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# ***K*-THEORY AND TOPOLOGICAL CYCLIC HOMOLOGY OF HENSELIAN PAIRS**

DUSTIN CLAUSEN, AKHIL MATHEW, AND MATTHEW MORROW

ABSTRACT. Given a henselian pair  $(R, I)$  of commutative rings, we show that the relative  $K$ -theory and relative topological cyclic homology with finite coefficients are identified via the cyclotomic trace  $K \rightarrow \mathrm{TC}$ . This yields a generalization of the classical Gabber–Gillet–Thomason–Suslin rigidity theorem (for mod  $n$  coefficients, with  $n$  invertible in  $R$ ) and McCarthy’s theorem on relative  $K$ -theory (when  $I$  is nilpotent).

We deduce that the cyclotomic trace is an equivalence in large degrees between  $p$ -adic  $K$ -theory and topological cyclic homology for a large class of  $p$ -adic rings. In addition, we show that  $K$ -theory with finite coefficients satisfies continuity for complete noetherian rings which are  $F$ -finite modulo  $p$ . Our main new ingredient is a basic finiteness property of  $\mathrm{TC}$  with finite coefficients.

## CONTENTS

1. Introduction	2
1.1. Rigidity results	2
1.2. Consequences	3
Conventions	5
Acknowledgments	5
2. The finiteness property of $\mathrm{TC}$	6
2.1. Reminders on cyclotomic spectra	6
2.2. Cocontinuity of $\mathrm{TC}/p$	8
2.3. Further finiteness of $\mathrm{TC}/p$	13
2.4. $\mathrm{TC}/p$ via de Rham–Witt	14
3. Henselian rings and pairs	17
3.1. Nonunital rings	17
3.2. Henselian pairs	20
4. The main rigidity result	23
4.1. Generalities on pseudocoherence	23
4.2. An axiomatic rigidity argument	26
4.3. Pseudocoherence of $K, \mathrm{TC}$	29
4.4. Equal characteristic case	31
4.5. Proof of the main result	33
5. Continuity and pro statements in algebraic $K$ -theory	35
5.1. Continuity and $K$ -theory	35
5.2. $p$ -adic continuity and $K$ -theory	39
5.3. Pro Geisser–Levine theorems	42

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6. Comparisons of $K$ and TC	47
6.1. Étale $K$ -theory is TC	48
6.2. Asymptotic comparison of $K$ and TC	49
6.3. Examples	50
6.4. Split injectivity	51
References	54

## 1. INTRODUCTION

**1.1. Rigidity results.** The purpose of this paper is to study the (connective) algebraic  $K$ -theory  $K(R)$  of a commutative ring  $R$ , by means of the cyclotomic trace [13]

$$K(R) \rightarrow \mathrm{TC}(R),$$

from  $K$ -theory to topological cyclic homology  $\mathrm{TC}(R)$ . The cyclotomic trace is known to be an extremely useful tool in studying  $K$ -theory. On the one hand,  $\mathrm{TC}(R)$  is often easier to calculate directly than  $K(R)$  and has various arithmetic interpretations. For instance, according to work of Bhatt–Morrow–Scholze [9],  $\mathrm{TC}(R)$  for  $p$ -adic rings is a form of syntomic cohomology of  $\mathrm{Spec}R$ . On the other hand, the cyclotomic trace is often an effective approximation to algebraic  $K$ -theory. It is known that the cyclotomic trace is a  $p$ -adic equivalence in nonnegative degrees for finite algebras over the Witt vectors over a perfect field [40], and the cyclotomic trace has been used in several fundamental calculations of algebraic  $K$ -theory such as [41].

Our main theorem extends the known range of situations in which  $K$ -theory is close to TC. To formulate our results cleanly, we introduce the following notation.

**Definition 1.1.** For a ring  $R$ , we write  $K^{\mathrm{inv}}(R)$  for the homotopy fiber of the cyclotomic trace  $K(R) \rightarrow \mathrm{TC}(R)$ .

Thus  $K^{\mathrm{inv}}(R)$  measures the difference between  $K$  and TC. The following fundamental result (preceded by the rational version due to Goodwillie [34] and the  $p$ -adic version proved by McCarthy [57] and generalized by Dundas [18]) shows that  $K^{\mathrm{inv}}$  is nil-invariant.

**Theorem 1.2** (Dundas–Goodwillie–McCarthy [20]). Let  $R \rightarrow R'$  be a map of rings which is a surjection with nilpotent kernel.<sup>1</sup> Then the map  $K^{\mathrm{inv}}(R) \rightarrow K^{\mathrm{inv}}(R')$  is an equivalence.

The above result is equivalent to the statement that for a nilpotent ideal, the relative  $K$ -theory is identified with relative topological cyclic homology. Our main result is an extension of Theorem 1.2 with finite coefficients to a more general class of surjections. We use the following classical definition in commutative algebra (see also Definition 3.11 below).

**Definition 1.3.** Let  $R$  be a commutative ring and  $I \subset R$  an ideal. Then  $(R, I)$  is said to be a *henselian pair* if given a polynomial  $f(x) \in R[x]$  and a root  $\bar{\alpha} \in R/I$  of  $\bar{f} \in (R/I)[x]$  with  $\bar{f}'(\bar{\alpha})$  being a unit of  $R/I$ , then  $\bar{\alpha}$  lifts to a root  $\alpha \in R$  of  $f$ .

Examples of henselian pairs include pairs  $(R, I)$  where  $R$  is  $I$ -adically complete (by Hensel’s lemma) and pairs  $(R, I)$  where  $I$  is locally nilpotent.

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<sup>1</sup>Or, more generally, a map of connective  $\mathbb{E}_1$ -ring spectra such that  $\pi_0 R \rightarrow \pi_0 R'$  is a surjection with nilpotent kernel

Algebraic  $K$ -theory with finite coefficients prime to the characteristic is known to interact well with henselian pairs; one has the following result of Gabber [25], preceded by work of Suslin [72] and Gillet–Thomason [33]. See also [69, Sec. 4.6] for a textbook reference.

**Theorem 1.4** (Gabber [25]). Let  $(R, I)$  be a henselian pair. Suppose  $n$  is invertible in  $R$ . Then the map  $K(R)/n \rightarrow K(R/I)/n$  is an equivalence of spectra.

The main result of this paper is the following common extension of Theorem 1.4 and the commutative case of Theorem 1.2.

**Theorem A.** Let  $(R, I)$  be a henselian pair. Then for any  $n$ , the map  $K^{\text{inv}}(R)/n \rightarrow K^{\text{inv}}(R/I)/n$  is an equivalence.

What is the significance of such rigidity results in  $K$ -theory? Computing the algebraic  $K$ -theory of rings and schemes is a fundamental and generally very difficult problem. One of the basic tools in doing so is *descent*: that is, reducing the computation of the  $K$ -theory of certain rings to that of other (usually easier) rings built from them. As is well-known, algebraic  $K$ -theory generally does not satisfy descent for the étale topology. On the other hand, a general result of Thomason–Trobaugh [74] (see also [73] for a survey) states that algebraic  $K$ -theory of rings and (reasonable) schemes satisfies descent for the *Nisnevich* topology, which is quite well-behaved for a noetherian scheme of finite Krull dimension. In the Nisnevich topology, the points are given by the spectra of *henselian* local rings. Up to a descent spectral sequence, algebraic  $K$ -theory can thus be computed if it is understood for henselian local rings. When  $n$  is invertible, Theorem 1.4 enables one to reduce the calculation (with mod  $n$  coefficients) to the  $K$ -theory of fields. Our main result extends this to the case where  $n$  is not assumed invertible, but with the additional term coming from TC. All of this only uses only the local case of Theorem A; invoking the general case also gives further information.

**1.2. Consequences.** As a consequence of Theorem A, we deduce various global structural properties about algebraic  $K$ -theory and topological cyclic homology, especially  $p$ -adic  $K$ -theory of  $p$ -adic rings. In many cases, we are able to extend known properties in the smooth case to provide results on the  $K$ -theory of singular schemes.

The first main consequence of our results is a general statement that  $p$ -adic algebraic  $K$ -theory and TC agree in large enough degrees for reasonable  $p$ -torsion schemes, or affine schemes on which  $(p)$  is henselian.

**Theorem B** (Asymptotic comparison of  $K$ , TC). Let  $R$  be a ring henselian along  $(p)$  and such that  $R/p$  has finite Krull dimension. Let  $d = \sup_{x \in \text{Spec}(R/p)} \log_p [k(x) : k(x)^p]$  where  $k(x)$  denotes the residue field at  $x$ . Then the map  $K(R)/p^i \rightarrow \text{TC}(R)/p^i$  is an equivalence in degrees  $\geq \max(d, 1)$  for each  $i \geq 1$ .

Theorem B specializes to a number of existing results and calculations of algebraic  $K$ -theory, and enables new ones.

- (1) For finitely generated algebras over a perfect field, the result was shown in the *smooth* case by Geisser–Levine [32] and Geisser–Hesselholt [28]: in fact, both  $K$ -theory and TC vanish mod  $p$  in sufficiently large degrees.
- (2) For singular curves, the result appears in Geisser–Hesselholt [29].
- (3) Our approach also applies to any semiperfect or semiperfectoid ring, where it shows that  $K/p$  is the connective cover of  $\text{TC}/p$ . This recovers calculations of Hesselholt [38] of the

$K$ -theory of the ring  $\mathcal{O}_{\mathbb{C}_p}$  of integers in the completed algebraic closure  $\mathbb{C}_p$  of  $\mathbb{Q}_p$ . See [9, Sec. 7.4] for some recent applications.

- (4) If  $R$  is any noetherian ring henselian along  $(p)$  and such that  $R/p$  is  $F$ -finite (i.e., the Frobenius map on  $R/p$  is finite), then the above result applies:  $p$ -adic  $K$ -theory and TC agree in sufficiently large degrees.

Another application of our results is to show that  $p$ -adic *étale*  $K$ -theory is identified with topological cyclic homology under quite general situations; this is shown in [28] in the smooth case. As a consequence, we may regard Theorem B as a type of  $p$ -adic Lichtenbaum–Quillen statement.

**Theorem C** (*Étale  $K$ -theory is TC at points of characteristic  $p$* ). Let  $R$  be a strictly henselian local ring with residue field of characteristic  $p > 0$ . Then  $K^{\text{inv}}(R)/p = 0$ , i.e., the map  $K(R) \rightarrow \text{TC}(R)$  is a  $p$ -adic equivalence.

In addition, we are able to obtain a general split injectivity statement about the cyclotomic trace of local  $\mathbb{F}_p$ -algebras.

**Theorem D** (*Split injectivity of the cyclotomic trace*). For any local  $\mathbb{F}_p$ -algebra  $R$  and any  $i \geq 1$ , the cyclotomic trace  $K(R)/p^i \rightarrow \text{TC}(R)/p^i$  is split injective on homotopy groups.

In fact, the splitting is functorial for ring homomorphisms, and one can identify the complementary summand in terms of de Rham–Witt cohomology: see Proposition 6.12.

Next, our results also imply statements internal to  $K$ -theory itself, especially the  $K$ -theory of  $\mathbb{F}_p$ -algebras. Let  $R$  be a local  $\mathbb{F}_p$ -algebra. If  $R$  is regular, there is a simple formula for the mod  $p$  algebraic  $K$ -theory of  $R$  given in the work of Geisser–Levine [32]. We let  $\Omega_{R, \log}^i$  denote the subgroup of  $\Omega_R^i$  consisting of elements which can be written as sums of products of forms  $dx/x$  for  $x$  a unit.

**Theorem 1.5** (Geisser–Levine [32]). The mod  $p$   $K$ -groups  $K_i(R; \mathbb{Z}/p\mathbb{Z}) = \pi_i(K(R)/p)$  are identified with the logarithmic forms  $\Omega_{R, \log}^i$  for  $i \geq 0$ .

Algebraic  $K$ -theory of singular  $\mathbb{F}_p$ -algebras is generally much more complicated. However, using results of [59, 58], we are able to prove a pro-version of the Geisser–Levine theorem (extending results of Morrow [59]).

**Theorem E** (Pro Geisser–Levine). For any regular local  $F$ -finite  $\mathbb{F}_p$ -algebra  $R$  and ideal  $I \subset R$ , there is an isomorphism of pro abelian groups  $\{K_i(R/I^n; \mathbb{Z}/p\mathbb{Z})\}_{n \geq 1} \simeq \left\{ \Omega_{R/I^n, \log}^i \right\}_{n \geq 1}$  for each  $i \geq 0$ .

Finally, we study the continuity question in algebraic  $K$ -theory for complete rings, considered by various authors including [30, 18, 72, 59, 61]. That is, when  $R$  is a ring and  $I$  an ideal, we study how close the map  $K(R) \rightarrow \varprojlim K(R/I^n)$  is to being an equivalence. Using results of Dundas–Morrow [22] on topological cyclic homology, we prove a general continuity statement in  $K$ -theory.

**Theorem F** (Continuity criterion for  $K$ -theory). Let  $R$  be a noetherian ring and  $I \subset R$  an ideal. Suppose  $R$  is  $I$ -adically complete and  $R/p$  is  $F$ -finite. Then the map  $K(R) \rightarrow \varprojlim K(R/I^n)$  is a  $p$ -adic equivalence.

In fact, an argument using Popescu’s approximation theorem shows that this continuity result is essentially equivalent to our main theorem on henselian pairs, see Remark 5.6.

In general, our methods do not control the *negative*  $K$ -theory. In addition, they are essentially limited to the affine case: that is, we do not treat henselian pairs of general schemes. For example, given a complete local ring  $(R, \mathfrak{m})$  and a smooth proper scheme  $X \rightarrow \text{Spec} R$ , our methods do not let

us compare the  $K$ -theory of  $X$  to its special fiber. Note that questions of realizing formal  $K$ -theory classes as algebraic ones are expected to be very difficult, cf. [10].

Theorem A depends essentially on a finiteness property of topological cyclic homology with mod  $p$  coefficients. In characteristic zero, negative cyclic homology is not a finitary invariant: that is, it does not commute with filtered colimits. The main technical tool we use in this paper is the observation that the situation is better modulo  $p$ . Topological cyclic homology of a ring  $R$  is built from the topological Hochschild homology  $\mathrm{THH}(R)$  and its natural structure as a cyclotomic spectrum. We use the Nikolaus–Scholze [60] description of the homotopy theory  $\mathrm{CycSp}$  of cyclotomic spectra to observe (Theorem 2.7) that topological cyclic homology modulo  $p$  commutes with filtered colimits.

**Theorem G** ( $\mathrm{TC}/p$  is finitary). The construction  $R \mapsto \mathrm{TC}(R)/p$ , from rings  $R$  to spectra, commutes with filtered colimits.

Theorem A also relies on various tools used in the classical rigidity results, reformulated in a slightly different form as the finiteness of certain functors, as well as the theory of non-unital henselian rings. In addition, it relies heavily on the calculations of Geisser–Levine [28] and Geisser–Hesselholt [28] of  $p$ -adic algebraic  $K$ -theory and topological cyclic homology for smooth schemes in characteristic  $p$ . We do not know if it is possible to give a proof of our result without using all of these techniques. However, in many cases (e.g.,  $\mathbb{F}_p$ -algebras) Theorem A can be proved without the full strength of the finiteness and spectral machinery; see Remark 4.37.

**Conventions.** In this paper, we will use the language of  $\infty$ -categories and higher algebra [51, 52]. On the one hand, the constructions of algebraic  $K$ -theory and of the cyclotomic trace are of course much older than the theory of  $\infty$ -categories. Many of our arguments use standard homotopical techniques that could be carried out in a modern model category of spectra, and we have tried to minimize the use of newer technology in the exposition when possible. On the other hand, we rely crucially on the Nikolaus–Scholze [60] approach to cyclotomic spectra, which is  $\infty$ -categorical in nature.

We will let  $\mathrm{Sp}$  denote the  $\infty$ -category of spectra, and write  $\otimes$  for the smash product of spectra. The sphere spectrum will be denoted by  $S^0 \in \mathrm{Sp}$ . In a stable  $\infty$ -category  $\mathcal{C}$ , we will write  $\mathrm{Hom}_{\mathcal{C}}(\cdot, \cdot) \in \mathrm{Sp}$  for the mapping spectrum between any two objects of  $\mathcal{C}$ . We will generally write  $\varprojlim$ ,  $\varinjlim$  for inverse and direct limits in  $\mathrm{Sp}$ ; in a model categorical approach, these would be typically called *homotopy* limits and colimits and written  $\mathrm{holim}$ ,  $\mathrm{hocolim}$ . Given an integer  $r$ , we let  $\mathrm{Sp}_{\geq r}$  (resp.  $\mathrm{Sp}_{< r}$ ) for the subcategory of  $X \in \mathrm{Sp}$  such that  $\pi_i(X) = 0$  for  $i < r$  (resp.  $i > r$ ). We let  $\tau_{\geq r}, \tau_{< r}$  denote the associated truncation functors.

We let  $K$  denote connective  $K$ -theory and  $\mathbb{K}$  denote its nonconnective analog;  $\mathrm{TC}$  denotes topological cyclic homology. Given a ring  $R$  and an ideal  $I \subset R$ , we let  $K(R, I)$  (resp.  $\mathbb{K}(R, I)$ ) denote the relative  $K$ -theory (resp. relative nonconnective  $K$ -theory), defined as  $\mathrm{fib}(K(R) \rightarrow K(R/I))$  and  $\mathrm{fib}(\mathbb{K}(R) \rightarrow \mathbb{K}(R/I))$ . We define  $K^{\mathrm{inv}}(R, I), \mathrm{TC}(R, I)$  similarly. Finally, following traditional practice in algebraic  $K$ -theory, we will sometimes write  $K_*(R; \mathbb{Z}/p^r\mathbb{Z})$  for the mod  $p^r$  homotopy of the spectrum  $K(R)$  (i.e.,  $\pi_*(K(R)/p^r)$ ) and similarly for  $\mathrm{TC}$ .

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## 2. THE FINITENESS PROPERTY OF TC

Given a ring  $R$ , we can form both its Hochschild homology  $\mathrm{HH}(R/\mathbb{Z})$  and its topological Hochschild homology spectrum  $\mathrm{THH}(R)$ . The main structure on Hochschild homology is the circle action, enabling one to form the *negative cyclic homology*  $\mathrm{HC}^-(R/\mathbb{Z}) = \mathrm{HH}(R/\mathbb{Z})^{hS^1}$ . However, there is additional information contained in topological Hochschild homology, encoded in the structure of a *cyclotomic spectrum*. Formally, cyclotomic spectra form a symmetric monoidal, stable  $\infty$ -category  $(\mathrm{CycSp}, \otimes, \mathbf{1})$ , and  $\mathrm{THH}(R) \in \mathrm{CycSp}$ . An object of  $\mathrm{CycSp}$  is in particular a spectrum with a circle action but also further information. This structure enables one to form the topological cyclic homology  $\mathrm{TC}(R) = \mathrm{Hom}_{\mathrm{CycSp}}(\mathbf{1}, \mathrm{THH}(R))$ . Introduced originally by Bökstedt–Hsiang–Madsen [13], nowadays there are several treatments of  $\mathrm{CycSp}$  by various authors, including Blumberg–Mandell [12], Barwick–Glasman [6], Nikolaus–Scholze [60], and Ayala–Mazel–Gee–Rozenblyum [5]. In particular, the paper [60] shows that the structure of  $\mathrm{CycSp}$  admits a dramatic simplification in the bounded-below case.

The purpose of this section is to prove a structural property of  $\mathrm{TC}$ . The datum of a circle action is essentially infinitary in nature: for example, forming  $S^1$ -homotopy fixed points is an infinite homotopy limit, and the construction  $R \mapsto \mathrm{HC}^-(R/\mathbb{Z})$  does not commute with filtered homotopy colimits. Classical presentations of the homotopy theory of cyclotomic spectra and  $\mathrm{TC}$  usually involve infinitary limits, such as a homotopy limit over a fixed point tower. However, in this section, we prove the slightly surprising but fundamental property that  $\mathrm{TC}/p$  commutes with filtered colimits. Our proof is based on the simplification to the theory  $\mathrm{CycSp}$  in the bounded-below case demonstrated by Nikolaus–Scholze [60].

**2.1. Reminders on cyclotomic spectra.** We start by recalling some facts about the  $\infty$ -category of ( $p$ -complete) cyclotomic spectra, following [60].

**Definition 2.1** (Nikolaus–Scholze [60]). A *cyclotomic spectrum* is a tuple

$$(X, \{\varphi_{X,p} : X \rightarrow X^{tC_p}\}_{p=2,3,5,\dots}),$$

where  $X \in \mathrm{Fun}(BS^1, \mathrm{Sp})$  is a spectrum with an  $S^1$ -action which is moreover equipped with a map  $\varphi_p : X \rightarrow X^{tC_p}$  (the *cyclotomic Frobenius* at  $p$ ), for each prime number  $p$ , which is required to be equivariant with respect to the  $S^1$ -action on  $X$  and the  $S^1 \simeq S^1/C_p$ -action on the Tate construction  $X^{tC_p}$ . By a standard abuse of notation we will occasionally denote a cyclotomic spectrum simply by the underlying spectrum  $X$ .

For the precise definition of the  $\infty$ -category  $\mathrm{CycSp}$  of cyclotomic spectra (whose objects are as above) as a lax equalizer, we refer the reader to [60, II.1]. See [60, IV.2] for a treatment of the symmetric monoidal structure.

The  $\infty$ -category  $\mathrm{CycSp}$  naturally has the structure of a presentably symmetric monoidal  $\infty$ -category and the tensor product recovers the smash product on the level of underlying spectra with  $S^1$ -action. Moreover, the forgetful functor  $\mathrm{CycSp} \rightarrow \mathrm{Sp}$  reflects equivalences, is exact, and preserves all colimits. These properties may be deduced from the general formalism of lax equalizers; see [60, II.1].

**Definition 2.2.** Given  $X \in \text{CycSp}$ , its *topological cyclic homology*  $\text{TC}(X)$  is defined as the mapping spectrum  $\text{TC}(X) = \text{Hom}_{\text{CycSp}}(\mathbf{1}, X)$ , where  $\mathbf{1} \in \text{CycSp}$  is the unit (cf. Definition 2.9). We moreover write

$$\text{TC}^-(X) = X^{hS^1}, \quad \text{TP}(X) = X^{tS^1}$$

for its *negative topological cyclic homology* and *periodic topological cyclic homology*, the latter having been particularly studied by Hesselholt [39].

The theory of cyclotomic spectra studied more classically using equivariant stable homotopy theory [12] (building on ideas introduced by Bökstedt–Hsiang–Madsen [13]) agrees with that of Nikolaus–Scholze in the bounded-below case [60, Thm. II.6.9]. It is the bounded-below case that is of interest to us, and it will be convenient to introduce the following notation.

**Definition 2.3.** Given  $n \in \mathbb{Z}$ , let  $\text{Sp}_{\geq n}$  denote the full subcategory of  $\text{Sp}$  consisting of those spectra  $X$  that satisfy  $\pi_i(X) = 0$  for  $i < n$ . Let  $\text{CycSp}_{\geq n}$  denote the full subcategory of  $\text{CycSp}$  consisting of those cyclotomic spectra  $(X, \{\varphi_{X,p}\}_{p=2,3,5,\dots})$  such that the underlying spectrum  $X$  belongs to  $\text{Sp}_{\geq n}$ .

In the case in which  $X \in \text{CycSp}$  is bounded below and  $p$ -complete, the theory simplifies in several ways. Firstly, the Tate constructions  $X^{tC_q}$  vanish for all primes  $q \neq p$  [60, Lem. I.2.9], and therefore we do not need to specify the maps  $\varphi_{X,q}$ : the only required data is that of the  $S^1$ -action and the  $S^1 \simeq S^1/C_p$ -equivariant Frobenius map  $\varphi_X = \varphi_{X,p} : X \rightarrow X^{tC_p}$ . Secondly, there is a basic simplification to the formula for  $\text{TC}(X)$ , using the two maps

$$\text{can}_X, \varphi_X : \text{TC}^-(X) \rightrightarrows \text{TP}(X).$$

Here  $\text{can}_X$  is the canonical map from homotopy fixed points to the Tate construction, while  $\varphi_X$  arises from taking  $S^1$ -homotopy-fixed points in the cyclotomic structure map  $X \rightarrow X^{tC_p}$  and using a version of the Tate orbit lemma to identify  $X^{tS^1} \simeq (X^{tC_p})^{h(S^1/C_p)}$  since  $X$  is bounded below and  $p$ -complete [60, Lem. II.4.2]. Using this two maps one has the fundamental formula [60, Prop II.1.9]

$$(1) \quad \text{TC}(X) = \text{fib}(\text{TC}^-(X) \xrightarrow{\text{can}_X - \varphi_X} \text{TP}(X)),$$

or equivalently  $\text{TC}(X)$  is the equalizer of  $\text{can}_X, \varphi_X$ .

A basic observation which goes into proving the main result of this section is that the above formula makes  $\text{TC}$  have an additional finiteness property that theories such that  $\text{TC}^-$  and  $\text{TP}$  do not enjoy. Roughly speaking the idea is the following: if in the definition above were to replace  $\text{can}_X - \varphi_X$  with just  $\text{can}_X$ , then the fiber would be  $X_{hS^1}$ , and this certainly commutes with colimits. We will see that  $\varphi_X$  is close enough to vanishing modulo  $p$  to deduce the same conclusion for  $\text{TC}(X)/p$ . As another example of such finiteness phenomena, compare for instance [3], where it was shown that the topological cyclic homology of a smooth and proper dg category over a finite field is a perfect  $H\mathbb{Z}_p$ -module.

**Example 2.4.** A key source of examples of cyclotomic spectra arises as follows. If  $R$  is a ring (or, more generally, a structured ring spectrum), then one can form its topological Hochschild homology  $\text{THH}(R)$  as an object of  $\text{CycSp}$ , following [60, §III.2]. As usual, we will write  $\text{TC}(R), \text{TC}^-(R), \text{TP}(R)$  in place of  $\text{TC}(\text{THH}(R)), \text{TC}^-(\text{THH}(R)), \text{TP}(\text{THH}(R))$ .

We will also use facts from the classical approach to topological cyclic homology (cf. [12, §6]) in order to verify a connectivity statement below. In particular, given  $X \in \text{CycSp}$  which is  $p$ -complete,



one extracts genuine fixed point spectra  $\mathrm{TR}^r(X; p) := X^{C_p^{r-1}}$  for each  $r \geq 1$ , which are related by Restriction ( $R$ ), Frobenius ( $F$ ), and Verschiebung ( $V$ ) maps:

$$R, F : \mathrm{TR}^{r+1}(X; p) \rightarrow \mathrm{TR}^r(X; p), \quad V : \mathrm{TR}^r(X; p) \rightarrow \mathrm{TR}^{r+1}(X; p).$$

Setting  $\mathrm{TR}(X; p) := \varprojlim_R \mathrm{TR}^r(X; p)$ , then the topological cyclic homology of  $X$  is classically defined as

$$(2) \quad \mathrm{TC}(X) = \mathrm{fib}(\mathrm{TR}(X; p) \xrightarrow{\mathrm{id}-F} \mathrm{TR}(X; p)).$$

For a comparison between this description of  $\mathrm{TC}$  and that given in Definition 2.2, we refer to [60, §II.4] (see also [60, Rem. II.1.3] for the observation that, in the  $p$ -complete case,  $p$ -cyclotomic spectra and cyclotomic spectra are the same.)

It is the classical approach to topological cyclic homology in which important structures such as the de Rham–Witt complex appear [37], and in which the sequence (3) below plays a fundamental role. Using the classical approach we also obtain the following lemma. In Remark 2.14 below, we will also give a proof purely based on the Nikolaus–Scholze approach.

**Lemma 2.5.** Suppose  $X \in \mathrm{CycSp}_{\geq n}$  is  $p$ -complete. Then  $\mathrm{TC}(X) \in \mathrm{Sp}_{\geq n-1}$ .

*Proof.* Using formula (2), it suffices to show that  $\mathrm{TR}(X; p) \in \mathrm{Sp}_{\geq n}$ ; using the Milnor sequence, it is therefore enough to prove that  $\mathrm{TR}^r(X; p) \in \mathrm{Sp}_{\geq n}$  and that the Restriction maps  $R : \pi_n \mathrm{TR}^{r+1}(X; p) \rightarrow \pi_n \mathrm{TR}^r(X; p)$  are surjective for all  $r \geq 1$ . But this inductively follows from the basic cofiber sequence (cf. [37, Th. 1.2])

$$(3) \quad X_{hC_p^r} \rightarrow \mathrm{TR}^{r+1}(X; p) \xrightarrow{R} \mathrm{TR}^r(X; p)$$

and the observation that  $X_{hC_p^r} \in \mathrm{Sp}_{\geq n}$ . □

**Corollary 2.6.** The functor  $\mathrm{TC}/p : \mathrm{CycSp}_{\geq 0} \rightarrow \mathrm{Sp}$ ,  $X \mapsto \mathrm{TC}(X)/p$  commutes with geometric realizations.

*Proof.* A geometric realization of connective spectra agrees with its  $n^{\mathrm{th}}$  partial geometric realization in degrees  $< n$ . Since  $X \mapsto \mathrm{TC}(X)/p = \mathrm{TC}(X/p)$  decreases connectivity at most by one by Lemma 2.5, to show that  $\mathrm{TC}/p$  commutes with geometric realizations it therefore suffices to show that it commutes with  $n^{\mathrm{th}}$  partial geometric realizations for all  $n$ . But these are finite colimits, and are thus preserved by  $\mathrm{TC}/p$  since the latter is an exact functor. □

**2.2. Cocontinuity of  $\mathrm{TC}/p$ .** In this subsection we prove that  $\mathrm{TC}/p$  commutes with all colimits:

**Theorem 2.7.** The functor  $\mathrm{TC}/p : \mathrm{CycSp}_{\geq 0} \rightarrow \mathrm{Sp}$ , commutes with all colimits.

Since  $\mathrm{TC}/p : \mathrm{CycSp} \rightarrow \mathrm{Sp}$  is an exact functor between stable  $\infty$ -categories, it automatically commutes with finite colimits; thus the essence of the above theorem is that it commutes with filtered colimits when restricted to  $\mathrm{CycSp}_{\geq 0}$ .

**Remark 2.8.** There is also a dual assertion which is significantly easier. Suppose we have a tower  $\cdots \rightarrow X_3 \rightarrow X_2 \rightarrow X_1$  in  $\mathrm{CycSp}_{\geq 0}$ . Then the inverse limit of the tower  $\{X_i\}$  exists in  $\mathrm{CycSp}$  (in fact, in  $\mathrm{CycSp}_{\geq -1}$ ) and is preserved by the forgetful functor  $\mathrm{CycSp} \rightarrow \mathrm{Sp}$ . Indeed, we have  $(\varprojlim X_i)^{tC_p} \simeq \varprojlim X_i^{tC_p}$  because the  $X_i$ 's are uniformly bounded-below, and taking  $C_p$ -homotopy orbits behaves as a finite colimit in any range of degrees. Then we can appeal to the description of  $\mathrm{CycSp}$  as a lax equalizer and [60, Prop. II.1.5]. In particular, it follows that if  $X = \varprojlim X_i$  on the level of underlying spectra, then  $\mathrm{TC}(X) \simeq \varprojlim \mathrm{TC}(X_i)$ .

The proof of Theorem 2.7 will proceed by reduction to the case of cyclotomic spectra over the “trivial” cyclotomic spectrum  $H\mathbb{F}_p^{\text{triv}}$ . We therefore begin by recalling the definition of such trivial cyclotomic spectra, before studying  $H\mathbb{F}_p^{\text{triv}}$  and its modules in further detail.

**Definition 2.9.** We have a presentably symmetric monoidal  $\infty$ -category  $\text{CycSp}$ , so it receives a unique symmetric monoidal, cocontinuous functor

$$\text{Sp} \rightarrow \text{CycSp}, \quad X \mapsto X^{\text{triv}}$$

whose right adjoint is the functor  $\text{TC} : \text{CycSp} \rightarrow \text{Sp}$ .

More explicitly, this functor  $X \mapsto X^{\text{triv}}$  can be identified as follows:  $X^{\text{triv}}$  has underlying spectrum  $X$  equipped with the trivial  $S^1$ -action; there is therefore a resulting  $S^1$ -equivariant map  $X \rightarrow X^{hC_p} = F(BC_{p+}, X)$ , and the cyclotomic Frobenius at  $p$  for  $X^{\text{triv}}$  is the composition  $\varphi_{X,p} : X \rightarrow X^{hC_p} \rightarrow X^{tC_p}$ . (Notation: when  $X$  is a  $p$ -complete spectrum for a particular prime number  $p$ , we will tend to drop the  $p$  from  $\varphi_{X,p}$ .) Compare the discussion on [60, p. 126].

For example, taking  $X$  to be the sphere spectrum  $S^0 \in \text{Sp}$  obtains the *cyclotomic sphere spectrum*, which is moreover the unit  $\mathbf{1} \in \text{CycSp}$ .

We will need some results about the cyclotomic spectrum  $H\mathbb{F}_p^{\text{triv}}$ . Firstly, recall that the Tate cohomology ring  $\pi_*(H\mathbb{F}_p^{tC_p}) = \widehat{H}^{-*}(C_p; \mathbb{F}_p)$  is

- ( $p > 2$ ) the tensor product of an exterior algebra on a degree  $-1$  class with a Laurent polynomial algebra on a degree  $-2$  class;
- ( $p = 2$ ) a Laurent polynomial algebra on a degree  $-1$  class.

Calculating the  $S^1$ -homotopy fixed points and Tate construction of  $\mathbb{F}_p$ , we find that

$$\text{TC}_*(H\mathbb{F}_p^{\text{triv}}) = \mathbb{F}_p[x], \quad \text{TP}_*(H\mathbb{F}_p^{\text{triv}}) = \mathbb{F}_p[x^{\pm 1}], \quad \text{where } |x| = -2.$$

The canonical map  $\text{TC}^-(H\mathbb{F}_p^{\text{triv}}) \rightarrow \text{TP}(H\mathbb{F}_p^{\text{triv}})$  carries  $x$  to  $x$  (i.e., it is the localization inverting  $x$ ).

The cyclotomic Frobenius  $\varphi_{H\mathbb{F}_p} = \varphi_{H\mathbb{F}_p^{\text{triv}}} : H\mathbb{F}_p \rightarrow H\mathbb{F}_p^{tC_p}$  is an isomorphism on  $\pi_0$ . An essential point is that the induced Frobenius  $\varphi_{H\mathbb{F}_p}^{hS^1} : \text{TC}^-(H\mathbb{F}_p^{\text{triv}}) \rightarrow \text{TP}(H\mathbb{F}_p^{\text{triv}})$  (which we will abusively also denote by  $\varphi_{H\mathbb{F}_p}$  to avoid clutter, when the source and target are clear) kills the class  $x$ , as we now check.

**Lemma 2.10.** The Frobenius map  $\varphi_{H\mathbb{F}_p} : \text{TC}_*(H\mathbb{F}_p^{\text{triv}}) \rightarrow \text{TP}_*(H\mathbb{F}_p^{\text{triv}})$  annihilates  $x$ .

*Proof.* There is a commutative diagram of spectra

$$\begin{array}{ccc} \text{TC}^-(H\mathbb{F}_p^{\text{triv}}) & \xrightarrow{\varphi_{H\mathbb{F}_p}} & \text{TP}(H\mathbb{F}_p) \\ \downarrow & & \downarrow \\ H\mathbb{F}_p & \xrightarrow{\varphi_{H\mathbb{F}_p}} & H\mathbb{F}_p^{tC_p} \end{array}$$

where the top row is obtained by taking  $S^1$ -homotopy fixed points on the bottom row via the Tate orbit lemma, [60, Lem. II.4.2]. The right vertical map  $\text{TP}(H\mathbb{F}_p^{\text{triv}}) \rightarrow H\mathbb{F}_p^{tC_p}$  is an isomorphism in degree  $-2$ : indeed, the degree  $-2$  generator is invertible in the source by the above calculation of  $\text{TP}(H\mathbb{F}_p^{\text{triv}})$ , and therefore necessarily maps to a generator of  $\pi_{-2}(H\mathbb{F}_p^{tC_p})$  by multiplicativity. Going both ways around the diagram and noting that of course  $\pi_{-2}(H\mathbb{F}_p) = 0$ , we find that  $\varphi_{H\mathbb{F}_p}$  must be zero on  $\pi_{-2}(\text{TC}^-(H\mathbb{F}_p^{\text{triv}}))$ .  $\square$

The previous proof is modeled on the analysis of the cyclotomic spectrum  $\mathrm{THH}(\mathbb{F}_p)$  [60, Sec. IV-4]. In fact, noting that  $\mathrm{TC}(\mathbb{F}_p)$  receives a map  $H\mathbb{Z}_p \rightarrow \mathrm{TC}(\mathbb{F}_p)$  (the source is the connective cover of the target) we obtain a map of cyclotomic spectra  $H\mathbb{Z}_p^{\mathrm{triv}} \rightarrow \mathrm{THH}(\mathbb{F}_p)$ , which can be used to recover Lemma 2.10 from the more precise assertions about the cyclotomic spectrum  $\mathrm{THH}(\mathbb{F}_p)$ .

We next need to recall a classical fact about circle actions. Compare [54, Sec. 7] for more general versions over complex-oriented bases.

**Lemma 2.11.** Let  $X \in \mathrm{Mod}_{H\mathbb{F}_p}(\mathrm{Sp}^{BS^1})$  be an  $H\mathbb{F}_p$ -module spectrum equipped with an  $S^1$ -action. Then there is a functorial cofiber sequence of  $H\mathbb{F}_p$ -module spectra

$$\Sigma^{-2}X^{hS^1} \xrightarrow{x} X^{hS^1} \longrightarrow X,$$

where the first map is given by multiplication by the generator  $x \in \pi_{-2}(H\mathbb{F}_p^{hS^1}) = H^2(\mathbb{C}\mathbb{P}^\infty; \mathbb{F}_p)$ . There is moreover a functorial cofiber sequence

$$\Sigma^{-2}X^{hS^1} \xrightarrow{x\mathrm{can}_X} X^{tS^1} \longrightarrow X_{hS^1},$$

where  $\mathrm{can}_X : X^{hS^1} \rightarrow X^{tS^1}$  is the canonical map.

*Proof.* The first assertion is classical in the theory of cyclic homology, and is formulated in this language in [54, Sec. 7]. The second assertion follows from the cofiber sequence  $X^{hS^1} \xrightarrow{\mathrm{can}_X} X^{tS^1} \rightarrow \Sigma^2 X_{hS^1}$  and the fact that multiplication by  $x$  is an equivalence on  $X^{tS^1}$ .  $\square$

**Proposition 2.12.** For  $X \in \mathrm{Mod}_{H\mathbb{F}_p^{\mathrm{triv}}}(\mathrm{CycSp})$ , there exists a functorial fiber sequence

$$(4) \quad \mathrm{TC}(X) \longrightarrow X \longrightarrow X_{hS^1}.$$

*Proof.* We show first that, for  $X \in \mathrm{Mod}_{H\mathbb{F}_p^{\mathrm{triv}}}(\mathrm{CycSp})$ , the composition

$$(5) \quad \Sigma^{-2}\mathrm{TC}^-(X) \xrightarrow{x} \mathrm{TC}^-(X) \xrightarrow{\varphi_X} \mathrm{TP}(X)$$

is functorially nullhomotopic. In fact, since  $X$  is an  $H\mathbb{F}_p^{\mathrm{triv}}$ -module, we find that  $\mathrm{TC}^-(X)$  is an  $\mathrm{TC}^-(H\mathbb{F}_p^{\mathrm{triv}})$ -module,  $\mathrm{TP}(X)$  is a  $\mathrm{TP}(H\mathbb{F}_p^{\mathrm{triv}})$ -module, and the map  $\varphi_X$  is  $\varphi_{H\mathbb{F}_p}$ -linear. We obtain a resulting functorial commutative diagram, where the base-changes are along  $\varphi_{H\mathbb{F}_p} : \mathrm{TC}^-(\mathbb{F}_p^{\mathrm{triv}}) \rightarrow \mathrm{TP}(\mathbb{F}_p^{\mathrm{triv}})$ ,

$$(6) \quad \begin{array}{ccc} \Sigma^{-2}\mathrm{TC}^-(X) & \xrightarrow{x} & \mathrm{TC}^-(X) \\ \downarrow & & \downarrow \\ \Sigma^{-2}\mathrm{TC}^-(X) \otimes_{\mathrm{TC}^-(\mathbb{F}_p^{\mathrm{triv}})} \mathrm{TP}(\mathbb{F}_p^{\mathrm{triv}}) & \xrightarrow{\varphi_{H\mathbb{F}_p}(x)} & \mathrm{TC}^-(X) \otimes_{\mathrm{TC}^-(\mathbb{F}_p^{\mathrm{triv}})} \mathrm{TP}(\mathbb{F}_p^{\mathrm{triv}}) \end{array} \xrightarrow{\varphi_X} \mathrm{TP}(X).$$

However  $\varphi_{H\mathbb{F}_p}(x)$  vanishes, by Lemma 2.10, and so the composite (5) is functorially nullhomotopic as desired.

It now follows that there is a functorial commutative diagram

$$(7) \quad \begin{array}{ccc} \Sigma^{-2}\mathrm{TC}^-(X) & \xrightarrow{\mathrm{id}} & \Sigma^{-2}\mathrm{TC}^-(X) \\ x \downarrow & & \downarrow x\mathrm{can}_X \\ \mathrm{TC}^-(X) & \xrightarrow{\mathrm{can}_X - \varphi_X} & \mathrm{TP}(X) \end{array}$$

in which the fiber of the bottom arrow is  $\mathrm{TC}(X)$  by formula (1). Therefore  $\mathrm{TC}(X)$  is also the fiber of the natural map  $X \rightarrow X_{hS^1}$  between the cofibers of the vertical arrows in the above diagram, where we identify the cofibers using Lemma 2.11.  $\square$

See also [60, Prop. IV.3.4] for a closely related result for  $p$ -cyclotomic spectra with Frobenius lifts.

**Corollary 2.13.** The functor  $\mathrm{Mod}_{H\mathbb{F}_p^{\mathrm{triv}}}(\mathrm{CycSp}) \rightarrow \mathrm{Sp}$  given by  $X \mapsto \mathrm{TC}(X)$  commutes with all colimits. That is, the unit of the presentably symmetric monoidal stable  $\infty$ -category  $\mathrm{Mod}_{H\mathbb{F}_p^{\mathrm{triv}}}(\mathrm{CycSp})$  is compact.

*Proof.* Since passage to homotopy orbits commutes with all colimits, this follows from Proposition 2.12.  $\square$

**Remark 2.14.** Let  $X \in \mathrm{CycSp}_{\geq 0}$  be  $p$ -complete. Using the functorial fiber sequence of Proposition 2.12, we can give an independent proof of Lemma 2.5 (and thus of Corollary 2.6) which is independent of the classical approach to  $\mathrm{TC}$  (i.e., based entirely on the formula (1)). Namely, we argue that  $\mathrm{TC}(X) \in \mathrm{Sp}_{\geq -1}$ .

First, suppose that  $X$  is an  $H\mathbb{F}_p^{\mathrm{triv}}$ -module. In this case, we find that  $\mathrm{TC}(X) \in \mathrm{Sp}_{\geq -1}$  thanks to Proposition 2.12. More generally, if  $X$  is a  $p$ -complete  $H\mathbb{Z}^{\mathrm{triv}}$ -module, then we find that  $X/p$  is an  $H\mathbb{F}_p^{\mathrm{triv}}$ -module so that  $\mathrm{TC}(X/p) \simeq \mathrm{TC}(X)/p$  belongs to  $\mathrm{Sp}_{\geq -1}$ ; by  $p$ -completeness we deduce that  $\mathrm{TC}(X) \in \mathrm{Sp}_{\geq -1}$ .

In general, let  $X \in \mathrm{CycSp}_{\geq 0}$  be an arbitrary  $p$ -complete object. Then, we consider the Adams tower for  $X$  with respect to the commutative algebra object  $H\mathbb{Z}^{\mathrm{triv}} \in \mathrm{CycSp}$ , i.e., the tower for the totalization of the cosimplicial object  $X \otimes (H\mathbb{Z}^{\mathrm{triv}})^{\otimes \bullet + 1}$  (cf. [54, Sec. 2.1] for a reference in this generality). That is, we consider the tower  $\{X_n\} \in \mathrm{CycSp}$  defined inductively such that  $X_0 = X$  and  $X_{n+1}$  is the fiber of the map  $X_n \rightarrow X_n \otimes H\mathbb{Z}^{\mathrm{triv}}$ . One sees easily that  $X_n \in \mathrm{CycSp}_{\geq n}$ , and that  $X_n$  is  $p$ -complete since  $\pi_i(S^0)$  is finite for  $i > 0$ .

Since  $X_n/X_{n+1}$  is a  $H\mathbb{Z}^{\mathrm{triv}}$ -module, it follows from the previous analysis that  $\mathrm{TC}(X_n/X_{n+1}) \in \mathrm{Sp}_{\geq (n-1)}$  for each  $n$ . Inductively, it follows now that  $\mathrm{TC}(X/X_{n+1}) = \mathrm{TC}(X_0/X_{n+1}) \in \mathrm{Sp}_{\geq -1}$  for each  $n$  and that the maps  $\mathrm{TC}(X/X_{n+1}) \rightarrow \mathrm{TC}(X/X_n)$  are surjective on  $\pi_0$ . Since  $X \rightarrow X/X_{n+1}$  is an equivalence in degrees  $\leq n$ , it follows (as in [60, Lem. I.2.6]) that

$$X^{hS^1} \simeq \varprojlim (X/X_n)^{hS^1}, \quad X^{tS^1} \simeq \varprojlim (X/X_n)^{tS^1}$$

and therefore by (1)

$$\mathrm{TC}(X) \simeq \varprojlim \mathrm{TC}(X/X_n).$$

Using the Milnor exact sequence and the connectivity properties of  $\mathrm{TC}(X/X_{n+1})$ , one concludes that  $\mathrm{TC}(X) \in \mathrm{Sp}_{\geq -1}$ .

We are now prepared to prove the main result of the subsection.

*Proof of Theorem 2.7.* Consider an  $X \in \mathrm{CycSp}_{\geq 0}$  which is  $p$ -complete. Since  $\mathrm{TC}(X \otimes (-)^{\mathrm{triv}})$  is an exact functor from spectra to spectra with value  $\mathrm{TC}(X)$  on  $S^0$ , we get a natural transformation

$$\mathrm{TC}(X) \otimes (-) \rightarrow \mathrm{TC}(X \otimes (-)^{\mathrm{triv}}),$$

uniquely characterized by being an equivalence on  $S^0$ , hence on any finite spectrum.<sup>2</sup> By Corollary 2.6, it is also an equivalence on any geometric realization of finite connective spectra, i.e., for any connective spectrum whose homology groups are finitely generated, e.g.,  $H\mathbb{F}_p$ . So we have shown that, for any  $X \in \text{CycSp}_{\geq 0}$  which is  $p$ -complete, there is a natural equivalence

$$\text{TC}(X) \otimes H\mathbb{F}_p \simeq \text{TC}(X \otimes H\mathbb{F}_p^{\text{triv}}).$$

Replacing  $X$  with  $X/p$ , the same conclusion follows for an arbitrary  $X \in \text{CycSp}_{\geq 0}$ . Then from Corollary 2.13 we deduce that the functor  $X \mapsto \text{TC}(X) \otimes H\mathbb{F}_p$  commutes with colimits in  $\text{CycSp}_{\geq 0}$ . Hence, given a diagram  $I \rightarrow \text{CycSp}_{\geq 0}$ , the induced map of spectra

$$(8) \quad \varinjlim_I \text{TC}(X_i) \rightarrow \text{TC}(X)$$

becomes an equivalence after smashing with  $H\mathbb{F}_p$ . Since both sides are bounded below mod  $p$  by Lemma 2.5, it follows that (8) is an equivalence after smashing with  $S^0/p$ .  $\square$

The following consequence concerning the topological cyclic homology of rings is of particular interest. Let  $\text{Alg}(\text{Sp}_{\geq 0})$  be the  $\infty$ -category of connective associative ring spectra.

**Corollary 2.15.** The functor  $\text{Alg}(\text{Sp}_{\geq 0}) \rightarrow \text{Sp}$ ,  $R \mapsto \text{TC}(R)/p$  commutes with sifted colimits.

*Proof.* This follows from Theorem 2.7 since  $\text{TC}(R) = \text{TC}(\text{THH}(R))$ , where we note that the topological Hochschild homology functor  $\text{THH} : \text{Alg}(\text{Sp}_{\geq 0}) \rightarrow \text{CycSp}_{\geq 0}$  commutes with sifted colimits (as geometric realizations and tensor products do).  $\square$

Certain special cases of Corollary 2.15 appear in the literature (or can be extracted from it).

**Example 2.16.** For commutative  $\mathbb{F}_p$ -algebras, Corollary 2.15 can be deduced using the description of Hesselholt [37] of  $\text{THH}$  of smooth  $\mathbb{F}_p$ -algebras in terms of the de Rham–Witt complex; we detail this in subsection 2.4 below.

**Example 2.17.** In the case of square-zero extensions, the verification (called the  $p$ -limit axiom) of Corollary 2.15 plays an important role in the proof that relative  $K$ -theory and  $\text{TC}$  with finite coefficients agree for nilimmersions of rings in McCarthy [57].

**Example 2.18.** Let  $M$  be an  $\mathbb{E}_1$ -monoid in the  $\infty$ -category of spaces, so that one can form the spherical monoid ring  $\Sigma_+^\infty M$ . The formula for  $\text{TC}$  of spherical monoid rings [60, Lemma IV.3.1] (preceded by the formula for spherical group rings appearing already in [13, Eq. (0.3)]) shows that for any such  $R$ , the cyclotomic structure map  $\text{THH}(R) \rightarrow \text{THH}(R)^{tC_p}$  factors  $S^1$ -equivariantly through  $\text{THH}(R)^{hC_p}$ , which is called a *Frobenius lift* in [60]. For cyclotomic spectrum  $X$  with a Frobenius lift, the construction of ( $p$ -complete)  $\text{TC}$  simplifies: one has a fiber square [60, Prop. IV-3.4] relating  $\text{TC}(X)$  to  $\Sigma X_{hS^1}$  and two copies of  $X$ . It follows from this that the construction  $M \mapsto \text{TC}(\Sigma_+^\infty M)/p$  commutes with filtered colimits in  $M$ . Since the free associative algebra is a spherical monoid ring, and since  $\text{TC}$  is already known to commute with geometric realizations, one can also deduce Corollary 2.15 in this way.

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<sup>2</sup>This is a standard “assembly map”. To construct it rigorously, note that left Kan extension of the restriction of the right hand side to finite spectra is uniquely characterized by its value on  $S^0$ , by the universal property of spectra among presentable stable  $\infty$ -categories, and therefore identifies with the left hand side.

**2.3. Further finiteness of  $\mathrm{TC}/p$ .** Here we use the material of subsection 2.2 to deduce further finiteness properties of topological cyclic homology.

Firstly we show that, in any given range,  $\mathrm{TC}/p$  can be approximated well by functors finitely built from taking  $S^1$ -homotopy orbits. Let  $\mathrm{Fun}(\mathrm{CycSp}_{\geq 0}, \mathrm{Sp})$  denote the  $\infty$ -category of functors  $\mathrm{CycSp}_{\geq 0} \rightarrow \mathrm{Sp}$ . For our purposes below, we will need this strengthening and not only Theorem 2.7.

**Proposition 2.19.** For any given integer  $n$ , there exists a functor  $F \in \mathrm{Fun}(\mathrm{CycSp}_{\geq 0}, \mathrm{Sp})$  with the following properties:

- (1)  $F$  belongs to the thick subcategory of  $\mathrm{Fun}(\mathrm{CycSp}_{\geq 0}, \mathrm{Sp})$  generated by the functor  $X \mapsto (X \otimes H\mathbb{F}_p^{\mathrm{triv}})_{hS^1}$ ;
- (2) There exists an equivalence  $\tau_{\leq n} F \simeq \tau_{\leq n}(\mathrm{TC}(X)/p)$ .

*Proof.* We claim that for each  $n$ , the functor  $X \mapsto \mathrm{TC}(X \otimes \tau_{\leq n+1}(S^0/p)^{\mathrm{triv}})$  has the desired properties. Indeed, it belongs to the thick subcategory generated by  $X \mapsto (X \otimes H\mathbb{F}_p^{\mathrm{triv}})_{hS^1}$  thanks to Proposition 2.12, while the connectivity assertion follows from Lemma 2.5.  $\square$

The second finiteness result concerns the “pro” structure of topological cyclic homology and will be used later in the paper. Given  $X \in \mathrm{CycSp}_{\geq 0}$ , the classical approach to topological cyclic homology (cf. subsection 2.1) involves the spectra

$$\mathrm{TC}^r(X; p) = \mathrm{fib}(\mathrm{TR}^r(X; p) \xrightarrow{R-F} \mathrm{TR}^{r-1}(X; p))$$

for all  $r \geq 1$ . The system  $\{\mathrm{TC}^r(X; p)\}_{r \geq 1}$  naturally forms a tower of spectra under the Restriction (or, equivalently, Frobenius) maps, and  $\mathrm{TC}(X; p) = \varprojlim_r \mathrm{TC}^r(X; p)$ . Note that  $\mathrm{TC}^r(X; p) \in \mathrm{Sp}_{\geq -1}$  for all  $r \geq 1$  (by the proof of Lemma 2.5), and that each functor  $\mathrm{CycSp}_{\geq 0} \rightarrow \mathrm{Sp}$ ,  $X \mapsto \mathrm{TC}^r(X; p)$  commutes with all colimits (by induction using (3)).

Thus the failure of  $\mathrm{TC}$  to commute with filtered colimits arises from the infinite tower  $\{\mathrm{TC}^r(-; p)\}_r$ . We will now prove a restatement of Theorem 2.7 to the effect that this tower is pro-constant modulo  $p$ . First we need a couple of general lemmas on inverse systems.

**Lemma 2.20.** Let  $\mathcal{C}$  be a category admitting countable coproducts, and consider a tower of functors  $\mathcal{C} \rightarrow \mathrm{Ab}$ , say  $\dots \rightarrow F_r \rightarrow F_{r-1} \rightarrow \dots \rightarrow F_1$ . Suppose that:

- (1) For each  $X \in \mathcal{C}$  we have  $\varprojlim_r F_r(X) = \varprojlim_r^1 F_r(X) = 0$ .
- (2) For each  $r \in \mathbb{N}$ , the functor  $F_r$  commutes with countable coproducts.

Then the tower  $\{F_r\}_{r \geq 1}$  is pro-zero, i.e. it is zero as an object of  $\mathrm{Pro}(\mathrm{Fun}(\mathcal{C}, \mathrm{Ab}))$ ; or, equivalently, for all  $r \in \mathbb{N}$  there is an  $s > r$  such that the morphism  $F_s \rightarrow F_r$  is 0.

*Proof.* First we show the weaker claim that for all  $X \in \mathcal{C}$ , the tower  $\{F_r(X)\}_r$  of abelian groups is pro-zero. Set  $M_r = F_r(X)$  for brevity. From the hypothesis we find that an infinite direct sum of copies of the tower  $\{M_r\}_r$  has vanishing  $\varprojlim^1$ ; thus [24, Cor. 6] implies that  $\{M_r\}_r$  is Mittag-Leffler, i.e., for each  $r$ , the descending sequence  $\{\mathrm{im}(M_s \rightarrow M_r)\}_{s \geq r}$  of submodules of  $M_r$  stabilizes. If this stable submodule were nonzero for some  $r$ , we would deduce the existence of a nonzero element of  $\varprojlim_s M_s$ , contradicting the hypothesis. Thus the stable value is 0 for all  $r$ , which exactly means that  $\{M_r\}_r$  is pro-zero.

Now, suppose the claim of the lemma does not hold, i.e., that  $\{F_r\}_r$  is not pro-zero. Then for each  $r \in \mathbb{N}$  and  $s > r$ , we can find an  $Y \in \mathcal{C}$  such that  $F_s(Y) \rightarrow F_r(Y)$  is nonzero. Let  $X$  denote the coproduct over all pairs  $s > r$  of a choice of such a  $Y$ . Then for every  $s > r$  the map  $F_s(X) \rightarrow F_r(X)$  is nonzero, so we deduce that the tower  $\{F_r(X)\}$  is not pro-zero, in contradiction to what was established above.  $\square$

**Lemma 2.21.** Let  $\mathcal{C}$  be an  $\infty$ -category admitting countable coproducts, and consider a tower of functors  $\mathcal{C} \rightarrow \mathrm{Sp}$ , say  $\dots \rightarrow F_r \rightarrow F_{r-1} \dots \rightarrow F_1$ . Suppose that:

- (1) For each  $X \in \mathcal{C}$  we have  $\varprojlim_r F_r(X) = 0$ .
- (2) For each  $r \in \mathbb{N}$ , the functor  $F_r$  commutes with countable coproducts.
- (3) The homotopy groups of each  $F_r(X)$  are zero outside some fixed interval  $[a, b]$ , independent of  $X$  and  $r$ .

Then the tower  $\{F_r\}_{r \geq 1}$  is pro-zero, i.e. it is zero as an object of  $\mathrm{Pro}(\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}))$ .

*Proof.* The Milnor sequence and induction up the Postnikov tower show that (1) implies  $\varprojlim_r \pi_i F_r(X) = \varprojlim_r^1 \pi_i F_r(X) = 0$  for all  $i \in \mathbb{Z}$ . Since  $\pi_i$  commutes with coproducts, we can apply the previous lemma to  $\{\pi_i F_r\}_r$  and conclude that each tower  $\{\pi_i F_r\}_r$  is pro-zero. Thus each Postnikov section of  $\{F_r\}_r$  is pro-zero, and therefore so is  $\{F_r\}_r$  itself, by devissage up the (finite) Postnikov tower.  $\square$

**Proposition 2.22.** Fix an integer  $k$ . Then the tower of objects of  $\mathrm{Fun}(\mathrm{CycSp}_{\geq 0}, \mathrm{Sp})$  given by  $\{\tau_{\leq k}(\mathrm{TC}^r(-)/p)\}_r$  is pro-constant with value  $\tau_{\leq k}(\mathrm{TC}(-)/p)$ .

*Proof.* Setting  $F_r(X) \stackrel{\mathrm{def}}{=} \tau_{\leq k} \mathrm{cofib}(\mathrm{TC}(X)/p \rightarrow \mathrm{TC}^r(X)/p)$ , we must show that the tower  $\{F_r(-)\}_r$  is pro-zero. Certainly  $\varprojlim_r F_r(X) = 0$ , and  $F_r$  commutes with arbitrary coproducts by Theorem 2.7, so this follows from Lemma 2.21.  $\square$

**2.4. TC/p via de Rham–Witt.** In the previous subsections, we proved that  $\mathrm{TC}/p$  commutes with filtered colimits on the  $\infty$ -category  $\mathrm{CycSp}_{\geq 0}$ . In practice, one is usually interested in  $\mathrm{TC}$  of rings, i.e.,  $\mathrm{TC}$  of the cyclotomic spectra  $\mathrm{THH}(R)$  for rings  $R$ .

The purpose of this subsection is twofold. The first is to review explicitly the apparatus of  $\mathrm{THH}$ ,  $\mathrm{TC}$ , and the de Rham–Witt complex in the case of smooth  $\mathbb{F}_p$ -algebras, which we will need in the sequel. The second is to use this formalism to present a more direct approach to proving that  $\mathrm{TC}/p$  commutes with filtered colimits on the category of (discrete, commutative)  $\mathbb{F}_p$ -algebras. While this is much weaker than what we have already established (and will be insufficient for some of our applications), it can be proved without the new approach to cyclotomic spectra of [60], and is historically the first case which was known to some experts.

We begin with a review of the relevant algebraic concepts. Let  $R$  be an  $\mathbb{F}_p$ -algebra. We let  $\Omega_R^n$  denote the  $n$ -forms of  $R$  (over  $\mathbb{F}_p$ ) and let  $\Omega_R^*$  denote the algebraic de Rham complex of  $R$ .

**Definition 2.23.** The *inverse Cartier operator*  $C^{-1} : \Omega_R^n \rightarrow H^n(\Omega_R^*) \subset \Omega_R^n/d\Omega_R^{n-1}$  is the multiplicative operator uniquely characterized by the formulas  $C^{-1}(a) = a^p$ ,  $C^{-1}(db) = b^{p-1}db$ , for  $a, b \in R$ .

Note that the construction  $C^{-1}$  is not well-defined as an operator  $\Omega_R^n \rightarrow \Omega_R^n$ ; it is only well-defined modulo boundaries. It has image in the cohomology of the de Rham complex; if we further assume that  $R$  is ind-smooth, i.e. can be written as a filtered colimit of smooth  $\mathbb{F}_p$ -algebras<sup>3</sup>, then  $C^{-1}$  provides a natural (Frobenius semi-linear) isomorphism between  $\Omega_R^n$  and  $H^n(\Omega_R^*)$  by the classical *Cartier isomorphism*.

**Definition 2.24** (The de Rham–Witt complex). Let  $R$  be an  $\mathbb{F}_p$ -algebra. For  $r \geq 1$ , we denote as usual by

$$W_r(R) \xrightarrow{d} W_r\Omega_R^1 \xrightarrow{d} W_r\Omega_R^2 \xrightarrow{d} \dots$$

<sup>3</sup>Since smooth algebras are finitely presented, it follows that the category of ind-smooth algebras indeed identifies with  $\mathrm{Ind}$  of the category of smooth algebras.

the classical de Rham–Witt complex (more precisely, differential graded algebras) of Bloch–Deligne–Illusie [44]; this is the usual de Rham complex  $\Omega_R^\bullet$  in the case  $r = 1$ . We recall that the individual de Rham–Witt groups are equipped with Restriction and Frobenius maps  $R, F : W_r\Omega_R^n \rightarrow W_{r-1}\Omega_R^n$  (we will not use the Verschiebung). We let  $W\Omega_R^*$  denote the inverse limit of the tower  $\{W_r\Omega_R^*\}$  under the Restriction maps.

The operator  $F$  acts as a type of divided Frobenius. For example, one has a commutative diagram

$$(9) \quad \begin{array}{ccc} W\Omega_R^* & \xrightarrow{F} & W\Omega_R^* \\ \downarrow & & \downarrow \\ \Omega_R^* & \xrightarrow{C^{-1}} & \Omega_R^*/d\Omega_R^{*-1}. \end{array}$$

Here the vertical maps are induced by the natural projections to the first term of the inverse limit defining  $W\Omega_R^*$ .

In the approach to THH via equivariant stable homotopy theory, the de Rham–Witt complex plays a fundamental role thanks to the following result of Hesselholt, a version of the classical Hochschild–Kostant–Rosenberg (HKR) theorem for Hochschild homology.

**Theorem 2.25** (Hesselholt [37]). Let  $R$  be a smooth  $\mathbb{F}_p$ -algebra. Then, for each  $s \geq 1$ , we have an isomorphism of graded rings

$$(10) \quad \mathrm{TR}_*^s(R; p) \simeq W_s\Omega_R^* \otimes_{\mathbb{Z}/p^s} \mathbb{Z}/p^s[\sigma_s], \quad |\sigma_s| = 2.$$

The Restriction maps are determined by the Restriction maps on the de Rham–Witt complex and send  $\sigma_s \mapsto p\sigma_{s-1}$ . Therefore, we have an isomorphism

$$(11) \quad \mathrm{TR}_*(R; p) \simeq W\Omega_R^*,$$

and the  $F$  map on  $\mathrm{TR}(R; p)$  induces the  $F$  map on  $W\Omega_R^*$ .

We can now prove the finiteness property of  $\mathrm{TC}/p$  for  $\mathbb{F}_p$ -algebras. While the result is not logically necessary now, the diagram (12) will play a role in the sequel.

**Proposition 2.26.** The functor  $\mathrm{Alg}_{/\mathbb{F}_p} \rightarrow \mathrm{Sp}$ ,  $R \mapsto \mathrm{TC}(R)/p$  commutes with filtered colimits.

*Proof.* Let  $R = \mathrm{colim}_i R_i$  be a filtered colimit of  $\mathbb{F}_p$ -algebras. By functorially picking simplicial resolutions of all the terms by ind-smooth  $\mathbb{F}_p$ -algebras, and recalling that  $\mathrm{TC}/p$  commutes with geometric realizations of rings, we reduce to the case in which all the  $R_i$  (hence also  $R$  itself) are ind-smooth over  $\mathbb{F}_p$ .

Next we use Hesselholt’s HKR theorem. Since each  $\mathrm{TR}^s(-; p)$  commutes with filtered colimits (induction on (2)), as does  $W_s\Omega_R^n$ , the formula (10) remains valid for ind-smooth  $\mathbb{F}_p$ -algebras. By taking the inverse limit over  $s$ , we deduce that  $\pi_*(\mathrm{TR}(R; p)) \simeq W\Omega_R^*$  for any ind-smooth  $\mathbb{F}_p$ -algebra  $R$ , i.e., (11) is valid in the ind-smooth case as well. Note here that each  $W\Omega_R^n$  is  $p$ -torsion-free, by a basic property of the de Rham–Witt complex for smooth  $\mathbb{F}_p$ -algebras [45, Cor. 3.6] which can easily be extended to the ind-smooth case as it derives from the stronger assertion that each Restriction map  $W_s\Omega_R^* \rightarrow W_{s-1}\Omega_R^*$  annihilates every element  $x \in W_s\Omega_R^*$  with  $px = 0$ . As a result, we also conclude that  $\pi_*(\mathrm{TR}(R; p)/p) \simeq W\Omega_R^*/p$ .

Taking fixed points for the Frobenius, we find that  $\pi_n(\mathrm{TC}(R)/p)$  fits into an exact sequence

$$W\Omega_R^{n+1}/p \xrightarrow{F-1} W\Omega_R^{n+1}/p \longrightarrow \pi_n(\mathrm{TC}(R)/p) \longrightarrow W\Omega_R^n/p \xrightarrow{F-1} W\Omega_R^n/p,$$



and similarly for each  $R_i$ . To complete the proof it is enough to show, for each  $n \geq 0$ , that the kernel and cokernel of  $F - 1 : W\Omega^n/p \rightarrow W\Omega^n/p$  commutes with our filtered colimit  $R = \text{colim}_i R_i$ ; we stress that  $W\Omega^n/p$  does not commute with filtered colimits.

For any ind-smooth  $\mathbb{F}_p$ -algebra  $R$ , consider the commutative diagram

$$(12) \quad \begin{array}{ccc} W\Omega_R^n/p & \xrightarrow{F-1} & W\Omega_R^n/p \\ \downarrow & & \downarrow \\ \Omega_R^n & \xrightarrow{C^{-1}_{-1}} & \Omega_R^n/d\Omega_R^{n-1} \end{array} .$$

Here  $C^{-1}$  is the inverse Cartier operator. To complete the proof we will show that the two horizontal arrows have isomorphic kernels (resp. cokernels), i.e., the square is both cartesian and cocartesian.

To see this, it suffices to show that the map induced between the kernels of the vertical maps (which are surjective) is an isomorphism. The kernel of the map  $W\Omega_R^n \rightarrow \Omega_R^n$  is generated by the images of  $V, dV$  by [45, Proposition I.3.21]. Note that the citation is for smooth  $\mathbb{F}_p$ -algebras, but the statement clearly passes to an ind-smooth algebra as the terms  $W_s\Omega^n$  commute with filtered colimits and  $W\Omega^n = \varinjlim W_s\Omega^n$ . It follows that the kernel of the first vertical map in (12) is spanned by the images of  $V, dV$  while the kernel of the second vertical map is spanned by the images of  $V, d$ .

Thus, we need to show that the map

$$(13) \quad F - 1 : \text{im} \left( W\Omega_R^n/p \oplus W\Omega_R^{n-1}/p \xrightarrow{V+dV} W\Omega_R^n/p \right) \rightarrow \text{im} \left( W\Omega_R^n/p \oplus W\Omega_R^{n-1}/p \xrightarrow{V+d} W\Omega_R^n/p \right)$$

is an isomorphism.

First, we show that  $F - 1$  is surjective. We record the following identities in  $W\Omega_R^n/p$ ,

$$(F - 1)V = -V, \quad (F - 1)d \sum_{i>0} V^i = d;$$

using it, we see immediately that  $F - 1$  is surjective in (13).

To see injectivity, we will use an important property of the de Rham–Witt complex: the image of  $F$  on  $W\Omega_R^n$  consists precisely of those elements  $w$  with  $dw$  divisible by  $p$ , thanks to [45, (3.21.1.5)]. Again, the reference is stated in the smooth case, but it also passes to the ind-smooth case. In fact, the necessary assertion at finite levels is that the Frobenius  $F : W_s\Omega_R^* \rightarrow W_{s-1}\Omega_R^*$  has image consisting of those  $x \in W_{s-1}\Omega_R^*$  such that  $dx$  is divisible by  $p$ , and that the Restriction  $R : W_s\Omega_R^* \rightarrow W_{s-1}\Omega_R^*$  annihilates every  $x \in W_s\Omega_R^*$  with  $px = 0$ . From this, the assertion about  $W\Omega_R^*$  follows.

Suppose  $x \in W\Omega_R^n$  can be written as  $x = Vy + dVz$  and  $(F - 1)x$  is divisible by  $p$ ; we show that  $x$  is divisible by  $p$  in  $W\Omega_R^n$ . This will hold if  $y, z \in \text{im}(F)$ . First,  $-dx \equiv d((F - 1)x)$  modulo  $p$ , so  $dx$  is divisible by  $p$ . It follows that  $dVy$ , and hence  $dy = FdVy$  is divisible by  $p$ , so that  $y$  belongs to the image of  $F$ . In particular, modulo  $p$ , we have  $x \equiv dVz$  and  $(F - 1)dVz = d(z - Vz)$  is divisible by  $p$ . Therefore,  $z - Vz$  belongs to the image of  $F$ . Since the operator  $V$  is topologically nilpotent on  $W\Omega_R^n$ , we find that  $(1 - V)$  is invertible and  $z \in \text{im}(F)$ . Therefore,  $x$  is divisible by  $p$  as desired.  $\square$

An alternate approach to the de Rham–Witt complex is developed in [8] based on the theory of *strict Dieudonné complexes*, which are essentially a linear version of Witt complexes (at least in the case of algebras over  $\mathbb{F}_p$ ).

A *saturated* Dieudonné complex  $(X^*, d, F)$  is a  $p$ -torsion-free complex equipped with an operator  $F : X^* \rightarrow X^*$  such that  $dF = pFd$ . We require in addition that  $F : X^n \rightarrow X^n$  is injective, and its image consists precisely of those  $x \in X^n$  such that  $p$  divides  $dx$ ; this implies that one can define uniquely an operator  $V$  such that  $FV = VF = p$ . A saturated Dieudonné complex  $(X^*, d, F)$  is called *strict* if  $X^*$  is in addition complete for the filtration defined by  $\{\text{im}(V^n, dV^n)\}$ . The de Rham–Witt complex of an ind-smooth algebra is a strict Dieudonné complex, and the functor  $A \mapsto W\Omega_A^*$  commutes with filtered colimits from ind-smooth  $\mathbb{F}_p$ -algebras to the category of strict Dieudonné complexes (where colimits in the latter involve a completion process). Thus, the result can be deduced once one knows:

**Proposition 2.27.** The functors from the category of strict Dieudonné complexes to graded abelian groups given by

$$(X^*, d, F) \mapsto \ker(F - 1 : X^*/p \rightarrow X^*/p), \quad \text{coker}(F - 1 : X^*/p \rightarrow X^*/p)$$

commute with filtered colimits.

Proposition 2.27 can be proved in an entirely analogous manner as above.

### 3. HENSELIAN RINGS AND PAIRS

**3.1. Nonunital rings.** In this section we will study various categories of non-unital rings.

**Definition 3.1.** Given a commutative base ring  $R$ , a *nonunital  $R$ -algebra* is an  $R$ -module  $I$  equipped with a multiplication map  $I \otimes_R I \rightarrow I$  which is commutative and associative. We denote the category of nonunital  $R$ -algebras by  $\text{Ring}_R^{\text{nu}}$ . By contrast, when we say “ $R$ -algebra,” we assume the existence of a unit.

Given an  $R$ -algebra  $S$ , any ideal  $I \subset S$  is a nonunital  $R$ -algebra. Conversely, given a nonunital  $R$ -algebra  $J$ , we can adjoin to it a unit and thus form the  $R$ -algebra  $S = R \rtimes J$ , in which  $J$  is embedded as an ideal. This latter construction establishes an equivalence between the category of nonunital  $R$ -algebras and the category of augmented  $R$ -algebras.

Note that the category  $\text{Ring}_R^{\text{nu}}$  has all limits and colimits, and that the forgetful functor to sets preserves all limits and sifted colimits. We next describe the free objects in  $\text{Ring}_R^{\text{nu}}$ .

**Example 3.2.** Let  $R[x_1, \dots, x_n]^+$  denote the ideal  $(x_1, \dots, x_n) \subset R[x_1, \dots, x_n]$  in the polynomial ring  $R[x_1, \dots, x_n]$ . Then  $R[x_1, \dots, x_n]^+$  is the free object of  $\text{Ring}_R^{\text{nu}}$  on  $n$  generators.

**Example 3.3.** We say that a sequence  $I' \rightarrow I \rightarrow \bar{I}$  in  $\text{Ring}_R^{\text{nu}}$  is a *short exact sequence* if it is a short exact sequence of underlying abelian groups. That is,  $I \rightarrow \bar{I}$  is surjective, and we can recover  $I'$  as the pullback  $0 \times_{\bar{I}} I$  in the category  $\text{Ring}_R^{\text{nu}}$ .

Next we define the notion of a local nonunital  $R$ -algebra.

**Definition 3.4.** A non-unital  $R$ -algebra  $I$  is *local* if the following equivalent conditions hold:

- (1) For any  $x \in I$ , there exists  $y \in I$  (necessarily unique) such that  $x + y + xy = 0$ .
- (2) Whenever  $I$  embeds as an ideal in an  $R$ -algebra  $S$ , then  $I$  is contained in the Jacobson radical of  $S$ .
- (3)  $I \subset R \rtimes I$  is contained in the Jacobson radical of  $R \rtimes I$ .

The equivalence of the above conditions follows because, given an ideal  $I \subset S$  in a commutative ring  $S$ , then  $I$  belongs to the Jacobson radical of  $S$  if and only if  $1 + I$  consists of units of  $S$ .

We let  $\text{Ring}_R^{\text{nu,loc}} \subset \text{Ring}_R^{\text{nu}}$  denote the full subcategory of nonunital local  $R$ -algebras.

Since the element  $y$  in condition (1) is unique, the subcategory  $\text{Ring}_R^{\text{nu,loc}}$  is closed under limits and sifted colimits in  $\text{Ring}_R^{\text{nu}}$ , both of which are computed at the level of underlying sets.

To describe all colimits, we need to localize further by observing that the inclusion  $\text{Ring}_R^{\text{nu,loc}} \subset \text{Ring}_R^{\text{nu}}$  admits a left adjoint. Namely, given a nonunital  $R$ -algebra  $I$ , we can build a local nonunital  $R$ -algebra  $I[(1+I)^{-1}]$  as follows: the elements formally written “ $1+x$ ” for  $x \in I$  form a commutative monoid under multiplication, and this monoid acts on  $I$  by multiplication. This makes  $I$  into a non-unital algebra over the monoid ring  $R[1+I]$ , and we can set

$$(14) \quad I[(1+I)^{-1}] := I \otimes_{R[1+I]} R[(1+I)^{gp}],$$

where  $(1+I)^{gp}$  is the group completion.

Equivalently, if  $I$  is embedded as an ideal in an  $R$ -algebra  $S$ , then we can form the localization  $S[(1+I)^{-1}]$  in the usual sense of commutative algebra, and then realize  $I[(1+I)^{-1}]$  as the kernel of the augmentation  $S[(1+I)^{-1}] \rightarrow S/I$ . The equivalence of this description with the previous one follows from the exactness of localizations.

In any case, this construction  $\text{Ring}_R^{\text{nu}} \rightarrow \text{Ring}_R^{\text{nu,loc}}$ ,  $I \mapsto I[(1+I)^{-1}]$  is the desired left adjoint. To compute a colimit in  $\text{Ring}_R^{\text{nu,loc}}$ , one computes the colimit in  $\text{Ring}_R^{\text{nu}}$  and then applies this left adjoint.

**Example 3.5.** Let  $R[x_1, \dots, x_n]_{(x_1, \dots, x_n)}^+ \in \text{Ring}_R^{\text{nu,loc}}$  be the image of  $R[x_1, \dots, x_n]^+ \in \text{Ring}_R^{\text{nu}}$  under the left adjoint explained above. In other words,  $R[x_1, \dots, x_n]_{(x_1, \dots, x_n)}^+$  is the ideal  $(x_1, \dots, x_n)$  of the localization  $R[x_1, \dots, x_n]_{1+(x_1, \dots, x_n)}$  of the polynomial ring  $R[x_1, \dots, x_n]$  at its multiplicative subset  $1 + (x_1, \dots, x_n)$ .

It follows that  $R[x_1, \dots, x_n]_{(x_1, \dots, x_n)}^+$  is a local nonunital  $R$ -algebra, and it is the free object on  $n$  generators.

**Remark 3.6.** Stated more formally, the preceding discussion shows that the inclusion  $\text{Ring}_R^{\text{nu,loc}} \subset \text{Ring}_R^{\text{nu}}$  is the right adjoint of a *localization* functor on the category  $\text{Ring}_R^{\text{nu}}$ . Specifically,  $\text{Ring}_R^{\text{nu,loc}}$  consists of those objects in  $I \in \text{Ring}_R^{\text{nu}}$  which are orthogonal [14, Sec. 5.4] to the map  $R[x]^+ \rightarrow R[x]_{1+(x)}^+$ , in the sense that such that any map  $R[x]^+ \rightarrow I$  extends uniquely over  $R[x]_{1+(x)}^+$ . Moreover, one sees easily from the construction of the localization (14) that it is independent of the ground ring  $R$ .

Analogous statements will be true when we restrict further to the subcategory of henselian nonunital rings; see the proof of Lemma 3.8.

We now make the following definition following the discussion in [25]. This is surely known to experts, but for the convenience of the reader we spell out some details.

**Definition 3.7.** A nonunital  $R$ -algebra  $I$  is *henselian* if for every  $n \geq 1$  and every polynomial  $g(x) \in I[x]$ , the equation

$$(15) \quad x(1+x)^{n-1} + g(x) = 0$$

has a solution in  $I$ . We let  $\text{Ring}_R^{\text{nu,h}} \subset \text{Ring}_R^{\text{nu}}$  denote the full subcategory of henselian nonunital  $R$ -algebras.

We note that if  $I$  is a henselian nonunital  $R$ -algebra then:

- (1) Considering the equation  $x + xy + y = 0$  for  $y \in I$ , we find that  $I$  is local, i.e.,  $\text{Ring}_R^{\text{nu,h}} \subset \text{Ring}_R^{\text{nu,loc}}$ .

- (2) The root of the equation (15) is necessarily unique: denoting the equation by  $f(x)$  for simplicity, if  $\alpha, \alpha' \in I$  are both roots then we have  $f'(\alpha) \in 1 + I$  and so the Taylor expansion shows that

$$0 = f(\alpha) - f(\alpha') \in \alpha - \alpha' + I(\alpha - \alpha'),$$

whence  $\alpha = \alpha'$  since  $I$  is local.

Since the solution of (15) is unique if it exists, the category of henselian nonunital rings has all limits and sifted colimits, both of which are computed at the level of underlying sets. Moreover, if  $I \rightarrow I'$  is a surjection in  $\text{Ring}_R^{\text{nu}}$  and  $I \in \text{Ring}_R^{\text{nu,h}}$ , then  $I' \in \text{Ring}_R^{\text{nu,h}}$  too. Finally, the condition that a nonunital  $R$ -algebra be henselian does not depend on the base ring  $R$ , i.e., we might as well take  $R = \mathbb{Z}$  in the definition.

Now we prove the existence of a left adjoint  $\text{Ring}_R^{\text{nu}} \rightarrow \text{Ring}_R^{\text{nu,h}}$ , which will be called *henselization*.<sup>4</sup> We will see below in Corollary 3.19 that the henselization does not depend on the base ring  $R$ .

**Lemma 3.8.** The category  $\text{Ring}_R^{\text{nu,h}}$  is presentable and the inclusion  $\text{Ring}_R^{\text{nu,h}} \subset \text{Ring}_R^{\text{nu}}$  admits a left adjoint.

*Proof.* It follows from the above discussion that  $\text{Ring}_R^{\text{nu,h}}$  is the orthogonal [14, Sec. 5.4] of  $\text{Ring}_R^{\text{nu}}$  with respect to the maps

$$f_{n,t} : R[x_0, \dots, x_t]^+ \rightarrow R[x_0, \dots, x_t, y]^+ / (y(1+y)^{n-1} + x_0 + x_1y + \dots + x_t y^t),$$

for  $n, t \geq 1$ . That is, a given object  $X \in \text{Ring}_R^{\text{nu}}$  belongs to  $\text{Ring}_R^{\text{nu,h}}$  if and only if

$$\text{Hom}_{\text{Ring}_R^{\text{nu}}}(R[x_0, \dots, x_t, y]^+ / (y(1+y)^{n-1} + x_0 + x_1y + \dots + x_t y^t), X) \xrightarrow{\circ f_{n,t}} \text{Hom}_{\text{Ring}_R^{\text{nu}}}(R[x_0, \dots, x_t]^+, X)$$

is an isomorphism for all  $n, t \geq 1$ . It now follows formally that  $\text{Ring}_R^{\text{nu,h}}$  is presentable and the desired left adjoint exists [14, Cor. 5.4.8].

Alternatively, one can appeal to the theory of henselian pairs and define the left adjoint directly by taking the henselization of the pair  $(R \times I, I)$ ; see the forthcoming Construction 3.15.  $\square$

**Example 3.9.** Let  $R\{x_1, \dots, x_n\}^+ \in \text{Ring}_R^{\text{nu,h}}$  denote the henselization of  $R[x_1, \dots, x_n]^+ \in \text{Ring}_R^{\text{nu}}$ . By construction,  $R\{x_1, \dots, x_n\}^+ \in \text{Ring}_R^{\text{nu,h}}$  is the free object on  $n$  generators.

**Remark 3.10.** The categories  $\text{Ring}_R^{\text{nu}}$ ,  $\text{Ring}_R^{\text{nu,loc}}$  and  $\text{Ring}_R^{\text{nu,h}}$  are all examples of models for a *Lawvere* or *algebraic theory* [1, Ch. 3] [15, Ch. 3].

Let  $\mathcal{C}$  be a category satisfying the following conditions:

- (Law)  $\mathcal{C}$  has all limits and colimits, and is equipped with a functor  $U : \mathcal{C} \rightarrow \text{Sets}$  which is conservative, preserves sifted colimits, and admits a left adjoint  $F : \text{Sets} \rightarrow \mathcal{C}$ .

For example,  $\mathcal{C}$  might be the category of rings, nonunital rings, groups, etc. In this case, one takes for  $U$  the forgetful functor taking the underlying set, and its right adjoint  $F$  is the free ring, nonunital ring, group, etc. on the given set. Denoting by  $\{1\}$  a one-point set, the element  $F(\{1\})$  is therefore both compact projective (i.e.,  $\text{Hom}_{\mathcal{C}}(F(\{1\}), \cdot)$  commutes with sifted colimits) and a strong generator (i.e.,  $\text{Hom}_{\mathcal{C}}(F(\{1\}), \cdot)$  is faithful and conservative). The *free object of  $\mathcal{C}$  on  $n$  generators* is by definition  $\bigsqcup_n F(\{1\}) \simeq F(\{1, \dots, n\})$ .

<sup>4</sup>It follows from this that the category of henselian nonunital rings is the category of algebras over a *Lawvere theory*, cf. Remark 3.10. The free algebras in this theory are the henselizations of polynomial rings, Example 3.9.

Let  $\mathcal{C}' \subset \mathcal{C}$  be a full subcategory, and assume that  $\mathcal{C}'$  is closed under limits (in  $\mathcal{C}$ ) and that the inclusion  $\mathcal{C}' \subset \mathcal{C}$  is a right adjoint. Then clearly  $\mathcal{C}'$  also satisfies conditions (Law), with  $U'$  given by the restriction of  $U$  to  $\mathcal{C}'$ . Moreover, the free objects of  $\mathcal{C}'$  are given by applying the left adjoint  $\mathcal{C} \rightarrow \mathcal{C}'$  to the free objects of  $\mathcal{C}$ . In practice, this is how the “free objects” in the above categories are constructed.

According to general results of Lawvere theory (in fact, we do not use the results in this paragraph, but the point of view may be helpful), the full subcategory  $\mathcal{C}_\Sigma \subset \mathcal{C}$  of compact projective objects of  $\mathcal{C}$  is the idempotent completion of the full subcategory consisting of the free objects  $\{\bigsqcup_n F(\{1\}) : n \geq 0\}$ . Furthermore,  $\mathcal{C}$  can be identified with the category of presheaves on  $\mathcal{C}_\Sigma$  which commute with finite products. In particular, objects of  $\mathcal{C}$  can be identified with sets equipped with various “operations” arising from maps between the free objects satisfying various relations.

**3.2. Henselian pairs.** Following Gabber [25], we now discuss the connection between henselian nonunital rings and the more familiar notion of henselian pairs [70, Tag 09XD]. We will thus deduce that the constructions of the previous subsection do not depend on the base ring  $R$ .

**Definition 3.11.** A *pair* is the data  $(S, I)$  where  $S$  is a commutative ring and  $I \subset S$  is an ideal. The collection of pairs forms a category in the obvious manner.

The pair  $(S, I)$  is said to be *henselian* if the following equivalent conditions hold:

- (1) Given a polynomial  $f(x) \in S[x]$  and a root  $\bar{\alpha} \in S/I$  of  $\bar{f} \in (S/I)[x]$  with  $\bar{f}'(\bar{\alpha})$  being a unit of  $S/I$ , then  $\bar{\alpha}$  lifts to a root  $\alpha \in S$  of  $f$ . Note that the lifted root  $\alpha \in S$  is necessarily unique by the same argument as we gave after Definition 3.7.
- (2) The ideal  $I$  is contained in the Jacobson radical of  $S$ , and the same condition as (1) holds for *monic* polynomials  $f(x) \in S[x]$ .
- (3) Given any commutative diagram

$$(17) \quad \begin{array}{ccc} A & \longrightarrow & S \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & S/I \end{array}$$

with  $A \rightarrow B$  étale, there exists a lift as in the dotted arrow.

- (4) Same condition as (2), but one also requires the lift to be unique.

We may also say that the surjective map  $S \rightarrow S/I$  is a henselian pair, if there is no risk of confusion.

If  $S$  is local with maximal ideal  $\mathfrak{m}$ , then  $S$  is said to be a *henselian local ring* if  $(S, \mathfrak{m})$  is a henselian pair.

We refer the reader to [70, Tag 09XD] as a basic general source for topics related to henselian pairs. We also record for future reference the following property of henselian pairs with respect to smooth morphisms.

**Theorem 3.12** (Elkik [23, Sec. II]). Let  $(S, I)$  be a henselian pair. Then  $S \rightarrow S/I$  has the right lifting property with respect to smooth maps. That is, any diagram as in (17) with  $A \rightarrow B$  smooth (rather than étale) admits a lift as in the dotted arrow.

We crucially need the following observation of Gabber that the condition that  $(S, I)$  be henselian depends only on  $I$  as a nonunital ring.

**Proposition 3.13** ([25, Prop. 1]). Let  $(S, I)$  be a pair. Then  $(S, I)$  is a henselian pair if and only if  $I$  is henselian as a nonunital ring.

**Corollary 3.14.** Let  $(S, I)$  be a pair. Then  $(S, I)$  is a henselian pair if and only if  $(\mathbb{Z} \times I, I)$  is a henselian pair.

We recall the following basic construction (see [70, Tag 09XD]) in the theory of henselian pairs.

**Construction 3.15.** Given a pair  $(S, I)$ , there is a pair  $(S^h, I^h)$  and a map  $(S, I) \rightarrow (S^h, I^h)$ , called the *henselization* of the original pair, with the following properties.

- (1)  $(S^h, I^h)$  is a henselian pair and is the initial henselian pair receiving a map from  $(S, I)$ . That is, the construction  $(S, I) \mapsto (S^h, I^h)$  is the left adjoint to the forgetful functor from henselian pairs to pairs.
- (2)  $S^h$  is a filtered colimit of étale  $S$ -algebras and  $I^h = IS^h = I \otimes_S S^h$ .
- (3) The map  $S/I \rightarrow S^h/I^h$  is an isomorphism.

**Remark 3.16.** Let  $(S, I)$  be a pair. Suppose given a factorization  $S \rightarrow \tilde{S} \rightarrow S/I$  such that  $\tilde{S}$  is a filtered colimit of étale  $S$ -algebras and such that  $\tilde{S} \rightarrow S/I$  has the right lifting property with respect to étale morphisms (i.e., the surjection  $\tilde{S} \rightarrow S/I$  is a henselian pair). Then  $(\tilde{S}, I\tilde{S})$  is the henselization of  $(S, I)$ . To see this, consider a henselian pair  $(A, J)$  and a map  $(S, I) \rightarrow (A, J)$ . Considering the commutative diagram

$$\begin{array}{ccccc} S & \longrightarrow & \tilde{S} & \longrightarrow & S/I \\ \downarrow & & \swarrow \text{dotted} & & \downarrow \\ A & \longrightarrow & & \longrightarrow & A/J \end{array}$$

the ind-étaleness of  $S \rightarrow \tilde{S}$  implies the existence of a unique dotted arrow  $\tilde{S} \rightarrow A$  making the diagram commute. This verifies the universal property of the henselization.

In particular, one can construct  $(S^h, I^h)$  by appealing to Quillen’s small object argument [43, Th. 2.1.14] to factor  $S \rightarrow S/I$  as the composite of a filtered colimit of étale morphisms and a morphism that has the right lifting property with respect to étale morphisms. Note that in (17), it suffices to take  $A, B$  finitely generated over  $\mathbb{Z}$ , by the structure theory for étale morphisms.

We now review the relation between henselizations of pairs and henselizations of nonunital rings. Most of this is implicit in [25], but we spell out the details.

**Definition 3.17.** A *Milnor square* of commutative rings is a diagram

$$(18) \quad \begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ S' & \longrightarrow & T' \end{array}$$

such that the vertical arrows are surjective and such that the diagram is both cartesian and cocartesian in the category of commutative rings. It follows in particular that if  $I \subset S, J \subset T$  are the respective kernels of the vertical maps  $S \rightarrow S', T \rightarrow T'$ , then  $f|_I : I \xrightarrow{\cong} J$  establishes an isomorphism of the ideals  $I, J$ .

**Lemma 3.18.** Consider a Milnor square as in (18) with respect to the ideals  $I \subset S$ ,  $J \subset T$ . Then the henselizations  $S^h$ ,  $T^h$  of  $S$ ,  $T$  along  $I$ ,  $J$  fit into a Milnor square

$$\begin{array}{ccc} S^h & \longrightarrow & T^h \\ \downarrow & & \downarrow \\ S' & \longrightarrow & T' \end{array} .$$

Furthermore,  $T^h \simeq S^h \otimes_S T$ .

*Proof.* Since  $S^h$  is flat over  $S$ , we can base-change (18) along  $S \rightarrow S^h$  to obtain a new Milnor square

$$\begin{array}{ccc} S^h & \longrightarrow & T \otimes_S S^h \\ \downarrow & & \downarrow \\ S' \otimes_S S^h & \longrightarrow & T' \otimes_S S^h \end{array} .$$

Note that  $S' \otimes_S S^h \simeq S'$  by the properties of the henselization, and similarly for  $T$ . Therefore the bottom row can be rewritten as  $S' \rightarrow T'$ . Since this is a Milnor square and the left vertical arrow is a henselian pair, so is the right vertical arrow thanks to Proposition 3.13. We have a factorization  $T \rightarrow T \otimes_S S^h \rightarrow T'$  as the composite of an ind-étale map and a map having the right lifting property with respect to étale maps. Thus, the result follows in view of Remark 3.16.  $\square$

**Corollary 3.19.** Let  $(S, I)$  be a pair and let  $(S^h, I^h)$  be its henselization. Let  $R$  be any ring mapping to  $S$ . Then  $I^h$  is the henselization of  $I$  as an object of  $\text{Ring}_R^{\text{nu}}$ . In particular, the henselization of  $I$  as an object of  $\text{Ring}_R^{\text{nu}}$  does not depend on the base ring  $R$ .

*Proof.* To distinguish the possible henselizations which a priori might not coincide, we temporarily write  $I^{\text{nuh}}$  to denote the henselization of  $I$  as an object of  $\text{Ring}_R^{\text{nu}}$ . The equivalence between  $\text{Ring}_R^{\text{nu}}$  and augmented  $R$ -algebras easily implies that  $(R \times I^{\text{nuh}}, I^{\text{nuh}})$  is the henselization of the pair  $(R \times I, I)$ . Applying Proposition 3.18 to the Milnor square

$$\begin{array}{ccc} R \times I & \longrightarrow & S \\ \downarrow & & \downarrow \\ R & \longrightarrow & S/I \end{array}$$

then reveals that the square

$$\begin{array}{ccc} R \times I^{\text{nuh}} & \longrightarrow & S^h \\ \downarrow & & \downarrow \\ R & \longrightarrow & S^h/I^h \end{array}$$

is also Milnor, i.e., that  $I^{\text{nuh}} \xrightarrow{\cong} I^h$ , as desired.  $\square$

Finally we recall Gabber's result that free henselian nonunital  $\mathbb{Q}$ -algebras are colimits of such  $\mathbb{Z}$ -algebras. For each  $N > 0$ , denote by

$$[N] : \mathbb{Z}\{x_1, \dots, x_n\}^+ \rightarrow \mathbb{Z}\{x_1, \dots, x_n\}^+$$

the endomorphism in  $\text{Ring}_{\mathbb{Z}}^{\text{nu,h}}$  which sends  $x_i \mapsto Nx_i$ .

**Corollary 3.20** ([25, Prop. 3]). Let  $n > 0$ . The filtered colimit of the endomorphisms  $[N]$  on  $\mathbb{Z}\{x_1, \dots, x_n\}^+$ , indexed over natural numbers  $N$  ordered by divisibility, is  $\mathbb{Q}\{x_1, \dots, x_n\}^+$ .

*Proof.* Henselization commutes with filtered colimits, and the filtered colimit of the analogous maps  $[N] : \mathbb{Z}[x_1, \dots, x_n]^+$  is given by  $\mathbb{Q}[x_1, \dots, x_n]^+$ . We now use Corollary 3.19 that henselization does not depend on the base ring.  $\square$

#### 4. THE MAIN RIGIDITY RESULT

In this section, we prove the main result of the paper (Theorem 4.35 below), which states that the relative  $K$ -theory and relative topological cyclic homology of a henselian pair agree after profinite completion.

Our proof will rely on several steps: a direct verification for smooth algebras in equal characteristic (where it is a corollary of the deep calculations of Geisser–Levine [32] of  $p$ -adic  $K$ -theory and of Geisser–Hesselholt [28] of topological cyclic homology), a finiteness property of  $K$ -theory and topological cyclic homology (expressed in the language of “pseudocoherent functors” below), and an imitation of the main steps of the proof of Gabber rigidity [25].

**4.1. Generalities on pseudocoherence.** In this subsection, we describe a basic finiteness property (called *pseudocoherence*) for spectrum-valued functors that will be necessary in the proof of the main theorem. Later, we will show that  $K$ -theory and TC satisfy this property.

**Definition 4.1.** Let  $\mathcal{C}$  be a small category and  $F : \mathcal{C} \rightarrow \text{Ab}$  a functor. We say that  $F$  is *finitely generated* if there exist finitely many objects  $X_1, \dots, X_n \in \mathcal{C}$  and a surjection of functors

$$(19) \quad \bigoplus_{i=1}^t \mathbb{Z}[\text{Hom}_{\mathcal{C}}(X_i, \cdot)] \twoheadrightarrow F.$$

We now want to introduce analogous concepts when  $\text{Ab}$  is replaced by the  $\infty$ -category  $\text{Sp}$  of spectra. We consider the  $\infty$ -category  $\text{Fun}(\mathcal{C}, \text{Sp})$  of functors from  $\mathcal{C}$  to spectra, and recall that  $\text{Fun}(\mathcal{C}, \text{Sp})$  is a presentable, stable  $\infty$ -category in which limits and colimits are computed targetwise. A family of compact generators of  $\text{Fun}(\mathcal{C}, \text{Sp})$  is given by the corepresentable functors  $\Sigma_{\dagger}^{\infty} \text{Hom}_{\mathcal{C}}(C, \cdot)$ , for  $C \in \mathcal{C}$ .

**Definition 4.2.** Let  $\mathcal{C}$  be a small category and  $F : \mathcal{C} \rightarrow \text{Sp}$  a functor.

- (1) We say that  $F$  is *perfect* if  $F$  belongs to the thick subcategory of  $\text{Fun}(\mathcal{C}, \text{Sp})$  generated by the functors  $\Sigma_{\dagger}^{\infty} \text{Hom}_{\mathcal{C}}(C, \cdot)$  for  $C \in \mathcal{C}$  (or equivalently is a compact object of  $\text{Fun}(\mathcal{C}, \text{Sp})$ , by a thick subcategory argument).
- (2) We say that  $F$  is *pseudocoherent* if, for each  $n \in \mathbb{Z}$ , there exists a perfect functor  $F'$  and a map  $F' \rightarrow F$  such that  $\tau_{\leq n} F'(C) \rightarrow \tau_{\leq n} F(C)$  is an equivalence for all  $C \in \mathcal{C}$ .

In the setting of structured ring spectra, the analog of a perfect functor is a perfect module and the analog of a pseudocoherent functor is an almost perfect module. We refer to [52, 7.2.4] for a detailed account of the theory in that setting; therefore, in our setting of functors, we will only sketch the proofs of the basic properties. Note in the next result that pseudocoherent functors are automatically bounded below, so the initial hypothesis is no loss of generality.

**Lemma 4.3.** The following conditions on a functor  $F \in \text{Fun}(\mathcal{C}, \text{Sp}_{\geq 0}) \subset \text{Fun}(\mathcal{C}, \text{Sp})$  are equivalent:

- (1)  $F$  is pseudocoherent.
- (2) For each  $n \in \mathbb{Z}$ , the functor  $\tau_{\leq n} F$  is a compact object of the  $\infty$ -category  $\text{Fun}(\mathcal{C}, \text{Sp}_{\leq n})$ .



- (3) There exists a sequence  $G_{-1} = 0 \rightarrow G_0 \rightarrow G_1 \rightarrow G_2 \rightarrow \dots$  such that  $G_i/G_{i-1}$  is a finite direct sum of functors of the form  $\Sigma^i \Sigma_+^\infty \text{Hom}_{\mathcal{C}}(C, \cdot)$  for  $C \in \mathcal{C}$  and such that  $\varinjlim_i G_i \simeq F$ .

*Proof.* For (1)  $\Rightarrow$  (2), suppose that  $F \in \text{Fun}(\mathcal{C}, \text{Sp})$  (we do not need  $F$  connective) is pseudocoherent and let  $n \in \mathbb{Z}$ . Then by definition there is a perfect functor  $F'$  and a map  $F' \rightarrow F$  inducing an equivalence  $\tau_{\leq n} F' \simeq \tau_{\leq n} F$ . Since  $\tau_{\leq n} : \text{Fun}(\mathcal{C}, \text{Sp}) \rightarrow \text{Fun}(\mathcal{C}, \tau_{\leq n} \text{Sp})$  preserves compact objects (as its right adjoint preserves filtered colimits), it follows that  $\tau_{\leq n} F \in \text{Fun}(\mathcal{C}, \tau_{\leq n} \text{Sp})$  is compact.

For (2)  $\Rightarrow$  (3), suppose that  $F \in \text{Fun}(\mathcal{C}, \text{Sp}_{\geq 0})$  and  $\tau_{\leq n} F \in \text{Fun}(\mathcal{C}, \tau_{\leq n} \text{Sp})$  is compact for each  $n$ . Then, by assumption  $\tau_{\leq 0} F \in \text{Fun}(\mathcal{C}, \text{Ab}) \subset \text{Fun}(\mathcal{C}, \text{Sp}_{\leq 0})$  is a compact object and thus  $\pi_0 F$  is a finitely generated functor. We can thus find a functor  $G \in \text{Fun}(\mathcal{C}, \text{Sp}_{\geq 0})$  which is a direct sum of  $\Sigma_+^\infty \text{Hom}_{\mathcal{C}}(X, \cdot)$ , for finitely many  $X \in \mathcal{C}$ , and a map  $G \rightarrow F$  which is a surjection on  $\pi_0$ . Let  $F_1$  be the resulting cofiber and observe that  $F_1$  has the same property as  $F$  (namely  $\tau_{\leq n} F_1 \in \text{Fun}(\mathcal{C}, \tau_{\leq n} \text{Sp})$  is compact for each  $n$ , since compact objects are closed under pushouts) and that  $F_1$  is connected. Continuing in this way, we find a sequence of functors  $F \rightarrow F_1 \rightarrow F_2 \rightarrow \dots$  such that  $F_n$  is concentrated in degrees  $\geq n$  and such that the cofiber of each  $F_n \rightarrow F_{n+1}$  is a finite direct sum of shifts of representables. We obtain the desired sequence  $G_n$  by setting  $G_n = \text{Fib}(F \rightarrow F_{n+1})$ .

For (3)  $\Rightarrow$  (1), note that each  $G_n$  is perfect by induction on  $n$ , and that  $G_n \rightarrow F$  is an equivalence on  $\tau_{\leq n}$ .  $\square$

Before stating some more properties, we introduce a further generalization (for a category relative to a subcategory) that will also be necessary in the sequel.

**Definition 4.4.** Let  $\mathcal{D} \subset \mathcal{C}$  be a small full subcategory of a (possibly large, but locally small) category  $\mathcal{C}$ . Let  $F : \mathcal{C} \rightarrow \text{Sp}$  be a functor.

- (1) We say that  $F$  is  $\mathcal{D}$ -perfect if  $F$  belongs to the thick subcategory of  $\text{Fun}(\mathcal{C}, \text{Sp})$  generated by the functors  $\Sigma_+^\infty \text{Hom}_{\mathcal{C}}(D, \cdot)$  for  $D \in \mathcal{D}$ .
- (2) We say that a functor  $F : \mathcal{C} \rightarrow \text{Sp}$  is called  $\mathcal{D}$ -pseudocoherent if for each  $n \in \mathbb{Z}$ , there exists a  $\mathcal{D}$ -perfect functor  $F'$  and a map  $F' \rightarrow F$  such that  $\tau_{\leq n} F'(C) \rightarrow \tau_{\leq n} F(C)$  is an equivalence for all  $C \in \mathcal{C}$ .
- (3) A functor  $F_0 : \mathcal{C} \rightarrow \text{Ab}$  is called  $\mathcal{D}$ -finitely generated if there is a surjection as in (19) with  $X_i \in \mathcal{D}$  for each  $i$ .

Next, we recall a basic construction that lets us reduce  $\mathcal{D}$ -pseudocoherence to (unrelative) pseudocoherence.

**Construction 4.5.** Let  $\mathcal{D} \subset \mathcal{C}$  be a full subcategory. Given a functor  $G : \mathcal{D} \rightarrow \text{Sp}$ , we can form the *left Kan extension*  $\text{Lan}(G) : \mathcal{C} \rightarrow \text{Sp}$  (cf. [51, Sec. 4.3]). By definition, for  $C \in \mathcal{C}$ ,  $\text{Lan}(G)(C)$  is the colimit

$$\text{Lan}(G)(C) = \varinjlim_{D \rightarrow C, D \in \mathcal{D}} G(D) \in \text{Sp}.$$

The construction  $F \mapsto \text{Lan}(F)$  is the left adjoint of the forgetful functor  $\text{Fun}(\mathcal{C}, \text{Sp}) \rightarrow \text{Fun}(\mathcal{D}, \text{Sp})$ . A functor  $F \in \text{Fun}(\mathcal{C}, \text{Sp})$  is said to be *left Kan extended* from  $\mathcal{D}$  if the natural map  $\text{Lan}(F|_{\mathcal{D}}) \rightarrow F$  is an equivalence in  $\text{Fun}(\mathcal{C}, \text{Sp})$ . The objects of  $\text{Fun}(\mathcal{C}, \text{Sp})$  which are left Kan extended from  $\mathcal{D}$  form a subcategory equivalent to  $\text{Fun}(\mathcal{D}, \text{Sp})$  (via restriction).

**Lemma 4.6.** A functor  $F : \mathcal{C} \rightarrow \text{Sp}$  is  $\mathcal{D}$ -pseudocoherent if and only if it is left Kan extended from  $\mathcal{D}$  and  $F|_{\mathcal{D}} : \mathcal{D} \rightarrow \text{Sp}$  is pseudocoherent.

*Proof.* The functor  $\Sigma_+^\infty \text{Hom}_{\mathcal{C}}(D, \cdot) \in \text{Fun}(\mathcal{C}, \text{Sp})$  for  $D \in \mathcal{D}$  is left Kan extended from  $\mathcal{D}$ , so a thick subcategory argument shows that any  $\mathcal{D}$ -perfect functor is left Kan extended from  $\mathcal{D}$ . To get the same for any  $\mathcal{D}$ -pseudocoherent functor, note that the localizations  $\text{Sp} \rightarrow \tau_{\leq n} \text{Sp}$  are a conservative family of colimit-preserving functors, so a functor to  $\text{Sp}$  is left Kan extended if and only if its image in each  $\tau_{\leq n} \text{Sp}$  is.

Conversely, suppose that  $F$  is left Kan extended from  $\mathcal{D}$  and  $F|_{\mathcal{D}} : \mathcal{D} \rightarrow \text{Sp}$  is pseudocoherent. Then  $\tau_{\leq n} F|_{\mathcal{D}} \simeq \tau_{\leq n} G|_{\mathcal{D}}$  for some  $G : \mathcal{C} \rightarrow \text{Sp}$  in the thick subcategory generated by the  $\Sigma_+^\infty \text{Hom}_{\mathcal{C}}(D, \cdot)$ , and we deduce  $\tau_{\leq n} F \simeq \tau_{\leq n} G$  by left Kan extension.  $\square$

**Proposition 4.7.** Let  $\mathcal{D} \subset \mathcal{C}$  be a small full subcategory. Let  $F : \mathcal{C} \rightarrow \text{Sp}_{\geq 0}$  be a functor. Then the following are equivalent:

- (1)  $F$  is  $\mathcal{D}$ -pseudocoherent.
- (2) There exists a sequence  $G_{-1} = 0 \rightarrow G_0 \rightarrow G_1 \rightarrow G_2 \rightarrow \dots$  such that  $G_i/G_{i-1}$  is a finite direct sum of functors of the form  $\Sigma^i \Sigma_+^\infty \text{Hom}_{\mathcal{C}}(D, \cdot)$  for  $D \in \mathcal{D}$  and such that  $\varinjlim_i G_i \simeq F$ .

*Proof.* Combine Lemmas 4.6 and 4.3.  $\square$

**Proposition 4.8.** Let  $\mathcal{C}$  be a category and  $\mathcal{D} \subset \mathcal{C}$  a small full subcategory.

- (1) The subcategory of  $\text{Fun}(\mathcal{C}, \text{Sp})$  spanned by the  $\mathcal{D}$ -pseudocoherent functors is thick.
- (2) Let  $d \in \mathbb{Z}$ , let  $K$  be a simplicial set, and let  $f : K \rightarrow \text{Fun}(\mathcal{C}, \text{Sp}_{\geq d})$  be a  $K$ -indexed diagram of functors  $\mathcal{C} \rightarrow \text{Sp}_{\geq d}$ . Suppose that the  $n$ -skeleton  $\text{sk}_n K$  is a finite simplicial set for every  $n \in \mathbb{Z}$  and that for each vertex  $k_0 \in K_0$ , the functor  $f(k_0) \in \text{Fun}(\mathcal{C}, \text{Sp})$  is  $\mathcal{D}$ -pseudocoherent. Then the functor  $\varinjlim_K f \in \text{Fun}(\mathcal{C}, \text{Sp})$  is  $\mathcal{D}$ -pseudocoherent.
- (3) Let  $d \in \mathbb{Z}$ , and let  $F_\bullet : \Delta^{\text{op}} \rightarrow \text{Fun}(\mathcal{C}, \text{Sp}_{\geq d})$  be a simplicial object in the category of functors  $\mathcal{C} \rightarrow \text{Sp}_{\geq d}$ . Suppose that  $F_i$  is  $\mathcal{D}$ -pseudocoherent for every  $i \geq 0$ . Then the geometric realization  $|F_\bullet| : \mathcal{C} \rightarrow \text{Sp}$  is  $\mathcal{D}$ -pseudocoherent.
- (4) Let  $d \in \mathbb{Z}$ , and let  $F \in \text{Fun}(\mathcal{C}, \text{Sp})$  be a  $\mathcal{D}$ -pseudocoherent functor such that  $\pi_n F = 0$  for all  $n < d$ . Then  $\pi_d F : \mathcal{C} \rightarrow \text{Ab}$  is a  $\mathcal{D}$ -finitely generated functor.
- (5) If  $\mathcal{C}$  has finite coproducts and  $\mathcal{D}$  is closed under them, then the subcategory of  $\mathcal{D}$ -pseudocoherent functors in  $\text{Fun}(\mathcal{C}, \text{Sp})$  is closed under smash products.

*Proof.* Claims (1) through (4) follow directly from Lemmas 4.3 and 4.6 by standard compactness arguments. Note in particular that in the situation of assertions (3) or (4), the colimit in question becomes finite after applying  $\tau_{\leq n}$  for any  $n$ . For (5), it suffices to observe that the smash products of functors of the form  $\{\Sigma_+^\infty \text{Hom}_{\mathcal{C}}(X, \cdot)\}_{X \in \mathcal{D}}$  are still of this form under the assumption that  $\mathcal{C}$  has finite coproducts and  $\mathcal{D} \subset \mathcal{C}$  is closed under them. It then follows by a thick subcategory argument that the subcategory of  $\mathcal{D}$ -perfect functors is closed under smash products, which implies the analogous assertion for  $\mathcal{D}$ -pseudocoherent functors.  $\square$

In order to analyze algebraic  $K$ -theory below, it will be useful to use both the  $K$ -theory space and spectrum simultaneously. Therefore we first prove a useful tool (Proposition 4.10) that for functors  $F$  taking values in connected spectra, pseudocoherence of  $F$  is equivalent to that of  $\Sigma^\infty \Omega^\infty F$ . To prove it, we need the following general result from Goodwillie calculus. Compare [2, Cor. 1.3], for instance. The tower  $\{P_n(\Sigma^\infty \Omega^\infty)\}$  is a special case of the Goodwillie tower of an arbitrary functor introduced in [35].

**Proposition 4.9.** There is a natural tower  $\{P_n = P_n(\Sigma^\infty \Omega^\infty)\}_{n \geq 0}$  of functors  $\text{Sp} \rightarrow \text{Sp}$  receiving a map from  $\Sigma^\infty \Omega^\infty$ ,

$$\Sigma^\infty \Omega^\infty X \rightarrow \{\dots \rightarrow P_n(X) \rightarrow P_{n-1}(X) \rightarrow \dots \rightarrow P_1(X)\}$$

such that:

- (1)  $P_1(X) \simeq X$  and the fiber of  $P_n(X) \rightarrow P_{n-1}(X)$  is naturally equivalent to  $(X^{\otimes n})_{h\Sigma_n}$ .
- (2) If  $X \in \mathrm{Sp}_{\geq 1}$ , the map  $\Sigma^\infty \Omega^\infty X \rightarrow \varprojlim_n P_n(X)$  is an equivalence and the connectivity of  $\Sigma^\infty \Omega^\infty X \rightarrow P_n(X)$  tends to  $\infty$  with  $n$ .

We can now prove the following tool for pseudocoherence.

**Proposition 4.10.** Let  $\mathcal{C}$  be a category with finite coproducts and let  $\mathcal{D} \subset \mathcal{C}$  be a small subcategory closed under finite coproducts. Let  $F : \mathcal{C} \rightarrow \mathrm{Sp}_{\geq 1}$  a functor. Then the following are equivalent:

- (1)  $F$  is  $\mathcal{D}$ -pseudocoherent.
- (2)  $\Sigma^\infty \Omega^\infty F$  is  $\mathcal{D}$ -pseudocoherent.
- (3)  $\Sigma_+^\infty \Omega^\infty F$  is  $\mathcal{D}$ -pseudocoherent.

*Proof.* The equivalence of (2) and (3) follows because  $\Sigma^\infty, \Sigma_+^\infty$  differ by the constant functor at  $S^0$ , which is corepresentable since  $\mathcal{C}$  has an initial object in  $\mathcal{D}$ .

Suppose  $F$  is  $\mathcal{D}$ -pseudocoherent. Using Proposition 4.9, we conclude that  $\Sigma^\infty \Omega^\infty F$  is the homotopy limit of a tower whose associated graded is given by  $F, (F^{\otimes 2})_{h\Sigma_2}, (F^{\otimes 3})_{h\Sigma_3}, \dots$ . Since  $F$  takes values in  $\mathrm{Sp}_{\geq 1}$ , the connectivity of the associated graded terms tends to  $\infty$ , so this tower stabilizes in any given finite range. As  $F$  is  $\mathcal{D}$ -pseudocoherent, so are all the graded terms in view of Proposition 4.8. It follows that  $\Sigma^\infty \Omega^\infty F$  is  $\mathcal{D}$ -pseudocoherent.

Conversely, suppose  $\Sigma^\infty \Omega^\infty F$  is  $\mathcal{D}$ -pseudocoherent. First, recall that the adjunction  $(\Sigma^\infty, \Omega^\infty)$  between the  $\infty$ -categories of pointed spaces and connective spectra is monadic since  $\Omega^\infty|_{\mathrm{Sp}_{\geq 0}}$  commutes with sifted colimits [52, Prop. 1.4.3.9]; alternatively, this follows explicitly from delooping machinery going back to [56]. As a result of monadicity, for any  $X \in \mathrm{Sp}_{\geq 0}$ , one has a natural simplicial spectrum (the bar resolution)  $(\Sigma^\infty \Omega^\infty)^{\bullet+1} X$  whose geometric realization is equivalent to  $X$ . Therefore, in our case, we can resolve  $F$  via the iterates  $(\Sigma^\infty \Omega^\infty)^k F$  for  $k \geq 1$ . By the previous direction and our assumption, each of these is  $\mathcal{D}$ -pseudocoherent. Thus we have written  $F$  as a geometric realization of  $\mathcal{D}$ -pseudocoherent functors  $\mathcal{C} \rightarrow \mathrm{Sp}_{\geq 1}$ , so  $F$  itself is  $\mathcal{D}$ -pseudocoherent.  $\square$

Finally, we observe that for functors into connective spectra, pseudocoherence can be tested after smashing with  $H\mathbb{Z}$ .

**Proposition 4.11.** Let  $\mathcal{D} \subset \mathcal{C}$  be a small subcategory. Let  $F : \mathcal{C} \rightarrow \mathrm{Sp}$  be a functor. If  $F$  is  $\mathcal{D}$ -pseudocoherent, then  $H\mathbb{Z} \otimes F$  is  $\mathcal{D}$ -pseudocoherent. Conversely, if  $F : \mathcal{C} \rightarrow \mathrm{Sp}$  is uniformly bounded-below and  $H\mathbb{Z} \otimes F$  is  $\mathcal{D}$ -pseudocoherent, so is  $F$ .

*Proof.* The first direction follows because we can approximate  $H\mathbb{Z}$  in any range by a finite spectrum. The second direction follows because a thick subcategory argument now implies that if  $H\mathbb{Z} \otimes F$  is  $\mathcal{D}$ -pseudocoherent, so is  $\tau_{\leq n} S^0 \otimes F$  for each  $n$ , and we can approximate  $F$  in any given range by  $\tau_{\leq n} S^0 \otimes F$ .  $\square$

**4.2. An axiomatic rigidity argument.** In this subsection, we present an axiomatic form, in the language of finitely generated functors, of the argument used by Gabber [25] to deduce rigidity for henselian pairs from the case of henselizations of smooth points on varieties. We will use the notion of pseudocoherence in the case where  $\mathcal{C} = \mathrm{Ring}_R^{\mathrm{nu}, \mathrm{h}}$  and  $\mathcal{D} = (\mathrm{Ring}_R^{\mathrm{nu}, \mathrm{h}})_\Sigma$  is the category of compact projective objects obtained by idempotent completing the subcategory  $\{R\{x_1, \dots, x_n\}^+ : n \geq 0\}$  of  $\mathrm{Ring}_R^{\mathrm{nu}, \mathrm{h}}$ .

**Definition 4.12.** (1) A functor  $\mathrm{Ring}_R^{\mathrm{nu}, \mathrm{h}} \rightarrow \mathrm{Ab}$  is *projectively finitely generated* if it is  $\mathcal{D}$ -finitely generated for  $\mathcal{D} = (\mathrm{Ring}_R^{\mathrm{nu}, \mathrm{h}})_\Sigma$ .

- (2) A functor  $\text{Ring}_R^{\text{nu,h}} \rightarrow \text{Sp}$  is *projectively pseudocoherent* if it is  $\mathcal{D}$ -pseudocoherent for  $\mathcal{D} = (\text{Ring}_R^{\text{nu,h}})_\Sigma$ .

**Example 4.13.** For each  $n$ , the functor  $I \mapsto \Sigma_+^\infty I^n$  (where  $I^n$  here denotes the cartesian product of  $n$  copies of  $I$ ) is projectively pseudocoherent. Indeed, it is the suspension spectrum of the corepresentable associated to  $R\{x_1, \dots, x_n\}^+ \in (\text{Ring}_R^{\text{nu,h}})_\Sigma$ .

**Example 4.14.** A projectively pseudocoherent functor  $\text{Ring}_R^{\text{nu,h}} \rightarrow \text{Sp}$  commutes with filtered colimits. In fact, the corepresentable functors  $I \mapsto \Sigma_+^\infty I^n$  clearly do, and the result for arbitrary projectively pseudocoherent functors follows now from an approximation argument. See also Corollary 4.18 below.

The following is the main technical step used in the proof of our rigidity result.

**Proposition 4.15** (Axiomatic rigidity argument). Let  $F : \text{Ring}^{\text{nu,h}} \rightarrow \text{Sp}$  be a projectively pseudocoherent functor. Suppose that:

- (1) For each prime field  $k$  and each  $n$ , we have  $F(k\{x_1, \dots, x_n\}^+) = 0$ .
- (2) Given a short exact sequence  $I' \rightarrow I \rightarrow \bar{I}$  in  $\text{Ring}^{\text{nu,h}}$ , the sequence  $F(I') \rightarrow F(I) \rightarrow F(\bar{I})$  is a fiber sequence of spectra.
- (3) If  $I \in \text{Ring}^{\text{nu,h}}$  is nilpotent, then  $F(I) = 0$ .

Then  $F = 0$ .

*Proof.* Let  $R$  be a prime field and restrict  $F$  to  $\text{Ring}_R^{\text{nu,h}}$  to obtain a functor  $F_R : \text{Ring}_R^{\text{nu,h}} \rightarrow \text{Sp}$ . We observe first that  $F$  is projectively pseudocoherent as a functor  $\text{Ring}_R^{\text{nu,h}} \rightarrow \text{Sp}$  (since the building blocks  $I \mapsto \Sigma_+^\infty I^n$  for projectively pseudocoherent functors  $\text{Ring}_R^{\text{nu,h}} \rightarrow \text{Sp}$  are independent of the base ring  $R$ ). In particular,  $F_R$  is left Kan extended from the subcategory  $(\text{Ring}_R^{\text{nu,h}})_\Sigma$ . By assumption,  $F_R$  annihilates  $(\text{Ring}_R^{\text{nu,h}})_\Sigma$ , so that  $F_R = 0$  on  $\text{Ring}_R^{\text{nu,h}}$ .

It follows that  $F$  annihilates any nonunital henselian algebra over a field. Moreover, if  $I \in \text{Ring}^{\text{nu,h}}$  is such that there exists  $N \in \mathbb{Z}_{>0}$  with  $NI = 0$ , it follows that  $F(I) = 0$ : in fact, since  $F$  preserves finite products we reduce to the case where  $N = p^r$  for some  $r$ , and then use the short exact sequence  $pI \rightarrow I \rightarrow I/pI$  where  $pI$  is nilpotent. The resulting fiber sequence then shows that  $F(I) \simeq F(I/pI) = 0$ .

Suppose  $F$  is not the zero functor. Let  $d$  be minimal such that  $\pi_d F \neq 0$ . Then  $\pi_d F$  is a projectively finitely generated functor  $\text{Ring}^{\text{nu,h}} \rightarrow \text{Ab}$  which commutes with filtered colimits; furthermore,  $\pi_d F$  annihilates any  $I \in \text{Ring}^{\text{nu,h}}$  which is either of bounded torsion or a  $\mathbb{Q}$ -vector space. Using Lemma 4.16 below, we find that  $\pi_d F = 0$ , a contradiction.  $\square$

We now give the argument for functors into abelian groups, which is simply a slight reformulation of the approach taken by Gabber [25].

**Lemma 4.16.** Let  $F_0 : \text{Ring}^{\text{nu,h}} \rightarrow \text{Ab}$  be a projectively finitely generated functor. Suppose that:

- (1)  $F_0(I) = 0$  if  $I \in \text{Ring}^{\text{nu,h}}$  is annihilated by an integer  $N > 0$ .
- (2)  $F_0(I) = 0$  if  $I \in \text{Ring}^{\text{nu,h}}$  is a nonunital  $\mathbb{Q}$ -algebra.
- (3) Given a short exact sequence  $I' \rightarrow I \rightarrow \bar{I}$  in  $\text{Ring}^{\text{nu,h}}$ , the sequence  $F_0(I') \rightarrow F_0(I) \rightarrow F_0(\bar{I}) \rightarrow 0$  of abelian groups is exact.
- (4)  $F_0$  commutes with filtered colimits.

Then  $F_0 = 0$ .

*Proof.* We will prove the following claim: there exists  $N \geq 1$  such that, for any  $I \in \text{Ring}^{\text{nu,h}}$ , the inclusion map  $i_N : NI \rightarrow I$  induces the zero map  $0 = (i_N)_* : F_0(NI) \rightarrow F_0(I)$ . Once this is done, the statement of the lemma will follow by applying (1) and (3) to the short exact sequence  $NI \rightarrow I \rightarrow I/NI$ .

As in Corollary 3.20, let  $[N] : \mathbb{Z}\{x_1, \dots, x_n\}^+ \rightarrow \mathbb{Z}\{x_1, \dots, x_n\}^+$  be the operator that multiplies each  $x_i$  by  $N$ , so that the colimit of the  $[N]$ , indexed over  $N \geq 1$  ordered by divisibility, is  $\mathbb{Q}\{x_1, \dots, x_n\}^+$ . By assumption  $F_0$  commutes with filtered colimits and annihilates any nonunital henselian  $\mathbb{Q}$ -algebra; so, given any element  $u \in F_0(\mathbb{Z}\{x_1, \dots, x_n\}^+)$ , it follows that there exists  $N > 0$  such that  $F_0([N])(u) = 0$ .

Now we use that  $F_0$  is projectively finitely generated. This means that there exists a surjection

$$\bigoplus_{i=1}^t \mathbb{Z}[\text{Hom}_{\text{Ring}^{\text{nu,h}}}(\mathbb{Z}\{x_1, \dots, x_{n_i}\}^+, \cdot)] \rightarrow F_0$$

of functors for some  $n_1, \dots, n_t \geq 1$ . For each  $i = 1, \dots, t$ , let  $g_i \in F_0(\mathbb{Z}\{x_1, \dots, x_{n_i}\}^+)$  be the image of  $\text{id} \in \text{Hom}_{\text{Ring}^{\text{nu,h}}}(\mathbb{Z}\{x_1, \dots, x_{n_i}\}^+, \mathbb{Z}\{x_1, \dots, x_{n_i}\}^+)$ . Then the above surjection concretely means the following: given any  $I \in \text{Ring}^{\text{nu,h}}$  and any element  $y \in F_0(I)$ , then  $y$  is a finite sum of elements of  $F_0(I)$  obtained by pushing forward the  $g_i$  along various maps  $\phi : \mathbb{Z}\{x_1, \dots, x_{n_i}\}^+ \rightarrow I$ .

By the second paragraph, there exists an integer  $N > 0$  such that  $[N]_*(g_i) = 0$  for all the generators  $g_i$ . From this we can easily complete the proof of the claim, as follows. Given any  $I \in \text{Ring}^{\text{nu,h}}$ , every map  $\phi : \mathbb{Z}\{x_1, \dots, x_{n_i}\}^+ \rightarrow NI$  has the property that the composite  $i_N \circ \phi : \mathbb{Z}\{x_1, \dots, x_{n_i}\}^+ \rightarrow I$  factors through  $[N]$ , i.e., one has a commutative diagram in  $\text{Ring}^{\text{nu,h}}$

$$\begin{array}{ccc} \mathbb{Z}\{x_1, \dots, x_{n_i}\}^+ & \xrightarrow{\phi} & NI \\ \downarrow [N] & & \downarrow i_N \\ \mathbb{Z}\{x_1, \dots, x_{n_i}\}^+ & \longrightarrow & I \end{array}$$

By choice of  $N$  it follows that the composite

$$\mathbb{Z}[\text{Hom}_{\text{Ring}^{\text{nu,h}}}(\mathbb{Z}\{x_1, \dots, x_{n_i}\}^+, NI)] \rightarrow F_0(NI) \rightarrow F_0(I)$$

is zero. But we have shown that every element of  $F_0(NI)$  is a finite sum of elements of the form  $\phi_*(g_i)$  for various maps  $\phi$ . Since  $g_i \in F_0(\mathbb{Z}\{x_1, \dots, x_{n_i}\}^+)$  is annihilated by  $[N]$ , it follows that  $(i_N)_* : F_0(NI) \rightarrow F_0(I)$  is zero, as claimed.  $\square$

This completes the proofs of the main results from the present subsection. For the convenience of the reader, we recall that the process of Kan extension which appeared in our notion of projective pseudocoherence has an alternative description via simplicial resolutions (which will not be used in the sequel).

**Construction 4.17.** Let  $\mathcal{E}$  be an  $\infty$ -category that has sifted colimits, fix a base ring  $R$  and a functor

$$F : \text{Ring}_R^{\text{nu,h}} \rightarrow \mathcal{E},$$

and consider the following condition on  $F$ :

(Kan)  $F$  is left Kan extended from the subcategory  $(\text{Ring}_R^{\text{nu,h}})_\Sigma \subset \text{Ring}_R^{\text{nu,h}}$ .

Suppose  $F$  satisfies (Kan). Then one can compute  $F$  from its values on  $(\text{Ring}_R^{\text{nu,h}})_\Sigma$  as follows. First,  $F$  commutes with filtered colimits, and hence its value is determined on all free objects in  $\text{Ring}_R^{\text{nu,h}}$ . Next, if  $I$  is an arbitrary nonunital henselian  $R$ -algebra, then there exists an augmented simplicial object  $I_\bullet : \Delta_+^{\text{op}} \rightarrow \text{Ring}_R^{\text{nu,h}}$  such that  $I_{-1} = I$ , each  $I_i$  is a free algebra in  $\text{Ring}_R^{\text{nu,h}}$ , and  $|I_\bullet| \simeq I_{-1} = I$ , i.e.,  $I_\bullet$  is a simplicial resolution of  $I$  by free objects; then the value of  $F$  on  $I$  is given by  $F(I) = |F(I_\bullet)|$ .

The construction of such simplicial resolutions (and the independence of choices) is a general homotopical technique going back to Quillen [65], originally developed to build the cotangent complex via simplicial commutative rings. Using the results of [65] (which work in particular for any category satisfying (Law) from Remark 3.10, by [65, Rmk. 1, pg. II.4.2]), we can make the category  $\text{Fun}(\Delta^{\text{op}}, \text{Ring}_R^{\text{nu}})$  of simplicial nonunital henselian  $R$ -algebras into a model category where the weak equivalences and fibrations are those of underlying simplicial sets. The resolution  $I_\bullet$  of  $I$  is then obtained as a cofibrant replacement in this model category of the constant simplicial object  $I$ . This can also be phrased using the language of nonabelian derived  $\infty$ -categories [51, Sec. 5.5.8].

We record these observations for the future in the following corollary which does not mention simplicial non-unital henselian rings.

**Corollary 4.18.** Let  $\mathcal{E}$  be an  $\infty$ -category with sifted colimits, and let  $F : \text{Ring}_R^{\text{nu,h}} \rightarrow \mathcal{E}$  be a functor. Then the following are equivalent:

- (1)  $F$  is left Kan extended from  $(\text{Ring}_R^{\text{nu,h}})_\Sigma \subset \text{Ring}_R^{\text{nu,h}}$ .
- (2) Whenever  $I_\bullet$  is a simplicial object in  $\text{Ring}_R^{\text{nu,h}}$  whose geometric realization is equivalent to  $I_{-1} \in \text{Ring}_R^{\text{nu,h}}$  (in particular, the homotopy type of the underlying simplicial set of  $I_\bullet$  is discrete), then the map  $|F(I_\bullet)| \rightarrow F(I_{-1})$  is an equivalence. Furthermore,  $F$  commutes with filtered colimits.

**4.3. Pseudocoherence of  $K, \text{TC}$ .** In this subsection, we show that  $K$ -theory and  $\text{TC}$  satisfy the projective pseudocoherence property studied in subsection 4.2. Given a space  $X$ , we write  $C_*(X; \mathbb{Z}) = H\mathbb{Z} \otimes \Sigma_+^\infty X$  for the singular chains on  $X$  with  $\mathbb{Z}$ -coefficients, viewed as a (generalized) Eilenberg-MacLane spectrum.

**Proposition 4.19.** The functor  $\text{Ring}^{\text{nu,h}} \rightarrow \text{Sp}$  given by  $I \mapsto C_*(BGL_\infty(\mathbb{Z} \rtimes I); \mathbb{Z})$  is projectively pseudocoherent.

*Proof.* First, by the homological stability results of Maazen and van der Kallen [75], we can approximate  $C_*(BGL_\infty(\mathbb{Z} \rtimes I); \mathbb{Z})$  by  $C_*(BGL_n(\mathbb{Z} \rtimes I); \mathbb{Z})$  in any given finite range. It is crucial for us that this stability range is independent of the choice of  $I$ : namely, the stability range depends on the Krull dimension of the maximal spectrum of  $\mathbb{Z} \rtimes I$ , but since  $I$  is contained in the Jacobson radical this is the maximal spectrum of  $\mathbb{Z}$ .

Thus, it suffices to show that for any  $n$ , the functor  $I \mapsto C_*(BGL_n(\mathbb{Z} \rtimes I); \mathbb{Z})$  is projectively pseudocoherent. We use the short exact sequence of groups

$$1 \rightarrow GL_n(1 + I) \rightarrow GL_n(\mathbb{Z} \rtimes I) \rightarrow GL_n(\mathbb{Z}) \rightarrow 1,$$

where  $GL_n(1 + I)$  is, by definition, the kernel of the second map in the above sequence. It follows that there is a  $GL_n(\mathbb{Z})$ -action on  $C_*(BGL_n(1 + I); \mathbb{Z})$  and we have

$$(20) \quad C_*(BGL_n(\mathbb{Z} \rtimes I); \mathbb{Z}) \simeq C_*(BGL_n(1 + I); \mathbb{Z})_{hGL_n(\mathbb{Z})}.$$

We next argue that the functor  $I \mapsto C_*(GL_n(1 + I); \mathbb{Z})$  is projectively pseudocoherent. Note that as a set,  $GL_n(1 + I)$  is naturally isomorphic to the cartesian product  $I^{n^2}$  since  $I$  is contained in the

radical of  $\mathbb{Z} \times I$ . Now we observe that using the classical bar construction and the isomorphism (of sets)  $GL_n(1+I) \simeq I^{n^2}$ , that  $C_*(GL_n(1+I); \mathbb{Z})$  is a geometric realization of functors of the form  $I \mapsto \mathbb{Z}[I^{n^2 i}]$  for  $i \geq 0$ . Since the functor  $I \mapsto \mathbb{Z}[I^{n^2 i}] \simeq H\mathbb{Z} \otimes \Sigma_+^\infty \text{Hom}_{\text{Ring}^{\text{nu,h}}}(\mathbb{Z}\{x_1, \dots, x_{n^2 i}\}^+, I)$  is clearly projectively pseudocoherent, it follows that  $I \mapsto C_*(GL_n(1+I); \mathbb{Z})$  is projectively pseudocoherent as desired.

Finally, the group  $GL_n(\mathbb{Z})$  admits a finite index normal subgroup  $N \leq GL_n(\mathbb{Z})$  such that  $BN$  has the homotopy type of a finite CW complex, by the existence of the Borel-Serre compactification (see [67] for a survey). Since  $(-)_h GL_n(\mathbb{Z}) \simeq ((-)_h N)_{h(G/N)}$ , we conclude by two applications of Proposition 4.8 that taking  $GL_n(\mathbb{Z})$ -homotopy orbits preserves projective pseudocoherence. Therefore, we conclude in view of the previous paragraph and (20).  $\square$

The following lemma is well-known:

**Lemma 4.20.** Let  $(R, I)$  be a henselian pair. Then  $K_0(R) \simeq K_0(R/I)$ .

*Proof.* In fact, we claim that isomorphism classes of finitely generated projective  $R$  modules and isomorphism classes of finitely generated projective  $R/I$ -modules agree. For one direction, if  $M, N$  are finitely generated projective  $R$ -modules, then any isomorphism  $M/IM \simeq N/IM$  can be lifted to a map  $M \rightarrow N$ , which is necessarily an isomorphism. For the other direction, by lifting idempotents any projective  $R/I$ -module lifts to  $R$ , cf. [70, Tag 0D49].  $\square$

**Proposition 4.21.** The functor  $\text{Ring}^{\text{nu,h}} \rightarrow \text{Sp}$  given by  $I \mapsto K(\mathbb{Z} \times I)$  is projectively pseudocoherent.

*Proof.* Since  $I$  is henselian, we have  $K_0(\mathbb{Z} \times I) = \mathbb{Z}$  by Lemma 4.20; since the constant functor  $\mathbb{Z}$  is projectively pseudocoherent, it remains to see that the functor  $I \mapsto \tau_{\geq 1} K(\mathbb{Z} \times I)$  is projectively pseudocoherent. By Propositions 4.10 and 4.11, it suffices to check that the functor

$$I \mapsto H\mathbb{Z} \otimes \Sigma_+^\infty \Omega^\infty \tau_{\geq 1} K(\mathbb{Z} \times I) \simeq C_*(GL_\infty(\mathbb{Z} \times I); \mathbb{Z})$$

is projectively pseudocoherent, which follows from Proposition 4.19 above.  $\square$

**Remark 4.22.** In particular, Proposition 4.21 shows that the functor  $\text{Ring}^{\text{nu,h}} \rightarrow \text{Sp}$ ,  $I \mapsto K(\mathbb{Z} \times I)$  commutes with simplicial resolutions in view of Construction 4.17. The same argument shows that this also holds in the larger category of nonunital local rings, even without the henselian condition. This observation, in various forms, plays an important role in the study of the local structure of  $K$ -theory. Compare, for instance [34, Lemma I.2.2] for the nilpotent case and [20, Ch. III, Prop. 1.4.2] for a more general assertion for radical pairs.

We next carry out analogous arguments for topological cyclic homology. This is considerably simpler and does not rely on tools such as homological stability; we will instead use the finiteness properties of cyclotomic spectra from Section 2.

**Lemma 4.23.** The Eilenberg–MacLane functor  $\text{Ring}^{\text{nu,h}} \rightarrow \text{Sp}$  given by  $I \mapsto HI$  is projectively pseudocoherent.

*Proof.* It is equivalent to check that the functor  $I \mapsto \Sigma HI$  is projectively pseudocoherent. For this, it suffices by Proposition 4.10 to check that the functor  $I \mapsto \Sigma_+^\infty \Omega^\infty(\Sigma HI) \otimes H\mathbb{Z} \simeq C_*(BI; \mathbb{Z})$  is projectively pseudocoherent. Now the bar construction of  $BI$  as a simplicial space implies the desired projective pseudocoherence: in fact,  $C_*(BI; \mathbb{Z})$  is a geometric realization of objects of the form  $\mathbb{Z}[I^n]$ .  $\square$

**Lemma 4.24.** The functor  $\text{Ring}^{\text{nu,h}} \rightarrow \text{Sp}$  given by  $I \mapsto \text{THH}(\mathbb{Z} \times I)$  is projectively pseudocoherent.

*Proof.* Using the cyclic bar construction for THH, this follows because all the terms  $(H(\mathbb{Z} \times I))^{\otimes k}$  (of which THH is a geometric realization) are projectively pseudocoherent in view of Lemma 4.23 and Proposition 4.8.  $\square$

**Proposition 4.25.** The functor  $\text{Ring}^{\text{nu,h}} \rightarrow \text{Sp}$  given by  $I \mapsto \text{TC}(\mathbb{Z} \times I)/p$  is projectively pseudocoherent.

*Proof.* This now follows from Lemma 4.24 and Proposition 2.19; indeed, for any of the functors  $F$  in the statement of the latter, we have that  $I \mapsto F(\text{THH}(\mathbb{Z} \times I))$  is projectively pseudocoherent, and then we can approximate  $\text{TC}/p$  in any range.  $\square$

**4.4. Equal characteristic case.** We next prove a special case of the rigidity result in equal characteristic. We begin by reviewing results of Geisser–Levine [32] and Geisser–Hesselholt [28] in a formulation that will be convenient for us. Compare also [27] for a survey treatment. Our main result here (Proposition 4.31) is a special case of the rigidity statement in the case of a smooth henselian pair over a perfect field of characteristic  $p$ . We keep the notation and terminology of the introduction. In particular, we will use the cyclotomic trace  $K(R) \rightarrow \text{TC}(R)$  for a ring  $R$ , and denote by  $K^{\text{inv}}(R)$  the homotopy fiber of this map. Recall also (Definition 2.23) the inverse Cartier operator on differential forms.

**Definition 4.26.** For an  $\mathbb{F}_p$ -algebra  $R$ , we let  $\nu^n(R) = \ker(1 - C^{-1} : \Omega_R^n \rightarrow \Omega_R^n/d\Omega_R^{n-1})$ . We will also write this as  $\Omega_{R,\log}^n$ . We also let  $\widehat{\nu}^n(R)$  be the cokernel of  $1 - C^{-1}$ .

The construction  $\nu^n$  thus defines a sheaf for the étale topology. To see this, we observe that it is a kernel of a map between two objects, both of which are quasi-coherent over the Frobenius twist by the next lemma. By contrast,  $\widehat{\nu}^n$  vanishes locally in the étale topology since  $1 - C^{-1}$  is surjective locally in the étale topology. Moreover,  $\nu^0$  is the constant sheaf  $\mathbb{F}_p$  and  $\nu^n(R) \subset \Omega_R^n$  is the subgroup generated étale locally by differential forms  $d \log x_1 \wedge \cdots \wedge d \log x_n$  for  $x_1, \dots, x_n$  units (cf. [45, Thm. 2.4.2] for the smooth case, and [59, Cor. 4.2] in general).

**Lemma 4.27.** Let  $R \rightarrow S$  be an étale map of  $\mathbb{F}_p$ -algebras. Let  $R^{(1)} \rightarrow R, S^{(1)} \rightarrow S$  be the Frobenius twists of  $R$  and  $S$ .<sup>5</sup> Then:

- (1) The map  $\Omega_R^n \otimes_R S \rightarrow \Omega_S^n$  is an isomorphism.
- (2) The de Rham differentials  $d : \Omega_R^{n-1} \rightarrow \Omega_R^n, \Omega_S^{n-1} \rightarrow \Omega_S^n$  are respectively  $R^{(1)}, S^{(1)}$ -linear so that the quotients  $\Omega_R^n/d\Omega_R^{n-1}, \Omega_S^n/d\Omega_S^{n-1}$  inherit the structure of  $R^{(1)}, S^{(1)}$ -modules respectively. The map  $\Omega_R^n/d\Omega_R^{n-1} \otimes_{R^{(1)}} S^{(1)} \rightarrow \Omega_S^n/d\Omega_S^{n-1}$  is an isomorphism.

*Proof.* Part (1) is standard. Since  $R \rightarrow S$  is étale, the natural square linking  $R^{(1)}, R, S^{(1)}$  and  $S$  is cocartesian, which now implies (2).  $\square$

We will need this definition in light of the following fundamental results about the structure of  $K$ -theory and TC for ind-smooth  $\mathbb{F}_p$ -algebras.

**Theorem 4.28.** Let  $R$  be an ind-smooth  $\mathbb{F}_p$ -algebra. Then, for each  $n \geq 0$ , one has:

- (1) (Geisser–Levine [32]) There is a natural map  $\pi_n(K(R)/p) \rightarrow \nu^n(R)$ , which is an isomorphism if  $R$  is local.

<sup>5</sup>Thus  $R^{(1)}$  consists of formal expressions of the form “ $a^p$ ”, and the map to  $R$  is “ $a^p \mapsto a^p$ ”. More straightforwardly, we can set  $R^{(1)} = R$  and take the map  $R^{(1)} = R \rightarrow R$  to be the Frobenius.



- (2) (Geisser–Hesselholt [28]) There is a functorial exact sequence  $0 \rightarrow \tilde{\nu}^{n+1}(R) \rightarrow \pi_n(\mathrm{TC}(R)/p) \rightarrow \nu^n(R) \rightarrow 0$ . Furthermore, under these identifications, the composite of  $\pi_n(K(R)/p) \rightarrow \pi_n(\mathrm{TC}(R)/p) \rightarrow \nu^n(R)$  (where the first map arises from the cyclotomic trace) is the map of (1).
- (3) If  $R$  is local, we have a functorial identification  $\pi_n(K^{\mathrm{inv}}(R)/p) \simeq \tilde{\nu}^{n+2}(R)$ .

*Proof.* First we note that it suffices to prove this theorem in the case where  $R$  is essentially of finite type over  $\mathbb{F}_p$ , by extending using filtered colimits. Then the first assertion follows from [32, Thm. 8.3], which shows that on smooth varieties over  $\mathbb{F}_p$ , the Zariski sheafification of the presheaf  $\pi_n(K(\cdot)/p)$  is given by  $\nu^n$ .

The exact sequence describing  $\pi_n(\mathrm{TC}(R)/p)$  actually follows from (12), though we will give a slightly different proof of the stronger second claim. Namely, we use the results of Geisser–Hesselholt [28, Thm. 4.2.6] that on smooth quasi-compact quasi-separated schemes over  $\mathbb{F}_p$ , the homotopy group sheaves in the *étale* topology of  $\mathrm{TC}/p$  identify with those of  $K/p$  via the cyclotomic trace, and hence identify with the  $\nu^n$ ; and moreover one has an *étale* descent spectral sequence starting from the *étale* cohomology of  $\nu^n$  and converging to  $\pi_*(\mathrm{TC}/p)$ . In the *étale* topology, there is a short exact sequence of sheaves  $0 \rightarrow \nu^n \rightarrow \Omega^n \xrightarrow{1-C^{-1}} \Omega^n/d\Omega^{n-1} \rightarrow 0$ , and the second and third terms are quasi-coherent (either over the structure sheaf or its Frobenius twist by Lemma 4.27), so have no higher cohomology on affines. It follows that  $H^0((\mathrm{Spec}R)_{\mathrm{ét}}, \nu^n) = \nu^n(R)$ ,  $H^1((\mathrm{Spec}R)_{\mathrm{ét}}, \nu^n) = \tilde{\nu}^n(R)$ , and the *étale* descent spectral sequence thus implies the second claim. The third claim follows from the first two.  $\square$

**Lemma 4.29.** Let  $(R, I)$  be a henselian pair. Let  $s \in I$  and let  $n \geq 1$ . Then the equation  $x - sx^n = 1$  can be solved in  $R$ .

*Proof.* The equation has a simple root in  $R/I$  (namely,  $x = 1$ ), which therefore admits a lift to  $R$  by definition of henselian.  $\square$

**Proposition 4.30.** Let  $(R, I)$  be a henselian pair of  $\mathbb{F}_p$ -algebras. Then for each  $n$ , the map  $\nu^n(R) \rightarrow \nu^n(R/I)$  is surjective and the map  $\tilde{\nu}^n(R) \rightarrow \tilde{\nu}^n(R/I)$  is an isomorphism.

*Proof.* We use the commutative diagram

$$\begin{array}{ccc}
 V_0 & \longrightarrow & V_1 \\
 \downarrow & & \downarrow \\
 \Omega_R^n & \xrightarrow{1-C^{-1}} & \Omega_R^n/d\Omega_R^{n-1} \\
 \downarrow & & \downarrow \\
 \Omega_{R/I}^n & \xrightarrow{1-C^{-1}} & \Omega_{R/I}^n/d\Omega_{R/I}^{n-1}
 \end{array}$$

where we define  $V_0, V_1$  to be the kernels of the surjective vertical maps  $\Omega_R^n \rightarrow \Omega_{R/I}^n$  and  $\Omega_R^n/d\Omega_R^{n-1} \rightarrow \Omega_{R/I}^n/d\Omega_{R/I}^{n-1}$ . We will show that the map  $V_0 \rightarrow V_1$  is surjective. This easily implies the desired conclusions about the maps  $\nu^n(R) \rightarrow \nu^n(R/I)$  and  $\tilde{\nu}^n(R) \rightarrow \tilde{\nu}^n(R/I)$  thanks to the snake lemma.

Consider a differential form  $\omega = adx_1dx_2 \dots dx_n \in \Omega_R^n$  such that one of  $\{a, x_1, \dots, x_n\}$  belongs to  $I$  (here  $n = 0$  is allowed). The image of such a class in  $\Omega_R^n/d\Omega_R^{n-1}$  belongs to  $V_1$ , and  $V_1$  is generated by such classes. For  $u \in R$ , we have

$$(1 - C^{-1})(u\omega) = (u - u^p a^{p-1} x_1^{p-1} \dots x_n^{p-1})\omega.$$

Since  $(R, I)$  is a henselian pair and  $a^{p-1}x_1^{p-1} \dots x_n^{p-1} \in I$ , we can choose  $u \in R$  such that  $u - u^p a^{p-1} x_1^{p-1} \dots x_n^{p-1} = 1$ , using Lemma 4.29. The class  $u\omega \in \Omega_R^n$  belongs to  $V_0$  and has image given by  $\omega$ ; so  $\omega$  is in the image of  $V_0 \rightarrow V_1$ , as desired.  $\square$

**Proposition 4.31.** Let  $(R, \mathfrak{m})$  be an ind-smooth henselian local  $\mathbb{F}_p$ -algebra with residue field  $k$ . Then the map  $K^{\text{inv}}(R) \rightarrow K^{\text{inv}}(k)$  becomes an equivalence modulo  $p$ .

*Proof.* By Theorem 4.28, it suffices to show that the map  $\tilde{v}^n(R) \rightarrow \tilde{v}^n(k)$  is an isomorphism. This follows from Proposition 4.30.  $\square$

**4.5. Proof of the main result.** In this subsection, we prove the main result of this paper, Theorem 4.35. Our goal is to show that the construction  $R \mapsto K^{\text{inv}}(R)$  with mod  $p$  coefficients is invariant under taking the quotient by a henselian ideal.

A key ingredient in the proof will be to use results of Geisser–Hesselholt [29] about excision in  $K$ -theory and topological cyclic homology to relate the setup of a general henselian pair to the case  $(\mathbb{Z} \times I, I)$ , where we can appeal to the finiteness results of the previous subsection.

We start by discussing these excision results. For this, we need to invoke the non-connective variant  $\mathbb{K}^{\text{inv}}$  of  $K^{\text{inv}}$ , defined as the fiber of the cyclotomic trace  $\mathbb{K} \rightarrow \text{TC}$  from non-connective  $K$ -theory. Recall also the standard notation  $K(R, I) = \text{fib}(K(R) \rightarrow K(R/I))$ , and similarly for any other functor on rings.

The following was proved in the rational case in [17], with finite coefficients in [29], and in general in [21]. Here commutativity is not necessary.

**Theorem 4.32** (Cortiñas; Geisser–Hesselholt; Dundas–Kittang). Suppose  $R$  is a unital associative ring,  $I \subset R$  a two-sided ideal, and  $f : R \rightarrow S$  a homomorphism such that  $f$  restricts to an isomorphism from  $I$  to a two-sided ideal  $J$  of  $S$  (so one has a Milnor square in the sense of Definition 3.17). Then the induced map

$$\mathbb{K}^{\text{inv}}(R, I) \rightarrow \mathbb{K}^{\text{inv}}(S, J)$$

is an equivalence.

This can be read as saying that  $\mathbb{K}^{\text{inv}}(R, I)$  “only depends on  $I$ .” To make this precise, one can define  $\mathbb{K}^{\text{inv}}(I) = \mathbb{K}^{\text{inv}}(\mathbb{Z} \times I, I)$  for any non-unital ring  $I$ . Clearly the new  $\mathbb{K}^{\text{inv}}$  restricts to the old one on unital rings; moreover, Theorem 4.32 implies that  $\mathbb{K}^{\text{inv}}$  sends short exact sequences  $I' \rightarrow I \rightarrow I''$  of non-unital rings to fiber sequences of spectra. In particular,  $\mathbb{K}^{\text{inv}}(I) \simeq \mathbb{K}^{\text{inv}}(R, I)$  whenever  $I$  embeds as a two-sided ideal in  $R$ .

For our purposes, we will need the analog of Theorem 4.32 for connective  $K$ -theory, in the context of non-unital henselian (commutative) rings. We can make the switch thanks to the following proposition, whose proof uses another excision theorem due to Bass–Milnor–Swan:

**Proposition 4.33.** Suppose  $(R, I)$  is a henselian pair, and  $f : R \rightarrow S$  is a ring homomorphism such that  $f$  restricts to an isomorphism from  $I$  to an ideal  $J$  of  $S$ . Then the square of spectra

$$(21) \quad \begin{array}{ccc} K(R, I) & \longrightarrow & K(S, J) \\ \downarrow & & \downarrow \\ \mathbb{K}(R, I) & \longrightarrow & \mathbb{K}(S, J) \end{array}$$

is cartesian.

*Proof.* Let  $F$  be the fiber of the top horizontal map and let  $\mathbb{F}$  be the fiber of the bottom horizontal map. Note that the maps  $K(R) \rightarrow \mathbb{K}(R)$ ,  $K(R/I) \rightarrow \mathbb{K}(R/I)$ , etc. are equivalences in degrees  $\geq 0$ ; thus, taking fibers, the maps  $K(R, I) \rightarrow \mathbb{K}(R, I)$  and  $K(S, J) \rightarrow \mathbb{K}(S, J)$  are equivalences in degrees  $\geq 0$ . Taking fibers again, we find that the map  $F \rightarrow \mathbb{F}$  is an equivalence in degrees  $\geq 0$ .

On the other hand,  $\mathbb{F}$  is concentrated in degrees  $\geq 0$  (even  $\geq 1$ ) by the excision theorem of Bass and Bass–Heller–Swan, [7, Thm. XII.8.3]. Thus it suffices to show that  $F$  is also concentrated in degrees  $\geq 0$ .

But since  $(R, I)$  and  $(S, J)$  are henselian pairs, it follows by Lemma 4.20 that the maps  $K_0(R) \rightarrow K_0(R/I)$ ,  $K_0(S) \rightarrow K_0(S/J)$  are isomorphisms and the maps  $K_1(R) \rightarrow K_1(R/I)$ ,  $K_1(S) \rightarrow K_1(S/J)$  (which are the abelianizations of  $GL_\infty(R) \rightarrow GL_\infty(R/I)$ ,  $GL_\infty(S) \rightarrow GL_\infty(S/J)$ ) are surjections; thus  $K(R, I)$  and  $K(S, J)$  are concentrated in degrees  $\geq 1$ , so that  $F$  is concentrated in degrees  $\geq 0$ , as desired.  $\square$

**Corollary 4.34.** Suppose  $(R, I)$  is a henselian pair, and  $f : R \rightarrow S$  is a ring homomorphism which restricts to an isomorphism from  $I$  to an ideal  $J \subset S$ . Then the map  $K^{\text{inv}}(R, I) \rightarrow K^{\text{inv}}(S, J)$  is an equivalence.

*Proof.* Combining with the homotopy cartesian square (21) between relative connective and non-connective  $K$ -theory, we find that the result now follows from Theorem 4.32.  $\square$

Again, this can be interpreted as saying that  $K^{\text{inv}}$  makes sense for non-unital henselian rings. We can now state and prove the main result of this paper.

**Theorem 4.35.** Let  $(R, I)$  be a henselian pair. Then for any prime number  $p$ , the map  $K^{\text{inv}}(R) \rightarrow K^{\text{inv}}(R/I)$  becomes an equivalence modulo  $p$ . Equivalently, the map  $K(R, I) \rightarrow \text{TC}(R, I)$  becomes an equivalence modulo  $p$ .

*Proof.* By Corollary 4.34, it suffices to consider the case where  $(R, I) = (\mathbb{Z} \times I, I)$  where  $I$  is a nonunital henselian ring. We now consider the functor

$$F : \text{Ring}^{\text{nu,h}} \rightarrow \text{Sp}, \quad F(I) = K^{\text{inv}}(\mathbb{Z} \times I, I)/p.$$

By Propositions 4.21 and 4.25,  $F$  is a projectively pseudocoherent functor. Implicit here is the statement (due to Quillen) that the  $K$ -groups of  $\mathbb{Z}$  are finitely generated, as are the mod  $p$  homotopy groups of  $\text{TC}(\mathbb{Z})$  (both of which follow from evaluating the projectively pseudocoherent functors  $I \mapsto K(\mathbb{Z} \times I)$ ,  $\text{TC}(\mathbb{Z} \times I)/p$  at  $I = 0$ ).

The assertion of the theorem is that  $F = 0$ . We will show this by checking that  $F$  satisfies the hypotheses of Proposition 4.15.

First observe that  $F$  sends a short exact sequence  $I' \rightarrow I \twoheadrightarrow \bar{I}$  of nonunital henselian rings to a fiber sequence of spectra. To see this, we consider the diagram

$$\begin{array}{ccccc} F(I') & \longrightarrow & F(I) & \longrightarrow & F(\bar{I}) \\ \downarrow & & \downarrow & & \downarrow \\ F(I') & \longrightarrow & K^{\text{inv}}(\mathbb{Z} \times I)/p & \longrightarrow & K^{\text{inv}}(\mathbb{Z} \times \bar{I})/p. \end{array}$$

Note that the right-hand square is homotopy cartesian, so the top row is a fiber sequence if and only if the bottom row is a fiber sequence. But the bottom row is a fiber sequence by Corollary 4.34 applied to the homomorphism  $(\mathbb{Z} \times I', I') \rightarrow (\mathbb{Z} \times I, I')$ .

Next, we claim that if  $I' \in \text{Ring}^{\text{nu,h}}$  is nilpotent, then  $F(I') = 0$ . This follows from Theorem 1.2 which shows that  $K^{\text{inv}}(\mathbb{Z} \times I', I') = 0$ .

To complete the proof, i.e., to verify the conditions of Proposition 4.15, we need to check that

$$F(k\{x_1, \dots, x_n\}^+) = 0,$$

whenever  $k$  is a prime field. Using Corollary 4.34, it suffices to show that the map  $K^{\text{inv}}(k\{x_1, \dots, x_n\}) \rightarrow K^{\text{inv}}(k)$  is an equivalence modulo  $p$ . There are two cases for this. If  $\text{char}(k) \neq p$ , this follows from Gabber rigidity. If  $\text{char}(k) = p$ , this follows from Proposition 4.31.  $\square$

**Remark 4.36.** Recall that Gabber’s proof [25] of rigidity is cleanly separated into two halves: the first half reduces the general case to the case of henselizations of smooth algebras over a field at a rational point, and the second half proves that case. In our proof above, we only need to invoke the second half of Gabber’s work, which is also covered by Gillet–Thomason [33].

However, our entire line of reasoning is in some sense modeled on the first half of Gabber’s proof. (The exceptions are Proposition 4.31, which is the new characteristic  $p$  ingredient, and the commutation of  $\text{TC}/p$  with filtered colimits, which is a necessary technical statement.) So we are not really avoiding the first half of Gabber’s proof, just explicating it in a modified context. Let us note in particular that the projective pseudocoherence of  $K$ -theory is our replacement for Suslin’s “method of universal homotopies” from [72], which has been crucial to many of the known rigidity results in  $K$ -theory.

**Remark 4.37.** While the the full strength of pseudocoherence appears to be necessary to obtain results for  $\mathbb{Z}$ -algebras, the result for henselian pairs of  $\mathbb{F}_p$ -algebras follows from Proposition 4.31 and the observation that  $K, \text{TC}/p$  commutes with filtered colimits of  $\mathbb{F}_p$ -algebras. One can then extend the result to  $\mathbb{Z}_p$ -algebras using the  $p$ -adic continuity statement of [30] (and a similar filtered colimit argument). Thus, for  $\mathbb{Z}_p$ -algebras at least, one can prove the main result purely using the classical approach to TC and ingredients predating [60].

## 5. CONTINUITY AND PRO STATEMENTS IN ALGEBRAIC $K$ -THEORY

In this section we consider various applications to the continuity problem in algebraic  $K$ -theory and to the related problem of describing the pro  $K$ -theory of formal schemes. In particular we show that, under mild hypotheses, algebraic  $K$ -theory with finite coefficients is continuous for complete noetherian rings (Theorem 5.5). We also show that algebraic  $K$ -theory satisfies a “derived” form of  $p$ -adic continuity for rings that are henselian along  $p$  (Theorem 5.22), extending results of Geisser–Hesselholt [30]. Finally, we prove a pro version of the Geisser–Levine [32] theorem on the  $p$ -adic  $K$ -theory of regular local  $\mathbb{F}_p$ -algebras to describe (under mild hypotheses) the pro abelian groups  $\{K_*(A/I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s$  when  $A$  is a regular local  $\mathbb{F}_p$ -algebra and  $I \subset A$  is any ideal, extending results of Morrow [59].

**5.1. Continuity and  $K$ -theory.** In this and the next subsection, we consider the following classical continuity question in  $K$ -theory.

**Question.** Let  $R$  be a ring and  $I$  be an ideal. How close is the map

$$(22) \quad K(R) \longrightarrow \varprojlim_s K(R/I^s)$$

to being an equivalence?

In order for this question to be reasonable, we should assume that  $R$  is  $I$ -adically complete, or at least that  $(R, I)$  forms a henselian pair. This question has been considered by various authors, and the above map has notably been shown to be an equivalence modulo  $p$  in the following cases, which we order historically:

- (1)  $R$  a complete discrete valuation ring of mixed characteristic  $(0, q)$ ,  $q \neq p$ , with  $I$  being the maximal ideal (Suslin [72]).
- (2)  $R$  a complete discrete valuation of mixed characteristic  $(0, p)$ , with  $I$  being the maximal ideal (Panin [61]);
- (3)  $(R, I)$  a henselian pair and  $p$  invertible in  $R$  (Gabber rigidity [25]; this subsumes case (1)).
- (4)  $R$  a complete discrete valuation ring of equal characteristic  $p$  with perfect residue field, with  $I$  being the maximal ideal, in degrees  $\leq 4$  (Dundas [19]).
- (5)  $R = A[[x_1, \dots, x_n]]$  and  $I = (x_1, \dots, x_n)$ , where  $A$  is any  $F$ -finite, regular, local  $\mathbb{F}_p$ -algebra (Geisser–Hesselholt [31]; this subsumes case (4)).
- (6)  $R$  any ring of finite stable rank in which  $p$  is a non-zero-divisor and which is henselian along  $I = pR$  (Geisser–Hesselholt [30])
- (7)  $R$  any  $F$ -finite, regular, local  $\mathbb{F}_p$ -algebra which is complete with respect to an ideal  $I \subseteq R$  such that  $R/I$  is “generalised normal crossings” (Morrow [59]; this subsumes case (5)).

In order to apply our earlier results to this question, we observe that if  $(R, I)$  is a henselian pair then Theorem 4.35 implies that the map (22) becomes an equivalence modulo  $p$  if and only if the corresponding map  $\mathrm{TC}(R) \rightarrow \varprojlim_s \mathrm{TC}(R/I^s)$  becomes an equivalence modulo  $p$ . This latter question is often more tractable and has been carefully studied in the recent work of Dundas–Morrow [22], who show that such continuity for topological cyclic homology holds quite generally under the assumption of  $F$ -finiteness.

**Definition 5.1.** An  $\mathbb{F}_p$ -algebra  $R$  is said to be *F-finite* if the absolute Frobenius map  $R \rightarrow R$  is finite, in other words if  $R$  is a finitely generated module over its subring of  $p^{\mathrm{th}}$ -powers.

Under  $F$ -finiteness, many additional finiteness properties follow; we refer to [22] for more details. For example, if  $R$  is an  $F$ -finite, noetherian  $\mathbb{F}_p$ -algebra, then the homotopy groups  $\pi_n L_{R/\mathbb{F}_p}$  of the cotangent complex are finitely generated  $R$ -modules for all  $n$  [22, Cor. 3.8]; in particular, the (algebraic) module of Kähler differentials  $\Omega_{\mathbb{F}_p[[t]]/\mathbb{F}_p}^1$  is a free  $\mathbb{F}_p[[t]]$ -module of rank one, whereas the analogous construction for a characteristic zero field is much harder to control and is not  $t$ -adically separated.

In this subsection, we simply combine the Dundas–Morrow results on topological cyclic homology with Theorem 4.35 to give a general answer to the above continuity question in  $K$ -theory. Since the results of [22] are stated only for  $\mathbb{Z}_{(p)}$ -algebras, we begin with a brief detour.

**Lemma 5.2.** Let  $R \rightarrow R'$  be a map of commutative rings such that the map  $R \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p \rightarrow R' \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p$  is an equivalence. Then  $\mathrm{THH}(R) \rightarrow \mathrm{THH}(R')$  is a mod  $p$  equivalence.

*Proof.* The hypothesis is equivalent to saying that  $HR \rightarrow HR'$  is a mod  $p$  equivalence of spectra. It follows that  $(HR)^{\otimes n} \rightarrow (HR')^{\otimes n}$  is a mod  $p$  equivalence for all  $n \geq 0$ , and hence that  $\mathrm{THH}(R) \rightarrow \mathrm{THH}(R')$  is a mod  $p$  equivalence, since mod  $p$  equivalences are preserved under colimits.  $\square$

**Lemma 5.3.** Let  $R$  be a noetherian ring and let  $\widehat{R}_p$  be its  $p$ -adic completion. Then the map  $\mathrm{THH}(R) \rightarrow \mathrm{THH}(\widehat{R}_p)$  is a mod  $p$  equivalence.

*Proof.* A noetherian ring has bounded  $p$ -torsion, which implies that the map  $R \rightarrow \widehat{R}_p$  induces an equivalence  $R \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p \simeq \widehat{R}_p \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p$ . Thus the statement follows from Lemma 5.2.  $\square$

The following is a slight extension of the continuity results of Dundas–Morrow [22], who only considered the case in which  $R$  is a  $\mathbb{Z}_{(p)}$  algebra.

**Proposition 5.4.** Let  $R$  be a noetherian ring which is complete along an ideal  $I \subseteq R$ , and suppose that  $R/pR$  is  $F$ -finite. Then the map  $\mathrm{THH}(R) \rightarrow \varprojlim_s \mathrm{THH}(R/I^s)$  is an equivalence modulo  $p$ .

*Proof.* Standard facts in commutative algebra show that  $\widehat{R}_p$  is  $I\widehat{R}_p$ -adically complete. Therefore, the canonical map

$$\mathrm{THH}(\widehat{R}_p)/p \longrightarrow \varprojlim_s \mathrm{THH}(\widehat{R}_p/I^s \widehat{R}_p)/p$$

is an equivalence by [22, Thm. 4.5]. But, again by standard facts in commutative algebra, each quotient  $\widehat{R}_p/I^s \widehat{R}_p$  coincides with the  $p$ -adic completion of  $R/I^s$ , so that applying Lemma 5.3 to  $R$  and to each  $R/I^s$  completes the proof.  $\square$

We can now state and prove our main result of the subsection, which resolves the continuity question in algebraic  $K$ -theory for all complete noetherian rings satisfying an  $F$ -finiteness hypothesis.

**Theorem 5.5.** Let  $R$  be a noetherian ring which is complete along an ideal  $I$ , and suppose that  $R/pR$  is  $F$ -finite. Then the map  $K(R) \rightarrow \varprojlim_s K(R/I^s)$  is an equivalence modulo  $p$ .

*Proof.* Using Theorem 4.35, the result reduces to the statement that  $\mathrm{TC}(R)/p \rightarrow \varprojlim_s \mathrm{TC}(R/I^s)/p$  is an equivalence. By Remark 2.8, this in turn follows from the analogous statement for  $\mathrm{THH}$ , which is given by Proposition 5.4.  $\square$

**Remark 5.6.** One can see that Theorem 5.5 is in fact equivalent to Theorem 4.35 (our main theorem on henselian pairs). Indeed, suppose Theorem 5.5 is known, and let  $(R, I)$  be a henselian pair. We want to show that  $K^{\mathrm{inv}}(R)/p \rightarrow K^{\mathrm{inv}}(R/I)/p$  is an equivalence. Since  $K$  and  $\mathrm{TC}/p$  commute with filtered colimits, so does  $K^{\mathrm{inv}}/p$ , hence we can assume  $(R, I)$  is the henselization of a finite type  $\mathbb{Z}$ -algebra at an ideal.

Let  $\widehat{R}$  denote the  $I$ -adic completion of  $R$ . By Néron–Popescu desingularization [62, 63] (see also [70, Tag 07BW]), applicable in view of the geometric regularity of  $R \rightarrow \widehat{R}$  (see [36, 7.8.3(v)]), the map  $R \rightarrow \widehat{R}$  is ind-smooth, i.e., we can write  $\widehat{R}$  as a filtered colimit of smooth  $R$ -algebras. Given a smooth  $R$ -algebra  $A$  and a map  $A \rightarrow \widehat{R}$  of  $R$ -algebras, it follows that the map  $R \rightarrow A$  admits a section by Elkik’s theorem (Theorem 3.12). Therefore, we deduce that  $R \rightarrow \widehat{R}$  is a filtered colimit of split injections. Thus the claim for  $(R, I)$  (i.e., that  $K^{\mathrm{inv}}(R)/p \rightarrow K^{\mathrm{inv}}(R/I)/p$  is an equivalence) will follow from the claim for  $(\widehat{R}, I\widehat{R})$ . (This is a standard Artin approximation argument.)

However, by the above continuity in  $K$  and  $\mathrm{TC}$ , we have

$$K^{\mathrm{inv}}(\widehat{R})/p \xrightarrow{\cong} \varprojlim K^{\mathrm{inv}}(R/I^n)/p.$$

On the other hand, the right hand side is a constant limit with value  $K^{\mathrm{inv}}(R/I)/p$ , by Theorem 1.2. This gives the claim for  $(\widehat{R}, I\widehat{R})$ , and therefore in general, by the above argument.

We finish the subsection by checking that the  $F$ -finiteness hypothesis in the above theorem appears to be necessary. This arises from a well-known problem in Milnor  $K$ -theory, namely that the complexity of symbols modulo powers of the ideal in question can increase without bound; to make this precise we closely follow the exposition of [10, App. B], where Bloch–Esnault–Kerz proved an analogous discontinuity result in characteristic zero.

**Theorem 5.7.** Let  $k$  be a field of characteristic  $p$  which is not  $F$ -finite, i.e., any  $p$ -basis of  $k$  has infinite cardinality. Then the map  $K(k[[t]])/p \rightarrow \varprojlim_s K(k[t]/(t^s))/p$  is not an equivalence; more precisely, the map on  $\pi_2$  is not surjective.

*Proof.* We begin with several straightforward reductions to Milnor  $K$ -theory. Firstly, since  $k[[t]]^\times$  is  $p$ -torsion-free, the canonical map  $K_2(k[[t]])/p \rightarrow \pi_2(K(k[[t]])/p)$  is an isomorphism. The  $p$ -torsion in the pro abelian group  $\{k[t]/(t^s)^\times\}_s$  is also zero, since the transition map  $k[t]/(t^{ps})^\times \rightarrow k[t]/(t^s)^\times$  clearly kills all  $p$ -torsion in the domain, and therefore  $\{K_2(k[t]/(t^s))/p\}_s \rightarrow \{\pi_2(K(k[t]/(t^s))/p)\}_s$  is an isomorphism of pro abelian groups. Secondly,  $k[[t]]$  and  $k[t]/(t^s)$  are local rings with infinite residue field, whence classical results in algebraic  $K$ -theory tell us that  $K_2^M(k[[t]]) \xrightarrow{\cong} K_2(k[[t]])$  and  $K_2^M(k[t]/(t^s)) \xrightarrow{\cong} K_2(k[t]/(t^s))$ . Finally, recall that the canonical map  $\pi_2(\varprojlim_s K(k[t]/(t^s))/p) \rightarrow \varprojlim_s \pi_2(K(k[t]/(t^s))/p)$  is surjective, by the Milnor sequence. In conclusion, to prove the theorem it is sufficient to show that the canonical map  $K_2^M(k[[t]])/p \rightarrow \varprojlim_s K_2^M(k[t]/(t^s))/p$  is not surjective.

To do this, we will detect symbols using the dlog maps from  $K_2^M$  to absolute Kähler differentials  $\Omega_-^2 := \Omega_-^2/\mathbb{F}_p$ :

$$\begin{array}{ccc} \Omega_{k[[t]]}^2 & \longrightarrow & \varprojlim_s \Omega_{k[t]/(t^s)}^2 \\ \uparrow \text{dlog} & & \uparrow \text{dlog} \\ K_2^M(k[[t]])/p & \longrightarrow & \varprojlim_s K_2^M(k[t]/(t^s))/p \end{array}$$

It remains to construct an element in the bottom right of the diagram whose image in the top right does not come from the top left.

We now closely follow Bloch–Esnault–Kerz, with the necessary modifications to deal with the fact that we will eventually need to restrict to dlog forms. Given any map of rings  $R \rightarrow S$  and differential form  $\tau \in \Omega_{S/R}^2$ , define its *weight*  $w(\tau)$  to be the smallest integer  $n$  for which it is possible to write  $\tau = \sum_{i=1}^n a_i db_i \wedge dc_i$  for some  $a_i, b_i, c_i \in S$ . If  $\ell$  is a subfield of  $k$  and  $b_1, \dots, b_n, c_1, \dots, c_n \in k$  form part of a  $p$ -basis for  $k$  relative to  $\ell$ , then the elements  $db_1, \dots, db_n, dc_1, \dots, dc_n$  are linearly independent in the  $k$ -vector space  $\Omega_{k/\ell}^1$  and so this element  $\sum_{i=1}^n db_i \wedge dc_i$  has weight  $\geq n$  in  $\Omega_{k/\ell}^2$  (Lemma 5.8 below).

Next consider the derivation

$$k[t]/(t^s) \rightarrow \Omega_k^1 \otimes_k k[t]/(t^s)$$

which “holds  $t$  constant”, so  $\sum_n c_n t^n \mapsto \sum_n (dc_n) t^n$ . This extends by multiplicativity to a map

$$\Omega_{k[t]/(t^s)}^* \rightarrow \Omega_k^* \otimes_k k[t]/(t^s)$$

which splits the canonical map  $\Omega_k^* \otimes_k k[t]/(t^s) \rightarrow \Omega_{k[t]/(t^s)}^*$ . Restricting to degree  $* = 2$  and passing to the limit as  $s \rightarrow \infty$  defines

$$e : \varprojlim_s \Omega_{k[t]/(t^s)}^2 \longrightarrow \varprojlim_s \Omega_k^2 \otimes_k k[t]/(t^s),$$

where each element on the right may be expressed as  $\sum_{j \geq 0} \tau_j t^j$  for some unique  $\tau_0, \tau_1, \dots \in \Omega_k^2$  (the  $t$ -adic coefficients of the element). Bloch–Esnault–Kerz [10, Lem. B2] show that the image of the composition

$$\Omega_{k[[t]]}^2 \longrightarrow \varprojlim_s \Omega_{k[t]/(t^s)}^2 \longrightarrow \varprojlim_s \Omega_k^2 \otimes_k k[t]/(t^s)$$

lands inside the set of those elements  $\sum_{j \geq 0} \tau_j t^j$  whose  $t$ -adic coefficients satisfy the following: there exists  $N \geq 0$  such that  $w(\tau_j) \leq N \binom{j+2}{j}$  for all  $j \geq 0$ .

We are now prepared to complete the proof by constructing a bad element of  $\varprojlim_s K_2^M(k[t]/(t^s))/p$ . First pick a sequence  $0 < w_1 < w_2 < \dots$  of integers growing sufficiently fast such that no value of  $N$  satisfies  $w_j \leq N \binom{2j}{2j-2}$  for all  $j \geq 1$ . Then pick a sequence of subfields  $\mathbb{F}_p = k_0 \subset k_1 \subset k_2 \subset \dots$  of  $k$

such that any  $p$ -basis for  $k_j$  relative to  $k_{j-1}$  has  $\geq 2w_j$  elements; let  $b_1^{(j)}, \dots, b_{w_j}^{(j)}, c_1^{(j)}, \dots, c_{w_j}^{(j)} \in k_i$  be part of such a relative  $p$ -basis. Set

$$f_s := \sum_{j=1}^{s-1} \sum_{i=1}^{w_j} \{1 + t^j b_i^{(j)}, 1 + t^j c_i^{(j)}\} \in K_2^M(k[t]/(t^s))/p$$

and note that the transition map  $K_2^M(k[t]/(t^s))/p \rightarrow K_2^M(k[t]/(t^{s-1}))/p$  exactly kills the  $j = s - 1$  part of the sum and therefore sends  $f_s$  to  $f_{s-1}$ ; we therefore may define  $f := \varprojlim_s f_s \in \varprojlim_s K_2^M(k[t]/(t^s))/p$ .

Let  $\sum_{i \geq 1} \tau_i t^i$  be the expansion of  $e(\mathrm{dlog} f) \in \varprojlim_s \Omega_k^2 \otimes_k k[t]/(t^s)$ . Noting that

$$\mathrm{dlog}(1 + t^j b) \wedge \mathrm{dlog}(1 + t^j c) \equiv t^{2j} da \wedge db \pmod{dt, t^{3j}}$$

with all the higher order terms given by various expressions in  $b, c$ , we see that

$$\tau_{2s-2} \equiv \sum_{i=1}^{w_s} b_i^{(s)} \wedge c_i^{(s)} \pmod{\Omega_{k_{s-1}}^2}.$$

By the second paragraph of the proof we deduce that the image of  $\tau_{2s-2}$  in  $\Omega_{k/k_{s-1}}^2$  has weight  $\geq w_s$ , whence a fortiori  $w(\tau_{2s-2}) \geq w_s$ . By choice of the sequence  $0 < w_1 < \dots$ , it follows that  $\mathrm{dlog} f$  cannot be lifted to  $\Omega_{k[[t]]}^2$ , which completes the proof.  $\square$

Let  $V$  be a vector space over a field  $k$ . Given a 2-form  $\omega \in \wedge^2 V$ , we define the *weight* of  $\omega$  to be the minimal  $n$  such that there exist elements  $\{x_i, y_i | 1 \leq i \leq n\} \subset V$  such that  $\omega = \sum_{i=1}^n x_i \wedge y_i$ . The following linear algebra lemma was used in the above proof.

**Lemma 5.8.** Let  $\{u_1, \dots, u_n, v_1, \dots, v_n\}$  be linearly independent in  $V$ . Then the form  $\sum_{i=1}^n u_i \wedge v_i$  has weight  $n$ .

*Proof.* Recall that we have an *interior product*  $V^\vee \otimes \wedge^2 V \rightarrow V$  given by  $v \otimes (x \wedge y) \mapsto \langle v, x \rangle y - \langle v, y \rangle x$ . Given  $\omega \in \wedge^2 V$ , we obtain a map  $f_\omega : V^\vee \rightarrow V$ . If  $\omega$  has weight  $w$ , then the rank of  $f_\omega$  is at most  $2w$ . Completing  $\{u_1, \dots, u_n, v_1, \dots, v_n\}$  to a basis of  $V$  and forming the dual basis, one now sees that the rank of  $f_{\omega_0}$  for  $\omega_0 = \sum_{i=1}^n u_i \wedge v_i$  is  $2n$ ; together, this implies the lemma.  $\square$

**5.2.  $p$ -ADIC CONTINUITY AND  $K$ -THEORY.** We now specialize the continuity question to the case where the ideal is  $(p)$ . In this case, we can often obtain stronger pro isomorphisms rather than simply isomorphisms in inverse limits and, secondly, hypotheses such as noetherianness and  $F$ -finiteness are no longer necessary.

**Definition 5.9.** Let  $R$  be a commutative ring. We say that  $K$ -theory is  *$p$ -adically continuous* for  $R$  if the map of spectra  $K(R) \rightarrow \varprojlim_i K(R/p^i R)$  is an equivalence modulo  $p$ .

One has the following general result.

**Theorem 5.10** (Geisser–Hesselholt [30]). If  $R$  is a local ring which is  $p$ -torsion-free and henselian along  $(p)$ , then  $K$ -theory is  $p$ -adically continuous at  $R$ . Moreover, the map  $K(R)/p \rightarrow \{K(R/p^i R)/p\}_{i \geq 1}$  induces an isomorphism of pro abelian groups upon applying  $\pi_j$  for any  $j$ .

Although the previous theorem concerns  $p$ -adic rings, its proof uses Gabber rigidity for  $\mathbb{Z}[1/p]$ -algebras, via a clever trick due to Suslin. Here we will obtain a generalization of Theorem 5.10 using our version of rigidity, which directly applies to  $p$ -adic rings. We observe also that the proof naturally yields something slightly stronger than an equivalence of pro abelian groups.



In this section, it will be convenient to work not only with commutative rings, but also with more general ring spectra. Hence, we will use the following variant.

**Variation.** The functors  $K, \mathrm{TC}$  are defined not only for ordinary rings  $R$ , but more generally for arbitrary  $\mathbb{E}_1$ -ring spectra  $R$ . For such  $R$ , we define  $K^{\mathrm{inv}}(R)$  similarly, as the fiber of the cyclotomic trace  $K(R) \rightarrow \mathrm{TC}(R)$ .

Part of the theorem of Dundas–Goodwillie–McCarthy [20] states that if  $R$  is a connective  $\mathbb{E}_1$ -algebra, then the map  $K^{\mathrm{inv}}(R) \rightarrow K^{\mathrm{inv}}(\pi_0 R)$  is an equivalence. In this sense, there is no extra generality afforded by the above variant. On the other hand, we will find that it is often easier to control  $\mathrm{TC}$  for appropriately “derived” constructions than underived constructions.

We next review some facts about nilpotent towers; compare [53].

**Definition 5.11.** Let  $\{A_i\}_{i \geq 1}$  be a tower of abelian groups. We say that the tower is *nilpotent* if there exists  $N > 0$  such that all the maps  $A_{i+N} \rightarrow A_i$  are zero. We say that the tower is *quickly converging* if there exists  $r \in \mathbb{Z}_{>0}$  such that the tower  $\{\mathrm{im}(A_{i+r} \rightarrow A_i)\}_{i \geq 1}$  is eventually constant.

By [53, Lemma 3.10], it follows that the collection of towers of abelian groups which are quickly converging forms an abelian subcategory of the category of towers which is closed under extensions. It thus follows that if  $\{A_i\}$  is a quickly converging tower and  $A$  is the inverse limit, then the kernel and cokernel of the map of towers  $\{A\} \rightarrow \{A_i\}$  are both nilpotent. In particular, the category of quickly converging towers is the smallest abelian subcategory containing the nilpotent towers and the constant towers which is closed under extensions.

**Definition 5.12.** Let  $\{X_i\}_{i \geq 1}$  be a tower in  $\mathrm{Sp}$ , i.e.,  $\{X_i\}_{i \geq 1} \in \mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{Sp})$ . We say that the tower is *nilpotent* if there exists  $N > 0$  such that all the maps  $X_{i+N} \rightarrow X_i$  are nullhomotopic. The collection of nilpotent towers forms a thick subcategory of the stable  $\infty$ -category of towers of spectra. We say that a tower  $\{X_i\}$  is *quickly converging* if the cofiber of the map of towers  $\{X\} \rightarrow \{X_i\}$  is nilpotent, where  $X := \varprojlim_i X_i$ , or equivalently if  $\{X_i\}$  lies in the smallest thick subcategory of towers containing the constant towers and the nilpotent towers.

The subcategories of nilpotent towers and quickly converging towers form thick subcategories of the category of towers. We also note the following basic observation.

**Lemma 5.13.** Let  $\{X_i\}_{i \geq 1}$  be a tower in  $\mathrm{Sp}$ . If  $\{X_i\}$  is quickly converging, the tower  $\{\pi_j X_i\}$  of abelian groups is quickly converging for each  $j$ . The converse holds if we suppose that each  $X_i$  has homotopy groups concentrated in the fixed range  $[a, b]$ .

*Proof.* It follows from [53, Lemma 3.11] that if  $\{X_i\}$  is quickly converging, then the towers of homotopy groups are quickly converging. Indeed, if  $\{X_i\}$  is nilpotent or constant, then the tower of homotopy groups is clearly quickly converging, and the referenced result shows that the condition on homotopy groups is stable in cofiber sequences of towers. For the converse direction, a dévissage reduces to the case where  $a = b$ . In this case, it follows from the paragraph following Definition 5.11: that is, quickly convergent towers of abelian groups are built up from constant and nilpotent towers.  $\square$

**Definition 5.14.** We say that a tower  $\{X_i\}$  of spectra is *almost nilpotent* if for each  $n \in \mathbb{Z}$ , the truncated tower  $\{\tau_{\leq n} X_i\}$  is nilpotent, or equivalently if each tower  $\{\pi_n(X_i)\}$  of abelian groups is nilpotent. We say that a tower is *almost quickly converging* if for each  $n$ , the tower  $\{\tau_{\leq n} X_i\}$  is quickly converging. If the tower is uniformly bounded below, then it is almost quickly converging if and only if each tower  $\{\pi_n(X_i)\}$  of abelian groups is quickly convergent.

The following then is a direct consequence of [53, Lemma 3.11].

**Lemma 5.15.** The almost nilpotent and almost quickly converging towers form thick subcategories of  $\text{Tow}(\text{Sp})$  including the nilpotent and quickly converging towers respectively.

From this we get:

**Lemma 5.16.** Let  $X_{\bullet,*}$  be a simplicial object in the  $\infty$ -category of towers of connective spectra. Suppose each tower  $\cdots \rightarrow X_{j,2} \rightarrow X_{j,1}$  is almost quickly convergent, with limit  $X_j$ . Then the geometric realization  $\cdots \rightarrow |X_{\bullet,2}| \rightarrow |X_{\bullet,1}|$  is almost quickly converging and has limit  $|X_{\bullet}|$  via the natural comparison map.

*Proof.* This follows from approximating the geometric realization with an  $n$ -skeletal geometric realization in any range of degrees.  $\square$

We now include a basic observation that given a tower of cyclotomic spectra whose underlying tower of spectra is almost quickly converging, the tower of  $\text{TC}/p$  is also almost quickly converging.

**Lemma 5.17.** Let  $\{X_i\}$  be a tower in  $\text{CycSp}_{\geq 0}$ . Suppose that the underlying tower of spectra  $\{X_i/p\}$  is almost quickly converging. Then the tower of spectra  $\{\text{TC}(X_i)/p\}$  is almost quickly converging.

*Proof.* This follows in a similar fashion as Proposition 2.19. See also Remark 2.8 for the identification of the inverse limit of the above tower in  $\text{CycSp}$ .  $\square$

**Lemma 5.18.** Let  $\{X_i\}$  and  $\{Y_i\}$  be two towers of connective spectra. If  $\{X_i\}$  and  $\{Y_i\}$  are almost quickly converging with inverse limits  $X$  and  $Y$ , then  $\{X_i \otimes Y_i\}$  is almost quickly converging with inverse limit  $X \otimes Y$ .

*Proof.* Note that  $\tau_{\leq m}(X_i \otimes Y_i) \simeq \tau_{\leq m}(\tau_{\leq m}X_i \otimes Y_i)$ . So if  $\{\tau_{\leq m}X_i\}$  is nilpotent, then so is  $\{\tau_{\leq m}(X_i \otimes Y_i)\}$ , and the limit is zero. Symmetrically, we get the same if  $\{\tau_{\leq m}Y_i\}$  is nilpotent. On the other hand, if  $\{\tau_{\leq m}X_i\}$  and  $\{\tau_{\leq m}Y_i\}$  are both constant, we see that  $\tau_{\leq m}(X_i \otimes Y_i)$  is constant, and the map from  $X \otimes Y$  to the limit is an isomorphism degrees  $\leq m-1$ . A thick subcategory argument lets us conclude.  $\square$

**Theorem 5.19.** Suppose that  $\{R_i\}_{i \geq 1}$  is a tower of connective  $\mathbb{E}_1$ -ring spectra, and  $R$  is another connective  $\mathbb{E}_1$ -ring spectrum with a comparison map  $R \rightarrow \varprojlim_i R_i$ . If this comparison map is an equivalence modulo  $p$  and the tower of spectra  $\{R_i/p\}$  is almost quickly converging, then the comparison map

$$\text{TC}(R) \rightarrow \varprojlim_i \text{TC}(R_i)$$

is an equivalence modulo  $p$  and the tower  $\{\text{TC}(R_i)/p\}$  is almost quickly converging.

*Proof.* Note that a tower is almost quickly converging modulo  $p$  if and only if it is almost quickly converging after smashing with  $X$ , for any choice of spectrum  $X$  generating the same thick subcategory as  $S^0/p$ . For example one can take  $X = S^0/p \otimes S^0/p$ . Keeping this in mind, an inductive application of Lemma 5.18 shows that  $\{(HR_i)^{\otimes n}\}_i$  is almost quickly converging modulo  $p$  for any  $n \geq 0$ , and has mod  $p$  limit  $(HR)^{\otimes n}/p$ . Then by Lemma 5.16 we deduce that  $\{\text{THH}(R_i)/p\}$  is almost quickly converging with limit  $\text{THH}(R)/p$ . From this, Lemma 5.17 lets us conclude.  $\square$

**Remark 5.20.** The ring spectra above do not quite need to be connective: it's enough for them to have connective THH, or even uniformly bounded below THH.

In the  $\mathbb{Z}$ -linear case, this immediately implies the following general  $p$ -adic continuity result for TC. We denote by  $\text{Mod}_{\mathbb{Z}}$  the symmetric monoidal  $\infty$ -category of  $\mathbb{Z}$ -module spectra, or equivalently the derived  $\infty$ -category of  $\mathbb{Z}$ .

**Theorem 5.21.** Let  $R$  be a connective  $\mathbb{E}_1$ -algebra in  $\text{Mod}_{\mathbb{Z}}$ . Then the map

$$\text{TC}(R) \rightarrow \varprojlim_i \text{TC}(R \otimes_{\mathbb{Z}} \mathbb{Z}/p^i)$$

is a  $p$ -adic equivalence. Moreover the tower on the right-hand-side is almost quickly converging modulo  $p$ .

*Proof.* It suffices to remark that the transition maps in the tower of fibers  $\{\text{fib}(R \rightarrow R \otimes_{\mathbb{Z}} \mathbb{Z}/p^i \mathbb{Z})\}_n$  identify with multiplication by  $p$  maps, hence are zero modulo  $p$ , so that Proposition 5.19 applies.  $\square$

We now obtain a general result on derived  $p$ -adic continuity of  $K$ -theory, and as a corollary a generalization of Theorem 5.10.

**Theorem 5.22.** Let  $R$  be a connective  $\mathbb{E}_1$ -algebra in  $\text{Mod}_{\mathbb{Z}}$  such that  $\pi_0(R)$  is commutative and henselian along  $(p)$ . Then the tower  $\{K(R \otimes_{\mathbb{Z}} \mathbb{Z}/p^i \mathbb{Z})/p\}_{i \geq 1}$  is almost quickly converging, and

$$K(R) \rightarrow \varprojlim_i K(R \otimes_{\mathbb{Z}} \mathbb{Z}/p^i \mathbb{Z})$$

is an equivalence modulo  $p$ .

*Proof.* By Theorem 4.35 we see  $K^{\text{inv}}(R) = K^{\text{inv}}(R \otimes_{\mathbb{Z}} \mathbb{Z}/p^i \mathbb{Z})$  for each  $i$ . This reduces us to the analogous result for TC, which is Theorem 5.21.  $\square$

There is also an underived version under a very mild hypothesis, which generalizes Geisser–Hesselholt’s Theorem 5.10.

**Theorem 5.23.** Let  $R$  be a commutative ring which is henselian along  $(p)$ , and suppose the  $p$ -power-torsion in  $R$  is bounded. Then  $K$ -theory is  $p$ -adically continuous at  $R$ . Moreover, the tower  $\{K(R/p^i R)/p\}$  is almost quickly converging.

*Proof.* The map of towers  $\{R \otimes_{\mathbb{Z}} \mathbb{Z}/p^i \mathbb{Z}\} \rightarrow \{R/p^i R\}$  has fiber  $\{\Sigma R[p^i]\}$  with transition maps multiplication by  $p$ . Because of the bounded  $p$ -power torsion hypothesis, this tower  $\{\Sigma R[p^i]\}$  is almost nilpotent modulo  $p$ . So we deduce  $p$ -adic continuity for TC as in Proposition 5.21, and then  $p$ -adic continuity for  $K$ -theory as in Proposition 5.22.  $\square$

**Remark 5.24.** Without assuming  $\mathbb{Z}$ -linear structure, one can get similar results by replacing  $- \otimes_{\mathbb{Z}} \mathbb{Z}/p^i \mathbb{Z}$  with  $- \otimes S_i$  for any reasonable tower  $\{S_i\}$  of which  $p$ -adically approximates the sphere spectrum, for example the tower coming from the usual cosimplicial Adams resolution associated to  $S^0 \rightarrow H\mathbb{F}_p$ .

**5.3. Pro Geisser–Levine theorems.** We begin by recalling from subsection 4.4 that one classically associates to any  $\mathbb{F}_p$ -algebra  $R$  the abelian group  $\nu^n(R) = \Omega_{R, \log}^n$ , defined either as  $\ker(1 - C^{-1} : \Omega_R^n \rightarrow \Omega_R^n / d\Omega_R^{n-1})$  or as the subgroup of  $\Omega_R^n$  which is generated étale locally by dlog forms. As we recalled in Theorem 4.28, when  $R$  is moreover local and ind-smooth then Geisser and Levine established isomorphisms  $K_n(R; \mathbb{Z}/p\mathbb{Z}) \cong \Omega_{R, \log}^n$  for all  $n \geq 0$ . The goal of this section is to establish an entirely analogous description of the pro abelian groups  $\{K_n(R/I^s; \mathbb{Z}/p\mathbb{Z})\}_s$  whenever  $I \subseteq R$  is an ideal.

It is convenient to work not only modulo  $p$  but more generally modulo  $p^r$  with  $r > 1$ . This necessitates introducing some standard notation surrounding de Rham–Witt groups, in particular the logarithmic subgroup which serves as a mod  $p^r$  lift of  $\Omega_{R,\log}^n$ .

**Definition 5.25** (Logarithmic de Rham–Witt groups). Let  $R$  be an  $\mathbb{F}_p$ -algebra. We recall the de Rham–Witt complex  $\{W_r\Omega_R^*\}$  as in [45] (see also Definition 2.24). Letting  $[\cdot] : R \rightarrow W_r(R)$  denote the Teichmüller lift, there is a resulting group homomorphism  $\mathrm{dlog}[\cdot] : R^\times \rightarrow W_r\Omega_R^1$ ,  $\alpha \mapsto \mathrm{dlog}[\alpha] = \frac{d[\alpha]}{[\alpha]}$ , and more generally

$$\mathrm{dlog}[\cdot] : R^{\times \otimes n} \rightarrow W_r\Omega_R^n, \quad \alpha_1 \otimes \cdots \otimes \alpha_n \mapsto \mathrm{dlog}[\alpha_1] \wedge \cdots \wedge \mathrm{dlog}[\alpha_n].$$

When  $R$  is local, the image of this map will be denoted by  $W_r\Omega_{R,\log}^n$ .

**Definition 5.26** (Globalization). Given an  $\mathbb{F}_p$ -scheme  $X$ , one defines  $W_r\Omega_X^n$  to be the Zariski (or étale, depending on the context) sheaf obtained by sheafifying  $U \mapsto W_r\Omega_{\mathcal{O}_X(U)}^n$ . When  $X = \mathrm{Spec} R$  is affine, this sheaf (in either topology) has no higher cohomology and has global sections  $W_r\Omega_R^n$ ; this follows from flat descent and the facts that if  $R \rightarrow S$  is étale then so is  $W_r(R) \rightarrow W_r(S)$  [76, Thm. 2.4] and moreover the canonical map  $W_r\Omega_R^n \otimes_{W_r(R)} W_r(S) \rightarrow W_r\Omega_S^n$  is an isomorphism [49, Prop. 1.7].

We define  $W_r\Omega_{R,\log}^n$  to be the subgroup of  $W_r\Omega_R^n$  consisting of elements which can be written Zariski locally as sums of  $\mathrm{dlog}$  forms, i.e.,  $H^0(\mathrm{Spec} R, -)$  of the image (as a Zariski sheaf) of

$$\mathrm{dlog}[\cdot] : \mathbb{G}_{m,\mathrm{Spec} R}^{\otimes n} \rightarrow W_r\Omega_{\mathrm{Spec} R}^n, \quad \alpha_1 \otimes \cdots \otimes \alpha_n \mapsto \mathrm{dlog}[\alpha_1] \wedge \cdots \wedge \mathrm{dlog}[\alpha_n].$$

We note that  $W_r\Omega_{R,\log}^n$  would be unchanged if we were to replace Zariski by étale in the previous sentence, by [59, Cor. 4.1(iii)]; in particular,  $W_1\Omega_{R,\log}^n$  really is the same as  $\Omega_{R,\log}^n$  as defined at the start of the subsection. Similarly, if  $X$  is a scheme, we let  $W_r\Omega_{X,\log}^n$  denote the sheaf  $\mathrm{Spec} R \mapsto W_r\Omega_{R,\log}^n$ .

In the case that  $R$  is an ind-smooth local  $\mathbb{F}_p$ -algebra, Geisser–Levine [32] established isomorphisms  $K_n(R; \mathbb{Z}/p^r\mathbb{Z}) \cong W_r\Omega_{R,\log}^n$  (given by the map  $\mathrm{dlog}[\cdot]$  on symbols) for all  $n \geq 0$ ,  $r \geq 1$ , and proved that each group  $K_n(R)$  is  $p$ -torsion-free. The primary goal of this subsection is to establish the following pro version of this theorem:

**Theorem 5.27.** Let  $R$  be a  $F$ -finite, regular, local noetherian  $\mathbb{F}_p$ -algebra and  $I \subseteq R$  an ideal; fix  $n, r \geq 0$ . Then the pro abelian group  $\{K_n(R/I^s)\}_s$  is  $p$ -torsion-free and there is a natural isomorphism of pro abelian groups

$$\{K_n(R/I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s \xrightarrow{\cong} \{W_r\Omega_{R/I^s,\log}^n\}_s$$

given by  $\mathrm{dlog}[\cdot]$  on symbols.

This theorem was proved in [59] under the assumption that  $\mathrm{Spec} R/I$  was sufficiently regular (the terminology used there was “generalised normal crossing”), and some applications were given to higher dimensional class field theory and to the deformation theory of algebraic cycles. Here we combine the results of [59] with our main rigidity theorem to prove the theorem in general as well as a similar result in a non-local, relative case (Theorem 5.32).

In fact, Theorem 5.32 below (which works for an arbitrary henselian ideal in a regular  $F$ -finite  $\mathbb{F}_p$ -algebra) is more fundamental. We can view this statement as a pro-version of the following calculation.

**Proposition 5.28.** Let  $(R, I)$  be a henselian pair of  $\mathbb{F}_p$ -algebras. Suppose that  $R$  and  $R/I$  are ind-smooth. Then we have a natural isomorphism

$$(23) \quad K_n(R, I; \mathbb{Z}/p) \simeq \Omega_{(R, I), \log}^n.$$

*Proof.* Indeed, this follows from our main result (which identifies the relative  $K$ -theory with relative TC), together with the Geisser–Hesselholt calculations of relative TC. We have the short exact sequence

$$0 \rightarrow \tilde{\nu}^{n+1}(R) \rightarrow \pi_n(\mathrm{TC}(R)/p) \rightarrow \nu^n(R) \rightarrow 0$$

as in Theorem 4.28 (see also (12)). Using Proposition 4.30, the result follows.  $\square$

We begin with a couple of lemmas concerning relative log de Rham–Witt groups.

**Definition 5.29.** Given an  $\mathbb{F}_p$ -algebra  $R$  and ideal  $I \subseteq R$ , we set  $W_r \Omega_{(R, I), \log}^n := \ker(W_r \Omega_{R, \log}^n \rightarrow W_r \Omega_{R/I, \log}^n)$ .

**Lemma 5.30.** Let  $R$  be an  $\mathbb{F}_p$ -algebra which is henselian along an ideal  $I \subseteq R$ . Then  $R - F : W_r \Omega_{(R, I)}^n \rightarrow W_{r-1} \Omega_{(R, I)}^n$  is surjective.

*Proof.* If  $R$  is  $F$ -finite, noetherian, and  $I$ -adically complete then this was proved in [59, Prop. 2.20], and we will reduce the general case to this one. Firstly, by taking a filtered colimit we may suppose that  $R$  is the henselization of a finite type  $\mathbb{F}_p$ -algebra along some ideal, and in particular that  $R$  is  $F$ -finite and excellent. Then the  $I$ -adic completion  $\widehat{R}$  is geometrically regular over  $R$  by [36, 7.8.3(v)], whence it is a filtered colimit of smooth  $R$ -algebras  $A$  by Néron–Popescu. But each structure map  $R \rightarrow A$  admits a splitting as in Remark 5.6. Therefore, taking a filtered colimit shows that

$$\mathrm{Coker}(W_r \Omega_{(R, I)}^n \xrightarrow{R-F} W_{r-1} \Omega_{(R, I)}^n) \longrightarrow \mathrm{Coker}(W_r \Omega_{(\widehat{R}, I\widehat{R})}^n \xrightarrow{R-F} W_{r-1} \Omega_{(\widehat{R}, I\widehat{R})}^n)$$

is injective. Since the right side vanishes by the first sentence of the paragraph, the proof is complete.  $\square$

A important result of Illusie states that if  $X$  is a smooth variety over a perfect field of characteristic  $p$ , then the canonical map of étale sheaves  $W_s \Omega_{X, \log}^n / p^r \rightarrow W_r \Omega_{X, \log}^n$  is an isomorphism whenever  $s \geq r$  and  $n \geq 0$ , and therefore

$$0 \longrightarrow \{W_s \Omega_{X, \log}^n\}_s \xrightarrow{p^r} \{W_s \Omega_{X, \log}^n\}_s \xrightarrow{\{R^{s-r}\}_s} W_r \Omega_{X, \log}^n \longrightarrow 0$$

is an exact sequence of pro sheaves on  $X_{\text{ét}}$  [45, §I.5.7]. We will require the following relative form of this result for henselian ideals:

**Lemma 5.31.** Let  $R$  be an  $F$ -finite, regular, noetherian  $\mathbb{F}_p$ -algebra which is henselian along an ideal  $I \subseteq R$ ; fix  $n \geq 0$ . Then the sequence of pro abelian groups

$$0 \longrightarrow \{W_s \Omega_{(R, I^s), \log}^n\}_s \xrightarrow{p^r} \{W_s \Omega_{(R, I^s), \log}^n\}_s \xrightarrow{\{R^{s-r}\}_s} \{W_r \Omega_{(R, I^s), \log}^n\}_s \longrightarrow 0$$

is exact for any  $r \geq 1$ .

*Proof.* We will need to argue via the étale topology, so we begin by introducing the necessary sheaves. Let  $X := \mathrm{Spec} R$ , with closed subschemes  $Y_s := \mathrm{Spec} R/I^s$  for  $s \geq 1$ ; let  $W_r \Omega_X^n$ ,  $W_r \Omega_{Y_s}^n$ , and  $W_r \Omega_{(X, Y_s)}^n := \ker(W_r \Omega_X^n \rightarrow W_r \Omega_{Y_s}^n)$  be the corresponding étale sheaves on  $X$ . As explained in Definition 5.25, these sheaves have no higher étale cohomology (since  $X$  is affine) and their global sections are respectively  $W_r \Omega_R^n$ ,  $W_r \Omega_{R/I^s}^n$ , and  $W_r \Omega_{(R, I^s)}^n$ .

Next let  $W_r\Omega_{X,\log}^n$  be the image in the étale topology of  $\mathrm{dlog}[\cdot] : \mathbb{G}_{m,X}^{\otimes n} \rightarrow W_r\Omega_X^n$ , and similarly for each  $Y_s$ , and set  $W_r\Omega_{(X,Y_s),\log}^n := \ker(W_r\Omega_{X,\log}^n \rightarrow W_r\Omega_{Y_s,\log}^n)$ . As we already mentioned in Definition 5.25, the subgroup of  $\mathrm{dlog}$  forms may be defined using either the Zariski or étale topology by [59, Cor. 4.1(iii)], and therefore the global sections of these sheaves are respectively  $W_r\Omega_{R,\log}^n$ ,  $W_r\Omega_{R/I^s,\log}^n$ , and  $W_r\Omega_{(R,I^s),\log}^n$ .

We claim that  $\{H_{\text{ét}}^1(X, W_r\Omega_{(X,Y_s),\log}^n)\}_r = 0$  for each  $s \geq 1$ . After replacing  $I$  by  $I^s$  we may as well assume  $s = 1$  and write  $Y = Y_s$  for simplicity of notation; the key will be that  $X$  is an affine scheme, henselian along  $Y$ . Appealing to [59, Cor. 4.1(iii)] for both  $X$  and  $Y$  gives rise to short exact sequences of pro sheaves on  $X_{\text{ét}}$ , in which the three vertical arrows are surjective:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \{W_r\Omega_{X,\log}^n\}_r & \longrightarrow & \{W_r\Omega_X^n\}_r & \xrightarrow{R-F} & \{W_{r-1}\Omega_X^n\}_r \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \{W_r\Omega_{Y,\log}^n\}_r & \longrightarrow & \{W_r\Omega_Y^n\}_r & \xrightarrow{R-F} & \{W_{r-1}\Omega_Y^n\}_r \longrightarrow 0 \end{array}$$

Taking the kernels of the vertical arrows gives us a short exact sequence of relative terms

$$(24) \quad 0 \longrightarrow \{W_r\Omega_{(X,Y),\log}^n\}_r \longrightarrow \{W_r\Omega_{(X,Y)}^n\}_r \xrightarrow{R-F} \{W_{r-1}\Omega_{(X,Y)}^n\}_r \longrightarrow 0$$

Since the middle and right terms have no higher cohomology, it is enough to check that  $R - F$  is surjective on global sections; but this is even true for each fixed level  $r$  by Lemma 5.30.

Finally, appealing to Illusie's result recalled immediately before the lemma (which has been extended to arbitrary regular  $\mathbb{F}_p$ -schemes by A. Shiho [68, Cor. 2.13]) and to an analogous pro version for the formal completion of  $X$  along  $Y_1$  [59, Cor. 4.8] gives rise to short exact sequences of pro sheaves on  $X_{\text{ét}}$ , in which the three vertical arrows are again surjective:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \{W_s\Omega_{X,\log}^n\}_s & \xrightarrow{p^r} & \{W_s\Omega_{X,\log}^n\}_s & \xrightarrow{\{R^{r-s}\}_s} & W_r\Omega_{X,\log}^n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \{W_s\Omega_{Y_s,\log}^n\}_s & \xrightarrow{p^r} & \{W_s\Omega_{Y_s,\log}^n\}_s & \xrightarrow{\{R^{r-s}\}_s} & \{W_r\Omega_{Y_s,\log}^n\}_s \longrightarrow 0 \end{array}$$

Taking kernels gives a short exact sequence of relative terms

$$0 \longrightarrow \{W_s\Omega_{(X,Y_s),\log}^n\}_s \xrightarrow{p^r} \{W_s\Omega_{(X,Y_s),\log}^n\}_s \xrightarrow{\{R^{r-s}\}_s} \{W_r\Omega_{(X,Y_s),\log}^n\}_s \longrightarrow 0.$$

Taking global sections completes the proof since the previous paragraph showed that the left term has no  $H_{\text{ét}}^1$ .  $\square$

The following is our relative form of Theorem 5.27 for henselian ideals (and a pro-version of the identification (23)), from which Theorem 5.27 will immediately follow.

**Theorem 5.32.** Let  $R$  be a  $F$ -finite, regular, noetherian  $\mathbb{F}_p$ -algebra which is henselian along an ideal  $I \subseteq R$ ; fix  $n, r \geq 0$ . Then the trace map induces a natural isomorphism of pro abelian groups

$$\{K_n(R, I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s \xrightarrow{\cong} \{W_r\Omega_{(R,I^s),\log}^n\}_s.$$

*Proof.* The proof is similar to the techniques of [59, §5.1], but we repeat the necessary details here. In particular, we begin with the same recollections on Hochschild–Kostant–Rosenberg theorems for the spectra  $\mathrm{TR}^s$ . If  $R$  is any  $\mathbb{F}_p$ -algebra, then the pro graded ring  $\{\mathrm{TR}_*(R; p)\}_s$  is a  $p$ -typical Witt complex with respect to its operators  $F, V, R$ ; by universality of the de Rham–Witt complex, there

are therefore natural maps of graded  $W_s(R)$ -algebras [37, Prop. 1.5.8]  $\lambda_{s,R} : W_s\Omega_R^* \rightarrow \mathrm{TR}_*^s(R; p)$  for  $s \geq 0$ , which are compatible with the Frobenius, Verschiebung, and Restriction maps (in other words, a morphism of  $p$ -typical Witt complexes).

From now on in the proof assume that  $R$  is regular, noetherian, and  $F$ -finite, and let  $I \subseteq R$  be any ideal. Hesselholt's HKR theorem [37, Thm. B] implies that the resulting map of pro abelian groups

$$\lambda_R : \{W_s\Omega_R^n\}_s \longrightarrow \{\mathrm{TR}_n^s(R; p)\}_s$$

is an isomorphism for each  $n \geq 1$ ; similarly, the pro HKR theorem of Dundas–Morrow [22, Cor. 4.15] implies that

$$\text{(pro-HKR)} \quad \lambda_{R/I^\infty} : \{W_s\Omega_{R/I^s}^n\}_s \longrightarrow \{\mathrm{TR}_n^s(R/I^s; p)\}_s$$

is an isomorphism of pro abelian groups for each  $n \geq 0$ . Since  $W_s\Omega_R^n \rightarrow W_s\Omega_{R/I^s}^n$  is surjective for all  $s, n \geq 0$ , it follows that the long exact sequence associated to  $\{\mathrm{TR}^s(R, I^s; p)\}_s \rightarrow \{\mathrm{TR}^s(R; p)\}_s \rightarrow \{\mathrm{TR}^s(R/I^s; p)\}_s$  breaks into short exact sequences and there are therefore natural induced isomorphisms of relative theories

$$\{W_s\Omega_{(R, I^s)}^n\}_s \xrightarrow{\cong} \{\mathrm{TR}_n^s(R, I^s; p)\}_s$$

for all  $n \geq 0$ .

Now assume that  $R$  is henselian along  $I$ . By Lemma 5.30, the map  $R - F : W_s\Omega_{(R, I^s)}^n \rightarrow W_{s-1}\Omega_{(R, I^s)}^n$  is surjective for all  $n, s \geq 0$ , and therefore the previous isomorphism shows that the long exact sequence associated to  $\{\mathrm{TC}^s(R, I^s; p)\}_s \rightarrow \{\mathrm{TR}^s(R, I^s; p)\}_s \xrightarrow{R-F} \{\mathrm{TR}^s(R, I^s; p)\}_s$  breaks into short exacts and thereby induces natural isomorphisms

$$\{\mathrm{TC}_n^s(R, I^s; p)\}_s \xrightarrow{\cong} \{\ker(W_s\Omega_{(R, I^s)}^n \xrightarrow{R-F} W_{s-1}\Omega_{(R, I^s)}^n)\}_s$$

The right side of the previous line is precisely  $\{W_s\Omega_{(R, I^s), \log}^n\}_s$  by applying [59, Cor. 4.1(iii)] to  $R$  and  $R/I^s$  for each  $s \geq 1$  (in fact, we have already made this argument: just take global sections in line (24) for each  $Y_s$  and then pass to the resulting diagonal of the  $\mathbb{N}^2$ -indexed pro abelian group); moreover, this latter pro abelian group is  $p$ -torsion-free by the injectivity in Lemma 5.31 (this is anyway easy: the pro abelian group is contained in  $\{W_s\Omega_R^n\}_s$ , which is  $p$ -torsion-free thanks to the equality of the  $p$ - and canonical-filtrations on the de Rham–Witt groups of any regular  $\mathbb{F}_p$ -algebra: see [45, Prop. I.3.2 & I.3.4] for the case of a smooth algebra over a perfect field, then apply Néron–Popescu), so passing to finite coefficients gives us isomorphisms

$$\{\mathrm{TC}_n^s(R, I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s \cong \{W_s\Omega_{(R, I^s), \log}^n/p^r\}_s$$

for each fixed  $r \geq 1$ . The right side is  $\{W_r\Omega_{(R, I^s), \log}^n\}_s$  by (the hard part of) Lemma 5.31.

Finally, pro-constancy of  $\{\mathrm{TC}^s(-; \mathbb{Z}/p^r)\}_s$  (Proposition 2.22) implies that the left side of the previous line is  $\{\mathrm{TC}_n(R, I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s$ , which by Theorem 4.35 identifies with  $\{K_n(R, I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s$  via the trace map.  $\square$

**Corollary 5.33.** Theorem 5.27 is true.

*Proof.* Let  $R$  be as in the statement of Theorem 5.27. Without loss of generality  $R$  is henselian along  $I$ , whence  $R$  is also local, and so the result follows by combining Theorem 5.32 with the isomorphism  $K_n(R; \mathbb{Z}/p^r\mathbb{Z}) \xrightarrow{\cong} W_r\Omega_{R, \log}^n$  of Geisser–Levine [32]. Note that the two results are compatible since the trace map is given by  $\mathrm{dlog}[\cdot]$  on symbols by [28, Lem. 4.2.3 & Cor. 6.4.1].

We establish the  $p$ -torsion-freeness of  $\{K_n(R/I^s)\}_s$  in the next corollary.  $\square$

In the following corollary  $K_n^M$  denotes Milnor  $K$ -theory (either the classical version or Kerz' improved variant [46]; the corollary holds for both):

**Corollary 5.34** (Comparison to Milnor  $K$ -theory;  $p$ -torsion-freeness). Let  $R$  be an  $F$ -finite, regular, noetherian  $\mathbb{F}_p$ -algebra and  $I \subseteq R$  an ideal such that  $R/I$  is local. Then, for all  $n, r \geq 0$ :

- (1) the canonical map  $\{K_n^M(R/I^s)/p^r\}_s \rightarrow \{K_n(R/I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s$  is surjective and has the same kernel as  $\text{dlog}[\cdot] : \{K_n^M(R/I^s)/p^r\}_s \rightarrow \{W_r\Omega_{R/I^s}^n\}_s$ ;
- (2) the pro abelian group  $\{K_n(R/I^s)\}_s$  is  $p$ -torsion-free.

*Proof.* Part (1) is an immediate consequence of the commutative diagram

$$\begin{array}{ccc} \{K_n^M(R/I^s)/p^r\}_s & \longrightarrow & \{K_n(R/I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s \xrightarrow{\cong} \{W_r\Omega_{R/I^s, \log}^n\}_s \\ & \searrow \text{dlog}[\cdot] & \nearrow \end{array}$$

which summarises the statement of Theorem 5.27, since the bendy arrow is surjective.

Part (1) clearly implies that  $\{K_n(R/I^s)\}_s \rightarrow \{K_n(R/I^s; \mathbb{Z}/p\mathbb{Z})\}_s$  is surjective, whence the usual Milnor sequence implies it is an isomorphism and that  $\{K_{n-1}(R/I^s)[p]\}_s = 0$ .  $\square$

**Corollary 5.35** (Lifting classes). Let  $R$  be an  $F$ -finite, regular, noetherian, local  $\mathbb{F}_p$ -algebra and  $I \subseteq R$  an ideal. Then, for all  $n, r \geq 0$ :

- (1) the sequences

$$0 \longrightarrow \{K_n(R, I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s \longrightarrow K_n(R; \mathbb{Z}/p^r\mathbb{Z}) \longrightarrow \{K_n(R/I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s \longrightarrow 0$$

are short exact;

- (2) there exists  $s \geq 1$  with the following property: if an element of  $K_n(R/I; \mathbb{Z}/p^r\mathbb{Z})$  lifts to  $K_n(R/I^s; \mathbb{Z}/p^r\mathbb{Z})$ , then it is symbolic and hence lifts to  $K_n(R)$ .

*Proof.* As we have just seen in Corollary 5.34,  $\{K_n(R/I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s$  is entirely symbolic; since  $R \rightarrow R/I$  is surjective on units (using the assumption  $I \subseteq \text{Jac}(R)$ ), we deduce that  $K_n(R; \mathbb{Z}/p^r\mathbb{Z}) \rightarrow \{K_n(R/I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s$  is surjective for all  $n \geq 0$ . Therefore the long exact relative sequences breaks into the desired short exact sequences.

Part (2) is a consequence of the surjectivity arguments of the previous paragraph, unravelling what it means for a map of pro abelian groups to be surjective.  $\square$

**Corollary 5.36** (Relative pro Geisser–Levine). Let  $R$  be an  $F$ -finite, regular, noetherian, local  $\mathbb{F}_p$ -algebra and  $I \subseteq R$  an ideal. Then, for all  $n, r \geq 0$ , the trace map induces a natural isomorphism of pro abelian groups  $\{K_n(R, I^s; \mathbb{Z}/p^r\mathbb{Z})\}_s \cong \{W_r\Omega_{(R, I^s), \log}^n\}_s$

*Proof.* This follows from the short exact sequence of Corollary 5.35(1) by applying usual Geisser–Levine to the middle term and Theorem 5.27 to the right term.  $\square$

## 6. COMPARISONS OF $K$ AND TC

In this section, we prove two main general results: the comparison of  $K$ -theory and TC in large degrees under mild finiteness hypotheses (Theorem 6.5) and the étale local comparison of  $K$  and TC (Theorem 6.1). We give various examples in subsection 6.3. Finally, we prove an injectivity result for the cyclotomic trace on local  $\mathbb{F}_p$ -algebras (Theorem 6.11).



**6.1. Étale  $K$ -theory is TC.** The results of [28] show that if  $R$  is a smooth algebra over a perfect field of characteristic  $p$ , then the  $p$ -adic étale  $K$ -theory of  $R$  agrees with  $\mathrm{TC}(R)$ . We extend this result to all commutative rings which are henselian along  $(p)$ .

**Theorem 6.1.** Let  $R$  be a strictly henselian local ring of residue characteristic  $p$ . Then  $K^{\mathrm{inv}}(R)/p = 0$ , i.e., the map  $K(R) \rightarrow \mathrm{TC}(R)$  is a  $p$ -adic equivalence.

*Proof.* By Theorem 1.2, we can assume that  $R$  is discrete, and then by Theorem 4.35, we may assume that  $R$  is a field itself. Thus we need to show that if  $k$  is a separably closed field of characteristic  $p > 0$ , then the map  $K(k) \rightarrow \mathrm{TC}(k)$  is a  $p$ -adic equivalence.

As  $\mathbb{F}_p$  is perfect,  $k$  is an ind-smooth  $\mathbb{F}_p$ -algebra (e.g., choose a transcendence basis for  $k$ ). Thus it suffices to show that the terms  $\tilde{\nu}^n(k)$  vanish by Theorem 4.28. That is, we need to show that the map

$$1 - C^{-1} : \Omega_k^n \rightarrow \Omega_k^n / d\Omega_k^{n-1}$$

is surjective. Given a form  $\omega = adx_1 \dots dx_n \in \Omega_k^n$ , we have  $(1 - C^{-1})(u\omega) = (u - u^p a^{p-1} x_1^{p-1} \dots x_n^{p-1})\omega$ . Since  $k$  is separably closed, we can solve the equation  $u - u^p a^{p-1} x_1^{p-1} \dots x_n^{p-1} = 1$  in  $k$ . This implies that  $1 - C^{-1}$  is surjective as desired and completes the proof.  $\square$

In the proof above, rather than using Theorem 4.28 and the groups  $\tilde{\nu}^n(k)$ , one can instead follow Suslin's arguments from [71] to show invariance of  $K^{\mathrm{inv}}(-)/p$  under extensions of separably closed fields. This reduces to the case  $k = \overline{\mathbb{F}_p}$ , which is easy since  $K/p$  and  $\mathrm{TC}/p$  can be directly calculated, cf. [64] or Example 6.8. We record this alternative argument in the following proposition.

**Proposition 6.2.** Let  $F : \mathrm{Ring} \rightarrow \mathrm{Ab}$  be a functor from rings to abelian groups. Suppose that  $F$  commutes with filtered colimits and satisfies rigidity, i.e., for a henselian pair  $(R, I)$  we have  $F(R) \simeq F(R/I)$ . Then for an extension  $K \rightarrow L$  of separably closed fields, we have  $F(K) \simeq F(L)$ .

*Proof.* Let  $k$  be a separably closed field. The crux of Suslin's argument is to show that if  $q : X \rightarrow \mathrm{Spec}(k)$  is a connected smooth affine  $k$ -scheme of finite type, then for any class  $\alpha \in F(X)$  and section  $x : \mathrm{Spec}(k) \rightarrow X$ , the pullback  $x^*\alpha \in F(k)$  is independent of the choice of  $x$ . It suffices to fix a section  $x_0 : \mathrm{Spec}(k) \rightarrow X$ , and show (by connectedness, in view of the Zariski density of  $k$ -points in  $X$ , cf., e.g., [70, Tag 04QM]) that there exists a Zariski neighborhood  $U$  of  $x_0$  such that  $x_0^*\alpha = x^*\alpha$  for all sections  $x : \mathrm{Spec}(k) \rightarrow U$ . Replacing  $\alpha$  by  $\alpha - q^*x_0^*\alpha$ , we can assume  $x^*\alpha = 0$ . Then by the assumption of rigidity, applied to the henselization of  $X$  at  $x_0$ , and the commutation of  $F$  with filtered colimits, we deduce that there is an étale neighborhood  $Y \rightarrow X$  of  $x_0$  such that  $\alpha$  pulls back to 0 on  $Y$ . Then we can take  $U$  to be the image of  $Y \rightarrow X$ : as  $k$  is separably closed, every  $k$ -point of  $U$  lifts to  $Y$ .

Now we prove the proposition. It suffices to consider the case where  $K$  is the separable closure of a prime field, and so is in particular perfect. Therefore, the extension  $K \rightarrow L$  is ind-smooth, i.e.,  $L$  is a filtered colimit of smooth  $K$ -algebras  $A_\alpha$ , with  $\alpha$  running over a filtered poset  $I$ . Then each map  $K \rightarrow A_\alpha$  admits a retraction, as  $K$  is separably closed. It follows that  $F(K) \rightarrow F(A_\alpha)$  is injective for each  $\alpha$ , so that  $F(K) \rightarrow F(L)$  is injective.

We now argue surjectivity. Let  $u \in F(A_\alpha)$ ; we show that the image of  $u$  in  $F(L)$  belongs to the image of  $F(K)$ . Passing to a component if necessary, we can assume  $\mathrm{Spec}(A_\alpha)$  is connected. Then  $\mathrm{Spec}(A_\alpha \otimes_K L)$  is connected as well, since  $K$  is algebraically closed. Now consider the two maps

$$f_1, f_2 : A_\alpha \rightrightarrows L,$$

where the first map is the colimit structure map and where the second map is  $A_\alpha \xrightarrow{s} K \rightarrow L$ . By the first paragraph of the argument applied to the  $L$ -algebra  $A_\alpha \otimes_K L$ , we conclude that these two maps have the same effect on  $u$ , verifying the claim.  $\square$

We can rephrase Theorem 6.1 in terms homotopy group sheaves. Let  $\pi_n(\mathcal{K}/p)$  denote the étale sheafification of the functor  $\pi_n(K(-)/p)$  over an arbitrary scheme  $X$ , and let  $\pi_n(\mathcal{TC}/p)$  denote the same for  $\pi_n(\mathrm{TC}(-)/p)$ . Then if  $i$  denotes the closed inclusion  $(X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p))_{\mathrm{ét}} \rightarrow X_{\mathrm{ét}}$ , Theorem 6.1 (plus the trivial fact that  $\mathrm{TC}/p$  vanishes on rings where  $p$  is invertible) is equivalent to the statement that for all  $n$ , there is an equivalence

$$(25) \quad \pi_n(\mathcal{TC}/p) \simeq i_* i^* \pi_n(\mathcal{K}/p)$$

adjoint to the map  $i^* \pi_n(\mathcal{K}/p) \rightarrow i^* \pi_n(\mathcal{TC}/p)$  (also an equivalence) given by the restriction of the cyclotomic trace. We can clearly also replace  $(-)/p$  with  $(-)/p^s$  for any  $s$ .

As a consequence, one obtains the following result, which will be discussed in more detail in the forthcoming paper [16]. In the smooth case, this is one of the main results of [28].

**Theorem 6.3.** Let  $X$  be a scheme proper over  $\mathrm{Spec}R$  for a ring  $R$  henselian along  $(p)$ . Denote by  $K^{\mathrm{ét}}(-)$  the étale Postnikov sheafification of the  $K$ -theory presheaf, meaning the limit over the étale sheafified Postnikov tower of  $K(-)$ .<sup>6</sup> Then the natural map

$$K^{\mathrm{ét}}(X) \rightarrow \mathrm{TC}(X)$$

is an equivalence mod  $p$ .

*Proof.* The comparison map is induced the cyclotomic trace, given that  $\mathrm{TC}$  is an étale Postnikov sheaf (for affines, see [28, Sec. 3]; in the semi-separated case see [11] for the conclusion of Nisnevich descent). Now assume  $X$  is proper over  $\mathrm{Spec}(R)$  henselian along  $(p)$ . Then the proper base change theorem together with Gabber's affine analog of the proper base change theorem [26] combine to show  $H_{\mathrm{ét}}^j(X, \mathcal{F}) \simeq H_{\mathrm{ét}}^j(X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p), i^* \mathcal{F})$  for all torsion abelian sheaves  $\mathcal{F}$  on  $X_{\mathrm{ét}}$ . In particular,  $X_{\mathrm{ét}}$  has finite  $p$ -cohomological dimension by induction on an affine cover and the following Lemma 6.4, so by comparing descent spectral sequences for  $K^{\mathrm{ét}}(X)$  and  $\mathrm{TC}(X)$  it suffices to show that  $\pi_n(\mathcal{K}/p)$  and  $\pi_n(\mathcal{TC}/p)$  have the same cohomology on  $X_{\mathrm{ét}}$ . But by the same base change results, this follows from the fact (see above) that they have the same restriction to  $X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p)$ .  $\square$

Above we used the following standard lemma.

**Lemma 6.4.** Let  $R$  be an  $\mathbb{F}_p$ -algebra. Then the mod  $p$  étale cohomological dimension of  $R$  is  $\leq 1$ .

*Proof.* This is classical in the noetherian case from the Artin-Schreier sequence (see [4, Exp. X, Thm. 5.1]). To obtain the non-noetherian case, we use the criterion from [4, Exp. IX, Prop. 5.5] together with the fact that cohomology commutes with filtered colimits [70, Tag 03Q4].  $\square$

**6.2. Asymptotic comparison of  $K$  and  $\mathrm{TC}$ .** Next, we show that  $K/p$  and  $\mathrm{TC}/p$  agree in large degrees for  $p$ -adic rings satisfying mild finiteness conditions. In view of Theorems 6.3 and 4.35, this yields a general  $p$ -adic Lichtenbaum–Quillen statement for rings which are henselian along  $(p)$ . Note that for smooth algebras, the result follows from the calculations of Geisser–Levine and Geisser–Hesselholt (Theorem 4.28) and for singular curves, a slight strengthening of this result appears in [29].

<sup>6</sup>This is what Thomason's étale hypercohomology construction implements. We refer to [50, Sec. 1.3] as a source for sheaves of spectra and the associated  $t$ -structure.

**Theorem 6.5.** Let  $R$  be a commutative ring and  $p$  be a prime number. Suppose that  $d \geq 1$  and:

- (1)  $R$  is henselian along  $(p)$ .
- (2) The ring  $R/p$  has finite Krull dimension.
- (3) For any  $x \in \text{Spec}(\pi_0(R)/p)$ , the residue field  $k(x)$  has the property that  $[k(x) : k(x)^p] \leq p^d$ .

The map  $K(R)/p^r \rightarrow \text{TC}(R)/p^r$  is an equivalence in degrees  $\geq d$  for any  $r$ .

*Proof.* We use throughout the following basic observation: if  $T \rightarrow T'$  is a map of spectra which is an equivalence in degrees  $\geq d$ , then for any spectrum  $T''$  with a map  $T'' \rightarrow T'$ , the map  $T \times_{T'} T'' \rightarrow T''$  is an equivalence in degrees  $\geq d$ . Using the pullback square from Theorem 1.2, we may reduce to the case where  $R$  is discrete. Using the pullback square from Theorem 4.35 (involving the spectra  $K(R)/p^r, \text{TC}(R)/p^r, K(R/p)/p^r, \text{TC}(R/p)/p^r$ ), we reduce to the case where  $R$  is an  $\mathbb{F}_p$ -algebra. It now suffices to see that  $K(R)/p^r \rightarrow \text{TC}(R)/p^r$  is an equivalence in degrees  $\geq d$ .

Note also that the result is clearly equivalent if we replace  $K(R)/p$  with  $\mathbb{K}(R)/p$ . By the theorems of Thomason–Trobaugh [74] and Blumberg–Mandell [11] respectively,  $\mathbb{K}(-)/p^r$  and  $\text{TC}(-)/p^r$  are Nisnevich sheaves on  $\text{Spec}R$  with values in the  $\infty$ -category  $\text{Sp}$ .

In the forthcoming paper [16], it will be proved that the Nisnevich topos of  $\text{Spec}R$  has *finite homotopy dimension* in the sense of [51, Def. 7.2.2.1]. In the case where  $R$  is noetherian, the result appears in [50, Thm. 3.7.7.1]. As a consequence, Postnikov towers in the  $\infty$ -category of Nisnevich sheaves of spectra on  $\text{Spec}R$  are convergent and one has a descent spectral sequence. Therefore, it suffices to see that the maps on stalks induce isomorphisms in degrees  $\geq d$ . The maps on stalks are

$$\mathbb{K}(A)/p^r \rightarrow \text{TC}(A)/p^r,$$

as  $A$  ranges over the henselizations of  $R$  at prime ideals. Since the map  $K(A)/p \rightarrow \mathbb{K}(A)/p$  is an equivalence in degrees  $\geq 1$ , it suffices to see that  $K(A)/p \rightarrow \text{TC}(A)/p$  is an equivalence in degrees  $\geq d$  for each such henselization  $A$ . This in turn follows from the fiber square of Theorem 4.35 (applied to the henselian local ring  $A$ ) and the fact that if  $k$  is a field of characteristic  $p$ , then the map  $K(k)/p^r \rightarrow \text{TC}(k)/p^r$  is an equivalence in degrees  $\geq \log_p[k : k^p]$  (which follows from Theorem 4.28 and the theory of  $p$ -bases, which implies  $\dim \Omega_k^1 \leq d$  and so  $\Omega_k^n = 0$  for  $n > d$  [55, Thm. 26.5]).  $\square$

We immediately conclude the following  $p$ -adic Lichtenbaum–Quillen isomorphism.

**Corollary 6.6.** Let  $R$  be a commutative ring which is henselian along  $(p)$  with  $\text{Spec}(R/p)$  of finite Krull dimension, and suppose that  $d \geq 1$  is such that  $[k(x) : k(x)^p] \leq p^d$  for all  $x \in \text{Spec}(R/p)$ . Then the map  $K(R)/p^r \rightarrow K^{\text{ét}}(R)/p^r$  is an equivalence in degrees  $\geq d$  for any  $r$ .

*Proof.* Combine Theorems 6.5 and 6.3.  $\square$

**6.3. Examples.** Finally, we include several explicit examples of comparisons between  $K$  and  $\text{TC}$ . We begin with an example of Theorem 6.5. In particular, the standard rings occurring in algebra or algebraic geometry over a perfect field of characteristic  $p$  are covered by this example.

**Example 6.7.** Let  $R$  be a noetherian  $\mathbb{F}_p$ -algebra which is *F-finite*, i.e., the Frobenius map is finite (Definition 5.1). Then  $R$  has finite Krull dimension [48, Prop. 1.1]. Moreover, if  $R$  is generated as a module by  $p^d$  elements over the Frobenius, this passes to any localization. It follows that if  $k$  is any residue field of  $R$ , then  $[k : k^p] \leq p^d$ . Therefore, the map  $K(R)/p \rightarrow \text{TC}(R)/p$  is an equivalence in degrees  $\geq d$  thanks to Theorem 6.5. Note that Theorem 6.5 assumes that  $d \geq 1$ , but if  $d = 0$  then  $R$  is a finite product of perfect fields, so that the result follows from Example 6.8 below.

We can also show that  $K$ -theory and  $\text{TC}$  agrees in connective degrees for certain large rings.

**Example 6.8.** Let  $R$  be a perfect  $\mathbb{F}_p$ -algebra, i.e., such that the Frobenius is an isomorphism. Then the map  $K(R)/p^r \rightarrow \mathrm{TC}(R)/p^r$  is an equivalence on connective covers for each  $r \geq 0$ .

In fact,  $\mathrm{TC}_i(R)$  vanishes for  $i > 0$ . This follows as in the calculation of  $\mathrm{TC}(\mathbb{F}_p)$  in [60, Sec. IV-4]. Indeed, by a variant of Bökstedt’s calculation, one finds  $\mathrm{THH}(R)_* \simeq R[\sigma]$  for  $|\sigma| = 2$  and

$$\mathrm{TC}_*^-(R) = W(R)[x, \sigma]/(x\sigma - p), \quad \mathrm{TP}_*(R) \simeq W(R)[x^{\pm 1}], \quad |x| = -2.$$

If  $R$  has no nontrivial idempotents, it follows that  $\pi_0(\mathrm{TC}(R)/p^r) = \mathbb{Z}/p^r\mathbb{Z}$  and that  $\pi_{-1}(\mathrm{TC}(R)/p^r)$  is the cokernel of  $F - 1$  on  $W_r(R)$ . We refer to the work of Bhatt–Morrow–Scholze [9] for more details.

Moreover,  $K_i(R)$  is a  $\mathbb{Z}[1/p]$ -module for  $i > 0$ . This follows from the existence of Adams operations in higher  $K$ -theory and the fact that  $\psi^p$  is the Frobenius, cf. [42, 47]. If  $R$  has no nontrivial idempotents, it follows by Zariski descent that the kernel of the map  $K_0(R) \rightarrow \mathbb{Z}$  is a  $\mathbb{Z}[1/p]$ -module, so that  $K_0(R)/p \simeq \mathbb{Z}/p$ . Combining all these observations, the claim for  $R$  follows provided  $\mathrm{Spec}(R)$  is connected. To reduce the general case to that one, note that everything commutes with filtered colimits, so we can assume  $R$  is the perfection of a finite type  $\mathbb{F}_p$ -algebra. Then  $\mathrm{Spec}(R)$  is noetherian, hence a finite disjoint union of connected affines, giving the reduction.

We obtain the following corollary.

**Corollary 6.9.** Let  $R$  be a ring henselian along  $(p)$ . Suppose  $R/p$  is a *semiperfect* ring, i.e., an  $\mathbb{F}_p$ -algebra such that the Frobenius map on  $R/p$  is a surjection. The map  $K(R)/p^r \rightarrow \mathrm{TC}(R)/p^r$  is an equivalence in degrees  $\geq 0$  for any  $r$ .

*Proof.* Using Theorem 4.35, we immediately reduce to the case where  $R$  itself is a semiperfect  $\mathbb{F}_p$ -algebra. In this case, let  $I \subset R$  be the nilradical. It follows that  $R/I$  is perfect and that  $I$  is locally nilpotent, so that  $(R, I)$  is a henselian pair. By Theorem 4.35 and Example 6.8, it follows that the map  $K(R)/p^r \rightarrow \mathrm{TC}(R)/p^r$  is an equivalence in degrees  $\geq 0$ .  $\square$

**Example 6.10.** Let  $C$  be a complete nonarchimedean field whose residue field is perfect of characteristic  $p$ . Let  $\mathcal{O}_C \subset C$  be the ring of integers, and let  $\pi \in \mathcal{O}_C$  be a nonzero element of positive valuation. Then  $\mathcal{O}_C$  is  $\pi$ -adically complete, and the image of the maximal ideal  $\mathfrak{m}_C \subset \mathcal{O}_C$  is a locally nilpotent ideal in  $\mathcal{O}_C/\pi$ : in particular,  $\mathfrak{m}_C \subset \mathcal{O}_C$  is henselian. We conclude that the map  $K(\mathcal{O}_C)/p^r \rightarrow \mathrm{TC}(\mathcal{O}_C)/p^r$  exhibits the former as the connective cover of the latter for  $r \geq 0$  since we know the analogous statement for the residue field (as in Example 6.8).

Given a perfectoid field  $C$  (in the sense of [66]), the ring of integers  $\mathcal{O}_C$  has perfect residue field, so the above conclusion holds. When  $C = \mathbb{C}_p$  is the completed algebraic closure of  $\mathbb{Q}_p$ , the  $p$ -adic  $K$ -theory of  $\mathcal{O}_C$  was calculated by Hesselholt [38] and shown to agree with  $\mathrm{TC}$ . A description of  $\mathrm{TC}(\mathcal{O}_C)$  in general has been given in [9].

**6.4. Split injectivity.** As we recalled in Theorem 4.28, results of Geisser–Levine and Geisser–Hesselholt show that the trace map  $K(R)/p \rightarrow \mathrm{TC}(R)/p$  induces split injections on homotopy groups whenever  $R$  is an ind-smooth local  $\mathbb{F}_p$ -algebra. In fact, the same argument shows that this also holds with mod  $p^r$  coefficients for any  $r$ , if we use the logarithmic de Rham–Witt groups (see Definition 5.25) as the mod  $p^r$  generalizations of the  $\nu^n(R)$ . The splitting comes from the étale descent spectral sequence for  $\mathrm{TC}$ , so that the complementary summand of  $\pi_n(K(R)/p^r)$  in  $\pi_n(\mathrm{TC}(R)/p^r)$  is given by  $H^1(\mathrm{Spec}(R)_{\acute{e}t}; \pi_{n+1}(\mathcal{TC}/p^r))$ , where  $\pi_{n+1}(\mathcal{TC}/p^r)$  denotes the étale sheafification of the presheaf  $\pi_n(\mathrm{TC}(-)/p^r)$ .

We can use our main result to extend this to arbitrary local  $\mathbb{F}_p$ -algebras.

**Theorem 6.11.** Let  $R$  be any local  $\mathbb{F}_p$ -algebra,  $n \geq -1$ , and  $r \geq 1$ . Then the trace map  $\pi_n(K(R)/p^r) \rightarrow \pi_n(\mathrm{TC}(R)/p^r)$  is split injective. More precisely, there is a functor  $\widetilde{\nu}_r^{n+1} : \mathrm{CAlg}_{\mathbb{F}_p} \rightarrow \mathrm{Ab}$  and a natural transformation  $\widetilde{\nu}_r^{n+1}(-) \rightarrow \pi_n(\mathrm{TC}(-)/p^r)$  such that, for  $R$  local, the induced map

$$\widetilde{\nu}_r^{n+1}(R) \oplus \pi_n(K(R)/p^r) \longrightarrow \pi_n(\mathrm{TC}(R)/p^r)$$

is an isomorphism. This is also compatible with the natural transition maps as  $r$  varies.

Moreover,  $\widetilde{\nu}_r^{n+1}$  commutes with filtered colimits; in particular, the direct sum decomposition above holds for arbitrary  $R$  after Zariski sheafification.

*Proof.* Recall that if  $R$  is an  $\mathbb{F}_p$ -algebra, we can make a functorial surjection to  $R$  from a polynomial algebra, namely take the polynomial algebra  $\mathbb{F}_p[x_a]_{a \in R}$  on variables indexed by the elements of  $R$  with the map  $x_a \mapsto a$ . Let  $R' \rightarrow R$  denote the henselization of this surjection at its kernel; thus  $R'$  is a functorial ind-smooth  $\mathbb{F}_p$ -algebra surjecting onto  $R$  with henselian kernel. Moreover,  $R \mapsto R'$  evidently commutes with filtered colimits.

Define for  $m \geq 0$

$$\widetilde{\nu}_r^m(R) := H^1(\mathrm{Spec}(R')_{\acute{e}t}; \pi_m(\mathrm{TC}/p^r)).$$

Since the coefficient presheaf  $\pi_m(\mathrm{TC}(-)/p^r)$  commutes with filtered colimits by Corollary 2.15, this étale cohomology group commutes with filtered colimits in  $R'$  by standard cocontinuity arguments. Combining with the previous, we find that  $\widetilde{\nu}_r^m(-)$  commutes with filtered colimits.

Furthermore, since  $\mathrm{Spec}(R')$  has étale  $p$ -cohomological dimension  $\leq 1$ , the descent spectral sequence for  $\mathrm{TC}(-)/p^r$  gives a natural map  $\widetilde{\nu}_r^{n+1}(R) \rightarrow \pi_n(\mathrm{TC}(R')/p^r)$  (compare with the proofs of Theorem 4.28 and Theorem 6.3). Composing with the map on TC induced by  $R' \rightarrow R$  defines the desired natural transformation  $\widetilde{\nu}_r^{n+1}(R) \rightarrow \pi_n(\mathrm{TC}(R)/p^r)$ .

Now assume  $R$  is local. Then  $R'$  is too, since a henselian ideal is radical, and it is moreover ind-smooth. Thus the argument recalled before the statement of the theorem, based on the results of Geisser–Levine and Geisser–Hesselholt, shows that

$$\widetilde{\nu}_r^{n+1}(R) \oplus \pi_n(K(R')/p^r) \xrightarrow{\sim} \pi_n(\mathrm{TC}(R')/p^r).$$

On the other hand our main rigidity theorem, Theorem 4.35, gives a long exact sequence

$$\dots \rightarrow \pi_n(K(R')/p^r) \rightarrow \pi_n(\mathrm{TC}(R')/p^r) \oplus \pi_n(K(R)/p^r) \rightarrow \pi_n(\mathrm{TC}(R)/p^r) \rightarrow \dots$$

Combining shows both that this long exact sequence breaks up into short exact sequences and that

$$\widetilde{\nu}_r^{n+1}(R) \oplus \pi_n(K(R)/p^r) \xrightarrow{\sim} \pi_n(\mathrm{TC}(R)/p^r),$$

as claimed.  $\square$

We next proceed to identify these constructions  $\widetilde{\nu}_r^m$ . When  $r = 1$ , the proof of Theorem 4.28 shows that  $\widetilde{\nu}_1^m(R) = \widetilde{\nu}^m(R')$ , and this combines with Proposition 4.30 to give an identification

$$\widetilde{\nu}_1^m(R) = \widetilde{\nu}^m(R) := \mathrm{coker}(1 - C^{-1} : \Omega_R^m \rightarrow \Omega_R^m/d\Omega_R^{m-1})$$

for arbitrary  $\mathbb{F}_p$ -algebras  $R$  and  $m \geq 0$ . More generally, we can obtain a similar description of the  $\widetilde{\nu}_r^m$  for  $r > 1$  as follows. As in [59, Sec. 4], we have a natural map

$$\overline{F} : W_r \Omega_R^m \rightarrow W_r \Omega_R^m/dV^{r-1} \Omega_R^m$$

for an arbitrary  $\mathbb{F}_p$ -algebra  $R$  which factors the Frobenius  $F : W_{r+1}\Omega_R^m \rightarrow W_r\Omega_R^m$ . Let  $\pi : W_r\Omega_R^m \rightarrow W_r\Omega_R^m/dV^{r-1}\Omega_R^m$  be the natural projection and consider the map

$$(26) \quad W_r\Omega_R^m \xrightarrow{\pi - \overline{F}} W_r\Omega_R^m/(dV^{r-1}\Omega_R^m).$$

By [59, Cor. 4.2(iii)], the map  $\pi - \overline{F}$  has kernel given by the logarithmic forms, i.e., we have a short exact sequence

$$(27) \quad 0 \rightarrow W_r\Omega_{R,\log}^m \rightarrow W_r\Omega_R^m \xrightarrow{\pi - \overline{F}} W_r\Omega_R^m/(dV^{r-1}\Omega_R^m).$$

**Proposition 6.12.** For any  $\mathbb{F}_p$ -algebra  $R$ , we have a natural identification of graded abelian groups

$$\widetilde{\nu}_r^m(R) \simeq \text{coker}(\pi - \overline{F}).$$

*Proof.* We claim that the map  $\pi - \overline{F}$  is surjective locally in the étale topology and the cokernel satisfies rigidity for henselian pairs. The first claim follows because the composite map

$$W_{r+1}\Omega_R^m \xrightarrow{R} W_r\Omega_R^m \xrightarrow{\pi - \overline{F}} W_r\Omega_R^m/dV^{r-1}\Omega_R^m$$

actually lifts to the map  $W_{r+1}\Omega_R^m \xrightarrow{R-F} W_r\Omega_R^m$ , which is a surjection in the étale topology thanks to [59, Cor. 4.1(ii)].

Second, let  $(R, I)$  be a henselian pair of  $\mathbb{F}_p$ -algebras. To see that  $\text{coker}(\pi - \overline{F})$  is rigid, we can imitate the strategy of Proposition 4.30. Namely, we let  $L$  denote the kernel of the surjection  $W_r\Omega_R^m/dV^{r-1}\Omega_R^m \rightarrow W_r\Omega_{R/I}^m/dV^{r-1}\Omega_{R/I}^m$  and contemplate the commutative diagram

$$\begin{array}{ccccc} & & & & W_r\Omega_{(R,I)}^m \\ & & & & \downarrow \\ & & & & L \\ & & & & \downarrow \\ W_{r+1}\Omega_{(R,I)}^m & \xrightarrow{R} & W_r\Omega_{(R,I)}^m & \xrightarrow{\psi} & L \\ \downarrow & & \downarrow & & \downarrow \\ W_{r+1}\Omega_R^m & \xrightarrow{R} & W_r\Omega_R^m & \xrightarrow{\pi - \overline{F}} & W_r\Omega_R^m/dV^{r-1}\Omega_R^m \\ \downarrow & & \downarrow & & \downarrow \\ W_{r+1}\Omega_{R/I}^m & \xrightarrow{R} & W_r\Omega_{R/I}^m & \xrightarrow{\pi - \overline{F}} & W_r\Omega_{R/I}^m/dV^{r-1}\Omega_{R/I}^m \end{array}$$

To prove rigidity, it suffices by the snake lemma to show that  $\psi : W_r\Omega_{(R,I)}^m \rightarrow L$  is a surjection. Note that the kernel  $L$  is surjected upon by the relative forms  $W_r\Omega_{(R,I)}^m$ ; this follows easily from the fact that  $\Omega_R^m \rightarrow \Omega_{R/I}^m$  is a surjection. Thus, the surjectivity of  $\psi$  now follows from the surjectivity of  $R - F : W_{r+1}\Omega_{(R,I)}^m \rightarrow W_r\Omega_{(R,I)}^m$  given by Lemma 5.30.

Now we prove the proposition. Let  $R$  be an  $\mathbb{F}_p$ -algebra and recall that, by definition,

$$\widetilde{\nu}_r^m(R) = H^1(\text{Spec}(R')_{\text{et}}; \pi_m(\mathcal{TC}/p^r)),$$

where  $R' \rightarrow R$  is a surjective map from an ind-smooth  $\mathbb{F}_p$ -algebra with henselian kernel. Since  $\pi_m(\mathcal{TC}/p^r) \simeq W_r\Omega_{\log}^m$  (as étale sheaves) over the ind-smooth  $\text{Spec}(R')$  by the results of Geisser–Levine and Geisser–Hesselholt, we can use the resolution of (27) in the étale topology and the fact that the two terms of (27) have no higher cohomology on affines to calculate this étale cohomology group. It follows that  $\widetilde{\nu}_r^m(R)$  identifies with  $\text{coker}(\pi - \overline{F})$  on  $R'$ . Since we have just seen this cokernel satisfies rigidity, we can replace  $R'$  by  $R$ , whence the claim.  $\square$

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