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PERIODIC CORRECTIONS IN SECULAR MILANKOVITCH THEORY APPLIED TO PASSIVE DEBRIS REMOVAL

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Abstract: *Most cartographic stability maps advocated for use in the new passive debris removal ideology based on orbital resonances are obtained through crude averaging methods. This means that from an operational perspective, its not clear where in the osculating space one should actually target to place the satellite on a natural disposal trajectory. It is also not obvious what effects the short-periodic terms may have on these re-entry solutions. We will derive the periodic corrections terms for the dominant perturbations affecting Earth satellites and investigate these considerations.*

Keywords: *Medium-Earth Orbits, Chaos, Resonances, Stability Maps. Post Mission Disposal.*

1. Introduction

The basic idea in the averaging method is to obtain approximate equations for the system evolution that contain only slowly changing variables by exploiting the presence of a small dimensionless parameter ε that characterizes the size of the perturbation. The tacit assumption is that the perturbing forces are sufficiently weak enough that these approximate *secular* equations of motion can be used to describe the orbital evolution. The perturbation equations in celestial mechanics, relating the time variation of the orbit parameters to the perturbing accelerations, are nonlinear first-order differential equations of the general form

$$\dot{\boldsymbol{x}} = \varepsilon \boldsymbol{g}(\boldsymbol{x}, t), \tag{1}$$

where $\boldsymbol{g}(\boldsymbol{x}, t)$ is assumed to be T-periodic in t. Equation 1 is trivially solved when $\varepsilon = 0$, yielding the integrals (Keplerian elements) in the unperturbed problem. The method of averaging consists in replacing Equation 1 by the averaged autonomous system

$$\dot{\bar{\boldsymbol{x}}} = \varepsilon \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}}), \tag{2}$$

$$\bar{\boldsymbol{g}}(\bar{\boldsymbol{x}}) = \frac{1}{T}\int_0^T \boldsymbol{g}(\boldsymbol{x}, t)\,\mathrm{d}t, \tag{3}$$

in which the average is performed over time, and it is understood that $\boldsymbol{x}$ in the integrand is to be regarded as a constant during the averaging process. The basis for this approximation is the 'averaging principle' which states that in the general (non-resonant) case, the short period terms removed by averaging cause only small oscillations, which are superimposed on the long-term drift described by the averaged system.

The removal of time or (what amounts to the same thing) mean anomaly requires computing the quadrature of functions depending implicitly on this variable through the true anomaly. Given a quantity $\mathscr{F}(\boldsymbol{\alpha},M)$, defined as a function of the dimensionless time variable M in addition to the other orbital elements represented as $\boldsymbol{\alpha}$, the average is defined by

$$\overline{\mathscr{F}}(\boldsymbol{\alpha}) = \frac{1}{2\pi}\int_0^{2\pi} \mathscr{F}(\boldsymbol{\alpha},M)\,\mathrm{d}M, \tag{4}$$

where the orbital elements $\boldsymbol{\alpha}$ are held constant in the integration. Although the average is defined with respect to mean anomaly, it is often more easily calculated by means of the true or eccentric anomaly, using the differential relationships

$$\frac{\mathrm{d}M}{2\pi} = \frac{r\mathrm{d}E}{2\pi a} = \frac{r^2\mathrm{d}f}{2\pi ab}, \tag{5}$$

where $b = a\sqrt{1-e^2}$ is the semi-minor axis; yielding the equivalent forms for averaging:

$$\overline{\mathscr{F}}(\boldsymbol{\alpha}) = \frac{1}{2\pi}\int_0^{2\pi} \mathscr{F}(\boldsymbol{\alpha},M)\,\mathrm{d}M = \frac{1}{2\pi a}\int_0^{2\pi} \mathscr{F}(\boldsymbol{\alpha},E)r\,\mathrm{d}E = \frac{1}{2\pi ab}\int_0^{2\pi} \mathscr{F}(\boldsymbol{\alpha},f)r^2\,\mathrm{d}f. \tag{6}$$

Note that r can be expressed in terms of f and E as

$$r = \begin{cases} \frac{a(1-e^2)}{1+e\cos f}, \\ a(1-e\cos E). \end{cases} \tag{7}$$

The general form of Gauss perturbation equations of the element set $\boldsymbol{H}$ (angular momentum vector), $\boldsymbol{e}$(eccentricity vector), and l can be stated as

$$\dot{\boldsymbol{H}} = \widetilde{\boldsymbol{r}}\cdot\boldsymbol{a}_d, \tag{8a}$$

$$\dot{\boldsymbol{e}} = \frac{1}{\mu}\left(\widetilde{\boldsymbol{v}}\cdot\widetilde{\boldsymbol{r}} - \widetilde{\boldsymbol{H}}\right)\cdot\boldsymbol{a}_d, \tag{8b}$$

$$\dot{l} = \left(-\frac{e}{\mu(1+\sqrt{1-e^2})}\left[H(\hat{\boldsymbol{e}}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} + (r+p)(\hat{\boldsymbol{e}}\cdot\boldsymbol{v})\hat{\boldsymbol{\theta}}\right] - \frac{2}{na^2}\boldsymbol{r} + \frac{\hat{\boldsymbol{z}}\cdot\boldsymbol{r}}{H(H+\hat{\boldsymbol{z}}\cdot\boldsymbol{H})}\boldsymbol{H}\right)\cdot\boldsymbol{a}_d + n, \tag{8c}$$

where $\boldsymbol{r}$ and $\boldsymbol{v}$ are the position and velocity vectors; $\hat{\boldsymbol{\theta}} = \widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{r}}/H$.
In this work, using the direct approach of Kozai [1], we will derive the *short-period correction terms* in the vectorial formulation for the J_2 effect only (contributions due to other perturbing forces will be addressed in future research) and apply these to determine the appropriate initial conditions in the *osculating* sense for targeting disposal regions depicted in *mean element* space in stability maps.

2. Body of paper, give a title

We want to characterize the general form of Equations (8) to include the Earth oblateness disturbing effect. The quadrupole disturbing function arising from an oblate planet can be stated in a general vector expression as follows

$$\mathscr{R}_2 = \frac{\mu J_2 R^2}{2r^3}\left[1 - 3(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})^2\right], \tag{9}$$

where J_2 is the dominant oblateness gravity field coefficient, R is the mean equatorial radius of the planet, and $\hat{\boldsymbol{p}}$ is a unit vector aligned with the planet's rotation pole, assumed to be fixed in inertial space. The perturbing acceleration is then given by

$$\boldsymbol{a}_2 = -\frac{3\mu J_2 R^2}{2r^4}\left\{\left[1-5(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}})^2\right]\hat{\boldsymbol{r}}+2(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})\hat{\boldsymbol{p}}\right\}. \tag{10}$$

Therefore the perturbation equations in Gaussian form can be stated as

$$\begin{aligned}\dot{\boldsymbol{H}}_2 &= \widetilde{\boldsymbol{r}}\cdot\boldsymbol{a}_2 \\ &= -\frac{3\mu J_2 R^2}{r^3}(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}})\widetilde{\hat{\boldsymbol{r}}}\cdot\hat{\boldsymbol{p}},\end{aligned} \tag{11}$$

$$\begin{aligned}\dot{\boldsymbol{e}}_2 &= \frac{1}{\mu}\left(\widetilde{\boldsymbol{v}}\cdot\widetilde{\boldsymbol{r}}-\widetilde{\boldsymbol{H}}\right)\cdot\boldsymbol{a}_2 \\ &= -\frac{3J_2R^2}{2r^4}\left\{-\left[1-5(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})^2\right]\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{r}}+2(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})(\widetilde{\boldsymbol{v}}\cdot\widetilde{\boldsymbol{r}}-\widetilde{\boldsymbol{H}})\cdot\hat{\boldsymbol{p}}\right\}.\end{aligned} \tag{12}$$

The averaged equations of motion have the form

$$\dot{\overline{\boldsymbol{H}}}_2 = \frac{3\mu J_2 R^2}{2a^3h^3}(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{h}})\widetilde{\hat{\boldsymbol{h}}}\cdot\hat{\boldsymbol{p}}, \tag{13}$$

$$\dot{\overline{\boldsymbol{e}}}_2 = -\frac{3nJ_2R^2}{4a^2h^4}\left\{\left[1-5(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{h}})^2\right]\widetilde{\hat{\boldsymbol{h}}}+2(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{h}})\widetilde{\hat{\boldsymbol{p}}}\right\}\cdot\boldsymbol{e}. \tag{14}$$

The differential equations for the periodic term can be written by substracting Eq.(13) from Eq.(11)

$$\begin{aligned}\dot{\boldsymbol{H}}_2^{sp} &= \dot{\boldsymbol{H}}_2-\dot{\overline{\boldsymbol{H}}}_2 \\ &= -3\mu J_2R^2\hat{\boldsymbol{p}}\cdot\left[\frac{\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r^3}+\frac{\hat{\boldsymbol{h}}\hat{\boldsymbol{h}}}{2a^3h^3}\right]\cdot\widetilde{\hat{\boldsymbol{p}}},\end{aligned} \tag{15}$$

where sp denotes the short-period perturbation and the orbital elements are taken as constant. Note that the independent variable can be transformed from time to true anomaly by using the relation

$$\mathrm{d}t = \frac{r^2}{H}\mathrm{d}f. \tag{16}$$

The short-period perturbations of the first-order can be obtained as

$$\Delta\boldsymbol{H}_{sp} = -3\mu J_2R^2\hat{\boldsymbol{p}}\cdot\int\left(\frac{\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r^3}+\frac{\hat{\boldsymbol{h}}\hat{\boldsymbol{h}}}{2a^3h^3}\right)\mathrm{d}t\cdot\widetilde{\hat{\boldsymbol{p}}} \tag{17}$$

where

$$\hat{\boldsymbol{r}} = \cos f\hat{\boldsymbol{e}}+\sin f\hat{\boldsymbol{e}}_\perp, \tag{18}$$

Then the mean value with respect to M can be computed from

$$\overline{\Delta\boldsymbol{H}}_{sp} = \frac{1}{2\pi}\int_0^{2\pi}\Delta\boldsymbol{H}_{sp}\,\mathrm{d}M. \tag{19}$$

Let us solve the integral in Eq.17 dividing it into part A

$$A = \int_0^t \frac{\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r^3} dt = \frac{\mu}{H^3} \int_0^f \left[\cos^2 f\, \hat{\boldsymbol{e}}\hat{\boldsymbol{e}} + \cos f \sin f (\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}) + \sin^2 f\, \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\right] (1 + e\cos f)\, df$$

$$= \frac{\mu}{12H^3} (6f + 3\sin 2f + 9e\sin f + e\sin 3f)\, \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}$$

$$+ \frac{\mu}{12H^3} (6f - 3\sin 2f + 3e\sin f - e\sin 3f)\, \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp \tag{20}$$

$$+ \frac{\mu}{12H^3} (-3e\cos f - e\cos 3f - 3\cos 2f + 4e + 3)\, (\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}})$$

and part B

$$B = \int_0^t \frac{\hat{\boldsymbol{h}}\hat{\boldsymbol{h}}}{2a^3h^3} dt = \frac{\hat{\boldsymbol{h}}\hat{\boldsymbol{h}}}{2a^3h^3} t = \frac{\hat{\boldsymbol{h}}\hat{\boldsymbol{h}}}{2na^3h^3} M \tag{21}$$

Substituting Eq.(20) and Eq.(21) in Eq.(17) we get

$$\Delta \boldsymbol{H}_{sp} = -3\mu J_2 R^2 \hat{\boldsymbol{p}} \cdot (A + B) \cdot \widetilde{\hat{\boldsymbol{p}}} \tag{22}$$

To obtain the average short period correction, as stated in Eq.(19), we have to integrate again over the mean anomaly. The core integrals are given by the following equations Eqs.(23),(24),(25),(26):

$$B1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\hat{\boldsymbol{h}}\hat{\boldsymbol{h}}}{2na^3h^3} M = \frac{\pi}{2na^3h^3} \hat{\boldsymbol{h}}\hat{\boldsymbol{h}} \tag{23}$$

$$A1 = \frac{\mu}{12H^3} \frac{1}{2\pi} \int_0^{2\pi} (6f + 3\sin 2f + 9e\sin f + e\sin 3f)\, \hat{\boldsymbol{e}}\hat{\boldsymbol{e}} dM = 0 \tag{24}$$

$$A2 = \frac{\mu}{12H^3} \frac{1}{2\pi} \int_0^{2\pi} (6f - 3\sin 2f + 3e\sin f - e\sin 3f)\, \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp dM = 0 \tag{25}$$

Eq.(24) and Eq.(25) are equal to zero being the integrals of odd functions over an interval from 0 to 2π

$$\begin{aligned} A3 &= \frac{\mu}{12H^3} \frac{1}{2\pi} \int_0^{2\pi} (-3e\cos f - e\cos 3f - 3\cos 2f + 4e + 3)\, dM \\ &= -\frac{\mu}{12H^3} \left[3eX_0^{0,1} + 3X_0^{0,2} + eX_0^{0,3} - 4e - 3\right] \end{aligned} \tag{26}$$

Where the Hansen's coefficients of interest are

$$X_0^{0,m} = \frac{1}{2\pi}\int_0^{2\pi}\cos(mf)dM = \frac{(-e)^m(1+m\sqrt{1-e^2})}{(1+\sqrt{1-e^2})^m}; \quad m = 1,2,3,... \tag{27}$$

therefore we finally get the short period correction for the angular momentum vector

$$\overline{\Delta \boldsymbol{H}}_{sp} = \Delta \boldsymbol{H}_{sp} = -3\mu J_2 R^2 \hat{\boldsymbol{p}}\cdot\left[A3(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}) + \frac{\pi}{2na^3h^3}\hat{\boldsymbol{h}}\hat{\boldsymbol{h}}\right]\cdot\widetilde{\hat{\boldsymbol{p}}} \tag{28}$$

where the dyadic term can be written in classical notation as follows

$$\hat{\boldsymbol{p}}\cdot A3(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}})\cdot\widetilde{\hat{\boldsymbol{p}}} = A3\left[(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}})(\widetilde{\hat{\boldsymbol{p}}}\cdot\hat{\boldsymbol{e}}_\perp) + (\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp)(\widetilde{\hat{\boldsymbol{p}}}\cdot\hat{\boldsymbol{e}})\right] \tag{29}$$

The same procedure has to be applied on the eccentricity vector so we substract Eq.(14) from Eq.(12)

$$\dot{\boldsymbol{e}}_2^{sp} = \dot{\boldsymbol{e}}_2 - \dot{\bar{\boldsymbol{e}}}_2 \tag{30}$$

The differential equations for the periodic term can be written as

$$\begin{aligned}\dot{\boldsymbol{e}}_2^{sp} = \frac{3J_2R^2}{4}\Bigg\{ & 2\widetilde{\boldsymbol{H}}\cdot\frac{\hat{\boldsymbol{r}}}{r^4} - \frac{10}{r^4}(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})^2\hat{\boldsymbol{r}}\cdot\widetilde{\boldsymbol{H}} - \frac{4}{r^4}(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})(\widetilde{\boldsymbol{v}}\cdot\widetilde{\boldsymbol{r}} - \widetilde{\boldsymbol{H}})\cdot\hat{\boldsymbol{p}} \\ & + \frac{n}{a^2h^4}\left[1 - 5(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{h}})^2\right]\widetilde{\hat{\boldsymbol{h}}}\cdot\boldsymbol{e} + \frac{2n}{a^2h^4}(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{h}})\widetilde{\hat{\boldsymbol{p}}}\cdot\boldsymbol{e}\Bigg\}\end{aligned} \tag{31}$$

We have to integrate Eq.(31) over time (or true anomaly) to obtain the short-period perturbations of the first-order. To do that, let us rearrange separately some of its terms introducing a new notation that, through the use of dyadic and tryadic expressions, allows to isolate time dependent terms that have to be subsequently integrated.

$$C1 = \frac{\boldsymbol{r}}{r^5} \tag{32}$$

$$C2 = \frac{1}{r^4}(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})^2\hat{\boldsymbol{r}}\cdot\widetilde{\boldsymbol{H}} = \begin{matrix}\hat{\mathbf{p}} \\ \cdot \\ \hat{\mathbf{p}}\cdot\ \frac{\mathbf{rrr}}{r^7}\ \cdot\widetilde{\boldsymbol{H}}\end{matrix} \tag{33}$$

$$\begin{aligned}C3 &= \frac{1}{r^4}(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})(\widetilde{\boldsymbol{v}}\cdot\widetilde{\boldsymbol{r}} - \widetilde{\boldsymbol{H}})\cdot\hat{\boldsymbol{p}} = \frac{1}{r^4}(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})\left[-\frac{\mu}{H^2}(\boldsymbol{r}\boldsymbol{e} + \frac{\boldsymbol{r}\boldsymbol{r}}{r})\cdot\widetilde{\boldsymbol{H}} - \frac{\mu e}{H}(\boldsymbol{r}\cdot\hat{\boldsymbol{e}}_\perp)\overline{\overline{U}} - \widetilde{\boldsymbol{H}}\right]\cdot\hat{\boldsymbol{p}} \\ &= -\frac{\mu}{H^2}\left(\hat{\mathbf{p}}\cdot\frac{\mathbf{rr}}{r^5}\cdot\widetilde{\boldsymbol{H}}\right)(\boldsymbol{e}\cdot\hat{\mathbf{p}}) - \frac{\mu}{H^2}\begin{matrix}\widetilde{\boldsymbol{H}} \\ \cdot \\ \hat{\mathbf{p}}\cdot\ \frac{\mathbf{rrr}}{r^6}\ \cdot\hat{\mathbf{p}}\end{matrix} - \frac{\mu e}{H}(\hat{\mathbf{p}}\cdot\frac{\mathbf{rr}}{r^5}\cdot\hat{\boldsymbol{e}}_\perp)\hat{\mathbf{p}} - \left(\hat{\mathbf{p}}\cdot\frac{\mathbf{r}}{r^5}\right)(\widetilde{\boldsymbol{H}}\cdot\hat{\mathbf{p}})\end{aligned} \tag{34}$$

In Eq.(34) we considered:

$$\widetilde{\boldsymbol{v}} \cdot \widetilde{\boldsymbol{r}} = \boldsymbol{r}\boldsymbol{v} - (\boldsymbol{r} \cdot \boldsymbol{v})\overline{\overline{U}} \tag{35}$$

where

$$\boldsymbol{r}\boldsymbol{v} = -\frac{\mu}{H^2}(\boldsymbol{r}\boldsymbol{e} + \frac{\boldsymbol{r}\boldsymbol{r}}{r}) \cdot \widetilde{\boldsymbol{H}} \tag{36}$$

$$\boldsymbol{r} \cdot \boldsymbol{v} = -\frac{\mu}{H^2}\left[\boldsymbol{r} \cdot (\tilde{\boldsymbol{e}} \cdot \boldsymbol{H}) + \boldsymbol{r} \cdot (\frac{\tilde{\boldsymbol{r}}}{r} \cdot \tilde{\boldsymbol{H}})\right] = \frac{\mu}{H^2}\left[eH(\boldsymbol{r} \cdot \hat{\boldsymbol{e}}_\perp)\right] \tag{37}$$

$\overline{\overline{U}}$ is identity dyadic that has the general property: $\overline{\overline{U}} \cdot \boldsymbol{p} = \boldsymbol{p} \cdot \overline{\overline{U}} = \boldsymbol{p}$. In Eqs.(33); Eq.(34) the new notation means that the central part (a dyad or a tryad) is subjected to a scalar or vector product with the surrounding quantities. In this way we can recognize and isolate the core integrals of interest:

$$\begin{aligned}
I = \int_0^t \frac{\boldsymbol{r}\boldsymbol{r}\boldsymbol{r}}{r^7}dt &= \int_0^t \frac{1}{r^4}\left[\cos^3 f\,\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}} + \cos^2 f \sin f\,(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}})\right.\\
&\quad \left. + \cos f \sin^2 f\,(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}) + \sin^3 f\,\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\right]dt\\
&= \frac{\mu^2}{H^5}\int_0^f \left[\cos^3 f\,\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}} + \cos^2 f \sin f\,(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}})\right.\\
&\quad \left. + \cos f \sin^2 f\,(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}) + \sin^3 f\,\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\right](1+e\cos f)^2 df\\
&= \frac{\mu^2}{16H^5}\left[10e^2\sin f + \frac{5}{3}e^2\sin 3f + \frac{1}{5}e^2\sin 5f + 8e\sin 2f + e\sin 4f + 12\sin f + \frac{4}{3}\sin 3f + 12ef\right]\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\\
&\quad + \frac{\mu^2}{16H^5}\left[-2e^2\cos f - e^2\cos 3f - \frac{1}{5}e^2\cos 5f - 4e\cos 2f - e\cos 4f - 4\cos f - \frac{4}{3}\cos 3f\right.\\
&\quad \left. + \frac{16}{5}e^2 + 5e + \frac{16}{3}\right](\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}})\\
&\quad + \frac{\mu^2}{16H^5}\left[2e^2\sin f - \frac{1}{3}e^2\sin 3f - \frac{1}{5}e^2\sin 5f - e\sin 4f + 4\sin f - \frac{4}{3}\sin 3f + 4ef\right](\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}})\\
&\quad + \frac{\mu^2}{16H^5}\left[-2e^2\cos f - \frac{1}{3}e^2\cos 3f + \frac{1}{5}e^2\cos 5f - 4e\cos 2f + e\cos 4f - 12\cos f - \frac{4}{3}\cos 3f\right.\\
&\quad \left. + \frac{32}{15}e^2 + 3e + \frac{32}{3}\right]\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp
\end{aligned} \tag{38}$$

$$II = \int_0^t \frac{\boldsymbol{r}\boldsymbol{r}}{r^5}dt = A \tag{39}$$

$$
\begin{aligned}
III &= \int_0^t \frac{\boldsymbol{r}}{r^5}dt = \frac{1}{H}\int_0^f \frac{\hat{\boldsymbol{r}}}{r^2}df = \frac{\mu^2}{H^5}\int_0^f (\cos f\hat{\boldsymbol{e}} + \sin f\hat{\boldsymbol{e}}_\perp)(1+e\cos f)^2 df \\
&= \frac{\mu^2}{12H^5}(9e^2\sin f + e^2\sin 3f + 6e\sin 2f + 12\sin f + 12ef)\hat{\boldsymbol{e}} \\
&- \frac{\mu^2}{12H^5}(3e^2\cos f + e^2\cos 3f + 6e\cos 2f + 12\cos f - 4e^2 - 6e - 12)\hat{\boldsymbol{e}}_\perp
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
IV &= \int_0^t \frac{\boldsymbol{rrr}}{r^6}dt = \int_0^t \frac{1}{r^3}\left[\cos^3 f\,\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}} + \cos^2 f\sin f\,(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}})\right. \\
&\qquad\qquad \left. + \cos f\sin^2 f\,(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}) + \sin^3 f\,\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\right] dt \\
&= \frac{\mu}{H^3}\int_0^f \left[\cos^3 f\,\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}} + \cos^2 f\sin f\,(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}})\right. \\
&\qquad\qquad \left. + \cos f\sin^2 f\,(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}) + \sin^3 f\,\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\right](1+e\cos f)df \\
&= \frac{\mu}{8H^3}(2e\sin 2f + \frac{1}{4}e\sin 4f + 6\sin f + \frac{2}{3}\sin 3f + 3ef)\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}} \\
&+ \frac{\mu}{8H^3}(-e\cos 2f - \frac{1}{4}e\cos 4f - 2\cos f - \frac{2}{3}\cos 3f + \frac{5}{4}e + \frac{8}{3})\,(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}) \\
&+ \frac{\mu}{8H^3}(-\frac{1}{4}e\sin 4f + 2\sin f - \frac{2}{3}\sin 3f + ef)\,(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}) \\
&+ \frac{\mu}{8H^3}(-e\cos 2f + \frac{1}{4}e\cos 4f - 6\cos f + \frac{2}{3}\cos 3f + \frac{3}{4}e + \frac{16}{3})\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp
\end{aligned}
\tag{41}
$$

Equations (38);(39);(40);(41) are substituted into Eqs (32);(33);(34) that are subsequently plugged into Eq.(31) to obtain the first-order short-period perturbations.

$$
\begin{aligned}
\Delta\boldsymbol{e}_2^{sp} &= \frac{3J_2R^2}{4}\Bigg\{2\widetilde{\boldsymbol{H}}\cdot III - 10 \overset{\hat{\mathbf{p}}}{\overset{\cdot}{\hat{\mathbf{p}}\cdot\ I\ \cdot\widetilde{\boldsymbol{H}}}} \\
&+ \frac{4\mu}{H^2}\left(\hat{\mathbf{p}}\cdot II\cdot\widetilde{\boldsymbol{H}}\right)(\boldsymbol{e}\cdot\hat{\mathbf{p}}) + \frac{4\mu}{H^2} \overset{\widetilde{\boldsymbol{H}}}{\overset{\cdot}{\hat{\mathbf{p}}\cdot\ IV\ \cdot\hat{\mathbf{p}}}} + \frac{4\mu e}{H}(\hat{\mathbf{p}}\cdot II\cdot\hat{\boldsymbol{e}}_\perp)\hat{\mathbf{p}} + 4\,(\hat{\mathbf{p}}\cdot III)\,(\widetilde{\boldsymbol{H}}\cdot\hat{\mathbf{p}}) \\
&+ \frac{M}{a^2h^2}\left[1-5(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{h}})^2\right]\widetilde{\hat{\boldsymbol{h}}}\cdot\boldsymbol{e} + \frac{2M}{a^2h^2}(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{h}})\widetilde{\hat{\boldsymbol{p}}}\cdot\boldsymbol{e}\Bigg\}
\end{aligned}
\tag{42}
$$

Then the mean value with respect to the mean anomaly is calculated through the following integrals

$$
\begin{aligned}
\bar{I} &= \frac{1}{2\pi}\int_0^{2\pi} I\, dM = \frac{\mu^2}{16H^5}\left[-2e^2X_0^{0,1} - e^2X_0^{0,3} - \frac{1}{5}e^2X_0^{0,5} - 4eX_0^{0,2}\right. \\
&\left. - eX_0^{0,4} - 4X_0^{0,1} - \frac{4}{3}X_0^{0,3} + \frac{16}{5}e^2 + 5e + \frac{16}{3}\right](\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}) \\
&+ \frac{\mu^2}{16H^5}\left[-2e^2X_0^{0,1} - \frac{1}{3}e^2X_0^{0,3} + \frac{1}{5}e^2X_0^{0,5} - 4eX_0^{0,2}\right. \\
&\left. + eX_0^{0,4} - 12X_0^{0,1} - \frac{4}{3}X_0^{0,3} + \frac{32}{15}e^2 + 3e + \frac{32}{3}\right]\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp \\
&= I_a(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}) + I_b(\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp)
\end{aligned} \tag{43}
$$

$$
\overline{II} = \frac{1}{2\pi}\int_0^{2\pi} II\, dM = A3(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}) \tag{44}
$$

$$
\begin{aligned}
\overline{III} &= \frac{1}{2\pi}\int_0^{2\pi} III\, dM = -\frac{\mu^2}{12H^5}\left[3e^2X_0^{0,1} + e^2X_0^{0,3} + 6eX_0^{0,2} + 12X_0^{0,1} - 4e^2 - 6e - 12\right]\hat{\boldsymbol{e}}_\perp \\
&= III_a\hat{\boldsymbol{e}}_\perp
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
\overline{IV} &= \frac{1}{2\pi}\int_0^{2\pi} IV\, dM = +\frac{\mu}{8H^3}(-eX_0^{0,2} - \frac{1}{4}eX_0^{0,4} - 2X_0^{0,1} - \frac{2}{3}X_0^{0,3} + \frac{5}{4}e + \frac{8}{3})(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}) \\
&+ \frac{\mu}{8H^3}(-eX_0^{0,2} + \frac{1}{4}eX_0^{0,4} - 6X_0^{0,1} + \frac{2}{3}X_0^{0,3} + \frac{3}{4}e + \frac{16}{3})\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp \\
&= IV_a(\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp + \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}) + IV_b(\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp)
\end{aligned} \tag{46}
$$

Finally we obtain the short period correction for the eccentricity vector

$$
\begin{aligned}
\overline{\Delta \boldsymbol{e}}_2^{sp} = \frac{3J_2R^2}{4} \Bigg\{ & 2\widetilde{\boldsymbol{H}}\cdot\overline{III} - 10 \underset{\hat{\mathbf{p}}\cdot\ \overline{I}\ \cdot\widetilde{\boldsymbol{H}}}{\overset{\hat{\mathbf{p}}}{\cdot}} \\
& + \frac{4\mu}{H^2}\left(\hat{\mathbf{p}}\cdot\overline{II}\cdot\widetilde{\boldsymbol{H}}\right)(\boldsymbol{e}\cdot\hat{\mathbf{p}}) + \frac{4\mu}{H^2} \underset{\hat{\mathbf{p}}\cdot\ \overline{IV}\ \cdot\hat{\mathbf{p}}}{\overset{\widetilde{\boldsymbol{H}}}{\cdot}} + \frac{4\mu e}{H}(\hat{\mathbf{p}}\cdot\overline{II}\cdot\hat{\boldsymbol{e}}_\perp)\hat{\mathbf{p}} + 4\left(\hat{\mathbf{p}}\cdot\overline{III}\right)(\widetilde{\boldsymbol{H}}\cdot\hat{\mathbf{p}}) \\
& + \frac{\pi}{a^2h^2}\left[1-5(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{h}})^2\right]\widetilde{\hat{\boldsymbol{h}}}\cdot\boldsymbol{e} + \frac{2\pi}{a^2h^2}(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{h}})\widetilde{\hat{\boldsymbol{p}}}\cdot\boldsymbol{e} \Bigg\}
\end{aligned}
\tag{47}
$$

The notation we introduced simplify the final expression but, for sake of clarity and to allow an easier implementation, we also report below the expanded algebra written in classical notation summarizing the main steps we did in previous calculations.
In order, the first term to consider as it appears in the final expression of Eq.(47) is:

$$
\widetilde{\boldsymbol{H}}\cdot\overline{III} = III_a(\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{e}}_\perp) \tag{48}
$$

The second term is

$$
\frac{1}{r^4}(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})^2\hat{\boldsymbol{r}}\cdot\widetilde{\boldsymbol{H}} = \underset{\hat{\mathbf{p}}\cdot\ \frac{\mathbf{rrr}}{r^7}\ \cdot\widetilde{\boldsymbol{H}}}{\overset{\hat{\mathbf{p}}}{\cdot}} \tag{49}
$$

after integrating twice, it becomes:

$$
\begin{aligned}
\underset{\hat{\mathbf{p}}\cdot\ \overline{I}\ \cdot\widetilde{\boldsymbol{H}}}{\overset{\hat{\mathbf{p}}}{\cdot}} = & I_a\left[(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}})(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}})(\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{e}}_\perp) + (\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}})(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp)(\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{e}}) + (\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp)(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}})(\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{e}})\right] \\
& + I_b\left[(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp)(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp)(\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{e}}_\perp)\right]
\end{aligned}
\tag{50}
$$

Then

$$
\begin{aligned}
& \frac{1}{r^4}(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})\left[-\frac{\mu}{H^2}(\boldsymbol{re}+\frac{\boldsymbol{rr}}{r})\cdot\widetilde{\boldsymbol{H}}\right]\cdot\hat{\boldsymbol{p}} \\
& = -\frac{\mu}{H^2}\left(\hat{\mathbf{p}}\cdot\frac{\mathbf{rr}}{r^5}\cdot\widetilde{\boldsymbol{H}}\right)(\boldsymbol{e}\cdot\hat{\mathbf{p}}) - \frac{\mu}{H^2} \underset{\hat{\mathbf{p}}\cdot\ \frac{\mathbf{rrr}}{r^6}\ \cdot\hat{\mathbf{p}}}{\overset{\widetilde{\boldsymbol{H}}}{\cdot}}
\end{aligned}
\tag{51}
$$

Eq.(51) consists of two terms that, after the two integration processes, become as in Eqs(52);(53):

$$
\left(\hat{\mathbf{p}}\cdot\overline{II}\cdot\widetilde{\boldsymbol{H}}\right)(\boldsymbol{e}\cdot\hat{\mathbf{p}}) = A3\left[(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}})(\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{e}}_\perp) + (\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp)(\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{e}})\right](\boldsymbol{e}\cdot\hat{\mathbf{p}}) \tag{52}
$$

$$
\begin{aligned}
\underset{\hat{\mathbf{p}}\cdot\ \overline{IV}\ \cdot\hat{\mathbf{p}}}{\overset{\widetilde{\boldsymbol{H}}}{\cdot}} = & IV_a\left[(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}})(\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{e}})(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp) + (\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}})(\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{e}}_\perp)(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}) + (\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp)(\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{e}})(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}})\right] \\
& + IV_b\left[(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp)(\widetilde{\boldsymbol{H}}\cdot\hat{\boldsymbol{e}}_\perp)(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp)\right]
\end{aligned}
\tag{53}
$$

Another term is:

$$\frac{1}{r^4}(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})\left[-\frac{\mu e}{H}(\boldsymbol{r}\cdot\hat{\boldsymbol{e}}_\perp)\overline{\overline{U}}\right]\cdot\hat{\boldsymbol{p}} = -\frac{\mu e}{H}(\hat{\mathbf{p}}\cdot\frac{\mathbf{rr}}{r^5}\cdot\hat{\boldsymbol{e}}_\perp)\hat{\mathbf{p}} \tag{54}$$

after the double integration we have:

$$(\hat{\mathbf{p}}\cdot\overline{II}\cdot\hat{\boldsymbol{e}}_\perp)\hat{\mathbf{p}} = A3\left[(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}})(\hat{\boldsymbol{e}}_\perp\cdot\hat{\boldsymbol{e}}_\perp)+(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp)(\hat{\boldsymbol{e}}_\perp\cdot\hat{\boldsymbol{e}})\right]\hat{\mathbf{p}} = A3\left[(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}})+(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{e}}_\perp)(\hat{\boldsymbol{e}}_\perp\cdot\hat{\boldsymbol{e}})\right]\hat{\mathbf{p}} \tag{55}$$

The following is the last term to analyze:

$$\left(\hat{\mathbf{p}}\cdot\frac{\mathbf{r}}{r^5}\right)(\widetilde{\boldsymbol{H}}\cdot\hat{\mathbf{p}}) \tag{56}$$

In the final expression it becomes

$$\left(\hat{\mathbf{p}}\cdot\overline{III}\right)(\widetilde{\boldsymbol{H}}\cdot\hat{\mathbf{p}}) \tag{57}$$

3. Results

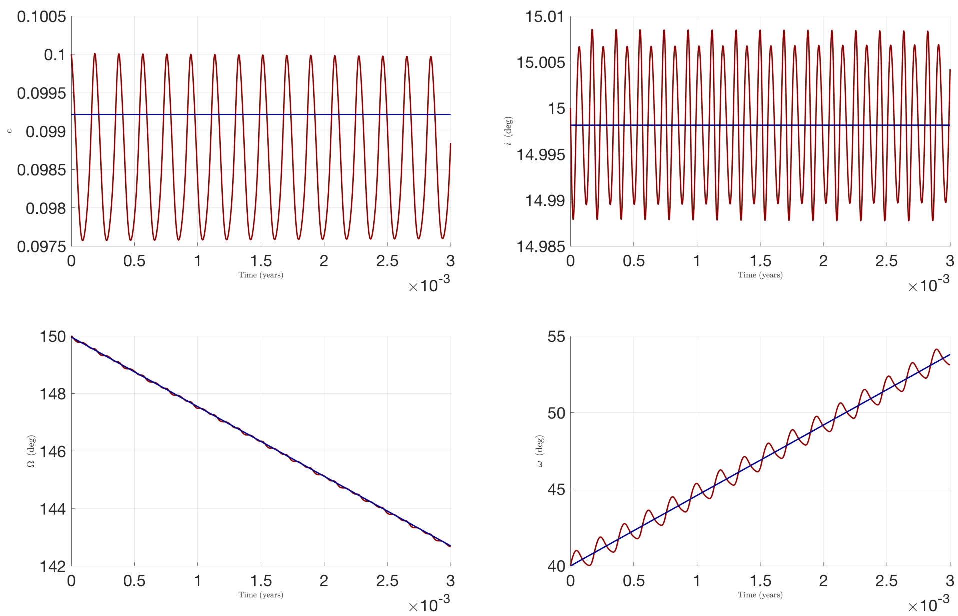
The short period correction curves are obtained correcting only the initial conditions. This corrected set represents the new initial conditions for the singly averaged model. A first-order averaged model, based on the Milankovitch vector formulation of perturbation theory [2] which govern the long-term evolution of satellite orbits is summarized in Table 3. where, for completeness, are reported the effects of solar radiation pressure and third-body perturbations, as well as that of the Earth oblateness, studied in this paper. Future work is intended to extend the results here provided to the other perturbations, too.

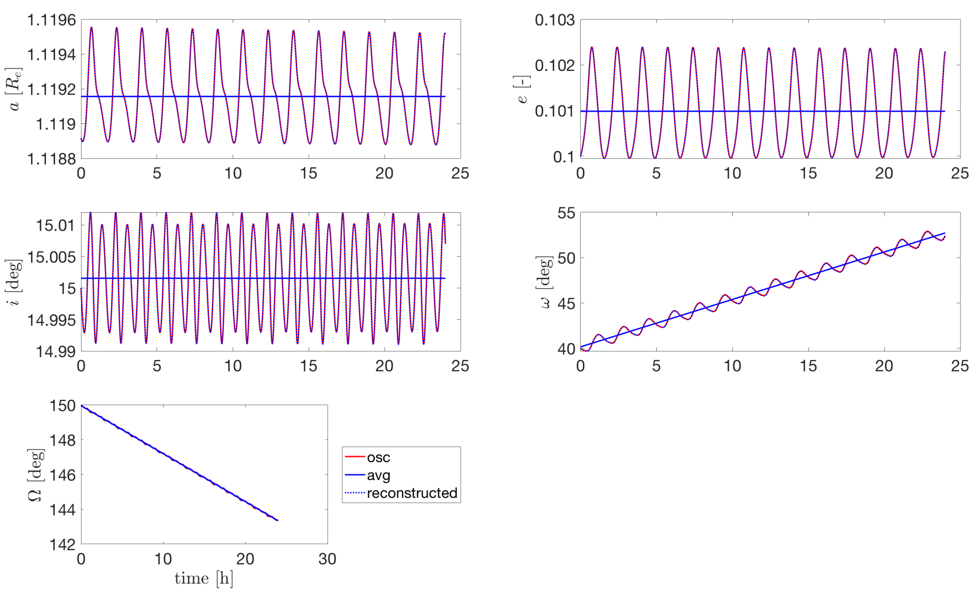
Table 1. Singly-averaged equations of motion governing solar radiation pressure, planetary oblateness, and third-body gravitational perturbations, where the notation follows from [3], to which we refer for the omitted details.

	SRP	Oblateness	Third-Body
$\dot{\boldsymbol{h}}$	$-\frac{H_s \tan\Lambda}{d_s^2}\widetilde{\hat{\boldsymbol{d}}}_s\cdot\boldsymbol{e}$	$-\frac{3nJ_2R^2}{2a^2h^5}(\hat{\boldsymbol{p}}\cdot\boldsymbol{h})\widetilde{\hat{\boldsymbol{p}}}\cdot\boldsymbol{h}$	$\frac{3\mu_p}{2nd_p^3}\hat{\boldsymbol{d}}_p\cdot(5\boldsymbol{ee}-\boldsymbol{hh})\cdot\widetilde{\hat{\boldsymbol{d}}}_p$
$\dot{\boldsymbol{e}}$	$-\frac{H_s \tan\Lambda}{d_s^2}\widetilde{\hat{\boldsymbol{d}}}_s\cdot\boldsymbol{h}$	$-\frac{3nJ_2R^2}{4a^2h^5}\left\{\left[1-\frac{5}{h^2}(\hat{\boldsymbol{p}}\cdot\boldsymbol{h})^2\right]\widetilde{\boldsymbol{h}}+2(\hat{\boldsymbol{p}}\cdot\boldsymbol{h})\widetilde{\hat{\boldsymbol{p}}}\right\}\cdot\boldsymbol{e}$	$\frac{3\mu_p}{2nd_p^3}\left[\hat{\boldsymbol{d}}_p\cdot(5\boldsymbol{eh}-\boldsymbol{he})\cdot\widetilde{\hat{\boldsymbol{d}}}_p-2\widetilde{\boldsymbol{h}}\cdot\boldsymbol{e}\right]$

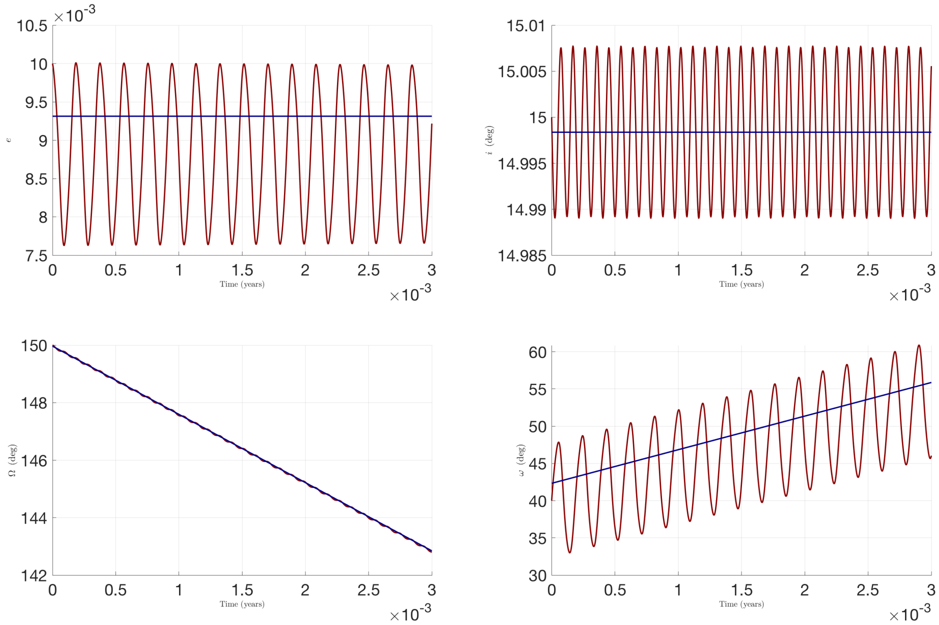
We test the equations on a reference orbits comparing our result with the following initial orbital elements (semi major axis, eccentricity, inclination, RAAN, argument of perigee and true longitude respectively):

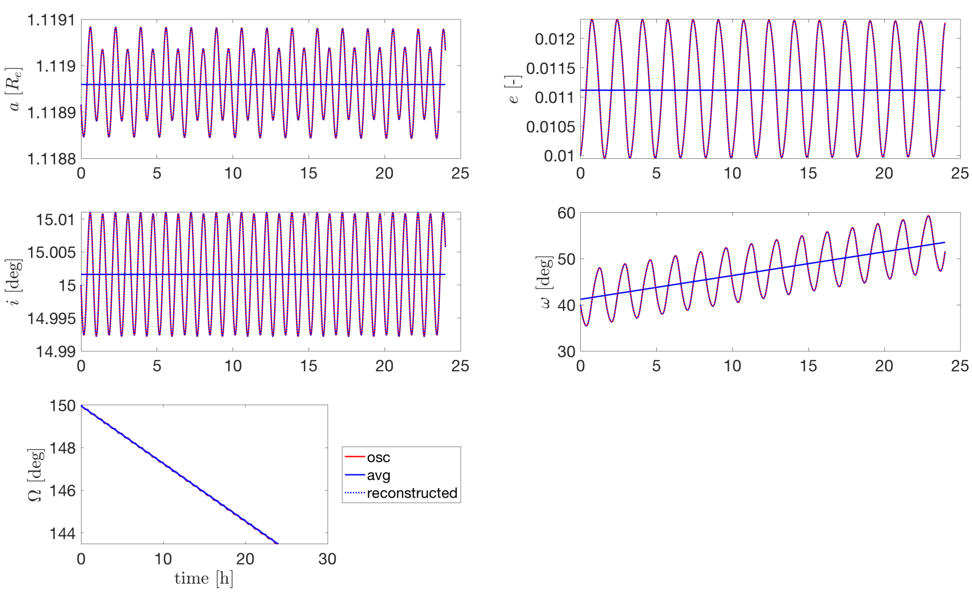
$$a = 7136.6\,\mathrm{km};\ \mathrm{e} = 0.1\,;\ \mathrm{i} = 15\,\mathrm{deg}\,;\ \Omega = 150\,\mathrm{deg}\,;\ \omega = 40\,\mathrm{deg}\,;\mathrm{L} = \Omega + \omega + \mathrm{f} = 210\,\mathrm{deg}$$



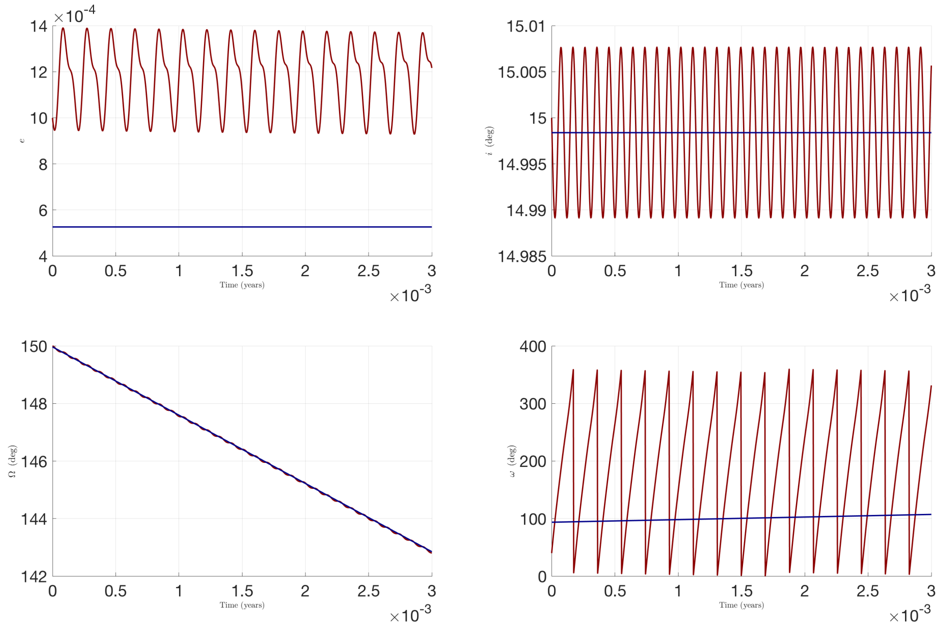


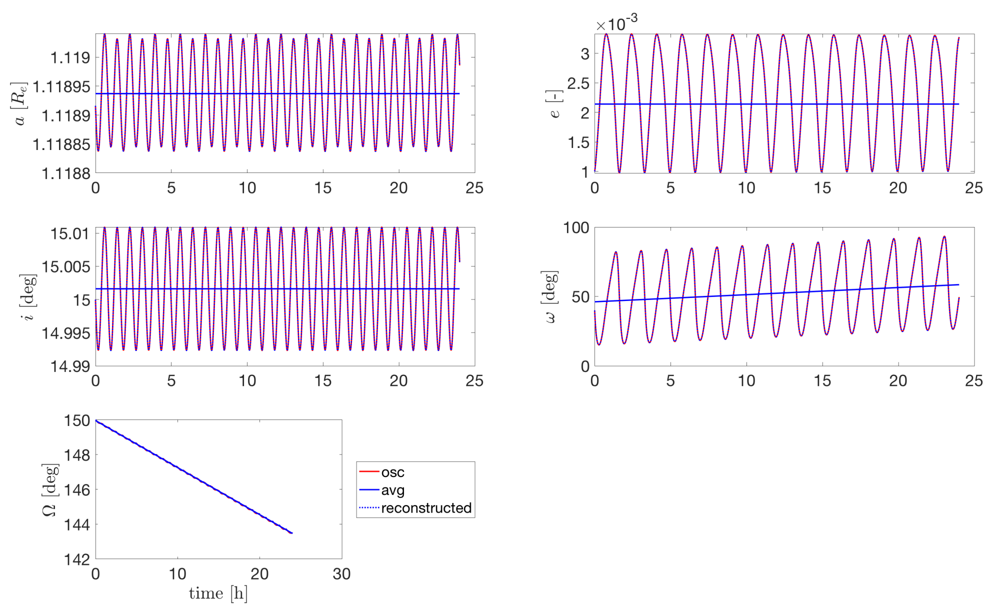
$$a = 7136.6\,\text{km};\ \text{e} = 0.01\,;\ \text{i} = 15\,\text{deg}\,;\ \Omega = 150\,\text{deg}\,;\ \omega = 40\,\text{deg}\,;\text{L} = \Omega + \omega + \text{f} = 210\,\text{deg}$$





$a = 7136.6\,\mathrm{km}$; $\mathrm{e} = 0.001$; $\mathrm{i} = 15\,\mathrm{deg}$; $\Omega = 150\,\mathrm{deg}$; $\omega = 40\,\mathrm{deg}$; $\mathrm{L} = \Omega + \omega + \mathrm{f} = 210\,\mathrm{deg}$





4. References

[1] Kozai, Y. “The Motion of a Close garEarth Satellite.” The Astronomical Journal, Vol. 64, pp. 367–377, 1959.

[2] Rosengren, A. J. and Scheeres, D. J. “On the Milankovitch Orbital Elements for Perturbed Keplerian Motion.” Celestial Mechanics and Dynamical Astronomy, Vol. 118, pp. 197–220, 2014.

[3] Rosengren, A. J. and Scheeres, D. J. “Long-Term Dynamics of High Area-to-Mass Ratio Objects in High-Earth Orbit.” Advances in Space Research, Vol. 52, pp. 1545–1560, 2013.