



**HAL**  
open science

# Consistency of the Frequency Domain Bootstrap for differentiable functionals

Patrice Bertail, Anna Dudek

► **To cite this version:**

Patrice Bertail, Anna Dudek. Consistency of the Frequency Domain Bootstrap for differentiable functionals. 2019. hal-02385774v1

**HAL Id: hal-02385774**

**<https://hal.science/hal-02385774v1>**

Preprint submitted on 29 Nov 2019 (v1), last revised 6 Jan 2021 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Consistency of the Frequency Domain Bootstrap for differentiable functionals\*

Patrice Bertail

*MODAL'X, UPL Université Paris Nanterre. e-mail: patrice.bertail@gmail.com*  
and

Anna E. Dudek

*Department of Applied Mathematics, AGH University of Science and Technology,  
al. Mickiewicza 30, 30-059 Krakow, Poland. e-mail: aedudek@agh.edu.pl*

**Abstract:** In this paper consistency of the Frequency Domain Bootstrap for differentiable functionals of spectral density function of a linear stationary time series is discussed. The notion of influence function in the time domain on spectral measures is introduced. Moreover, the Fréchet differentiability of functionals of spectral measures is defined. Sufficient and necessary conditions for consistency of the FDB in the considered problems are provided and the second order correctness is discussed for some functionals. Finally, validity of the FDB for the empirical processes is considered.

**Keywords and phrases:** empirical process, Fréchet differentiability, influence function, second order correctness, spectral density function, spectral measure.

## 1. Introduction

Bootstrap for dependent data has been developed over the last three decades. Most of existing bootstrap approaches are designed for the time domain. Widely applied for stationary time series are block bootstrap methods. For instance, the Moving Block Bootstrap (Künsch (1989), Liu and Singh (1992)), the Circular Block Bootstrap (Politis and Romano (1992)), the Stationary Bootstrap (Politis and Romano (1994)), the Tapered Block Bootstrap (Paparoditis and Politis (2001)), the Regenerative Block Bootstrap (Bertail and Cléménçon(1996)). Some of these techniques can be also adapted for nonstationary data. Additionally, there exist methods introduced for particular classes of nonstationary time series. Among them we have the Seasonal Block Bootstrap (Politis (2001)), the Periodic Block Bootstrap (Chan et al. (2004)), the Generalized Seasonal Block Bootstrap (Dudek et al. (2014)), the Generalized Seasonal Tapered Block Bootstrap (Dudek et al. (2016)), the Extension of Moving Block Bootstrap (Dudek (2015), Dudek (2018)). Sometimes in the parametric setting it is also possible to apply to dependent sequences the techniques designed for i.i.d. data like the

---

\*Anna Dudek was partially supported by the Faculty of Applied Mathematics AGH UST statutory tasks within subsidy of Ministry of Science and Higher Education. As far Patrice Bertail, this research has been conducted as part of the project Labex MME-DII (ANR11-LBX-0023-01).

i.i.d. bootstrap of Efron (1979) or wild bootstrap of Wu (1986) (see e.g., Lahiri (2003) and Shimzu (2010)).

Alternatively, one may bootstrap the time series in the frequency domain. In that case the usual approach is to apply the i.i.d. bootstrap to studentized periodogram estimates ( Hurvich and Zeger 1987, Franke and Härdle, 1992, Nordgaard 1992). The most classical example of this idea is the Frequency Domain Bootstrap (FDB). Other method called the Local Bootstrap (LB) was proposed by Paparoditis and Politis (1999). In this approach one bootstraps the periodogram ordinates locally around the frequency of interest. In contrary to the FDB, the LB does not require estimation of the spectral density function. Both methods share same limitations of applicability, i.e. they are consistent only for some classes of functionals. To extend applicability of the bootstrap in the frequency domain a few other bootstrap methods were proposed: the Autoregressive Aided Periodogram Bootstrap (Kreiss and Paparoditis (2003)), the Convolved Bootstrapped Periodograms of Subsamples (CBPS) (Meyer et al. (2018)) and the Time Frequency Toggle (TFT)-bootstrap (Kirsch and Politis (2011)). All these approaches are much more difficult to implement than the FDB. They depend on unknown tuning parameters. Moreover, in contrary to other techniques the TFT-bootstrap is not purely a frequency domain technique. Indeed, the idea is to bootstrap Fourier coefficients obtained after applying a fast Fourier transform to the considered time series, and at the end, to back-transformed these quantities to obtain a bootstrap sample in the time domain. It should be also noticed that the CBPS is asymptotically valid in quite general framework for linear functionals and could be adapted to the general functional considered in this paper, but it seems very challenging to study its second order properties.

In this paper we focus on the classical Frequency Domain Bootstrap. Till now its consistency/inconsistency was proven in some particular cases, mainly for stationary linear processes. Franke and Härdle (1992) considered the problem of spectral density estimation, while Dahlhaus and Janas (1996) obtained validity of the FDB for ratio statistics and Whittle estimator. Finally, Kim and Nordman (2013) extended its applicability for Whittle estimator to long-range dependent linear models. It is worth to indicate that the FDB works for Whittle estimator since the functional corresponding to the Whittle estimator may be expressed approximately as a ratio statistic as will be seen later. Moreover, it is known that the FDB is not consistent for some functionals e.g., the autocovariance function. This originates from the fact that the FDB assumes that periodogram ordinates are independent while for non-Gaussian processes this condition usually does not hold. As a result bootstrap variance for the considered statistics not always converges to the asymptotic one (see e.g., Janas and Dahlhaus (1994), Paparoditis (2002)). However, till now in the literature there is no result stating the form of the class of functionals for which the FDB is consistent, while for particular functional it is important to know if the simplest bootstrap approach is valid or some generalization is essential. The main advantage of the FDB method is the fact that it is based on the i.i.d. bootstrap and does not require choosing any tuning parameters like for instance block length.

Moreover, till now the second order correctness was proven only for the FDB in some specific cases.

The aim of this paper is twofold. At first, we revisit some well known the FDB consistency results for spectral density function. We focus on smooth functions of linear functionals of spectral density. In particular we give necessary and sufficient conditions for the validity of the FDB. Essentially, in the general (non-Gaussian) case, the FDB works for functions (of linear functionals), which are homogeneous of degree 0. We then generalize existing results to differentiable functionals of spectral density function in the framework of stationary linear processes. For that purpose we introduce a concept of influence function in the time domain analogously to the i.i.d. case, but on spectral measures instead of cumulative distribution functions. These influence functions behave quite differently than in the usual i.i.d. set-up and may not be automatically centered. Moreover, we define the notion of the Fréchet differentiability of functionals of spectral measures. The FDB is asymptotically valid if and only if the kurtosis of the process is 0 (for instance in the Gaussian case) or if the functional of interest has a centered influence function, which is the case for ratio statistics as well as some Whittle estimators. We then study under what conditions the empirical process in the frequency domain, as considered in Dahlhaus (1988), converges to some Gaussian process and when its bootstrap version is valid. We essentially show that this holds if and only if all the functions are centered with respect to the given spectral density. In other cases, the bootstrap will fail to give the correct asymptotic distributions.

Paper is organized as follows. In Section 2 notation is introduced and the FDB algorithm is recalled. Consistency of the FDB for differentiable functions of linear functionals and its second order correctness is discussed in Section 3. Influence function in the time domain is introduced in Section 4 and some examples are presented. Section 5 is dedicated to the sufficient and necessary conditions for asymptotic validity of the FDB for Fréchet differentiable functionals in the time domain. The analogous conditions for empirical processes are formulated in Section 6. Section 7 contains short discussion of the obtained results. All proofs can be found in the Appendix.

## 2. Problem formulation

Let  $\{X_t, t \in \mathbb{Z}\}$  be a real valued stationary time series. For simplicity we assume that  $EX_t = 0$ . Moreover, let  $X_t$  admit an infinite moving average representation

$$X_t = \sum_{j=0}^{\infty} a_j \zeta_{t-j} \text{ with } \sum_{j=1}^{\infty} j^2 |a_j| < \infty, a_0 = 1,$$

where  $(\zeta_t)_{t \in \mathbb{Z}}$  is an i.i.d. sequence with  $E\zeta_t^2 = \sigma^2$  and  $E\zeta_t^8 < \infty$ . Such conditions allow to verify easily the conditions for asymptotic normality of the estimators that we are going to consider (see Dahlhaus, 1985, Corollary 3.2)

Let  $R(k) = \text{Cov}(X_1, X_{1+k})$  be the autocovariance function of the process  $X_t$

and let

$$f(\omega) = 1/(2\pi) \sum_{k=-\infty}^{\infty} R(k) \exp(-ik\omega)$$

be its spectral density function.

By  $I_n(\omega)$  we denote the periodogram i.e.,

$$I_n(\omega) = \frac{1}{2\pi n} d_n(\omega) d_n(-\omega), \quad (1)$$

where

$$d_n(\omega) = \sum_{t=1}^n X_t \exp(-it\omega), \quad \omega \in [-\pi, \pi] \quad (2)$$

is the discrete Fourier transform of  $\{X_t, t \in \mathbb{Z}\}$ . It is known that  $I_n(\omega)$  is not a consistent estimator of the spectral density  $f$  but it is asymptotically unbiased and may serve as a basis for estimating many parameters.

In this paper we are interested in functionals of the spectral density  $T(f)$  and in particular, smooth functions  $g$  (second order differentiable) of linear functionals

$$T(f) = g(A(\xi, f)),$$

where

$$A(\xi, f) = \left( \int_0^\pi \xi_1(\omega) f(\omega) d\omega, \int_0^\pi \xi_2(\omega) f(\omega) d\omega, \dots, \int_0^\pi \xi_p(\omega) f(\omega) d\omega \right)'$$

and

$$\xi = (\xi_1, \dots, \xi_p) : [0, \pi] \rightarrow \mathbb{R}^p.$$

We usually estimate  $T(f)$  using the plug-in estimator  $T(I_n)$ . Its bootstrap counterpart is obtained using the Frequency Domain Bootstrap (FDB), which we recall in Section 2.2.

### 2.1. Discretized versions

In general, to compute integrals in  $A(\xi, f)$  we estimate the functional by using the Riemann approximation of the integral at specific frequencies, most of the time the Fourier frequencies  $\tilde{\lambda}_{jn} = 2\pi j/n$ ,  $j = 1, \dots, n_0$ , where  $n_0 = [n/2]$  is the integer part of  $n/2$ . We denote

$$\tilde{A}_n(\xi, f) = \left( \frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_1(\tilde{\lambda}_{jn}) f(\tilde{\lambda}_{jn}), \frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_2(\tilde{\lambda}_{jn}) f(\tilde{\lambda}_{jn}), \dots, \frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_p(\tilde{\lambda}_{jn}) f(\tilde{\lambda}_{jn}) \right)'$$

However, it is known that the error of approximation in this case is of order  $O(n^{-1})$

$$\frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_i(\tilde{\lambda}_{jn}) f(\tilde{\lambda}_{jn}) - \int_0^\pi \xi_i(\omega) f(\omega) d\omega \sim n^{-1} \pi (\xi_i(\pi) f(\pi) - \xi_i(0) f(0)) / 2.$$

Assume in addition that  $\xi_1(\omega)f(\omega)$  is twice differentiable, which, under the condition that we introduce later, will reduce to the assumption that the spectral density is twice differentiable. Then one can rather choose an approximation at the midpoint frequencies  $\lambda_{jn} = (2\pi j + \pi)/n$ . By the well known Polya's theorem, we have

$$\frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_i(\lambda_{jn})f(\lambda_{jn}) - \int_0^\pi \xi_i(\omega)f(\omega)d\omega \sim \frac{\pi^2}{24n^2} \left( (\xi_i f)'(\pi) - (\xi_i f)'(0) \right).$$

If the spectral density is not twice differentiable, then the approximation will be of order  $O(n^{-1})$ . Notice that the periodogram is infinitely differentiable as a function of  $\omega$ . Thus, when we replace  $f$  by  $I_n$  we automatically get

$$\frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_i(\lambda_{jn})I_n(\lambda_{jn}) - \int_0^\pi \xi_i(\omega)I_n(\omega)d\omega = O(n^{-2}).$$

As a consequence we will always gain in using the discretization at midpoint frequencies  $\lambda_{jn} = (2\pi j + \pi)/n$  and hence such  $\lambda_{jn}$  are chosen in the sequel. This is a minor point if we are interested only in the first order asymptotics but it can have important consequence for one and two sided confidence intervals. The approximation error in the case of the standardized version of  $n^{1/2} \left( \frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_i(\tilde{\lambda}_{jn})f(\tilde{\lambda}_{jn}) - \int_0^\pi \xi_i(\omega)f(\omega)d\omega \right)$  typically implies a bias of a size  $1/\sqrt{n}$ , when the standard Fourier frequencies are used. To our knowledge, this has not been discussed before in the bootstrap literature.

## 2.2. Frequency Domain Bootstrap

The main idea underlying the FDB is based on the observation that the periodogram values evaluated at different frequencies  $0 < \omega_1 < \omega_2 < \dots < \omega_k < \pi$ ,  $I_n(\omega_j)$ ,  $j = 1, \dots, k$  are asymptotically independent and exponentially distributed i.e., asymptotically we have

$$\frac{I_n(\omega_j)}{f(\omega_j)} \xrightarrow[n \rightarrow \infty]{d} \text{Exp}(1).$$

The FDB essentially consists in "bootstrapping" these standardized frequencies, once  $f$  is estimated by some convergent estimator.

Recall that for linear processes considered here, we have (see Priestley (1981), Lahiri (2003), p. 235)

$$\begin{aligned} \mathbb{E} \left( \frac{I_n(\omega_j)}{f(\omega_j)} \right) &= 1 + O(n^{-1}), \\ \text{Var} \left( \frac{I_n(\omega_j)}{f(\omega_j)} \right) &= 1 + O(n^{-1}), \\ \text{Cov} \left( \frac{I_n(\omega_j)}{f(\omega_j)}, \frac{I_n(\omega_k)}{f(\omega_k)} \right) &= n^{-1}k_4 + o(n^{-1}), \text{ for } j \neq k, \end{aligned}$$

where

$$k_4 = \mathbb{E}\zeta^4/\sigma^4 - 3$$

is the kurtosis of the innovations.

This means that even if the random variables are asymptotically independent, they are not independent for finite  $n$  and hence we expect the i.i.d. bootstrap of the frequencies to have problem when  $k_4 \neq 0$ .

Below we recall the FDB algorithm together with its simple modification and the consistency result.

**Step 0** Compute an estimator of the spectral density  $f$ , for instance the kernel estimator

$$\begin{aligned} \widehat{f}_n(\omega) &= \widehat{f}_n(\omega, h) = \frac{(2\pi)^2}{nh} \sum_{j=-n_0}^{n_0} k\left(\frac{\omega - \lambda_{jn}}{h}\right) I_n(\lambda_{jn}) \\ &= \frac{1}{nh} \sum_{j=-n_0}^{n_0} 2\pi k\left(\frac{\omega - \lambda_{jn}}{h}\right) |d_n(\lambda_{jn})|^2, \end{aligned} \quad (3)$$

where  $n_0 = \lfloor n/2 \rfloor$  and  $\lambda_{jn} = (2\pi j + \pi)/n$ . Moreover,  $k$  is a kernel on  $[-\pi, \pi]$ , that is a real valued, non-negative, symmetric function such that  $\int_{-\infty}^{\infty} k(x) dx = 1$ . The smoothing window parameter  $h = h_n$  converges to 0 as  $n \rightarrow \infty$ .

**Step 1** Approximate the functional

$$T(f) = g(A(\xi, f))$$

by the sequence of  $T_n = T_n(I_n) = g(A_n(\xi, I_n))$ , where

$$A_n(\xi, f) = \left( \frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_1(\lambda_{jn}) f(\lambda_{jn}), \frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_2(\lambda_{jn}) f(\lambda_{jn}), \dots, \frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_p(\lambda_{jn}) f(\lambda_{jn}) \right).$$

The correcting factor  $2\pi$  appears because we are evaluating the functions at "the Fourier frequencies".

**Step 2** Compute for  $j = 1, \dots, n_0$  the standardized periodogram ordinates

$$\widehat{\epsilon}_{jn} = \frac{I_n(\lambda_{jn})}{\widehat{f}_n(\lambda_{jn})}.$$

Because of the estimated standardization, the mean of  $\widehat{\epsilon}_{jn}$  may be neither equal nor close to 1. To solve this problem, compute the rescaled values

$$\widetilde{\epsilon}_{jn} = \frac{\widehat{\epsilon}_{jn}}{\widehat{\epsilon}_{\cdot n}}$$

with

$$\widehat{\epsilon}_{\cdot n} = \frac{1}{n_0} \sum_{i=1}^{n_0} \widehat{\epsilon}_{jn}.$$

**Step 3** Generate  $\epsilon_{jn}^*$  i.i.d. from the empirical distribution  $P_{n_0} = \frac{1}{n_0} \sum_{i=1}^{n_0} \delta_{\epsilon_{jn}}$  (by construction  $\mathbb{E}_{P_{n_0}} \epsilon_{jn}^* = 1$  and  $\text{Var}_{P_{n_0}}(\epsilon_{jn}^*) \rightarrow 1$ ) that is, draw the bootstrap values randomly with replacement from these rescaled values and compute the bootstrapped periodogram values

$$I_n^*(\lambda_{jn}) = \widehat{f}_n(\lambda_{jn}) \epsilon_{jn}^*, \quad j = 1, \dots, n_0.$$

In Step 0 of the FDB algorithm we used the standard kernel estimator with a normalization of the kernel to 1 on  $\mathbb{R}$  ( $\int_{-\infty}^{\infty} k(x) dx = 1$ ) instead of  $2\pi$  which may be found in Lahiri (2003) (see p. 299 below expression (9.20)) or other papers. This standardization by  $2\pi$  is a source of confusion in many applications and in several expressions, all the more than the conventions are not same in signal theory.

The  $(2\pi)^2$  in the expression for  $\widehat{f}_n(\omega)$  (see formula (3)) may be surprising, but is essentially due to the fact that the functions are taken at the Fourier frequencies. Thus,  $1/(nh_n) \sum_{j=-n_0}^{n_0} k\left(\frac{\omega - \lambda_{jn}}{h}\right)$  is a Riemann integral equivalent to  $1/(2\pi h) \int_{-1/2}^{1/2} k\left(\frac{\omega - 2\pi x}{h}\right) dx$ , which converges to  $1/(2\pi)$  as  $h \rightarrow 0$ . Integrating locally  $I_n(2\pi x)$  also results in an additional  $1/2\pi$  term.

In Step 1 in some cases it is possible to consider the estimator

$$T_n = g\left(A_n(\xi, \widehat{f}_n)\right),$$

where  $\widehat{f}_n$  is an estimator of  $f$ . In the parametric case one may use the parametric estimator

$$\widehat{f}_n(\omega) = f_{\widehat{\theta}_n}(\omega),$$

where  $\widehat{\theta}_n$  is a convergent estimator of the parameter  $\theta$  (for instance a Whittle estimator of  $\theta$ , under specific assumptions).

**Remark 2.1.** Notice that  $\widehat{\epsilon}_{jn} = \frac{I_n(\lambda_{jn})}{\widehat{f}_n(\lambda_{jn})}$  essentially aims at reproducing the behavior of  $\frac{I_n(\lambda_{jn})}{f(\lambda_{jn})}$  which is asymptotically  $\mathcal{E}xp(1)$ . If one is only interested in asymptotic result, then Step 2 can be skipped and Step 3 can be replaced by the following more parametric bootstrap procedure.

**Step 3\*** Generate  $\epsilon_{jn}^*$  i.i.d.  $\mathcal{E}xp(1)$ , then compute

$$I_n^*(\lambda_{jn}) = \widehat{f}_n(\lambda_{jn}) \epsilon_{jn}^*, \quad j = 1, \dots, n_0.$$

Once the frequencies are resampled, then compute the corresponding value of the statistics  $T_n$ , that is

$$T_n^* = T_n(I_n^*) = g(A_n(\xi, I_n^*)),$$

where

$$A_n(\xi, I_n^*) = \left( \frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_1(\lambda_{jn}) I_n^*(\lambda_{jn}), \frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_2(\lambda_{jn}) I_n^*(\lambda_{jn}), \dots, \frac{2\pi}{n} \sum_{j=1}^{n_0} \xi_p(\lambda_{jn}) I_n^*(\lambda_{jn}) \right)'$$



### 3. Consistency of FDB for differentiable functions of linear functionals

The following theorem, which summarizes the main known results, shows that the FDB is valid only for specific processes or specific functionals (see Lahiri (2003) ch. 9.2 and references therein).

**Theorem 3.1.** *Assume that*

(i)  $\{X_t, t \in \mathbb{Z}\}$  *is a stationary linear process of the form*

$$X_t = \sum_{j=0}^{\infty} a_j \zeta_{t-j} \text{ with } \sum_{j=0}^{\infty} j^2 |a_j| < \infty, \quad a_0 = 1,$$

where  $(\zeta_t)_{t \in \mathbb{Z}}$  *is an i.i.d. white noise with*  $E\zeta_t^2 = \sigma^2$  *and*  $E\zeta_t^8 < \infty$ ;

(ii) *the spectral density estimator*  $\hat{f}_n$  *converges to*  $f$  *uniformly over*  $[0, \pi]$ ;

(iii)  $\inf_{\lambda \in [0, \pi]} f(\lambda) > 0$ .

*Then we have*

$$\sqrt{n}(A_n(\xi, I_n) - A(\xi, f)) \xrightarrow[n \rightarrow \infty]{d} N(0, \Sigma_\xi),$$

where

$$\Sigma_\xi = \left[ 2\pi \int_0^\pi \xi_i(\omega) \xi_j(\omega) f(\omega)^2 d\omega + \frac{k_4}{\sigma^4} \int_0^\pi \xi_i(\omega) f(\omega) d\omega \int_0^\pi \xi_j(\omega) f(\omega) d\omega \right]_{\substack{1 \leq i \leq p \\ 1 \leq j \leq p}}.$$

Moreover, the frequency domain bootstrap satisfies

$$\sqrt{n}(A_n(\xi, I_n^*) - A_n(\xi, \hat{f}_n)) \xrightarrow[n \rightarrow \infty]{d^*} N(0, \Sigma_\xi^*),$$

where

$$\Sigma_\xi^* = \left[ 2\pi \int_0^\pi \xi_i(\omega) \xi_j(\omega) f(\omega)^2 d\omega \right]_{\substack{1 \leq i \leq p \\ 1 \leq j \leq p}}.$$

Notice that we have

$$\Sigma_\xi = \Sigma_\xi^* + \frac{k_4}{\sigma^4} A(\xi, f) A(\xi, f)'. \quad (4)$$

Thus, the bootstrap asymptotically works when either  $\kappa_4 = 0$  or if the functional of interest is such that the quantities  $\int_0^\pi \xi_i(\omega) f(\omega) d\omega = 0$ . The first condition is for instance satisfied in the Gaussian case, but is very restrictive in the framework of bootstrap.

Let us recall that we are interested in a smooth functional  $T(f) = g(A(\xi, f))$ ,

which is differentiable around  $A(\xi, f)$ . Using Slutsky's lemma (or the delta method) we derive the corresponding equation to (4). We have

$$\begin{aligned} & \nabla g(A(\xi, f))' \Sigma_\xi \nabla g(A(\xi, f)) \\ &= \nabla g(A(\xi, f)) \Sigma_\xi^* \nabla g(A(\xi, f))' + \frac{k_4}{\sigma^4} \|A(\xi, f)' \nabla g(A(\xi, f))\|^2, \end{aligned}$$

where  $\nabla$  is the gradient operator. Thus, the FDB has the correct asymptotic variance iff  $k_4 = 0$  or

$$A(\xi, f)' \nabla g(A(\xi, f)) = 0. \quad (5)$$

In that case we have that (under appropriate assumptions)

$$\begin{aligned} \sqrt{n}(A_n(\xi, I_n) - A(\xi, f)) &\xrightarrow[n \rightarrow \infty]{d} N(0, \nabla g'(A(\xi, f)) \Sigma_\xi \nabla g(A(\xi, f))), \\ \sqrt{n}(g(A_n(\xi, I_n^*)) - g(A_n(\xi, \hat{f}_n))) &\xrightarrow[n \rightarrow \infty]{d^*} N(0, \nabla g(A(\xi, f))' \Sigma_\xi \nabla g(A(\xi, f))). \end{aligned}$$

In particular it is easy to check that for  $p = 2$ , if  $g(x, y) = x/y$  then (5) holds immediately. This explains why the bootstrap is asymptotically valid for any ratio statistics

$$T(f) = \frac{\int_0^\pi \xi_1(\omega) f(\omega) d\omega}{\int_0^\pi \xi_2(\omega) f(\omega) d\omega},$$

for which the second coordinate of  $A(\xi, f)$  is nonzero (see Dahlhaus and Janas (1996), Lahiri (2003)). But actually this result covers more functionals for general  $p$ . Indeed, if we want to have

$$A(\xi, f)' \nabla g(A(\xi, f)) = 0$$

for any value of the parameter  $A(\xi, f)$  in an open set, then equivalently we have

$$x' \nabla g(x) = 0$$

for any  $x = (x_1, \dots, x_p)'$  in an open set. This equation is known as the Euler differential equation and is equivalent to the fact that  $g$  is homogeneous of degree zero. Let us recall that  $g$  is homogeneous of degree zero iff

$$g(\lambda x_1, \dots, \lambda x_p) = \lambda^0 g(x_1, \dots, x_p),$$

which yields by derivation in  $\lambda$  to

$$x' \nabla g(x) = 0.$$

Recall also that if at least one of the coordinates does take the value 0 (a.s), then the function  $g$  can always be expressed as a function of ratio (which explains why ratio plays such an important role in the validity of the FDB).

**Corollary 3.1.** *When  $k_4 \neq 0$  and  $g$  is differentiable in each component in the neighbourhood of  $A(\xi, f)$ , homogeneity of degree 0 of  $g$  is a necessary and sufficient condition for the FDB to work asymptotically for  $g(A(\xi, f))$  (provided that  $\nabla g(A(\xi, f)) \neq 0$ ).*

**Examples:**

1. the FDB for the spectral distribution function, of the variance estimator or of the autocovariance function  $R(k)$  fails unless the kurtosis of the innovations  $k_4 = 0$ ;
2. the FDB for autocorrelation, which can be expressed as ratio statistics, asymptotically works;
3. if  $T(f) = g(A(\xi, f))$  with  $\xi = (\xi_1, \dots, \xi_p)$  and  $g(x_1, \dots, x_p) = \prod_{i=1}^p x_i^{\alpha_i}$  with  $\sum_{i=1}^p \alpha_i = 0$ , then the FDB works asymptotically;
4. the FDB is also valid for functionals that are not directly ratios. For instance for  $p = 2$  one can take  $g(x, y) = \frac{xy}{(x^2+y^2)}$ , or for  $p = 3$ ,  $g(x, y, z) = \frac{xyz}{(x^2z+y^2x+z^2y)}$ . These functions are differentiable outside the set of points for which the denominator equals 0 and they are homogeneous of degree zero. Moreover, they all can be expressed as functions of different types of ratios.

**3.1. Second order theory for the FDB**

Janas and Dahlhaus (1996) proved the second order validity of the FDB in the particular case when the statistics of interest are ratio of linear functionals. Following their proof it is easy to show that the same result holds for any function of linear functionals that is smooth and homogenous of degree 0.

Recall that we consider the process with linear representation

$$X_t = \sum_{j=-\infty}^{\infty} a_j \zeta_{t-j} \text{ with } \sum_{j=-\infty}^{\infty} j^2 |a_j| < \infty, \quad a_0 = 1. \quad (6)$$

Its transfer function is given by

$$A(\omega) = \sum_{t=-\infty}^{\infty} a_t \exp(it\omega), \quad \omega \in [-\pi, \pi] \quad (7)$$

and the spectral density is such that for any  $\omega \in [-\pi, \pi]$ ,

$$f(\omega) = \frac{1}{2\pi} |A(\omega)|^2 > \eta, \text{ for some } \eta > 0.$$

Moreover, let  $\hat{f}$  be a tapered estimator of  $f$  of the form

$$\hat{f}(\omega) = \frac{1}{2\pi} |d_n(\omega)|^2$$

with

$$d_n(\omega) = \sum_{t=1}^n h^{(\rho)}\left(\frac{t}{n}\right) X_t \exp(-i\omega t), \quad (8)$$

where for some  $\rho \in (0, 1]$  (the proportion of tapered data), we define the taper function  $h^{(\rho)} : \mathbb{R} \rightarrow [0, 1]$  by

$$h^{(\rho)}(x) = u(x/\rho)\mathbb{I}_{\{x \in (0, \rho/2)\}} + \mathbb{I}_{\{x \in [\rho/2, 1-\rho/2]\}} + u((1-x)/\rho)\mathbb{I}_{\{x \in (1-\rho/2, 1)\}},$$

where  $u : [0, 1/2] \rightarrow [0, 1]$  is twice differentiable with bounded second order derivative,  $u(0) = 0$ ,  $u(1/2) = 1$ .

To obtain the second order validity of the FDB we consider the following assumptions.

- A1** The function  $g$  is twice differentiable, homogeneous of degree 0, such that  $\nabla g(A(\xi, f)) \neq 0$ .
- A2**  $(\zeta_t)_{t \in \mathbb{N}}$  (see (6)) are i.i.d r.v's with distribution  $P$  such that  $E_P \zeta_t = 0$ ,  $\text{Var}_P(\zeta_t) = 1$  and  $E_P \zeta_t^8 < \infty$ .
- A3**  $M_3 := E \zeta_t^3 = 0$ .
- A4** The spectral density function  $f(\omega)$  is such that

$$\inf_{\omega \in [0, \pi]} f(\omega) > 0.$$

Moreover, the tapered estimator  $\widehat{f}(\omega)$  is uniformly strongly consistent i.e.,

$$\sup_{\omega \in [0, \pi]} |\widehat{f}(\omega) - f(\omega)| \longrightarrow 0 \text{ a.s. as } n \longrightarrow \infty.$$

- A5**  $\xi = (\xi_1, \dots, \xi_p) : [0, \pi] \rightarrow \mathbb{R}^p$  is a vector of bounded functions having bounded variation, which are extended to the whole real line in a way that the extension is symmetric around 0 and periodic (with period  $2\pi$ ).
- A6** The proportion of tapered data  $\rho = \rho_n$  (see (8)) is such that

$$\rho_n \sim n^{-\delta}, \quad \delta < 1/6.$$

- A7** The filter coefficients  $\{a_j\}_{j \in \mathbb{Z}}$  and the Fourier coefficients  $\widehat{\xi}(\omega)$  of  $\xi(\omega)$  are decreasing exponentially that is, there exists  $C > 0$  such that

$$\begin{aligned} |a_j| &\leq \exp(-C|j|), \\ \|\widehat{\xi}(\omega)\|_2 &\leq \exp(-C|\omega|), \end{aligned}$$

for all large  $\omega$ .

- A8** The following Cramér condition holds: for some  $0 < \delta < 1$  and some  $M > 0$ , for any  $t = (t_1, t_2)$ ,  $\|t\| > M$ ,

$$|E_P \exp(it'(\xi_1, \xi_1^2))| < 1 - \delta.$$

- A9** Let  $W_{n,1} = n^{-1/2} (d_n(\frac{2\pi}{n} j_1), d_n(\frac{2\pi}{n} j_2), \dots, d_n(\frac{2\pi}{n} j_8))'$ ,  $(j_1, \dots, j_8) \in \{1, \dots, \frac{n}{2} - 1\}^8$  and  $W_{n,2} = \int (\xi(\omega)', 1)' I_n(\omega) d\omega$ . Then the limits of  $\text{Cov}(W_{n,1})$  and  $\text{Cov}(W_{n,2})$  exist and are nonsingular. Moreover,  $\int (\xi(\omega)', 1)' (\xi(\omega)', 1) f^2(\omega) d\omega$  is nonsingular.

Assumption **A3** is very strong. It is clearly satisfied in the Gaussian case but not for general processes. Condition **A8** is automatically satisfied if  $\xi$  has an absolutely continuous part with respect to Lebesgue measure on  $\mathbb{R}$ . Finally, **A9** ensures that the cumulants of order 4 are not degenerate and that their empirical versions are close to the true ones.

Theorem below states the second order correctness of the FDB for functions or ratio functionals of the spectral density (when standardized by the true variance).

**Theorem 3.2.** *Let **A1** – **A2** and **A4** – **A9** hold. Then we have almost surely as  $n \rightarrow \infty$  uniformly in  $x$*

$$\begin{aligned} & \Pr^* \left( \text{Var}^*(g(A_n(\xi, I_n^*)))^{-1/2} (g(A_n(\xi, I_n^*)) - g(A_n(\xi, \hat{f}_n))) \leq x \right) \\ & - \Pr \left( \text{Var}(g(A_n(\xi, I_n)))^{-1/2} (g(A_n(\xi, I_n)) - g(A(\xi, f))) \leq x \right) \\ & = -4\pi \frac{M_3^2}{\sigma^6 n^{1/2}} C(\xi) (x^2 - 1) + o(n^{-1/2}), \end{aligned}$$

where  $C(\xi)$  is a constant dependent of  $\xi$  and  $f$ . If in addition **A3** holds (the skewness of the residuals is 0), then the bootstrap is second order correct.

Theorem 3.2 shows that the FDB is asymptotically valid and/or second order correct under very specific conditions:

- for any smooth functional, the bootstrap will be second order correct if  $E\zeta_1^3 = 0$  and  $k_4 = E\zeta_1^4 / (E\zeta_1^2)^2 - 3 = 0$  that is typically in the Gaussian case;
- only homogeneous functions of degree 0 of linear functionals (including ratios of linear functionals) are candidates for the asymptotic validity when  $k_4 \neq 0$ ;
- only linear time series with i.i.d. innovations such that  $E\zeta_1^3 = 0$  can be second order correct without corrections.

Thus, one should be careful while applying the FDB method on specific functionals and should not expect second order corrections without some further modification of the procedure.

A solution to obtain second order valid confidence intervals via calibration of the quantile of the bootstrap distribution is to use the Edgeworth expansion inversion (see Abramovitz and Singh (1985)) when  $M_3 \neq 0$ . Indeed it is easy to see with their results that if one has estimator of the quantities  $M_3$ ,  $\sigma^2$  and  $C(\xi)$  say  $\widehat{M}_3$ ,  $\widehat{\sigma}^2$  and  $\widehat{C}(\xi)$  such that

$$P \left( \left| \frac{\widehat{M}_3^2}{\widehat{\sigma}^6} \widehat{C}(\xi) (T_n^2 - 1) - \frac{M_3^2}{\sigma^6} C(\xi) (T(f)^2 - 1) \right| > \varepsilon \right) = o(n^{-1/2}),$$

then we can correct either the original statistics or the bootstrap quantiles to get second order correction. However, such methods may require some complicated computations to obtain a valid estimator of  $C(\xi)$ .

#### 4. Influence function in the time domain

In this section we introduce a concept of an influence function in the time domain that will allow us later to state sufficient and necessary conditions for consistency of the FDB.

Note that the functional  $T(f) = g(A(\xi, f))$  can be seen as a functional of the spectral measure  $F$  on  $[0, \pi]$ . We have

$$A(\xi, f) = \left( \int_0^\pi \xi_1(\omega) F(d\omega), \int_0^\pi \xi_2(\omega) F(d\omega), \dots, \int_0^\pi \xi_p(\omega) F(d\omega) \right).$$

We denote  $A(\xi, f)$  and  $T(f)$  by  $\mathbb{A}(\xi, F)$  and  $\mathbb{T}(F)$ , respectively, to stress the dependence of  $F$  rather than of  $f$ . The natural estimator of  $\mathbb{T}(F)$  is simply  $\mathbb{T}(\widehat{F}_n)$ , where

$$\widehat{F}_n(\lambda) = \int_0^\lambda I_n(\omega) d\omega$$

may be also seen by extension as a positive measure

$$\widehat{F}_n([\lambda_1, \lambda_2]) = \int_{\lambda_1}^{\lambda_2} I_n(\omega) d\omega.$$

Since we know that it is easier to get asymptotic distribution of the process  $\sqrt{n}(\widehat{F}_n - F)$ , than of the corresponding process based on non-integrated periodogram, it is natural to try to study the differentiability property of  $T$  in the time domain, to get an analogue of the functional delta-method. In this case we will see that it is possible to introduce a contaminated version of  $F$  by some Dirac measure to compute an equivalent of the influence function but in the time domain.

Let  $\mathbb{T}$  be a functional defined on a vectorial space of positive measure on  $[0, \pi]$  including Dirac measures, denoted by  $\mathcal{F}$ . We define the following notion of influence function in the time domain analogously to the i.i.d. case, but on spectral measures instead of cdf's.

**Definition 4.1.** *Let  $\mathbb{T} : \mathcal{F} \rightarrow \mathbb{R}^K$  be a functional of spectral measures. The uncentered influence function (with value in  $\mathbb{R}^K$ ) or the first order gradient of  $\mathbb{T}$  in the periodic direction  $\omega_0$  is given by*

$$T^{(1)}(\omega_0, F) = \left. \frac{\partial \mathbb{T}(F + \varepsilon \delta_{\omega_0})}{\partial \varepsilon} \right|_{\varepsilon=0}.$$

It is known that the purely periodic process

$$\eta_t(\omega_0) = A \cos(\omega_0 t) + B \sin(\omega_0 t),$$

where  $EA = EB = 0$  and  $\text{Var}(A) = \text{Var}(B) = 1$ , has spectral measure given by

$$h_{\omega_0} = \frac{\delta_{-\omega_0} + \delta_{\omega_0}}{2}.$$

Then for a general process  $X_t$  with spectral measure  $F$  on  $[0, \pi]$ , the contaminated process  $X_t + \sqrt{2\varepsilon}\eta_t$ , where  $X_t$  and  $\eta_t$  are independent, has spectral measure  $F + \varepsilon\delta_{\omega_0}$  on  $[0, \pi]$ . Thus, the influence function can be interpreted as the infinitesimal variation of the functional  $\mathbb{T}(F)$  when  $F$  is contaminated by a purely periodic process with variance going to 0 at rate  $\sqrt{2\varepsilon}$ .

**Examples:**

1. The linear functional  $\mathbb{A}(\xi, F)$  has influence function given by

$$T^{(1)}(\omega_0, F) = (\xi_1(\omega_0), \dots, \xi_p(\omega_0)).$$

Notice that in general

$$\int_0^\pi T^{(1)}(\omega_0, f)F(d\omega) \neq 0.$$

A particular case is the covariance function of order  $k$

$$R(k) = \mathbb{E}X_t X_{t+k} = \int_0^\pi \cos(\omega k) f(\omega) d\omega,$$

for which we have

$$T^{(1)}(\omega_0, f) = \cos(\omega_0 k)$$

and for some  $k$

$$\int_0^\pi \cos(\omega k) f(\omega) d\omega \neq 0.$$

2. *Whittle estimators.*

For simplicity all the calculus are made for  $\theta \in \mathbb{R}$ , but the final result actually also holds for the multidimensional case. A Whittle estimator can be seen as a M-estimator that is a solution of the equation

$$\int_0^\pi \frac{\dot{f}_\theta(\omega)}{f_\theta(\omega)} \left( \frac{I_n(\omega) - f_\theta(\omega)}{f_\theta(\omega)} \right) d\omega = 0$$

or equivalently

$$\int_0^\pi \frac{\dot{f}_\theta(\omega)}{f_\theta(\omega)^2} \widehat{F}_n(d\omega) - \int_0^\pi \frac{\dot{f}_\theta(\omega)}{f_\theta(\omega)} d\omega = 0,$$

where

$$\dot{f}_\theta(\omega) = \frac{\partial f_\theta(\omega)}{\partial \theta}.$$

Thus, we can define  $\theta = \mathbb{T}(F)$ , the functional solution of the equation

$$\begin{aligned} 0 &= \int_0^\pi \frac{\dot{f}_\theta(\omega)}{f_\theta(\omega)^2} F(d\omega) - \int_0^\pi \frac{\dot{f}_\theta(\omega)}{f_\theta(\omega)} d\omega \\ &= \int_0^\pi \frac{\dot{f}_\theta(\omega)}{f_\theta(\omega)^2} F(d\omega) - \int_0^\pi \frac{\dot{f}_\theta(\omega)}{f_\theta(\omega)^2} F_\theta(d\omega) \end{aligned}$$

or similarly

$$\int_0^\pi \frac{\partial}{\partial \theta} \left( \frac{1}{f_\theta(\omega)} \right) (F - F_\theta)(d\omega) = 0.$$

To compute the influence function of  $\mathbb{T}(F)$ , we consider  $\theta_\varepsilon^\circ = \mathbb{T}(F + \varepsilon\delta_{\omega_0})$  the solution of the equation

$$\int_0^\pi \frac{\dot{f}_{\theta_\varepsilon^\circ}(\omega)}{f_{\theta_\varepsilon^\circ}(\omega)^2} F(d\omega) + \varepsilon \frac{\dot{f}_{\theta_\varepsilon^\circ}(\omega_0)}{f_{\theta_\varepsilon^\circ}(\omega_0)^2} - \int_0^\pi \frac{\dot{f}_{\theta_\varepsilon^\circ}(\omega)}{f_{\theta_\varepsilon^\circ}(\omega)} d\omega = 0.$$

Then calculating the derivative with respect to  $\varepsilon$ , we get that

$$\begin{aligned} & - \left( \int_0^\pi \frac{\partial^2}{\partial \theta^2} \left( \frac{1}{f_\theta(\omega)} \right) F(d\omega) \right) T^{(1)}(\omega_0, F) \\ & + \frac{\dot{f}_\theta(\omega_0)}{f_\theta(\omega_0)^2} - \left( \int_0^\pi \frac{\partial^2 \log(f_\theta(\omega))}{\partial \theta^2} d\omega \right) T^{(1)}(\omega_0, F) = 0, \end{aligned}$$

which yields the influence function

$$\begin{aligned} T^{(1)}(\omega_0, F) &= \left( \int_0^\pi \frac{\partial^2}{\partial \theta^2} \left( \frac{1}{f_\theta(\omega)} \right) F(d\omega) + \int_0^\pi \frac{\partial^2 \log(f_\theta(\omega))}{\partial \theta^2} d\omega \right)^{-1} \\ &\cdot \left( - \frac{\partial}{\partial \theta} \left( \frac{1}{f_\theta(\omega_0)} \right) \right). \end{aligned} \quad (9)$$

Moreover, one may note that in the particular case of ARMA or FARIMA models, we have  $\int_0^\pi \frac{\partial^2 \log(f_\theta(\omega))}{\partial \theta^2} d\omega = 0$  and  $\int_0^\pi \frac{\partial}{\partial \theta} \left( \frac{1}{f_\theta(\omega)} \right) F(d\omega) = 0$  (see the comment under Remark 1, p. 409 of Kim and Nordmann, 2013). Thus, we get

$$T^{(1)}(\omega_0, F_\theta) = - \left( \int_0^\pi \frac{\partial^2}{\partial \theta^2} \left( \frac{1}{f_\theta(\omega)} \right) F_\theta(d\omega) \right)^{-1} \frac{\partial}{\partial \theta} \left( \frac{1}{f_\theta(\omega_0)} \right).$$

### 3. General contrasts.

Notice that the Whittle estimators are a simple case of contrast estimators satisfying some estimating equation

$$\int_0^\pi \psi(\omega, F_\theta)(F - F_\theta)(d\omega) = 0.$$

Assuming that  $\psi$  is twice differentiable and that there exists a unique solution to this problem for any  $F$  in  $\mathcal{F}$ , it is easy to compute the corresponding influence function. Consider  $\theta_\varepsilon^\circ = \mathbb{T}(F + \varepsilon\delta_{\omega_0})$  the solution of

$$\begin{aligned} 0 &= \int_0^\pi \psi(\omega, F_{\theta_\varepsilon^\circ})(F + \varepsilon\delta_{\omega_0} - F_{\theta_\varepsilon^\circ})(d\omega) \\ &= \int_0^\pi \psi(\omega, F_{\theta_\varepsilon^\circ})F(d\omega) + \varepsilon\psi(\omega_0, F_{\theta_\varepsilon^\circ}) - \int_0^\pi \psi(\omega, F_{\theta_\varepsilon^\circ})F_{\theta_\varepsilon^\circ}(d\omega). \end{aligned}$$



By derivation, we get

$$0 = T^{(1)}(\omega_0, F) \int_0^\pi \psi'(\omega, F_\theta) F(d\omega) + \psi(\omega_0, F_\theta) \\ - T^{(1)}(\omega_0, F) \left( \int_0^\pi \psi'(\omega, F_\theta) F_\theta(d\omega) + \int_0^\pi \psi(\omega, F_\theta) \dot{f}_\theta(\omega) d\omega \right)$$

and hence the influence function is of the form

$$T^{(1)}(\omega_0, F) = - \left( \int_0^\pi \psi'(\omega, F_\theta) F(d\omega) - \int_0^\pi \psi'(\omega, F_\theta) F_\theta(d\omega) \\ - \int_0^\pi \psi(\omega, F_\theta) \dot{f}_\theta(\omega) d\omega \right)^{-1} \psi(\omega_0, F_\theta).$$

The main benefit of having the influence function is that it allows to linearize the functional of interest. Typically we expect that

$$\mathbb{T}(\widehat{F}_n) - \mathbb{T}(F) = \int_0^\pi T^{(1)}(\omega, F)(\widehat{F}_n - F)(d\omega) + R_n,$$

where  $R_n$  is a remainder, which needs to be controlled either by choosing an adequate metric or directly by hand. In many applications (e.g., Whittle estimator) this remainder is typically of order  $R_n = o_P(n^{-1/2})$ . As a consequence the limiting behavior of  $\sqrt{n}(\mathbb{T}(\widehat{F}_n) - \mathbb{T}(F))$  is determined by the linear part i.e.,  $\int_0^\pi T^{(1)}(\omega, F)(\widehat{F}_n - F)(d\omega)$ .

## 5. Sufficient and necessary conditions for asymptotic validity of FDB

To establish conditions for the consistency of the FDB, we introduce below a notion of Fréchet differentiability of functionals of spectral measures. For this purpose we first endow the space  $\mathcal{F}$  with a metric  $d$  between measures. We assume that this metric is compatible with the linear structure of the space i.e., that we have  $d(F + \varepsilon G, F) \leq |\varepsilon| C_{G,F}$  for some constant  $C_{G,F}$  depending on  $G$  and  $F$ .

**Definition 5.1.** Let  $\mathbb{T} : \mathcal{F} \rightarrow \mathbb{R}^K$  be a functional (with no constraint on the total mass of these measures).  $\mathbb{T}$  is said to be Fréchet differentiable on  $F \in \mathcal{F}$  for the metric  $d$ , with gradient  $g^{(1)}$  iff there exists a linear continuous operator  $D\mathbb{T}^{(1)} : \mathcal{F} \rightarrow \mathbb{R}^K$  and a continuous function  $r : \mathbb{R} \rightarrow \mathbb{R}^K$  with  $r(0) = 0$  such that for any  $G \in \mathcal{F}$

$$\mathbb{T}(G) - \mathbb{T}(F) = D\mathbb{T}^{(1)}(G - F) + r(d(G, F))d(G, F),$$

where

$$D\mathbb{T}^{(1)}(G - F) = \int_0^\pi g^{(1)}(\omega)(G - F)(d\omega).$$

**Remark 5.1.** *Exhibiting the correct metric, which makes a functional Fréchet differentiable, is a challenging task. In the i.i.d. case such a task was considered in Barbe and Bertail (1995), Dudley (1990) who proposed to use Zolotarev metrics indexed by class of functions. This idea can be also adapted in our framework since we know from Dahlhaus (1988) that an empirical process indexed by classes of functions in the frequency domain behaves like an empirical process for i.i.d. data under some entropy metric condition on the class  $\mathcal{F}$ . The question of the validity of the FDB then comes down to study separately the validity of the FDB of the linear part and the rate of convergence of the residual part (which is considered in Section 6).*

**Lemma 5.1.** *If  $\mathbb{T} : \mathcal{F} \rightarrow \mathbb{R}^K$  is Fréchet differentiable at  $F$  for the metric  $d$ , then  $T^{(1)}(\omega, F)$  is a gradient of  $\mathbb{T}$  and we have*

$$\mathbb{T}(\widehat{F}_n) - \mathbb{T}(F) = \int_0^\pi T^{(1)}(\omega, F)(\widehat{F}_n - F)(d\omega) + r(d(\widehat{F}_n, F))d(\widehat{F}_n, F).$$

A von Mises' type of theorem follows immediately from the representation above. Moreover, under an additional assumption controlling the behavior of the remainder evaluated at  $\widehat{F}_n^*$ , we establish a necessary and sufficient condition for the asymptotic validity of the FDB of a non-degenerate general functional.

**Theorem 5.1.** *Assume that the assumptions (i)-(iii) of Theorem 3.1 hold and that  $\mathbb{T} : \mathcal{F} \rightarrow \mathbb{R}^K$  is Fréchet differentiable at  $F$  for the metric  $d$ , with influence function  $T^{(1)}(\omega, F)$ . If  $d(\widehat{F}_n, F) = O_P(n^{-1/2})$  and  $0 < \int_0^\pi T^{(1)}(\omega, F_\theta)T^{(1)}(\omega, F_\theta)' f_\theta(\omega)^2 d\omega < \infty$ , then we have*

$$\begin{aligned} \sqrt{n}(\mathbb{T}(\widehat{F}_n) - \mathbb{T}(F)) &\xrightarrow[n \rightarrow \infty]{d} N\left(0, 2\pi \int_0^\pi T^{(1)}(\omega, F_\theta)T^{(1)}(\omega, F_\theta)' f_\theta(\omega)^2 d\omega \right. \\ &\quad \left. + \frac{k_4}{\sigma^4} \left( \int_0^\pi T^{(1)}(\omega, F_\theta) f_\theta(\omega) d\omega \right) \left( \int_0^\pi T^{(1)}(\omega, F) f(\omega) d\omega \right)'\right). \end{aligned}$$

If additionally  $d(\widehat{F}_n^*, F) = O_P(n^{-1/2})$  in probability along the sample, we have

$$\sqrt{n}(\mathbb{T}(\widehat{F}_n^*) - \mathbb{T}(\widehat{F}_n)) \xrightarrow[n \rightarrow \infty]{d} N\left(0, 2\pi \int_0^\pi T^{(1)}(\omega, F_\theta)T^{(1)}(\omega, F_\theta)' f_\theta(\omega)^2 d\omega\right)$$

in probability along the sample. Then the bootstrap is asymptotically valid iff

$$k_4 \int_0^\pi T^{(1)}(\omega, F_\theta) f_\theta(\omega) d\omega = 0. \quad (10)$$

**Remark 5.2.** *The condition (10) essentially means that either  $k_4 = 0$ , which is true in the Gaussian case, or the influence function is centered. As already noticed the latter may not be the case with our definition of the influence function as shown in the following.*

### Examples:

1. The FDB for a linear functional with only one function ( $p = 1$ ) i.e., for  $A(\xi_1, F)$ , is valid iff  $k_4 \int_0^\pi \xi_1(\omega) f_\theta(\omega) d\omega = 0$ . In particular one can notice that for the autocovariance the FDB does not work.
2. The influence function of the ratio  $T(F) = \int_0^\pi \xi_1(\omega) F(d\omega) / \int_0^\pi \xi_2(\omega) F(d\omega)$  is given by

$$\begin{aligned} T^{(1)}(\omega_0, F_\theta) &= \frac{d}{d\varepsilon} \left( \frac{\int_0^\pi \xi_1(\omega) F(d\omega) + \varepsilon \xi_1(\omega_0)}{\int_0^\pi \xi_2(\omega) F(d\omega) + \varepsilon \xi_2(\omega_0)} \right) \Bigg|_{\varepsilon=0} \\ &= \frac{\xi_1(\omega_0)}{\int_0^\pi \xi_2(\omega) F(d\omega)} - \frac{\xi_2(\omega_0) \int_0^\pi \xi_1(\omega) F(d\omega)}{\left( \int_0^\pi \xi_2(\omega) F(d\omega) \right)^2} \\ &= \frac{\xi_1(\omega_0) - T(F) \xi_2(\omega_0)}{\int_0^\pi \xi_2(\omega) F(d\omega)}. \end{aligned}$$

Note that the influence function is automatically centered.

3. *Whittle estimators.*

In this case, for ARMA or FARIMA models, we have the recentering property

$$\int_0^\pi T^{(1)}(\omega, F_\theta) F_\theta(d\omega) = 0.$$

$T^{(1)}(\omega_0, F_\theta)$  is precisely the linear part obtained in Dahlaus and Janas (1996). It is automatically centered under their assumptions. This explains why the limiting distribution does not depend on  $k_4$  and why the bootstrap works (asymptotically) in that case. However, notice that in models where the variance depends on  $\theta$ , the influence function given by (9) should be considered and may not be centered, so that the bootstrap may fail in that case!

4. *(continuation of Example 3 from Section 4)*

If the M-estimator is constructed in the way that

$$\int_0^\pi \psi(\omega, F_\theta) f_\theta(\omega) d\omega = 0,$$

then under the assumptions of Theorem 5.1 the FDB is valid. In particular notice that the Whittle estimator satisfies this property.

## 6. Invalidity/validity of the bootstrap for empirical processes in the time domain

Empirical spectral processes indexed by some class of real functions  $\mathcal{F}$  satisfying some integrability conditions are studied in Dahlhaus (1988). The framework in the time domain is a bit different from the usual one. We consider class of functions of the following form

$$\mathcal{F} = \left\{ h : [0, \pi] \rightarrow \mathbb{R} \text{ such that } \int_0^\pi h(w)^2 f(w)^2 dw < \infty \right\}.$$

We are interested in the behavior of the (infinite) dimensional vectors of the form

$$\left\{ n^{1/2} \left( \int h(\omega) I_n(\omega) - \int h(\omega) f(\omega) d\omega \right), h \in \mathcal{F} \right\}$$

and more precisely in the discretized version of this quantity at the Fourier frequencies. We put

$$Z_n(h) = \frac{2\pi}{n^{1/2}} \left( \sum_{j=1}^{n_0} h(\lambda_{jn}) I_n(\lambda_{jn}) - \sum_{j=1}^{n_0} h(\lambda_{jn}) f(\lambda_{jn}) \right), h \in \mathcal{F}.$$

Let us introduce  $d_{\mathcal{F}}$  the pseudo-distance between spectral densities defined by

$$d_{\mathcal{F}}(f_1, f_2) = \sup_{h \in \mathcal{F}} \left\{ \left| \int h(\omega) f_1(\omega) d\omega - \int h(\omega) f_2(\omega) d\omega \right| \right\}.$$

In the following, we will also be interested in the rate of convergence  $d_{\mathcal{F}}(I_n, f)$ . Indeed typically to ensure that Theorem 5.1 yields a CLT or to study general M-estimators including the Whittle estimators, we end-up with controlling  $d_{\mathcal{F}}(I_n, f)$  for a specific class of function (see examples in Dahlhaus and Polonik (2002)). Moreover, we want to check under which conditions we have convergence of the bootstrap versions to the same limit or at least when we have  $d(\widehat{F}_n^*, F) = O_P(n^{-1/2})$  as assumed in Theorem 5.1 for this kind of metrics. Additionally, we introduce the computable discretized version of  $d_{\mathcal{F}}(f_1, f_2)$  given by

$$d_{\mathcal{F},n}(f_1, f_2) = \frac{2\pi}{n} \sup_{h \in \mathcal{F}} \left| \sum_{j=1}^{n_0} h(\lambda_{jn}) (f_1(\lambda_{jn}) - f_2(\lambda_{jn})) \right|,$$

which obviously converges to  $d_{\mathcal{F}}(f_1, f_2)$  as  $n \rightarrow \infty$ .

As in Dahlhaus (1988) the process  $(Z_n(h), h \in \mathcal{F})$  is a random element of  $l^\infty(\mathcal{F})$  (the space of all bounded functions from  $\mathcal{F}$  to  $\mathbb{R}$ ) equipped with the metric,  $\|z\|_{\mathcal{F}} = \sup_{h \in \mathcal{F}} |z(h)|$ . Moreover, as in the usual case  $(l^\infty(\mathcal{F}), \|z\|_{\mathcal{F}})$  is a (generally non-separable) Banach space. It was proven in Dahlhaus (1988) that under the conditions discussed below  $(Z_n(h))_{h \in \mathcal{F}}$  converges to a Gaussian process in  $l^\infty(\mathcal{F})$ .

Define the semi-metric on  $\mathcal{F}$

$$\rho_2(h, g) = \int_0^\pi (h(\omega) - g(\omega))^2 f(\omega)^2 d\omega.$$

When  $f$  is bounded (as will be the case later), it is possible to use

$$\tilde{\rho}_2(h, g) = \int_0^\pi (h(\omega) - g(\omega))^2 d\omega$$

as in Dahlhaus and Polonik (2002). However, notice that if one wants to generalize the results presented below to fractional stationary times series with a singularity at 0, then  $\rho_2$  should be used.

The bracketing number  $N(\delta, \mathcal{F}, \rho_2)$  is defined as the smallest number  $m$  such that, there exist functions  $g_1, g_2, \dots, g_m \in \mathcal{F}$  such that for any  $g \in \mathcal{F}$ ,  $\inf_{1 \leq i \leq m} \rho_2(g, g_i) \leq \delta$ . At this point we refer the reader to the discussion in Dahlhaus and Polonik (2002) explaining the link between bracketing numbers and regular covering numbers (the number of balls needed to cover  $\mathcal{F}$  with balls of size  $\delta$ ) and providing examples of calculus of this quantity for many classes of functions. In many examples, when  $f$  is bounded,

$$N(\delta, \mathcal{F}, \rho_2) \leq N(\delta, \mathcal{F}, \tilde{\rho}_2)$$

and these quantities can be bounded by a polynomial  $C\delta^{-V}$  for some positive constants  $C$  and  $V$ .

Following Dahlhaus (1988) we assume the following conditions:

- B0** the process  $(X_t)_{t \in \mathbb{Z}}$  is strictly stationary and centered;
- B1** function  $f$  is continuous and Hölder of order  $k > 1/2$  (and thus bounded) that is, for some positive constant  $K$

$$|f(\omega_1) - f(\omega_2)| \leq K|\omega_1 - \omega_2|^k;$$

- B2** the fourth order spectrum is continuous and the spectrum of all order  $m \geq 2$  are bounded by  $C^m$ , where  $C$  is some positive constant;
- B3**  $(\mathcal{F}, \rho_2)$  is totally bounded and is a permissible subset of the set of all real functions such that  $\int_0^\pi h(w)^2 f(\omega)^2 d\omega < \infty$ . Moreover, there exists an envelop  $H$  such that  $|h(\omega)| \leq H(\omega)$ ,  $\omega \in [0, \pi]$  with

$$\int_0^\pi H(w)^2 f(\omega) d\omega < \infty;$$

- B4** the covering number satisfies the integrability condition

$$\int_0^1 (\log(N(\delta, \mathcal{F}, \rho_2)))^2 < \infty;$$

- B5** the spectral density estimator  $\hat{f}_n$  converges to  $f$  uniformly over  $[0, \pi]$ .

In this section similarly to Dahlhaus (1988) we make a stronger assumption **B0** on the generating process. The reason for this is that exponential inequalities (here on weighted sums of the periodogram at the Fourier frequencies) are essential to obtain maximal inequalities and to control increments of empirical processes as was done in the i.i.d. case. Condition **B4** comes from the paper by Dahlhaus and Polonik (2002) who have improved the original condition of Dahlhaus (1988). Notice that **B4** is stronger than the usual assumption (in the i.i.d. case) which would rather be of the form  $\int_0^1 (\log(N(\delta, \mathcal{F}, \rho_2)))^{1/2} < \infty$ . This

means that  $N(\delta, \mathcal{F}, \rho_2)$  should be of the order much smaller than  $\exp(\delta^{-1/2})$  (rather than the "Gaussian" rate  $\exp(\delta^{-2})$ ). In most applications (in particular if the bracketing number is polynomial) this condition will be satisfied. This restriction is due to the fact that the exponential inequalities obtained in this framework are typically proved with a suboptimal rate of order  $\exp(-t^{1/2})$  instead of the Gaussian rate  $\exp(-t^2)$ .

In the following we consider the bootstrap version of  $(Z_n(f), h \in \mathcal{F})$ , say  $(Z_n^*(f), h \in \mathcal{F})$ , obtained using the standard FDB procedure described before (either with a parametric estimator of the spectral density or with a non-parametric one), where

$$Z_n^*(h) = \frac{2\pi}{n^{1/2}} \left( \sum_{j=1}^{n_0} h(\lambda_{jn}) I_n^*(\lambda_{jn}) - \sum_{j=1}^{n_0} h(\lambda_{jn}) I_n(\lambda_{jn}) \right).$$

**Theorem 6.1.** *Under assumptions **B0-B5**, the empirical spectral process  $(Z_n(h), h \in \mathcal{F})$  converges in  $(l^\infty(\mathcal{F}), \|\cdot\|_{\mathcal{F}})$  to a centered Gaussian process with continuous sample paths and covariance operator, given, for all  $h \in \mathcal{F}$ ,  $g \in \mathcal{F}$  by*

$$c(h, g) = 2\pi \int_0^\pi h(\omega)g(\omega)f(\omega)^2 d\omega + \frac{k_4}{\sigma^4} \int_0^\pi h(\omega)f(\omega) d\omega \int_0^\pi g(\omega)f(\omega) d\omega.$$

*Moreover, the bootstrap empirical spectral process  $Z_n^*(h)$  also converges in  $(l^\infty(\mathcal{F}), \|\cdot\|_{\mathcal{F}})$  to a (different) centered Gaussian process with continuous sample paths and covariance operator given, for all  $h \in \mathcal{F}$ ,  $g \in \mathcal{F}$ ,*

$$c_1(h, g) = 2\pi \int_0^\pi h(\omega)g(\omega)f(\omega)^2 d\omega.$$

*As a consequence the FDB of the empirical spectral process is only asymptotically valid on classes of functions satisfying the additional conditions, for ALL  $h \in \mathcal{F}$*

$$k_4 \int_0^\pi h(\omega)f(\omega) d\omega = 0.$$

*Moreover, in any case we have*

$$d_{\mathcal{F}}(I_n, f) = O_P(n^{-1/2})$$

*and*

$$d_{\mathcal{F}}(I_n^*, I_n) = O_P(n^{-1/2}) \text{ in probability along the sample.}$$

## 7. Summary and conclusions

In this paper we provided sufficient and necessary conditions for the consistency of the FDB in the case of linear stationary time series. For this purpose we defined the influence function in the time domain on spectral measures, which

allowed us to linearize the functional of interest. Moreover, we introduced a notion of Fréchet differentiability of functionals of spectral measures. We discussed consistency of the FDB and its second order correctness for differentiable functionals of spectral density function. Finally, we stated sufficient and necessary conditions for the FDB validity in the case of empirical processes.

Our results allow to understand why the FDB is valid for some functionals (e.g., the Whittle estimator) or empirical processes and for which functionals or empirical processes it can be consistent. For instance only homogeneous functions of degree 0 of linear functionals (including ratios of linear functionals) are candidates for the asymptotic validity of the FDB when kurtosis of innovations is equal to zero. Moreover, we indicated that the second order correctness can be obtained only for linear time series with i.i.d. innovations such that  $E\zeta_1^3 = 0$ . Thus, one should carefully apply the FDB method for particular functionals.

## Appendix

### Proof of Theorem 3.2

The cumulants of the true distribution and the bootstrap one differ essentially in the term in  $M_3^2$  appearing in the cumulants of order three of the statistics of interest (see (3.9) and Lemma 2 of Dalhaus and Janas (1996)). As noticed in Theorem 1 of the same authors the other terms match in the expansion if  $M_3 = 0$ .  $\square$

### Proof of Lemma 3.2

Proof of Theorem 3.2 follows the same reasoning as presented in Dahlhaus and Janas (1996) and hence we skip the technical details.  $\square$

### Proof of Lemma 5.1

Since  $\mathbb{T}$  is Fréchet differentiable with gradient  $g^{(1)}$ , we get

$$\begin{aligned} \mathbb{T}(F + \varepsilon\delta_{\omega_0}) - \mathbb{T}(F) &= \varepsilon \int_0^\pi g^{(1)}(\omega)\delta_{\omega_0}(d\omega) + r(d(F + \varepsilon\delta_{\omega_0}, F))d(F + \varepsilon\delta_{\omega_0}, F) \\ &= \varepsilon g^{(1)}(\omega_0) + o(\varepsilon) \end{aligned}$$

and by definition

$$\frac{(\mathbb{T}(F + \varepsilon\delta_{\omega_0}) - \mathbb{T}(F))}{\varepsilon} \longrightarrow g^{(1)}(\omega_0) = T^{(1)}(\omega_0, F) \quad \text{as } \varepsilon \longrightarrow 0.$$

$\square$

### Proof of Theorem 5.1

The representation and the hypothesis on the metric imply that

$$\sqrt{n}(\mathbb{T}(\widehat{F}_n) - \mathbb{T}(F)) = \sqrt{n} \int_0^\pi T^{(1)}(\omega, F)(\widehat{F}_n - F)(d\omega) + o_P(1).$$

Then applying Corollary 4.1. from Kreiss and Paparoditis (2003) one gets the asymptotic distribution.

Similarly for the bootstrap, by applying the Fréchet differentiability assumption twice we have

$$\sqrt{n}((\mathbb{T}(\widehat{F}_n^*) - T(F)) + (T(F) - \mathbb{T}(\widehat{F}_n))) = \sqrt{n} \int_0^\pi T^{(1)}(\omega, F)(\widehat{F}_n^* - \widehat{F}_n)(d\omega) + o_P(1).$$

Applying Theorem 3.1 to the linear part one gets the limiting distribution

$$N\left(0, 2\pi \int_0^\pi T^{(1)}(\omega, F_\theta) T^{(1)}(\omega, F_\theta)' f_\theta(\omega)^2 d\omega\right),$$

which coincides with the distribution of  $\sqrt{n}(\mathbb{T}(\widehat{F}_n) - \mathbb{T}(F))$  iff

$$\left(\int_0^\pi T^{(1)}(\omega, F_\theta) f_\theta(\omega) d\omega\right) \left(\int_0^\pi T^{(1)}(\omega, F) f(\omega) d\omega\right)' = 0.$$

The last condition implies (by taking the trace) that

$$\left\|\int_0^\pi T^{(1)}(\omega, F_\theta) f_\theta(\omega) d\omega\right\|^2 = 0,$$

which ends the proof of the theorem.  $\square$

### Proof of Theorem 6.1

The proof follows standard arguments from the empirical process literature.

The marginal distribution converges obviously to the marginal distribution of the limit process by Theorem 3.1. The result concerning  $(Z_n(h), h \in \mathcal{F})$  is a special case of Dahlhaus (1988) (with no tapering and for the univariate time series). Thus, essentially we have to prove the result for the bootstrap empirical process.

Notice that we have

$$\begin{aligned} Z_n^*(h) &= \frac{2\pi}{n^{1/2}} \left( \sum_{j=1}^{n_0} h(\lambda_{jn}) \widehat{f}_n(\lambda_{jn})(\varepsilon_{jn}^* - 1) - \sum_{j=1}^{n_0} h(\lambda_{jn})(I_n(\lambda_{jn}) - \widehat{f}_n(\lambda_{jn})) \right) \\ &= I + II + III \end{aligned}$$

with

$$\begin{aligned} I &= \frac{2\pi}{n^{1/2}} \sum_{j=1}^{n_0} h(\lambda_{jn}) f(\lambda_{jn})(\varepsilon_{jn}^* - 1), \\ II &= \frac{2\pi}{n^{1/2}} \sum_{j=1}^{n_0} h(\lambda_{jn})(\widehat{f}_n(\lambda_{jn}) - f(\lambda_{jn}))(\varepsilon_{jn}^* - 1), \\ III &= \frac{2\pi}{n^{1/2}} \sum_{j=1}^{n_0} h(\lambda_{jn})(I_n(\lambda_{jn}) - \widehat{f}_n(\lambda_{jn})). \end{aligned}$$



We are going to show that  $II$  and  $III$  are uniformly small of order  $o_P(1)$ , so that the limiting distribution is essentially given by  $I$ . But notice that this is simply a process with i.i.d. random variables and hence it is sufficient to verify for instance the assumptions of Theorem 2.11.9 (p. 211) of van der Vaart and Wellner (1996) to get the convergence to the Gaussian process given before. We have that

$$|II| \leq \sup |\widehat{f}_n(\omega) - f(\omega)| \frac{2\pi}{n} \sum_{j=1}^{n_0} H(\lambda_{jn}) |\varepsilon_{jn}^* - 1|.$$

Since the  $\varepsilon_{jn}^*$  are i.i.d, we get that  $\frac{2\pi}{n} \sum_{j=1}^{n_0} H(\lambda_{jn}) |\varepsilon_{jn}^* - 1| = O_P(1)$  in probability along the sample and by condition **B5** we obtain the uniform convergence. Moreover,

$$\begin{aligned} \text{Var}(III) &= (2\pi)^2 \frac{1}{n^2} \sum_{j=1}^{n_0} h(\lambda_{jn})^2 \text{Var} \left( I_n(\lambda_{jn}) - \widehat{f}_n(\lambda_{jn}) \right) \\ &+ (2\pi)^2 \frac{1}{n^2} \sum_{j=1}^{n_0} \sum_{k=1, k \neq j}^{n_0} h(\lambda_{jn}) h(\lambda_{kn}) \text{Cov} \left( I_n(\lambda_{jn}) - \widehat{f}_n(\lambda_{jn}), I_n(\lambda_{kn}) - \widehat{f}_n(\lambda_{kn}) \right). \end{aligned}$$

Note that the the first term on the right-hand side in the above expression is bounded by

$$(2\pi)^2 \frac{1}{n^2} \sum_{j=1}^{n_0} H(\lambda_{jn})^2 \text{Var}(I_n(\lambda_{jn}) - \widehat{f}_n(\lambda_{jn})) = O(n^{-1}),$$

because  $\int H(\omega)^2 f(\omega)^2 d\omega < \infty$ .

One can easily show that that uniformly in  $j$  and  $k$

$$\text{Cov}(I_n(\lambda_{jn}) - \widehat{f}_n(\lambda_{jn}), I_n(\lambda_{kn}) - \widehat{f}_n(\lambda_{kn})) = o(1).$$

Since  $\frac{2\pi}{n^2} \sum_{j=1}^{n_0} \sum_{k=1, k \neq j}^{n_0} H(\lambda_{jn}) H(\lambda_{kn})$  converges to  $\int_0^\pi H(\omega) f(\omega) d\omega$ , we get that the second term on the right-hand side of  $\text{Var}(III)$  is of order  $o(1)$  uniformly in  $h$ .

Thus, it follows that uniformly in  $h$ , we have that  $II + III = o_P(1)$  along the sample.

To prove the conclusion of the theorem, now we investigate the behaviour of  $I$ . Note that the entropy condition is automatically satisfied under the stronger entropy condition (needed for  $Z_n(f)$ ) since we have by the Cauchy-Schwartz inequality

$$\int_0^1 (\log(N(\delta, \mathcal{F}, \rho_2)))^{1/2} < \left( \int_0^1 (\log(N(\delta, \mathcal{F}, \rho_2)))^2 \right)^{1/4} < \infty.$$

The Lindeberg-Feller condition of Theorem 2.11.9 p. 211 of van der Vaart and Wellner (1996) is fulfilled because the moments of order 3 of  $|\varepsilon_{jn}^* - 1|$  are finite

in probability along the sample. Thus, it is sufficient to verify an equicontinuity condition. Notice that we have

$$\begin{aligned} & \sup_{h,g \in \mathcal{F}, \rho_2(h,g) \leq \eta_n} \mathbf{E}^* |Z_n^*(h) - Z_n^*(g)|^2 \\ &= \sup_{h,g \in \mathcal{F}, \rho_2(h,g) \leq \eta_n} \frac{(2\pi)^2}{n} \sum_{j=1}^{n_0} (h(\lambda_{jn}) - g(\lambda_{jn}))^2 f(\lambda_{jn})^2 \mathbf{E}^* (\varepsilon_{jn}^* - 1)^2. \end{aligned}$$

Note that  $\mathbf{E}^*(\varepsilon_{jn}^* - 1)^2$  is converging to 1 in probability, it is bounded and as  $n \rightarrow \infty$

$$\frac{2\pi}{n} \sum_{j=1}^{n_0} (h(\lambda_{jn}) - g(\lambda_{jn}))^2 f(\lambda_{jn})^2 \rightarrow \rho_2(f, g).$$

This convergence is uniform over the set  $\mathcal{F}_{\eta_n} = \{h, g \in \mathcal{F}, \rho_2(h, g) \leq \eta_n\}$ , because by the same arguments as in van der Vaart and Wellner(1996) p. 128, we have that

$$N(\delta, \mathcal{F}_{\eta_n}, \rho_2) \leq 4N(\delta/2, \mathcal{F}, \rho_2)^2 < \infty,$$

ensuring the validity of the Glivenko-Cantelli theorem over the class  $\mathcal{F}_{\eta_n}$ . It follows that for  $n$  large enough, there exists a constant  $C > 0$  such that

$$\sup_{h,g \in \mathcal{F}, \rho_2(h,g) \leq \delta} \mathbf{E} |Z_n^*(h) - Z_n^*(g)|^2 \leq C \delta_n \quad \text{in probability,}$$

which converges to 0 when  $\delta_n \rightarrow 0$ .

Thus,  $Z_n^*(h)$  converges in  $(l^\infty(\mathcal{F}), \|\cdot\|_{\mathcal{F}})$  to a (different) centered Gaussian process with covariance given by the limit of covariance, for all  $h \in \mathcal{F}$ ,  $g \in \mathcal{F}$ ,

$$\begin{aligned} & \frac{1}{n} \text{Cov}^* \left( \sum_{j=1}^{n_0} h(\lambda_{jn}) f(\lambda_{jn}) (\varepsilon_{jn}^* - 1), \sum_{j=1}^{n_0} g(\lambda_{jn}) f(\lambda_{jn}) (\varepsilon_{jn}^* - 1) \right) \\ &= \frac{1}{n} \sum_{j=1}^{n_0} h(\lambda_{jn}) f(\lambda_{jn}) g(\lambda_{jn}) f(\lambda_{jn}) \mathbf{E}^* (\varepsilon_{jn}^* - 1)^2 \\ &\rightarrow c_1(h, g) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

The two limits of the empirical processes coincide iff the second term in the covariance  $c(h, g)$  vanishes that is iff  $k_4 \int_0^\pi h(\omega) f(\omega) d\omega = 0$ .  $\square$

## References

- [1] BARBE, P., and BERTAIL, P. (1995) *The Weighted Bootstrap*. Lecture Notes in Statistics, Springer N.Y.
- [2] BERTAIL, P. and CLEMENCON, S. (2006). Regenerative block bootstrap for Markov chains. *Bernoulli*, **12**, no. 4, 689–712.  
doi:10.3150/bj/1155735932. <https://projecteuclid.org/euclid.bj/1155735932>

- [3] CHAN, V., LAHIRI, S.N., and MEEKER, W.Q. (2004) Block bootstrap estimation of the distribution of cumulative outdoor degradation. *Technometrics*, **46**, 215-224.
- [4] DAHLHAUS, R. (1985) Asymptotic Normality of Spectral Estimates. *J. Multivariate Anal.*, **16**, 412-431.
- [5] DAHLHAUS, R. (1988) Empirical spectral processes and their applications to time series analysis. *Stoch. Proc. Appl.*, **30**, 69-83.
- [6] DAHLHAUS, R. and POLONIK W. (2002) Empirical Spectral Processes and Nonparametric Maximum Likelihood Estimation for Time Series. In: Dehling H., Mikosch T., Sørensen M. (eds) Empirical Process Techniques for Dependent Data. Birkhäuser, Boston, MA
- [7] DAHLHAUS, R. and JANAS, D. (1996) A frequency domain bootstrap for ratio statistics in time series analysis. *The Annals of Statistics*, **24**, 1934–1963.
- [8] DUDLEY, R.M. (1990) Nonlinear functionals of empirical measures and the bootstrap. In: E. Eberlein, J. Kuelbs and M.B. Marcus, eds., *Probability in Banach Spaces*, Vol. **7**, pp. 63-82, Birkhäuser, Boston.
- [9] EFRON, B. (1979) Bootstrap Methods: Another look at the Jackknife. *Ann Statist.*, **7**, 1-26.
- [10] DUDEK, A.E. (2015) Circular block bootstrap for coefficients of autocovariance function of almost periodically correlated time series. *Metrika*, **78(3)**, 313-335.
- [11] DUDEK, A.E. (2018) Block bootstrap for periodic characteristics of periodically correlated time series. *Journal of Nonparametric Statistics*, **30(1)**, 87-124.
- [12] DUDEK, A.E., LEŚKOW, J., PAPANODITIS, E. and POLITIS, D. (2014) A generalized block bootstrap for seasonal time series. *J. Time Ser. Anal.*, **35**, 89-114.
- [13] DUDEK, A.E., PAPANODITIS, E. and POLITIS, D. (2016) Generalized Seasonal Tapered Block Bootstrap, *Statistics and Probability Letters*, **115**, 27-35.
- [14] FRANKE, J. and HÄRDLE, W. (1992) On bootstrapping kernel spectral estimates. *Ann. Statist.*, **20**, 121–145.
- [15] HURVICH, C. M. and ZEGER, S. L. (1987) Frequency domain bootstrap methods for time series. Technical Report 87-115, Graduate School of Business Administration, New York Univ.
- [16] JANAS, D. and DAHLHAUS, R. (1994) A frequency domain bootstrap for time series. In *Computationally Intensive Statistical Methods. In Proceedings of the 26th Symposium on the Interface J. Sall and A. Lehman, eds.*, 423-425. Interface Foundation of North America, Fairfax Station, VA.
- [17] KIM, Y.M. and NORDMAN, D.J. (2013) A frequency domain bootstrap for Whittle estimation under long-range dependence. *Journal of Multivariate Analysis*, **115C**, 405-420.
- [18] KIRCH C. and POLITIS D.N., (2011) TFT-bootstrap: resampling time series in the frequency domain to obtain replicates in the time domain. *The Annals of Statistics*, **39**, 1427–1470.

- [19] KREISS, J.-P. and PAPANODITIS, E. (2003) Autoregressive aided periodogram bootstrap for time series. *Ann. Stat.*, **31**, 1923–1955.
- [20] KÜNSCH, H. (1989) The jackknife and the bootstrap for general stationary observations. *Ann. Statist.*, **17**, 1217–1241.
- [21] Lahiri, S.N. (2003) *Resampling Methods for Dependent Data*. Springer, New York.
- [22] LIU, R. and SINGH, K. (1992) Moving block jackknife and bootstrap capture weak dependence. *Exploring the Limits of Bootstrap*. Wiley Ser. Probab. Math. Statist. Probab. Math. Statist. Wiley, New York, pp 225–248.
- [23] MEYER, M., PAPANODITIS E. and KREISS J.-P. (2018) A Frequency Domain Bootstrap for General Stationary Processes. Preprint <https://arxiv.org/abs/1806.06523>.
- [24] NORDGAARD, A. (1992) Resampling stochastic processes using a bootstrap approach. In *Bootstrapping and Related Techniques. Lecture Notes in Econom. Math. Sys.* 376. Springer, Berlin.
- [25] PAPANODITIS, E. (2002) Frequency domain bootstrap for time series. In *Empirical Process Techniques for Dependent Data (H. Dehling et al., eds.)* 365–381. Birkhäuser, Boston.
- [26] PAPANODITIS, E. and POLITIS, D. N. (1999) The local bootstrap for periodogram statistics. *J. Time Ser. Anal.*, **20**, 193–222.
- [27] PAPANODITIS, E. and POLITIS, D. (2001) Tapered block bootstrap. *Biometrika*, **88**, 1105–1119.
- [28] POLITIS, D.N. and ROMANO, J.P. (1992) A circular block-resampling procedure for stationary data. *Exploring the Limits of Bootstrap*. Wiley Ser. Probab. Math. Statist. Probab. Math. Statist. Wiley, New York, pp 263–270.
- [29] POLITIS, D.N, ROMANO, J.P. (1994) The stationary bootstrap. *J. Amer. Statist. Assoc.*, **89**, 1303–1313.
- [30] PRIESTLEY, M.B., (1981) *Spectral Analysis and Time Series Analysis*. Academic Press, London.
- [31] SHIMIZU, K. (2010) *Bootstrapping Stationary ARMA-GARCH Models*. Vieweg+Teubner.
- [32] WU, C.F.J. (1986) Jackknife, bootstrap and other resampling methods in regression analysis (with discussions). *Annals of Statistics*, **14**, 1261–1350.