbb{Z}_L^d$; that is, to decompose functions $u(x)$ as Fourier series $u(x) = \sum_{\boldsymbol{k}\in\mathbb{Z}_L^d} v_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x}$. We modify the norms $|\cdot|_{h^p}$ accordingly:

$$\|u\|_p^2 = (2\pi L)^d \sum_{\boldsymbol{k}\in\mathbb{Z}_L^d} \left(|\boldsymbol{k}| \vee \frac{1}{L}\right)^{2p} |v_{\boldsymbol{k}}|^2 =: |v|_{h^p}^2.$$

Now, as in the introduction, the eigenvalues of the negative Laplacian are $\lambda_{\boldsymbol{k}} = |\boldsymbol{k}|^2$ and the damping coefficients are $\gamma_{\boldsymbol{k}} = f(\lambda_{\boldsymbol{k}})$.

In the v-coordinates the nonlinearity becomes the mapping $v \mapsto P^0(v)$, whose $\boldsymbol{k}$th component is

$$P^0_{\boldsymbol{k}}(v) = -\mathrm{i}\rho \sum_{\boldsymbol{k}_1,\dots,\boldsymbol{k}_{2q_*+1}\in\mathbb{Z}_L^d} v_{\boldsymbol{k}_1}\cdots v_{\boldsymbol{k}_{q_*+1}}\bar{v}_{\boldsymbol{k}_{q_*+2}}\cdots\bar{v}_{\boldsymbol{k}_{2q_*+1}}\delta^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol{k}}$$

(see (1.8)). Accordingly,

$$v_{\boldsymbol{k}}\cdot P^0_{\boldsymbol{k}} = \rho \sum_{\boldsymbol{k}_1,\dots,\boldsymbol{k}_{2q_*+1}\in\mathbb{Z}_L^d} \mathrm{Im}(v_{\boldsymbol{k}_1}\cdots v_{\boldsymbol{k}_{q_*+1}}\bar{v}_{\boldsymbol{k}_{q_*+2}}\cdots\bar{v}_{\boldsymbol{k}_{2q_*+1}}\bar{v}_{\boldsymbol{k}})\delta^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol{k}}. \quad (4.1)$$

In order to calculate the resonant average, we first notice that $v_{\boldsymbol{k}}\cdot P^0_{\boldsymbol{k}}$ can be written as a series (2.17), where $|C_{pql}| \leqslant 1$ and $|q| + |p| + |l| = 2q_* + 2$. In this case the sum

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on the left-hand side of (2.18) is bounded by

$$C\Big(\sum_{\boldsymbol{k}\in\mathbb{Z}_L^d}|v_{\boldsymbol{k}}|\Big)^{2q_*+2}\leqslant C_1(L)|v|_p^{q_*+1}\Big(\sum_{\boldsymbol{k}\in\mathbb{Z}_L^d}|\boldsymbol{k}|^{-2p}\Big)^{q_*+1}.$$

So condition (2.18) is met if $2p>d$.

Since the order of the resonance is $m=2q_*+2$, $\langle v_{\boldsymbol{k}}\cdot P^0_{\boldsymbol{k}}\rangle_\Lambda(v)$ equals

$$\rho\sum_{\boldsymbol{k}_1,\dots,\boldsymbol{k}_{2q_*+1}\in\mathbb{Z}_L^d}\operatorname{Im}(v_{\boldsymbol{k}_1}\cdots v_{\boldsymbol{k}_{q_*+1}}\bar v_{\boldsymbol{k}_{q_*+2}}\cdots\bar v_{\boldsymbol{k}_{2q_*+1}}\bar v_{\boldsymbol{k}})\delta^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol{k}}\delta(\lambda^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol{k}}),$$

(see (1.21)). This follows from (4.1) and (2.19) if one notes that the restriction $(q-l)\cdot\Lambda=0$ appearing there is now replaced by the factor $\delta(\lambda^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol{k}})$. In a similar way, we see that the quantity R^0_k, entering (3.26), takes the form

$$R^0_{\boldsymbol{k}}(v)=-\mathrm{i}\rho\sum_{\boldsymbol{k}_1,\dots,\boldsymbol{k}_{2q_*+1}\in\mathbb{Z}_L^d}v_{\boldsymbol{k}_1}\cdots v_{\boldsymbol{k}_{q_*+1}}\bar v_{\boldsymbol{k}_{q_*+2}}\cdots\bar v_{\boldsymbol{k}_{2q_*+1}}\times\delta^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol{k}}\delta(\lambda^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol{k}}).$$

Taking into account that $R^1_{\boldsymbol{k}}=-\gamma_{\boldsymbol{k}}v_{\boldsymbol{k}}$, we finally arrive at an explicit formula for the effective equation (3.26):

$$\mathrm{d}v_{\boldsymbol{k}}=\Big(-\gamma_{\boldsymbol{k}}v_{\boldsymbol{k}}-\mathrm{i}\rho\sum_{\boldsymbol{k}_1,\dots,\boldsymbol{k}_{2q_*+1}\in\mathbb{Z}_L^d}v_{\boldsymbol{k}_1}\cdots v_{\boldsymbol{k}_{q_*+1}}\bar v_{\boldsymbol{k}_{q_*+2}}\cdots\bar v_{\boldsymbol{k}_{2q_*+1}}\times\delta^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol{k}}\delta(\lambda^{1\cdots q_*+1}_{q_*+2\cdots 2q_*+1\boldsymbol{k}})\Big)\,\mathrm{d}\tau+b_{\boldsymbol{k}}\,\mathrm{d}\boldsymbol{\beta}^{\boldsymbol{k}}\tag{4.2}$$

with $\boldsymbol{k}\in\mathbb{Z}_L^d$. Due to (3.32),

$$R^0_k(v)=\mathrm{i}\rho\nabla_{v_k}\mathcal{H}^{\mathrm{res}}(v)=2\mathrm{i}\rho\frac{\partial}{\partial\bar v_k}\mathcal{H}^{\mathrm{res}}(v).$$

Therefore, (4.2) can be written as the damped–driven Hamiltonian system (1.18).

Examples. (a) If $q_*=1$, then (4.2) reads

$$\mathrm{d}v_{\boldsymbol{k}}=\Big(-\gamma_{\boldsymbol{k}}v_{\boldsymbol{k}}-\mathrm{i}\rho\sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}''\in\mathbb{Z}_L^d}v_{\boldsymbol{k}}v_{\boldsymbol{k}'}\bar v_{\boldsymbol{k}''}\delta_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{k}''+r}\delta_{\lambda_{\boldsymbol{k}}+\lambda_{\boldsymbol{k}'},\lambda_{\boldsymbol{k}''}+\lambda_{\boldsymbol{k}}}\Big)\,\mathrm{d}\tau+b_{\boldsymbol{k}}\,\mathrm{d}\boldsymbol{\beta}^{\boldsymbol{k}},$$

where $\boldsymbol{k}\in\mathbb{Z}_L^d$. If $f(t)=t+1$, then this equation looks similar to the complex Ginzburg–Landau equation

$$\dot u-\Delta u+u=\mathrm{i}|u|^2u+\frac{\mathrm{d}}{\mathrm{d}\tau}\sum b_{\boldsymbol{k}}\boldsymbol{\beta}^{\boldsymbol{k}}(\tau)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x},$$

written in the Fourier coefficients. The latter equation possesses nice analytical properties; for example, its stationary measure is unique for any d (see [16]).

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(b) Our results remain true if the Hamiltonian $\mathcal{H}$, corresponding to the nonlinearity in (1.5), has variable coefficients. In particular, let $d = 1$ and let the nonlinearity in (1.5) be replaced by $-\mathrm{i}p(x)|u|^2u$, where $p(x)$ is a sufficiently smooth function. Then the effective equation is

$$\mathrm{d}v_k = \Bigg(-\gamma_k v_k - \mathrm{i} \sum_{k_1,k_2,k_3,k_4\in\mathbb{Z}_L} v_{k_1} v_{k_2} \bar{v}_{k_3} p_{k_4} \delta_{k_1+k_2+k_4,k_3+k} \delta_{k_1^2+k_2^2,k_3^2+k^2}\Bigg)\,\mathrm{d}\tau + b_k\,\mathrm{d}\boldsymbol{\beta}^k,$$

where $k_L \in \mathbb{Z}_d$ and the p_ks are the Fourier coefficients of $p(x)$.

5. Main results

5.1. Averaging theorem for the initial-value problem

We recall that r is a fixed even integer such that $r \geqslant d/2 + 1$, and write

$$h^r = h, \quad C([0,T],h) = \mathcal{H}_a.$$

We provide $\mathcal{H}_a$ with the Borel σ-algebra and the natural filtration of the σ-algebras $\{\mathcal{F}_t,\ 0 \leqslant t \leqslant T\}$.

Let $v^\nu(\tau)$ be a solution of (3.1) such that $v^\nu(0) = v_0 = \mathcal{F}(u_0) \in h^r$, and consider the corresponding process $a^\nu(\tau)$. Due to (3.29), the process a^ν satisfies obvious analogies with estimates (2.3), (2.5) and (2.6). Since $(R+\mathcal{R})(a)$ is the nonlinearity $P(v)$, written in the a-variables,

$$|(R+\mathcal{R})(a)(\tau)|_h = |P(v)(\tau)|_h \leqslant C|v(\tau)|_h^{q_*+1} = C|a(\tau)|_h^{q_*+1}.$$

Therefore, all moments of $|(R+\mathcal{R})(a)|_{\mathcal{H}_a}$ are finite, and we get from (3.31) that $\mathbb{E}|a^\nu|_{C^{1/3}([0,T],h)} \leqslant \bar{C}$, uniformly in ν. Now, arguing as when proving lemma 3.4, we get that the set of laws $\mathcal{D}(a^\nu(\cdot))$, $0 < \nu \leqslant 1$, is tight in $\mathcal{H}_a$. Consider any limiting measure corresponding to the laws $\mathcal{D}(a^\nu(\cdot))$:

$$\mathcal{D}(a^{\nu_\ell}(\cdot)) \rightharpoonup \mathcal{Q}_a^0 \quad \text{as } \nu_\ell \to 0. \tag{5.1}$$

THEOREM 5.1. *There exists a unique weak solution $a(\tau)$ of effective equation (3.26), satisfying $a(0) = v_0$ a.s. The law of $a(\cdot)$ in the space $\mathcal{H}_a$ coincides with $\mathcal{Q}_a^0$. The convergence (5.1) holds as $\nu \to 0$.*

The proof of the theorem is presented at the end of this section.

Let $\mathcal{Q}^0$ be a measure in $\mathcal{H}_{I,V}$, as in (3.16). Since $(I,V)(v^\nu(\cdot)) = (I,V)(a^\nu(\cdot))$ for any $\nu > 0$, rewriting $a(\tau)$ as $v(\tau)$ we derive a corollary of the previous theorem.

THEOREM 5.2. *There exists a unique weak solution $v(\tau)$ of effective equation (3.26) satisfying $v(0) = v_0$ a.s. The law of $(I,V)(v(\cdot))$ in the space $\mathcal{H}_{I,V}$ coincides with $\mathcal{Q}^0$, and the convergence (3.16) holds as $\nu \to 0$. Moreover, for any vectors $\tilde{s}_1,\dots,\tilde{s}_m \in \mathbb{Z}_0^\infty$, perpendicular to Λ, we have the convergence*

$$\mathcal{D}(I, V^{\tilde{s}_1},\dots,V^{\tilde{s}_m})(v^\nu(\cdot)) \rightharpoonup \mathcal{D}(I, V^{\tilde{s}_1},\dots,V^{\tilde{s}_m})(v(\cdot)).$$

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By this result the Cauchy problem for the effective equation has a weak solution. Using lemma 3.5 and the Yamada–Watanabe argument (see [9, 20, 26]) we get that the equation is well posed.

Corollary 5.3. *For any $v_0 \in h^r$, (3.26) has a unique strong and a unique weak solution $v(\tau)$ such that $v(0) = v_0$. Its law satisfies (3.14).*

Now, let $\tilde{s} \in \mathbb{Z}_0^\infty$ be any non-zero vector, orthogonal to Λ, and consider $\varphi(v^\nu(\tau)) \cdot \tilde{s} = \varphi(V^{\tilde{s}}(v^\nu(\tau))) \in S^1$. Since $\varphi(V^{\tilde{s}})$ is a discontinuous function of $V^{\tilde{s}} \in \mathbb{C}$, in order to pass to a limit as $\nu \to 0$ we do the following. We identify S^1 with $\{v \in \mathbb{R}^2 \colon |v| = 1\}$, define $\lceil \tilde{s} \rceil = N$, and approximate the discontinuous function $V^N = (V_1, \dots, V_N) \mapsto \varphi(V^{\tilde{s}})$ by continuous functions

$$V^N \mapsto f_\delta([I(V^N)])\varphi(V^{\tilde{s}}) \in \mathbb{R}^2, \quad [I] = \min_{1 \leqslant k \leqslant N} I_k,\ 0 < \delta \ll 1,$$

where f_δ is continuous, $0 \leqslant f_\delta \leqslant 1$, $f_\delta(t) = 0$ for $t \leqslant \delta/2$ and $f_\delta = 1$ for $t \geqslant \delta$.

For any measure μ_τ in a complete metric space, which weakly continuously depends on τ, and any $\tau_1 < \tau_2$ we will define

$$\langle \mu_\tau \rangle_{\tau_1}^{\tau_2} = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \mu_\tau \, \mathrm{d}\tau.$$

Then the argument above, jointly with lemma 3.2, implies the following corollary.

Corollary 5.4. *Let $\tilde{s} \in \mathbb{Z}_0^\infty$ be any non-zero vector, orthogonal to Λ, and let $0 \leqslant \tau_1 < \tau_2 \leqslant T$. Then*

$$\langle \mathcal{D}(\varphi(v^\nu(\tau)) \cdot \tilde{s}) \rangle_{\tau_1}^{\tau_2} \rightharpoonup \langle \mathcal{D}(\varphi(v(\tau)) \cdot \tilde{s}) \rangle_{\tau_1}^{\tau_2} \quad \text{as } \nu \to 0.$$

On the contrary, if $s \cdot \Lambda \neq 0$, then by proposition 5.10 we get that

$$\langle \mathcal{D}(\varphi(v^\nu(\tau)) \cdot s) \rangle_{\tau_1}^{\tau_2} \rightharpoonup \text{đ}\varphi.$$

More generally, if vectors $\tilde{s}_1, \dots, \tilde{s}_M$ from $\mathbb{Z}_0^\infty$ are perpendicular to Λ and a vector s is not, then

$$\big\langle \mathcal{D}(I, \varphi \cdot \tilde{s}_1, \dots, \varphi \cdot \tilde{s}_M, \varphi \cdot s)(v^\nu(\tau)) \big\rangle_{\tau_1}^{\tau_2} \rightharpoonup \big\langle \mathcal{D}(I, \varphi \cdot \tilde{s}_1, \dots, \varphi \cdot \tilde{s}_M)(v(\tau)) \big\rangle_{\tau_1}^{\tau_2} \times \text{đ}\varphi.$$

We do not know of an equivalent description of the measure $\mathcal{Q}^0$ only in terms of the slow variables (I, V) of (3.1), but the following result holds true.

Proposition 5.5. *Consider the natural process on the space $\mathcal{H}_{I,V}$ with the measure $\mathcal{Q}^0$. If, for some $N \in \mathbb{N}$ and $\delta_* > 0$, stopping times $0 \leqslant \tau_1 < \tau_2 \leqslant T$ satisfy (3.27), then for $\tau \in [\tau_1, \tau_2]$ the process $(I, \Phi^{(N)})((I, V)(\tau))$ is a weak solution of the averaged equations (3.17) and (3.18)$|_{j \leqslant J}$. Here $\Phi^{(N)} = (\Phi_1, \dots, \Phi_{J(N)})$.*

Since the averaged quantities $\langle v_k \cdot P_k \rangle_\Lambda$ and $\langle \mathrm{i} v_k \cdot P_k \rangle_\Lambda$ are functions of I and Φ (see (2.21)), (3.17) and (3.18)$|_{j \leqslant J}$ form an underdetermined system of equations for the variables (I, Φ).

Proof of theorem 5.1. The proof follows the Khasminski scheme (see [6, 11, 17]). Its crucial step is given by the following lemma.

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LEMMA 5.6. *For any $k \geqslant 1$ one has*

$$\mathfrak{A}_k^\nu := \mathbb{E} \max_{0 \leqslant \tau \leqslant T} \left| \int_0^\tau \mathcal{R}_k(a^\nu(s), \nu^{-1}s)\, \mathrm{d}s \right| \to 0 \quad \textit{as } \nu \to 0. \tag{5.2}$$

The lemma is proved in § 5.4, following the arguments in [13, 17]. We now derive from it the theorem.

For $\tau \in [0, T]$ consider the processes

$$N_k^{\nu_l} = a_k^{\nu_l}(\tau) - \int_0^\tau R_k(a^{\nu_l}(s))\, \mathrm{d}s, \quad k \geqslant 1.$$

Due to (3.31) we can write $N_k^{\nu_l}$ as

$$N_k^{\nu_l}(\tau) = \tilde{N}_k^{\nu_l}(\tau) + \bar{N}_k^{\nu_l}(\tau),$$

where

$$\tilde{N}_k^{\nu_l}(\tau) = a^{\nu_l}(\tau) - \int_0^\tau (R_k(a^{\nu_l}(s)) + \mathcal{R}_k(a^{\nu_l}(s), \nu_l^{-1}s))\, \mathrm{d}s$$

is a $\mathcal{Q}_a^0$ martingale and the disparity $\bar{N}_k^{\nu_l}$ is

$$\bar{N}_k^{\nu_l}(\tau) = \int_0^\tau \mathcal{R}_k(a^{\nu_l}(s), \nu_l^{-1}s)\, \mathrm{d}s.$$

The convergence $\mathcal{D}(a^{\nu_l}) \rightharpoonup \mathcal{Q}_a^0$ and lemma 5.6 imply that the processes

$$N_k(\tau) = a_k(\tau) - \int_0^\tau R_k(a)\, \mathrm{d}s, \quad k \geqslant 1,$$

are $\mathcal{Q}_a^0$ martingales (for details, see [17, proposition 6.3]).

Similarly to (5.2), we find that

$$\mathbb{E} \max_{0 \leqslant \tau \leqslant T} \left| \int_0^\tau \mathcal{R}_k(a^\nu(s), \nu^{-1}s)\, \mathrm{d}s \right|^2 \to 0 \quad \text{as } \nu \to 0.$$

Then, using the same arguments as before, we see that the processes

$$N_{k_1}(\tau) N_{k_2}(\tau) - \int_0^\tau A_{k_1 k_2}\, \mathrm{d}s$$

are $\mathcal{Q}_a^0$ martingales, where $A_{k_1 k_2}$ denotes the diffusion matrix for system (3.26). That is, $\mathcal{Q}_a^0$ is a solution of the martingale problem with drift R_k and diffusion A. Hence, $\mathcal{Q}_a^0$ is a law of a weak solution of (3.26). Such a solution exists for any $v_0 \in h$. So by lemma 3.5 and the Yamada–Watanabe argument (see [9, 20, 26]), weak and strong solutions for (3.26) both exist and are unique. Hence, the limit in (3.16) does not depend on the sequence $\nu_l \to 0$, the convergence holds as $\nu \to 0$, and the theorem is proved. □

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5.2. Averaging theorem for stationary solutions

Let $v^\nu(\tau)$ be a stationary solution of (3.1) as at the end of § 2.1.[10] Solutions v^ν inherit the *a priori* estimates (2.3), (2.5), (2.6), and so the set of laws $\mathcal{D}(I(v^\nu(\cdot)), V(v^\nu(\cdot)))$, $0 < \nu \leqslant 1$, is still tight in $\mathcal{H}_{I,V}$ (see lemma 3.4). Consider any limit

$$\mathcal{D}(I(v^{\nu_\ell}(\cdot)), V(v^{\nu_\ell}(\cdot))) \rightharpoonup \mathcal{Q} \quad \text{as } \nu_\ell \to 0. \tag{5.3}$$

As before, the measure $\mathcal{Q}$ satisfies (3.14) (with the constants C_n, C', C'', corresponding to $v_0 = 0$). Moreover, it is stationary in τ.

THEOREM 5.7. *There exists a stationary solution $v(\tau)$ of the effective equation (3.26) such that $\mathcal{Q} = \mathcal{D}(I(v(\cdot)), V(v(\cdot)))$.*

Proof. Define $\mu^\nu = \mathcal{D}v^\nu(\tau)$. Estimate (2.5) with $2m = r$ and $n = 1$ implies that $\int |v|^2_{h^{r+1}} \mu^\nu(\mathrm{d}v) \leqslant C$ for all ν. So the set of measures μ^ν is tight in $\mathcal{H}^r$. Replacing, if necessary, the sequence $\{\nu_l\}$ by a subsequence, we obtain that

$$\mu^{\nu_l} \rightharpoonup \mu^0 \quad \text{as } \nu_l \to 0. \tag{5.4}$$

Clearly, $(I, V) \circ \mu^0$ is the marginal distribution for $\mathcal{Q}$ for $\tau = \text{const.}$, which we will denote by q (i.e. $q = \mathcal{Q}|_{\tau=\text{const.}}$).

Let $v^0(\tau)$, $\tau \geqslant 0$, be a solution of the effective equation (3.26) such that $\mathcal{D}v^0(0) = \mu^0$ (which exists by corollary 5.3 and the estimates on μ^0). Then, for the same reason as in § 5.1,

$$\mathcal{D}(I, V)(v^0(\tau))|_{\tau\in[0,T]} = \mathcal{Q},$$

and $\mathcal{D}(I, V)(v^0(\tau)) \equiv q$. We do not know if the solution v^0 is stationary, but from the Bogolyubov–Krylov argument we know that for a suitable sequence $T_j \to \infty$ we have the convergence

$$\frac{1}{T_j} \int_0^{T_j} \mathcal{D}(v^0(\tau))\, \mathrm{d}\tau \rightharpoonup m^0,$$

where m^0 is a stationary measure for (3.26). We still have that $(I, V) \circ m^0 = q$, and the measure m^0 satisfies the same *a priori* estimates as before. Let $v(\tau)$ be a solution for (3.26) such that $\mathcal{D}v(0) = m^0$. It is stationary and $\mathcal{D}(I, V)(v(\tau)) \equiv q$. Slightly modifying the argument above, we also get that $\mathcal{D}(I, V)(v(\cdot)) = \mathcal{Q}$. □

Writing the convergence (5.3) as $\mathcal{D}(I, V)(v^{\nu_l}(\cdot)) \rightharpoonup \mathcal{D}(I, V)(v(\cdot))$, we note that, as in § 5.1, we also have that

$$\mathcal{D}(I, V^{\tilde{s}_1}, \ldots, V^{\tilde{s}_m})(v^{\nu_l}(\tau)) \rightharpoonup \mathcal{D}(I, V^{\tilde{s}_1}, \ldots, V^{\tilde{s}_m})(v(\tau)) = (I, V^{\tilde{s}_1}, \ldots, V^{\tilde{s}_m}) \circ m^0$$

as $\nu_l \to 0$, for any m and any vectors $\tilde{s}_1, \ldots, \tilde{s}_m$ perpendicular to Λ. Since for stationary solutions $v^\nu(\tau)$ we have $\langle \mathcal{D}(v^\nu(\tau))\rangle_{\tau_1}^{\tau_2} = \mathcal{D}(v^\nu(\tau))$, arguing as when proving corollary 5.4 we also get that

$$\mathcal{D}(I, \Phi^{\tilde{s}_1}, \ldots, \Phi^{\tilde{s}_m})(v^{\nu_l}(\tau)) \rightharpoonup (I, \Phi^{\tilde{s}_1}, \ldots, \Phi^{\tilde{s}_m}) \circ m^0. \tag{5.5}$$

[10] Under certain restrictions on the equation it is known that its law (i.e. the stationary measure of the equation) is unique; see, for example, [24]. We will not discuss this now.

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Moreover, if $s \in \mathbb{Z}_0^\infty$ is such that $s \cdot \Lambda \neq 0$, then in view of proposition 5.10 and the stationarity of the solutions, we have

$$\mathcal{D}(I, \Phi^{\tilde{s}_1}, \dots, \Phi^{\tilde{s}_m}, \Phi^s)(v^{\nu_l}(\tau)) \rightharpoonup ((I, \Phi^{\tilde{s}_1}, \dots, \Phi^{\tilde{s}_m}) \circ m^0) \times đ\theta. \tag{5.6}$$

If (3.26) has a unique stationary measure m^0, then the convergences above hold as $\nu \to 0$. But in this case a stronger assertion holds.

THEOREM 5.8. *Let v^ν be a stationary solution of (3.1), let $\mathcal{D}(v^\nu(\tau)) \equiv \mu^\nu$ and assume that the effective equation (3.26) has a unique stationary measure m^0. Then*

$$\mu^\nu \rightharpoonup m^0 \quad \text{as } \nu \to 0. \tag{5.7}$$

Proof. (i) Consider again the convergence (5.4). We are going to show that the limiting measure μ^0 equals m^0. Then the limit in (5.4) does not depend on the sequence $\{\nu_l \to 0\}$, so it holds as $\nu \to 0$, and (5.7) follows.

(ii) Due to lemma 3.2, $\mu^\nu(\eth) = 0 = \mu^0(\eth)$, so we may regard μ^ν and μ^0 as measures on $h_I^r \times \mathbb{T}^\infty$. Let us fix any $n \in \mathbb{N}$ and consider measures $\mu^{\nu n}$, μ^{0n} and m^{0n}, which are images of the measures μ^ν, μ^0 and m^0 under the projection

$$\Pi^n \colon v \mapsto v^n = (v_1, \dots, v_n)$$

(see notation and agreement in § 1.5). We will regard them as measures on $\mathbb{R}_+^n \times \mathbb{T}^n = \{(I^n, \varphi^n)\}$. To prove that $\mu^0 = m^0$, it suffices to verify that $\mu^{0n} = m^{0n}$ for each n.

Let us define $\mathcal{A}(\Lambda^n) =: \mathcal{A}^n$, and let the vectors $\zeta^1, \dots, \zeta^n \in \mathbb{Z}^n$ and the unimodular matrix R be as in lemma 2.1 with $\mathcal{A} = \mathcal{A}^n$. Let $L = L_{\mathcal{A}^n} \colon \mathbb{T}^n \to \mathbb{T}^{n-1}$ be the operator in (2.9), i.e.

$$L \colon \mathbb{T}^n \ni \varphi^n \mapsto (\varphi^n \cdot \zeta^1, \dots, \varphi^n \cdot \zeta^{n-1})^{\mathrm{T}} \in \mathbb{T}^{n-1}. \tag{5.8}$$

Writing $R^{\mathrm{T}}(\varphi^n) = (y_1, \dots, y_n)^{\mathrm{T}} = (\boldsymbol{y}^{\mathrm{T}}, y_n)^{\mathrm{T}}$, where $\boldsymbol{y} = (y_1, \dots, y_{n-1})^{\mathrm{T}}$, we have $L(\varphi^n) = \boldsymbol{y}$. We will denote by π^1 the natural projection $y \mapsto \boldsymbol{y}$.

For later purposes we make the following observation. Let μ be a Borel measure on h. Consider its images under rotations $\Psi_{t\Lambda}$ and projections Π^n. In the (I, φ)-variables the mapping $\Psi_{t\Lambda}$ becomes $\mathrm{id} \times (\cdot + t\Lambda)$, so

$$\Pi^n \circ (\Psi_{t\Lambda} \circ \mu) = (\mathrm{id} \times (\cdot + t\Lambda^n)) \circ \Pi^n \circ \mu$$

(where $\Pi^n \circ \mu$ is written in the (I^n, φ^n)-variables). By (2.8) the transformation R^{T} of $\mathbb{T}^n$ conjugates the translation by the vector $t\Lambda^n$ with the translation by te^n. Therefore,

$$\mathcal{R}^{\mathrm{T}} \circ \Pi^n \circ (\Psi_{t\Lambda} \circ \mu) = (\mathrm{id} \times (\cdot + te^n)) \circ \mathcal{R}^{\mathrm{T}} \circ \Pi^n \circ \mu, \tag{5.9}$$

where $\mathcal{R}^{\mathrm{T}} = \mathrm{id} \times R^{\mathrm{T}}$.

(iii) Let us apply to the measures $\mu^{\nu n}$, μ^{0n}, m^{0n} the transformation $\mathcal{R}^{\mathrm{T}}$:

$$N^{\nu n} = \mathcal{R}^{\mathrm{T}} \circ \mu^{\nu n}, \qquad N^{0n} = \mathcal{R}^{\mathrm{T}} \circ \mu^{0n}, \qquad M^{0n} = \mathcal{R}^{\mathrm{T}} \circ m^{0n}. \tag{5.10}$$

Recall that by (5.4), $N^{\nu_l n} \rightharpoonup N^{0n}$ as $\nu_l \to 0$. Our first goal is to calculate the limiting measure N^{0n}. To do this let us disintegrate $N^{\nu n}$ and N^{0n} with respect to the mapping

$$\mathrm{id} \times \pi^1 \colon \mathbb{R}_+^n \times \mathbb{T}^n \to \mathbb{R}_+^n \times \mathbb{T}^{n-1}, \quad (I^n, (\boldsymbol{y}^{\mathrm{T}}, y_n)^{\mathrm{T}}) \mapsto (I^n, \boldsymbol{y}).$$

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That is (see [4, § 10.2]), write them as

$$N^{\nu n} = N^{\nu n}_{I^n,\boldsymbol{y}}(\mathrm{d}y_n)p^{\nu n}(\mathrm{d}I^n\mathrm{d}\boldsymbol{y}), \qquad N^{0n} = N^{0n}_{I^n,\boldsymbol{y}}(\mathrm{d}y_n)p^{0n}(\mathrm{d}I^n\mathrm{d}\boldsymbol{y}),$$

where $p^{\nu n} = (\mathrm{id}\times\pi^1)\circ N^{\nu n}$ and $p^{0n} = (\mathrm{id}\times\pi^1)\circ N^{0n}$. Since $\boldsymbol{y} = L(\varphi^n)$, we have $p^{\nu n} = \mathcal{D}(I^n\times(L\circ\varphi^n))(v^{\nu n}(\tau))$. As each vector ζ^j in (5.8) is perpendicular to Λ^n, in view of (5.5) we have

$$p^{0n} = \lim_{\nu_l\to 0}\mathcal{D}(I^n\times(L\circ\varphi^n))(v^{\nu_l n}(\tau)) = (I^n\times(L\circ\varphi^n))\circ m^{0n}. \tag{5.11}$$

To calculate N^{0n}, it remains to find the fibre measures $N^{0n}_{I^n,\boldsymbol{y}}$. To do this let us take any bounded continuous function f on $\mathbb{R}^n_+\times\mathbb{T}^{n-1}\times S^1$ and consider $\langle N^{\nu n}, f\rangle = \mathbb{E}f(I^n,\boldsymbol{y},y_n)(v^\nu(\tau))$. Since $\boldsymbol{y}(v) = L(\varphi^n)$ and $y_n(v) = v\cdot\eta^n$, where the vector η^n is not perpendicular to Λ, by (5.6),

$$\langle N^{\nu n}, f\rangle \to \int f(I^n,\boldsymbol{y},y_n)((I^n\times(L\circ\varphi^n))m^{0n})(\mathrm{d}I^n\mathrm{d}\boldsymbol{y})\,đy_n.$$

On the other hand, by (5.4),

$$\langle N^{\nu_l n}, f\rangle \to \langle N^{0n}, f\rangle = \int f(I^n,\boldsymbol{y},y_n)N^{0n}_{I^n,\boldsymbol{y}}(\mathrm{d}y_n)p^{0n}(\mathrm{d}I^n\mathrm{d}\boldsymbol{y}).$$

Since $p^{0n} = (I^n\times(L\circ\varphi^n))\circ m^{0n}$, we get from the two convergences above that for p^{0n}-almost all pairs $(I^n,\boldsymbol{y})$ we have $N^{0n}_{I^n,\boldsymbol{y}} = đy_n$. Accordingly,

$$N^{0n} = đy_n\times p^{0n}(\mathrm{d}I^n\mathrm{d}\boldsymbol{y}).$$

(iv) Consider the measure M^{0n}. Due to (5.11) its disintegration with respect to the mapping $\mathrm{id}\times\pi^1$ may be written as

$$M^{0n} = M^{0n}_{I^n,\boldsymbol{y}}(\mathrm{d}y_n)p^{0n}(\mathrm{d}I^n\mathrm{d}\boldsymbol{y}) \tag{5.12}$$

with some unknown fibre measures $M^{0n}_{I^n,\boldsymbol{y}}$. Now consider the rotated measure $\Psi_{t\Lambda}\circ m^0$, $t\geqslant 0$, and its n-dimensional projection. By (5.9),

$$\mathcal{R}^{\mathrm{T}}\circ\Pi^n\circ\Psi_{t\Lambda}\circ m^0 = (\mathrm{id}\times l_t)\circ\mathcal{R}^{\mathrm{T}}\circ m^{0n},$$

where $l_t(\boldsymbol{y},y_n) = (\boldsymbol{y},y_n+t)$. Due to (5.10) and (5.12), the measure on the right-hand side equals

$$M^{0n}_{I^n,\boldsymbol{y}}(\mathrm{d}y_n+t)p^{0n}(\mathrm{d}I^n\mathrm{d}\boldsymbol{y}).$$

But by corollary 3.10 the measure on the left-hand side does not depend on t. So $M^{0n}_{I^n,\boldsymbol{y}}(\mathrm{d}y_n)\equiv M^{0n}_{I^n,\boldsymbol{y}}(\mathrm{d}y_n+t)$ is a translation-invariant measure on S^1, and it must be equal to $đy_n$. Accordingly,

$$M^{0n} = đy_n\times p^{0n}(\mathrm{d}I^n\,\mathrm{d}y^n) = N^{0n}.$$

(v) We have established that $N^{\nu_l n}\rightharpoonup M^{0n}$ as $\nu_l\to 0$. So $\nu^{\nu_l n}\rightharpoonup m^{0n}$, which completes the proof. □

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5.3. Mixing in the effective equations

We start with the case in which the function $f(\lambda)$ has a linear growth. For simplicity of notation we suppose that $f(\lambda) = \lambda + 1$. We are also forced to assume that $q_* = 1$.

The effective equation (3.26)=(1.18) with $q_* = 1$ looks similar to $(1.6)_{\nu=\infty,\, q_*=1}$, studied in [16]. It turns out that the two equations are indeed similar, at least for $d \leqslant 3$, and that the proof of the mixing in [16, § 4], based on an abstract theorem from [19], applies to (3.26) with minimal changes. Indeed, the crucial step in [16] in order to apply the result from [19] is to establish for solutions of the equation the exponential estimate of the form

$$\mathbb{P}\bigg\{ \sup_{t\geqslant 0} \bigg(\int_0^t |u(s)|^2_{L_\infty}\,\mathrm{d}s - Kt \bigg) \geqslant \sigma \bigg\} \leqslant C' \exp(c_1 |u_0|^2_{L_\infty} - c_2\sigma) \quad \forall \sigma > 0, \tag{5.13}$$

with suitable constants K, C', c_1 and c_2. This estimate is important in the study of the mixing since it allows one to control divergence of trajectories $u_1(t)$ and $u_2(t)$, corresponding to the same realization of the random force, through the inequality[11]

$$|u_1(t) - u_2(t)|_{L_2} \leqslant |u_1(0) - u_2(0)|_{L_2} \exp\bigg(C \int_0^t (|u_1(s)|^2_{L_\infty} + |u_2(s)|^2_{L_\infty})\,\mathrm{d}s \bigg). \tag{5.14}$$

For (3.26) an analogy of (5.13) follows by applying the Itô formula to $[v]_1^2 = H_0(v) + H_1(v)$ (see (3.33)), since due to (3.33) we have that

$$\mathrm{d}[v(\tau)]_1^2 + 2\int_0^\tau [v(s)]_2^2\,\mathrm{d}s = 4\tau\mathcal{B} + 2\sum_{j=1}^\infty (\lambda_j + 1)(v_j(\tau)\cdot \mathrm{d}\boldsymbol{\beta}^j(\tau)),$$

where we define $[v]_2^2 = \sum(\lambda_j + 1)^2 |v_j|^2$ and $\mathcal{B} = \sum(\lambda_j + 1) b_j^2$. Applying to this relation the supermartingale inequality in the standard way (see, for example, [16, 19]), we get that

$$\mathbb{P}\bigg\{ \sup_{\tau\geqslant 0} \bigg(\int_0^\tau [v(s)]_2^2\,\mathrm{d}s - 2\mathcal{B}t \bigg) \geqslant \sigma \bigg\} \leqslant C' \exp(c_1 |v_0|_1^2 - c_2\sigma) \quad \forall \sigma > 0.$$

If $d \leqslant 3$, then by lemma 3.11 the divergence of two solutions for (3.26) with the same ω satisfies

$$|v_1(\tau) - v_2(\tau)|_{h^0} \leqslant |v_1(0) - v_2(0)|_{h^0} \exp\bigg(C \int_0^\tau ([v_1(s)]_2^2 + [v_2(s)]_2^2)\,\mathrm{d}s \bigg).$$

These last two estimates allow us to repeat literally for (3.26) the reduction to theorem 3.1.3 from [19], made in [16], and prove the following theorem.

THEOREM 5.9. *Let $q_* = 1$, $f(\lambda) = \lambda + 1$ and $d \leqslant 3$. Then the effective equation (3.26) has a unique stationary measure μ and is mixing. That is, every solution $v(\tau)$ satisfies $\mathcal{D}(v(\tau)) \rightharpoonup \mu$ as $\tau \to \infty$.*

[11] To match (5.13) and (5.14) we use, crucially, that $q_* \leqslant 1$.

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The presented proof uses that the nonlinearity in the effective equation is at most cubic. It also applies to the effective equations for (1.6), where the Hamiltonian $\mathcal{H}$ is one of the two functions $\mathcal{H}^3$ with cubic densities as at the end of § 1.1 (in this case the argument works if $d \leqslant 6$). The proof without changes applies to (1.6), where $q_* = 1$, $d \leqslant 3$ and $f(\lambda)$ grows superlinearly. The argument may also be adjusted to the case in which $q_* = 1$, d is any integer number and $f(\lambda) = c_1 + \lambda^{c_d}$, where c_d is sufficiently big. Based on the similarity with $(1.6)_{\nu=\infty,\, q_*=1}$, studied in [16] for any space dimension, we conjecture that for $q_* = 1$ and $f(\lambda) = \lambda + 1$ the effective equation is well posed and mixing for any d. But it is unknown how to prove the mixing for equations with $q_* \geqslant 2$ (in any space dimension).

5.4. Proof of lemma 5.6

For this proof we adopt notation from [17]. Namely, we denote by $\varkappa(t)$ various functions of t such that $\varkappa \to 0$ as $t \to \infty$, and denote by $\varkappa_\infty(t)$ functions satisfying $\varkappa(t) = o(t^{-N})$ for each N. We write $\varkappa(t, M)$ to indicate that $\varkappa(t)$ depends on a parameter M. In addition, for events Q and O and a random variable f we write $\mathbb{P}_O(Q) = \mathbb{P}(O \cap Q)$ and $\mathbb{E}_O(f) = \mathbb{E}(\chi_O f)$. Here and below, M stands for a suitable function of ν such that $M(\nu) \to \infty$ as $\nu \to 0$, but

$$\nu M^n \to 0 \quad \text{as } \nu \to 0 \quad \forall n.$$

Denote by $\Omega_M = \Omega_M^\nu$ the event $\Omega_M = \{\sup_{0 \leqslant \tau \leqslant T} |a^\nu(\tau)|_{h^r} \leqslant M\}$. Then, by (2.6), $\mathbb{P}(\Omega_M^c) \leqslant \varkappa_\infty(M)$ uniformly in ν, so that one has $\mathfrak{A}_k^\nu \leqslant \varkappa_\infty(M) + \mathfrak{A}_{k,M}^\nu$, where we have defined

$$\mathfrak{A}_{k,M}^\nu := \mathbb{E}_{\Omega_M} \max_{0 \leqslant \tau \leqslant T} \left| \int_0^\tau \mathcal{R}_k(a^\nu(s), \nu^{-1} s)\, \mathrm{d}s \right|. \tag{5.15}$$

So it remains to estimate $A_{k,M}^\nu$.

Consider a partition of $[0, T]$ by the points

$$\tau_n = nL, \quad 0 \leqslant n \leqslant K \sim T/L,$$

where τ_K is the last point τ_n in $[0, T)$. The diameter L of the partition is $L = \sqrt{\nu}$. Defining

$$\eta_l = \int_{\tau_l}^{\tau_{l+1}} \mathcal{R}_k(a^\nu(s), \nu^{-1} s)\, \mathrm{d}s, \quad 0 \leqslant l \leqslant K - 1, \tag{5.16}$$

we see that

$$\mathfrak{A}_{k,M}^\nu \leqslant LC(M) + \mathbb{E}_{\Omega_M} \sum_{l=0}^{K-1} |\eta_l|, \tag{5.17}$$

since for $\omega \in \Omega_M$ the integrand in (5.16) is smaller than a suitable $C(M)$ (see lemma 3.1 and (3.22)). For any l let us consider the event

$$\mathcal{F}_l = \Big\{ \sup_{\tau_l \leqslant \tau \leqslant \tau_{l+1}} |a^\nu(\tau) - a^\nu(\tau_l)|_h \geqslant P_1(M) L^{1/3} \Big\},$$

where $P_1(M)$ is a suitable polynomial. It is not hard to verify using the Doob

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inequality that for a suitable choice of P_1 the probability of $\mathbb{P}(\mathcal{F}_l)$ is less than $\varkappa_\infty(L^{-1};M)$ (see [17]). One gets

$$\sum_{l=0}^{K-1} |\mathbb{E}_{\Omega_M}|\eta_l| - \mathbb{E}_{\Omega_M\setminus\mathcal{F}_l}|\eta_l|| \leqslant C(M)L\sum_{l=0}^{K-1}\mathbb{P}(\mathcal{F}_l) \leqslant C(M)\varkappa_\infty(L^{-1};M), \tag{5.18}$$

so that it remains to estimate $\sum \mathbb{E}_{\Omega_M\setminus\mathcal{F}_l}|\eta_l|$.

We have

$$\begin{aligned}|\eta_l| \leqslant{}& \left|\int_{\tau_l}^{\tau_{l+1}} (\mathcal{R}_k(a^\nu(s),\nu^{-1}s) - \mathcal{R}_k(a^\nu(\tau_l),\nu^{-1}s))\,\mathrm{d}s\right| \\ &+ \left|\int_{\tau_l}^{\tau_{l+1}} (\mathcal{R}_k(a^\nu(\tau_l),\nu^{-1}s))\,\mathrm{d}s\right| \\ =:{}& \Upsilon_l^1 + \Upsilon_l^2.\end{aligned}$$

By the regularity of the integrand and the definition of $\mathcal{F}_l$,

$$\sum_l \mathbb{E}_{\Omega_M\setminus\mathcal{F}_l}\Upsilon_l^1 \leqslant \varkappa(L^{-1/3};M) = \varkappa(\nu^{-1/6};M). \tag{5.19}$$

So it remains to estimate the expectation of $\sum \Upsilon_l^2$. Defining $t=\nu\tau$ and making use of (3.30), we write Υ_l^2 as

$$\begin{aligned}\Upsilon_l^2 &= L\left|\frac{\nu}{L}\int_0^{\nu^{-1}L} \sum_{\substack{p,q,l\in\mathbb{Z}_{+0}^\infty,\\ q-l-e^k\notin\mathcal{A}(\Lambda,m),\\ |q|+|l|+1\leqslant m}} P_k^{0pql}(a)\exp(-\mathrm{i}t(\Lambda\cdot(q-l-e^k)))\,\mathrm{d}t\right| \\ &\leqslant LC(M)\frac{\nu}{L}\sup_{\substack{p,q,l\in\mathbb{Z}_{+0}^\infty,\\ q-l-e^k\notin\mathcal{A}(\Lambda,m),\\ |q|+|l|+1\leqslant m}} \frac{1}{\Lambda\cdot(q-l-e^k)} \leqslant L\varkappa(\nu^{-1}L;M),\end{aligned}$$

because the supremum in the second line is bounded by 1, since both Λ and $q-l-e^k$ are integer vectors. Therefore,

$$\sum_l \mathbb{E}_{\Omega_M\setminus\mathcal{F}_l}\Upsilon_l^2 \leqslant \varkappa(\nu^{-1/2};M). \tag{5.20}$$

Now (5.15) and (5.17)–(5.20) imply that

$$\mathfrak{A}_k^\nu \leqslant \varkappa_\infty(M) + \varkappa(\nu^{-1/2};M) + \varkappa_\infty(\nu^{-1};M) + \varkappa(\nu^{-1/6};M) + \varkappa(\nu^{-1/2};M).$$

Choosing first M large and then ν small, we make the right-hand side above arbitrarily small. This proves the lemma. □

An argument similar to the previous one (see appendix A) implies the following assertion.

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Proposition 5.10. *Let $s \in \mathbb{Z}_0^\infty$ be such that $s\cdot\Lambda \neq 0$ and let $G\colon \mathbb{R}_+^M \times \mathbb{T}^{J(M)} \times S^1 \to \mathbb{R}$ be a bounded Lipschitz-continuous function for some $M \geqslant 1$. Then*

$$\mathfrak{B}^\nu := \mathbb{E} \max_{0\leqslant\tau\leqslant T} \left| \int_0^\tau \left(G(I^{\nu M}(l), \Phi^{\nu(M)}(l), s\cdot\varphi^\nu(l)) - \int_{S^1} G(I^{\nu M}(l), \Phi^{\nu(M)}(l), \theta)\,\text{đ}\theta \right) \mathrm{d}l \right| \to 0 \quad \text{as } \nu \to 0.$$

In particular, taking for G Lipschitz functions on S^1, we get that $\langle \mathcal{D}(s\cdot\varphi^\nu(l))\rangle_0^t \rightharpoonup \mathrm{d}\theta$ as $\nu \to 0$ for any $t > 0$.

Appendix A. Proof of proposition 5.10

For this proof, as in § 5.4, we denote by $\varkappa(t)$ various functions of t such that $\varkappa \to 0$ as $t \to \infty$, and denote by $\varkappa_\infty(t)$ functions satisfying $\varkappa(t) = o(t^{-N})$ for each N. For events Q and $\mathcal{O}$ and a random variable f we write $\mathbb{P}_{\mathcal{O}}(Q) = \mathbb{P}(\mathcal{O}\cap Q)$ and $\mathbb{E}_{\mathcal{O}}(f) = \mathbb{E}(\chi_{\mathcal{O}} f)$. Without loss of generality we assume that $|G| \leqslant 1$ and $\operatorname{Lip} G \leqslant 1$.

Let us denote by R a suitable function of ν such that $R(\nu) \to \infty$ as $\nu \to 0$, but

$$\nu R^n \to 0 \quad \text{as } \nu \to 0 \quad \forall n.$$

Denote, moreover, by $\Omega_R = \Omega_R^\nu$ the event $\Omega_R = \{\sup_{0\leqslant\tau\leqslant T} |v^\nu(\tau)|_r \leqslant R\}$. Then, by (2.6), $\mathbb{P}(\Omega_R^c) \leqslant \varkappa_\infty(R)$ uniformly in ν.

Taking into account the boundedness of G, we get that

$$\mathfrak{B}^\nu \leqslant \varkappa_\infty(R) + \mathbb{E}_{\Omega_R} \max_{0\leqslant\tau\leqslant T} \left| \int_0^\tau \left(G(I^{\nu M}(l), \Phi^{\nu(M)}(l), s\cdot\varphi^\nu(l)) - \int_{S^1} G(I^{\nu M}(l), \Phi^{\nu(M)}(l), \theta)\,\text{đ}\theta \right) \mathrm{d}l \right|.$$

As in the proof of lemma 5.6, consider a partition of $[0,T]$ by the points

$$\tau_n = \tau_0 + nL, \quad 0 \leqslant n \leqslant K \sim T/L, \tag{A 1}$$

where τ_K is the last point τ_n in $[0,T)$. The diameter L of the partition is $L = \sqrt{\nu}$, and the non-random phase $\tau_0 \in [0,L)$ will be chosen later. Defining

$$\eta_n = \int_{\tau_n}^{\tau_{n+1}} \left(G(I^{\nu M}, \Phi^{\nu(M)}, s\cdot\varphi^\nu) - \int_{S^1} G(I^{\nu M}, \Phi^{\nu(M)}, \theta)\,\text{đ}\theta \right) \mathrm{d}l, \quad 0 \leqslant n \leqslant K-1, \tag{A 2}$$

we see that

$$\mathfrak{B}^\nu \leqslant \varkappa_\infty(R) + CL + \mathbb{E}_{\Omega_R} \sum_{n=0}^{K-1} |\eta_n|, \tag{A 3}$$

so it remains to estimate $\sum \mathbb{E}_{\Omega_R}|\eta_n|$. We write

$$\hat{G}(\psi; l) = G(I^{\nu M}(l), \Phi^{\nu(M)}(l), \psi), \quad \psi \in S^1,$$

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so that we have

$$
\begin{aligned}
|\eta_n| \leqslant & \left| \int_{\tau_n}^{\tau_{n+1}} (\hat{G}(s\cdot\varphi^\nu(l);l) - \hat{G}(s\cdot\varphi^\nu(\tau_n) + \nu^{-1}(s\cdot\Lambda)(l-\tau_n);\tau_n))\,\mathrm{d}l \right| \\
& + \left| \int_{\tau_n}^{\tau_{n+1}} \left(\hat{G}(s\cdot\varphi^\nu(\tau_n) + \nu^{-1}(s\cdot\Lambda)(l-\tau_n);\tau_n) - \int_{S^1} \hat{G}(\theta;\tau_n)\,\text{đ}\theta \right) \mathrm{d}l \right| \\
& + \left| \int_{\tau_l}^{\tau_{l+1}} \left(\int_{S^1} \hat{G}(\theta;\tau_n)\,\text{đ}\theta - \int_{S^1} \hat{G}(\theta;l)\,\text{đ}\theta \right) \mathrm{d}l \right| \\
=: & \ \Upsilon_n^1 + \Upsilon_n^2 + \Upsilon_n^3.
\end{aligned}
$$

To estimate the quantities $\Upsilon_n^{1,2,3}$, we first optimize the choice of the phase τ_0. A crucial point here is that if we set $N := M \vee \lceil s \rceil$, the function G depends only on v^N. So we consider the events $\mathcal{E}_n$, $1 \leqslant n \leqslant K$,

$$
\mathcal{E}_n = \{ I_k^\nu(\tau_n) \leqslant \varepsilon \text{ for some } k \leqslant N \}, \quad \text{where } \varepsilon \geqslant \nu^a,\ a = 1/10. \tag{A 4}
$$

Since for each $k \leqslant M$, by lemma 3.2, we have

$$
\int_0^L \sum_{n=0}^K \mathbb{P}(I_k^\nu(\bar{\tau}_n) \leqslant \varepsilon)\,\mathrm{d}\bar{\tau}_0 = \int_0^T \mathbb{P}(I_k^\nu(\tau) \leqslant \varepsilon)\,\mathrm{d}\tau = \varkappa(\varepsilon^{-1}; R, N)
$$

(here each $\bar{\tau}_n$ is regarded as a function of $\tau_0 = \bar{\tau}_0$, given by (A 1)), we can choose $\tau_0 \in [0, L)$ in such a way that

$$
K^{-1} \sum_{n=0}^{K-1} \mathbb{P}(\mathcal{E}_n) = \varkappa(\varepsilon^{-1}; R, N).
$$

For any n consider the event

$$
\mathcal{Q}_n = \Big\{ \sup_{\tau_n \leqslant \tau \leqslant \tau_{n+1}} |I^\nu(\tau) - I^\nu(\tau_n)|_{h_I} \geqslant P_1(R) L^{1/3} \Big\},
$$

where $P_1(R)$ is a suitable polynomial. It is not hard to verify using the Doob inequality that the probability satisfies $\mathbb{P}(\mathcal{Q}_n) \leqslant \varkappa_\infty(L^{-1})$ (see [17]). Setting $\mathcal{F}_n = \mathcal{E}_n \cup \mathcal{Q}_n$, $n = 0, \ldots, K-1$, we have that

$$
\frac{1}{K} \sum_{n=0}^{K-1} \mathbb{P}(\mathcal{F}_n) \leqslant \varkappa(\varepsilon^{-1}; R, N) + \varkappa_\infty(\nu^{-1/2}; N) =: \tilde{\varkappa}.
$$

Accordingly,

$$
\sum_{n=0}^{K-1} |(\mathbb{E}_{\mathcal{F}_n \cap \Omega_R}) \Upsilon_n^j| \leqslant CL \sum_{n=0}^{K-1} \mathbb{P}(\mathcal{F}_n) \leqslant C\tilde{\varkappa} := \tilde{\varkappa}_1, \quad j = 1, 2, 3. \tag{A 5}
$$

If $\omega \in \Omega_R \backslash \mathcal{F}_n$, then for $\tau \in [\tau_n, \tau_{n+1}]$ we have that $I_k^\nu(\tau) \geqslant \varepsilon - P_1(R) L^{1/3} \geqslant \frac{1}{2}\varepsilon$. On the other hand, by lemma 3.1, for any positive δ we have the estimate

$$
||v_k|^{-2}(\mathrm{i}v_k \cdot P_k(v))\chi_{\{|v_k| > \delta\}}| \leqslant \delta^{-1} Q_k(|v|_{h^r}),
$$

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where Q_k is a polynomial. These relations and (3.6) imply that

$$\mathbb{P}_{\Omega_R\setminus\mathcal{F}_n}\{|\varphi^{\nu N}(l)-(\varphi^{\nu M}(\tau_n)+\nu^{-1}\Lambda^N(l-\tau_n))|\geqslant\nu^a \text{ for some } l\in[\tau_n,\tau_{n+1}]\}\leqslant\varkappa_\infty(\nu^{-1};R,N)$$

(see the estimate of $\mathbb{P}(\mathcal{Q}_n)$). Therefore,

$$\mathbb{P}_{\Omega_R\setminus\mathcal{F}_n}\{|s\cdot\varphi^{\nu N}(l)-(s\cdot\varphi^{\nu N}(\tau_n)+\nu^{-1}(s\cdot\Lambda)(l-\tau_n))|\geqslant\nu^a \text{ for some } l\in[\tau_n,\tau_{n+1}]\}\leqslant\varkappa_\infty(\nu^{-1};R,N,s,\Lambda)$$

and

$$\mathbb{P}_{\Omega_R\setminus\mathcal{F}_n}\{|\Phi^{\nu(M)}(l)-\Phi^{\nu(M)}(\tau_n)|\geqslant\nu^a \text{ for some } l\in[\tau_n,\tau_{n+1}]\}\leqslant\varkappa_\infty(\nu^{-1};R,N).$$

From here and the definition of the events $\mathcal{Q}_n$ we find that

$$\sum_l \mathbb{E}_{\Omega_R\setminus\mathcal{F}_n}\Upsilon_n^1\leqslant C(R)\nu^{1/6}+C(R)\nu^a+\varkappa_\infty(\nu^{-1};R,N,s\cdot\Lambda). \tag{A 6}$$

For the same reason also

$$\sum_l \mathbb{E}_{\Omega_R\setminus\mathcal{F}_n}\Upsilon_n^3\leqslant C(R)\nu^{1/6}+C(R)\nu^a+\varkappa_\infty(\nu^{-1};R,N,s\cdot\Lambda). \tag{A 7}$$

So it remains to estimate the expectation of $\sum\Upsilon_n^2$. Defining $t=\nu(l-\tau_n)$ we write Υ_n^2 as

$$\begin{aligned}\Upsilon_n^2&=\left|\int_{\tau_n}^{\tau_{n+1}}\hat{G}(s\cdot\varphi^\nu(\tau_n)+\nu^{-1}(s\cdot\Lambda)(l-\tau_n);\tau_n)\,\mathrm{d}l-L\int_{S^1}\hat{G}(\theta;\tau_n)\,đ\theta\right|\\&=L\left|\frac{\nu}{L}\int_0^{\nu^{-1}L}\hat{G}(s\cdot\varphi^\nu(\tau_n)+s\cdot\Lambda t;\tau_n)\,\mathrm{d}t-\int_{S^1}\hat{G}(\theta;\tau_n)\,đ\theta\right|.\end{aligned}$$

Let us expand $\hat{G}(\psi;\tau_n)$ as a Fourier series $\hat{G}(\psi)=\sum g_k\mathrm{e}^{\mathrm{i}k\psi}$, where each g_k is a random variable and $g_0=\int_{S^1}\hat{G}(\theta;\tau_n)\,đ\theta$ (we discard the dependence on τ_n, which is fixed throughout the argument). Then

$$\left|\frac{1}{T}\int_0^T\hat{G}(\psi_0+t(s\cdot\Lambda))\,\mathrm{d}t-g_0\right|\leqslant\varepsilon\quad\forall T\geqslant T_\varepsilon,$$

for a suitable non-random T_ε. Indeed, for each non-zero k, one has

$$\left|\frac{1}{T}\int_0^T\mathrm{e}^{\mathrm{i}k(\psi_0+t(s\cdot\Lambda))}\,\mathrm{d}t\right|\leqslant\frac{2}{T|s\cdot\Lambda|},$$

so that[12]

$$\left|\frac{1}{T}\int_0^T\hat{G}(\psi_0+t(s\cdot\Lambda))\,\mathrm{d}t-g_0\right|\leqslant\frac{2}{T|s\cdot\Lambda|}\sum|g_k|\leqslant\frac{2C}{T|s\cdot\Lambda|}.$$

[12] By the Bernstein theorem, $\sum_{k=1}^\infty|g_k|\leqslant C$, where the constant $C=C(\hat{G})$ is finite if the function $\hat{G}(\psi)$ is Lipschitz continuous. The proof of the theorem (see, for example, [29, § VI.3]) easily implies that C depends only on the Lipschitz constant of $\hat{G}$, which equals 1 in our case.

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We have thus proved that $\Upsilon_n^2 \leqslant L\varkappa(\nu^{-1}L; R, N, \varepsilon, s\cdot\Lambda)$. Therefore,

$$\sum_l \mathbb{E}_{\Omega_R\setminus\mathcal{F}_l^c}\Upsilon_n^2 \leqslant \varkappa(\nu^{-1/2}; R, N, \varepsilon, s\cdot\Lambda). \tag{A 8}$$

Now (A 3), (A 5) and (A 6)–(A 8) imply that $\mathfrak{B}^\nu$ is bounded by

$$\varkappa_\infty(R) + \varkappa(\nu^{-a}; R, N) + \varkappa(\varepsilon^{-1}; R, N) + C(R)\nu^a \\ + C(R)\nu^{1/6} + \varkappa(\nu^{-1/2}; R, N, \varepsilon, s\cdot\Lambda).$$

Choosing first R large, and next ε and ν small in such a way that (A 4) holds, we make the quantity above arbitrarily small. This proves the required convergence.

The second assertion of the proposition follows from the first one, since to check the weak convergence of measures on a complete metric space it suffices to take for test functions the Lipschitz functions. □

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