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Acoustic wave propagation in effective graded fully-anisotropic fluid layers

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1 This work deals with the modeling of sound wave propagation in anisotropic and
2 heterogeneous media. The scattering problem considered in this work involves an
3 infinite layer of finite thickness containing an anisotropic fluid whose properties can
4 vary along the depth of the layer. The specular transmission and reflection of an
5 acoustic plane wave by such a layer is modeled through the state vector formalism
6 for the acoustic fields. This is solved using three different numerical techniques,
7 namely the transfer matrix method, Peano series and the transfer Green's function.
8 These three methods are compared to demonstrate the convergence of the numerical
9 solutions. Moreover, the implemented numerical procedures allow to retrieve the
10 internal acoustic fields and show their dependency along with the fluid's anisotropic
11 properties. Results are then presented to illustrate the changes in absorption that
12 can be achieved by tuning the anisotropy of the fluid as well as the variation of
13 these properties across the depth of the layer. The results presented are in very
14 good agreement across the different methods. Given that many porous materials
15 can be modeled as equivalent fluids, the results presented show the potential offered
16 by such numerical techniques, and can further give more insight on inhomogeneous
17 anisotropic porous materials.

Keywords: acoustic control, anisotropic fluid, heterogeneous fluid, graded porous
layer, absorption

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18 I. INTRODUCTION

19 Acoustic treatments involving porous materials are commonly used for sound absorption
20 purposes. The recent development of additive manufacturing provides more control on the
21 micro-structures of these porous materials. Hence, the anisotropic and graded properties of
22 such microstructures influence the wave propagation in the medium, which is numerically
23 described. A rigid-frame porous medium is usually modeled as an equivalent fluid, that can
24 display anisotropic and heterogeneous frequency dependent effective properties. One way
25 to describe these effective properties is the well-known Johnson–Champoux–Allard–Lafarge
26 (JCAL) model¹ which provides the thermal and viscous dynamic permeabilities of the prop-
27 agation medium. For a periodic porous material, formed by a repetition of a unit cell,
28 the JCAL model can rely on homogenized properties of this unit cell calculated using the
29 method of multiple scales². Since the viscous dissipation has been shown to be direction-
30 dependent^{3–5} in anisotropic media, the same considerations are used in the current paper.
31 Recent work⁶ have shown that anisotropic materials can have different apparent sound speed
32 depending on the direction of propagation, coupling viscous and inertial regimes. This is
33 especially visible at grazing angles of incidence, and can be exploited for absorption consid-
34 ering a diffuse field where all incidences are accounted for. The derivation of the equations
35 has been done recently to retrieve the effective properties of an anisotropic homogeneous
36 material⁷, and is recalled in the section regarding wave propagation.

37 The present work focuses on the modeling and analysis of inhomogeneous anisotropic
38 materials. The scattering problem considered here involves an infinite layer of finite thickness

39 containing an anisotropic fluid whose properties can vary across the depth of the layer. The
40 transmission and reflection of an acoustic plane wave by such a layer is modeled through
41 the state vector formalism, which is solved using three different techniques. First, the
42 layer is assumed piece-wise constant and the standard transfer matrix method (TMM)⁸
43 is employed. The other two methods are applicable to continuously graded media. The
44 Peano Series (PS) has previously been used for graded⁹⁻¹¹ and anisotropic materials¹², and
45 wave-splitting techniques for continuously graded media¹³⁻¹⁸. In addition, the internal fields
46 and dissipation rate of energy are estimated¹⁹ and shown to be dependent on the fluid's
47 effective properties. Other solution procedures can however be applied to approximate such
48 propagation problem, as Euler or Runge-Kutta iterative schemes, which are commonly used
49 for linear systems¹⁴.

50 The article is organized as follows, we first introduce the equivalent fluid model and
51 the propagation problem considered in this work. The different numerical approaches are
52 then presented, so as to solve for the acoustic fields inside the layer. Numerical results
53 of the scattering coefficients on such anisotropic graded material are presented for all the
54 methods considered, which show good agreement. Finally, further insight is provided into
55 the dissipation rate within the anisotropic material and in the role played by the orientation
56 of the micro-structure.

57 II. PROPAGATION IN GRADED ANISOTROPIC FLUID LAYERS

In this section the propagation of a plane wave through an anisotropic, heterogeneous
equivalent fluid is described. We set the reference in the Cartesian coordinate system $\mathcal{R}_0 =$

$(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ with the associated spatial coordinates vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. The fluid layer, denoted Ω , is a slab of finite thickness L and infinite extent in the (O, \mathbf{x}_\perp) plane, as illustrated in Fig. 1. The subscript \perp denotes the restriction of a vector to the (O, \mathbf{x}_\perp) plane with $\mathbf{x}_\perp = \{x_1, x_2\}$. The domain Ω is delimited by the plane boundaries at $x_3 = 0$ and $x_3 = L$ denoted Γ_0 and Γ_L respectively. We solve for the sound field in this layer Ω in the linear harmonic regime using the time convention $e^{-i\omega t}$ where ω is the angular frequency. The effective bulk modulus and density of the anisotropic heterogeneous fluid are denoted $B(x_3, \omega)$ and $\boldsymbol{\rho}(x_3, \omega)$. Note that these quantities are complex-valued, frequency dependent and can vary along the x_3 direction, moreover, while the bulk modulus of the medium is scalar, the density is a second order tensor accounting for anisotropic phenomena. The pressure p and velocity \mathbf{v} induced by the acoustic field in Ω are governed by the following linear equations for mass conservation and momentum conservation

$$i\omega \boldsymbol{\rho}(x_3, \omega) \mathbf{v}(\mathbf{x}, \omega) = \nabla p(\mathbf{x}, \omega) , \quad (1a)$$

$$i\omega B^{-1}(x_3, \omega) p(\mathbf{x}, \omega) = \nabla \cdot \mathbf{v}(\mathbf{x}, \omega) . \quad (1b)$$

58

The exterior of the domain Ω is denoted Ω_0 and contains an homogeneous isotropic fluid, taken to be air in this case. The density of air is $\rho_0 = 1.213 \text{ kg.m}^{-3}$ and its bulk modulus $B_0 = \gamma P_0$ with $\gamma = 1.4$ the ratio of specific heat and $P_0 = 101\,325 \text{ Pa}$ the atmospheric pressure. The sound field in the exterior domain Ω_0 satisfies

$$i\omega \rho_0 \mathbf{v}(\mathbf{x}, \omega) = \nabla p(\mathbf{x}, \omega) , \quad (2a)$$

$$i\omega B_0^{-1} p(\mathbf{x}, \omega) = \nabla \cdot \mathbf{v}(\mathbf{x}, \omega) . \quad (2b)$$

While the density of the isotropic fluid in Ω_0 is described by the scalar ρ_0 , the anisotropy of the fluid in the layer Ω is described by the tensor density $\boldsymbol{\rho}$. This tensor accounts for the fact that the properties of the waves in Ω depend on the direction of propagation. The density tensor $\boldsymbol{\rho}$ is diagonal in the special case where its principal directions are aligned with the coordinate system \mathcal{R}_0 . But in the general case it is full, symmetric and can be written

$$\boldsymbol{\rho} = \mathbf{R} \begin{bmatrix} \rho_{11} & 0 & 0 \\ 0 & \rho_{22} & 0 \\ 0 & 0 & \rho_{33} \end{bmatrix}_{\mathcal{R}_\Omega} \mathbf{R}^T, \quad (3)$$

with \mathbf{R} the complete rotation matrix accounting for the yaw, pitch and roll angles, respectively (u_1, u_2, u_3) along $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. For the sake of simplicity and since the particle velocity depends on the inverse of the density tensor, the second-order tensor $\mathbf{H} = \boldsymbol{\rho}^{-1}$ remains symmetric and will be used instead of $\boldsymbol{\rho}$ in the remainder of this work.

In the upper region of Ω_0 , $x_3 \geq L$, we define an incident plane wave with unit amplitude:

$$p^i(\mathbf{x}, \omega) = e^{ik_1x_1 + ik_2x_2 - ik_3(x_3 - L)},$$

where the components of the wave-vector \mathbf{k}^i are given by

$$k_1 = -k_0 \cos(\theta) \cos(\psi), \quad (4)$$

$$k_2 = -k_0 \cos(\theta) \sin(\psi), \quad (5)$$

$$k_3 = k_0 \sin(\theta), \quad (6)$$

with ψ and θ the polar and elevation angles, respectively. $k_0 = \omega/c_0$ is the free-field acoustic wave-number.

The presence of the anisotropic layer Ω gives rise to a reflected wave p^r in the upper region of Ω_0 and to a transmitted wave p^t in the lower region of Ω_0 , $x_3 \leq 0$. These are written

$$p^r(\mathbf{x}, \omega) = \tilde{R}e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp + ik_3(x_3 - L)}, \quad (7)$$

$$p^t(\mathbf{x}, \omega) = \tilde{T}e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp - ik_3x_3}, \quad (8)$$

71 where \tilde{R} and \tilde{T} are the specular coefficients of reflection and transmission and $\mathbf{k}_\perp = \{k_1, k_2\}$
 72 and $\mathbf{x}_\perp = \{x_1, x_2\}$. As the incident wave could physically come from $x_3 < 0$, it is important
 73 to be explicit about the scattering coefficients which are \tilde{R}^\pm and \tilde{T}^\pm , depending on the
 74 sign of wave incidence. The system being reciprocal we reach $\tilde{T} = \tilde{T}^+ = \tilde{T}^-$, whereas the
 75 distinction has to be made for the reflection since the heterogeneity of the medium can be
 76 non-symmetric. Without any specific considerations about the effective properties of the
 77 medium, $\tilde{R}^+ \neq \tilde{R}^-$ in the inhomogeneous case. For the sake of simplicity and as reversing
 78 the layer Ω between its interfaces is equivalent to propagating in the opposite direction, we
 79 use the notation $\tilde{R} = \tilde{R}^+$ when the incident waves comes from the upper region $x_3 \geq L$.
 80 However, the solution procedures developed further are valuable for all scattering coefficients.

82

The incident plane wave p^i also induces a sound field in the anisotropic and graded layer Ω . Given that (i) the properties of this layer are independent of x_1 and x_2 and (ii) the incident field has an harmonic spatial dependence $e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}$, it is clear that the wave field in

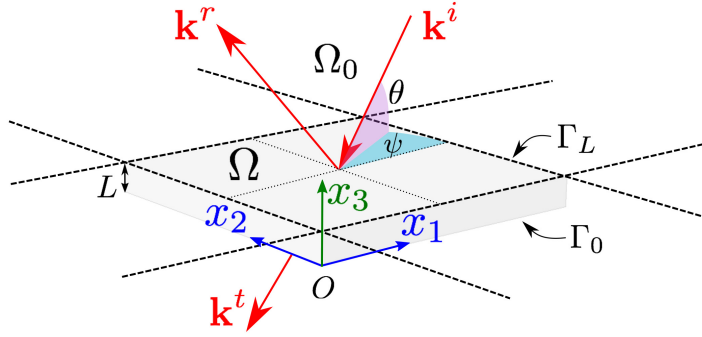


FIG. 1. [Color online] Schematic representation of the propagation problem in Ω_0 and Ω . A fluid layer of finite thickness L along x_3 , with infinite dimension in the (O, \mathbf{x}_\perp) plane and interfaces Γ_0 and Γ_L . Incident, reflected and transmitted wave-vectors are represented with red arrows. The elevation and azimuthal angles θ and ψ are shown respectively in purple and cyan.

the layer Ω retains the same harmonic spatial dependence:

$$p(\mathbf{x}, \omega) = p(x_3)e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}, \quad (9a)$$

$$\mathbf{v}(\mathbf{x}, \omega) = \mathbf{v}(x_3)e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}. \quad (9b)$$

$$(9c)$$

The derivation of the governing equations Eqs. (1) has recently been done for retrieval techniques and applied to fully-anisotropic porous materials⁷. The process is recalled as follows, and leads to the state-vector equation for pressure and normal particle velocity. From the conservation equations Eqs. (1), the transverse and normal components of the

fields are expanded,

$$i\omega \mathbf{v}_\perp = i\mathbf{H}_\perp \cdot \mathbf{k}_\perp p + H_{33} \mathbf{q} \frac{\partial p}{\partial x_3}, \quad (10a)$$

$$i\omega H_{33}^{-1} v_3 = i\mathbf{k}_\perp \cdot \mathbf{q} p + \frac{\partial p}{\partial x_3}, \quad (10b)$$

$$i\omega B^{-1} p = i\mathbf{k}_\perp \cdot \mathbf{v}_\perp + \frac{\partial v_3}{\partial x_3}, \quad (10c)$$

84 where we have again used the notation $\mathbf{v}_\perp = \{v_1, v_2\}$. We have also introduced the coupling
 85 vector $\mathbf{q} = \{H_{13}/H_{33}, H_{23}/H_{33}\}$ and the 2×2 matrix $\mathbf{H}_\perp = H_{mn} \forall (m, n) \in \{1, 2\}^2$. From
 86 Eqs. (10a) and (10c) we get

$$i\omega B^{-1} p = i\mathbf{k}_\perp \cdot \left[i(\mathbf{H}_\perp \cdot \mathbf{k}_\perp) \frac{p}{i\omega} + \frac{H_{33}}{i\omega} \mathbf{q} \frac{\partial p}{\partial x_3} \right] + \frac{\partial v_3}{\partial x_3}. \quad (11)$$

87 Together with the momentum conservation in Eq. (10b), leads to

$$i\omega B^{-1} p = H_{33} (\mathbf{k}_\perp \cdot \mathbf{q})^2 \frac{p}{i\omega} - \mathbf{k}_\perp \cdot (\mathbf{H}_\perp \cdot \mathbf{k}_\perp) \frac{p}{i\omega} + i(\mathbf{k}_\perp \cdot \mathbf{q}) v_3 + \frac{\partial v_3}{\partial x_3}, \quad (12)$$

88 where after rearranging the pressure terms, emerges the equivalent bulk modulus:

$$B_{eq}^{-1} = B^{-1} + [H_{33} (\mathbf{k}_\perp \cdot \mathbf{q})^2 - \mathbf{k}_\perp \cdot (\mathbf{H}_\perp \cdot \mathbf{k}_\perp)] / \omega^2. \quad (13)$$

89 Yields the following equation of mass conservation, where B_{eq} relates the compressibility
 90 effects of the equivalent fluid, accounting for anisotropic dependencies and oblique consid-
 91 erations,

$$i\omega B_{eq}^{-1} p = i\mathbf{k}_\perp \cdot \mathbf{q} v_3 + \frac{\partial v_3}{\partial x_3}, \quad (14)$$

92 and with Eq. 10b, they characterize the sound field in the layer Ω , with equivalent density
 93 H_{33}^{-1} and bulk modulus B_{eq} . They can be written using a state-vector formulation

$$\frac{d\mathbf{W}}{dx_3} = \mathbf{A}(x_3) \mathbf{W}, \quad (15)$$

94 where we have introduced the state vector $\mathbf{W} = \{p, v_3\}^T$ (with T the non-conjugate trans-
 95 pose), and the matrix

$$\mathbf{A}(x_3) = \begin{bmatrix} -i\mathbf{k}_\perp \cdot \mathbf{q} & i\omega H_{33}^{-1} \\ i\omega B_{eq}^{-1} & -i\mathbf{k}_\perp \cdot \mathbf{q} \end{bmatrix}. \quad (16)$$

96 At the interfaces Γ_0 and Γ_L between the anisotropic layer and the surrounding fluid, the
 97 continuity of pressure and normal velocity is imposed as boundary conditions. As a conse-
 98 quence, the state vector at both interfaces reads

$$\mathbf{W}_L = \begin{Bmatrix} 1 + \tilde{R} \\ Z_e^{-1}(\tilde{R} - 1) \end{Bmatrix} \quad \text{and} \quad \mathbf{W}_0 = \begin{Bmatrix} \tilde{T} \\ -Z_e^{-1}\tilde{T} \end{Bmatrix}, \quad (17)$$

99 with $Z_e = Z_0/\sin(\theta)$ the apparent impedance of the air in domain Ω_0 with respect to the
 100 unit outward normal vector $\mathbf{n} = \mathbf{e}_3$ at interface Γ_L . Note that in the case where the layer is
 101 rigidly backed (absorption problem), the boundary term at Γ_0 simplifies to $\mathbf{W}_0 = \{p(0), 0\}^T$
 102 since the Neumann condition involves zero normal velocity on the rigid layer backing.

103 III. SOLUTION PROCEDURES

104 The state-vector Eq. (15) can be solved using a variety of numerical techniques. In this
 105 section three different methods are presented. The well-known TMM is first described, then
 106 two other approaches are presented for continuously graded media.

107 **A. Transfer matrix method**

108 The heterogeneous fluid layer Ω can be approximated by a succession of N homogeneous
 109 layers. The propagation of the waves through each homogeneous layer can be solved exactly
 110 using the TMM⁸. This approximation is accurate provided that the thickness of each homo-
 111 geneous layer is small compared to the wavelength. We introduce the start- and end-points
 112 of the successive homogeneous layers as $x_3^{(i)}$ so that $x_3^{(0)} = 0$ and $x_3^{(N)} = L$. The state vectors
 113 on either sides of the i th homogeneous layer can be related as follows

$$\mathbf{W} \left(x_3^{(i+1)} \right) = \mathbf{M} \left(x_3^{(i+1)}, x_3^{(i)} \right) \mathbf{W} \left(x_3^{(i)} \right), \quad (18)$$

114 where \mathbf{M} is the matricant which can be written in terms of the constant matrix \mathbf{A}_i associated
 115 with the i th homogeneous layer:

$$\mathbf{A}_i = \mathbf{A} \left(\frac{x_3^{(i+1)} + x_3^{(i)}}{2} \right). \quad (19)$$

116 To do so we first diagonalize this matrix by writing $\mathbf{A}_i = \mathbf{V}_i^{-1} \boldsymbol{\lambda}_i \mathbf{V}_i$ with $\boldsymbol{\lambda}_i$ the diagonal
 117 matrix of eigenvalues and \mathbf{V}_i the matrix of eigenvectors. The state-vector formulation Eq.
 118 (15) in the i th layer can be transformed into two decoupled ordinary differential equations:

$$\frac{d}{dx_3} (\mathbf{V}_i \mathbf{W}) = \boldsymbol{\lambda}_i (\mathbf{V}_i \mathbf{W}). \quad (20)$$

119 These can be readily solved to obtain the matricant:

$$\mathbf{M} \left(x_3^{(i+1)}, x_3^{(i)} \right) = \mathbf{V}_i^{-1} \begin{pmatrix} e^{\lambda_1 l_i} & 0 \\ 0 & e^{\lambda_2 l_i} \end{pmatrix} \mathbf{V}_i, \quad (21)$$

120 with $l_i = x_3^{(i+1)} - x_3^{(i)}$. This expression can be directly written as a matrix exponential²⁰:

$$\mathbf{M} \left(x_3^{(i+1)}, x_3^{(i)} \right) = \exp(\mathbf{A}_i l_i). \quad (22)$$

121 The overall transfer matrix \mathbf{M} relating the state vectors at the two interfaces Γ_0 and Γ_L is
 122 defined as the product of the matricants of all the homogeneous layers:

$$\mathbf{W}_L = \mathbf{M}\mathbf{W}_0, \quad \mathbf{M} = \prod_{i=0}^{N-1} e^{\mathbf{A}_i l_i}. \quad (23)$$

123 The discretization of domain Ω is chosen to be linear across $N = 40$ positions, and will serve
 124 as comparison with two different methods which follow.

125 B. Peano Series

126 Another approach to solve Eq. (15) is to use the PS which have previously been used for
 127 continuously graded isotropic materials⁹. In the homogeneous case, i.e. when \mathbf{A} is constant,
 128 the PS can be shown to be equivalent to the product of matrix exponentials in Eq. (23). In
 129 the present case of x_3 dependent properties, the matrix \mathbf{A} does not commute with itself for
 130 different values of x_3 , so $\forall(x'_3, x''_3) \in [0, L]^2, x'_3 \neq x''_3, [\mathbf{A}(x'_3)\mathbf{A}(x''_3) - \mathbf{A}(x''_3)\mathbf{A}(x'_3)] \neq 0$ and
 131 the matricant is no longer defined by matrix exponentials, but rather by the Peano series.
 132 Using this formalism, the matricant \mathbf{M} defined by Eq. (23) is written as an infinite series
 133 of integrals^{18,20}:

$$\mathbf{M}(0, x_3) = \mathbb{I}_d + \int_0^{x_3} \mathbf{A}(\xi) d\xi + \int_0^{x_3} \mathbf{A}(\xi) \int_0^\xi \mathbf{A}(\xi_1) d\xi d\xi_1 + \dots \quad (24)$$

134 In practice this is calculated through the use of the following recurrence relations¹¹,

$$\begin{cases} \mathbf{M}^{\{0\}}(0, L) = \mathbb{I}_d \\ \mathbf{M}^{\{n\}}(0, L) = \mathbb{I}_d + \int_0^L \mathbf{A}(x_3) \mathbf{M}^{\{n-1\}}(x_3) dx_3 \end{cases} \quad (25)$$

135 and the state vector relation at both interfaces now reads,

$$\mathbf{W}_L = \lim_{n \rightarrow \infty} \mathbf{M}^{\{n\}}(0, L) \mathbf{W}_0. \quad (26)$$

136

137 An approximate solution is obtained by truncating this infinite series. In fact, unlike
 138 the TMM where the matricant of the system is assembled piece by piece, each term of the
 139 integral series accounts for the whole domain $0 < x_3 < L$. The integral itself is estimated
 140 by the trapezoidal method at each iteration, using the same unit spacing L/N . Hence, any
 141 additional term of the truncated series tends to refine the solution given by this method.
 142 The recurrence relation is chosen to be expanded up to 50 terms, a sufficient number for the
 143 series to converge.

144 C. Wave-Splitting, Transfer Green Functions

145 The wave-splitting method relies on the separation of the overall acoustic field into for-
 146 ward and backward propagative waves^{14,18}. Since the effective properties of the medium are
 147 inhomogeneous along x_3 , the wave-splitting applied in the current paper is not related to Ω ,
 148 but rather with respect to the domain Ω_0 ^{15,16}. The wave-splitting matrix is independent of
 149 the graded parameters (tensorial density $\boldsymbol{\rho}$ and equivalent bulk modulus B_{eq}), which ensures
 150 the split fields to be continuous across any x_3 -plane in the medium Ω ¹⁷. These are defined
 151 as $s^\pm = (p \pm Z_e \mathbf{v} \cdot \mathbf{n})/2$ where the \pm sign indicates the direction of propagation relative to
 152 the unit vector \mathbf{n} . Although they only have a physical sense in Ω_0 according to the wave-
 153 splitting transformation, the associated change of basis remains valid. It is then possible to

154 introduce a new vector $\mathbf{S} = \{s^+, s^-\}^T$ which is related to the original vector \mathbf{W} by,

$$\mathbf{S}(x_3, \omega) = \mathbf{Z}\mathbf{W}(x_3, \omega), \quad \text{with } \mathbf{Z} = \frac{1}{2} \begin{bmatrix} 1 & Z_e \\ 1 & -Z_e \end{bmatrix}. \quad (27)$$

155 Introducing this definition in the state vector formulation Eq. (15), it is straightforward to

156 obtain:

$$\frac{d}{dx_3} \mathbf{S} = \mathbf{B}(x_3) \mathbf{S}, \quad (28)$$

157 with

$$\mathbf{B}(x_3) = \mathbf{Z}\mathbf{A}(x_3)\mathbf{Z}^{-1} = \begin{bmatrix} U^+ & U^- \\ -U^- & -U^+ \end{bmatrix} - i(\mathbf{k}_\perp \cdot \mathbf{q})\mathbb{I}_d, \quad (29)$$

\mathbb{I}_d being the identity matrix, and

$$U^\pm(x_3, \omega) = \frac{i\omega}{2} [Z_e B_{eq}^{-1}(x_3, \omega) \pm H_{33}^{-1}(x_3, \omega) Z_e^{-1}].$$

The differential equations Eq. (28) can be solved using the transfer Green's functions (TGF)¹⁹ method by writing the forward and backward internal fields in terms of the transmitted wave $s^-(0, \omega)$ as follows:

$$s^\pm(x_3, \omega) = G^\pm(x_3, \omega) s^-(0, \omega),$$

158 where G^\pm denote the two Green's functions. They are solutions of the following differential

159 equations

$$\frac{d\mathbf{G}}{dx_3} = \mathbf{B}\mathbf{G}, \quad (30)$$

160 with $\mathbf{G} = \{G^+, G^-\}$.

161 In the case of an absorption problem (rigid backing at Γ_0) the boundary condition for the
162 Green functions Eq. (30) reads $\mathbf{G}_0 = \{1, 1\}$ as a total specular reflection. In the case of a
163 transmission problem, we must have a total transmission at the interface Γ_0 , corresponding
164 to $\mathbf{G}_0 = \{1, 0\}$. The fluid layer heterogeneity being of macroscopic scale (the order of L),
165 the spatial discretization is easily achieved. The continuous graded properties along x_3 in
166 the domain Ω are split linearly into $N = 40$ positions. The differential system of equations,
167 Eq. (30) is solved numerically.

168 IV. RESULTS AND DISCUSSIONS

169 This section deals with the numerical validation of the proposed models. The scattering
170 coefficients are retrieved with all three different methods and applied to an heterogeneous
171 anisotropic porous material.

172 A. Scattering coefficients

With the TMM and the PS the reflection and transmission coefficients are readily avail-
able as part of the solution procedures. From the relation $\mathbf{W}_L = \mathbf{M}\mathbf{W}_0$ from Eq. (23) one
can derive the following expressions for these coefficients:

$$\tilde{T} = 2Z_e^{-1}[Z_e^{-1}\text{Tr}(\mathbf{M}) - Z_e^{-2}M_{12} - M_{21}]^{-1}, \quad (31a)$$

$$\tilde{R} = M_{11}\tilde{T} - Z_e^{-1}M_{12}\tilde{T} - 1, \quad (31b)$$

173 where $\text{Tr}(\mathbf{M})$ is the trace of the square matrix \mathbf{M} and M_{ij} are the coefficients of the matrix.
 174 Note that \tilde{R} and \tilde{T} are functions of the angular frequency ω and the incidence angles (the
 175 polar and elevation angles ψ and θ , respectively).

For the wave-splitting method, the reflection and transmission coefficients are recovered from the solutions for the Green's functions G^+ and G^- as follows^{14,17,18}:

$$\tilde{T} = 1/G^-(0) , \quad (32a)$$

$$\tilde{R} = G^+(L)/G^-(L) . \quad (32b)$$

To quantify the acoustic dissipation inside the layer Ω we calculate the absorption coefficient. As mentioned earlier, as the scattering coefficients depend from the direction of incidence, the absorption coefficient follows the same dependency,

$$\alpha^\pm(\omega) = 1 - |\tilde{R}^\pm(\omega)|^2 - |\tilde{T}(\omega)|^2 .$$

176 It will vary between 0 and 1 and can also be calculated when the layer is rigidly backed
 177 so $\tilde{T} = 0$. The different computing methods have been compared to the transfer matrix
 178 method. For a similar spatial sampling (linear with $N = 40$) the relative error between each
 179 method is below 0.2% and the average computation time per frequency is $t_{tgf} \approx 0.21$ s for
 180 Green's functions (and mainly depends on absolute and relative tolerances of the numerical
 181 integration), while $t_{ps} \approx 0.05$ s for 50 terms of Peano Series and $t_{tmm} \leq 0.01$ s for TMM.
 182 These results are obtained by averaging the computing time over 100 frequency points, the
 183 overall comparison for the three different methods can also be done in parallel. Moreover,
 184 other numerical differentiation procedures can be set up to reach the scattering coefficients,
 185 such as Runge-Kutta schemes¹⁴.

TABLE I. Homogenized JCAL parameters for the anisotropic unit cell with characteristic size ℓ_c in the coordinate system \mathcal{R}_0 .

	ϕ (1)	Λ' (m)	\mathcal{K}'_0 (m ²)	τ^∞ (1)	Λ (m)	\mathcal{K}_0 (m ²)
Ω	0.7210	$0.533 \ell_c$	$0.0214 \ell_c^2$	-	-	-
\mathbf{e}_1	-	-	-	2.987	$0.129 \ell_c$	$5.74 \cdot 10^{-4} \ell_c^2$
\mathbf{e}_2	-	-	-	1.089	$0.448 \ell_c$	$1.56 \cdot 10^{-2} \ell_c^2$
\mathbf{e}_3	-	-	-	1.487	$0.273 \ell_c$	$4.83 \cdot 10^{-3} \ell_c^2$

186 **B. Porous material**

187 The anisotropic fluid layer considered as an example in the present work is a periodic
188 porous material. The unit cell that is periodically distributed to form this periodic material
189 is a rigid cube of length ℓ_c from which an ellipsoid with semi-axes of different lengths is
190 carved out, see Fig 2(c). The effective properties of this unit cell are obtained using the
191 multiple-scale method outlined in Ref. 2 and 7. The resulting parameters of the JCAL
192 model¹ are listed in Table I as functions of the unit cell size ℓ_c and in the coordinate system
193 \mathcal{R}_0 . Some of these parameters are scalar quantities (porosity ϕ , characteristic thermal length
194 Λ' and static thermal permeability \mathcal{K}'_0) while others are tensorial (high-frequency tortuosity
195 τ^∞ , characteristic viscous length Λ and static viscous permeability \mathcal{K}_0).

197 To obtain an inhomogeneous material the unit cell size ℓ_c is varied along the x_3 direction.
198 As a consequence the effective JCAL parameters will also vary along this direction. The

199 profile chosen as an example in this work is the ‘ramp’ shown in Fig. 2(a). The unit cell size
 200 ℓ_c is varied continuously from 0.1 mm at the base of the layer ($x_3 = 0$) to 2 mm at the top of
 201 the layer ($x_3 = L$). This profile was chosen to achieve an impedance matching between the
 202 exterior domain and the porous material. The layer thickness is $L = 50$ mm and achieves
 203 perfect absorption at the frequency $f_0 = 2500$ Hz.

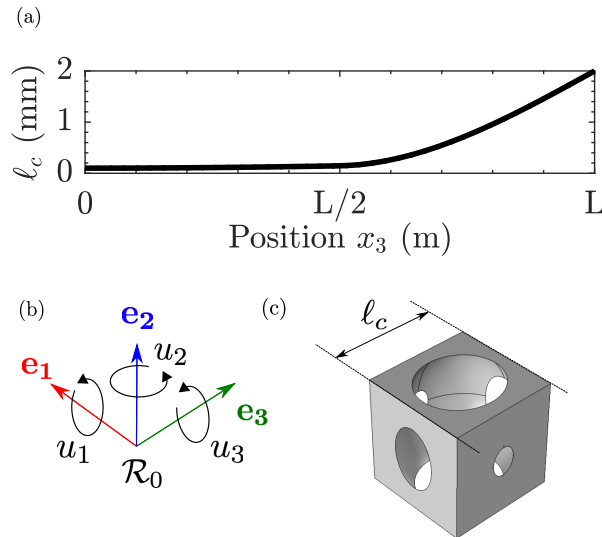


FIG. 2. [Color online] (a) Variation of the unit cell size ℓ_c along the depth of the porous material layer between Γ_0 and Γ_L . (b) Cartesian coordinate system \mathcal{R}_0 with its associated orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and the rotation angles (u_1, u_2, u_3) . (c) Unit cell for the periodic anisotropic porous material. Shown here is the rigid skeleton and a fluid region formed by a body-centered ellipsoid with semi-axes $r_1 = 0.51\ell_c$, $r_2 = 0.7\ell_c$ and $r_3 = 0.55\ell_c$.

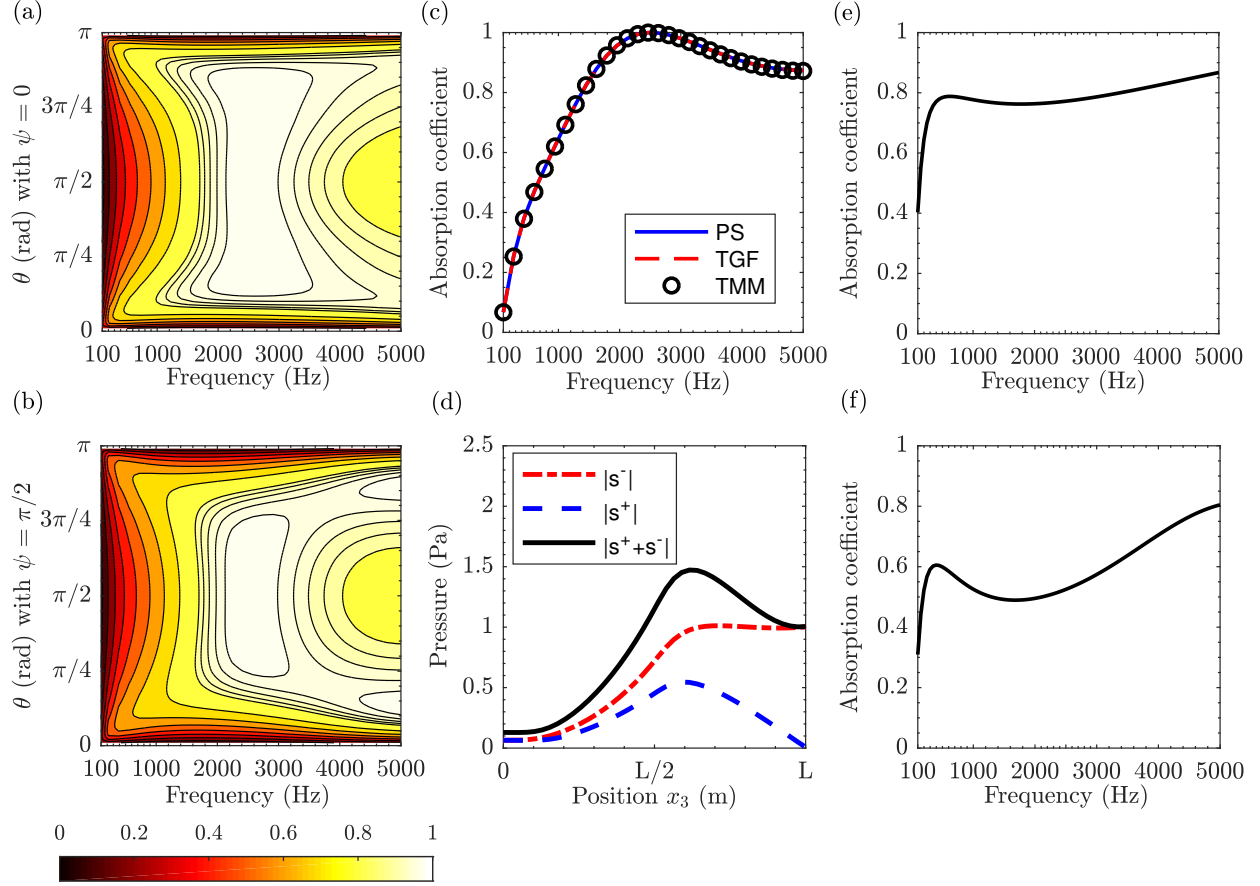


FIG. 3. [Color online] Absorption coefficient at oblique incidence, on the frequency range 100Hz – 5kHz and for elevation angle θ from 0 to π , with $\psi = 0$ (a) and $\psi = \pi/2$ (b). (c) Absorption coefficient at normal incidence, on the same frequency range, using the TMM, PS and TGF methods. (d) Magnitude of the split fields in the porous layer \mathbf{S} , for perfect absorption frequency f_0 at normal incidence. Absorption coefficient at grazing incidence, on the frequency range 100Hz – 5kHz and for elevation angle $\theta = \pi/20$, with $\psi = 0$ (e) and $\psi = \pi/2$ (f).

204 C. Influence of wave incidence

205 We begin by considering the case of a plane wave at oblique incidence with $\mathbf{k}^i =$
 206 (k_1, k_2, k_3) . Results are shown for an absorption problem, when the layer is rigidly backed

207 at Γ_0 . Fig. 3(a) shows the absorption coefficient as a function of frequency between 100 Hz
 208 and 5 kHz. The second axis spans the values of elevation angle, while the polar angle of
 209 incidence is $\psi = 0$ in Fig.3(a) and $\psi = \pi/2$ in Fig.3(b). While the absorption is limited at
 210 low frequency, this material is able to achieve a perfect absorption ($\alpha = 1$) for a frequency
 211 close to $f_0 = 2500$ Hz. However, we can observe a notable change in the absorption depend-
 212 ing on the polar angle of incidence. Fig.3(c) also shows that the three solution procedures
 213 presented here (namely the TMM, PS and TGF) are in excellent agreement over the whole
 214 range of frequencies. Fig.3(d) shows the evolution of the forward and backward components
 215 $s^\pm(x_3)$ in the layer Ω for the frequency where the perfect absorption f_0 is achieved. It is
 216 clear that the magnitude of the backward wave $s^+(x_3, \omega) = (p + Z_e v_3)/2$ vanishes on the
 217 upper side of the layer ($x_3 = L$), which is consistent with the fact that there are no reflected
 218 wave at this frequency. Also visible in Fig.3(d) is the strong absorption of the forward
 219 propagating component s^- when it reaches the more resistive part of the porous layer (i.e.
 220 where ℓ_c is small). Concerning the dependence of α with the elevation angle θ , the system
 221 tends towards total reflection for grazing incidences and the anisotropic properties of the
 222 fluid layer are clearly visible, as shown in Figs. 3(e) and 3(f).

223 D. Effects of anisotropic coupling

224 In the results above the unit cell has been aligned with the coordinate system, as shown in
 225 Fig.3. To illustrate the effects of the anisotropy of the material, one can rotate the unit cell
 226 using the expression given in Eq. (3). This is shown in Fig.4 for the absorption coefficient

227 α . Depending on the rotation components (u_1, u_2, u_3) involved in the density tensor, the
 228 acoustic behavior of the fluid layer is significantly impacted, especially at high frequencies.

To provide further insight into the losses occurring within the layer, we derive the balance
 of acoustic energy in an anisotropic fluid. From the governing Eq. (1) in Ω , one can derive,

$$i\omega \mathbf{v}^* \boldsymbol{\rho} \mathbf{v} - \mathbf{v}^* \cdot \nabla p = 0, \quad (33a)$$

$$i\omega B^{-1} |p|^2 - \bar{p} \nabla \cdot \mathbf{v} = 0, \quad (33b)$$

229 where we have introduced the conjugated transposed velocity \mathbf{v}^* and the conjugated pressure
 230 \bar{p} . As depicted in Eq.(3), the density tensor is complex and symmetric and emerges from the
 231 dynamic viscous permeability of the medium Ω . It can be split into its complex components
 232 from the Toeplitz decomposition²¹ so, $\boldsymbol{\rho} = \boldsymbol{\rho}_R + i\boldsymbol{\rho}_I$ with $\boldsymbol{\rho}_R = (\boldsymbol{\rho} + \boldsymbol{\rho}^*)/2$ and $\boldsymbol{\rho}_I =$
 233 $(\boldsymbol{\rho} - \boldsymbol{\rho}^*)/2i$. In the general case of a non-symmetric $\boldsymbol{\rho}$ tensor, both Hermitian matrices $\boldsymbol{\rho}_R$
 234 and $\boldsymbol{\rho}_I$ remain complex-valued, however, in our case of symmetric tensor density, $\boldsymbol{\rho}_R$ and $\boldsymbol{\rho}_I$
 235 are real. Taking the sum of both of the equations (33) yields,

$$\frac{1}{2}(i\omega) (\mathbf{v}^*(\boldsymbol{\rho}_R + i\boldsymbol{\rho}_I)\mathbf{v} + B^{-1}|p|^2) = \frac{1}{2} (\mathbf{v}^* \cdot \nabla p + \bar{p} \nabla \cdot \mathbf{v}), \quad (34)$$

236 which after expansion of the complex terms and reads,

$$\frac{1}{2}\omega (i\mathbf{v}^* \boldsymbol{\rho}_R \mathbf{v} - \mathbf{v}^* \boldsymbol{\rho}_I \mathbf{v} + iB^{-1}|p|^2) = \frac{1}{2} (\mathbf{v}^* \cdot \nabla p + \bar{p} \nabla \cdot \mathbf{v}). \quad (35)$$

237 Now considering the real part of this equality, it yields to the time average of the acoustic
 238 instantaneous intensity²², as the products $\mathbf{v}^* \boldsymbol{\rho}_R \mathbf{v}$ and $\mathbf{v}^* \boldsymbol{\rho}_I \mathbf{v}$ are real-valued,

$$\frac{1}{2}\omega (\mathbf{v}^* \boldsymbol{\rho}_I \mathbf{v} + \text{Im}\{B^{-1}\}|p|^2) = -\frac{1}{2}\text{Re}\{\mathbf{v}^* \cdot \nabla p + \bar{p} \nabla \cdot \mathbf{v}\}, \quad (36)$$

239 where from the product rule of the divergence we now reach,

$$\nabla \cdot \left(\frac{1}{2} \text{Re}\{\mathcal{P}\} \right) = -\frac{1}{2} \omega (\mathbf{v}^* \boldsymbol{\rho}_I \mathbf{v} + \text{Im}\{B^{-1}\} |p|^2) . \quad (37)$$

240 The left-hand side of this equation is the divergence of the Poynting vector $\mathcal{P} = p\mathbf{v}^*$, since
 241 the porous layer is purely lossy, we expect this term to be strictly negative. This quantity
 242 is homogeneous to the dissipation rate of acoustic energy at each infinitesimal point $x_3 \in \Omega$
 243 and is expressed in W.m^{-3} . Although, it is estimated as only dependent of the normal
 244 direction x_3 since the acoustic fields in Eq. (9) show an harmonic spatial dependence.

245 It highlights the role of the coupling vector \mathbf{q} and its effect on the fully-anisotropic
 246 behavior of such medium. The total energy lost in the system can be retrieved by spatial
 247 integration between boundaries Γ_0 and Γ_L . As all three components of the particle velocity
 248 are involved, the transverse part of \mathbf{v} is derived from Eqs. (10a) and (10b).

249 Inside the domain Ω , the transverse components of particle velocity read

$$\mathbf{v}_\perp = (\mathbf{H}_\perp \cdot \mathbf{k}_\perp - H_{33} \mathbf{q} (\mathbf{k}_\perp \cdot \mathbf{q})) p / \omega + v_3 \mathbf{q}. \quad (38)$$

250 It is worth noting that even at normal incidence, with $\mathbf{k}_\perp = (0, 0)$, the coupling still occurs
 251 from the term $v_3 \mathbf{q}$. In order to illustrate this effect, Fig.4 shows the absorption coefficient
 252 when the fluid is taken out of its principal directions. Also considering normal incidence
 253 and with $\mathcal{R}_0 \equiv \mathcal{R}_\Omega$, a sole rotation around \mathbf{e}_3 cannot impact the acoustic properties of the
 254 fluid. First, the dependence on the rotation angle around \mathbf{e}_1 is shown in Fig.4(a), which is
 255 π -periodic. Then on Fig.4(b) the absorption coefficient varies as the cell is rotated around
 256 the \mathbf{e}_2 unit vector.

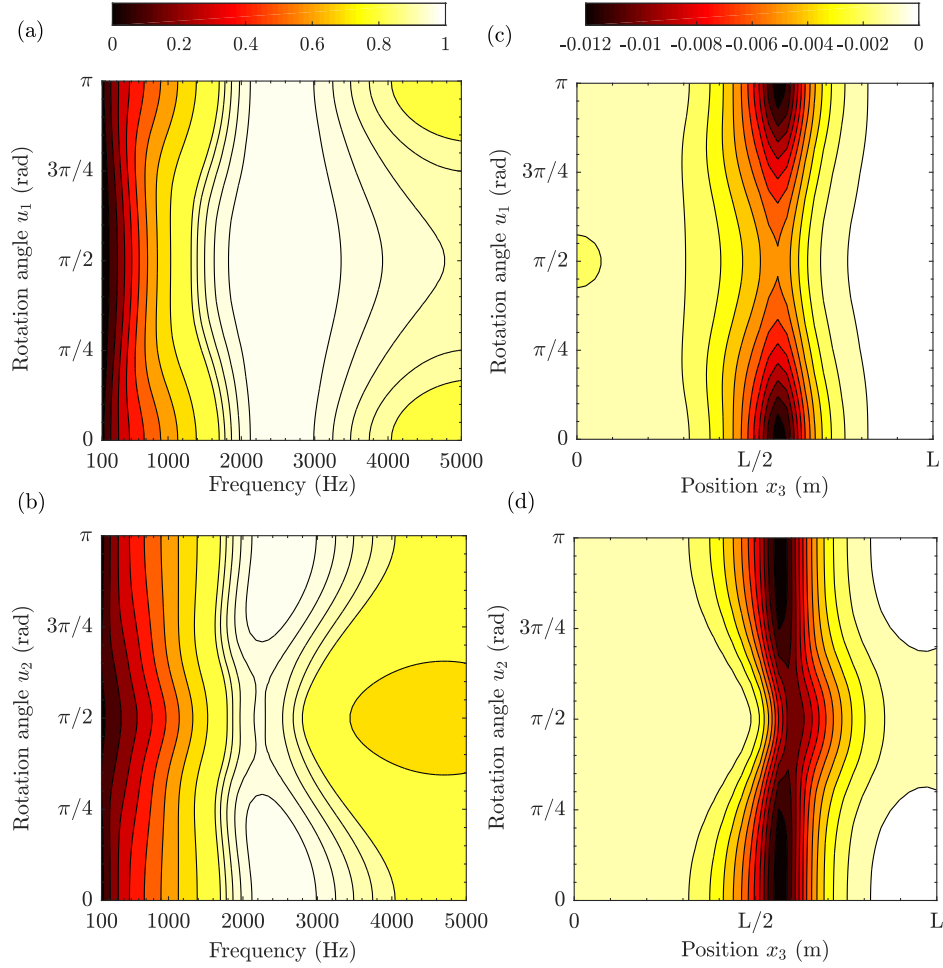


FIG. 4. [Color online] Absorption coefficient at normal incidence as a function of the circular frequency ω , the rotation angle u_1 in (a) and u_2 in (b). Energy dissipation rate at normal incidence between $x_3 = 0$ and L for rotation angles u_1 (c) and u_2 (d), from 0 to π at the perfect absorption frequency f_0 .

257 As depicted in Eq. (35), the estimated dissipation rate directly depend on the rotations
 258 applied to the density tensor. Figures 4(c) and 4(d) display the estimated dissipation inside
 259 the domain Ω , at frequency f_0 . As previously, the dependence on the rotation angles (u_1, u_2)
 260 affects the losses in the fluid, hence on the absorption properties. We notice that most of

261 the energy losses in the domain are localized where the pore size becomes small, which is
262 correlated to the total pressure profile in Fig.3(d).

263 **E. Diffuse field absorption**

264 Instead of a single wave with a specific incidence angle, one can also consider a diffuse
265 field where all wave directions are present, but uncorrelated with the same intensity. The
266 corresponding absorption coefficient accounts for the absorption averaged over all possible
267 angles of incidence:

$$\alpha_{diff}(\omega) = \frac{1}{2\pi} \int_0^\pi \int_0^\pi \alpha(\omega, \theta, \psi) \cos(\theta) d\theta d\psi, \quad (39)$$

268 with $(\theta, \psi) \in [0, \pi]^2$ and frequency ω . The averaging process is done accounting for the solid
269 angle associated to each direction of incidence, which induces the weight $\cos(\theta)$. This diffuse
270 field absorption coefficient is shown in Fig.5 as a function of frequency using 400 plane wave
271 direction to compute the average. As pictured in Fig.5, the graded anisotropic materials
272 is able to provide good diffuse absorption over a wide range of frequencies. However its
273 absorption is limited at low frequencies. Unlike the absorption of the plane wave at normal
274 incidence which is perfect around 2500Hz (see Fig.3), the diffuse field case is unable to reach
275 a perfect absorption. This is explained by the contributions of the plane waves with grazing
276 incidence which can only be partially absorbed. But as oblique incidences weight a lot
277 in this considerations, the anisotropic properties firmly impact the diffuse field absorption
278 coefficient.

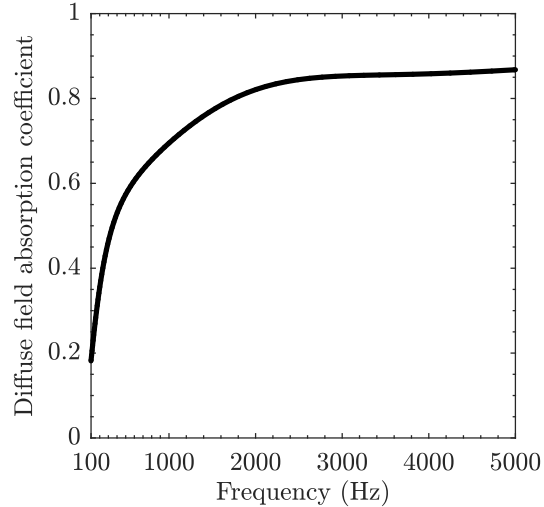


FIG. 5. Diffuse field absorption coefficient as a function of frequency.

280 V. CONCLUSIONS

281 In this work, the propagation of acoustic waves through a graded layer of anisotropic fluid
 282 has been modeled to calculate the transmission and reflection coefficients. This approach
 283 is applicable to a wide range of porous materials that are described by their effective bulk
 284 modulus and density tensor, and in this case is developed for non-symmetric heterogeneous
 285 systems. Three different numerical techniques have been presented and compared to solve
 286 for the sound field in such a layer. Two of the solution procedures account for the contin-
 287 uous macro-modulated effective properties of the anisotropic medium, and altogether show
 288 excellent agreement with the more traditional TMM approach. In addition, the knowledge
 289 of the pressure and velocity fields inside the anisotropic fluid provides useful insight into the
 290 losses occurring within the layer.

291 The dependence of the absorption coefficient with frequency (over the range 100Hz –
 292 5kHz), angles of incidence and orientation of the micro-structure has been discussed in

293 detail. All the results demonstrates the complex interplay between these parameters and
294 the fact that the anisotropy plays a significant role in the absorption achieved by this kind
295 of materials. The absorption of a diffuse field was also considered.

296 The use of anisotropic and heterogeneous materials drastically enhances the potential for
297 efficient acoustic control in scattering and absorption problems. The next step on this topic
298 would be to perform a full optimization of both the anisotropy and the heterogeneity of a
299 porous layer, so as to maximize the acoustic absorption in specific applications.

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