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SUCCESSIVE MAXIMA OF THE NON-GENUS PART OF CLASS NUMBERS

GEORGES GRAS

ABSTRACT. Some PARI programs have brought out a property for the non-genus part of the class number of the imaginary quadratic fields, with respect to $(\sqrt{D})^\varepsilon$, where D is the absolute value of the discriminant and $\varepsilon \in]0, 1[$, in relation with the ε -conjecture. The general Conjecture 3.1, restricted to quadratic fields, states that, for $\varepsilon \in]0, 1[$, the successive maxima, as D increases, of $\frac{H}{2^{N-1} \cdot (\sqrt{D})^\varepsilon}$, where H is the class number and N the number of ramified primes, occur only for prime discriminants (i.e., H odd); we perform computations giving some obviousness in the selected intervals. For degree $p > 2$ cyclic fields, we define a “mean value” of the non-genus parts of the class numbers of the fields having the same conductor and obtain an analogous property on the successive maxima. In Theorem 2.6 we prove, under an assumption (true for $p = 2, 3$), that the sequence of successive maxima of $\frac{H}{p^{N-1} \cdot (\sqrt{D})^\varepsilon}$ is infinite. Finally we consider cyclic or non-cyclic abelian fields of degrees 4, 8, 6, 9, 10, 30 to test the Conjecture 3.1. The successive maxima of $\frac{H}{(\sqrt{D})^\varepsilon}$ are also analyzed.

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1. INTRODUCTION AND MAIN CONJECTURE

Let \mathcal{K}_G^s be the set of Galois number fields of *abelian* Galois group G and signature s ; for $K \in \mathcal{K}_G^s$, of discriminant D_K , let H_K be the class number of K and $g_{K/\mathbb{Q}}$ the “genus part” of H_K (formula (1)). The purpose of this paper is to suggest that for any $\varepsilon \in]0, 1[$, the successive maxima of a suitable mean of the $\frac{H_K}{g_{K/\mathbb{Q}} \cdot (\sqrt{|D_K|})^\varepsilon}$ over the set $\mathcal{F}_{G,f}^s$ of fields $K \in \mathcal{K}_G^s$ of conductor f , have some smooth behavior, regarding the number $N := \omega(f)$ of ramified primes, these sets of fields being ordered by increasing conductors f ; of course, for $G \simeq \mathbb{Z}/2\mathbb{Z}$ and s fixed, a quadratic field of conductor f is unique.

By “successive maxima” of a sequence $(X_i)_{i \geq 1}$ of positive numbers, we mean the sequence $(Y_n)_{n \geq 1}$ inductively defined by: $Y_1 = X_1$, $Y_{n+1} = X_{i_{n+1}}$, for the smallest index i_{n+1} such that $X_{i_{n+1}} > Y_n$; thus $Y_{n+1} > Y_n$ for all n , the sequence (Y_n) being finite or infinite.

Such a Y_n will also be called a *local maximum*.

For instance, the case of imaginary quadratic fields will give the following details of computation with $\varepsilon = 0.05$ (see §4.1.1). The sequence $X := \frac{H}{2^{N-1} \cdot (\sqrt{D})^\varepsilon}$ is indexed by the absolute value of the discriminants and the successive maxima Y are written with $\langle\langle \dots \rangle\rangle$ (all are with $N = 1$):

$\langle\langle D_K = -3 \rangle\rangle$	$Y = 0.972908$	$N = 1$	$H = 1$
$D_K = -4$	$X = 0.965936$	$N = 1$	$H = 1$
$D_K = -7$	$X = 0.952516$	$N = 1$	$H = 1$
$D_K = -8$	$X = 0.949342$	$N = 1$	$H = 1$
$D_K = -11$	$X = 0.941814$	$N = 1$	$H = 1$
$D_K = -15$	$X = 0.934539$	$N = 2$	$H = 2$
$D_K = -19$	$X = 0.929033$	$N = 1$	$H = 1$
$D_K = -20$	$X = 0.927842$	$N = 2$	$H = 2$
$\langle\langle D_K = -23 \rangle\rangle$	$Y = 2.773818$	$N = 1$	$H = 3$
$D_K = -24$	$X = 0.923622$	$N = 2$	$H = 2$
$D_K = -31$	$X = 2.753196$	$N = 1$	$H = 3$
$D_K = -35$	$X = 0.914951$	$N = 2$	$H = 2$
$D_K = -39$	$X = 1.824960$	$N = 2$	$H = 4$
$D_K = -40$	$X = 0.911902$	$N = 2$	$H = 2$
$D_K = -43$	$X = 0.910255$	$N = 1$	$H = 1$
$\langle\langle D_K = -47 \rangle\rangle$	$Y = 4.541167$	$N = 1$	$H = 5$
$D_K = -51$	$X = 0.906380$	$N = 2$	$H = 2$
$D_K = -52$	$X = 0.905941$	$N = 2$	$H = 2$
$D_K = -55$	$X = 1.809343$	$N = 2$	$H = 4$

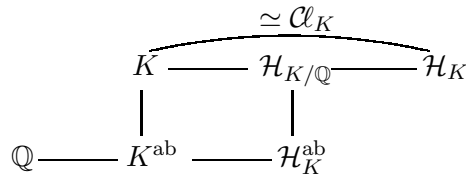
D_K=-56	X=1.808528	N=2	H=4
D_K=-59	X=2.709255	N=1	H=3
D_K=-67	X=0.900218	N=1	H=1
D_K=-68	X=1.799771	N=2	H=4
<<D_K=-71	Y=6.292403>>	N=1	H=7
D_K=-79	X=4.482593	N=1	H=5
D_K=-83	X=2.686236	N=1	H=3
(...)			
D_K=-163	X=0.880430	N=1	H=1
D_K=-164	X=3.521185	N=2	H=8
<<D_K=-167	Y=9.678872>>	N=1	H=11
D_K=-168	X=0.879766	N=3	H=4
(...)			

For short, one can say that the expression under consideration is locally maximum only when N_K ($K \in \mathcal{F}_{G,f}^s$) is minimum, which is natural in some sense since, as we know, $N_K \gg 0$ gives a large genus number dividing H_K (and a large discriminant D_K). The study performed in [14], for the degree p cyclic fields, shows that the genus part, and more precisely the p -part of H_K , may be an obstruction regarding the ε -conjectures on the behavior of $\frac{H_K}{(\sqrt{|D_K|})^\varepsilon}$ as $|D_K| \rightarrow \infty$, at least for sparse families of fields when considering the Koymans–Pagano density results [16] (see Section 2).

Remark 1.1. The genus formula (1), recalled below, yields, in the abelian case, $g_{K/\mathbb{Q}} = \frac{1}{[K:\mathbb{Q}]} \cdot \prod_\ell e_\ell$, where e_ℓ is the ramification index of ℓ ; it depends simply on the number N_K of ramified primes only in the case $G \simeq \mathbb{Z}/p\mathbb{Z}$ where $g_{K/\mathbb{Q}} = p^{N_K-1}$. Otherwise, many “subcases” may occur, which make more difficult some statements. Indeed, it is clear that the structure of G has a strong influence on $N_K = \omega(f_K)$, thus on $g_{K/\mathbb{Q}}$; but the non-genus part of H_K is exactly the “non-predictible” part of the class number.

The difficulty is to take into account these remarks and to give precise and relevant definitions then a right analysis of the growth of these functions of the class numbers regarding $(\sqrt{|D_K|})^\varepsilon$; so we shall restrict ourselves to some arbitrary contexts, hoping that some general properties will emerge from these experiments and will strengthen the Conjecture 1.3.

1.1. Genus field and genus number $g_{K/\mathbb{Q}}$. For simplicity, to recall genus theory, we restrict ourselves to the base field $k = \mathbb{Q}$. The genus field, according to an extension K/\mathbb{Q} , is the maximal subextension $\mathcal{H}_{K/\mathbb{Q}}$ of the Hilbert class field \mathcal{H}_K of K (in the restricted sense) equal to the compositum of K with an abelian extension A of \mathbb{Q} . Clearly we can take $A = \mathcal{H}_K^{\text{ab}}$ (the maximal abelian subextension of \mathcal{H}_K):



The genus number is $g_{K/\mathbb{Q}} := [\mathcal{H}_{K/\mathbb{Q}} : K]$. In the general Galois case, we have (in the restricted sense for class groups):

$$(1) \quad g_{K/\mathbb{Q}} = \frac{\prod_\ell e_\ell^{\text{ab}}}{[K^{\text{ab}} : \mathbb{Q}]},$$

where e_ℓ^{ab} is the ramification index of the prime ℓ in K^{ab}/\mathbb{Q} . A general non-Galois formula does exist (e.g., [12, Theorem IV.4.2 & Corollaries]).

We shall call $[\mathcal{H}_K : \mathcal{H}_{K/\mathbb{Q}}]$ the non-genus part of the class number $H_K := \#\mathcal{C}_K = [\mathcal{H}_K : K]$. So we shall put, in the abelian case:

$$h_K = \frac{H_K}{g_{K/\mathbb{Q}}} = \frac{H_K}{[\mathcal{H}_{K/\mathbb{Q}} : K]} = \frac{[K : \mathbb{Q}] \cdot H_K}{\prod_{\ell} e_{\ell}}.$$

When K/\mathbb{Q} is cyclic, the genus number $g_{K/\mathbb{Q}}$ is equal to the number of invariant classes by $G := \text{Gal}(K/\mathbb{Q})$ (Chevalley's formula [2]).

1.2. Practical framework – Main conjecture. Since the classification of Galois number fields, of given non-abelian Galois group, by increasing conductors or discriminants, is too complex, it is assumed in all the sequel that the extensions K/\mathbb{Q} are abelian of Galois group G (and of signature s); which allows a classification using the conductors to list the suitable fields, even if a conductor may correspond to a finite number of fields.

For prime degree p cyclic fields, the classifications via increasing discriminants is equivalent to that using the conductors f_K since $D_K = f_K^{p-1}$.

Definition 1.2. Denote by \mathcal{K}_G^s the set of Galois number fields K , of given abelian Galois group G and signature $s = \pm 1$ (i.e., real or imaginary fields).

The conductor of the field K is denoted f_K and its discriminant D_K ; then $\omega(f_K) = \omega(D_K) =: N_K$ denotes the number of ramified primes. We then define $\mathcal{F}_{G,f}^s := \{K \in \mathcal{K}_G^s, f_K = f\}$.

If G is cyclic of order d and if s is fixed, we shall put $\mathcal{K}_G^s = \mathcal{K}_d^-$ or \mathcal{K}_d^+ and in the same way for $\mathcal{F}_{G,f}^s =: \mathcal{F}_{d,f}^s$, $s = \pm 1$.

We shall in general remove the index K for the above objects, giving the notations f , D , N , H , h .

If $G \simeq \mathbb{Z}/p\mathbb{Z}$, the genus number only depends on the number N of prime divisors of the conductor f and one gets:

$$h = \frac{H}{p^{N-1}}.$$

It is clear that the structure of abelian group of G leads to some constraints on $\text{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q}) \simeq (\mathbb{Z}/f\mathbb{Z})^\times$, then on N (e.g., if for some prime number p , $\dim_{\mathbb{F}_p}(G/G^p) \geq 2$, the conductor f of K cannot be a prime number $q \neq p$). Moreover, the wild ramification gives some trouble as the case of abelian 2-extensions with even conductors since the 2-rank of $(\mathbb{Z}/2^n\mathbb{Z})^\times$ ($n \geq 2$) may be 1 or 2; the case of $p > 2$ is easier since $(\mathbb{Z}/p^n\mathbb{Z})^\times \otimes \mathbb{Z}_p$ ($n \geq 2$) is cyclic. The mixed case where G is a product of p -groups for at least two distinct primes p depends on several sub-cases.

If two fields, in the same family, have the same discriminant, they have the same conductor, except possibly for the wild parts; but in our examples, we will have equality.

Of course, a list by increasing discriminants may be very different from the same list by increasing conductors; for instance, the degree 4 cyclic fields K_1 and K_2 such that $D_{K_1} = 5^3 17^2 = 36125$ and $D_{K_2} = 17^3 3^2 = 44217 > D_{K_1}$ are of conductors $f_{K_1} = 85$ and $f_{K_2} = 51$, respectively.

So, it makes sense to consider, instead, a classification by increasing conductors, which seems to be a better classification with regard to Galois structure and ramification. But, considering the ε -conjectures, the class numbers are usually compared with suitable functions of the discriminant, as $(\sqrt{|D_K|})^\varepsilon$.

Whence the following conjecture, strengthened by the various computations that we shall perform in the forthcoming sections:

Conjecture 1.3. *Let G be an abelian group and consider the family \mathcal{K}_G^s (see Definition 1.2). Let $\mathcal{F}_{G,f}^s$ (resp. $\mathcal{F}_{G,f}'^s$) be the set of abelian fields $K \in \mathcal{K}_G^s$ of conductor f (resp. of any conductor $f' \mid f$). For any $\varepsilon \in]0, 1[$, let:*

$$C_\varepsilon(\mathcal{F}_{G,f}^s) := \left[\prod_{K \in \mathcal{F}_{G,f}^s} \frac{\#\mathcal{C}_K}{g_{K/\mathbb{Q}} \cdot (\sqrt{|D_K|})^\varepsilon} \right]^{\frac{1}{\#\mathcal{F}_{G,f}^s}},$$

$$C'_\varepsilon(\mathcal{F}_{G,f}'^s) := \left[\prod_{K \in \mathcal{F}_{G,f}'^s} \frac{\#\mathcal{C}_K}{g_{K/\mathbb{Q}} \cdot (\sqrt{|D_K|})^\varepsilon} \right]^{\frac{1}{\#\mathcal{F}_{G,f}'^s}},$$

$$C_\varepsilon(K) := \frac{\#\mathcal{C}_K}{g_{K/\mathbb{Q}} \cdot (\sqrt{|D_K|})^\varepsilon}, \text{ for any } K \in \mathcal{K}_G^s.$$

Let $r(G)$ be the minimal number of generators of G and let $\#G = \prod_{i=1}^t p_i^{n_i}$ be the prime decomposition of the order of G , then put $R(G) := \sum_{i=1}^t n_i$.

For instance $r(\mathbb{Z}/4\mathbb{Z}) = r(\mathbb{Z}/6\mathbb{Z}) = 1$ and $R(\mathbb{Z}/4\mathbb{Z}) = R(\mathbb{Z}/6\mathbb{Z}) = 2, \dots$

We then have conjecturally¹:

(i) *The successive maxima of $C_\varepsilon(\mathcal{F}_{G,f}^s)$, $C'_\varepsilon(\mathcal{F}_{G,f}'^s)$, $C_\varepsilon(K)$, as f increases from its minimum, occur for conductors f such that $r(G) \leq \omega(f) \leq R(G)$, with probability 1.² This property does not depend on the choice of $\varepsilon \in]0, 1[$.*

(ii) *For any $\varepsilon \in]0, 1[$, the sequences of successive maxima are infinite.*

Remarks 1.4. (i) The functions C_ε and C'_ε are multiplicative means over the fields of same conductor f (resp. of any conductor $f' \mid f$). The computation of the successive maxima of the function C_ε over \mathcal{K}_G^s does not consider any mean and is equivalent to take the maximum of $\frac{\#\mathcal{C}_K}{g_{K/\mathbb{Q}} \cdot (\sqrt{|D_K|})^\varepsilon}$ over $\mathcal{F}_{G,f}^s$, for each f , then the successive maxima of these numbers.

One has $\#\mathcal{F}_{G,f}^s = 1$, for all f , only for the family of quadratic fields, in which case, $C_\varepsilon(\mathcal{F}_{G,f}^s) = C_\varepsilon(K)$ for the unique $K \in \mathcal{F}_{G,f}^s$.

(ii) The sequences are not necessarily the same as ε varies, but the practice shows that they differ very little.

(iii) It is possible that “except for a number of conductors of density 0” may be replaced by “except a finite number of conductors”, but the numerical experiments are not sufficient to be more precise.

These kind of functions are related to some less known results concerning minorations of class numbers and to the ε -conjectures that we recall now.

2. CLASSICAL RESULTS AND ε -CONJECTURES

2.1. Upper and lower bounds for class numbers. The order of magnitude of the class number $\#\mathcal{C}_K$ of a number field K is very irregular but is in part governed by the absolute value of the discriminant D_K and the signature of the field K ; of course, this is to be precised and a method is to compare the class number with $(\sqrt{|D_K|})^\varepsilon$, for any $\varepsilon \in]0, 1[$ fixed.

Using the family \mathcal{K}_p^s of degree p cyclic fields for simplicity ($p \geq 2$ prime), let's recall that there exist an absolute constant \mathcal{C} and, for all $\eta > 0$, \mathcal{C}_η such

¹Recall that when $\#G$ is odd, the prime 2 is unramified in the abelian extension K/\mathbb{Q} . When $\#G$ is even and 2 ramifies with $8 \mid f$, one must preferably replace $\omega(f)$ by $\omega(f) + 1$; for simplicity, we shall neglect this exception in the statements and comments.

²This means the exactness except possibly for a set of conductors of zero density. However, the probabilistic behaviour of $\omega(f)$ in $[r(G), R(G)]$ is unclear.

that (Siegel, Landau; see, e.g., [24, Chapter 4, §1, Theorem 4.4]):

$$(2) \quad \begin{aligned} (i) \quad & \#\mathcal{C}_K \leq \mathfrak{C} \cdot \sqrt{|D_K|} \cdot \log(\sqrt{|D_K|}), \\ (ii) \quad & \#\mathcal{C}_K \leq \mathfrak{C}_\eta \cdot (\sqrt{|D_K|})^{1+\eta}. \end{aligned}$$

Under GRH (Ankeny–Brauer–Chowla; see more comments in [5, 6, 8]), we have the following result for quadratic fields which will be of some significance for successive maxima:

Proposition 2.1. *For any $\eta > 0$, there exist infinitely many $K \in \mathcal{K}_2^s$ such that $\#\mathcal{C}_K \geq (\sqrt{|D_K|})^{1-\eta}$.*

Moreover, Littlewood proves the existence of constant \mathfrak{C}' , \mathfrak{C}'' , such that:

$$(3) \quad \begin{aligned} (i) \quad & \#\mathcal{C}_K \leq \mathfrak{C}' \cdot \sqrt{|D_K|} \cdot \log_2(\sqrt{|D_K|}), \text{ for all } K \in \mathcal{K}_2^-, \\ (ii) \quad & \#\mathcal{C}_K \geq \mathfrak{C}'' \cdot \sqrt{|D_K|} \cdot \log_2(\sqrt{|D_K|}), \text{ for infinitely many } K \in \mathcal{K}_2^-. \end{aligned}$$

For more information and generalization of such phenomena, including real fields, see [5, 6, 8, 15, 17, 18, 19, 20, 23, 27].

2.2. The ε -conjectures. The concept of ε -conjecture comes from the work of Ellenberg–Venkatesh [10], in close relation with the heuristics and conjectures of Cohen–Lenstra–Martinet. Many developments have followed as [1, 7, 9, 11, 14, 16, 21, 22, 26, 28], to give an order of magnitude of various invariants attached to the class group of a number field K , according to the function $(\sqrt{|D_K|})^\varepsilon$ of its discriminant, for any $\varepsilon > 0$.

We shall now emphasize some of these ε -conjectures and precise our purpose, although this paper is not concerned by the q -class groups $\mathcal{C}_K \otimes \mathbb{Z}_q$, for each prime q , except that genus theory involves the q -parts of the class group when $q \mid d$, as for the degree p cyclic extensions for which the p -Sylow subgroup $\mathcal{C}_K \otimes \mathbb{Z}_p$ may be very large:

(i) The p -rank ε -conjecture for number fields claims that for all $\varepsilon > 0$:

$$\#(\mathcal{C}_K \otimes \mathbb{F}_p) \ll_{d,p,\varepsilon} (\sqrt{|D_K|})^\varepsilon, \quad \text{for all } K \in \mathcal{K}_d^s.$$

We have proved the p -rank ε -conjecture for the family of degree p cyclic extensions [14, Theorem 2.5].

Many authors study and prove for \mathcal{K}_d^s inequalities of the form:

$$(4) \quad \#(\mathcal{C}_K \otimes \mathbb{F}_p) \ll_{d,p,\eta} (\sqrt{|D_K|})^{c+\eta}, \quad \eta > 0,$$

with a constant $c \in]0, 1[$ as small as possible (e.g., $c = 1 - \frac{1}{p(d-1)}$) since the equality does exist for $c = 1$ (relation (2) (ii)).

(ii) The strong (p, ε) -conjecture that we have considered in [14] is:

$$(5) \quad \#(\mathcal{C}_K \otimes \mathbb{Z}_p) \ll_{d,p,\varepsilon} (\sqrt{|D_K|})^\varepsilon, \quad \text{for all } K \in \mathcal{K}_d^s.$$

The results of density of [16, Theorem 1.1] prove, under GRH, that the strong (p, ε) -conjecture for degree p cyclic fields is true, *except possibly for sparse subfamilies of fields of zero density*, probably in relation with that we brought out in §2.1.

2.3. Non-genus number in the degree p cyclic case. We have shown in [14], using the family of degree p cyclic extensions K/\mathbb{Q} , that the genus subgroup of \mathcal{C}_K , may be an obstruction or at least a difficulty, to prove inequality (5) and one may for instance replace the whole class group by its non-genus part in the above (p, ε) -conjecture.

Consider a family \mathcal{K}_p^s of prime degree $p \geq 2$ cyclic fields K .

Let $G = \text{Gal}(K/\mathbb{Q}) =: \langle \sigma \rangle$ and let H , be the class number of K in the restricted sense when this makes sense (e.g., case of real quadratic fields).

We consider the quotient $\mathcal{C}_K/\mathcal{C}_K^G \simeq \mathcal{C}_K^{1-\sigma}$ (of order $h := \frac{H}{g_{K/\mathbb{Q}}}$), of \mathcal{C}_K by the subgroup of invariant (or ambiguous) classes; this group measures the set of “exceptional p -classes” and that of classes of prime to p orders.

So we may consider the non-genus (p, ε) -conjecture for this family:

$$(6) \quad \#(\mathcal{C}_K \otimes \mathbb{Z}_p)^{1-\sigma} \ll_{p,\varepsilon} \cdot (\sqrt{|D_K|})^\varepsilon,$$

except for subfamilies of zero density.

2.4. Successive maxima – Main Theorem for degree p cyclic fields.

We have (to our knowledge) no information on the successive maxima of $\frac{\#\mathcal{C}_K}{(\sqrt{|D_K|})^\varepsilon}$ or $\frac{\#\mathcal{C}_K}{g_{K/\mathbb{Q}} \cdot (\sqrt{|D_K|})^\varepsilon}$, as K varies in some defined family ordered by increasing discriminants and considering that several fields may have the same discriminant, which forces to give the more precise definitions used in the Conjecture 1.3.

Remark 2.2. A main problem is then that Proposition 2.1 (or the lower bound of (3)) will be in contradiction with an unconditional global abelian “strong ε -conjecture”, this strong ε -conjecture being that for all $\varepsilon \in]0, 1[$:

$$(7) \quad \#\mathcal{C}_K \ll_{G,\varepsilon} \cdot (\sqrt{|D_K|})^\varepsilon,$$

for the family \mathcal{K}_G^s of field of abelian Galois group G and signature s .

To overcome this difficulty, we must conjecture that this inequality (7) holds *except for a subfamily of zero density*. This caution shall be necessary whatever the context as shown by the work [16] of Koymans–Pagano on densities in the family of degree p cyclic fields for which the results are true with “probability 1”, in the framework of the above description on the possible existence of sparse “bad subfamilies”.

To take into account these phenomena, we base our study on the following which holds for $G \simeq \mathbb{Z}/2\mathbb{Z}$ and $G \simeq \mathbb{Z}/3\mathbb{Z}$:

Hypothesis 2.3. *We assume that Proposition 2.1 does exist in \mathcal{K}_G^s for all non-trivial abelian group G , in other words that for all $\eta \in]0, 1[$ there exist infinitely many $K \in \mathcal{K}_G^s$ such that $\#\mathcal{C}_K \geq (\sqrt{|D_K|})^{1-\eta}$.*

Remark 2.4. This statement is equivalent to: “for all $\eta \in]0, 1[$ there exist $\mathcal{C}_\eta > 0$ and infinitely many $K \in \mathcal{K}_G^s$ such that $\#\mathcal{C}_K \geq \mathcal{C}_\eta \cdot (\sqrt{|D_K|})^{1-\eta}$ ”. Indeed, let $\eta \in]0, 1[$ be given and let $\theta > 0$ be such that $\eta_0 := \eta - \theta \in]0, 1[$; we obtain, with η_0 and the existence of \mathcal{C}_{η_0} , an infinite sequence of fields $K_i \in \mathcal{K}_G^s$ such that:

$$\begin{aligned} \log(\mathcal{C}_{K_i}) &\geq \log(\mathcal{C}_{\eta_0}) + (1 - (\eta - \theta)) \cdot \log(\sqrt{|D_{K_i}|}) \\ &\geq \log(\mathcal{C}_{\eta_0}) + \theta \cdot \log(\sqrt{|D_{K_i}|}) + (1 - \eta) \cdot \log(\sqrt{|D_{K_i}|}) \\ &\geq (1 - \eta) \cdot \log(\sqrt{|D_{K_i}|}) \text{ for all } i \geq i_0, i_0 \text{ suitable.} \end{aligned}$$

A consequence of this is that for any $\varepsilon \in]0, 1[$, the sequence of successive maxima of $\frac{\#\mathcal{C}_K}{(\sqrt{|D_K|})^\varepsilon}$ is infinite since $\frac{\#\mathcal{C}_K}{(\sqrt{|D_K|})^\varepsilon} \geq (\sqrt{|D_K|})^{1-\eta-\varepsilon}$ will be unbounded, due to infinitely many fields $K_i \in \mathcal{K}_G^s$, as soon as η is chosen small enough to get $1 - \eta - \varepsilon > 0$.

For the family of degree p cyclic fields (of given signature for $p = 2$), we will have some information (Theorem 2.6) on the successive maxima of:

$$C_\varepsilon(K) := \frac{\#(\mathcal{C}_K^{1-\sigma})}{(\sqrt{|D_K|})^\varepsilon} =: \frac{H}{p^{N-1} \cdot (\sqrt{D})^\varepsilon}, \quad N = \omega(D),$$

for which we have proposed the Conjecture 1.3. This is related to (6) in the following manner for $G \simeq \mathbb{Z}/p\mathbb{Z}$:

Remark 2.5. The strong ε -conjecture states that, “for almost all K ”, the sequence $\frac{\#\mathcal{C}_K}{(\sqrt{|D_K|})^\varepsilon}$ is bounded by a constant \mathcal{C}_ε (which increases as ε decreases). For $\varepsilon \in]0, 1[$, this constant is enormous (and unbounded as $\varepsilon \rightarrow 0$). Thus the sequence obtained by some discriminants D_{K_i} giving the successive maxima, is limited by \mathcal{C}_ε , up to some huge unreachable discriminant; but if the sequence of the successive maxima is infinite, the “exceptional cases” of the ε -conjecture may be some local maximua, but cannot be computed in practice when ε is too small.

Using these lower bounds and our results in [14] proving (unconditionally) the p -rank ε -conjecture, we can state the following main result for the family \mathcal{K}_p^s of prime degree p cyclic fields; \mathcal{K}_2^s may be \mathcal{K}_2^- or \mathcal{K}_2^+ then of course $\mathcal{K}_p^s = \mathcal{K}_p^+$ for $p > 2$, and where $\#\mathcal{F}_{p,f}^s = (p-1)^{N-1}$ for $N := \omega(f)$:

Theorem 2.6. *Let $\varepsilon \in]0, 1[$ be given. Under Hypothesis 2.3 in the family \mathcal{K}_p^s of degree p cyclic number fields ordered by increasing conductors f (or discriminants $D = f^{p-1}$), the sequence of the successive maxima of:*

$$C_\varepsilon(K) = \frac{H}{p^{N-1} \cdot (\sqrt{D})^\varepsilon}, \quad K \in \mathcal{K}_p^s, \quad H := \#\mathcal{C}_K, \quad N := \omega(f),$$

is infinite.

Proof. By assumption, for all $\eta \in]0, 1[$, there exist an infinite sequence $(f_i)_{i \geq 1}$ and at least a degree p cyclic fields K_i , of conductor f_i , such that:

$$C_\varepsilon(K_i) = \frac{\#\mathcal{C}_{K_i}}{p^{N_i-1} \cdot (\sqrt{|D_i|})^\varepsilon} \geq \frac{1}{p^{N_i-1}} (\sqrt{|D_i|})^{1-\eta-\varepsilon}, \quad N_i := \omega(f_i), \quad |D_i| = f_i^{p-1}.$$

The claim consists in proving that the right member is unbounded, in other words that the sequence (which depends on the choice of η):

$$A_i^{(\eta)} := (1 - \eta - \varepsilon) \cdot \log(\sqrt{|D_i|}) - (N_i - 1) \cdot \log(p)$$

is unbounded (which amounts to prove that the N_i are not too large). This is only a sufficient condition. To simplify, we delete the index i , we fix η and such an indice i , hence the corresponding field K of huge class number, and the integers N, f, D ; we put $A_i^{(\eta)} =: A$.

From [14, § 2.3], we have obtained the existence, for all $\theta > 0$, of a constant $C_{p,\theta} > 0$ such that, whatever the field $K \in \mathcal{K}_p^s$:

$$p^{(p-1)(N-1)} \leq C_{p,\theta} \cdot (\sqrt{D_N})^\theta, \quad D_N = f_N^{p-1},$$

where f_N is the product of the N first consecutive prime numbers congruent to 1 modulo p , so that $|D| > D_N$ since for the field K , $\omega(D) = \omega(D_N) = N$.

Which yields:

$$(N-1) \cdot \log(p) \leq \frac{\log(C_{p,\theta})}{p-1} + \frac{\theta \cdot \log(\sqrt{D_N})}{p-1}.$$

Then:

$$\begin{aligned}
 A &\geq (1 - \eta - \varepsilon) \cdot \log(\sqrt{|D|}) - \frac{\log(C_{p,\theta})}{p-1} - \frac{\theta \cdot \log(\sqrt{D_N})}{p-1} \\
 (8) \quad &\geq (1 - \eta - \varepsilon) \cdot \log(\sqrt{|D|}) - \frac{\log(C_{p,\theta})}{p-1} - \frac{\theta \cdot \log(\sqrt{|D|})}{p-1} \\
 &\geq \left[1 - \eta - \varepsilon - \frac{\theta}{p-1}\right] \cdot \log(\sqrt{|D|}) - \frac{\log(C_{p,\theta})}{p-1}.
 \end{aligned}$$

The parameters $\eta > 0$ and $\theta > 0$ may be chosen arbitrarily close to 0. Then as soon as $0 < \eta < 1 - \varepsilon - \frac{\theta}{p-1}$, the corresponding sequence $(A_i^{(\eta)})_i$ is unbounded, which yields the claim. \square

Remarks 2.7. (i) It is more difficult to prove the same result for the successive maxima of the means $C_\varepsilon(\mathcal{F}_{p,f}^s)$ (except for $p = 2$) or $C'_\varepsilon(\mathcal{F}'_{p,f})$. However, since all the experiments show that the successive maxima are obtained with very small values of N , there is no doubt for the infiniteness of the corresponding sequences of maxima (perhaps for $0 < \varepsilon \ll 1$ since an ε close to 0 allows less restrictive values of θ and η and easier numerical data).

(ii) The case $\varepsilon = 0$ makes sense for the study of the successive maxima of the function $C_0(K) := \frac{\#\mathcal{C}_K}{g_{K/\mathbb{Q}} \cdot (\sqrt{|D_K|})^0}$ equal to $\frac{H}{p^{N-1}}$ in the degree p cyclic case, but there is no 0-conjecture since the class number is unbounded in any infinite family \mathcal{K}_d^s . At the opposite case, for any $\eta > 0$, we know that $\#\mathcal{C}_K \ll_\eta \cdot (\sqrt{|D_K|})^{1+\eta}$, which will give some information (see § 4.1.3).

(iii) The inequality (4), with $\varepsilon = c + \eta$, which is proved in some cases, and the relations (2) with Proposition 2.1, show that the problem of the strong ε -conjectures is only concerned with $\varepsilon \in]0, 1[$ (cf. Theorem 2.6).

3. STUDY OF THE QUADRATIC FIELDS

We shall first illustrate the property under consideration by means of the family $\mathcal{K}_2 = \mathcal{K}_2^- \cup \mathcal{K}_2^+$ of quadratic fields.

3.1. Ordered list of discriminants. Let $K \in \mathcal{K}_2^s$ identified by means of its signature and the absolute value D of its discriminant D_K (which is also its conductor f_K). We shall use first the case of imaginary quadratic fields since the class numbers are more rapidly computed by PARI/GP [25], which allows more important bounds for the discriminants; then we shall see that the results are similar in the real case.

Any list of quadratic fields K will be ordered by increasing D as the following program does in the imaginary case, where $[bD, BD]$ is the interval for D , H is the class number of K and $h = \frac{H}{2^{N-1}}$, N being the number of ramified primes and 2^{N-1} the genus number (for the real case, see Section 5):

```

{bD=1;BD=10^4;for(D=bD,BD,e2=valuation(D,2);d=D/2^e2;if(e2==1||e2>3||
core(d)!=d||(e2==0&&Mod(-d,4)!=1)||e2==2&&Mod(-d,4)!=-1),next);
H=qfbclassno(-D);N=omega(D);h=H/2^(N-1);
print("D_K=", -D, " H=", H, " h=", h, " N=", N)}}

```

D_K=-3	H=1	h=1	N=1
D_K=-4	H=1	h=1	N=1
D_K=-7	H=1	h=1	N=1
D_K=-8	H=1	h=1	N=1
D_K=-11	H=1	h=1	N=1
D_K=-15	H=2	h=1	N=2
D_K=-19	H=1	h=1	N=1
(...)			

D_K=-10000000003	H=10538	h=5269	N=2
D_K=-10000000004	H=40944	h=20472	N=2
D_K=-10000000007	H=95488	h=23872	N=3
D_K=-10000000011	H=20264	h=5066	N=3
D_K=-10000000015	H=39520	h=2470	N=5
D_K=-10000000019	H=39809	h=39809	N=1
D_K=-10000000020	H=42368	h=1324	N=6
(...)			
D_K=-100000000000000000003	H=1442333424	h=360583356	N=3
D_K=-100000000000000000004	H=7297756288	h=456109768	N=5
D_K=-100000000000000000007	H=8734748716	h=2183687179	N=3
D_K=-100000000000000000011	H=2377640288	h=148602518	N=5
D_K=-100000000000000000015	H=5235364148	h=1308841037	N=3
D_K=-100000000000000000019	H=3301267440	h=1650633720	N=2
D_K=-100000000000000000020	H=3240597760	h=50634340	N=7
(...)			

As can be seen, the numbers H and h grow globally, but with many local decreases, the case of $N = \omega(D)$ being completely chaotic compared to the growth of D ; so we intend to look at the successive extrema (especially the successive maxima) of some function of these parameters.

3.2. Conjecture 1.3 for quadratic fields. Let $K \in \mathcal{K}_2^s$ be any quadratic field. Let $G = \text{Gal}(K/\mathbb{Q}) =: \langle \sigma \rangle$ and let \mathcal{C}_K , of order H , be the class group of K (in the restricted sense for $K \in \mathcal{K}_2^+$). We consider the quotient $\mathcal{C}_K/\mathcal{C}_K^G \simeq \mathcal{C}_K^{1-\sigma}$ (of order $h := \frac{H}{2^{N-1}}$). Since $\mathcal{F}_{2,f}^s$ is reduced to a single field identified by its conductor $f = D := |D_K|$, we put:

$$(9) \quad C_\varepsilon(D) = \frac{h}{(\sqrt{D})^\varepsilon} = \frac{H}{2^{N-1} \cdot (\sqrt{D})^\varepsilon}, \quad \varepsilon \in]0, 1[.$$

We shall compute the *successive maxima* of the function $C_\varepsilon(D)$ when D increases.

The following conjecture, suggested by some computations in [14], is the Conjecture 1.3 for quadratic fields since $r(G) = R(G) = 1$, which will be strengthened from numerical experiments given in the next sections.

Conjecture 3.1. *Let $\varepsilon \in]0, 1[$. For each quadratic field K (of a selected signature $s = \pm 1$) of discriminant D_K , we denote by $N := \omega(D_K) = \omega(f_K)$ the number of ramified primes, by H the class number in the restricted sense, and we put $D := |D_K|$.*

Then, the successive maxima of $C_\varepsilon(D) := \frac{H}{2^{N-1} \cdot (\sqrt{D})^\varepsilon}$, as D increases from its minimum, only occur for prime conductors $f_K = \ell \equiv s \pmod{4}$ (i.e., $N = 1$, H odd), except possibly for a finite number. These properties do not depend on the choice of ε .

The sequence of successive maxima is infinite from Theorem 2.6, and only depends of ε .

For the imaginary quadratic fields, the numerical experiments show that all the successive maxima are given by prime discriminants.

Denote by $s \ell_i \equiv 1 \pmod{4}$ this conjectural sequence of prime discriminants giving these successive maxima; then:

$$(10) \quad C_\varepsilon(\ell_i) = \frac{H_i}{(\sqrt{\ell_i})^\varepsilon} \quad \text{for all } i, \text{ where } H_i = \#\mathcal{C}_{\mathbb{Q}(\sqrt{s\ell_i})}.$$

As opposed to the conjectural inequality (7) (except a set of fields of zero density), the fields $K_i := \mathbb{Q}(\sqrt{s\ell_i})$ would perhaps show some examples for the minorations given by Proposition 2.1

3.3. Some heuristics for imaginary quadratic fields. Using the program 4.1.1, we start by giving some insights about the behaviour of the class numbers in various situations.

3.3.1. Large genus numbers. We shall give some calculations showing that, contrary to the common sense, a great lot of ramified primes in K (whence a large genus number) do not give *systematically* a large class number H (see a numerical example in the next subsection).

For instance, we may use the family of fields $K = \mathbb{Q}(\sqrt{-2 \cdot 3 \cdot 5 \cdot 7 \dots q_n})$, using the sequence of the n first odd prime numbers or any similar one as $\mathbb{Q}(\sqrt{-q_{n_0} \dots q_{n_0+n}})$. The following results are very convincing ($D = |D_K|$):

```

D=[2,3] H=1
D=[2,3;3] H=2
D=[2,3;3;5] H=4
D=[2,3;3;5;7] H=8
D=[2,3;3;5;7;11] H=32
D=[2,3;3;5;7;11;13] H=128
D=[2,3;3;5;7;11;13;17] H=448
D=[2,3;3;5;7;11;13;17;19] H=2048
D=[2,3;3;5;7;11;13;17;19;23] H=10240
D=[2,3;3;5;7;11;13;17;19;23;29] H=44544
D=[2,3;3;5;7;11;13;17;19;23;29;31] H=264192
D=[2,3;3;5;7;11;13;17;19;23;29;31;37] H=1908736
D=[2,3;3;5;7;11;13;17;19;23;29;31;37;41] H=11059200
D=[2,3;3;5;7;11;13;17;19;23;29;31;37;41;43] H=76988416
D=[2,3;3;5;7;11;13;17;19;23;29;31;37;41;43;47] H=436731904
D=[2,3;3;5;7;11;13;17;19;23;29;31;37;41;43;47;53] H=3745513472
(...)
D=[3] H=1
D=[3;5] H=2
D=[2,2;3;5;7] H=8
D=[3;5;7;11] H=8
D=[3;5;7;11;13] H=96
D=[3;5;7;11;13;17] H=256
D=[2,2;3;5;7;11;13;17;19] H=1536
D=[3;5;7;11;13;17;19;23] H=2688
D=[3;5;7;11;13;17;19;23;29] H=39424
D=[2,2;3;5;7;11;13;17;19;23;29;31] H=193536
D=[2,2;3;5;7;11;13;17;19;23;29;31;37] H=1210368
D=[2,2;3;5;7;11;13;17;19;23;29;31;37;41] H=7471104
D=[3;5;7;11;13;17;19;23;29;31;37;41;43] H=57925632
D=[2,2;3;5;7;11;13;17;19;23;29;31;37;41;43;47] H=364101632
D=[2,2;3;5;7;11;13;17;19;23;29;31;37;41;43;47;53] H=2738487296
D=[3;5;7;11;13;17;19;23;29;31;37;41;43;47;53;59] H=19836665856
(...)
    
```

3.3.2. Successive maxima of $\frac{H}{2^{N-1} \cdot (\sqrt{|D_K|})^{0.05}}$ (see Table I, Appendix 9.1). As expected, the computation of the successive maxima of $\frac{H}{2^{N-1} \cdot (\sqrt{|D_K|})^{0.05}}$ in an interval containing such a huge discriminant gives much larger non-genus class numbers as the following computation shows with the discriminant D_0 :

$$D_0 = -892371480 = -2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23,$$

for which $\frac{H}{2^{N-1} \cdot (\sqrt{|D_K|})^{0.05}} = 23.89441208$ while the order of magnitude of the successive maxima for such order of magnitude of discriminants is 39326. ... ; it is clear that $H = 10240$ for $D_K = -892371480$, is far to give a local maximum (see line ***):

```

eps=0.05
D_K=-891819431 H=64723 h=64723 N=1 C=38663.54897528134436
D_K=-891972311 H=65807 h=65807 N=1 C=39310.92910296295340
    
```

D_K=-892371479	H=65833	h=65833	N=1	C=39326.02076918814615
D_K=-892371480	H=10240	h=40	N=9	C=23.89441208396179319 ***
D_K=-892909079	H=66789	h=66789	N=1	C=39896.49651358522089
D_K=-898496759	H=68083	h=68083	N=1	C=40663.12685277796768
D_K=-900119351	H=68387	h=68387	N=1	C=40842.85100395781405

Similarly, for the last discriminant -961380175077106319535 above, with $H = 19836665856$, $\frac{H}{2^{N-1} \cdot (\sqrt{|D_K|})^{0.05}} = 180903.24774117$ for a local maximum > 23715549673 at this place:

D_K=-961380175077106313759	H=73527737200	N=5	C=21972467046.501154082
D_K=-961380175077106315439	H=74348146642	N=2	C=22217631934.140092600
D_K=-961380175077106319279	H=76863968144	N=2	C=22969440804.542458758
D_K=-961380175077106319519	H=79360715400	N=3	C=23715549673.045788420

3.3.3. *Successive maxima of $\frac{H}{(\sqrt{D})^\varepsilon}$ (see Table II, Appendix 9.2).* In terms of successive maxima of $\frac{H}{(\sqrt{D})^\varepsilon}$, we observe that they occur for fields K such $N \in \{1, 2, 3\}$ for $D \leq 153116519$ and that for very large discriminants this set is $\{1, 2, 3, 4\}$ (but mainly 1, 2). We give below some excerpts of the whole table for $\varepsilon = 0.05$:

```
{eps=0.05;bd=1;BD=10^9;Cm=0;for(D=bD,BD,e2=valuation(D,2);d=D/2^e2;
if(e2==1||e2>3||core(d)!=d||e2==0&Mod(-d,4)!=1)||e2==2&Mod(-d,4)!=-1),
next);H=qfbcassno(-D);C=H/(D^(eps/2));if(C>Cm,Cm=C;
print("D_K=", -D, " H=", H, " N=", omega(D), " C=", precision(C,1)))}}
```

```
eps=0.05
D_K=-3          H=1      N=1      C=0.972908434869468710
D_K=-15         H=2      N=2      C=1.869079241783001633
D_K=-23         H=3      N=1      C=2.773818617890694607
D_K=-39         H=4      N=2      C=3.649920257029810563
D_K=-47         H=5      N=1      C=4.541167885124564221
D_K=-71         H=7      N=1      C=6.292403751297605636
D_K=-95         H=8      N=2      C=7.139156409132720178
(...)
D_K=-1036031    H=1908   N=2      C=1349.565761315970158
D_K=-1062311    H=1927   N=1      C=1362.151530472142189
D_K=-1086791    H=1965   N=1      C=1388.221940001361239
D_K=-1101239    H=1970   N=2      C=1391.294878906636963
D_K=-1127759    H=1982   N=2      C=1398.937276305459193
D_K=-1138199    H=1984   N=3      C=1400.026360256248040
D_K=-1139519    H=2027   N=1      C=1430.328227182419060
(...)
D_K=-100682039  H=22178  N=2      C=13990.99428148920046
D_K=-102628679  H=22235  N=1      C=14020.23894496263475
D_K=-104716751  H=22315  N=1      C=14063.59940393199907
D_K=-105734879  H=22530  N=2      C=14195.66476857999654
D_K=-106105271  H=23077  N=1      C=14539.04651797868699
D_K=-110487599  H=23275  N=1      C=14648.96196034653848
D_K=-110869391  H=23356  N=3      C=14698.67458157540117
(...)
D_K=-10000000000000391  H=206713870  N=2  C=32761940.53502909525
D_K=-10000000000013399  H=213240384  N=2  C=33796323.29593711028
D_K=-10000000000024271  H=221061984  N=4  C=35035963.35535050720
D_K=-10000000000026959  H=227089538  N=2  C=35991266.28552908299
D_K=-10000000000070519  H=231231712  N=4  C=36647756.62297995465
D_K=-10000000000123559  H=235020092  N=2  C=37248174.39021544804
D_K=-10000000000141511  H=245552727  N=1  C=38917484.54122868952
D_K=-10000000001180279  H=251465706  N=2  C=39854628.55744758011
(...)
```

3.3.4. *Decomposition of the prime numbers.* We see that (except for few cases at the beginning of the algorithm) the discriminants are such $D_K \equiv 1$

(mod 8), which corresponds to the splitting of 2 in K ; similarly, the small unramified prime numbers 3, 5, 7, 11, ... splits in K . For $D_K = -1432580159$, all primes p , $2 \leq p \leq 67$, split. For $D_K = -2017348511$ all primes p , $2 \leq p \leq 107$, split, except $p = 79$ (Table I, Appendix 9.1).

This is not too surprising since the class of a prime ideal of small norm p is of high order since we must have a solution to $x^2 + D \cdot y^2 = p^n$, which yields approximatively $n > \frac{D}{\log(p)}$. In the references [5, 6, 8, 15, 17, 18, 19, 20, 23, 27], using complex L -functions, it is the process to get large class numbers $H = \frac{L(1, \chi)}{\pi} \sqrt{D}$, where χ is the Dirichlet character modulo D . This is also the principle of the paper of Ellenberg–Venkatesh for the p -rank ε -conjecture (see [10, § 1.1]).

4. PROGRAMS AND TABLES FOR IMAGINARY QUADRATIC FIELDS

The two following programs compute the successive maxima of $\frac{H}{2^{N-1} \cdot (\sqrt{D})^\varepsilon}$. Program I works with a given ε and the computations of N, H as the discriminant D increases arbitrarily. Program II computes, once for all, a list of triples (D, N, H) ; then, for comparison of the results, we can use this list for a great lot of values of ε for faster computations of the successive maxima.

4.1. Successive maxima of $\frac{H}{2^{N-1} \cdot (\sqrt{D})^\varepsilon}$.

4.1.1. Program I – Successive maxima of $\frac{H}{2^{N-1} \cdot (\sqrt{D})^\varepsilon}$ with fixed ε . One must choose ε in `eps` and the interval `[bD, BD]` for D . The successive maxima of $C_\varepsilon(D) = \frac{H}{2^{N-1} \cdot (\sqrt{D})^\varepsilon}$ are given in `C` with the corresponding values of H (in `H`), h (in `h`) and N (in `N`). We write some excerpts of the table with various ε , $0 < \varepsilon \ll 1$, before giving a more complete table in § 4.1.2 and Appendix 9.1 (all the missing results below are with $N = 1$).

```
{eps=0.05;bD=1;BD=5*10^7;Cm=0;for(D=bD,BD,e2=valuation(D,2);d=D/2^e2;
if(e2==1||e2>3||core(d)!=d||e2==0&Mod(-d,4)!=1)||e2==2&Mod(-d,4)!=-1,
next);H=qfbclassno(-D);N=omega(D);h=H/2^(N-1);C=h/(D^(eps/2));if(C>Cm,Cm=C;
print("D_K=",D," H=",H," h=",h," N=",N," C=",precision(C,1)))}
eps=0.05
D_K=-3          H=1          h=1          N=1          C=0.972908434869468710
D_K=-23         H=3          h=3          N=1          C=2.773818617890694607
D_K=-47         H=5          h=5          N=1          C=4.541167885124564221
D_K=-71         H=7          h=7          N=1          C=6.292403751297605636
D_K=-167        H=11         h=11         N=1          C=9.678872599268429560
D_K=-191        H=13         h=13         N=1          C=11.40033250135200530
(...)
D_K=-1006799    H=1853       h=1853       N=1          C=1311.601334671650505
D_K=-1032071    H=1857       h=1857       N=1          C=1313.618222552253122
D_K=-1062311    H=1927       h=1927       N=1          C=1362.151530472142189
D_K=-1086791    H=1965       h=1965       N=1          C=1388.221940001361239
D_K=-1139519    H=2027       h=2027       N=1          C=1430.328227182419060
D_K=-1190591    H=2051       h=2051       N=1          C=1445.678077927788222
(...)
D_K=-10057031   H=6425       h=6425       N=1          C=4293.499209586390154
D_K=-10289639   H=6563       h=6563       N=1          C=4383.211236938274762
D_K=-10616759   H=6583       h=6583       N=1          C=4393.130011813243966
D_K=-10865279   H=6725       h=6725       N=1          C=4485.297628751901855
D_K=-10984991   H=6767       h=6767       N=1          C=4512.073690323125874
D_K=-11101151   H=6777       h=6777       N=1          C=4517.553301414761614
(...)
D_K=-30706079   H=12017      h=12017      N=1          C=7809.360152349024611
D_K=-32936879   H=12151      h=12151      N=1          C=7882.608544620325792
```

```

D_K=-34103471 H=12285 h=12285 N=1 C=7962.605439538764212
D_K=-34867271 H=12531 h=12531 N=1 C=8117.555770269597220
D_K=-35377319 H=12549 h=12549 N=1 C=8126.265293030949160
D_K=-36098039 H=12673 h=12673 N=1 C=8202.426446333352781
(...)
D_K=-68514599 H=17485 h=17485 N=1 C=11137.07280268520213
D_K=-68621159 H=17763 h=17763 N=1 C=11313.70536648692712
D_K=-69671159 H=18455 h=18455 N=1 C=11749.99613226375525
D_K=-75038519 H=18611 h=18611 N=1 C=11827.35419243076771
D_K=-75865919 H=18653 h=18653 N=1 C=11850.79599797629443
D_K=-76708271 H=19191 h=19191 N=1 C=12189.23782258446658
(...)

```

Taking intervals with huge bounds, still with $C_m = 0$, it is normal that the first maximum is not necessarily large enough to initiate the process which quickly stabilizes with prime discriminants (i.e., $N = 1$):

```

D_K=-100000000003 H=31057 h=31057 N=1 C=16487.67818446943949
D_K=-100000000004 H=150192 h=37548 N=3 C=19933.64911196521700
D_K=-100000000007 H=294590 h=147295 N=2 C=78196.62421286658525
D_K=-100000000019 H=160731 h=160731 N=1 C=85329.58760528567330
D_K=-100000000091 H=225785 h=225785 N=1 C=119865.7442379667594
D_K=-100000000103 H=350895 h=350895 N=1 C=186284.6970536358269
D_K=-100000000319 H=405637 h=405637 N=1 C=215346.3732873515489
D_K=-100000001111 H=520065 h=520065 N=1 C=276094.4184123037331
(...)
D_K=-100000000000000000391 H=12650126133 h=12650126133 N=1 C=4000321126.86
D_K=-100000000000000000559 H=13816100521 h=13816100521 N=1 C=4369034602.82
D_K=-1000000000000000004159 H=15675687007 h=15675687007 N=1 C=4957087483.00
D_K=-1000000000000000004519 H=17188889951 h=17188889951 N=1 C=5435604269.51
D_K=-1000000000000000005071 H=18354412823 h=18354412823 N=1 C=5804174963.56
D_K=-10000000000000000010039 H=21539854655 h=21539854655 N=1 C=6811500117.88
D_K=-100000000000000000375799 H=22001805625 h=22001805625 N=1 C=6957581841.13
D_K=-100000000000000940831 H=22578025429 h=22578025429 N=1 C=7139798542.48
(...)

```

There are huge intervals between two successive maxima; this probably means that the density of these successive maxima is zero. Perhaps this comes easily from analytical computations as those given in [5, 6, 8, 17] and others.

4.1.2. Distribution of the values of N for the successive maxima of $\frac{H}{2^{N-1}(\sqrt{D})^\varepsilon}$.

We give the statistical distribution of the values of N for the successive maxima of $C_\varepsilon(D) = \frac{H}{2^{N-1}(\sqrt{D})^\varepsilon}$ with $\varepsilon = 0.02$ (where N_i denotes the number of $N = i$ at each new maximum; N_D is the total number of consider discriminants); see Appendix 9.1:

```

{eps=0.02; bD=1; BD=10^12; ND=0; N1=0; N2=0; N3=0; N4=0; N5=0; N6=0;
Cm=0; for (D=bD, BD, e2=valuation(D, 2); d=D/2^e2;
if (e2==1 || e2>3 || core(d)!=d || (e2==0 & Mod(-d, 4)!=1) || (e2==2 & Mod(-d, 4)!=-1),
next); ND=ND+1; H=qfbclassno(-D); N=omega(D); h=H/2^(N-1); C=h/D^(eps/2);
if (C>Cm, Cm=C; if (N==1, N1=N1+1); if (N==2, N2=N2+1); if (N>=3, N3=N3+1); print();
print("D_K=", -D, " H=", H, " h=", h, " N=", N, " C=", precision(C, 1));
print("ND=", ND, " N1=", N1, " N2=", N2, " N3=", N3))}

```

4.1.3. *Computations with $\varepsilon \approx 1$.* The case $\varepsilon = 1$ or $\varepsilon = 1 \pm \eta$ (for any $\eta > 0$ close to 0 as we recalled in § 2.1) has some interest because of some known upper and lower bounds for H with classical inequalities in \mathcal{K}_2^- :

$\#\mathcal{C}_K \leq \mathcal{C}_\eta \cdot (\sqrt{D})^{1+\eta}$, for all D (inequality (2) (ii)), for some constant \mathcal{C}_η , and the inverse inequality $\#\mathcal{C}_K \geq (\sqrt{D})^{1-\eta}$, for infinitely many D (Proposition 2.1), giving (in some sense with $\varepsilon = 1 + \eta$): $C_{1+\eta}(D) \leq \frac{\mathcal{C}_\eta}{2^{N-1}}$, for all D , and proving that $\liminf_D (C_{1+\eta}(D)) = 0$ and that $\limsup_D (C_{1+\eta}(D)) < \infty$.

A priori, the second inequality is more interesting when $N = 1$ (then $D_K = s\ell \equiv 1 \pmod{4}$), yielding $C_{1+\eta}(\ell) \gg_{\eta} 0$. This gives some information regarding the conjectural sequence ℓ_i defined by (10).

```
{eps=1;bd=1;BD=10^8;Cm=0;for(D=bD,BD,e2=valuation(D,2);d=D/2^e2;
if(e2==1||e2>3|core(d)!=d|(e2==0&Mod(-d,4)!=1)||e2==2&Mod(-d,4)!=-1),
next);H=qfbclassno(-D);N=omega(D);h=H/2^(N-1);C=h/(D^(eps/2));if(C>Cm,Cm=C;
print("D_K=",D," H=",H," h=",h," N=",N," C=",precision(C,1)))}
```

D_K	H	h	N	C
D_K=-3	H=1	h=1	N=1	C=0.577350269189625764
D_K=-23	H=3	h=3	N=1	C=0.625543242171224288
D_K=-47	H=5	h=5	N=1	C=0.729324957489472779
D_K=-71	H=7	h=7	N=1	C=0.830747160735697329
D_K=-167	H=11	h=11	N=1	C=0.851205555787550539
D_K=-191	H=13	h=13	N=1	C=0.940646986880148131
D_K=-239	H=15	h=15	N=1	C=0.970269341029726350
D_K=-311	H=19	h=19	N=1	C=1.077391156535111118
D_K=-479	H=25	h=25	N=1	C=1.142279155962384611
D_K=-719	H=31	h=31	N=1	C=1.156104917973712355
D_K=-1151	H=41	h=41	N=1	C=1.208498720419083129
D_K=-1319	H=45	h=45	N=1	C=1.239053663199498814
D_K=-1511	H=49	h=49	N=1	C=1.260560945801805027
D_K=-1559	H=51	h=51	N=1	C=1.291656751946608433
D_K=-2351	H=63	h=63	N=1	C=1.299314349827718283
D_K=-2999	H=73	h=73	N=1	C=1.333013744070415380
D_K=-3671	H=81	h=81	N=1	C=1.336881232618676007
D_K=-5711	H=109	h=109	N=1	C=1.442349199551804946
D_K=-6551	H=117	h=117	N=1	C=1.445546485075073189
D_K=-9239	H=139	h=139	N=1	C=1.446113287777516307
D_K=-10391	H=153	h=153	N=1	C=1.500938018707737875
D_K=-14951	H=185	h=185	N=1	C=1.512991916312454831
D_K=-15791	H=195	h=195	N=1	C=1.551778729394620610
D_K=-18191	H=213	h=213	N=1	C=1.579251567061564281
D_K=-31391	H=289	h=289	N=1	C=1.631155133694937910
D_K=-38639	H=325	h=325	N=1	C=1.653371403795025953
D_K=-63839	H=423	h=423	N=1	C=1.674161419982429269
D_K=-88919	H=505	h=505	N=1	C=1.693534674808400201
D_K=-95471	H=533	h=533	N=1	C=1.725009428818167194
D_K=-147671	H=667	h=667	N=1	C=1.735714238040899734
D_K=-191231	H=761	h=761	N=1	C=1.740225421963015830
D_K=-250799	H=875	h=875	N=1	C=1.747210185407398438
D_K=-284231	H=943	h=943	N=1	C=1.768788765798006676
D_K=-289511	H=963	h=963	N=1	C=1.789755692477908166
D_K=-312311	H=1001	h=1001	N=1	C=1.791184972640468990
D_K=-366791	H=1121	h=1121	N=1	C=1.850956776350963554
D_K=-514751	H=1347	h=1347	N=1	C=1.877452665000664534
D_K=-628319	H=1495	h=1495	N=1	C=1.886040855881855214
D_K=-701399	H=1581	h=1581	N=1	C=1.887770950411945618
D_K=-819719	H=1725	h=1725	N=1	C=1.905270304565445333
D_K=-1412759	H=2311	h=2311	N=1	C=1.944311765270786403
D_K=-2470079	H=3079	h=3079	N=1	C=1.959089455884703822
D_K=-3068831	H=3485	h=3485	N=1	C=1.989373351795705735
D_K=-3190151	H=3593	h=3593	N=1	C=2.011646181366134258
D_K=-4305479	H=4227	h=4227	N=1	C=2.037142972901196004
D_K=-10289639	H=6563	h=6563	N=1	C=2.045984480430942894
D_K=-11935871	H=7165	h=7165	N=1	C=2.073906326984584401
D_K=-12537719	H=7457	h=7457	N=1	C=2.105983077270287202
D_K=-16368959	H=8565	h=8565	N=1	C=2.116980398599532524
D_K=-29036999	H=11421	h=11421	N=1	C=2.119474846049997785
D_K=-29858111	H=11885	h=11885	N=1	C=2.175043874892598601
D_K=-48796439	H=15195	h=15195	N=1	C=2.175237288881570956
D_K=-58057991	H=16631	h=16631	N=1	C=2.182666393650392413
D_K=-69671159	H=18455	h=18455	N=1	C=2.210993836560839743


```
D_K=-82636319 H=20299 h=20299 N=1 C=2.233002255956684257
D_K=-106105271 H=23077 h=23077 N=1 C=2.240324202281881086
(...)
```

For $\varepsilon = 1.1$ we have found the sequence of prime discriminants:

```
D_K=-3 H=1 h=1 N=1 C=0.5464913722529576510
D_K=-47 H=5 h=5 N=1 C=0.6016115735959143380
D_K=-71 H=7 h=7 N=1 C=0.6712834625402534319
D_K=-191 H=13 h=13 N=1 C=0.7233941632706684037
D_K=-239 H=15 h=15 N=1 C=0.7378573598045812696
D_K=-311 H=19 h=19 N=1 C=0.8086030720739909916
D_K=-479 H=25 h=25 N=1 C=0.8389874728227036802
D_K=-1151 H=41 h=41 N=1 C=0.8495568162708291688
D_K=-1319 H=45 h=45 N=1 C=0.8651230358188882253
D_K=-1511 H=49 h=49 N=1 C=0.8741795137245637057
D_K=-1559 H=51 h=51 N=1 C=0.8943444427816406492
D_K=-5711 H=109 h=109 N=1 C=0.9359115563608913749
D_K=-10391 H=153 h=153 N=1 C=0.9452134463623177874
D_K=-15791 H=195 h=195 N=1 C=0.9569942159441953143
D_K=-18191 H=213 h=213 N=1 C=0.9670712843980043869
D_K=-31391 H=289 h=289 N=1 C=0.9719748099741753952
D_K=-38639 H=325 h=325 N=1 C=0.9750325810206494251
D_K=-366791 H=1121 h=1121 N=1 C=0.9753833631207420025
```

For $\varepsilon = 1.2$ one only finds a sequence of 7 solutions in the selected interval:

```
D_K=-3 H=1 h=1 N=1 C=0.5172818579717865581
D_K=-71 H=7 h=7 N=1 C=0.5424291630211148068
D_K=-191 H=13 h=13 N=1 C=0.5563182817283039269
D_K=-239 H=15 h=15 N=1 C=0.5611158267043577307
D_K=-311 H=19 h=19 N=1 C=0.6068723733265467012
D_K=-479 H=25 h=25 N=1 C=0.6162241303969013829
D_K=-1559 H=51 h=51 N=1 C=0.6192449976582213274
```

For $\varepsilon = 1.25$, the successive maxima obtained is:

```
D_K=-3 H=1 h=1 N=1 C=0.5032678828257016665
D_K=-311 H=19 h=19 N=1 C=0.5257488819694962654
D_K=-479 H=25 h=25 N=1 C=0.5281172008961688794
```

But for $\varepsilon = 1.3$ one gets an empty sequence in the interval $[3, 10^8]$ (i.e., the initialization $D_K = -3$, $H = 1$, $h = 1$, $N = 1$, $C = 0.4896335682000245806$). After some adjustments giving the limit value $\text{eps} \approx 1.2690005098$, we have the same property (still in the interval $[3, 10^8]$).

4.1.4. *Program II – Successive maxima of $\frac{H}{2^{N-1}(\sqrt{D})^\varepsilon}$ with variable ε .* Now

we intend to vary $\varepsilon \in]0, 1[$; the program computes first the list of all discriminants in the selected interval, with the values of N and H and builds the corresponding lists LD, LN, LH :

```
{LD=List;LN=List;LH=List;NL=0;bD=1;BD=10^6;
for(D=bD,BD,e2=valuation(D,2);d=D/2^e2;if(e2==1||e2>3||core(d)!=d
||e2==0&Mod(-d,4)!=1)||e2==2&Mod(-d,4)!=-1),next);NL=NL+1;
listput(LD,D);N=omega(D);listput(LN,N);H=qfbcassno(-D);listput(LH,H)}
```

Once we have built the lists LD, LN, LH of the NL discriminants $D \in [bD, BD]$, the numbers N of ramified primes and the class numbers H , respectively, one uses the following instruction with adequate choices of ε , which is very fast (the program only writes the solutions with $N > 1$, if any):

```
{for(k=0,100,eps=k*0.0005;print(eps);Cm=0;for(j=1,NL,D=LD[j];N=LN[j];H=LH[j];
C=H/(2^(N-1)*D^(eps/2)));if(C>Cm,Cm=C;if(N>1,h=H/2^(N-1));
print("D_K=",D," H=",H," h=",h," N=",N," C=",precision(C,1))))}
```

```
{for(k=0,100,eps=k*0.005;print(eps);Cm=0;for(j=1,NL,D=LD[j];N=LN[j];H=LH[j];
C=H/(2^(N-1)*D^(eps/2)));if(C>Cm,Cm=C;if(N>1,h=H/2^(N-1));
print("D_K=",D," H=",H," h=",h," N=",N," C=",precision(C,1))))}
```

and so on at will. To obtain the complete table for each ε , one drops the test on N :

```
{for(k=0,100,eps=k*0.005;print(eps);Cm=0;for(j=1,NL,D=LD[j];N=LN[j];H=LH[j];
C=H/(2^(N-1)*D^(eps/2));if(C>Cm,Cm=C;h=H/2^(N-1);
print("D_K=",D," H=",H," h=",h," N=",N," C=",precision(C,1))))}
```

Remark 4.1. When ε varies, the conjecture is fulfilled but the lists of successive maxima may be different as illustrated by the following excerpt:

eps=0.2

D_K=-61151	H=395	h=395	N=1	C=131.2069195353264980
D_K=-61871	H=399	h=399	N=1	C=132.3805503137288383
D_K=-63839	H=423	h=423	N=1	C=139.9045245218128567
D_K=-80471	H=455	h=455	N=1	C=147.0440503223027966
D_K=-87071	H=469	h=469	N=1	C=150.3784103142821144

eps=0.05

D_K=-61151	H=395	h=395	N=1	C=299.8727528609150616
D_K=-61871	H=399	h=399	N=1	C=302.8208103554958095
D_K=-63839	H=423	h=423	N=1	C=320.7843818658118495
D_K=-75479	H=429	h=429	N=1	C=323.9751045905628560
D_K=-80471	H=455	h=455	N=1	C=343.0602594282148802
D_K=-87071	H=469	h=469	N=1	C=352.9197837212165526

where for $\varepsilon = 0.05$ we obtain a supplementary maximum for the discriminant $D_K = -75479$.

4.2. **Successive maxima of $\frac{H}{(\sqrt{D})^\varepsilon}$.** When we suppress the genus factor 2^{N-1} , we obtain many successive maxima with composite discriminants. Before giving a more complete table in §4.2.3 and Appendix 9.2, we give some examples with various ε

4.2.1. *First examples with various ε .*

```
{eps=0.1;bd=1;BD=10^8;Cm=0;for(D=bd,BD,e2=valuation(D,2);d=D/2^e2;
if(e2==1||e2>3||core(d)!=d||e2==0&Mod(-d,4)!=1)||e2==2&Mod(-d,4)!=-1),
next);H=qfbclassno(-D);N=omega(D);h=H/2^(N-1);C=H/D^(eps/2);
if(C>Cm,Cm=C; \\ extrema test
if(N>1,print("D_K=",D," H=",H," h=",h," N=",N," C=",precision(C,1))))}
```

eps=0.1

D_K=-15	H=2	h=1	N=2	C=1.746728606032060139
D_K=-39	H=4	h=2	N=2	C=3.330479470669139602
D_K=-95	H=8	h=4	N=2	C=6.370944279257599436
D_K=-119	H=10	h=5	N=2	C=7.874493918573016203
D_K=-215	H=14	h=7	N=2	C=10.70301447267741998
D_K=-551	H=26	h=13	N=2	C=18.96338088337081395
D_K=-671	H=30	h=15	N=2	C=21.66631878049052680
D_K=-791	H=32	h=16	N=2	C=22.92140074278638768
D_K=-959	H=36	h=18	N=2	C=25.53945176699087213
D_K=-1679	H=52	h=26	N=2	C=35.87160500976424895
D_K=-1991	H=56	h=28	N=2	C=38.30314759825849873
D_K=-2159	H=60	h=30	N=2	C=40.87319792142757059
D_K=-2519	H=64	h=32	N=2	C=43.26319266328724219
D_K=-2831	H=68	h=34	N=2	C=45.69954953825021836
(...)				
D_K=-4199	H=88	h=22	N=3	C=57.98629737295230403
D_K=-4991	H=92	h=23	N=3	C=60.10055034028705556
(...)				
D_K=-201431	H=808	h=404	N=2	C=438.7377920576307146
D_K=-230879	H=824	h=206	N=3	C=444.3835663948799569
D_K=-239279	H=880	h=220	N=3	C=473.7371498813250101

```

D_K=-250631 H=892 h=223 N=3 C=479.0855974383289828
D_K=-259871 H=894 h=447 N=2 C=479.2913905823451286
D_K=-277031 H=918 h=459 N=2 C=490.5872564296189357
D_K=-277679 H=930 h=465 N=2 C=496.9421066431691019
D_K=-301391 H=974 h=487 N=2 C=518.3253419873442213
D_K=-307271 H=992 h=496 N=2 C=527.3944964507170297
D_K=-321551 H=1008 h=504 N=2 C=534.6850471107224307
D_K=-328319 H=1024 h=512 N=2 C=542.6067047266572173
D_K=-331679 H=1048 h=524 N=2 C=555.0414075965477244
D_K=-346631 H=1054 h=527 N=2 C=556.9897983860860616
(...)
D_K=-10000335791 H=224228 h=56057 N=3 C=70907.00047065937018
D_K=-10000430831 H=225362 h=112681 N=2 C=71265.56829114792618
D_K=-10000494479 H=230920 h=115460 N=2 C=73023.13519081427577
D_K=-10000950671 H=232228 h=58057 N=3 C=73436.59259212560310
D_K=-10001585879 H=234304 h=117152 N=2 C=74092.84302164243799
D_K=-10005286079 H=234340 h=58585 N=3 C=74102.85661222034478
D_K=-10005638639 H=236064 h=118032 N=2 C=74647.88735802392332
D_K=-10005946271 H=239138 h=119569 N=2 C=75619.82786456386412
D_K=-10014676559 H=239668 h=119834 N=2 C=75784.11884852224786
D_K=-10017718031 H=240986 h=120493 N=2 C=76199.71955113559173
D_K=-10042787999 H=244463 h=244463 N=1 C=77289.48656509497127
D_K=-10085398391 H=245575 h=245575 N=1 C=77624.62225923972404
(...)

```

Taking $\varepsilon = 0.20$ then $\varepsilon = 0.005$, we obtain, locally, the following variation of the results giving the successive maxima of $\frac{H}{D^{\varepsilon/2}}$:

```

eps=0.20                                eps =0.005
D_K=-34151 H=292 h=73 N=3 C=102.811    D_K=-34151 H=292 h=73 N=3 C=284.478
D_K=-34271 H=300 h=150 N=2 C=105.59     D_K=-34271 H=300 h=150 N=2 C=292.269
D_K=-45239 H=352 h=176 N=2 C=120.501    D_K=-45239 H=352 h=176 N=2 C=342.691
D_K=-56759 H=384 h=192 N=2 C=128.507    D_K=-48551 H=354 h=177 N=2 C=344.578
                                           D_K=-55319 H=376 h=94 N=3 C=365.873
                                           D_K=-56759 H=384 h=192 N=2 C=373.633
D_K=-60359 H=398 h=199 N=2 C=132.375    D_K=-60359 H=398 h=199 N=2 C=387.196
D_K=-65159 H=424 h=212 N=2 C=139.948    D_K=-65159 H=424 h=212 N=2 C=412.411

```

4.2.2. *Successive maxima of $\frac{H}{\sqrt{D}}$.* The computation of the successive maxima of $C = \frac{H}{\sqrt{D}}$, which corresponds to take the value $\varepsilon = 1$, gives the following (two days of computer):

```

{eps=1;Cm=0;bD=2;BD=10^15;for(D=bD,BD,e2=valuation(D,2);d=D/2^e2;
if(e2==1||e2>3||core(d)!=d||e2==0&Mod(-d,4)!=1)||e2==2&Mod(-d,4)!=-1),
next);H=qfbclassno(-D);C=H/D^(eps/2);if(C>Cm,Cm=C; \\ maxima test
print("D_K=", -D, " H=", H, " N=", omega(D), " C=", precision(C,1))}}

```

```

eps=1
D_K=-3 H=1 N=1 C=0.5773502691896257646
D_K=-23 H=3 N=1 C=0.6255432421712242881
D_K=-39 H=4 N=2 C=0.6405126152203485340
D_K=-47 H=5 N=1 C=0.7293249574894727793
D_K=-71 H=7 N=1 C=0.8307471607356973298
D_K=-119 H=10 N=2 C=0.9166984970282112951
D_K=-191 H=13 N=1 C=0.9406469868801481313
D_K=-215 H=14 N=2 C=0.9547920752586628931
D_K=-239 H=15 N=1 C=0.9702693410297263506
D_K=-311 H=19 N=1 C=1.0773911565351111182
D_K=-479 H=25 N=1 C=1.1422791559623846113
D_K=-671 H=30 N=2 C=1.1581371547232743925
D_K=-959 H=36 N=2 C=1.1625006306971278460
D_K=-1151 H=41 N=1 C=1.2084987204190831295
D_K=-1319 H=45 N=1 C=1.2390536631994988148
D_K=-1511 H=49 N=1 C=1.2605609458018050271

```

D_K=-1559	H=51	N=1	C=1.2916567519466084330
D_K=-2351	H=63	N=1	C=1.2993143498277182834
D_K=-2999	H=73	N=1	C=1.3330137440704153804
D_K=-3071	H=76	N=2	C=1.3714301220316550780
D_K=-5711	H=109	N=1	C=1.4423491995518049464
D_K=-6551	H=117	N=1	C=1.4455464850750731891
D_K=-8399	H=134	N=2	C=1.4621468997251127841
D_K=-10391	H=153	N=1	C=1.5009380187077378750
D_K=-13439	H=174	N=2	C=1.5009484314636654405
D_K=-13991	H=178	N=2	C=1.5048583556514348538
D_K=-14951	H=185	N=1	C=1.5129919163124548319
D_K=-15791	H=195	N=1	C=1.5517787293946206107
D_K=-18191	H=213	N=1	C=1.5792515670615642815
D_K=-31391	H=289	N=1	C=1.6311551336949379106
D_K=-38639	H=325	N=1	C=1.6533714037950259530
D_K=-45239	H=352	N=2	C=1.6549549083889909590
D_K=-63839	H=423	N=1	C=1.6741614199824292696
D_K=-88919	H=505	N=1	C=1.6935346748084002013
D_K=-95471	H=533	N=1	C=1.7250094288181671943
D_K=-118271	H=606	N=2	C=1.7621119228622864136
D_K=-201431	H=808	N=2	C=1.8003137828933471731
D_K=-331679	H=1048	N=2	C=1.8197104745663294562
D_K=-366791	H=1121	N=1	C=1.8509567763509635548
D_K=-514751	H=1347	N=1	C=1.8774526650006645342
D_K=-628319	H=1495	N=1	C=1.8860408558818552140
D_K=-701399	H=1581	N=1	C=1.8877709504119456186
D_K=-819719	H=1725	N=1	C=1.9052703045654453332
D_K=-890951	H=1818	N=2	C=1.9260473919033585268
D_K=-1238639	H=2150	N=2	C=1.9318174505729727404
D_K=-1339439	H=2262	N=2	C=1.9544795494902961276
D_K=-2155919	H=2968	N=2	C=2.0213786607241667950
D_K=-4305479	H=4227	N=1	C=2.0371429729011960035
D_K=-6077111	H=5092	N=2	C=2.0655694919245702532
D_K=-11915279	H=7206	N=2	C=2.0875753006426724498
D_K=-12537719	H=7457	N=1	C=2.1059830772702872018
D_K=-16368959	H=8565	N=1	C=2.1169803985995325238
D_K=-20357039	H=9632	N=2	C=2.1348097104711738560
D_K=-29858111	H=11885	N=1	C=2.1750438748925986011
D_K=-48796439	H=15195	N=1	C=2.1752372888815709557
D_K=-54694631	H=16208	N=2	C=2.1915787444351673315
D_K=-63434159	H=17546	N=2	C=2.2030103210049586306
D_K=-69671159	H=18455	N=1	C=2.2109938365608397427
D_K=-82636319	H=20299	N=1	C=2.2330022559566842564
D_K=-106105271	H=23077	N=1	C=2.2403242022818810857
D_K=-131486759	H=25817	N=1	C=2.2514627416787898558
D_K=-173540351	H=29674	N=2	C=2.2525573483717439197
D_K=-197317559	H=32062	N=2	C=2.2824840043420033386
D_K=-242230271	H=35666	N=2	C=2.2916073035266200847
D_K=-258363551	H=36865	N=1	C=2.2934993203392119001
D_K=-354544511	H=43212	N=3	C=2.2949275753614584471
D_K=-469058399	H=49913	N=1	C=2.3046247166517230619
D_K=-499769591	H=51733	N=1	C=2.3141033452776863919
D_K=-507138551	H=52152	N=2	C=2.3158352241187728477
D_K=-625378991	H=58028	N=3	C=2.3204165730228952904
D_K=-693126671	H=61506	N=2	C=2.3362062349259721311
D_K=-826771391	H=67332	N=2	C=2.3416867288431861497
D_K=-967603199	H=73018	N=2	C=2.3473685708435125091
D_K=-1001354639	H=74768	N=3	C=2.3627719510460623112
D_K=-1017054191	H=75498	N=2	C=2.3673550657766015599
(...)			

which is infinite.

4.2.3. Distribution of the values of N for the successive maxima of $\frac{H}{(\sqrt{D})^\varepsilon}$.

We give now the statistical distribution of the values of N for the successive maxima of $C = \frac{H}{(\sqrt{D})^\varepsilon}$ with $\varepsilon = 0.02$; see Appendix 9.2 for the whole table:

```
{eps=0.02; bD=1; BD=10^12; ND=0; N1=0; N2=0; N3=0; N4=0; N5=0; N6=0;
Cm=0; for(D=bD, BD, e2=valuation(D, 2); d=D/2^e2;
if(e2==1 || e2>3 || core(d)!=d || (e2==0&Mod(-d, 4)!=1) || (e2==2&Mod(-d, 4)!=-1),
next); ND=ND+1; H=qfbclassno(-D); N=omega(D); h=H/2^(N-1); C=H/D^(eps/2);
if(C>Cm, Cm=C; if(N==1, N1=N1+1); if(N==2, N2=N2+1); if(N==3, N3=N3+1);
if(N==4, N4=N4+1); if(N==5, N5=N5+1); if(N>=6, N6=N6+1); print();
print("D_K=", -D, " H=", H, " h=", h, " N=", N, " C=", precision(C, 1));
print("ND=", ND, " N1=", N1, " N2=", N2, " N3=", N3,
" N4=", N4, " N5=", N5, " N6=", N6))}
(...)
```

D_K=-4079 H=85 h=85 N=1 C=78.21920339356804770
ND=1241 N1=21 N2=17 N3=0 N4=0 N5=0 N6=0

D_K=-4199 H=88 h=22 N=3 C=80.95640481885140620
ND=1278 N1=21 N2=17 N3=1 N4=0 N5=0 N6=0
(...)

D_K=-2819879 H=3220 h=1610 N=2 C=2775.578833830112931
ND=857149 N1=101 N2=100 N3=16 N4=0 N5=0 N6=0

D_K=-2843231 H=3290 h=1645 N=2 C=2835.683632670836421
ND=864246 N1=101 N2=101 N3=16 N4=0 N5=0 N6=0

D_K=-2919479 H=3305 h=3305 N=1 C=2847.858521758496226
ND=887419 N1=102 N2=101 N3=16 N4=0 N5=0 N6=0

D_K=-2954591 H=3464 h=866 N=3 C=2984.509115202034352
ND=898089 N1=102 N2=101 N3=17 N4=0 N5=0 N6=0

D_K=-3068831 H=3485 h=3485 N=1 C=3001.463405484378354
ND=932809 N1=103 N2=101 N3=17 N4=0 N5=0 N6=0

D_K=-3190151 H=3593 h=3593 N=1 C=3093.279082652959006
ND=969674 N1=104 N2=101 N3=17 N4=0 N5=0 N6=0

D_K=-3312839 H=3632 h=1816 N=2 C=3125.675125147948027
ND=1006982 N1=104 N2=102 N3=17 N4=0 N5=0 N6=0

D_K=-3406751 H=3636 h=1818 N=2 C=3128.242922219554128
ND=1035530 N1=104 N2=103 N3=17 N4=0 N5=0 N6=0

D_K=-3511199 H=3640 h=1820 N=2 C=3130.738753931376549
ND=1067278 N1=104 N2=104 N3=17 N4=0 N5=0 N6=0

D_K=-3524351 H=3714 h=1857 N=2 C=3194.266213425503417
ND=1071274 N1=104 N2=105 N3=17 N4=0 N5=0 N6=0
(...)

D_K=-73396319 H=18648 h=9324 N=2 C=15558.779203424189371
ND=22309821 N1=168 N2=191 N3=34 N4=0 N5=0 N6=0

D_K=-74175191 H=18672 h=2334 N=4 C=15577.158967342971961
ND=22546587 N1=168 N2=191 N3=34 N4=1 N5=0 N6=0
(...)

The first crossing of the “curves” $N = 1$ and $N = 2$ ” occurs approximatively for $D = 2843231$. The first $N = 3$ is obtained for $D_K = -4199 = 13 \cdot 17 \cdot 19$.

The first $N = 4$ is obtained for $D_K = -74175191 = -41 \cdot 71 \cdot 83 \cdot 307$ (for $N_D = 22546587$).

4.3. **Successive minima of $\frac{H}{2^{N-1} \cdot (\sqrt{|D_K|})^\epsilon}$.** Without any comments, we give the program computing the *successive minima* of the function

$$c_\epsilon(K) := \frac{H}{2^{N-1} \cdot (\sqrt{|D_K|})^\epsilon}$$

up to $5 \cdot 10^7$, for imaginary quadratic fields.

From (2) (ii), the sequence $\frac{H}{(\sqrt{|D_K|})^\epsilon}$ shall be finite for $\epsilon = 1 + \eta$, $\eta > 0$, and infinite for $\epsilon < 1$ and a fortiori for $\frac{H}{2^{N-1} \cdot (\sqrt{|D_K|})^\epsilon}$ since N is unbounded. We

test the case $\epsilon = 1$:

```

{eps=1; bD=1; BD=10^8; cM=1.0; for(D=bD, BD, e=valuation(D, 2); d=D/2^e;
if(e==1 || e>3 || core(d)!=d || (e==0 & Mod(d, 4)!=-1) || (e==2 & Mod(d, 4)!=1),
next); N=omega(D); H=qfbclassno(-D); h=H/2^(N-1); c=h/(sqrt(D)^eps);
if(c<cM, cM=c; \\ minima test
print("D_K=", -D, " H=", H, " h=", h, " N=", N, " c=", precision(c, 1)))}
eps=1
D_K=-3      H=1      h=1      N=1      C=0.5773502691896257646
D_K=-4      H=1      h=1      N=1      C=0.5000000000000000000
D_K=-7      H=1      h=1      N=1      C=0.3779644730092272272
D_K=-8      H=1      h=1      N=1      C=0.3535533905932737622
D_K=-11     H=1      h=1      N=1      C=0.3015113445777636227
D_K=-15     H=2      h=1      N=2      C=0.2581988897471611257
D_K=-19     H=1      h=1      N=1      C=0.22941573387056176592
D_K=-20     H=2      h=1      N=2      C=0.22360679774997896964
D_K=-24     H=2      h=1      N=2      C=0.20412414523193150818
D_K=-35     H=2      h=1      N=2      C=0.16903085094570331551
D_K=-40     H=2      h=1      N=2      C=0.15811388300841896661
D_K=-43     H=1      h=1      N=1      C=0.15249857033260466633
D_K=-51     H=2      h=1      N=2      C=0.14002800840280098035
D_K=-52     H=2      h=1      N=2      C=0.13867504905630728051
D_K=-67     H=1      h=1      N=1      C=0.12216944435630522344
D_K=-84     H=4      h=1      N=3      C=0.10910894511799619063
D_K=-88     H=2      h=1      N=2      C=0.10660035817780521715
D_K=-91     H=2      h=1      N=2      C=0.10482848367219182958
D_K=-115    H=2      h=1      N=2      C=0.09325048082403137657
D_K=-120    H=4      h=1      N=3      C=0.09128709291752768558
D_K=-123    H=2      h=1      N=2      C=0.09016696346674322897
D_K=-132    H=4      h=1      N=3      C=0.08703882797784891909
D_K=-148    H=2      h=1      N=2      C=0.08219949365267864445
D_K=-163    H=1      h=1      N=1      C=0.07832604499879573413
D_K=-168    H=4      h=1      N=3      C=0.07715167498104595513
D_K=-187    H=2      h=1      N=2      C=0.07312724241271306502
D_K=-195    H=4      h=1      N=3      C=0.07161148740394328805
D_K=-228    H=4      h=1      N=3      C=0.06622661785325219033
D_K=-232    H=2      h=1      N=2      C=0.06565321642986127833
D_K=-235    H=2      h=1      N=2      C=0.06523280730534421982
D_K=-267    H=2      h=1      N=2      C=0.06119900613621045639
D_K=-280    H=4      h=1      N=3      C=0.05976143046671968200
D_K=-312    H=4      h=1      N=3      C=0.05661385170722978753
D_K=-340    H=4      h=1      N=3      C=0.05423261445466404300
D_K=-372    H=4      h=1      N=3      C=0.05184758473652126342
D_K=-403    H=2      h=1      N=2      C=0.04981354813867178958
D_K=-408    H=4      h=1      N=3      C=0.04950737714883371546
D_K=-420    H=8      h=1      N=4      C=0.04879500364742665897
D_K=-427    H=2      h=1      N=2      C=0.04839339184958272744
D_K=-435    H=4      h=1      N=3      C=0.04794633014853841427
D_K=-483    H=4      h=1      N=3      C=0.04550157551932900739
D_K=-520    H=4      h=1      N=3      C=0.04385290096535146074
D_K=-532    H=4      h=1      N=3      C=0.04335549847620599959
D_K=-555    H=4      h=1      N=3      C=0.04244763599780088816
D_K=-595    H=4      h=1      N=3      C=0.04099600308453938812

```

D_K=-627	H=4	h=1	N=3	C=0.03993615319154358665
D_K=-660	H=8	h=1	N=4	C=0.03892494720807614855
D_K=-708	H=4	h=1	N=3	C=0.03758230140014144273
D_K=-715	H=4	h=1	N=3	C=0.03739787960033828712
D_K=-760	H=4	h=1	N=3	C=0.03627381250550058360
D_K=-795	H=4	h=1	N=3	C=0.03546634510659543259
D_K=-840	H=8	h=1	N=4	C=0.03450327796711771089
D_K=-1012	H=4	h=1	N=3	C=0.03143473067309657382
D_K=-1092	H=8	h=1	N=4	C=0.030261376633440120992
D_K=-1155	H=8	h=1	N=4	C=0.029424494316824984322
D_K=-1320	H=8	h=1	N=4	C=0.027524094128159015140
D_K=-1380	H=8	h=1	N=4	C=0.026919095102908275320
D_K=-1428	H=8	h=1	N=4	C=0.026462806201248157256
D_K=-1435	H=4	h=1	N=3	C=0.026398183867422732984
D_K=-1540	H=8	h=1	N=4	C=0.025482359571881277232
D_K=-1848	H=8	h=1	N=4	C=0.023262105259961771688
D_K=-1995	H=8	h=1	N=4	C=0.022388683141982250148
D_K=-3003	H=8	h=1	N=4	C=0.018248296715045297629
D_K=-3315	H=8	h=1	N=4	C=0.017368336857296871107
D_K=-5460	H=16	h=1	N=5	C=0.013533299049019169228
D_K=-31395	H=32	h=2	N=5	C=0.011287551685861596711
D_K=-33915	H=32	h=2	N=5	C=0.010860106519162152420
D_K=-40755	H=32	h=2	N=5	C=0.009906940323254157974
D_K=-94395	H=48	h=3	N=5	C=0.009764426918800617707
D_K=-106260	H=96	h=3	N=6	C=0.009203146787097731965
D_K=-145860	H=96	h=3	N=6	C=0.007855125898384663556
D_K=-264180	H=128	h=4	N=6	C=0.007782336824316934453
D_K=-298452	H=128	h=4	N=6	C=0.007321882321555281918
D_K=-304980	H=128	h=4	N=6	C=0.007243097165890304815
D_K=-323323	H=64	h=4	N=5	C=0.007034636991172515786
D_K=-435435	H=128	h=4	N=6	C=0.006061754228857454068
D_K=-915915	H=160	h=5	N=6	C=0.005224472151550085263
D_K=-1141140	H=320	h=5	N=7	C=0.004680589348015776317
D_K=-1381380	H=320	h=5	N=7	C=0.004254156107243915719
D_K=-3598980	H=512	h=8	N=7	C=0.004216967659631715196
D_K=-4555915	H=256	h=8	N=6	C=0.003748022424187745904
D_K=-6561555	H=512	h=8	N=7	C=0.003123105105098544600
D_K=-8843835	H=576	h=9	N=7	C=0.003026371188649244116
D_K=-16041795	H=768	h=12	N=7	C=0.002996089378541778906
D_K=-17852835	H=768	h=12	N=7	C=0.002840060884694504730
D_K=-19399380	H=1536	h=12	N=8	C=0.002724503191342580514
D_K=-23483460	H=1536	h=12	N=8	C=0.002476282585134087042
D_K=-33093060	H=1664	h=13	N=8	C=0.002259825416804010321
D_K=-40562340	H=1536	h=12	N=8	C=0.001884168528832614627
(...)				

5. PROGRAMS AND TABLES FOR REAL QUADRATIC FIELDS

5.1. **Successive maxima of $\frac{H}{2^{N-1}(\sqrt{|D_K|})^\varepsilon}$ with various ε .** We have analogous properties for real quadratic fields and the reader may use the following program to test various situations; here H is the class number in the *restricted sense* to get $g_{K/\mathbb{Q}} = 2^{N-1}$:

```
{LD=List;LN=List;LH=List;NL=0;bd=2;BD=10^5;
for(D=bd,BD,e2=valuation(D,2);d=D/2^e2;if(e2==1||e2>3||
core(d)!=d||(e2==0 & Mod(d,4)!=1)||e2==2 & Mod(d,4)!=-1),next);
NL=NL+1;listput(LD,D);N=omega(D);listput(LN,N);P=x^2-D;
K=bnfinit(P,1);H=bnfnarrow(K)[1];listput(LH,H)}
```

As for the imaginary case, one then uses any interval of values of ε :

```
{for(k=0,100,eps=k*0.005;print(eps);Cm=0;for(j=1,NL,D=LD[j];N=LN[j];H=LH[j];
C=H/(2^(N-1)*(D^(eps/2)))));if(C>Cm,Cm=C;if(N>1,h=H/2^(N-1);
print("D_K=",D," H=",H," h=",h," N=",N," C=",precision(C,1))))}
```

```
{for(k=0,50,eps=k*0.01;print(eps);Cm=0;for(j=1,NL,D=LD[j];N=LN[j];H=LH[j];
C=H/(2^(N-1)*(D^(eps/2)));if(C>Cm,Cm=C;h=H/2^(N-1);
print("D_K=",D," H=",H," h=",h," N=",N," C=",precision(C,1))))}
```

When $\varepsilon < 0.42$, one finds only the exceptional counterexample of composite discriminant:

$$D_K = 136 \quad H = 4 \quad h = 2 \quad N = 2 \quad C = 0.730557694812702258$$

and no example with $N > 1$ for larger ε (in the selected interval):

```
eps=0.41
D_K=5      H=1      h=1      N=1      C=0.718970628883420458
D_K=136    H=4      h=2      N=2      C=0.730557694812702258
D_K=229    H=3      h=3      N=1      C=0.984815739249975242
D_K=401    H=5      h=5      N=1      C=1.463273164377664277
D_K=577    H=7      h=7      N=1      C=1.901327800418162251
D_K=1129   H=9      h=9      N=1      C=2.130297468803580754
D_K=1297   H=11     h=11     N=1      C=2.530696066538532698
D_K=7057   H=21     h=21     N=1      C=3.413910809035837466
D_K=8761   H=27     h=27     N=1      C=4.198945711923644210
D_K=14401  H=43     h=43     N=1      C=6.039457956003442814
D_K=41617  H=57     h=57     N=1      C=6.440570637045096139
D_K=57601  H=63     h=63     N=1      C=6.659665172069346510
D_K=90001  H=87     h=87     N=1      C=8.392639118660242231
D_K=176401 H=153    h=153    N=1      C=12.85756466018982938
D_K=650281 H=207    h=207    N=1      C=13.31326654631031378
D_K=921601 H=235    h=235    N=1      C=14.07137574672269499
(...)
```

```
eps=0.42
D_K=5      H=1      h=1      N=1      C=0.713208152908235133
D_K=229    H=3      h=3      N=1      C=0.958419857815418096
D_K=401    H=5      h=5      N=1      C=1.420069782057472125
D_K=577    H=7      h=7      N=1      C=1.841836686780795434
D_K=1129   H=9      h=9      N=1      C=2.056727626379498179
D_K=1297   H=11     h=11     N=1      C=2.441604350895807169
D_K=7057   H=21     h=21     N=1      C=3.265946529044727296
D_K=8761   H=27     h=27     N=1      C=4.012614990621811735
D_K=14401  H=43     h=43     N=1      C=5.757129506348588758
D_K=41617  H=57     h=57     N=1      C=6.107001038822202258
D_K=57601  H=63     h=63     N=1      C=6.304494128338600865
D_K=90001  H=87     h=87     N=1      C=7.927336842220217845
D_K=176401 H=153    h=153    N=1      C=12.10392466127156707
D_K=650281 H=207    h=207    N=1      C=12.45142685877934641
D_K=921601 H=235    h=235    N=1      C=13.13753377133558054
(...)
```

If $\varepsilon \rightarrow 0$, we find analogous sub-lists:

```
eps=0.000001
D_K=5      H=1      h=1      N=1      C=0.999999919528107616
D_K=136    H=4      h=2      N=2      C=1.999999508734571761
D_K=229    H=3      h=3      N=1      C=2.999999184941810187
D_K=401    H=5      h=5      N=1      C=4.999998501509867721
D_K=577    H=7      h=7      N=1      C=6.999997774755560416
D_K=1129   H=9      h=9      N=1      C=8.999996836911151974
D_K=1297   H=11     h=11     N=1      C=10.99999605770565506
D_K=7057   H=21     h=21     N=1      C=20.99999069513798489
D_K=8761   H=27     h=27     N=1      C=26.99998774461458228
D_K=14401  H=43     h=43     N=1      C=42.99997941364113356
D_K=32401  H=45     h=45     N=1      C=44.99997663163079519
D_K=41617  H=57     h=57     N=1      C=56.99996968665561316
D_K=57601  H=63     h=63     N=1      C=62.99996547192955773
D_K=90001  H=87     h=87     N=1      C=86.99995037705828937
D_K=176401 H=153    h=153    N=1      C=152.9999075840874610
```



```
D_K=650281 H=207 h=207 N=1 C=206.9998614636418471
D_K=921601 H=235 h=235 N=1 C=234.9998386271104724
(...)
```

For the real quadratic fields the conjecture may have only the exception $D_K = 136$ (with $N = 2$), but $\varepsilon = 0.0000001$ and we will now go to the limit. For a more complete table, see Table III, Appendix 9.3.

5.2. Successive maxima of H and $\frac{H}{2^{N-1}}$. In this subsection we put $\varepsilon = 0$ to study the successive maxima of H (with the the corresponding value of h), then the successive maxima of h :

Successive maxima of H

```
{Hm=0;bd=2;BD=10^9;for(D=bD,BD,e2=valuation(D,2);d=D/2^e2;
if(e2==1||e2>3|core(d)!=d|(e2==0&Mod(d,4)!=1)|(e2==2&Mod(d,4)!=-1),
next);P=x^2-D;K=bnfinit(P,1);H=bnfnarrow(K)[1];if(H>Hm,Hm=H;N=omega(D);
print("D_K=",D," H=",H," h=",H/2^(N-1)," N=",N))}
```

D_K=5	H=1	h=1	N=1
D_K=12	H=2	h=1	N=2
D_K=60	H=4	h=1	N=3
D_K=316	H=6	h=3	N=2
D_K=505	H=8	h=4	N=2
D_K=817	H=10	h=5	N=2
D_K=940	H=12	h=3	N=3
D_K=1596	H=16	h=2	N=4
D_K=3129	H=20	h=5	N=3
D_K=4009	H=22	h=11	N=2
D_K=4345	H=24	h=6	N=3
D_K=6097	H=28	h=7	N=3
D_K=11289	H=36	h=9	N=3
D_K=13321	H=52	h=13	N=3
D_K=17889	H=60	h=15	N=3
D_K=34689	H=64	h=16	N=3
D_K=44065	H=68	h=17	N=3
D_K=49321	H=92	h=23	N=3
D_K=54769	H=100	h=25	N=3
D_K=104017	H=104	h=26	N=3
D_K=108889	H=140	h=35	N=3
D_K=176401	H=153	h=153	N=1
D_K=181689	H=160	h=40	N=3
D_K=247921	H=240	h=60	N=3
D_K=318049	H=244	h=61	N=3
D_K=365521	H=252	h=63	N=3
D_K=381049	H=262	h=131	N=2
D_K=685561	H=268	h=67	N=3
D_K=697921	H=376	h=94	N=3
D_K=767449	H=392	h=196	N=2
D_K=1299505	H=448	h=112	N=3
D_K=1483321	H=456	h=114	N=3
D_K=1686481	H=474	h=237	N=2
D_K=1713961	H=478	h=239	N=2
D_K=1758289	H=492	h=123	N=3
D_K=2191249	H=504	h=126	N=3
D_K=2259961	H=596	h=149	N=3
D_K=2419321	H=600	h=150	N=3
D_K=2500681	H=612	h=306	N=2
D_K=2782641	H=616	h=154	N=3
D_K=2784721	H=632	h=158	N=3
D_K=3755521	H=752	h=47	N=5
D_K=3992041	H=780	h=195	N=3
D_K=4232449	H=880	h=220	N=3
D_K=5247841	H=984	h=246	N=3
D_K=5674321	H=1128	h=564	N=2
D_K=7064521	H=1180	h=295	N=3

(...)

We note the exceptional case of $D_K = 3755521 = 7 \cdot 11 \cdot 17 \cdot 19 \cdot 151$ for which $H = 752$, $h = 47$, $N = 5$.

Successive maxima of h

```
{hm=0;bd=2;BD=10^9;for(D=bd,BD,e2=valuation(D,2);d=D/2^e2;
if(e2==1||e2>3||core(d)!=d||(e2==0&Mod(d,4)!=1)||e2==2&Mod(d,4)!=-1),
next);P=x^2-D;K=bnfinit(P,1);H=bnfnarrow(K)[1];N=omega(D);h=H/2^(N-1);
if(h>hm,hm=h;print("D_K=",D," H=",H," h=",h," N=",N))}
D_K=5      H=1      h=1      N=1
D_K=136    H=4      h=2      N=2
D_K=229    H=3      h=3      N=1
D_K=401    H=5      h=5      N=1
D_K=577    H=7      h=7      N=1
D_K=1129   H=9      h=9      N=1
D_K=1297   H=11     h=11     N=1
D_K=7057   H=21     h=21     N=1
D_K=8761   H=27     h=27     N=1
D_K=14401  H=43     h=43     N=1
D_K=32401  H=45     h=45     N=1
D_K=41617  H=57     h=57     N=1
D_K=57601  H=63     h=63     N=1
D_K=90001  H=87     h=87     N=1
D_K=176401 H=153    h=153    N=1
D_K=650281 H=207    h=207    N=1
D_K=921601 H=235    h=235    N=1
D_K=1299601 H=357    h=357    N=1
D_K=2944657 H=377    h=377    N=1
D_K=3686401 H=455    h=455    N=1
D_K=3920401 H=575    h=575    N=1
D_K=7290001 H=655    h=655    N=1
(...)
```

6. CYCLIC CUBIC FIELDS – SUCCESSIVE MAXIMA OF $\frac{H}{3^{N-1} \cdot (\sqrt{D})^\varepsilon}$

The program computing the list of cyclic cubic fields with increasing conductors f (in f) is for instance given in [13, §6.1]. We then obtain the following program for the computation of the successive maxima of $\frac{H}{3^{N-1} \cdot f^\varepsilon}$ (using $\sqrt{D} = f$), where H is the class number in the ordinary sense and $N = \omega(f)$; this is equivalent to compute the successive maxima of the numbers $C_\varepsilon(D) = \sup_{K \in \mathcal{F}_{3,f}} \left(\frac{H}{3^{N-1} \cdot f^\varepsilon} \right)$ over increasing conductors.

When $N > 1$, we do not consider in this program the mean of the class numbers h and H for the 2^{N-1} fields of same conductor f and compute the maxima inside this set of fields $\mathcal{F}_{3,f}$; which shall give some defects of the method, but a forthcoming program, for degree p cyclic fields, $p \geq 3$ (see §7.1), shall compute the means $C_\varepsilon(\mathcal{F}_{3,f})$ of the $\frac{H}{2^{N-1} \cdot (\sqrt{D})^\varepsilon}$, $K \in \mathcal{F}_{3,f}$.

```
{eps=0.1;bf=7;Bf=10^8;Cm=0; \\ successive conductors:
for(f=bf,Bf,phi=eulerphi(f);if(Mod(phi,3)!=0,next);
e3=valuation(f,3);ff=f/3^e3;if(e3==1||e3>2||core(ff)!=ff,next);
\\ test of q|ff congruent to 1 (mod 3):
F=factor(ff);Div=component(F,1);S=0;dF=matsize(F)[1];
for(j=1,dF,q=Div[j];if(Mod(q,3)!=1,S=1;break));
\\ list of degree 3 polynomials of conductor f:
if(S==0,N=omega(f); \\ f is the conductor to be put as (a^2+27*b^2)/4
for(b=1,sqrt(4*f/27),if(e3==2 & Mod(b,3)==0,next);
A=4*f-27*b^2;if(issquare(A,&a)==1, \\ expression of a defining polynomial:
if(e3==0,if(Mod(a,3)==1,a=-a);P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(e3==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);
K=bnfinit(P,1);H=K.no;h=H/3^(N-1);C=h/f^eps; if(C>Cm,Cm=C;
print("f_K=",f," H=",H," h=",h," N=",N," C=",precision(C,1))))))}
```

```

eps=0.1
f=7      H=1      h=1      N=1      C=0.823171253993044237
f=163    H=4      h=4      N=1      C=2.403483975167433545
f=313    H=7      h=7      N=1      C=3.940429903816599872
f=1063   H=13     h=13     N=1      C=6.475749278819724722
f=1489   H=19     h=19     N=1      C=9.150906778915694477
f=2763   H=63     h=21     N=2      C=9.507825584310405539
f=4219   H=28     h=28     N=1      C=12.15170110906513844
f=4867   H=228    h=76     N=2      C=32.51527696336010219
f=11491  H=109    h=109    N=1      C=42.79477185917101752
f=15013  H=127    h=127    N=1      C=48.54639357681429304
f=23173  H=349    h=349    N=1      C=127.7400849965886920
f=74419  H=688    h=688    N=1      C=224.0887506125577062
f=165889 H=2352   h=784    N=2      C=235.6862811297153681
f=191413 H=961    h=961    N=1      C=284.7909895844608044
f=241117 H=1216   h=1216   N=1      C=352.1362960501321708
f=294313 H=2107   h=2107   N=1      C=598.1134808281728111
f=348457 H=3391   h=3391   N=1      C=946.4831145084927291
f=888313 H=3484   h=3484   N=1      C=3436.604704785608734
f=1066063 H=4087   h=4087   N=1      C=4030.666398029350791
f=1074607 H=4681   h=4681   N=1      C=4616.442084136778108
f=1339813 H=4819   h=4819   N=1      C=4751.490684118511988
f=1543813 H=6223   h=6223   N=1      C=6134.952524706830718
f=1632013 H=6277   h=6277   N=1      C=6187.844694562183552
f=1653919 H=15307  h=15307  N=1      C=15089.38598754532585
f=3046471 H=43216  h=43216  N=1      C=42575.59929298335085
f=6575641 H=46108  h=46108  N=1      C=45389.80807553668707
(...)

```

For the conductor $f = 165889 = 19 \cdot 8731$, we have the following information for the two cubic fields of conductor f , which perhaps explains the exception and suggests the method that we will perform in the next section:

```

x^3+x^2-55296*x+3809303  H=3      h=1      N=2      C=0.300620256543004
x^3+x^2-55296*x-1996812  H=2352   h=784    N=2      C=235.6862811297153

```

Indeed, the program considers the large local maximum $H = 2352$ while the other field has class number 3.

The same program for $p = 3$, computing the mean values over the sets of fields K having the same conductor f , would be the following (the new list obtained does not contain data with $N = 2$):

```

{eps=0.1;bf=7;Bf=10^8;Cm=0; \\ successive conductors:
for(f=bf,Bf,phi=eulerphi(f);if(Mod(phi,3)!=0,next);
e3=valuation(f,3);ff=f/3^e3;if(e3==1||e3>2||core(ff)!=ff,next);
\\ test of q|ff congruent to 1 (mod p):
F=factor(ff);Div=component(F,1);S=0;dF=matsize(F)[1];
for(j=1,dF,q=Div[j];if(Mod(q,3)!=1,S=1;break));
\\ list of degree 3 polynomials of conductor f:
if(S==0,N=omega(f);HH=1;hh=1;CC=1.0;
for(b=1,sqrt(4*f/27),if(e3==2 & Mod(b,3)==0,next);
A=4*f-27*b^2;if(issquare(A,&a)==1,
if(e3==0,if(Mod(a,3)==1,a=-a);P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(e3==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);
K=bnfinit(P,1);D=K.disc;H=K.no;h=H/3^(N-1);C=h/f^eps;
HH=HH*H;hh=hh*h;CC=CC*C));nK=2^(N-1);
C=CC^(1/nK); \\ mean value of CC and extrema test:
if(C>Cm,Cm=C;H=HH^(1/nK);h=hh^(1/nK); \\ mean values of H,h
print("f=",factor(f)," P=",P," H=",precision(H,1),
" h=",precision(h,1)," N=",N," C=",precision(C,1))}}

```

One verifies that this program gives the same results and is faster than the one we will give in the next subsection for any $p \geq 3$.

7. GENERAL APPROACH OF THE DEGREE p CYCLIC CASE, $p > 2$

For these degree p cyclic fields K (thus real), except the cubic case, a defining polynomial of K corresponding to increasing conductors f is only possible, via PARI/GP, by using `polsubcyclo(f, p)`, and selecting the polynomials P giving the conductor f .

Remark 7.1. We know that $f = (p^2)^\delta \cdot q_1 \dots q_n$, for primes $q_i \equiv 1 \pmod{p}$ and $\delta \in \{0, 1\}$. If $N = n + \delta$ is the number of ramified primes, there are $\frac{p^N - 1}{p - 1}$ fields whose conductor f' divides f and $(p - 1)^{N-1}$ fields K of exact conductor f , a priori indistinguishable. So there are various methods to build a “mean value” (in the multiplicative sense) for the computations of successive maxima of C_ε .

Let $K(f)$ be the compositum of the degree p cyclic fields of conductors $f' \mid f$; then $K(f)$ is the subfield of $\mathbb{Q}(\mu_f)$ fixed by Γ^p , where $\Gamma = \text{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q})$. Considering that the set of rational characters χ of order p of Γ defines the set of degree p cyclic subfields $K_\chi = K^{\text{Ker}(\chi)}$ of $K(f)$, we get that, roughly speaking, the class number of $K(f)$ is equal to the product of the class numbers of the K_χ (this is true for the non- p -parts), which gives a serious reason to take into account, for any family of numbers $(X_K)_{K \in \mathcal{F}'_{p,f}}$, the

standard mean value $\left[\prod_{K \in \mathcal{F}'_{p,f}} X_K \right]^{\frac{1}{\#\mathcal{F}'_{p,f}}}$, or the analogous one replacing $\mathcal{F}'_{p,f}$ by $\mathcal{F}_{p,f}$ (cf. Conjecture 3.1).

In the programs, `p` is the prime considered, `factor(f)` gives the set of ramified primes, `H` is a defining polynomial of the field, `H` its class number in the ordinary sense, `h := \frac{H}{p^{N-1}}` the non-genus part, and `C = C_\varepsilon(D)`. The programs compute the two previous multiplicative mean values of `H`, `h` and `C`, taking into account the two following remarks:

(i) We consider, first, only the fields of exact conductor f ; so in that case the number N (then the genus number) is the same for all these fields and makes sense in the results; but if a local maximum is reached by a composite conductor, the mean values of `H` and `h` can be non-integers. Otherwise, if the maximum is reached by a prime conductor, one may write the corresponding polynomial which is unique.

(ii) Then we consider all the fields whose conductor f' divides f ; in that case N and P depend on each field and the program gives only the mean values of `H`, `h` and `C`.

7.1. Fields of conductor f : mean value of $\frac{H}{p^{N-1}(\sqrt{D})^\varepsilon}$. Thus $\sqrt{D} = f^{\frac{p-1}{2}}$; since PARI/GP gives the $\frac{p^N - 1}{p - 1}$ polynomials (defining all the fields of degree p contained in $\mathbb{Q}(\mu_f)$) in the vector `V = Mat(polsubcyclo(f, p))`, we must test, by means of the discriminants, using the function `nfdisc`, if the field defined by P is of exact conductor f .

The results show that one always get $N = 1$ and for instance, for $p = 3$, the conductor $165889 = 19 \cdot 8731$ does not appear as a maximum. Since all results are with prime conductors, we write the corresponding (unique) polynomial; this makes sense only in that case:

```
{eps=0.01;p=3;Cm=0; \ \ successive conductors:
for(f=7,10^6,phi=eulerphi(f);if(Mod(phi,p)!=0,next);
ep=valuation(f,p);ff=f/p^ep;if(ep==1||ep>2||core(ff)!=ff,next);
\ \ test of q|ff congruent to 1 (mod p):
F=factor(ff);Div=component(F,1);S=0;dF=matsize(F)[1];
for(j=1,dF,q=Div[j];if(Mod(q,p)!=1,S=1;break));
\ \ list of degree p polynomials in the fth cyclotomic field:
```

```

if(S==0,V=Mat(polsubcyclo(f,p));dV=matsize(V)[2];HH=1;hh=1;CC=1.0;
for(k=1,dV,P=component(V,k)[1];D=nfdisc(P);N=omega(f);
\\ test conductor(K)=f:
if(omega(D)==N,K=bnfinit(P,1);H=K.no;h=H/p^(N-1);C=h/D^(eps/2);
HH=HH*H;hh=hh*h;CC=CC*C);nK=(p-1)^(N-1);
C=CC^(1/nK); \\ mean value of CC and extrema test:
if(C>Cm,Cm=C;H=HH^(1/nK);h=hh^(1/nK));\\ mean values of H,h
print("f=",factor(f)," P=",P," H=",precision(H,1),
" h=",precision(h,1)," N=",N," C=",precision(C,1))))}

```

```

p=3
f=[7,1] P=x^3+x^2-2*x-1 H=1 h=1 N=1 C=0.9807290047229
f=[163,1] P=x^3+x^2-54*x-169 H=4 h=4 N=1 C=3.8013522515881
f=[313,1] P=x^3+x^2-104*x+371 H=7 h=7 N=1 C=6.6091041630333
f=[1063,1] P=x^3+x^2-354*x+2441 H=13 h=13 N=1 C=12.124895929808
f=[1489,1] P=x^3+x^2-496*x+4081 H=19 h=19 N=1 C=17.661380777478
f=[4219,1] P=x^3+x^2-1406*x-625 H=28 h=28 N=1 C=25.757632116142
f=[5119,1] P=x^3+x^2-1706*x+26543 H=31 h=31 N=1 C=28.462290210252
f=[7351,1] P=x^3+x^2-2450*x-1089 H=49 h=49 N=1 C=44.826271355692
f=[9511,1] P=x^3+x^2-3170*x+65168 H=73 h=73 N=1 C=66.610178628175
f=[11491,1] P=x^3+x^2-3830*x-89800 H=109 h=109 N=1 C=99.271119537133
f=[15013,1] P=x^3+x^2-5004*x+134561 H=127 h=127 N=1 C=115.35569659944
f=[23173,1] P=x^3+x^2-7724*x-258336 H=349 h=349 N=1 C=315.62805883284
f=[74419,1] P=x^3+x^2-24806*x-11025 H=688 h=688 N=1 C=614.99502127609
f=[191413,1] P=x^3+x^2-63804*x+6181931 H=961 h=961 N=1 C=850.94927649437
f=[241117,1] P=x^3+x^2-80372*x-35721 H=1216 h=1216 N=1 C=1074.2646788161
f=[294313,1] P=x^3+x^2-98104*x+11794321 H=2107 h=2107 N=1 C=1857.7036788965
f=[348457,1] P=x^3+x^2-116152*x-15190144 H=3391 h=3391 N=1 C=2984.7385706495
(...)

```

Same program with $\varepsilon = 0.4$:

```

f=[7,1] P=x^3+x^2-2*x-1 H=1 h=1 N=1 C=0.4591565499594
f=[163,1] P=x^3+x^2-54*x-169 H=4 h=4 N=1 C=0.5214167154550
f=[313,1] P=x^3+x^2-104*x+371 H=7 h=7 N=1 C=0.7028785742753
f=[1063,1] P=x^3+x^2-354*x+2441 H=13 h=13 N=1 C=0.8004423281900
f=[1489,1] P=x^3+x^2-496*x+4081 H=19 h=19 N=1 C=1.0223408675783
f=[7351,1] P=x^3+x^2-2450*x-1089 H=49 h=49 N=1 C=1.3920585783307
f=[9511,1] P=x^3+x^2-3170*x+65168 H=73 h=73 N=1 C=1.8708216514284
f=[11491,1] P=x^3+x^2-3830*x-89800 H=109 h=109 N=1 C=2.5899022210956
f=[15013,1] P=x^3+x^2-5004*x+134561 H=127 h=127 N=1 C=2.7115444434617
f=[23173,1] P=x^3+x^2-7724*x-258336 H=349 h=349 N=1 C=6.2637226912587
f=[74419,1] P=x^3+x^2-24806*x-11025 H=688 h=688 N=1 C=7.7431006805904
f=[241117,1] P=x^3+x^2-80372*x-35721 H=1216 h=1216 N=1 C=8.5515028017316
f=[294313,1] P=x^3+x^2-98104*x+11794321 H=2107 h=2107 N=1 C=13.681721863115
f=[348457,1] P=x^3+x^2-116152*x-15190144 H=3391 h=3391 N=1 C=20.581077709540
(...)

```

7.2. Fields of conductors $f' | f$: mean value of $\frac{H}{p^{N-1}(\sqrt{D})^\varepsilon}$. The program is the same as the previous one without testing the conductors.

The mean value C_ε (in C) is computed over all the $C_\varepsilon(D) = \frac{H}{p^{N-1}(\sqrt{D})^\varepsilon}$.

If H and h are the corresponding mean values, we will see that $H = h$ for the successive maxima of these mean values in the selected intervals, which indicates that the maxima are reached for prime conductors (i.e., $dV = 1$):

```

{eps=0.02;p=3;Cm=0; \\ successive conductors:
for(f=7,10^6,phi=eulerphi(f);if(Mod(phi,p)!=0,next);
ep=valuation(f,p);ff=f/p^ep;if(ep==1||ep>2||core(ff)!=ff,next);
\\ test of q|ff congruent to 1 (mod p):
F=factor(ff);Div=component(F,1);S=0;dF=matsize(F)[1];
for(j=1,dF,q=Div[j];if(Mod(q,p)!=1,S=1;break));
\\ list of degree p polynomials in the fth cyclotomic field:
if(S==0,V=Mat(polsubcyclo(f,p));dV=matsize(V)[2];HH=1;hh=1;CC=1;
for(k=1,dV,P=component(V,k)[1];K=bnfinit(P,1);

```

```

D=K.disc;N=omega(D);H=K.no;h=H/p^(N-1);C=h/D^(eps/2);
HH=HH*H;hh=hh*h;CC=CC*C);
C=CC^(1/dV); \\ mean value of CC and extrema test:
if(C>Cm,Cm=C;H=HH^(1/dV);h=hh^(1/dV); \\ mean values of H,h
print("f=",factor(f)," H=",precision(H,1),
" h=",precision(h,1)," C=",precision(C,1))))}
p=3
f=[7,1]      H=1      h=1      C=0.961829380704799441
f=[163,1]    H=4      h=4      C=3.612569735163622387
f=[313,1]    H=7      h=7      C=6.240036833974918259
f=[1063,1]   H=13     h=13     C=11.30870010066782027
f=[1489,1]   H=19     h=19     C=16.41707215616319439*
f=[4219,1]   H=28     h=28     C=23.69484329394764450***
f=[5119,1]   H=31     h=31     C=26.13232141976207791***
f=[7351,1]   H=49     h=49     C=41.00805313579896863*
f=[9511,1]   H=73     h=73     C=60.77966982023849296
f=[11491,1]  H=109    h=109    C=90.41059792803446483
f=[15013,1]  H=127    h=127    C=104.7790294326226146
f=[23173,1]  H=349    h=349    C=285.4471963397972172
f=[74419,1]  H=688    h=688    C=549.7367386546242885
f=[191413,1] H=961    h=961    C=753.5012186954210082
f=[241117,1] H=1216   h=1216   C=949.0498356513206700
f=[294313,1] H=2107   h=2107   C=1637.903634829544508
f=[348457,1] H=3391   h=3391   C=2627.149612245101090
(...)
```

Same program wit h $\varepsilon = 0.4$:

```

f=[7,1]      H=1      h=1      C=0.459156549959434092
f=[163,1]    H=4      h=4      C=0.521416715455083946
f=[313,1]    H=7      h=7      C=0.702878574275380209
f=[1063,1]   H=13     h=13     C=0.800442328190010161
f=[1489,1]   H=19     h=19     C=1.022340867578304918*
f=[7351,1]   H=49     h=49     C=1.392058578330795015*
f=[9511,1]   H=73     h=73     C=1.870821651428443619
f=[11491,1]  H=109    h=109    C=2.589902221095671686
f=[15013,1]  H=127    h=127    C=2.711544443461731055
f=[23173,1]  H=349    h=349    C=6.263722691258787681
f=[74419,1]  H=688    h=688    C=7.743100680590427658
f=[241117,1] H=1216   h=1216   C=8.551502801731692495
f=[294313,1] H=2107   h=2107   C=13.68172186311554962
f=[348457,1] H=3391   h=3391   C=20.58107770954063231
(...)
```

We remark an example of different lists according to ε (cf. 1489 and 7351).

7.3. Successive maxima of the mean value of $\frac{H}{(\sqrt{D})^\varepsilon}$. In that case, the parameters N , P do not make sense for non-prime f .

7.3.1. $G \simeq \mathbb{Z}/p\mathbb{Z}$. Fields of conductor f and mean value of $\frac{H}{(\sqrt{D})^\varepsilon}$.

```

{eps=0.02;p=3;Cm=0; \\ successive conductors:
for(f=1,10^6,phi=eulerphi(f);if(Mod(phi,p)!=0,next);
ep=valuation(f,p);ff=f/p^ep;if(ep==1||ep>2||core(ff)!=ff,next);
\\ test of q|ff congruent to 1 (mod p):
F=factor(ff);Div=component(F,1);S=0;dF=matsize(F)[1];
for(j=1,dF,q=Div[j];if(Mod(q,p)!=1,S=1;break));
\\ list of degree p polynomials in the fth cyclotomic field:
if(S==0,V=Mat(polsubcyclo(f,p));dV=matsize(V)[2];HH=1;hh=1;CC=1;
for(k=1,dV,P=component(V,k)[1];D=nfdisc(P);N=omega(f);
\\ test of ramification:
if(omega(D)==N,K=bnfinit(P,1);H=K.no;C=H/D^(eps/2);
HH=HH*H;CC=CC*C);nK=(p-1)^(N-1);
C=CC^(1/nK); \\ mean value of CC and extrema test:
if(C>Cm,Cm=C;H=HH^(1/nK); \\ mean of H
print("f=",factor(f)," H=",precision(H,1)," C=",precision(C,1))))}
```

```

p=3
f=[7,1]          H=1      C=0.961829380704799441
f=[3,2; 7,1]    H=3.0    C=2.761432575434841409
f=[163,1]       H=4      C=3.612569735163622387
f=[313,1]       H=7      C=6.240036833974918259
f=[3,2; 73,1]   H=9.0    C=7.904805747328513653
f=[1063,1]      H=13     C=11.30870010066782027
f=[7,1; 181,1]  H=18.0   C=15.60331847344150489
f=[1489,1]      H=19     C=16.41707215616319439
f=[3,2; 307,1]  H=23.811761799581315314
                  C=20.32188136664178736
f=[4219,1]      H=28     C=23.69484329394764450
f=[5119,1]      H=31     C=26.13232141976207791
f=[7351,1]      H=49     C=41.00805313579896863
f=[9511,1]      H=73     C=60.77966982023849296
f=[11491,1]     H=109    C=90.41059792803446483
f=[15013,1]     H=127    C=104.7790294326226146
f=[23173,1]     H=349    C=285.4471963397972172
f=[74419,1]     H=688    C=549.7367386546242885
f=[191413,1]    H=961    C=753.5012186954210082
f=[241117,1]    H=1216   C=949.0498356513206700
f=[294313,1]    H=2107   C=1637.903634829544508
f=[348457,1]    H=3391   C=2627.149612245101090
(...)

```

Same program with $\varepsilon = 0.2$:

```

f=[7,1]          H=1      C=0.677610913400480949
f=[3,1; 7,1]    H=3      C=1.309945251235621361
f=[163,1]       H=4      C=1.444183804721662077
f=[313,1]       H=7      C=2.218141118127442645
f=[3,1; 73,1]   H=9      C=2.458835278103877640
f=[1063,1]      H=13     C=3.225794517087244988
f=[7,1; 181,1]  H=18     C=4.312381578109574420
f=[1489,1]      H=19     C=4.407320782968695309
f=[3,1; 307,1]  H=23.811761799581315314
                  C=4.881073557098234152
f=[4219,1]      H=28     C=5.273708565859104133
f=[5119,1]      H=31     C=5.617261834937099453
f=[7351,1]      H=49     C=8.258987246521752304
f=[9511,1]      H=73     C=11.68631595303996494
f=[11491,1]     H=109    C=16.80176604108711584
f=[15013,1]     H=127    C=18.55710495523587440
f=[23173,1]     H=349    C=46.75509832359800418
f=[74419,1]     H=688    C=72.98803510333878161
f=[191413,1]    H=961    C=84.39740660613575690
f=[241117,1]    H=1216   C=101.9736603584756001
f=[294313,1]    H=2107   C=169.7863008772629543
f=[348457,1]    H=3391   C=264.1787927011785785
(...)

```

As expected, there are some composite conductors giving successive maxima.

7.3.2. $G \simeq \mathbb{Z}/p\mathbb{Z}$. Fields of conductors $f' \mid f$ and mean value of $\frac{H}{(\sqrt{D})^\varepsilon}$.

```

{eps=0.02;p=3;Cm=0;
for(f=7,10^6,phi=eulerphi(f);if(Mod(phi,p)!=0,next);
ep=valuation(f,p);ff=f/p^ep;if(ep==1||ep>2||core(ff)!=ff,next);
F=factor(ff);Div=component(F,1);S=0;dF=matsize(F)[1];
for(j=1,dF,q=Div[j];if(Mod(q,p)!=1,S=1;break));
if(S==0,V=Mat(polsubcyclo(f,p));dV=matsize(V)[2];HH=1;CC=1;
for(k=1,dV,P=component(V,k)[1];K=bnfinit(P,1);
D=K.disc;H=K.no;C=H/D^(eps/2);HH=HH*H;CC=CC*C);
C=CC^(1/dV); \\ mean value of CC and extrema test:
if(C>Cm,Cm=C;H=HH^(1/dV);
print("f=",factor(f)," H=",precision(H,1)," C=",precision(C,1))))}

```

p=3

f=[7,1]	H=1	C=0.961829380704799441
f=[3,2; 7,1]	H=1.7320508075688772936	C=1.627685591700590660
f=[163,1]	H=4	C=3.612569735163622387
f=[313,1]	H=7	C=6.240036833974918259
f=[1063,1]	H=13	C=11.30870010066782027
f=[1489,1]	H=19	C=16.41707215616319439
f=[4219,1]	H=28	C=23.69484329394764450
f=[5119,1]	H=31	C=26.13232141976207791
f=[7351,1]	H=49	C=41.00805313579896863
f=[9511,1]	H=73	C=60.77966982023849296
f=[11491,1]	H=109	C=90.41059792803446483
f=[15013,1]	H=127	C=104.7790294326226146
f=[23173,1]	H=349	C=285.4471963397972172
f=[74419,1]	H=688	C=549.7367386546242885
f=[191413,1]	H=961	C=753.5012186954210082
f=[241117,1]	H=1216	C=949.0498356513206700
f=[294313,1]	H=2107	C=1637.903634829544508
f=[348457,1]	H=3391	C=2627.149612245101090
(...)		

Same program with $\varepsilon = 0.2$:

f=[7,1]	H=1	C=0.677610913400480949
f=[3,1; 7,1]	H=1.7320508075688772936	C=0.930378241131994665
f=[163,1]	H=4	C=1.444183804721662077
f=[313,1]	H=7	C=2.218141118127442645
f=[1063,1]	H=13	C=3.225794517087244988
f=[1489,1]	H=19	C=4.407320782968695309
f=[4219,1]	H=28	C=5.273708565859104133
f=[5119,1]	H=31	C=5.617261834937099453
f=[7351,1]	H=49	C=8.258987246521752304
f=[9511,1]	H=73	C=11.68631595303996494
f=[11491,1]	H=109	C=16.80176604108711584
f=[15013,1]	H=127	C=18.55710495523587440
f=[23173,1]	H=349	C=46.75509832359800418
f=[74419,1]	H=688	C=72.98803510333878161
f=[191413,1]	H=961	C=84.39740660613575690
f=[241117,1]	H=1216	C=101.9736603584756001
f=[294313,1]	H=2107	C=169.7863008772629543
f=[348457,1]	H=3391	C=264.1787927011785785

7.4. **Analogous tables for $p = 5$.** The computations need more time but we obtain similar results. The polynomial P makes sense because $N = 1$ in all the successive maxima given by the program.

7.4.1. $G \simeq \mathbb{Z}/p\mathbb{Z}$. Fields of conductor f and mean value of $\frac{H}{p^{N-1}(\sqrt{D})^\varepsilon}$.

p=5 eps=0.01

f=[11,1]	P=x^5+x^4-4*x^3-3*x^2+3*x+1	H=1	h=1	N=1	C=0.953173909654138686
f=[191,1]	P=x^5+x^4-76*x^3-359*x^2-437*x-155	H=11	h=11	N=1	C=9.903119457621479703
f=[941,1]	P=x^5+x^4-376*x^3+3877*x^2-13445*x+15271	H=16	h=16	N=1	C=13.95237660139343057
f=[3931,1]	P=x^5+x^4-1572*x^3+36637*x^2-309637*x+899921	H=256	h=256	N=1	C=216.9451294678943811
f=[5051,1]	P=x^5+x^4-2020*x^3-55965*x^2-558681*x-1920311	H=1451	h=1451	N=1	C=1223.488425196746388
f=[80251,1]	P=x^5+x^4-32100*x^3-3617715*x^2-151594781*x-2241837391	H=37631	h=37631	N=1	C=30023.18416657978783
f=[154291,1]	P=x^5+x^4-61716*x^3+9584557*x^2-555677185*x+11397220121	H=108691	h=108691	N=1	C=85590.73829767367053

$f=[349211,1]$ $P=x^5+x^4-139684x^3-32923613x^2-2897797577x-90346698359$
 $H=186091$ $h=186091$ $N=1$ $C=144166.2531422279504$
 $f=[555671,1]$ $P=x^5+x^4-222268x^3+65858127x^2-7299765465x+286894118701$
 $H=721151$ $h=721151$ $N=1$ $C=553515.5874018525546$
 $f=[641491,1]$ $P=x^5+x^4-256596x^3-82033869x^2-9804476597x-415497361495$
 $H=1566401$ $h=1566401$ $N=1$ $C=1198834.245498212652$
 $f=[1464901,1]$ $P=x^5+x^4-585960x^3+282608701x^2-51039799381x+3272126756879$
 $H=4628591$ $h=4628591$ $N=1$ $C=3484437.195260249301$

$p=5$ $\text{eps}=0.400$

$f=[11,1]$ $P=x^5+x^4-4x^3-3x^2+3x+1$
 $H=1$ $h=1$ $N=1$ $C=0.146854024200198000$
 $f=[191,1]$ $P=x^5+x^4-76x^3-359x^2-437x-155$
 $H=11$ $h=11$ $N=1$ $C=0.164651613663662086$
 $f=[3931,1]$ $P=x^5+x^4-1572x^3+36637x^2-309637x+899921$
 $H=256$ $h=256$ $N=1$ $C=0.340908264873088935$
 $f=[5051,1]$ $P=x^5+x^4-2020x^3-55965x^2-558681x-1920311$
 $H=1451$ $h=1451$ $N=1$ $C=1.581122495380098063$
 $f=[80251,1]$ $P=x^5+x^4-32100x^3-3617715x^2-151594781x-2241837391$
 $H=37631$ $h=37631$ $N=1$ $C=4.487303013765233752$
 $f=[154291,1]$ $P=x^5+x^4-61716x^3+9584557x^2-555677185x+11397220121$
 $H=108691$ $h=108691$ $N=1$ $C=7.682827980583928749$
 $f=[349211,1]$ $P=x^5+x^4-139684x^3-32923613x^2-2897797577x-90346698359$
 $H=186091$ $h=186091$ $N=1$ $C=144166.2531422279504$
 $f=[555671,1]$ $P=x^5+x^4-222268x^3+65858127x^2-7299765465x+286894118701$
 $H=721151$ $h=721151$ $N=1$ $C=553515.5874018525546$
 $f=[641491,1]$ $P=x^5+x^4-256596x^3-82033869x^2-9804476597x-415497361495$
 $H=1566401$ $h=1566401$ $N=1$ $C=1198834.2454982126520$
 $f=[1464901,1]$ $P=x^5+x^4-585960x^3+282608701x^2-51039799381x+3272126756879$
 $H=4628591$ $h=4628591$ $N=1$ $C=3484437.195260249301$

7.4.2. $G \simeq \mathbb{Z}/p\mathbb{Z}$. Fields of conductors $f' \mid f$ and mean value of $\frac{H}{p^{N-1}(\sqrt{D})^\varepsilon}$.

$p=5$ $\text{eps}=0.02$

$f=[11,1]$ $H=1$ $h=1$ $C=0.908540502045356139$
 $f=[191,1]$ $H=11$ $h=11$ $C=8.915615908356468213$
 $f=[941,1]$ $H=16$ $h=16$ $C=12.16680080169443102$
 $f=[3931,1]$ $H=256$ $h=256$ $C=183.8483953118806834$
 $f=[5051,1]$ $H=1451$ $h=1451$ $C=1031.649846030609568$
 $f=[80251,1]$ $H=37631$ $h=37631$ $C=23953.43167867894208$
 $f=[154291,1]$ $H=108691$ $h=108691$ $C=67400.00995796213464$
 (...)

$p=5$ $\text{eps}=0.40$

$f=[11,1]$ $H=1$ $h=1$ $C=0.146854024200198000$
 $f=[191,1]$ $H=11$ $h=11$ $C=0.164651613663662086$
 $f=[3931,1]$ $H=256$ $h=256$ $C=0.340908264873088935$
 $f=[5051,1]$ $H=1451$ $h=1451$ $C=1.581122495380098063$
 $f=[80251,1]$ $H=37631$ $h=37631$ $C=4.487303013765233752$
 $f=[154291,1]$ $H=108691$ $h=108691$ $C=7.682827980583928749$
 (...)

7.4.3. $G \simeq \mathbb{Z}/p\mathbb{Z}$. Fields of conductors $f' \mid f$ and mean value of $\frac{H}{(\sqrt{D})^\varepsilon}$.

$p=5$ $\text{eps}=0.020$

$f=[11,1]$ $H=1$ $C=0.908540502045356139$
 $f=[191,1]$ $H=11$ $C=8.915615908356468213$
 $f=[941,1]$ $H=16$ $C=12.16680080169443102$
 $f=[3931,1]$ $H=256$ $C=183.8483953118806834$
 $f=[5051,1]$ $H=1451$ $C=1031.649846030609568$
 $f=[80251,1]$ $H=37631$ $C=23953.43167867894208$
 $f=[154291,1]$ $H=108691$ $C=67400.00995796213464$
 (...)

$p=5$ $\text{eps}=0.200$

$f=[11,1]$ $H=1$ $C=0.383215375735627288$
 $f=[191,1]$ $H=11$ $C=1.345796325712134995$
 $f=[3931,1]$ $H=256$ $C=9.341976011931885039$

```
f=[5051,1]    H=1451    C=47.89789912716968310
f=[80251,1]   H=37631   C=410.9278521967080188
f=[154291,1]  H=108691  C=913.8130312255608229
(...)
```

8. OTHER REAL OR IMAGINARY ABELIAN FIELDS

We will try to illustrate the Conjecture 1.3 about the nature of G and especially about the influence of the number of generators of G . As we have explained in the introduction, we have chosen to classify by increasing conductors instead of discriminants. The programs eliminate the integers f which are not conductors for trivial reasons, then test that the current polynomial P of degree $d = \#G$ (given by `polsubcyclo(f,d)`) defines a field K of exact conductor f ; for this, we consider the base field \mathbb{Q} defined via `Q = bnfinit(y - 1, 1)` and the programs compute the conductor of the (abelian) extension defined by P using `rnfconductor(Q,P)`.

For this, we will consider for $p = 2, 3$, $G \simeq \mathbb{Z}/4\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$, $(\mathbb{Z}/2\mathbb{Z})^3$, $(\mathbb{Z}/3\mathbb{Z})^2$, then $G \simeq \mathbb{Z}/6\mathbb{Z}$. In an heuristic point of view, we must observe that the successive maxima, for the non-genus parts, are obtained for almost all conductors f fulfilling the inequalities $r(G) \leq \omega(f) \leq R(G)$. The number of fields K fulfilling the conditions (Galois group, conductor, signature, etc.) is denoted `nK` in the programs, for the computation of the means.

8.1. Imaginary quartic cyclic fields of conductor f . The program uses `G1 = polgalois(polcyclo(5))` to have the group $\mathbb{Z}/4\mathbb{Z}$ and test the Galois groups of order 4 given by `PARI/GP` via `polgalois(P)`.

8.1.1. $G \simeq \mathbb{Z}/4\mathbb{Z}$. *Successive maxima of the mean of $\frac{H}{g_{K/\mathbb{Q}}(\sqrt{D})^\varepsilon}$.*

```
{eps=0.01;Cm=0;G1=polgalois(polcyclo(5));Q=bnfinit(y-1,1);
\\ successive conductors:
for(f=1,10^8,phi=eulerphi(f);if(Mod(phi,4)!=0,next);
f2=valuation(f,2);ff=f/(2^f2);if(f2==1||f2>4||core(ff)!=ff,next);
\\ list of degree 4 polynomials in the fth cyclotomic field:
V=Mat(polsubcyclo(f,4));dV=matsize(V)[2];nK=0;HH=1;hh=1;CC=1.0;
for(k=1,dV,P=component(V,k)[1];D=nfdisc(P);
\\ test of ramification and Galois group:
if(omega(D)==omega(f) & polgalois(P)==G1,fK=rnfconductor(Q,P);
if(component(fK[1][1],1)[1]==f,K=bnfinit(P,1); \\ test fK=f
if(K.sign[2]!=0, \\ test K imaginary
\\ computation of the genus number:
Df=factor(f);df=matsize(Df)[1];Div=component(Df,1);gK=1;for(j=1,df,
q=Div[j];F=idealfactor(K,q);eq=component(F,1)[1][3];gK=gK*eq);gK=gK/4;
nK=nK+1;H=K.no;h=H/gK;C=h/D^(eps/2);HH=HH*H;hh=hh*h;CC=CC*C));
\\ mean of CC and extrema test:
if(nK!=0,C=CC^(1/nK);if(C>Cm,Cm=C;H=HH^(1/nK);h=hh^(1/nK); \\ mean of H,h
print("f=",f,"=",Df," H=",precision(H,1),
" C=",precision(C,1)," omega(f)=",omega(f)," nK=",nK))})}
```

f=5=[5,1]	H=1	C=0.976147508055066522	omega(f)=1	nK=1
f=51=[3,1; 17,1]	H=10	C=4.739604006983370970	omega(f)=2	nK=1
f=109=[109,1]	H=17	C=15.84482780824856713	omega(f)=1	nK=1
f=123=[3,1; 41,1]	H=34	C=15.90325237458455595	omega(f)=2	nK=1
f=181=[181,1]	H=25	C=23.12463199169500313	omega(f)=1	nK=1
f=229=[229,1]	H=51	C=47.00809414205433725	omega(f)=1	nK=1
f=511=[7,1; 73,1]	H=226	C=103.9148512284847603	omega(f)=2	nK=1
f=723=[3,1; 241,1]	H=244	C=111.1369214637977656	omega(f)=2	nK=1
f=733=[733,1]	H=135	C=122.2805008461487052	omega(f)=1	nK=1
f=829=[829,1]	H=145	C=131.0960735251626679	omega(f)=1	nK=1
f=1093=[1093,1]	H=925	C=832.8416392829036284	omega(f)=1	nK=1
f=2029=[2029,1]	H=1183	C=1055.298969187637205	omega(f)=1	nK=1

f=3027=[3,1; 1009,1]	H=9100	C=4056.783364201576760	omega(f)=2	nK=1
f=4516=[2,2; 1129,1]	H=10404	C=4616.994304602346489	omega(f)=2	nK=1
f=7573=[7573,1]	H=7569	C=6619.870380619097221	omega(f)=1	nK=1
f=10376=[2,3; 1297,1]	H=21692	C=9539.917095634037343	omega(f)=2	nK=1
f=10957=[10957,1]	H=14119	C=12280.28831693695852	omega(f)=1	nK=1
f=13271=[23,1; 577,1]	H=52780	C=23249.01055510957600	omega(f)=2	nK=1
f=14267=[11,1; 1297,1]	H=107470	C=47113.91854137649729	omega(f)=2	nK=1
f=31333=[31333,1]	H=69331	C=59359.11736777416773	omega(f)=1	nK=1
f=40967=[71,1; 577,1]	H=201600	C=87807.23696987217193	omega(f)=2	nK=1
f=43387=[43,1; 1009,1]	H=205254	C=89098.12848060980410	omega(f)=2	nK=1
f=48851=[11,1; 4441,1]	H=208090	C=89556.08009624690970	omega(f)=2	nK=1
f=54987=[3,1; 18329,1]	H=258990	C=110543.8804978991659	omega(f)=2	nK=1
f=59053=[59053,1]	H=257125	C=218059.8330367149523	omega(f)=1	nK=1
f=61327=[7,1; 8761,1]	H=1721952	C=736888.4677682894188	omega(f)=2	nK=1
f=150031=[7,1; 21433,1]	H=3365766	C=1421139.460131018697	omega(f)=2	nK=1
f=197341=[197341,1]	H=1927305	C=1605174.264796961130	omega(f)=1	nK=1
f=245029=[245029,1]	H=2147103	C=1782438.755437137458	omega(f)=1	nK=1
f=267707=[11,1; 24337,1]	H=4458500	C=1870470.412521304054	omega(f)=2	nK=1
f=319611=[3,1; 106537,1]	H=4964300	C=2063689.257767956675	omega(f)=2	nK=1
f=356411=[11,1; 32401,1]	H=12376170	C=5169921.722394828539	omega(f)=2	nK=1
(...)				

To get the table for real fields, one has to replace the test $\text{if}(K.\text{sign}[2]! = 0$ by $\text{if}(K.\text{sign}[2] == 0$, which gives the short excerpt:

f=15=[3,1; 5,1]	H=1	C=0.482741062227355455	omega(f)=2	nK=1
f=16=[2,4]	H=1	C=0.962594443101751380	omega(f)=1	nK=1
f=212=[2,2; 53,1]	H=5	C=2.323032419483186330	omega(f)=2	nK=1
f=257=[257,1]	H=3	C=2.760401519960496457	omega(f)=1	nK=1
f=401=[401,1]	H=5	C=4.570069858094963666	omega(f)=1	nK=1
f=577=[577,1]	H=7	C=6.363270756545588592	omega(f)=1	nK=1
f=916=[2,2; 229,1]	H=15	C=6.817782326293629939	omega(f)=2	nK=1
f=1129=[1129,1]	H=9	C=8.099386061052115192	omega(f)=1	nK=1
f=1297=[1297,1]	H=11	C=9.878672447143713655	omega(f)=1	nK=1
f=1697=[1697,1]	H=17	C=15.20560462285605392	omega(f)=1	nK=1
f=2005=[5,1; 401,1]	H=80	C=35.97684919154074598	omega(f)=2	nK=1
f=4616=[2,3; 577,1]	H=112	C=49.85853230401655734	omega(f)=2	nK=1
f=7753=[7753,1]	H=75	C=65.57211873846320180	omega(f)=1	nK=1
f=8116=[2,2; 2029,1]	H=175	C=76.98005594903680110	omega(f)=2	nK=1
f=10376=[2,3; 1297,1]	H=286	C=125.7798400032885248	omega(f)=2	nK=1
f=24337=[24337,1]	H=185	C=158.9928970375704956	omega(f)=1	nK=1
f=25096=[2,3; 3137,1]	H=1152	C=499.9699077725989672	omega(f)=2	nK=1
f=54772=[2,2; 13693,1]	H=1755	C=750.2034311196041105	omega(f)=2	nK=1
f=56456=[2,3; 7057,1]	H=4074	C=1746.750303432222082	omega(f)=2	nK=1
f=115208=[2,3; 14401,1]	H=4472	C=1896.989860579581230	omega(f)=2	nK=1
(...)				

8.1.2. $G \simeq \mathbb{Z}/4\mathbb{Z}$. *Successive maxima of the mean of $\frac{H}{(\sqrt{D})^\epsilon}$.* Considering the same program for the successive maxima of the whole class numbers H , the number $\omega(f)$ of primes dividing f tends to be larger in a statistical point of view (for instance some $\omega(f) = 3$):

```
{eps=0.1;Cm=0;G1=polgalois(polcyclo(5));Q=bnfinit(y-1,1);
\\ successive conductors:
for(f=1,10^6,phi=eulerphi(f);if(Mod(phi,4)!=0,next);
f2=valuation(f,2);ff=f/(2^f2);if(f2==1||f2>4||core(ff)!=ff,next);
\\ list of degree 4 polynomials in the fth cyclotomic field:
V=Mat(polsubcyclo(f,4));dV=matsize(V)[2];nK=0;HH=1;CC=1.0;
for(k=1,dV,P=component(V,k)[1];D=nfdisc(P);
\\ test of ramification and Galois group:
if(omega(D)==omega(f) & polgalois(P)==G1,fK=rnfconductor(Q,P);
if(component(fK[1][1],1)[1]==f,K=bnfinit(P,1); \\ test fK=f
if(K.sign[2]!=0, \\ test K imaginary
nK=nK+1;H=K.no;C=H/abs(D)^(eps/2);HH=HH*H;CC=CC*C))));
```

```

\\ mean of CC and extrema test:
if(nK!=0,C=CC^(1/nK);if(C>Cm,Cm=C;H=HH^(1/nK); \\ mean of H
print("f=",f,"=",factor(f)," H=",precision(H,1),
" C=",precision(C,1)," omega(f)=",omega(f)," nK=",nK)))}

f=5=[5,1] H=1 C=0.785515030231764334 omega(f)=1 nK=1
f=40=[2,3; 5,1] H=2 C=1.276072931359182746 omega(f)=2 nK=1
f=51=[3,1; 17,1] H=10 C=5.857611124383549536 omega(f)=2 nK=1
f=109=[109,1] H=17 C=8.410754307693637111 omega(f)=1 nK=1
f=123=[3,1; 41,1] H=34 C=17.45216109112128972 omega(f)=2 nK=1
f=187=[11,1; 17,1] H=34 C=17.48929528433495368 omega(f)=2 nK=1
f=229=[229,1] H=51 C=22.57328962224578600 omega(f)=1 nK=1
f=291=[3,1; 97,1] H=68 C=30.67474966040826406 omega(f)=2 nK=1
f=411=[3,1; 137,1] H=74 C=31.69651460480555020 omega(f)=2 nK=1
f=435=[3,1; 5,1; 29,1] H=115.37764081484765738
C=49.00093067514680456 omega(f)=3 nK=2
f=511=[7,1; 73,1] H=226 C=97.74627279797258465 omega(f)=2 nK=1
f=1015=[5,1; 7,1; 29,1] H=326.3372488699382291
C=127.3361291154895528 omega(f)=3 nK=2
f=1093=[1093,1] H=925 C=323.8535846044137371 omega(f)=1 nK=1
f=1604=[2,2; 401,1] H=1060 C=375.5157960003103599 omega(f)=2 nK=1
f=1799=[7,1; 257,1] H=1398 C=500.6190813114782884 omega(f)=2 nK=1
f=3027=[3,1; 1009,1] H=9100 C=2888.987028762488449 omega(f)=2 nK=1
f=4516=[2,2; 1129,1] H=10404 C=3155.660350206402243 omega(f)=2 nK=1
f=8655=[3,1; 5,1; 577,1] H=15064 C=4427.461660644001969 omega(f)=3 nK=1
f=10376=[2,3; 1297,1] H=21692 C=6012.424012890724163 omega(f)=2 nK=1
f=12419=[11,1; 1129,1] H=24858 C=6814.324702892086861 omega(f)=2 nK=1
f=13271=[23,1; 577,1] H=52780 C=14863.47066425293806 omega(f)=2 nK=1
f=14267=[11,1; 1297,1] H=107470 C=28854.06354818241425 omega(f)=2 nK=1
f=40967=[71,1; 577,1] H=201600 C=50721.05997730998500 omega(f)=2 nK=1
f=54987=[3,1; 18329,1] H=258990 C=53223.08625892298533 omega(f)=2 nK=1
f=61327=[7,1; 8761,1] H=1721952 C=363185.2356760567566 omega(f)=2 nK=1
f=150031=[7,1; 21433,1] H=3365766 C=620742.7707399591228 omega(f)=2 nK=1
f=267707=[11,1; 24337,1] H=4458500 C=771097.7682083982741 omega(f)=2 nK=1
f=319611=[3,1; 106537,1] H=4964300 C=783467.2078602062662 omega(f)=2 nK=1
f=331223=[23,1; 14401,1] H=4510700 C=783993.1400401131476 omega(f)=2 nK=1
f=356411=[11,1; 32401,1] H=12376170 C=2050516.345595339414 omega(f)=2 nK=1
(...)

```

8.2. **Imaginary biquadratic fields of conductor f .** For non-cyclic fields, except for $K = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$, the conductor is necessarily such that $\omega(f) \geq 2$, and the “exceptional” cases should give $\omega(f) = 3, 4, \dots$

We put $G2 = \text{polgalois}(\text{polcyclo}(12))$ to have the group $(\mathbb{Z}/2\mathbb{Z})^2$:

8.2.1. $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. *Successive maxima of the mean of $\frac{H}{g_{K/\mathbb{Q}} \cdot (\sqrt{D})^\epsilon}$.*

```

{eps=0.01;Cm=0;G2=polgalois(polcyclo(12));Q=bnfinit(y-1,1);
\\ successive conductors:
for(f=1,10^6,phi=eulerphi(f);if(Mod(phi,4)!=0,next);
f2=valuation(f,2);ff=f/(2^f2);if(f2==1||f2>4||core(ff)!=ff,next);
\\ list of degree 4 polynomials in the fth cyclotomic field:
V=Mat(polsubcyclo(f,4));dV=matsize(V)[2];nK=0;HH=1;hh=1;CC=1.0;
for(k=1,dV,P=component(V,k)[1];D=nfdisc(P);
\\ test of ramification and Galois group:
if(omega(D)==omega(f) & polgalois(P)==G2,fK=rnfconductor(Q,P);
if(component(fK[1][1],1)[1]==f,K=bnfinit(P,1); \\ test fK=f
if(K.sign[2]!=0, \\ test K imaginary
\\ computation of the genus number:
Df=factor(f);df=matsize(Df)[1];Div=component(Df,1);gK=1;for(j=1,df,
q=Div[j];F=idealfactor(K,q);eq=component(F,1)[1][3];gK=gK*eq);gK=gK/4;
nK=nK+1;H=K.no;h=H/gK;C=h/abs(D)^(eps/2);HH=HH*H;hh=hh*h;CC=CC*C));
\\ mean of CC and extrema test:
if(nK!=0,C=CC^(1/nK);if(C>Cm,Cm=C;H=HH^(1/nK);h=hh^(1/nK); \\ mean of H,h

```

```
print("f=",f,"=",Df," H=",precision(H,1),
      " C=",precision(C,1)," omega(f)=",omega(f)," nK=",nK))})
```

f=8=[2,3]	H=1	C=0.972654947412285518	omega(f)=1	nK=1
f=12=[2,2; 3,1]	H=1	C=0.975457130078795936	omega(f)=2	nK=1
f=39=[3,1; 13,1]	H=2	C=1.928054694106823799	omega(f)=2	nK=1
f=69=[3,1; 23,1]	H=3	C=2.875628398112372534	omega(f)=2	nK=1
f=95=[5,1; 19,1]	H=4	C=3.821930235323273249	omega(f)=2	nK=1
f=119=[7,1; 17,1]	H=5	C=4.766663944449055461	omega(f)=2	nK=1
f=155=[5,1; 31,1]	H=6	C=5.704898650119474394	omega(f)=2	nK=1
f=213=[3,1; 71,1]	H=7	C=6.63459235562253982	omega(f)=2	nK=1
f=295=[5,1; 59,1]	H=12	C=11.33660528868662035	omega(f)=2	nK=1
f=316=[2,2; 79,1]	H=15	C=14.16101517671179429	omega(f)=2	nK=1
f=391=[17,1; 23,1]	H=21	C=19.78324489680824008	omega(f)=2	nK=1
f=527=[17,1; 31,1]	H=27	C=25.35979029775623778	omega(f)=2	nK=1
f=655=[5,1; 131,1]	H=30	C=28.11634357939838355	omega(f)=2	nK=1
f=695=[5,1; 139,1]	H=36	C=33.71961852330659701	omega(f)=2	nK=1
f=755=[5,1; 151,1]	H=42	C=39.30699295336691860	omega(f)=2	nK=1
f=1055=[5,1; 211,1]	H=54	C=50.36875720346152355	omega(f)=2	nK=1
f=1195=[5,1; 239,1]	H=60	C=55.89559343232499964	omega(f)=2	nK=1
f=1207=[17,1; 71,1]	H=63	C=58.68450919759415447	omega(f)=2	nK=1
f=1343=[17,1; 79,1]	H=85	C=79.09302129521269893	omega(f)=2	nK=1
f=1751=[17,1; 103,1]	H=120	C=111.3649136315469349	omega(f)=2	nK=1
f=2155=[5,1; 431,1]	H=126	C=116.6906535951326993	omega(f)=2	nK=1
f=2159=[17,1; 127,1]	H=150	C=138.9148686560359562	omega(f)=2	nK=1
f=2567=[17,1; 151,1]	H=154	C=142.3726150732244520	omega(f)=2	nK=1
f=3239=[41,1; 79,1]	H=175	C=161.4113014454594570	omega(f)=2	nK=1
f=3247=[17,1; 191,1]	H=208	C=191.8441285757209858	omega(f)=2	nK=1
f=3791=[17,1; 223,1]	H=238	C=219.1741945424862956	omega(f)=2	nK=1
f=4471=[17,1; 263,1]	H=286	C=262.9432257122401507	omega(f)=2	nK=1
f=5095=[5,1; 1019,1]	H=312	C=286.4726411861354221	omega(f)=2	nK=1
f=5287=[17,1; 311,1]	H=323	C=296.4629523337229678	omega(f)=2	nK=1
f=6071=[13,1; 467,1]	H=336	C=307.9687675618586813	omega(f)=2	nK=1
f=6155=[5,1; 1231,1]	H=378	C=346.4172575739003240	omega(f)=2	nK=1
f=6239=[17,1; 367,1]	H=405	C=371.1110393200093025	omega(f)=2	nK=1
f=6995=[5,1; 1399,1]	H=432	C=395.3992758795738976	omega(f)=2	nK=1
f=7295=[5,1; 1459,1]	H=440	C=402.5524030522415731	omega(f)=2	nK=1
f=7463=[17,1; 439,1]	H=495	C=452.7683540573952683	omega(f)=2	nK=1
f=7895=[5,1; 1579,1]	H=504	C=460.7411639385067668	omega(f)=2	nK=1
f=8551=[17,1; 503,1]	H=525	C=479.5557849789485247	omega(f)=2	nK=1
f=9655=[5,1; 1931,1]	H=546	C=498.1327779099005440	omega(f)=2	nK=1
f=9895=[5,1; 1979,1]	H=644	C=587.3969800430742345	omega(f)=2	nK=1
f=10183=[17,1; 599,1]	H=800	C=729.4763720843714494	omega(f)=2	nK=1
f=12359=[17,1; 727,1]	H=884	C=804.5118246192972826	omega(f)=2	nK=1
f=12431=[31,1; 401,1]	H=1155	C=1051.082786075149247	omega(f)=2	nK=1
f=13271=[23,1; 577,1]	H=1386	C=1260.474876339380179	omega(f)=2	nK=1
f=17663=[17,1; 1039,1]	H=1426	C=1293.149930556092194	omega(f)=2	nK=1
f=22559=[17,1; 1327,1]	H=1560	C=1411.209202757567127	omega(f)=2	nK=1
f=24463=[17,1; 1439,1]	H=1833	C=1656.827780210867060	omega(f)=2	nK=1
f=24599=[17,1; 1447,1]	H=1978	C=1787.792507027724221	omega(f)=2	nK=1
f=27119=[47,1; 577,1]	H=3045	C=2749.505294205692715	omega(f)=2	nK=1
f=29831=[23,1; 1297,1]	H=3432	C=3095.997498110343389	omega(f)=2	nK=1
f=40967=[71,1; 577,1]	H=4116	C=3701.272193141451084	omega(f)=2	nK=1
f=47423=[47,1; 1009,1]	H=4235	C=3802.712781426917957	omega(f)=2	nK=1
f=53063=[47,1; 1129,1]	H=5760	C=5166.239854556097149	omega(f)=2	nK=1
f=60959=[47,1; 1297,1]	H=9130	C=8177.497107751319985	omega(f)=2	nK=1
f=80159=[71,1; 1129,1]	H=10710	C=9566.431099491350923	omega(f)=2	nK=1
f=96359=[167,1; 577,1]	H=14784	C=13181.14176742287956	omega(f)=2	nK=1
f=137903=[239,1; 577,1]	H=22050	C=19589.02663140455669	omega(f)=2	nK=1
f=151751=[263,1; 577,1]	H=26754	C=23745.28612625392000	omega(f)=2	nK=1
f=207143=[359,1; 577,1]	H=34713	C=30713.50960296288906	omega(f)=2	nK=1
f=216599=[167,1; 1297,1]	H=37268	C=32959.41709369729731	omega(f)=2	nK=1
f=248687=[431,1; 577,1]	H=43071	C=38038.94157342156453	omega(f)=2	nK=1

f=313799=[311,1; 1009,1] H=58121 C=51211.38983446847804 omega(f)=2 nK=1
 f=351119=[311,1; 1129,1] H=83106 C=73143.85340462042777 omega(f)=2 nK=1
 (...)

8.2.2. $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Successive maxima of the mean of $\frac{H}{(\sqrt{D})^\varepsilon}$. Same computations for the successive maxima of the whole class number; we observe similar results as for the cyclic case with few $\omega(D) = 3$:

```
{eps=0.01;Cm=0;G2=polgalois(polcyclo(12));Q=bnfinit(y-1,1);
\\ successive conductors:
for(f=1,10^6,phi=eulerphi(f);if(Mod(phi,4)!=0,next);
f2=valuation(f,2);ff=f/(2^f2);if(f2==1||f2>3||core(ff)!=ff,next);
\\ list of degree 4 polynomials in the fth cyclotomic field:
V=Mat(polsubcyclo(f,4));dV=matsize(V)[2];nK=0;HH=1;hh=1;CC=1.0;
for(k=1,dV,P=component(V,k)[1];D=nfdisc(P);
\\ test of ramification and Galois group:
if(omega(D)==omega(f) & polgalois(P)==G2,fK=rnfconductor(Q,P);
if(component(fK[1][1],1)[1]==f,K=bnfinit(P,1); \\ test fK=f
if(K.sign[2]!=0, \\ test K imaginary
nK=nK+1;H=K.no;C=H/D^(eps/2);HH=HH*H;CC=CC*C));)
\\ mean of CC and extrema test:
if(nK!=0,C=CC^(1/nK);if(C>Cm,Cm=C;H=HH^(1/nK); \\ mean of H
print("f=",f,"=",factor(f)," H=",precision(H,1),
" C=",precision(C,1)," omega(f)=",omega(f)," nK=",nK)))}
```

f=8=[2,3]	H=1	C=0.972654947412285518	omega(f)=1	nK=1
f=12=[2,2; 3,1]	H=1	C=0.975457130078795936	omega(f)=2	nK=1
f=24=[2,3; 3,1]	H=1.4142135623730950488	C=1.365236011752337206	omega(f)=2	nK=4
f=39=[3,1; 13,1]	H=2	C=1.928054694106823799	omega(f)=2	nK=1
f=56=[2,3; 7,1]	H=2.378414230005442134	C=2.276671989463074733	omega(f)=2	nK=4
f=69=[3,1; 23,1]	H=3	C=2.875628398112372534	omega(f)=2	nK=1
f=95=[5,1; 19,1]	H=4	C=3.821930235323273249	omega(f)=2	nK=1
f=119=[7,1; 17,1]	H=5	C=4.766663944449055461	omega(f)=2	nK=1
f=136=[2,3; 17,1]	H=5.656854249492380195	C=5.357742969863248032	omega(f)=2	nK=4
f=155=[5,1; 31,1]	H=6	C=5.704898650119474394	omega(f)=2	nK=1
f=213=[3,1; 71,1]	H=7	C=6.63459235562253982	omega(f)=2	nK=1
f=255=[3,1; 5,1; 17,1]	H=9.118028227819110568	C=8.626517954608540449	omega(f)=3	nK=4
f=295=[5,1; 59,1]	H=12	C=11.33660528868662035	omega(f)=2	nK=1
f=316=[2,2; 79,1]	H=15	C=14.16101517671179429	omega(f)=2	nK=1
f=391=[17,1; 23,1]	H=21	C=19.78324489680824008	omega(f)=2	nK=1
f=527=[17,1; 31,1]	H=27	C=25.35979029775623778	omega(f)=2	nK=1
f=655=[5,1; 131,1]	H=30	C=28.11634357939838355	omega(f)=2	nK=1
f=695=[5,1; 139,1]	H=36	C=33.71961852330659701	omega(f)=2	nK=1
f=755=[5,1; 151,1]	H=42	C=39.30699295336691860	omega(f)=2	nK=1
f=1055=[5,1; 211,1]	H=54	C=50.36875720346152355	omega(f)=2	nK=1
f=1195=[5,1; 239,1]	H=60	C=55.89559343232499964	omega(f)=2	nK=1
f=1207=[17,1; 71,1]	H=63	C=58.68450919759415447	omega(f)=2	nK=1
f=1343=[17,1; 79,1]	H=85	C=79.09302129521269893	omega(f)=2	nK=1
f=1751=[17,1; 103,1]	H=120	C=111.3649136315469349	omega(f)=2	nK=1
f=2155=[5,1; 431,1]	H=126	C=116.6906535951326993	omega(f)=2	nK=1
f=2159=[17,1; 127,1]	H=150	C=138.9148686560359562	omega(f)=2	nK=1
f=2567=[17,1; 151,1]	H=154	C=142.3726150732244520	omega(f)=2	nK=1
f=3239=[41,1; 79,1]	H=175	C=161.4113014454594570	omega(f)=2	nK=1
f=3247=[17,1; 191,1]	H=208	C=191.8441285757209858	omega(f)=2	nK=1
f=3791=[17,1; 223,1]	H=238	C=219.1741945424862956	omega(f)=2	nK=1
f=4471=[17,1; 263,1]	H=286	C=262.9432257122401507	omega(f)=2	nK=1
f=5095=[5,1; 1019,1]	H=312	C=286.4726411861354221	omega(f)=2	nK=1
f=5287=[17,1; 311,1]	H=323	C=296.4629523337229678	omega(f)=2	nK=1
f=6071=[13,1; 467,1]	H=336	C=307.9687675618586813	omega(f)=2	nK=1
f=6155=[5,1; 1231,1]	H=378	C=346.4172575739003240	omega(f)=2	nK=1

f=6239=[17,1; 367,1]	H=405	C=371.1110393200093025	omega(f)=2	nK=1
f=6995=[5,1; 1399,1]	H=432	C=395.3992758795738976	omega(f)=2	nK=1
f=7295=[5,1; 1459,1]	H=440	C=402.5524030522415731	omega(f)=2	nK=1
f=7463=[17,1; 439,1]	H=495	C=452.7683540573952683	omega(f)=2	nK=1
f=7895=[5,1; 1579,1]	H=504	C=460.7411639385067668	omega(f)=2	nK=1
f=8551=[17,1; 503,1]	H=525	C=479.5557849789485247	omega(f)=2	nK=1
f=9655=[5,1; 1931,1]	H=546	C=498.1327779099005440	omega(f)=2	nK=1
f=9895=[5,1; 1979,1]	H=644	C=587.3969800430742345	omega(f)=2	nK=1
f=10183=[17,1; 599,1]	H=800	C=729.4763720843714494	omega(f)=2	nK=1
f=12359=[17,1; 727,1]	H=884	C=804.5118246192972826	omega(f)=2	nK=1
f=12431=[31,1; 401,1]	H=1155	C=1051.082786075149247	omega(f)=2	nK=1
f=13271=[23,1; 577,1]	H=1386	C=1260.474876339380179	omega(f)=2	nK=1
f=17663=[17,1; 1039,1]	H=1426	C=1293.149930556092194	omega(f)=2	nK=1
f=22559=[17,1; 1327,1]	H=1560	C=1411.209202757567127	omega(f)=2	nK=1
f=24463=[17,1; 1439,1]	H=1833	C=1656.827780210867060	omega(f)=2	nK=1
f=24599=[17,1; 1447,1]	H=1978	C=1787.792507027724221	omega(f)=2	nK=1
f=27119=[47,1; 577,1]	H=3045	C=2749.505294205692715	omega(f)=2	nK=1
f=29831=[23,1; 1297,1]	H=3432	C=3095.997498110343389	omega(f)=2	nK=1
f=40967=[71,1; 577,1]	H=4116	C=3701.272193141451084	omega(f)=2	nK=1
f=47423=[47,1; 1009,1]	H=4235	C=3802.712781426917957	omega(f)=2	nK=1
f=53063=[47,1; 1129,1]	H=5760	C=5166.239854556097149	omega(f)=2	nK=1
f=60959=[47,1; 1297,1]	H=9130	C=8177.497107751319985	omega(f)=2	nK=1
f=80159=[71,1; 1129,1]	H=10710	C=9566.431099491350923	omega(f)=2	nK=1
f=96359=[167,1; 577,1]	H=14784	C=13181.14176742287956	omega(f)=2	nK=1
f=137903=[239,1; 577,1]	H=22050	C=19589.02663140455669	omega(f)=2	nK=1
f=151751=[263,1; 577,1]	H=26754	C=23745.28612625392000	omega(f)=2	nK=1
f=207143=[359,1; 577,1]	H=34713	C=30713.50960296288906	omega(f)=2	nK=1
f=216599=[167,1; 1297,1]	H=37268	C=32959.41709369729731	omega(f)=2	nK=1
f=248687=[431,1; 577,1]	H=43071	C=38038.94157342156453	omega(f)=2	nK=1
f=313799=[311,1; 1009,1]	H=58121	C=51211.38983446847804	omega(f)=2	nK=1
f=351119=[311,1; 1129,1]	H=83106	C=73143.85340462042777	omega(f)=2	nK=1

(...)

8.3. Experiment for multi-quadratic fields of rank 3. We give the case of imaginary multi-quadratic fields. For the 2-rank 3, we obtain the following results with many $\omega(f) = 3 = r(G) = R(G)$, which are, as for the case of the 3-rank 2, very in accordance with the Conjecture 3.1 (all the cases in the selected interval are such that $h = H$ since $nK = 1$); note that the cases $\omega(f) = 2$ correspond to conductors $f = 2^3 \cdot \ell$ for which the wild ramification of 2 “counts for two generator” of $\text{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q})$:

```
{eps=0.01;Cm=0;G3=polgalois(polcyclo(24));Q=bnfinit(y-1,1);
\\ successive conductors:
for(f=1,10^6,phi=eulerphi(f);if(Mod(phi,8)!=0,next);
f2=valuation(f,2);ff=f/(2^f2);if(f2==1||f2>4||core(ff)!=ff,next);
\\ list of degree 8 polynomials in the fth cyclotomic field:
V=Mat(polsubcyclo(f,8));dV=matsize(V)[2];nK=0;HH=1;hh=1;CC=1.0;
for(k=1,dV,P=component(V,k)[1];D=nfdisc(P);
\\ test of ramification and Galois group:
if(omega(D)==omega(f) & polgalois(P)==G3,fK=rnfconductor(Q,P);
if(component(fK[1][1],1)[1]==f,K=bnfinit(P,1); \\ test fK=f
if(K.sign[2]!=0, \\ test K imaginary
\\ computation of the genus number:
Df=factor(f);df=matsize(Df)[1];Div=component(Df,1);gK=1;for(j=1,df,
q=Div[j];F=idealfactor(K,q);eq=component(F,1)[1][3];gK=gK*eq);gK=gK/8;
nK=nK+1;H=K.no;h=H/gK;C=h/abs(D)^(eps/2);HH=HH*H;hh=hh*h;CC=CC*C));
\\ mean of CC and extrema test:
if(nK!=0,C=CC^(1/nK);if(C>Cm,Cm=C;H=HH^(1/nK);h=hh^(1/nK); \\ mean of H,h
print("f=",f,"=",Df," H=",H,
" C=",precision(C,1)," omega(f)=",omega(f)," nK=",nK))}}}
```

f=24=[2,3; 3,1]	H=1	C=0.925497340811340481	omega(f)=2	nK=1
f=56=[2,3; 7,1]	H=2	C=1.819892080922239686	omega(f)=2	nK=1

f=104=[2,3; 13,1]	H=3	C=2.696248944021590053	omega(f)=2	nK=1
f=136=[2,3; 17,1]	H=4	C=3.575762069753877514	omega(f)=2	nK=1
f=184=[2,3; 23,1]	H=6	C=5.331314392329002593	omega(f)=2	nK=1
f=248=[2,3; 31,1]	H=12	C=10.59916401240363877	omega(f)=2	nK=1
f=328=[2,3; 41,1]	H=16	C=14.05341612030092425	omega(f)=2	nK=1
f=376=[2,3; 47,1]	H=20	C=17.51885180427532467	omega(f)=2	nK=1
f=568=[2,3; 71,1]	H=42	C=36.48729968846260794	omega(f)=2	nK=1
f=632=[2,3; 79,1]	H=60	C=52.01352759088901895	omega(f)=2	nK=1
f=904=[2,3; 113,1]	H=64	C=55.08533633618848662	omega(f)=2	nK=1
f=1016=[2,3; 127,1]	H=120	C=103.0440149547490389	omega(f)=2	nK=1
f=1495=[5,1; 13,1; 23,1]	H=120	C=103.6784324181077920	omega(f)=3	nK=1
f=1595=[5,1; 11,1; 29,1]	H=160	C=138.0590144834850848	omega(f)=3	nK=1
f=1784=[2,3; 223,1]	H=336	C=285.2927796818069852	omega(f)=2	nK=1
f=2056=[2,3; 257,1]	H=384	C=325.1248480613886757	omega(f)=2	nK=1
f=2755=[5,1; 19,1; 29,1]	H=416	C=355.0511294873608592	omega(f)=3	nK=1
f=3055=[5,1; 13,1; 47,1]	H=450	C=383.2765860472549375	omega(f)=3	nK=1
f=3512=[2,3; 439,1]	H=750	C=628.2457702585119454	omega(f)=2	nK=1
f=3835=[5,1; 13,1; 59,1]	H=792	C=671.5059640041688235	omega(f)=3	nK=1
f=4355=[5,1; 13,1; 67,1]	H=990	C=837.2505305391947183	omega(f)=3	nK=1
f=4495=[5,1; 29,1; 31,1]	H=1344	C=1135.911966664581723	omega(f)=3	nK=1
f=4616=[2,3; 577,1]	H=1568	C=1306.291624871039468	omega(f)=2	nK=1
f=4879=[7,1; 17,1; 41,1]	H=2730	C=2303.541446342435365	omega(f)=3	nK=1
f=6088=[2,3; 761,1]	H=3600	C=2982.582105985213100	omega(f)=2	nK=1
f=8555=[5,1; 29,1; 59,1]	H=5376	C=4485.541300328763516	omega(f)=3	nK=1
f=10295=[5,1; 29,1; 71,1]	H=7168	C=5958.616976839243116	omega(f)=3	nK=1
f=11455=[5,1; 29,1; 79,1]	H=7200	C=5972.450989307247585	omega(f)=3	nK=1
f=11635=[5,1; 13,1; 179,1]	H=7680	C=6368.628152049485991	omega(f)=3	nK=1
f=12536=[2,3; 1567,1]	H=9900	C=8084.467416708329516	omega(f)=2	nK=1
f=12935=[5,1; 13,1; 199,1]	H=10368	C=8579.454193948520892	omega(f)=3	nK=1
f=14495=[5,1; 13,1; 223,1]	H=12250	C=10113.73859193621600	omega(f)=3	nK=1
f=15215=[5,1; 17,1; 179,1]	H=12480	C=10293.64404441184357	omega(f)=3	nK=1
f=15535=[5,1; 13,1; 239,1]	H=14040	C=11575.52994096647780	omega(f)=3	nK=1
f=16031=[17,1; 23,1; 41,1]	H=31248	C=25746.78542880426499	omega(f)=3	nK=1
f=21607=[17,1; 31,1; 41,1]	H=55080	C=45113.03465787028027	omega(f)=3	nK=1
f=30095=[5,1; 13,1; 463,1]	H=59598	C=48491.07352541684482	omega(f)=3	nK=1
f=30595=[5,1; 29,1; 211,1]	H=70848	C=57625.48395127060845	omega(f)=3	nK=1
f=32759=[17,1; 41,1; 47,1]	H=140400	C=114040.8629779094895	omega(f)=3	nK=1
f=47995=[5,1; 29,1; 331,1]	H=155232	C=125128.8153786116152	omega(f)=3	nK=1
f=49487=[17,1; 41,1; 71,1]	H=341334	C=274972.7946695323715	omega(f)=3	nK=1
(...)				

8.4. Experiment for multi-cubic fields of rank 2. The program is easily deduced from the previous ones. For the 3-rank 2, we obtain the following results, still in accordance with the Conjecture 3.1 (all cases in the selected interval are such that $nK = 1$, which means $h = H$):

```
{eps=0.01;Cm=0;G2=polgalois(polsubcyclo(63,9));Q=bnfinit(y-1,1);
\\ successive conductors:
for(f=1,10^6,phi=eulerphi(f);if(Mod(phi,9)!=0,next);
f3=valuation(f,3);ff=f/(3^f3);if(f3==1||f3>2||core(ff)!=ff,next);
\\ list of degree 9 polynomials in the fth cyclotomic field:
V=Mat(polsubcyclo(f,9));dV=matsize(V)[2];nK=0;HH=1;hh=1;CC=1.0;
for(k=1,dV,P=component(V,k)[1];D=nfdisc(P);
\\ test of ramification and Galois group:
if(omega(D)==omega(f) & polgalois(P)==G2,fK=rnfconductor(Q,P);
if(component(fK[1][1],1)[1]==f,K=bnfinit(P,1); \\ test fK=f
\\ computation of the genus number:
Df=factor(f);df=matsize(Df)[1];Div=component(Df,1);gK=1;for(j=1,df,
q=Div[j];F=idealfactor(K,q);eq=component(F,1)[1][3];gK=gK*eq);gK=gK/9;
nK=nK+1;H=K.no;h=H/gK;C=h/abs(D)^(eps/2);HH=HH*H;hh=hh*h;CC=CC*C));
\\ mean of CC and extrema test:
if(nK!=0,C=CC^(1/nK);if(C>Cm,Cm=C;H=HH^(1/nK);h=hh^(1/nK); \\ mean of H,h
print("f=",f,"=",Df," H=",precision(H,1),
" C=",precision(C,1)," omega(f)=",omega(f)," nK=",nK))}
```


f=63=[3,2; 7,1]	H=1	C=0.883120128476567309	omega(f)=2	nK=1
f=657=[3,2; 73,1]	H=3	C=2.469416204162289171	omega(f)=2	nK=1
f=679=[7,1; 97,1]	H=4	C=3.289303128491503489	omega(f)=2	nK=1
f=1261=[13,1; 97,1]	H=7	C=5.650366096932072126	omega(f)=2	nK=1
f=1267=[7,1; 181,1]	H=12	C=9.684962591947700189	omega(f)=2	nK=1
f=1687=[7,1; 241,1]	H=13	C=10.40231258099219766	omega(f)=2	nK=1
f=1899=[3,2; 211,1]	H=16	C=12.75746054059586620	omega(f)=2	nK=1
f=2119=[13,1; 163,1]	H=28	C=22.25225881928152337	omega(f)=2	nK=1
f=2817=[3,2; 313,1]	H=112	C=88.25197174283300061	omega(f)=2	nK=1
f=5947=[19,1; 313,1]	H=196	C=151.0174501642113834	omega(f)=2	nK=1
f=7441=[7,1; 1063,1]	H=325	C=248.7335806711150378	omega(f)=2	nK=1
f=12439=[7,1; 1777,1]	H=432	C=325.5668539139437488	omega(f)=2	nK=1
f=14527=[73,1; 199,1]	H=637	C=477.8308156077793569	omega(f)=2	nK=1
f=36667=[37,1; 991,1]	H=729	C=531.8623730500720912	omega(f)=2	nK=1
f=38503=[139,1; 277,1]	H=1296	C=944.1481890324295829	omega(f)=2	nK=1
f=45229=[31,1; 1459,1]	H=1417	C=1027.323779063204492	omega(f)=2	nK=1
f=51457=[7,1; 7351,1]	H=2548	C=1840.162134135815576	omega(f)=2	nK=1
f=53809=[7,1; 7687,1]	H=59584	C=42973.82464267023290	omega(f)=2	nK=1

8.5. Degree 6 imaginary cyclic fields of conductor f . We consider the more complex case of an abelian imaginary extension of degree 6. In the following program, the odd prime number p may be arbitrary, giving the results for degree $2p$ abelian fields. Since 2 does not ramify in the degree p subfield of K , the 2-part of f_K is that of the conductor of the quadratic field. Then the p -part may be p or p^2 . The signature of K is given by the sign of the discriminant D . Recall that $r(G) = 1$ and $R(G) = 2$:

8.5.1. $G \simeq \mathbb{Z}/6\mathbb{Z}$. Successive maxima of the mean of $\frac{H}{g_{K/\mathbb{Q}}(\sqrt{D})^\varepsilon}$.

```
{eps=0.1;Cm=0;Q=bnfinit(y-1,1);p=3; \ successive conductors:
for(f=1,10^6,phi=eulerphi(f);if(Mod(phi,2*p)!=0,next);
f2=valuation(f,2);fp=valuation(f,p);ff=f/(2^f2*p^fp);
if(f2==1||f2>3||fp>2||core(ff)!=ff,next);
\ list of degree 2*p polynomials in the fth cyclotomic field:
V=Mat(polsubcyclo(f,2*p));dV=matsize(V)[2];nK=0;HH=1;hh=1;CC=1.0;
for(k=1,dV,P=component(V,k)[1];D=nfdisc(P); \ discriminant
if(omega(D)==omega(f)&sign(D)==-1,fK=rnfconductor(Q,P);
if(component(fK[1][1],1)[1]==f,K=bnfinit(P,1); \ test fK=f
\ computation of the genus number:
Df=factor(f);df=matsize(Df)[1];Div=component(Df,1);gK=1;for(j=1,df,
q=Div[j];F=idealfactor(K,q);eq=component(F,1)[1][3];gK=gK*eq);gK=gK/(2*p);
nK=nK+1;H=K.no;h=H/gK;C=h/abs(D)^(eps/2);HH=HH*H;hh=hh*h;CC=CC*C));
\ mean of CC and extrema test:
if(nK!=0,C=CC^(1/nK);if(C>Cm,Cm=C;H=HH^(1/nK);h=hh^(1/nK); \ means of H,h
print("f=",f,"=",Df," H=",precision(H,1),
" C=",precision(C,1)," omega(f)=",omega(f)," nK=",nK))}
```

f=7=[7,1]	H=1	C=0.614788152951264365	omega(f)=1	nK=1
f=31=[31,1]	H=9	C=3.814187916735195017	omega(f)=1	nK=1
f=127=[127,1]	H=65	C=19.36254438931795110	omega(f)=1	nK=1
f=215=[5,1; 43,1]	H=266	C=40.79801435406385917	omega(f)=2	nK=1
f=223=[223,1]	H=301	C=77.89153810527680943	omega(f)=1	nK=1
f=439=[439,1]	H=405	C=88.47873262719499548	omega(f)=1	nK=1
f=527=[17,1; 31,1]	H=1026	C=142.1378046706661584	omega(f)=2	nK=1
f=607=[607,1]	H=832	C=167.6201224272746063	omega(f)=1	nK=1
f=919=[919,1]	H=1273	C=231.2062265704086934	omega(f)=1	nK=1
f=1063=[1063,1]	H=3211	C=562.3503754553501155	omega(f)=1	nK=1
f=1399=[1399,1]	H=5184	C=847.6382852373517913	omega(f)=1	nK=1
f=1535=[5,1; 307,1]	H=13034	C=1222.975132851082342	omega(f)=2	nK=1
f=1831=[1831,1]	H=46417	C=7095.859076727868026	omega(f)=1	nK=1
f=4219=[4219,1]	H=136080	C=16884.64442622664510	omega(f)=1	nK=1
f=7295=[5,1; 1459,1]	H=378560	C=24057.21799929730949	omega(f)=2	nK=1

f=8287=[8287, 1]	H=233415	C=24464.11035100730691	omega(f)=1	nK=1
f=9127=[9127, 1]	H=355167	C=36337.12762910420034	omega(f)=1	nK=1
f=9511=[9511, 1]	H=9535041	C=965530.5493702800350	omega(f)=1	nK=1
f=32359=[32359, 1]	H=58428335	C=4356369.789422838646	omega(f)=1	nK=1
f=82351=[82351, 1]	H=131613755	C=7769338.805955799822	omega(f)=1	nK=1
f=97039=[97039, 1]	H=152805744	C=8657711.338391957094	omega(f)=1	nK=1
f=110359=[110359, 1]	H=267362135	C=14668913.96712886411	omega(f)=1	nK=1
f=110735=[5, 1; 22147, 1]	H=749451690	C=24129005.39563081245	omega(f)=2	nK=1
(...)				

For instance, the field of conductor $f = 9511$ is such that $\#\mathcal{C}_K = 9535041 = 69 \cdot 73 \cdot (3 \cdot 631)$, where the classe number of the quadratic (resp. cubic) field is 69 (resp. 73), the large number of “relative” classes being $1893 = 3 \cdot 631$.

8.5.2. $G \simeq \mathbb{Z}/6\mathbb{Z}$. *Successive maxima of the mean of $\frac{H}{(\sqrt{D})^\epsilon}$.* Without the genus factor, we obtain supplementary conductors, as $f = 46631$ for which $\omega(f) = 3$ and $f = 62909$ with the first $\omega(f) = 4$:

```
{eps=0.1;Cm=0;Q=bnfinit(y-1,1);p=3;
for(f=1,10^6,phi=eulerphi(f);if(Mod(phi,2*p)!=0,next);
f2=valuation(f,2);fp=valuation(f,p);ff=f/(2^f2*p^fp);
if(f2==1||f2>3||fp>2||core(ff)!=ff,next);
V=Mat(polsubcyclo(f,2*p));dV=matsize(V)[2];nK=0;HH=1;hh=1;CC=1.0;
for(k=1,dV,P=component(V,k)[1];D=nfdisc(P);
if(omega(D)==omega(f)&sign(D)==-1,fK=rnfconductor(Q,P);
if(component(fK[1][1],1)[1]==f,K=bnfinit(P,1);
nK=nK+1;H=K.no;C=H/abs(D)^(eps/2);HH=HH*H;CC=CC*C));
if(nK!=0,C=CC^(1/nK);if(C>Cm,Cm=C;H=HH^(1/nK);
print("f=",f,"=",factor(f)," H=",precision(H,1),
" C=",precision(C,1)," omega(f)=",omega(f)," nK=",nK))))}
f=7=[7,1] H=1 C=0.614788152951264365 omega(f)=1 nK=1
f=31=[31,1] H=9 C=3.814187916735195017 omega(f)=1 nK=1
f=117=[3,2; 13,1] H=13.637704533357613719
C=4.854098763584681809 omega(f)=2 nK=5
f=127=[127,1] H=65 C=19.36254438931795110 omega(f)=1 nK=1
f=215=[5,1; 43,1] H=266 C=81.59602870812771834 omega(f)=2 nK=1
f=439=[439,1] H=405 C=88.47873262719499548 omega(f)=1 nK=1
f=527=[17,1; 31,1] H=1026 C=284.2756093413323168 omega(f)=2 nK=1
f=815=[5,1; 163,1] H=1440 C=316.5702435490889628 omega(f)=2 nK=1
f=1023=[3,1; 11,1; 31,1] H=2416 C=606.0094764864390741 omega(f)=3 nK=1
f=1399=[1399,1] H=5184 C=847.6382852373517913 omega(f)=1 nK=1
f=1535=[5,1; 307,1] H=13034 C=2445.950265702164685 omega(f)=2 nK=1
f=1831=[1831,1] H=46417 C=7095.859076727868026 omega(f)=1 nK=1
f=4219=[4219,1] H=136080 C=16884.64442622664510 omega(f)=1 nK=1
f=7295=[5,1; 1459,1] H=378560 C=48114.43599859461900 omega(f)=2 nK=1
f=9511=[9511,1] H=9535041 C=965530.5493702800350 omega(f)=1 nK=1
f=32359=[32359,1] H=58428335 C=4356369.789422838646 omega(f)=1 nK=1
f=82351=[82351,1] H=131613755 C=7769338.805955799822 omega(f)=1 nK=1
f=97039=[97039,1] H=152805744 C=8657711.338391957094 omega(f)=1 nK=1
f=110359=[110359,1] H=267362135 C=14668913.96712886411 omega(f)=1 nK=1
f=110735=[5,1; 22147,1] H=749451690 C=24129005.39563081245 omega(f)=2 nK=1
(...)
```

8.6. Degree 10 imaginary cyclic fields of conductor f .

8.6.1. $G \simeq \mathbb{Z}/10\mathbb{Z}$. *Successive maxima of the mean of $\frac{H}{g_{K/\mathbb{Q}}(\sqrt{D})^\epsilon}$.*

f=11=[11,1]	H=1	C=0.339917318183292738	omega(f)=1	nK=1
f=31=[31,1]	H=3	C=0.639747345421760414	omega(f)=1	nK=1
f=71=[71,1]	H=7	C=1.028091534474586170	omega(f)=1	nK=1
f=77=[7,1; 11,1]	H=5	C=1.177981365155155362	omega(f)=2	nK=1
f=88=[2,3; 11,1]	H=10.488088481701515470	C=1.591537796213539734	omega(f)=2	nK=2
f=123=[3,1; 41,1]	H=22.627416997969520780	C=2.508456244067394871	omega(f)=2	nK=2

f=124=[2,2; 31,1]	H=41	C=7.340475295096265777	omega(f)=2	nK=1
f=151=[151,1]	H=1967	C=205.7144190804658850	omega(f)=1	nK=1
f=431=[431,1]	H=161931	C=10563.67974232756675	omega(f)=1	nK=1
f=1091=[1091,1]	H=262667	C=11281.95533783001965	omega(f)=1	nK=1
f=1255=[5,1; 251,1]	H=426432	C=11863.80817047845353	omega(f)=2	nK=1
f=2111=[2111,1]	H=411649	C=13137.33757891912240	omega(f)=1	nK=1
f=2351=[2351,1]	H=22273713	C=677218.6915129510617	omega(f)=1	nK=1
f=3931=[3931,1]	H=43298816	C=1044599.109023261171	omega(f)=1	nK=1
f=5051=[5051,1]	H=704444539	C=15181930.21787324823	omega(f)=1	nK=1
(...)				

8.6.2. $G \simeq \mathbb{Z}/10\mathbb{Z}$. Successive maxima of the mean of $\frac{H}{(\sqrt{D})^\varepsilon}$.

f=11=[11,1]	H=1	C=0.339917318183292738	omega(f)=1	nK=1
f=31=[31,1]	H=3	C=0.639747345421760414	omega(f)=1	nK=1
f=55=[5,1; 11,1]	H=4	C=0.909265644114650678	omega(f)=2	nK=1
f=71=[71,1]	H=	C=1.028091534474586170	omega(f)=1	nK=1
f=77=[7,1; 11,1]	H=5	C=1.177981365155155362	omega(f)=2	nK=1
f=88=[2,3; 11,1]	H=10.488088481701515470			
		C=2.250774336434575013	omega(f)=2	nK=2
f=123=[3,1; 41,1]	H=22.627416997969520780			
		C=3.547492840979584473	omega(f)=2	nK=2
f=124=[2,2; 31,1]	H=41	C=7.340475295096265777	omega(f)=2	nK=1
f=132=[2,2; 3,1; 11,1]	H=44	C=8.035828447368965852	omega(f)=3	nK=1
f=143=[11,1; 13,1]	H=50	C=8.950709414213818126	omega(f)=2	nK=1
f=151=[151,1]	H=1967	C=205.7144190804658850	omega(f)=1	nK=1
f=431=[431,1]	H=161931	C=10563.67974232756675	omega(f)=1	nK=1
f=1091=[1091,1]	H=262667	C=11281.95533783001965	omega(f)=1	nK=1
f=1255=[5,1; 251,1]	H=426432	C=23727.61634095690706	omega(f)=2	nK=1
f=2351=[2351,1]	H=22273713	C=677218.6915129510617	omega(f)=1	nK=1
f=3931=[3931,1]	H=43298816	C=1044599.109023261171	omega(f)=1	nK=1
f=5051=[5051,1]	H=704444539	C=15181930.21787324823	omega(f)=1	nK=1
f=8831=[8831,1]	H=9494084864	C=159128753.6488153656	omega(f)=1	nK=1
(...)				

8.7. **Some additional data.** We indicate the numerical results for some other abelian groups (limited by PARI/GP to groups G of order less than 12), for the mean of $\frac{H}{g_{K/\mathbb{Q}}(\sqrt{D})^\varepsilon}$ with $\varepsilon = 0.01$. These results strengthen somewhat the conjectural inequality $r(G) \leq \omega(f) \leq R(G)$ of Conjecture 1.3.

8.7.1. $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

f=15=[3,1; 5,1]	H=1	h=1	C=0.9321557326415818	omega(f)=2	nK=1
f=39=[3,1; 13,1]	H=2	h=2	C=1.8116290245333817	omega(f)=2	nK=1
f=51=[3,1; 17,1]	H=5	h=5	C=4.4927692286025652	omega(f)=2	nK=1
f=123=[3,1; 41,1]	H=17	h=17	C=14.877260946454665	omega(f)=2	nK=1
f=187=[11,1; 17,1]	H=17	h=17	C=14.883586644228593	omega(f)=2	nK=1
f=272=[2,4; 17,1]	H=43.886403	h=17.416330	C=14.929910722942526	omega(f)=2	nK=9
f=287=[7,1; 41,1]	H=35	h=35	C=30.114979221003303	omega(f)=2	nK=1
f=291=[3,1; 97,1]	H=68	h=68	C=57.991365573835759	omega(f)=2	nK=1
f=327=[3,1; 109,1]	H=102	h=102	C=86.683203264487910	omega(f)=2	nK=1
f=391=[17,1; 23,1]	H=105	h=105	C=90.581868448183013	omega(f)=2	nK=1
f=411=[3,1; 137,1]	H=111	h=111	C=93.686914633082859	omega(f)=2	nK=1
f=451=[11,1; 41,1]	H=111	h=111	C=94.648036619685123	omega(f)=2	nK=1
f=511=[7,1; 73,1]	H=791	h=791	C=668.92101009616677	omega(f)=2	nK=1
f=623=[7,1; 89,1]	H=1199	h=1199	C=1007.9419532931184	omega(f)=2	nK=1
f=791=[7,1; 113,1]	H=1280	h=1280	C=1068.3551688844457	omega(f)=2	nK=1
f=1371=[3,1; 457,1]	H=5340	h=5340	C=4347.1171682925243	omega(f)=2	nK=1
f=1679=[23,1; 73,1]	H=8268	h=8268	C=6827.5710056916921	omega(f)=2	nK=1
f=1799=[7,1; 257,1]	H=17475	h=17475	C=14230.404002409721	omega(f)=2	nK=1
f=3027=[3,1; 1009,1]	H=27300	h=27300	C=21702.186282280434	omega(f)=2	nK=1
f=3431=[47,1; 73,1]	H=36465	h=36465	C=29684.830601395338	omega(f)=2	nK=1

$f=3503=[31, 1; 113, 1]$ $H=43836$ $h=43836$ $C=35514.960098930545$ $\omega(f)=2$ $nK=1$
 $f=4372=[2, 2; 1093, 1]$ $H=60125$ $h=60125$ $C=47408.398500381492$ $\omega(f)=2$ $nK=1$
 $f=4616=[2, 3; 577, 1]$
 $H=129838.811331$ $h=77202.619119$ $C=60564.691595960546$ $\omega(f)=2$ $nK=4$
 $f=5183=[71, 1; 73, 1]$ $H=89600$ $h=89600$ $C=72340.774959541852$ $\omega(f)=2$ $nK=1$
 $f=5311=[47, 1; 113, 1]$ $H=112355$ $h=112355$ $C=90273.045356601456$ $\omega(f)=2$ $nK=1$
 $f=7031=[79, 1; 89, 1]$ $H=197100$ $h=197100$ $C=157852.83359289318$ $\omega(f)=2$ $nK=1$
 $f=7251=[3, 1; 2417, 1]$ $H=267325$ $h=267325$ $C=207013.58713575608$ $\omega(f)=2$ $nK=1$
 $f=8116=[2, 2; 2029, 1]$ $H=502775$ $h=502775$ $C=389147.29273104160$ $\omega(f)=2$ $nK=1$
 $f=10376=[2, 3; 1297, 1]$
 $H=1048152.694144$ $h=623235.320742$ $C=477184.80861414767$ $\omega(f)=2$ $nK=4$
 $f=13271=[23, 1; 577, 1]$ $H=5225220$ $h=5225220$ $C=4055409.8285235355$ $\omega(f)=2$ $nK=1$

8.7.2. $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

$f=31=[31, 1]$ $H=9$ $h=9$ $C=0.061911033427912298$ $\omega(f)=1$ $nK=1$
 $f=77=[7, 1; 11, 1]$ $H=35.777087$ $h=35.777087$
 $C=0.188405690668671425$ $\omega(f)=2$ $nK=2$
 $f=93=[3, 1; 31, 1]$ $H=755$ $h=755$ $C=2.705201622444429614$ $\omega(f)=2$ $nK=1$
 $f=124=[2, 2; 31, 1]$ $H=5084$ $h=5084$ $C=14.68095059019253155$ $\omega(f)=2$ $nK=1$
 $f=151=[151, 1]$ $H=238007$ $h=238007$ $C=164.8440046936183581$ $\omega(f)=1$ $nK=1$
 $f=211=[211, 1]$ $H=467523$ $h=467523$ $C=199.3395258269297336$ $\omega(f)=1$ $nK=1$
 $f=244=[2, 2; 61, 1]$ $H=900747.730538$ $h=636924.828402$
 $C=643.3512814804028363$ $\omega(f)=2$ $nK=2$
 $f=248=[2, 3; 31, 1]$ $H=731438.520484$ $h=517205.137855$
 $C=814.9949665105118220$ $\omega(f)=2$ $nK=2$
 $f=331=[331, 1]$ $H=10805967$ $h=10805967$
 $C=2398.352674944173664$ $\omega(f)=1$ $nK=1$

8.7.3. $G = \mathbb{Z}/9\mathbb{Z}$.

$f=19=[19, 1]$ $H=1$ $h=1$ $C=0.888893756893086074$ $\omega(f)=1$ $nK=1$
 $f=163=[163, 1]$ $H=4$ $h=4$ $C=3.262665022855041198$ $\omega(f)=1$ $nK=1$
 $f=1063=[1063, 1]$ $H=13$ $h=13$ $C=9.837438305141874497$ $\omega(f)=1$ $nK=1$
 $f=1141=[7, 1; 163, 1]$
 $H=52.306787$ $h=17.435595$ $C=13.41517512486724241$ $\omega(f)=2$ $nK=2$
 $f=1153=[1153, 1]$ $H=19$ $h=19$ $C=14.33112977180156483$ $\omega(f)=1$ $nK=1$
 $f=1459=[1459, 1]$ $H=247$ $h=247$ $C=184.5587931331744690$ $\omega(f)=1$ $nK=1$
 $f=3547=[3547, 1]$ $H=16777$ $h=16777$ $C=12098.17485260014863$ $\omega(f)=1$ $nK=1$

8.7.4. $G = \mathbb{Z}/8\mathbb{Z}$.

$f=32=[2, 5]$ $H=1$ $h=1$ $C=0.898132372883934$ $\omega(f)=1$ $nK=1$
 $f=68=[2, 2; 17, 1]$ $H=4$ $h=2$ $C=1.761664016490415$ $\omega(f)=2$ $nK=1$
 $f=73=[73, 1]$ $H=89$ $h=89$ $C=76.59028869662640$ $\omega(f)=1$ $nK=1$
 $f=89=[89, 1]$ $H=113$ $h=113$ $C=96.57168147147798$ $\omega(f)=1$ $nK=1$
 $f=233=[233, 1]$ $H=1433$ $h=1433$ $C=1184.100959851340$ $\omega(f)=1$ $nK=1$
 $f=339=[3, 1; 113, 1]$ $H=4274$ $h=2137$ $C=1771.757792528851$ $\omega(f)=2$ $nK=1$
 $f=601=[601, 1]$ $H=34153$ $h=34153$ $C=27300.35230848298$ $\omega(f)=1$ $nK=1$
 $f=1609=[1609, 1]$ $H=189473$ $h=189473$ $C=146324.7766845321$ $\omega(f)=1$ $nK=1$
 $f=1731=[3, 1; 577, 1]$ $H=460656$ $h=230328$ $C=180369.5234991963$ $\omega(f)=2$ $nK=1$
 $f=2441=[2441, 1]$ $H=387433$ $h=387433$ $C=294870.7790203549$ $\omega(f)=1$ $nK=1$
 $f=3449=[3449, 1]$ $H=549225$ $h=549225$ $C=412981.9168610820$ $\omega(f)=1$ $nK=1$
 $f=4201=[4201, 1]$ $H=647593$ $h=647593$ $C=483598.3404675706$ $\omega(f)=1$ $nK=1$
 $f=4297=[4297, 1]$ $H=1747657$ $h=1747657$ $C=1304053.513066895$ $\omega(f)=1$ $nK=1$
 $f=4409=[4409, 1]$ $H=8637921$ $h=8637921$ $C=6439576.759445647$ $\omega(f)=1$ $nK=1$
 $f=7753=[7753, 1]$ $H=98996475$ $h=98996475$ $C=72358288.91356542$ $\omega(f)=1$ $nK=1$

We have no justification for these results and hope that analytic tools may help. In this direction see [5, 6, 8, 15, 17, 18, 19, 20, 27] among a wide literature on the subject in the spirit of Brauer–Siegel theorems and Cohen–Lenstra–Martinet–Malle heuristics.

9. APPENDIX – NUMERICAL TABLES FOR QUADRATIC FIELDS

9.1. Table I: Imaginary case – Successive maxima of $\frac{H}{2^{N-1}(\sqrt{D})^\varepsilon}$.

```
{eps=0.02;bD=1;BD=10^10;ND=0;N1=0;N2=0;N3=0;N4=0;N5=0;N6=0;
Cm=0;for(D=bD,BD,e2=valuation(D,2);d=D/2^e2;
if(e2==1||e2>3||core(d)!=d||(e2==0&Mod(-d,4)!=1)||e2==2&Mod(-d,4)!=-1),
next);ND=ND+1;H=qfbclassno(-D);N=omega(D);h=H/2^(N-1);C=h/D^(eps/2);
if(C>Cm,Cm=C;if(N==1,N1=N1+1);if(N==2,N2=N2+1);if(N>=3,N3=N3+1);print();
print("D_K=", -D, " H=", H, " h=", h, " N=", N, " C=", precision(C,1));
print("ND=", ND, " N1=", N1, " N2=", N2, " N3=", N3))}
```

```
eps=0.02
```

```
D_K=-3      H=1 h=1      N=1      C=0.9890740041721707131
ND=1      N1=1      N2=0      N3=0
```

```
D_K=-23     H=3 h=3      N=1      C=2.907394579154063316
ND=9      N1=2      N2=0      N3=0
```

```
D_K=-47     H=5 h=5      N=1      C=4.811151422352073885
ND=16     N1=3      N2=0      N3=0
```

```
D_K=-71     H=7 h=7      N=1      C=6.707882653447388596
ND=24     N1=4      N2=0      N3=0
```

```
D_K=-167    H=11 h=11     N=1      C=10.451184640094270919
ND=54     N1=5      N2=0      N3=0
```

```
D_K=-191    H=13 h=13     N=1      C=12.334825747149741887
ND=60     N1=6      N2=0      N3=0
```

```
D_K=-239    H=15 h=15     N=1      C=14.200619147352270997
ND=74     N1=7      N2=0      N3=0
```

```
D_K=-311    H=19 h=19     N=1      C=17.940146990449301159
ND=98     N1=8      N2=0      N3=0
```

```
D_K=-431    H=21 h=21     N=1      C=19.763985292405135024
ND=133    N1=9      N2=0      N3=0
```

```
D_K=-479    H=25 h=25     N=1      C=23.503722641804738888
ND=148    N1=10     N2=0      N3=0
```

```
D_K=-719    H=31 h=31     N=1      C=29.026482150819988158
ND=221    N1=11     N2=0      N3=0
```

```
D_K=-839    H=33 h=33     N=1      C=30.851502556545188090
ND=259    N1=12     N2=0      N3=0
```

```
D_K=-1031   H=35 h=35     N=1      C=32.65393002076587744
ND=316    N1=13     N2=0      N3=0
```

```
D_K=-1151   H=41 h=41     N=1      C=38.20965386322569539
ND=352    N1=14     N2=0      N3=0
```

```
D_K=-1319   H=45 h=45     N=1      C=41.88032717792313869
ND=403    N1=15     N2=0      N3=0
```

```
D_K=-1511   H=49 h=49     N=1      C=45.54109150907451145
ND=460    N1=16     N2=0      N3=0
```

```
D_K=-1559   H=51 h=51     N=1      C=47.38509055810921595
ND=475    N1=17     N2=0      N3=0
```

D_K=-2351 ND=717	H=63 N1=18	h=63 N2=0	N=1 N3=0	C=58.29455926353563301
D_K=-2999 ND=911	H=73 N1=19	h=73 N2=0	N=1 N3=0	C=67.38342712710593338
D_K=-3671 ND=1120	H=81 N1=20	h=81 N2=0	N=1 N3=0	C=74.61689535356539441
D_K=-3911 ND=1191	H=83 N1=21	h=83 N2=0	N=1 N3=0	C=76.41088226472032525
D_K=-4079 ND=1241	H=85 N1=22	h=85 N2=0	N=1 N3=0	C=78.21920339356804770
D_K=-4679 ND=1424	H=91 N1=23	h=91 N2=0	N=1 N3=0	C=83.62571843882399364
D_K=-5351 ND=1633	H=93 N1=24	h=93 N2=0	N=1 N3=0	C=85.34903184030047657
D_K=-5519 ND=1682	H=97 N1=25	h=97 N2=0	N=1 N3=0	C=88.99244325400892204
D_K=-5591 ND=1704	H=99 N1=26	h=99 N2=0	N=1 N3=0	C=90.81556718893487486
D_K=-5711 ND=1739	H=109 N1=27	h=109 N2=0	N=1 N3=0	C=99.96762544713008421
D_K=-6551 ND=1996	H=117 N1=28	h=117 N2=0	N=1 N3=0	C=107.15755251862864416
D_K=-8111 ND=2465	H=121 N1=29	h=121 N2=0	N=1 N3=0	C=110.58459361583971130
D_K=-8951 ND=2724	H=135 N1=30	h=135 N2=0	N=1 N3=0	C=123.25798194656118673
D_K=-9239 ND=2813	H=139 N1=31	h=139 N2=0	N=1 N3=0	C=126.86988627904520537
D_K=-10391 ND=3163	H=153 N1=32	h=153 N2=0	N=1 N3=0	C=139.48414907932407178
D_K=-12239 ND=3723	H=161 N1=33	h=161 N2=0	N=1 N3=0	C=146.53737808857220506
D_K=-12911 ND=3928	H=165 N1=34	h=165 N2=0	N=1 N3=0	C=150.09780643352709822
D_K=-13751 ND=4185	H=171 N1=35	h=171 N2=0	N=1 N3=0	C=155.45788955518033181
D_K=-14159 ND=4308	H=173 N1=36	h=173 N2=0	N=1 N3=0	C=157.23013137872983161
D_K=-14759 ND=4488	H=181 N1=37	h=181 N2=0	N=1 N3=0	C=164.43263091549638326
D_K=-14951 ND=4544	H=185 N1=38	h=185 N2=0	N=1 N3=0	C=168.04477981706382148

D_K=-15791 ND=4801	H=195 N1=39	h=195 N2=0	N=1 N3=0	C=177.03148607782637142
D_K=-18119 ND=5511	H=205 N1=40	h=205 N2=0	N=1 N3=0	C=185.85425945032247040
D_K=-18191 ND=5533	H=213 N1=41	h=213 N2=0	N=1 N3=0	C=193.09945040387485726
D_K=-22679 ND=6893	H=217 N1=42	h=217 N2=0	N=1 N3=0	C=196.29240443750101460
D_K=-23399 ND=7115	H=229 N1=43	h=229 N2=0	N=1 N3=0	C=207.08255245619442654
D_K=-25679 ND=7807	H=239 N1=44	h=239 N2=0	N=1 N3=0	C=215.92459807818457596
D_K=-26879 ND=8175	H=245 N1=45	h=245 N2=0	N=1 N3=0	C=221.24423003919512892
D_K=-27191 ND=8271	H=247 N1=46	h=247 N2=0	N=1 N3=0	C=223.02456525251780566
D_K=-27431 ND=8344	H=259 N1=47	h=259 N2=0	N=1 N3=0	C=233.83921675412191926
D_K=-31391 ND=9549	H=289 N1=48	h=289 N2=0	N=1 N3=0	C=260.5732269028509740
D_K=-35279 ND=10729	H=305 N1=49	h=305 N2=0	N=1 N3=0	C=274.6785060757030752
D_K=-38639 ND=11744	H=325 N1=50	h=325 N2=0	N=1 N3=0	C=292.4240597709833585
D_K=-43391 ND=13188	H=329 N1=51	h=329 N2=0	N=1 N3=0	C=295.6799671969475978
D_K=-44519 ND=13528	H=331 N1=52	h=331 N2=0	N=1 N3=0	C=297.4010789447064907
D_K=-46199 ND=14042	H=337 N1=53	h=337 N2=0	N=1 N3=0	C=302.6798953356657547
D_K=-49871 ND=15157	H=357 N1=54	h=357 N2=0	N=1 N3=0	C=320.3979549191792761
D_K=-50111 ND=15229	H=359 N1=55	h=359 N2=0	N=1 N3=0	C=322.1774337434058561
D_K=-50951 ND=15485	H=361 N1=56	h=361 N2=0	N=1 N3=0	C=323.9184419334974895
D_K=-53231 ND=16182	H=375 N1=57	h=375 N2=0	N=1 N3=0	C=336.3331087168986246
D_K=-57119 ND=17358	H=381 N1=58	h=381 N2=0	N=1 N3=0	C=341.4736287549717613
D_K=-58439 ND=17764	H=387 N1=59	h=387 N2=0	N=1 N3=0	C=346.7719313457439729

D_K=-61151 ND=18591	H=395 h=395 N1=60 N2=0	N=1 N3=0	C=353.7798227363277427
D_K=-61871 ND=18812	H=399 h=399 N1=61 N2=0	N=1 N3=0	C=357.3205751223884702
D_K=-63839 ND=19416	H=423 h=423 N1=62 N2=0	N=1 N3=0	C=378.6949438056025608
D_K=-75479 ND=22949	H=429 h=429 N1=63 N2=0	N=1 N3=0	C=383.4237680124604160
D_K=-80471 ND=24461	H=455 h=455 N1=64 N2=0	N=1 N3=0	C=406.4012196838752623
D_K=-87071 ND=26468	H=469 h=469 N1=65 N2=0	N=1 N3=0	C=418.5757917504236169
D_K=-88079 ND=26773	H=487 h=487 N1=66 N2=0	N=1 N3=0	C=434.5905089090455506
D_K=-88919 ND=27028	H=505 h=505 N1=67 N2=0	N=1 N3=0	C=450.6106298053805055
D_K=-95471 ND=29019	H=533 h=533 N1=68 N2=0	N=1 N3=0	C=475.2569694226818385
D_K=-104711 ND=31828	H=555 h=555 N1=69 N2=0	N=1 N3=0	C=494.4166185804803582
D_K=-115079 ND=34981	H=559 h=559 N1=70 N2=0	N=1 N3=0	C=497.5100373011850563
D_K=-119591 ND=36349	H=589 h=589 N1=71 N2=0	N=1 N3=0	C=524.0084734955214091
D_K=-128519 ND=39063	H=609 h=609 N1=72 N2=0	N=1 N3=0	C=541.4116782358050222
D_K=-142271 ND=43245	H=639 h=639 N1=73 N2=0	N=1 N3=0	C=567.5050027257323546
D_K=-145991 ND=44386	H=651 h=651 N1=74 N2=0	N=1 N3=0	C=578.0131617801329047
D_K=-147671 ND=44895	H=667 h=667 N1=75 N2=0	N=1 N3=0	C=592.1515654661281541
D_K=-158759 ND=48253	H=669 h=669 N1=76 N2=0	N=1 N3=0	C=593.4972817749738278
D_K=-162839 ND=49494	H=673 h=673 N1=77 N2=0	N=1 N3=0	C=596.8943663973072745
D_K=-166319 ND=50554	H=699 h=699 N1=78 N2=0	N=1 N3=0	C=619.8230990005178269
D_K=-174599 ND=53080	H=713 h=713 N1=79 N2=0	N=1 N3=0	C=631.9302021831613947
D_K=-178559 ND=54282	H=729 h=729 N1=80 N2=0	N=1 N3=0	C=645.9660759734119445

D_K=-191231 ND=58137	H=761 N1=81	h=761 N2=0	N=1 N3=0	C=673.8590631029059800
D_K=-199559 ND=60664	H=777 N1=82	h=777 N2=0	N=1 N3=0	C=687.7336994546257594
D_K=-211559 ND=64312	H=797 N1=83	h=797 N2=0	N=1 N3=0	C=705.0241706302228067
D_K=-217439 ND=66096	H=805 N1=84	h=805 N2=0	N=1 N3=0	C=711.9057585857295015
D_K=-225671 ND=68601	H=821 N1=85	h=821 N2=0	N=1 N3=0	C=725.7856873792070544
D_K=-236519 ND=71896	H=831 N1=86	h=831 N2=0	N=1 N3=0	C=734.2811229892052112
D_K=-250799 ND=76230	H=875 N1=87	h=875 N2=0	N=1 N3=0	C=772.7069082881262912
D_K=-267671 ND=81356	H=883 N1=88	h=883 N2=0	N=1 N3=0	C=779.2641380783919147
D_K=-283799 ND=86262	H=929 N1=89	h=929 N2=0	N=1 N3=0	C=819.3804645078858601
D_K=-284231 ND=86395	H=943 N1=90	h=943 N2=0	N=1 N3=0	C=831.7158507650857724
D_K=-289511 ND=88002	H=963 N1=91	h=963 N2=0	N=1 N3=0	C=849.1993174564820835
D_K=-312311 ND=94927	H=1001 N1=92	h=1001 N2=0	N=1 N3=0	C=882.0398458290537064
D_K=-329111 ND=100037	H=1007 N1=93	h=1007 N2=0	N=1 N3=0	C=886.8619995858851371
D_K=-333791 ND=101464	H=1033 N1=94	h=1033 N2=0	N=1 N3=0	C=909.6316759228785982
D_K=-366791 ND=111500	H=1121 N1=95	h=1121 N2=0	N=1 N3=0	C=986.1918880026803627
D_K=-397799 ND=120929	H=1125 N1=96	h=1125 N2=0	N=1 N3=0	C=988.9079894017006404
D_K=-407879 ND=123986	H=1133 N1=97	h=1133 N2=0	N=1 N3=0	C=995.6910339871266643
D_K=-422231 ND=128346	H=1169 N1=98	h=1169 N2=0	N=1 N3=0	C=1026.9729627090228374
D_K=-425519 ND=129347	H=1175 N1=99	h=1175 N2=0	N=1 N3=0	C=1032.1639273771506496
D_K=-461639 ND=140320	H=1205 N1=100	h=1205 N2=0	N=1 N3=0	C=1057.6549882045300230
D_K=-465071 ND=141359	H=1217 N1=101	h=1217 N2=0	N=1 N3=0	C=1068.1085355213607564

D_K=-470279 ND=142946	H=1221 h=1221 N1=102 N2=0	N=1 N3=0	C=1071.4998339723808177
D_K=-475151 ND=144426	H=1235 h=1235 N1=103 N2=0	N=1 N3=0	C=1083.6739686040697105
D_K=-491279 ND=149334	H=1259 h=1259 N1=104 N2=0	N=1 N3=0	C=1104.3645264170961013
D_K=-493919 ND=150138	H=1299 h=1299 N1=105 N2=0	N=1 N3=0	C=1139.3904991221287309
D_K=-504479 ND=153350	H=1311 h=1311 N1=106 N2=0	N=1 N3=0	C=1149.6728111565926667
D_K=-514751 ND=156480	H=1347 h=1347 N1=107 N2=0	N=1 N3=0	C=1181.0046939377777554
D_K=-572711 ND=174082	H=1391 h=1391 N1=108 N2=0	N=1 N3=0	C=1218.2818517395393811
D_K=-618719 ND=188063	H=1461 h=1461 N1=109 N2=0	N=1 N3=0	C=1278.6017114226840154
D_K=-628319 ND=190979	H=1495 h=1495 N1=110 N2=0	N=1 N3=0	C=1308.1555579500644967
D_K=-683231 ND=207671	H=1509 h=1509 N1=111 N2=0	N=1 N3=0	C=1319.3000051805935002
D_K=-701399 ND=213196	H=1581 h=1581 N1=112 N2=0	N=1 N3=0	C=1381.8860048652978316
D_K=-748271 ND=227448	H=1587 h=1587 N1=113 N2=0	N=1 N3=0	C=1386.2333331523428212
D_K=-770591 ND=234229	H=1629 h=1629 N1=114 N2=0	N=1 N3=0	C=1422.5018667232365208
D_K=-777479 ND=236321	H=1641 h=1641 N1=115 N2=0	N=1 N3=0	C=1432.8531882616800175
D_K=-795791 ND=241895	H=1659 h=1659 N1=116 N2=0	N=1 N3=0	C=1448.2328550493903471
D_K=-819719 ND=249166	H=1725 h=1725 N1=117 N2=0	N=1 N3=0	C=1505.4018638206122492
D_K=-880871 ND=267753	H=1733 h=1733 N1=118 N2=0	N=1 N3=0	C=1511.2956753346931059
D_K=-913799 ND=277766	H=1737 h=1737 N1=119 N2=0	N=1 N3=0	C=1514.2281362689274317
D_K=-952199 ND=289440	H=1755 h=1755 N1=120 N2=0	N=1 N3=0	C=1529.2899820327188184
D_K=-953039 ND=289693	H=1829 h=1829 N1=121 N2=0	N=1 N3=0	C=1593.7588109630809336
D_K=-1006799 ND=306038	H=1853 h=1853 N1=122 N2=0	N=1 N3=0	C=1613.7861784770214299

D_K=-1032071 ND=313725	H=1857 N1=123	h=1857 N2=0	N=1 N3=0	C=1616.8689014845503871
D_K=-1062311 ND=322911	H=1927 N1=124	h=1927 N2=0	N=1 N3=0	C=1677.3326387265150602
D_K=-1086791 ND=330349	H=1965 N1=125	h=1965 N2=0	N=1 N3=0	C=1710.0196241642265837
D_K=-1139519 ND=346386	H=2027 N1=126	h=2027 N2=0	N=1 N3=0	C=1763.1389228900476362
D_K=-1190591 ND=361907	H=2051 N1=127	h=2051 N2=0	N=1 N3=0	C=1783.2327616010918186
D_K=-1212119 ND=368450	H=2061 N1=128	h=2061 N2=0	N=1 N3=0	C=1791.6061278221333682
D_K=-1244591 ND=378316	H=2073 N1=129	h=2073 N2=0	N=1 N3=0	C=1801.5612647406824590
D_K=-1286711 ND=391117	H=2093 N1=130	h=2093 N2=0	N=1 N3=0	C=1818.3371768677714209
D_K=-1330031 ND=404278	H=2115 N1=131	h=2115 N2=0	N=1 N3=0	C=1836.8418010349795952
D_K=-1332119 ND=404915	H=2131 N1=132	h=2131 N2=0	N=1 N3=0	C=1850.7084994714225218
D_K=-1347191 ND=409496	H=2149 N1=133	h=2149 N2=0	N=1 N3=0	C=1866.1309843916244010
D_K=-1360631 ND=413581	H=2203 N1=134	h=2203 N2=0	N=1 N3=0	C=1912.8331671898703248
D_K=-1387151 ND=421649	H=2221 N1=135	h=2221 N2=0	N=1 N3=0	C=1928.0900842598358388
D_K=-1412759 ND=429424	H=2311 N1=136	h=2311 N2=0	N=1 N3=0	C=2005.8537496212781478
D_K=-1579751 ND=480170	H=2395 N1=137	h=2395 N2=0	N=1 N3=0	C=2076.441171091851336
D_K=-1614311 ND=490676	H=2421 N1=138	h=2421 N2=0	N=1 N3=0	C=2098.528720649882136
D_K=-1642031 ND=499109	H=2457 N1=139	h=2457 N2=0	N=1 N3=0	C=2129.371038564755560
D_K=-1669799 ND=507548	H=2471 N1=140	h=2471 N2=0	N=1 N3=0	C=2141.145120228567370
D_K=-1750271 ND=532010	H=2567 N1=141	h=2567 N2=0	N=1 N3=0	C=2223.283350513854295
D_K=-1884791 ND=572912	H=2669 N1=142	h=2669 N2=0	N=1 N3=0	C=2309.914695388376059
D_K=-2005559 ND=609619	H=2695 N1=143	h=2695 N2=0	N=1 N3=0	C=2330.968556138095747

D_K=-2042039 ND=620701	H=2735 N1=144	h=2735 N2=0	N=1 N3=0	C=2365.139113782900517
D_K=-2092991 ND=636188	H=2739 N1=145	h=2739 N2=0	N=1 N3=0	C=2368.014506993711335
D_K=-2121479 ND=644854	H=2753 N1=146	h=2753 N2=0	N=1 N3=0	C=2379.796514005747629
D_K=-2159951 ND=656555	H=2757 N1=147	h=2757 N2=0	N=1 N3=0	C=2382.825982972243361
D_K=-2172551 ND=660388	H=2815 N1=148	h=2815 N2=0	N=1 N3=0	C=2432.812840594557905
D_K=-2198759 ND=668348	H=2829 N1=149	h=2829 N2=0	N=1 N3=0	C=2444.618934174549674
D_K=-2307191 ND=701314	H=2901 N1=150	h=2901 N2=0	N=1 N3=0	C=2505.629729302251923
D_K=-2336039 ND=710072	H=2921 N1=151	h=2921 N2=0	N=1 N3=0	C=2522.590501305530441
D_K=-2420519 ND=735762	H=2983 N1=152	h=2983 N2=0	N=1 N3=0	C=2575.219002858894226
D_K=-2449919 ND=744706	H=2999 N1=153	h=2999 N2=0	N=1 N3=0	C=2588.719222098567801
D_K=-2470079 ND=750829	H=3079 N1=154	h=3079 N2=0	N=1 N3=0	C=2657.556952933138179
D_K=-2611391 ND=793789	H=3105 N1=155	h=3105 N2=0	N=1 N3=0	C=2678.507615445246235
D_K=-2693591 ND=818769	H=3139 N1=156	h=3139 N2=0	N=1 N3=0	C=2706.998400628945415
D_K=-2798351 ND=850609	H=3219 N1=157	h=3219 N2=0	N=1 N3=0	C=2774.929505870438035
D_K=-2882351 ND=876141	H=3231 N1=158	h=3231 N2=0	N=1 N3=0	C=2784.450420774047030
D_K=-2888591 ND=878035	H=3277 N1=159	h=3277 N2=0	N=1 N3=0	C=2824.031787422810485
D_K=-2919479 ND=887419	H=3305 N1=160	h=3305 N2=0	N=1 N3=0	C=2847.858521758496226
D_K=-2958359 ND=899233	H=3311 N1=161	h=3311 N2=0	N=1 N3=0	C=2852.651195076490407
D_K=-3043871 ND=925219	H=3325 N1=162	h=3325 N2=0	N=1 N3=0	C=2863.896951763712271
D_K=-3068831 ND=932809	H=3485 N1=163	h=3485 N2=0	N=1 N3=0	C=3001.463405484378354
D_K=-3190151 ND=969674	H=3593 N1=164	h=3593 N2=0	N=1 N3=0	C=3093.279082652959006

D_K=-3362519 ND=1022082	H=3607 N1=165	h=3607 N2=0	N=1 N3=0	C=3103.698275030715432
D_K=-3654719 ND=1110901	H=3755 N1=166	h=3755 N2=0	N=1 N3=0	C=3228.355865089873748
D_K=-3760511 ND=1143061	H=3807 N1=167	h=3807 N2=0	N=1 N3=0	C=3272.128934247471760
D_K=-3842999 ND=1168121	H=3881 N1=168	h=3881 N2=0	N=1 N3=0	C=3335.008460658430075
D_K=-4025999 ND=1223764	H=3911 N1=169	h=3911 N2=0	N=1 N3=0	C=3359.224888464010862
D_K=-4031639 ND=1225479	H=3927 N1=170	h=3927 N2=0	N=1 N3=0	C=3372.920344194204803
D_K=-4080359 ND=1240283	H=3997 N1=171	h=3997 N2=0	N=1 N3=0	C=3432.631349672983710
D_K=-4145951 ND=1260219	H=4065 N1=172	h=4065 N2=0	N=1 N3=0	C=3490.473203669014561
D_K=-4305479 ND=1308714	H=4227 N1=173	h=4227 N2=0	N=1 N3=0	C=3628.206801079341630
D_K=-4785791 ND=1454704	H=4255 N1=174	h=4255 N2=0	N=1 N3=0	C=3648.379670904952236
D_K=-4796039 ND=1457819	H=4355 N1=175	h=4355 N2=0	N=1 N3=0	C=3734.043150040430546
D_K=-4867799 ND=1479633	H=4399 N1=176	h=4399 N2=0	N=1 N3=0	C=3771.209295058015080
D_K=-4962359 ND=1508374	H=4465 N1=177	h=4465 N2=0	N=1 N3=0	C=3827.053922733881144
D_K=-5124479 ND=1557647	H=4505 N1=178	h=4505 N2=0	N=1 N3=0	C=3860.097710924855682
D_K=-5154551 ND=1566791	H=4551 N1=179	h=4551 N2=0	N=1 N3=0	C=3899.284533343267754
D_K=-5416199 ND=1646328	H=4611 N1=180	h=4611 N2=0	N=1 N3=0	C=3948.736699572478907
D_K=-5473991 ND=1663902	H=4641 N1=181	h=4641 N2=0	N=1 N3=0	C=3974.006083741482663
D_K=-5645279 ND=1715966	H=4785 N1=182	h=4785 N2=0	N=1 N3=0	C=4096.048478009798967
D_K=-5811599 ND=1766532	H=4797 N1=183	h=4797 N2=0	N=1 N3=0	C=4105.128556468933615
D_K=-5826911 ND=1771176	H=4829 N1=184	h=4829 N2=0	N=1 N3=0	C=4132.404459586926310
D_K=-5892311 ND=1791068	H=4913 N1=185	h=4913 N2=0	N=1 N3=0	C=4203.818020286060312

D_K=-6211631 ND=1888129	H=5009 N1=186	h=5009 N2=0	N=1 N3=0	C=4283.699278839097414
D_K=-6429671 ND=1954414	H=5097 N1=187	h=5097 N2=0	N=1 N3=0	C=4357.453347377080273
D_K=-6729791 ND=2045628	H=5163 N1=188	h=5163 N2=0	N=1 N3=0	C=4411.863927391362439
D_K=-6915191 ND=2101968	H=5167 N1=189	h=5167 N2=0	N=1 N3=0	C=4414.082233728897430
D_K=-7038191 ND=2139350	H=5227 N1=190	h=5227 N2=0	N=1 N3=0	C=4464.552040331372863
D_K=-7130471 ND=2167415	H=5307 N1=191	h=5307 N2=0	N=1 N3=0	C=4532.292242701995445
D_K=-7325471 ND=2226681	H=5349 N1=192	h=5349 N2=0	N=1 N3=0	C=4566.928815798262702
D_K=-7475159 ND=2272191	H=5359 N1=193	h=5359 N2=0	N=1 N3=0	C=4574.541298297046086
D_K=-7592471 ND=2307840	H=5453 N1=194	h=5453 N2=0	N=1 N3=0	C=4654.056660470069008
D_K=-7747871 ND=2355071	H=5557 N1=195	h=5557 N2=0	N=1 N3=0	C=4741.858311704268116
D_K=-7790639 ND=2368073	H=5595 N1=196	h=5595 N2=0	N=1 N3=0	C=4774.021384799018032
D_K=-8150159 ND=2477368	H=5637 N1=197	h=5637 N2=0	N=1 N3=0	C=4807.689085944958822
D_K=-8213399 ND=2496583	H=5669 N1=198	h=5669 N2=0	N=1 N3=0	C=4834.607570611023407
D_K=-8263631 ND=2511849	H=5793 N1=199	h=5793 N2=0	N=1 N3=0	C=4940.055400147202968
D_K=-8282039 ND=2517447	H=5819 N1=200	h=5819 N2=0	N=1 N3=0	C=4962.116821165968662
D_K=-8670791 ND=2635585	H=5931 N1=201	h=5931 N2=0	N=1 N3=0	C=5055.304697269465203
D_K=-8851751 ND=2690579	H=5999 N1=202	h=5999 N2=0	N=1 N3=0	C=5112.208640776301944
D_K=-8947319 ND=2719631	H=6027 N1=203	h=6027 N2=0	N=1 N3=0	C=5135.518076756355998
D_K=-9067439 ND=2756142	H=6127 N1=204	h=6127 N2=0	N=1 N3=0	C=5220.030420270431426
D_K=-9646199 ND=2932093	H=6177 N1=205	h=6177 N2=0	N=1 N3=0	C=5259.373807277047663
D_K=-9679799 ND=2942301	H=6233 N1=206	h=6233 N2=0	N=1 N3=0	C=5306.870176633808615

D_K=-9858551 ND=2996634	H=6343 h=6343 N1=207 N2=0	N=1 N3=0	C=5399.537733875210128
D_K=-10057031 ND=3056961	H=6425 h=6425 N1=208 N2=0	N=1 N3=0	C=5468.250912707056314
D_K=-10289639 ND=3127669	H=6563 h=6563 N1=209 N2=0	N=1 N3=0	C=5584.424231397030922
D_K=-10616759 ND=3227091	H=6583 h=6583 N1=210 N2=0	N=1 N3=0	C=5599.689361281579953
D_K=-10865279 ND=3302640	H=6725 h=6725 N1=211 N2=0	N=1 N3=0	C=5719.155170318675930
D_K=-10984991 ND=3339026	H=6767 h=6767 N1=212 N2=0	N=1 N3=0	C=5754.242753006635690
D_K=-11101151 ND=3374344	H=6777 h=6777 N1=213 N2=0	N=1 N3=0	C=5762.139994939395548
D_K=-11526191 ND=3503550	H=6855 h=6855 N1=214 N2=0	N=1 N3=0	C=5826.269922830374966
D_K=-11689079 ND=3553057	H=6873 h=6873 N1=215 N2=0	N=1 N3=0	C=5840.748968739509412
D_K=-11935871 ND=3628069	H=7165 h=7165 N1=216 N2=0	N=1 N3=0	C=6087.621659502384111
D_K=-12537719 ND=3811025	H=7457 h=7457 N1=217 N2=0	N=1 N3=0	C=6332.598559043377796
D_K=-13518119 ND=4109018	H=7665 h=7665 N1=218 N2=0	N=1 N3=0	C=6504.336429150969353
D_K=-13665791 ND=4153909	H=7697 h=7697 N1=219 N2=0	N=1 N3=0	C=6530.781274097722047
D_K=-13999079 ND=4255221	H=7719 h=7719 N1=220 N2=0	N=1 N3=0	C=6547.869967583099803
D_K=-14669591 ND=4459049	H=7759 h=7759 N1=221 N2=0	N=1 N3=0	C=6578.722563570647347
D_K=-14810351 ND=4501830	H=7831 h=7831 N1=222 N2=0	N=1 N3=0	C=6639.136082107079643
D_K=-14875559 ND=4521658	H=7847 h=7847 N1=223 N2=0	N=1 N3=0	C=6652.408651094394416
D_K=-15097631 ND=4589150	H=7863 h=7863 N1=224 N2=0	N=1 N3=0	C=6664.985172778287978
D_K=-15301079 ND=4650998	H=7907 h=7907 N1=225 N2=0	N=1 N3=0	C=6701.384213051951199
D_K=-15360791 ND=4669151	H=7953 h=7953 N1=226 N2=0	N=1 N3=0	C=6740.107862655148867
D_K=-15495479 ND=4710080	H=7967 h=7967 N1=227 N2=0	N=1 N3=0	C=6751.383330767298783

D_K=-15633791 ND=4752121	H=7995 N1=228	h=7995 N2=0	N=1 N3=0	C=6774.509015428199387
D_K=-15815159 ND=4807257	H=8063 N1=229	h=8063 N2=0	N=1 N3=0	C=6831.340364891819332
D_K=-15973151 ND=4855291	H=8303 N1=230	h=8303 N2=0	N=1 N3=0	C=7033.980045599390247
D_K=-16178399 ND=4917664	H=8373 N1=231	h=8373 N2=0	N=1 N3=0	C=7092.375743095949030
D_K=-16368959 ND=4975593	H=8565 N1=232	h=8565 N2=0	N=1 N3=0	C=7254.160441114470195
D_K=-17368511 ND=5279386	H=8653 N1=233	h=8653 N2=0	N=1 N3=0	C=7324.349807661736943
D_K=-17445119 ND=5302686	H=8757 N1=234	h=8757 N2=0	N=1 N3=0	C=7412.054608680323031
D_K=-18159671 ND=5519865	H=8779 N1=235	h=8779 N2=0	N=1 N3=0	C=7427.693410511578252
D_K=-18410831 ND=5596187	H=8931 N1=236	h=8931 N2=0	N=1 N3=0	C=7555.258975257590923
D_K=-18690671 ND=5681264	H=9035 N1=237	h=9035 N2=0	N=1 N3=0	C=7642.085775940775483
D_K=-19620719 ND=5963983	H=9047 N1=238	h=9047 N2=0	N=1 N3=0	C=7648.520607269092972
D_K=-19632791 ND=5967645	H=9361 N1=239	h=9361 N2=0	N=1 N3=0	C=7913.934013830225939
D_K=-20711591 ND=6295581	H=9541 N1=240	h=9541 N2=0	N=1 N3=0	C=8061.795199927889133
D_K=-21542351 ND=6548087	H=9779 N1=241	h=9779 N2=0	N=1 N3=0	C=8259.647537643789714
D_K=-22295831 ND=6777117	H=9807 N1=242	h=9807 N2=0	N=1 N3=0	C=8280.449991561518506
D_K=-22527599 ND=6847559	H=9825 N1=243	h=9825 N2=0	N=1 N3=0	C=8294.790277974804695
D_K=-23016551 ND=6996169	H=9849 N1=244	h=9849 N2=0	N=1 N3=0	C=8313.267112625059942
D_K=-23123231 ND=7028594	H=9853 N1=245	h=9853 N2=0	N=1 N3=0	C=8316.258830656193340
D_K=-23347271 ND=7096703	H=9869 N1=246	h=9869 N2=0	N=1 N3=0	C=8328.960217632602576
D_K=-23572799 ND=7165251	H=10257 N1=247	h=10257 N2=0	N=1 N3=0	C=8655.581380133549148
D_K=-24789959 ND=7535212	H=10423 N1=248	h=10423 N2=0	N=1 N3=0	C=8791.236827340836172

D_K=-25155191 ND=7646234	H=10445 N1=249	h=10445 N2=0	N=1 N3=0	C=8808.504248124283417
D_K=-25489439 ND=7747851	H=10503 N1=250	h=10503 N2=0	N=1 N3=0	C=8856.247861295519608
D_K=-26060999 ND=7921584	H=10631 N1=251	h=10631 N2=0	N=1 N3=0	C=8962.191253873202892
D_K=-26644679 ND=8099003	H=10661 N1=252	h=10661 N2=0	N=1 N3=0	C=8985.491515130042387
D_K=-26719391 ND=8121720	H=10699 N1=253	h=10699 N2=0	N=1 N3=0	C=9017.266848485055928
D_K=-26795999 ND=8145016	H=10741 N1=254	h=10741 N2=0	N=1 N3=0	C=9052.405859245128726
D_K=-27113831 ND=8241635	H=10821 N1=255	h=10821 N2=0	N=1 N3=0	C=9118.753757417788628
D_K=-27441959 ND=8341358	H=10853 N1=256	h=10853 N2=0	N=1 N3=0	C=9144.619759875459149
D_K=-27655031 ND=8406137	H=10863 N1=257	h=10863 N2=0	N=1 N3=0	C=9152.337738740142472
D_K=-27701519 ND=8420270	H=10885 N1=258	h=10885 N2=0	N=1 N3=0	C=9170.719234478690844
D_K=-28107791 ND=8543764	H=11039 N1=259	h=11039 N2=0	N=1 N3=0	C=9299.111737302289309
D_K=-28464719 ND=8652246	H=11105 N1=260	h=11105 N2=0	N=1 N3=0	C=9353.528930926324437
D_K=-28586351 ND=8689219	H=11169 N1=261	h=11169 N2=0	N=1 N3=0	C=9407.033783680700862
D_K=-29036999 ND=8826184	H=11421 N1=262	h=11421 N2=0	N=1 N3=0	C=9617.775029814718636
D_K=-29261231 ND=8894360	H=11447 N1=263	h=11447 N2=0	N=1 N3=0	C=9638.928458613975357
D_K=-29858111 ND=9075782	H=11885 N1=264	h=11885 N2=0	N=1 N3=0	C=10005.725007723046170
D_K=-30706079 ND=9333539	H=12017 N1=265	h=12017 N2=0	N=1 N3=0	C=10114.020220045565043
D_K=-32936879 ND=10011607	H=12151 N1=266	h=12151 N2=0	N=1 N3=0	C=10219.630562353580258
D_K=-34103471 ND=10366212	H=12285 N1=267	h=12285 N2=0	N=1 N3=0	C=10328.735949610411367
D_K=-34867271 ND=10598395	H=12531 N1=268	h=12531 N2=0	N=1 N3=0	C=10533.229588133498493
D_K=-35377319 ND=10753433	H=12549 N1=269	h=12549 N2=0	N=1 N3=0	C=10546.828160902883529

D_K=-36098039 ND=10972497	H=12673 N1=270	h=12673 N2=0	N=1 N3=0	C=10648.896322497746060
D_K=-36415991 ND=11069163	H=12719 N1=271	h=12719 N2=0	N=1 N3=0	C=10686.612105631847528
D_K=-36705191 ND=11157063	H=12727 N1=272	h=12727 N2=0	N=1 N3=0	C=10692.487943350604692
D_K=-36830351 ND=11195107	H=12817 N1=273	h=12817 N2=0	N=1 N3=0	C=10767.734180759178032
D_K=-37883039 ND=11515082	H=12933 N1=274	h=12933 N2=0	N=1 N3=0	C=10862.125829607070081
D_K=-38316911 ND=11646958	H=13177 N1=275	h=13177 N2=0	N=1 N3=0	C=11065.795525619164629
D_K=-39268319 ND=11936150	H=13253 N1=276	h=13253 N2=0	N=1 N3=0	C=11126.889493067954242
D_K=-40101071 ND=12189269	H=13757 N1=277	h=13757 N2=0	N=1 N3=0	C=11547.611851413558250
D_K=-44075279 ND=13397288	H=13791 N1=278	h=13791 N2=0	N=1 N3=0	C=11565.217581854933994
D_K=-44197271 ND=13434363	H=13969 N1=279	h=13969 N2=0	N=1 N3=0	C=11714.165696255019868
D_K=-44706191 ND=13589069	H=14281 N1=280	h=14281 N2=0	N=1 N3=0	C=11974.432562055311119
D_K=-45145799 ND=13722710	H=14317 N1=281	h=14317 N2=0	N=1 N3=0	C=12003.443470478537163
D_K=-45995039 ND=13980817	H=14537 N1=282	h=14537 N2=0	N=1 N3=0	C=12185.621394292240343
D_K=-46789679 ND=14222365	H=14587 N1=283	h=14587 N2=0	N=1 N3=0	C=12225.439542120647805
D_K=-48077279 ND=14613761	H=14591 N1=284	h=14591 N2=0	N=1 N3=0	C=12225.472656815986102
D_K=-48749879 ND=14818189	H=14691 N1=285	h=14691 N2=0	N=1 N3=0	C=12307.550411679042075
D_K=-48796439 ND=14832355	H=15195 N1=286	h=15195 N2=0	N=1 N3=0	C=12729.660556923636539
D_K=-50849399 ND=15456356	H=15351 N1=287	h=15351 N2=0	N=1 N3=0	C=12855.051280755096493
D_K=-51579239 ND=15678204	H=15381 N1=288	h=15381 N2=0	N=1 N3=0	C=12878.338107421950026
D_K=-53276471 ND=16194075	H=15525 N1=289	h=15525 N2=0	N=1 N3=0	C=12994.699900260789901
D_K=-55230551 ND=16788039	H=15663 N1=290	h=15663 N2=0	N=1 N3=0	C=13105.486704339469629

D_K=-55625951 ND=16908237	H=15925 N1=291	h=15925 N2=0	N=1 N3=0	C=13323.755868949020362
D_K=-56751551 ND=17250391	H=16107 N1=292	h=16107 N2=0	N=1 N3=0	C=13473.327962293666164
D_K=-57912791 ND=17603361	H=16141 N1=293	h=16141 N2=0	N=1 N3=0	C=13499.034042216498575
D_K=-58057991 ND=17647491	H=16631 N1=294	h=16631 N2=0	N=1 N3=0	C=13908.482343424443541
D_K=-62702111 ND=19059122	H=17083 N1=295	h=17083 N2=0	N=1 N3=0	C=14275.499700693736568
D_K=-62997479 ND=19148915	H=17167 N1=296	h=17167 N2=0	N=1 N3=0	C=14345.020573307382819
D_K=-64080719 ND=19478170	H=17349 N1=297	h=17349 N2=0	N=1 N3=0	C=14494.631319442176281
D_K=-65810351 ND=20003915	H=17379 N1=298	h=17379 N2=0	N=1 N3=0	C=14515.828922577369275
D_K=-68514599 ND=20825931	H=17485 N1=299	h=17485 N2=0	N=1 N3=0	C=14598.485587769954452
D_K=-68621159 ND=20858307	H=17763 N1=300	h=17763 N2=0	N=1 N3=0	C=14830.361429510711670
D_K=-69671159 ND=21177489	H=18455 N1=301	h=18455 N2=0	N=1 N3=0	C=15405.773880507805419
D_K=-75038519 ND=22809012	H=18611 N1=302	h=18611 N2=0	N=1 N3=0	C=15524.473007836241690
D_K=-75865919 ND=23060528	H=18653 N1=303	h=18653 N2=0	N=1 N3=0	C=15557.801389596738063
D_K=-76708271 ND=23316557	H=19191 N1=304	h=19191 N2=0	N=1 N3=0	C=16004.760643042570316
D_K=-77852711 ND=23664456	H=19493 N1=305	h=19493 N2=0	N=1 N3=0	C=16254.212955497627534
D_K=-80080439 ND=24341601	H=19521 N1=306	h=19521 N2=0	N=1 N3=0	C=16272.968995940832205
D_K=-82636319 ND=25118466	H=20299 N1=307	h=20299 N2=0	N=1 N3=0	C=16916.204782081083020
D_K=-87508511 ND=26599417	H=20473 N1=308	h=20473 N2=0	N=1 N3=0	C=17051.436949114886506
D_K=-89619911 ND=27241187	H=20715 N1=309	h=20715 N2=0	N=1 N3=0	C=17248.879671472887804
D_K=-93157679 ND=28316551	H=21507 N1=310	h=21507 N2=0	N=1 N3=0	C=17901.426862406590944
D_K=-98187911 ND=29845560	H=22043 N1=311	h=22043 N2=0	N=1 N3=0	C=18337.921971024075779

D_K=-102628679 ND=31195398	H=22235 N1=312	h=22235 N2=0	N=1 N3=0	C=18489.469322853509612
D_K=-104716751 ND=31830056	H=22315 N1=313	h=22315 N2=0	N=1 N3=0	C=18552.256048921703834
D_K=-106105271 ND=32252144	H=23077 N1=314	h=23077 N2=0	N=1 N3=0	C=19183.240892980024564
D_K=-110487599 ND=33584194	H=23275 N1=315	h=23275 N2=0	N=1 N3=0	C=19340.003764999075064
D_K=-115452431 ND=35093334	H=23501 N1=316	h=23501 N2=0	N=1 N3=0	C=19519.213378365246032
D_K=-119478719 ND=36317183	H=23553 N1=317	h=23553 N2=0	N=1 N3=0	C=19555.698211591970344
D_K=-119713151 ND=36388440	H=23643 N1=318	h=23643 N2=0	N=1 N3=0	C=19630.039052804821616
D_K=-120261671 ND=36555163	H=23715 N1=319	h=23715 N2=0	N=1 N3=0	C=19688.918290546147356
D_K=-120293879 ND=36564948	H=24503 N1=320	h=24503 N2=0	N=1 N3=0	C=20343.085515621533132
D_K=-129386039 ND=39328638	H=24921 N1=321	h=24921 N2=0	N=1 N3=0	C=20675.051070947983302
D_K=-131486759 ND=39967180	H=25817 N1=322	h=25817 N2=0	N=1 N3=0	C=21414.944566807835268
D_K=-136497191 ND=41490178	H=25891 N1=323	h=25891 N2=0	N=1 N3=0	C=21468.296640828674008
D_K=-139191191 ND=42309059	H=26033 N1=324	h=26033 N2=0	N=1 N3=0	C=21581.821730147519992
D_K=-143105639 ND=43498861	H=26295 N1=325	h=26295 N2=0	N=1 N3=0	C=21792.979367051871080
D_K=-147959159 ND=44974165	H=27335 N1=326	h=27335 N2=0	N=1 N3=0	C=22647.363965396484786
D_K=-155194511 ND=47173486	H=27981 N1=327	h=27981 N2=0	N=1 N3=0	C=23171.517035677590206
D_K=-163985279 ND=49845555	H=28275 N1=328	h=28275 N2=0	N=1 N3=0	C=23402.085675107539408
D_K=-165118799 ND=50190111	H=28675 N1=329	h=28675 N2=0	N=1 N3=0	C=23731.514850536918040
D_K=-173921879 ND=52865922	H=28925 N1=330	h=28925 N2=0	N=1 N3=0	C=23925.984962036670592
D_K=-175143359 ND=53237207	H=28953 N1=331	h=28953 N2=0	N=1 N3=0	C=23947.469764145756396
D_K=-175872311 ND=53458771	H=29037 N1=332	h=29037 N2=0	N=1 N3=0	C=24015.949953201369540

D_K=-178734839 ND=54328857	H=29153 N1=333	h=29153 N2=0	N=1 N3=0	C=24107.998761827923500
D_K=-179961911 ND=54701838	H=30069 N1=334	h=30069 N2=0	N=1 N3=0	C=24863.781414488596696
D_K=-190280471 ND=57838323	H=30093 N1=335	h=30093 N2=0	N=1 N3=0	C=24869.757063712171962
D_K=-192126359 ND=58399440	H=30151 N1=336	h=30151 N2=0	N=1 N3=0	C=24915.284532232042980
D_K=-193489799 ND=58813887	H=30439 N1=337	h=30439 N2=0	N=1 N3=0	C=25151.494731298727216
D_K=-193510511 ND=58820178	H=30831 N1=338	h=30831 N2=0	N=1 N3=0	C=25475.373830752769914
D_K=-198153359 ND=60231417	H=31003 N1=339	h=31003 N2=0	N=1 N3=0	C=25611.422815680774784
D_K=-202575551 ND=61575587	H=31395 N1=340	h=31395 N2=0	N=1 N3=0	C=25929.528346733144892
D_K=-204874799 ND=62274474	H=31419 N1=341	h=31419 N2=0	N=1 N3=0	C=25946.421734145115302
D_K=-205402991 ND=62435033	H=31755 N1=342	h=31755 N2=0	N=1 N3=0	C=26223.221868933470292
D_K=-205884839 ND=62581502	H=32529 N1=343	h=32529 N2=0	N=1 N3=0	C=26861.760265958781340
D_K=-216511151 ND=65811563	H=32661 N1=344	h=32661 N2=0	N=1 N3=0	C=26957.193432818245206
D_K=-221172311 ND=67228400	H=32917 N1=345	h=32917 N2=0	N=1 N3=0	C=27162.700178228734628
D_K=-221439599 ND=67309640	H=32979 N1=346	h=32979 N2=0	N=1 N3=0	C=27213.533128811988280
D_K=-223007879 ND=67786343	H=33015 N1=347	h=33015 N2=0	N=1 N3=0	C=27241.316970305374152
D_K=-225547559 ND=68558280	H=33345 N1=348	h=33345 N2=0	N=1 N3=0	C=27510.490924581303148
D_K=-227549999 ND=69166941	H=33405 N1=349	h=33405 N2=0	N=1 N3=0 N4=0 N5=0 N6=0	C=27557.556576757048114
D_K=-229115519 ND=69642773	H=33607 N1=350	h=33607 N2=0	N=1 N3=0	C=27722.296293539670298
D_K=-230284991 ND=69998259	H=34099 N1=351	h=34099 N2=0	N=1 N3=0	C=28126.713312157034486
D_K=-236550599 ND=71902784	H=34285 N1=352	h=34285 N2=0	N=1 N3=0	C=28272.545605000215690
D_K=-240868151 ND=73215135	H=35181 N1=353	h=35181 N2=0	N=1 N3=0	C=29006.169874263569012

D_K=-254944511 ND=77493861	H=35247 N1=354	h=35247 N2=0	N=1 N3=0	C=29044.085185605491394
D_K=-255613679 ND=77697256	H=35343 N1=355	h=35343 N2=0	N=1 N3=0	C=29122.427300947365024
D_K=-257054951 ND=78135333	H=35369 N1=356	h=35369 N2=0	N=1 N3=0	C=29142.212542340746850
D_K=-258363551 ND=78533109	H=36865 N1=357	h=36865 N2=0	N=1 N3=0	C=30373.296227365697868
D_K=-278299031 ND=84592762	H=37095 N1=358	h=37095 N2=0	N=1 N3=0	C=30540.086196359450206
D_K=-281846759 ND=85671133	H=37661 N1=359	h=37661 N2=0	N=1 N3=0	C=31002.143135995017932
D_K=-287943431 ND=87524304	H=38107 N1=360	h=38107 N2=0	N=1 N3=0	C=31362.573219348269590
D_K=-289226999 ND=87914446	H=38977 N1=361	h=38977 N2=0	N=1 N3=0	C=32077.168152646785502
D_K=-299120951 ND=90921866	H=39383 N1=362	h=39383 N2=0	N=1 N3=0	C=32400.396663769135542
D_K=-315073799 ND=95770930	H=40135 N1=363	h=40135 N2=0	N=1 N3=0	C=33001.91523031460096
D_K=-323812319 ND=98427082	H=40229 N1=364	h=40229 N2=0	N=1 N3=0	C=33070.16056024920297
D_K=-331406039 ND=100735328	H=40799 N1=365	h=40799 N2=0	N=1 N3=0	C=33530.95434353131384
D_K=-335858519 ND=102088707	H=40875 N1=366	h=40875 N2=0	N=1 N3=0	C=33588.93252826555030
D_K=-344397479 ND=104684235	H=41627 N1=367	h=41627 N2=0	N=1 N3=0	C=34198.29965317746712
D_K=-348309191 ND=105873262	H=41793 N1=368	h=41793 N2=0	N=1 N3=0	C=34330.79793085519642
D_K=-349879631 ND=106350632	H=42075 N1=369	h=42075 N2=0	N=1 N3=0	C=34560.89161534508842
D_K=-360963671 ND=109719806	H=42493 N1=370	h=42493 N2=0	N=1 N3=0	C=34893.35736507248580
D_K=-362303471 ND=110127045	H=42829 N1=371	h=42829 N2=0	N=1 N3=0	C=35167.96263930365671
D_K=-377980151 ND=114892188	H=42903 N1=372	h=42903 N2=0	N=1 N3=0	C=35213.80633544981956
D_K=-379064111 ND=115221623	H=43279 N1=373	h=43279 N2=0	N=1 N3=0	C=35521.40135260767573
D_K=-382331351 ND=116214768	H=43803 N1=374	h=43803 N2=0	N=1 N3=0	C=35948.39098607591136

D_K=-388743599 ND=118163864	H=43967 N1=375	h=43967 N2=0	N=1 N3=0	C=36076.98209829588629
D_K=-392969279 ND=119448314	H=44853 N1=376	h=44853 N2=0	N=1 N3=0	C=36800.00775604496330
D_K=-393250679 ND=119533852	H=45105 N1=377	h=45105 N2=0	N=1 N3=0	C=37006.49829594266727
D_K=-400088471 ND=121612318	H=45187 N1=378	h=45187 N2=0	N=1 N3=0	C=37067.38499305704961
D_K=-405247079 ND=123180321	H=45381 N1=379	h=45381 N2=0	N=1 N3=0	C=37221.75641332248223
D_K=-413266871 ND=125618064	H=46493 N1=380	h=46493 N2=0	N=1 N3=0	C=38126.35298530334046
D_K=-423536879 ND=128739777	H=46845 N1=381	h=46845 N2=0	N=1 N3=0	C=38405.58024460587069
D_K=-431854799 ND=131268144	H=47063 N1=382	h=47063 N2=0	N=1 N3=0	C=38576.80271596791083
D_K=-438040511 ND=133148323	H=47271 N1=383	h=47271 N2=0	N=1 N3=0	C=38741.78681212866816
D_K=-451427759 ND=137217629	H=47703 N1=384	h=47703 N2=0	N=1 N3=0	C=39084.07245235363204
D_K=-462942479 ND=140717655	H=47781 N1=385	h=47781 N2=0	N=1 N3=0	C=39138.12035762195311
D_K=-464781719 ND=141276725	H=47979 N1=386	h=47979 N2=0	N=1 N3=0	C=39298.74682092421613
D_K=-466176959 ND=141700841	H=48237 N1=387	h=48237 N2=0	N=1 N3=0	C=39508.88577063717407
D_K=-466926599 ND=141928695	H=48749 N1=388	h=48749 N2=0	N=1 N3=0	C=39927.60175886569612
D_K=-469058399 ND=142576710	H=49913 N1=389	h=49913 N2=0	N=1 N3=0	C=40879.10741974106891
D_K=-493193711 ND=149912933	H=50807 N1=390	h=50807 N2=0	N=1 N3=0	C=41590.42675030752527
D_K=-499769591 ND=151911758	H=51733 N1=391	h=51733 N2=0	N=1 N3=0	C=42342.83826103346939
D_K=-514027751 ND=156245727	H=51927 N1=392	h=51927 N2=0	N=1 N3=0	C=42489.67086792668054
D_K=-533571671 ND=162186376	H=51987 N1=393	h=51987 N2=0	N=1 N3=0	C=42522.89546584478885
D_K=-535659071 ND=162820854	H=52105 N1=394	h=52105 N2=0	N=1 N3=0	C=42617.74981814582018
D_K=-538939631 ND=163818017	H=52749 N1=395	h=52749 N2=0	N=1 N3=0	C=43141.85646898487245

D_K=-546655511 ND=166163401	H=53753 N1=396	h=53753 N2=0	N=1 N3=0	C=43956.74954353350691
D_K=-560493911 ND=170369788	H=54143 N1=397	h=54143 N2=0	N=1 N3=0	C=44264.60640241699044
D_K=-564673919 ND=171640342	H=54565 N1=398	h=54565 N2=0	N=1 N3=0	C=44606.29808323945608
D_K=-578671271 ND=175895037	H=55401 N1=399	h=55401 N2=0	N=1 N3=0	C=45278.63070931557859
D_K=-580284311 ND=176385362	H=55525 N1=400	h=55525 N2=0	N=1 N3=0	C=45378.71136702007682
D_K=-607252799 ND=184582780	H=55677 N1=401	h=55677 N2=0	N=1 N3=0	C=45482.26991935323304
D_K=-607732319 ND=184728536	H=55977 N1=402	h=55977 N2=0	N=1 N3=0	C=45726.97751295409225
D_K=-617328071 ND=187645298	H=56115 N1=403	h=56115 N2=0	N=1 N3=0	C=45832.52742670313938
D_K=-618864791 ND=188112396	H=56353 N1=404	h=56353 N2=0	N=1 N3=0	C=46025.77216246253766
D_K=-624612551 ND=189859477	H=56881 N1=405	h=56881 N2=0	N=1 N3=0	C=46452.71647458767550
D_K=-625267991 ND=190058703	H=56911 N1=406	h=56911 N2=0	N=1 N3=0	C=46476.72896924418979
D_K=-636716519 ND=193538627	H=57103 N1=407	h=57103 N2=0	N=1 N3=0	C=46625.06647931220268
D_K=-638829551 ND=194180894	H=57575 N1=408	h=57575 N2=0	N=1 N3=0	C=47008.90084085392771
D_K=-651914639 ND=198158273	H=57661 N1=409	h=57661 N2=0	N=1 N3=0	C=47069.57344403037362
D_K=-652927679 ND=198466200	H=57891 N1=410	h=57891 N2=0	N=1 N3=0	C=47256.59225302117987
D_K=-658469759 ND=200150774	H=57923 N1=411	h=57923 N2=0	N=1 N3=0	C=47278.71767401292502
D_K=-659512871 ND=200467843	H=57937 N1=412	h=57937 N2=0	N=1 N3=0	C=47289.39640287638571
D_K=-660489959 ND=200764864	H=58435 N1=413	h=58435 N2=0	N=1 N3=0	C=47695.16836321952654
D_K=-665316599 ND=202231975	H=58755 N1=414	h=58755 N2=0	N=1 N3=0	C=47952.86359793561205
D_K=-687997631 ND=209126173	H=59753 N1=415	h=59753 N2=0	N=1 N3=0	C=48751.03554809357154
D_K=-688333799 ND=209228365	H=60135 N1=416	h=60135 N2=0	N=1 N3=0	C=49062.46049037268961

D_K=-705390839 ND=214413071	H=60367 N1=417	h=60367 N2=0	N=1 N3=0	C=49239.68835185148739
D_K=-719300711 ND=218641138	H=61063 N1=418	h=61063 N2=0	N=1 N3=0	C=49797.67107964654530
D_K=-722463551 ND=219602520	H=61261 N1=419	h=61261 N2=0	N=1 N3=0	C=49956.95076433056784
D_K=-729175919 ND=221642846	H=62109 N1=420	h=62109 N2=0	N=1 N3=0	C=50643.79167902164205
D_K=-739541591 ND=224793655	H=62461 N1=421	h=62461 N2=0	N=1 N3=0	C=50923.62448618090835
D_K=-742821719 ND=225790677	H=62833 N1=422	h=62833 N2=0	N=1 N3=0	C=51224.64412975506203
D_K=-760087319 ND=231038772	H=63157 N1=423	h=63157 N2=0	N=1 N3=0	C=51476.95599183646263
D_K=-764386559 ND=232345598	H=63963 N1=424	h=63963 N2=0	N=1 N3=0	C=52130.95662224305213
D_K=-800408111 ND=243294868	H=64343 N1=425	h=64343 N2=0	N=1 N3=0	C=52416.52085213527187
D_K=-803460479 ND=244222658	H=64431 N1=426	h=64431 N2=0	N=1 N3=0	C=52486.21156217726975
D_K=-803886431 ND=244352141	H=65939 N1=427	h=65939 N2=0	N=1 N3=0	C=53714.36049914315880
D_K=-831654599 ND=252792730	H=66245 N1=428	h=66245 N2=0	N=1 N3=0	C=53945.30768401898009
D_K=-855732671 ND=260111533	H=67445 N1=429	h=67445 N2=0	N=1 N3=0	C=54906.83089380942748
D_K=-882313319 ND=268191111	H=68243 N1=430	h=68243 N2=0	N=1 N3=0	C=55539.48930900911650
D_K=-900119351 ND=273603480	H=68387 N1=431	h=68387 N2=0	N=1 N3=0	C=55645.56437743327347
D_K=-915655919 ND=278326059	H=69181 N1=432	h=69181 N2=0	N=1 N3=0	C=56281.99876483378738
D_K=-927961319 ND=282066410	H=69471 N1=433	h=69471 N2=0	N=1 N3=0	C=56510.38311561806443
D_K=-931529639 ND=283151011	H=70907 N1=434	h=70907 N2=0	N=1 N3=0	C=57676.26710774322579
D_K=-974742551 ND=296286212	H=71091 N1=435	h=71091 N2=0	N=1 N3=0	C=57799.71859570828810
D_K=-983274239 ND=298879506	H=71725 N1=436	h=71725 N2=0	N=1 N3=0	C=58310.10323354070866
D_K=-984452999 ND=299237789	H=73131 N1=437	h=73131 N2=0	N=1 N3=0	C=59452.42342116160936

D_K=-1022409911 ND=310775328	H=73399 N1=438	h=73399 N2=0	N=1 N3=0	C=59647.72616455519597
D_K=-1028010959 ND=312477876	H=73619 N1=439	h=73619 N2=0	N=1 N3=0	C=59823.24082115718328
D_K=-1041610391 ND=316611624	H=74705 N1=440	h=74705 N2=0	N=1 N3=0	C=60697.75342433928900
D_K=-1054820399 ND=320626969	H=74835 N1=441	h=74835 N2=0	N=1 N3=0	C=60795.71603058559750
D_K=-1068628679 ND=324824205	H=75161 N1=442	h=75161 N2=0	N=1 N3=0	C=61052.61655522508850
D_K=-1073223191 ND=326220765	H=76327 N1=443	h=76327 N2=0	N=1 N3=0	C=61997.08818330312358
D_K=-1111949159 ND=337992094	H=76467 N1=444	h=76467 N2=0	N=1 N3=0	C=62088.79091610950490
D_K=-1112301791 ND=338099270	H=76609 N1=445	h=76609 N2=0	N=1 N3=0	C=62203.89319898620866
D_K=-1113314879 ND=338407238	H=76687 N1=446	h=76687 N2=0	N=1 N3=0	C=62266.65966552959353
D_K=-1113352679 ND=338418716	H=76917 N1=447	h=76917 N2=0	N=1 N3=0	C=62453.38891082958095
D_K=-1118314391 ND=339926899	H=77395 N1=448	h=77395 N2=0	N=1 N3=0	C=62838.71065160483026
D_K=-1130984399 ND=343778105	H=77697 N1=449	h=77697 N2=0	N=1 N3=0	C=63076.80458027673475
D_K=-1139075159 ND=346237347	H=78141 N1=450	h=78141 N2=0	N=1 N3=0	C=63432.73556536265302
D_K=-1184068679 ND=359913706	H=81267 N1=451	h=81267 N2=0	N=1 N3=0	C=65944.78535195567069
D_K=-1229647319 ND=373767961	H=81419 N1=452	h=81419 N2=0	N=1 N3=0	C=66043.17723612411681
D_K=-1237871879 ND=376267963	H=82171 N1=453	h=82171 N2=0	N=1 N3=0	C=66648.72033098856270
D_K=-1250370239 ND=380066972	H=83503 N1=454	h=83503 N2=0	N=1 N3=0	C=67722.29891185607166
D_K=-1302462719 ND=395901232	H=83989 N1=455	h=83989 N2=0	N=1 N3=0	C=68088.65532478101015
D_K=-1324212119 ND=402512258	H=84537 N1=456	h=84537 N2=0	N=1 N3=0	C=68521.56227169522387
D_K=-1334332991 ND=405588628	H=84611 N1=457	h=84611 N2=0	N=1 N3=0	C=68576.32153935543911
D_K=-1338055991 ND=406720316	H=85633 N1=458	h=85633 N2=0	N=1 N3=0	C=69402.70796563996223

D_K=-1404765911	H=85743	h=85743	N=1	C=69458.05775887208565
ND=426997640	N1=459	N2=0	N3=0	
D_K=-1411243391	H=85753	h=85753	N=1	C=69462.96278484972745
ND=428966559	N1=460	N2=0	N3=0	
D_K=-1413028319	H=86635	h=86635	N=1	C=70176.52694691682354
ND=429509124	N1=461	N2=0	N3=0	
D_K=-1432580159	H=89145	h=89145	N=1	C=72199.76765816785209
ND=435452191	N1=462	N2=0	N3=0	
D_K=-1496626199	H=89523	h=89523	N=1	C=72474.21064917101757
ND=454919898	N1=463	N2=0	N3=0	
D_K=-1506721919	H=89569	h=89569	N=1	C=72506.57561160217400
ND=457988597	N1=464	N2=0	N3=0	
D_K=-1514970551	H=92891	h=92891	N=1	C=75191.64675412222550
ND=460495892	N1=465	N2=0	N3=0	
D_K=-1589390711	H=93485	h=93485	N=1	C=75636.18678387487827
ND=483116870	N1=466	N2=0	N3=0	
D_K=-1595386679	H=95093	h=95093	N=1	C=76934.27918746556795
ND=484939438	N1=467	N2=0	N3=0	
D_K=-1651031399	H=96025	h=96025	N=1	C=77661.67667650821250
ND=501853420	N1=468	N2=0	N3=0	
D_K=-1706994119	H=98209	h=98209	N=1	C=79401.54785829749918
ND=518863995	N1=469	N2=0	N3=0	
D_K=-1757493911	H=98735	h=98735	N=1	C=79803.54654106092957
ND=534214060	N1=470	N2=0	N3=0	
D_K=-1784227199	H=99313	h=99313	N=1	C=80258.60363995921571
ND=542339966	N1=471	N2=0	N3=0	
D_K=-1796222711	H=99821	h=99821	N=1	C=80663.73259141576146
ND=545986152	N1=472	N2=0	N3=0	
D_K=-1822315871	H=100245	h=100245	N=1	C=80994.67807804433189
ND=553917560	N1=473	N2=0	N3=0	
D_K=-1843192511	H=101379	h=101379	N=1	C=81901.58302795851589
ND=560263336	N1=474	N2=0	N3=0	
D_K=-1884403751	H=102519	h=102519	N=1	C=82804.24881790354091
ND=572790021	N1=475	N2=0	N3=0	
D_K=-1936851479	H=102869	h=102869	N=1	C=83064.13652916933369
ND=588732174	N1=476	N2=0	N3=0	
D_K=-1980242879	H=105537	h=105537	N=1	C=85199.60084691963854
ND=601921606	N1=477	N2=0	N3=0	
D_K=-2017348511	H=107005	h=107005	N=1	C=86368.67599016617290
ND=613200409	N1=478	N2=0	N3=0	

9.2. Table II: Imaginary case – Successive maxima of $\frac{H}{(\sqrt{D})^\epsilon}$.

```
{eps=0.02; bD=1; BD=10^10; ND=0; N1=0; N2=0; N3=0; N4=0; N5=0; N6=0;
Cm=0; for(D=bD, BD, e2=valuation(D, 2); d=D/2^e2;
if(e2==1 || e2>3 || core(d)!=d || (e2==0 & Mod(-d, 4)!=1) || (e2==2 & Mod(-d, 4)!=-1),
next); ND=ND+1; H=qfbclassno(-D); N=omega(D); h=H/2^(N-1); C=H/D^(eps/2);
if(C>Cm, Cm=C; if(N==1, N1=N1+1); if(N==2, N2=N2+1); if(N==3, N3=N3+1);
if(N==4, N4=N4+1); if(N==5, N5=N5+1); if(N>=6, N6=N6+1); print();
print("D_K=", -D, " H=", H, " h=", h, " N=", N, " C=", precision(C, 1));
print("ND=", ND, " N1=", N1, " N2=", N2, " N3=", N3,
" N4=", N4, " N5=", N5, " N6=", N6))}}
```

```
eps=0.02
```

```
D_K=-3      H=1 h=1      N=1      C=0.9890740041721707131
ND=1      N1=1      N2=0      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-15     H=2 h=1      N=2      C=1.9465657742816563006
ND=6      N1=1      N2=1      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-23     H=3 h=3      N=1      C=2.907394579154063316
ND=9      N1=2      N2=1      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-39     H=4 h=2      N=2      C=3.856109388213647598
ND=13     N1=2      N2=2      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-47     H=5 h=5      N=1      C=4.811151422352073885
ND=16     N1=3      N2=2      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-71     H=7 h=7      N=1      C=6.707882653447388596
ND=24     N1=4      N2=2      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-95     H=8 h=4      N=2      C=7.643860470646546498
ND=31     N1=4      N2=3      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-119    H=10 h=5     N=2      C=9.533327888898110922
ND=38     N1=4      N2=4      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-167    H=11 h=11    N=1      C=10.451184640094270919
ND=54     N1=5      N2=4      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-191    H=13 h=13    N=1      C=12.334825747149741887
ND=60     N1=6      N2=4      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-215    H=14 h=7     N=2      C=13.267944649325521866
ND=66     N1=6      N2=5      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-239    H=15 h=15    N=1      C=14.200619147352270997
ND=74     N1=7      N2=5      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-311    H=19 h=19    N=1      C=17.940146990449301159
ND=98     N1=8      N2=5      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-431    H=21 h=21    N=1      C=19.763985292405135024
ND=133    N1=9      N2=5      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-479    H=25 h=25    N=1      C=23.503722641804738888
ND=148    N1=10     N2=5      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-551    H=26 h=13    N=2      C=24.409665720156204002
ND=168    N1=10     N2=6      N3=0      N4=0      N5=0      N6=0
```

```
D_K=-671    H=30 h=15    N=2      C=28.109558827525112636
ND=206    N1=10     N2=7      N3=0      N4=0      N5=0      N6=0
```

D_K=-719	H=31	h=31	N=1	C=29.026482150819988158					
ND=221	N1=11	N2=7	N3=0	N4=0	N5=0	N6=0			
D_K=-791	H=32	h=16	N=2	C=29.934238425711375460					
ND=243	N1=11	N2=8	N3=0	N4=0	N5=0	N6=0			
D_K=-839	H=33	h=33	N=1	C=30.851502556545188090					
ND=259	N1=12	N2=8	N3=0	N4=0	N5=0	N6=0			
D_K=-959	H=36	h=18	N=2	C=33.61122295470954506					
ND=295	N1=12	N2=9	N3=0	N4=0	N5=0	N6=0			
D_K=-1151	H=41	h=41	N=1	C=38.20965386322569539					
ND=352	N1=13	N2=9	N3=0	N4=0	N5=0	N6=0			
D_K=-1319	H=45	h=45	N=1	C=41.88032717792313869					
ND=403	N1=14	N2=9	N3=0	N4=0	N5=0	N6=0			
D_K=-1511	H=49	h=49	N=1	C=45.54109150907451145					
ND=460	N1=15	N2=9	N3=0	N4=0	N5=0	N6=0			
D_K=-1559	H=51	h=51	N=1	C=47.38509055810921595					
ND=475	N1=16	N2=9	N3=0	N4=0	N5=0	N6=0			
D_K=-1679	H=52	h=26	N=2	C=48.27839644415276197					
ND=513	N1=16	N2=10	N3=0	N4=0	N5=0	N6=0			
D_K=-1991	H=56	h=28	N=2	C=51.90358005689658641					
ND=608	N1=16	N2=11	N3=0	N4=0	N5=0	N6=0			
D_K=-2159	H=60	h=30	N=2	C=55.56594746241438252					
ND=659	N1=16	N2=12	N3=0	N4=0	N5=0	N6=0			
D_K=-2351	H=63	h=63	N=1	C=58.29455926353563301					
ND=717	N1=17	N2=12	N3=0	N4=0	N5=0	N6=0			
D_K=-2519	H=64	h=32	N=2	C=59.17900954956148095					
ND=769	N1=17	N2=13	N3=0	N4=0	N5=0	N6=0			
D_K=-2831	H=68	h=34	N=2	C=62.80431946037447745					
ND=862	N1=17	N2=14	N3=0	N4=0	N5=0	N6=0			
D_K=-2999	H=73	h=73	N=1	C=67.38342712710593338					
ND=911	N1=18	N2=14	N3=0	N4=0	N5=0	N6=0			
D_K=-3071	H=76	h=38	N=2	C=70.13596779325028926					
ND=934	N1=18	N2=15	N3=0	N4=0	N5=0	N6=0			
D_K=-3671	H=81	h=81	N=1	C=74.61689535356539441					
ND=1120	N1=19	N2=15	N3=0	N4=0	N5=0	N6=0			
D_K=-3839	H=82	h=41	N=2	C=75.50429751127858116					
ND=1170	N1=19	N2=16	N3=0	N4=0	N5=0	N6=0			
D_K=-3911	H=83	h=83	N=1	C=76.41088226472032525					
ND=1191	N1=20	N2=16	N3=0	N4=0	N5=0	N6=0			
D_K=-4031	H=84	h=42	N=2	C=77.30812818364454794					
ND=1228	N1=20	N2=17	N3=0	N4=0	N5=0	N6=0			
D_K=-4079	H=85	h=85	N=1	C=78.21920339356804770					
ND=1241	N1=21	N2=17	N3=0	N4=0	N5=0	N6=0			

D_K=-4199 ND=1278	H=88 h=22 N1=21 N2=17	N=3 N3=1	C=80.95640481885140620 N4=0 N5=0 N6=0
D_K=-4679 ND=1424	H=91 h=91 N1=22 N2=17	N=1 N3=1	C=83.62571843882399364 N4=0 N5=0 N6=0
D_K=-4991 ND=1522	H=92 h=23 N1=22 N2=17	N=3 N3=2	C=84.49012480784070728 N4=0 N5=0 N6=0
D_K=-5351 ND=1633	H=93 h=93 N1=23 N2=17	N=1 N3=2	C=85.34903184030047657 N4=0 N5=0 N6=0
D_K=-5519 ND=1682	H=97 h=97 N1=24 N2=17	N=1 N3=2	C=88.99244325400892204 N4=0 N5=0 N6=0
D_K=-5591 ND=1704	H=99 h=99 N1=25 N2=17	N=1 N3=2	C=90.81556718893487486 N4=0 N5=0 N6=0
D_K=-5711 ND=1739	H=109 h=109 N1=26 N2=17	N=1 N3=2	C=99.96762544713008421 N4=0 N5=0 N6=0
D_K=-6431 ND=1959	H=114 h=57 N1=26 N2=18	N=2 N3=2	C=104.42922770558734590 N4=0 N5=0 N6=0
D_K=-6551 ND=1996	H=117 h=117 N1=27 N2=18	N=1 N3=2	C=107.15755251862864416 N4=0 N5=0 N6=0
D_K=-7391 ND=2245	H=120 h=60 N1=27 N2=19	N=2 N3=2	C=109.77266654930785907 N4=0 N5=0 N6=0
D_K=-8111 ND=2465	H=121 h=121 N1=28 N2=19	N=1 N3=2	C=110.58459361583971130 N4=0 N5=0 N6=0
D_K=-8279 ND=2515	H=126 h=63 N1=28 N2=20	N=2 N3=2	C=115.13059955191783711 N4=0 N5=0 N6=0
D_K=-8399 ND=2553	H=134 h=67 N1=28 N2=21	N=2 N3=2	C=122.42286040447331713 N4=0 N5=0 N6=0
D_K=-8951 ND=2724	H=135 h=135 N1=29 N2=21	N=1 N3=2	C=123.25798194656118673 N4=0 N5=0 N6=0
D_K=-9071 ND=2759	H=138 h=69 N1=29 N2=22	N=2 N3=2	C=125.98026998264445817 N4=0 N5=0 N6=0
D_K=-9239 ND=2813	H=139 h=139 N1=30 N2=22	N=1 N3=2	C=126.86988627904520537 N4=0 N5=0 N6=0
D_K=-10391 ND=3163	H=153 h=153 N1=31 N2=22	N=1 N3=2	C=139.48414907932407178 N4=0 N5=0 N6=0
D_K=-10919 ND=3324	H=156 h=78 N1=31 N2=23	N=2 N3=2	C=142.14865988042392703 N4=0 N5=0 N6=0
D_K=-12239 ND=3723	H=161 h=161 N1=32 N2=23	N=1 N3=2	C=146.53737808857220506 N4=0 N5=0 N6=0
D_K=-12911 ND=3928	H=165 h=165 N1=33 N2=23	N=1 N3=2	C=150.09780643352709822 N4=0 N5=0 N6=0
D_K=-13439 ND=4090	H=174 h=87 N1=33 N2=24	N=2 N3=2	C=158.22152960807372581 N4=0 N5=0 N6=0

D_K=-13991 ND=4259	H=178 N1=33	h=89 N2=25	N=2 N3=2	C=161.79366568964864186 N4=0	N5=0	N6=0
D_K=-14759 ND=4488	H=181 N1=34	h=181 N2=25	N=1 N3=2	C=164.43263091549638326 N4=0	N5=0	N6=0
D_K=-14951 ND=4544	H=185 N1=35	h=185 N2=25	N=1 N3=2	C=168.04477981706382148 N4=0	N5=0	N6=0
D_K=-15791 ND=4801	H=195 N1=36	h=195 N2=25	N=1 N3=2	C=177.03148607782637142 N4=0	N5=0	N6=0
D_K=-17111 ND=5201	H=202 N1=36	h=101 N2=26	N=2 N3=2	C=183.23929641057125087 N4=0	N5=0	N6=0
D_K=-18119 ND=5511	H=205 N1=37	h=205 N2=26	N=1 N3=2	C=185.85425945032247040 N4=0	N5=0	N6=0
D_K=-18191 ND=5533	H=213 N1=38	h=213 N2=26	N=1 N3=2	C=193.09945040387485726 N4=0	N5=0	N6=0
D_K=-21479 ND=6526	H=228 N1=38	h=114 N2=27	N=2 N3=2	C=206.35486224721466610 N4=0	N5=0	N6=0
D_K=-23231 ND=7062	H=234 N1=38	h=117 N2=28	N=2 N3=2	C=211.61925365047799378 N4=0	N5=0	N6=0
D_K=-23999 ND=7298	H=242 N1=38	h=121 N2=29	N=2 N3=2	C=218.78293015046375304 N4=0	N5=0	N6=0
D_K=-25871 ND=7866	H=248 N1=38	h=124 N2=30	N=2 N3=2	C=224.03896035171403702 N4=0	N5=0	N6=0
D_K=-27431 ND=8344	H=259 N1=39	h=259 N2=30	N=1 N3=2	C=233.83921675412191926 N4=0	N5=0	N6=0
D_K=-29279 ND=8905	H=264 N1=39	h=66 N2=30	N=3 N3=3	C=238.19813920798394606 N4=0	N5=0	N6=0
D_K=-30239 ND=9197	H=266 N1=39	h=133 N2=31	N=2 N3=3	C=239.92525352129298066 N4=0	N5=0	N6=0
D_K=-30719 ND=9346	H=272 N1=39	h=68 N2=31	N=3 N3=4	C=245.29846655724050968 N4=0	N5=0	N6=0
D_K=-31391 ND=9549	H=289 N1=40	h=289 N2=31	N=1 N3=4	C=260.5732269028509740 N4=0	N5=0	N6=0
D_K=-34151 ND=10385	H=292 N1=40	h=73 N2=31	N=3 N3=5	C=263.0563665762332205 N4=0	N5=0	N6=0
D_K=-34271 ND=10421	H=300 N1=40	h=150 N2=32	N=2 N3=5	C=270.2539105976938120 N4=0	N5=0	N6=0
D_K=-35279 ND=10729	H=305 N1=41	h=305 N2=32	N=1 N3=5	C=274.6785060757030752 N4=0	N5=0	N6=0
D_K=-38639 ND=11744	H=325 N1=42	h=325 N2=32	N=1 N3=5	C=292.4240597709833585 N4=0	N5=0	N6=0
D_K=-43391 ND=13188	H=329 N1=43	h=329 N2=32	N=1 N3=5	C=295.6799671969475978 N4=0	N5=0	N6=0

D_K=-44519 ND=13528	H=331 h=331 N1=44 N2=32	N=1 N3=5	C=297.4010789447064907 N4=0 N5=0 N6=0
D_K=-45239 ND=13748	H=352 h=176 N1=44 N2=33	N=2 N3=5	C=316.2186887837021396 N4=0 N5=0 N6=0
D_K=-48551 ND=14758	H=354 h=177 N1=44 N2=34	N=2 N3=5	C=317.7907706890234872 N4=0 N5=0 N6=0
D_K=-49871 ND=15157	H=357 h=357 N1=45 N2=34	N=1 N3=5	C=320.3979549191792761 N4=0 N5=0 N6=0
D_K=-50111 ND=15229	H=359 h=359 N1=46 N2=34	N=1 N3=5	C=322.1774337434058561 N4=0 N5=0 N6=0
D_K=-50951 ND=15485	H=361 h=361 N1=47 N2=34	N=1 N3=5	C=323.9184419334974895 N4=0 N5=0 N6=0
D_K=-53231 ND=16182	H=375 h=375 N1=48 N2=34	N=1 N3=5	C=336.3331087168986246 N4=0 N5=0 N6=0
D_K=-55319 ND=16812	H=376 h=94 N1=48 N2=34	N=3 N3=6	C=337.1002710487943399 N4=0 N5=0 N6=0
D_K=-56759 ND=17250	H=384 h=192 N1=48 N2=35	N=2 N3=6	C=344.1841581050617778 N4=0 N5=0 N6=0
D_K=-58439 ND=17764	H=387 h=387 N1=49 N2=35	N=1 N3=6	C=346.7719313457439729 N4=0 N5=0 N6=0
D_K=-60359 ND=18351	H=398 h=199 N1=49 N2=36	N=2 N3=6	C=356.5132306646911757 N4=0 N5=0 N6=0
D_K=-61871 ND=18812	H=399 h=399 N1=50 N2=36	N=1 N3=6	C=357.3205751223884702 N4=0 N5=0 N6=0
D_K=-63839 ND=19416	H=423 h=423 N1=51 N2=36	N=1 N3=6	C=378.6949438056025608 N4=0 N5=0 N6=0
D_K=-65159 ND=19816	H=424 h=212 N1=51 N2=37	N=2 N3=6	C=379.5125241767332615 N4=0 N5=0 N6=0
D_K=-68591 ND=20856	H=434 h=217 N1=51 N2=38	N=2 N3=6	C=388.2639407446393096 N4=0 N5=0 N6=0
D_K=-71231 ND=21657	H=440 h=110 N1=51 N2=38	N=3 N3=7	C=393.4830110419582534 N4=0 N5=0 N6=0
D_K=-75071 ND=22825	H=454 h=227 N1=51 N2=39	N=2 N3=7	C=405.7898042147335309 N4=0 N5=0 N6=0
D_K=-79439 ND=24153	H=468 h=117 N1=51 N2=39	N=3 N3=8	C=418.0666413199568179 N4=0 N5=0 N6=0
D_K=-84839 ND=25794	H=480 h=240 N1=51 N2=40	N=2 N3=8	C=428.5043962812660506 N4=0 N5=0 N6=0
D_K=-88079 ND=26773	H=487 h=487 N1=52 N2=40	N=1 N3=8	C=434.5905089090455506 N4=0 N5=0 N6=0
D_K=-88919 ND=27028	H=505 h=505 N1=53 N2=40	N=1 N3=8	C=450.6106298053805055 N4=0 N5=0 N6=0

D_K=-92039 ND=27979	H=510 h=255 N1=53 N2=41	N=2 N3=8	C=454.9152095400395493 N4=0 N5=0 N6=0
D_K=-95471 ND=29019	H=533 h=533 N1=54 N2=41	N=1 N3=8	C=475.2569694226818385 N4=0 N5=0 N6=0
D_K=-102119 ND=31041	H=534 h=267 N1=54 N2=42	N=2 N3=8	C=475.8282159204198926 N4=0 N5=0 N6=0
D_K=-104711 ND=31828	H=555 h=555 N1=55 N2=42	N=1 N3=8	C=494.4166185804803582 N4=0 N5=0 N6=0
D_K=-111359 ND=33854	H=556 h=278 N1=55 N2=43	N=2 N3=8	C=495.0026664362860457 N4=0 N5=0 N6=0
D_K=-113879 ND=34616	H=566 h=283 N1=55 N2=44	N=2 N3=8	C=503.7928444196468031 N4=0 N5=0 N6=0
D_K=-118271 ND=35951	H=606 h=303 N1=55 N2=45	N=2 N3=8	C=539.1924985221349878 N4=0 N5=0 N6=0
D_K=-128519 ND=39063	H=609 h=609 N1=56 N2=45	N=1 N3=8	C=541.4116782358050222 N4=0 N5=0 N6=0
D_K=-129191 ND=39266	H=620 h=155 N1=56 N2=45	N=3 N3=9	C=551.1621262850333144 N4=0 N5=0 N6=0
D_K=-140591 ND=42734	H=634 h=317 N1=56 N2=46	N=2 N3=9	C=563.1313203394783479 N4=0 N5=0 N6=0
D_K=-142271 ND=43245	H=639 h=639 N1=57 N2=46	N=1 N3=9	C=567.5050027257323546 N4=0 N5=0 N6=0
D_K=-145991 ND=44386	H=651 h=651 N1=58 N2=46	N=1 N3=9	C=578.0131617801329047 N4=0 N5=0 N6=0
D_K=-147671 ND=44895	H=667 h=667 N1=59 N2=46	N=1 N3=9	C=592.1515654661281541 N4=0 N5=0 N6=0
D_K=-152591 ND=46389	H=672 h=336 N1=59 N2=47	N=2 N3=9	C=596.3949863539554270 N4=0 N5=0 N6=0
D_K=-162839 ND=49494	H=673 h=673 N1=60 N2=47	N=1 N3=9	C=596.8943663973072745 N4=0 N5=0 N6=0
D_K=-166319 ND=50554	H=699 h=699 N1=61 N2=47	N=1 N3=9	C=619.8230990005178269 N4=0 N5=0 N6=0
D_K=-167639 ND=50960	H=708 h=354 N1=61 N2=48	N=2 N3=9	C=627.7540265818590824 N4=0 N5=0 N6=0
D_K=-171071 ND=52008	H=716 h=179 N1=61 N2=48	N=3 N3=10	C=634.7186484634092424 N4=0 N5=0 N6=0
D_K=-175031 ND=53211	H=726 h=363 N1=61 N2=49	N=2 N3=10	C=643.4361705320585957 N4=0 N5=0 N6=0
D_K=-178559 ND=54282	H=729 h=729 N1=62 N2=49	N=1 N3=10	C=645.9660759734119445 N4=0 N5=0 N6=0
D_K=-185279 ND=56332	H=756 h=378 N1=62 N2=50	N=2 N3=10	C=669.6433086128915913 N4=0 N5=0 N6=0

D_K=-191231 ND=58137	H=761 h=761 N1=63 N2=50	N=1 N3=10	C=673.8590631029059800 N4=0 N5=0 N6=0
D_K=-191351 ND=58173	H=762 h=381 N1=63 N2=51	N=2 N3=10	C=674.7403218736688293 N4=0 N5=0 N6=0
D_K=-199559 ND=60664	H=777 h=777 N1=64 N2=51	N=1 N3=10	C=687.7336994546257594 N4=0 N5=0 N6=0
D_K=-201431 ND=61237	H=808 h=404 N1=64 N2=52	N=2 N3=10	C=715.1054660787479531 N4=0 N5=0 N6=0
D_K=-225671 ND=68601	H=821 h=821 N1=65 N2=52	N=1 N3=10	C=725.7856873792070544 N4=0 N5=0 N6=0
D_K=-230879 ND=70178	H=824 h=206 N1=65 N2=52	N=3 N3=11	C=728.2715883026181575 N4=0 N5=0 N6=0
D_K=-236519 ND=71896	H=831 h=831 N1=66 N2=52	N=1 N3=11	C=734.2811229892052112 N4=0 N5=0 N6=0
D_K=-239279 ND=72733	H=880 h=220 N1=66 N2=52	N=3 N3=12	C=777.4878775306151773 N4=0 N5=0 N6=0
D_K=-250631 ND=76178	H=892 h=223 N1=66 N2=52	N=3 N3=13	C=787.7247780459091274 N4=0 N5=0 N6=0
D_K=-259871 ND=78983	H=894 h=447 N1=66 N2=53	N=2 N3=13	C=789.2052045502888466 N4=0 N5=0 N6=0
D_K=-277031 ND=84201	H=918 h=459 N1=66 N2=54	N=2 N3=13	C=809.8738896035974036 N4=0 N5=0 N6=0
D_K=-277679 ND=84395	H=930 h=465 N1=66 N2=55	N=2 N3=13	C=820.4413077339826125 N4=0 N5=0 N6=0
D_K=-284231 ND=86395	H=943 h=943 N1=67 N2=55	N=1 N3=13	C=831.7158507650857724 N4=0 N5=0 N6=0
D_K=-289511 ND=88002	H=963 h=963 N1=68 N2=55	N=1 N3=13	C=849.1993174564820835 N4=0 N5=0 N6=0
D_K=-301391 ND=91607	H=974 h=487 N1=68 N2=56	N=2 N3=13	C=858.5540754108198426 N4=0 N5=0 N6=0
D_K=-307271 ND=93396	H=992 h=496 N1=68 N2=57	N=2 N3=13	C=874.2516415002957115 N4=0 N5=0 N6=0
D_K=-312311 ND=94927	H=1001 h=1001 N1=69 N2=57	N=1 N3=13	C=882.0398458290537064 N4=0 N5=0 N6=0
D_K=-321551 ND=97740	H=1008 h=504 N1=69 N2=58	N=2 N3=13	C=887.9490224703820876 N4=0 N5=0 N6=0
D_K=-328319 ND=99798	H=1024 h=512 N1=69 N2=59	N=2 N3=13	C=901.8555795162252529 N4=0 N5=0 N6=0
D_K=-331679 ND=100819	H=1048 h=524 N1=69 N2=60	N=2 N3=13	C=922.8988458979801636 N4=0 N5=0 N6=0
D_K=-346631 ND=105366	H=1054 h=527 N1=69 N2=61	N=2 N3=13	C=927.7734429013765585 N4=0 N5=0 N6=0

D_K=-356879 ND=108490	H=1064 h=532 N1=69 N2=62	N=2 N3=13	C=936.3030067257179780 N4=0 N5=0 N6=0
D_K=-366791 ND=111500	H=1121 h=1121 N1=70 N2=62	N=1 N3=13	C=986.1918880026803627 N4=0 N5=0 N6=0
D_K=-386231 ND=117409	H=1126 h=563 N1=70 N2=63	N=2 N3=13	C=990.0791600209717668 N4=0 N5=0 N6=0
D_K=-407879 ND=123986	H=1133 h=1133 N1=71 N2=63	N=1 N3=13	C=995.6910339871266643 N4=0 N5=0 N6=0
D_K=-422231 ND=128346	H=1169 h=1169 N1=72 N2=63	N=1 N3=13	C=1026.9729627090228374 N4=0 N5=0 N6=0
D_K=-425519 ND=129347	H=1175 h=1175 N1=73 N2=63	N=1 N3=13	C=1032.1639273771506496 N4=0 N5=0 N6=0
D_K=-426191 ND=129554	H=1184 h=592 N1=73 N2=64	N=2 N3=13	C=1040.0534516460200533 N4=0 N5=0 N6=0
D_K=-442679 ND=134559	H=1204 h=602 N1=73 N2=65	N=2 N3=13	C=1057.2205535681095402 N4=0 N5=0 N6=0
D_K=-461159 ND=140175	H=1206 h=603 N1=73 N2=66	N=2 N3=13	C=1058.5437223123816383 N4=0 N5=0 N6=0
D_K=-465071 ND=141359	H=1217 h=1217 N1=74 N2=66	N=1 N3=13	C=1068.1085355213607564 N4=0 N5=0 N6=0
D_K=-469559 ND=142727	H=1250 h=625 N1=74 N2=67	N=2 N3=13	C=1096.9658593327904439 N4=0 N5=0 N6=0
D_K=-491279 ND=149334	H=1259 h=1259 N1=75 N2=67	N=1 N3=13	C=1104.3645264170961013 N4=0 N5=0 N6=0
D_K=-493919 ND=150138	H=1299 h=1299 N1=76 N2=67	N=1 N3=13	C=1139.3904991221287309 N4=0 N5=0 N6=0
D_K=-504479 ND=153350	H=1311 h=1311 N1=77 N2=67	N=1 N3=13	C=1149.6728111565926667 N4=0 N5=0 N6=0
D_K=-514751 ND=156480	H=1347 h=1347 N1=78 N2=67	N=1 N3=13	C=1181.0046939377777554 N4=0 N5=0 N6=0
D_K=-527999 ND=160503	H=1350 h=675 N1=78 N2=68	N=2 N3=13	C=1183.3342574503481345 N4=0 N5=0 N6=0
D_K=-572711 ND=174082	H=1391 h=1391 N1=79 N2=68	N=1 N3=13	C=1218.2818517395393811 N4=0 N5=0 N6=0
D_K=-592679 ND=180148	H=1402 h=701 N1=79 N2=69	N=2 N3=13	C=1227.4952443859443535 N4=0 N5=0 N6=0
D_K=-618719 ND=188063	H=1461 h=1461 N1=80 N2=69	N=1 N3=13	C=1278.6017114226840154 N4=0 N5=0 N6=0
D_K=-628319 ND=190979	H=1495 h=1495 N1=81 N2=69	N=1 N3=13	C=1308.1555579500644967 N4=0 N5=0 N6=0
D_K=-683231 ND=207671	H=1509 h=1509 N1=82 N2=69	N=1 N3=13	C=1319.3000051805935002 N4=0 N5=0 N6=0

D_K=-685439 ND=208340	H=1510 h=755 N1=82 N2=70	N=2 N3=13	C=1320.1316981510410208 N4=0 N5=0 N6=0
D_K=-698759 ND=212391	H=1516 h=758 N1=82 N2=71	N=2 N3=13	C=1325.1221917635417178 N4=0 N5=0 N6=0
D_K=-701399 ND=213196	H=1581 h=1581 N1=83 N2=71	N=1 N3=13	C=1381.8860048652978316 N4=0 N5=0 N6=0
D_K=-748271 ND=227448	H=1587 h=1587 N1=84 N2=71	N=1 N3=13	C=1386.2333331523428212 N4=0 N5=0 N6=0
D_K=-749759 ND=227900	H=1608 h=804 N1=84 N2=72	N=2 N3=13	C=1404.5487822068940143 N4=0 N5=0 N6=0
D_K=-770591 ND=234229	H=1629 h=1629 N1=85 N2=72	N=1 N3=13	C=1422.5018667232365208 N4=0 N5=0 N6=0
D_K=-777479 ND=236321	H=1641 h=1641 N1=86 N2=72	N=1 N3=13	C=1432.8531882616800175 N4=0 N5=0 N6=0
D_K=-795791 ND=241895	H=1659 h=1659 N1=87 N2=72	N=1 N3=13	C=1448.2328550493903471 N4=0 N5=0 N6=0
D_K=-816791 ND=248279	H=1664 h=832 N1=87 N2=73	N=2 N3=13	C=1452.2193276956789238 N4=0 N5=0 N6=0
D_K=-818039 ND=248656	H=1666 h=833 N1=87 N2=74	N=2 N3=13	C=1453.9425851690031328 N4=0 N5=0 N6=0
D_K=-819719 ND=249166	H=1725 h=1725 N1=88 N2=74	N=1 N3=13	C=1505.4018638206122492 N4=0 N5=0 N6=0
D_K=-880871 ND=267753	H=1733 h=1733 N1=89 N2=74	N=1 N3=13	C=1511.2956753346931059 N4=0 N5=0 N6=0
D_K=-883559 ND=268570	H=1768 h=884 N1=89 N2=75	N=2 N3=13	C=1541.7711157095196656 N4=0 N5=0 N6=0
D_K=-890951 ND=270819	H=1818 h=909 N1=89 N2=76	N=2 N3=13	C=1585.2411623379364011 N4=0 N5=0 N6=0
D_K=-929711 ND=282604	H=1834 h=917 N1=89 N2=77	N=2 N3=13	C=1598.5118188332398105 N4=0 N5=0 N6=0
D_K=-960671 ND=292014	H=1868 h=934 N1=89 N2=78	N=2 N3=13	C=1627.6129067320909398 N4=0 N5=0 N6=0
D_K=-1036031 ND=314928	H=1908 h=954 N1=89 N2=79	N=2 N3=13	C=1661.2104058064934824 N4=0 N5=0 N6=0
D_K=-1062311 ND=322911	H=1927 h=1927 N1=90 N2=79	N=1 N3=13	C=1677.3326387265150602 N4=0 N5=0 N6=0
D_K=-1086791 ND=330349	H=1965 h=1965 N1=91 N2=79	N=1 N3=13	C=1710.0196241642265837 N4=0 N5=0 N6=0
D_K=-1101239 ND=334743	H=1970 h=985 N1=91 N2=80	N=2 N3=13	C=1714.1444241231948453 N4=0 N5=0 N6=0
D_K=-1127759 ND=342813	H=1982 h=991 N1=91 N2=81	N=2 N3=13	C=1724.1755695639542890 N4=0 N5=0 N6=0

D_K=-1138199 ND=345987	H=1984 N1=91	h=496 N2=81	N=3 N3=14	C=1725.7563728536484838 N4=0	N5=0	N6=0
D_K=-1139519 ND=346386	H=2027 N1=92	h=2027 N2=81	N=1 N3=14	C=1763.1389228900476362 N4=0	N5=0	N6=0
D_K=-1190591 ND=361907	H=2051 N1=93	h=2051 N2=81	N=1 N3=14	C=1783.2327616010918186 N4=0	N5=0	N6=0
D_K=-1202759 ND=365601	H=2080 N1=93	h=1040 N2=82	N=2 N3=14	C=1808.2628034356133900 N4=0	N5=0	N6=0
D_K=-1238639 ND=376511	H=2150 N1=93	h=1075 N2=83	N=2 N3=14	C=1868.5684531060590882 N4=0	N5=0	N6=0
D_K=-1339439 ND=407137	H=2262 N1=93	h=1131 N2=84	N=2 N3=14	C=1964.3703539614856986 N4=0	N5=0	N6=0
D_K=-1412759 ND=429424	H=2311 N1=94	h=2311 N2=84	N=1 N3=14	C=2005.8537496212781478 N4=0	N5=0	N6=0
D_K=-1504751 ND=457386	H=2336 N1=94	h=1168 N2=85	N=2 N3=14	C=2026.2740941979355156 N4=0	N5=0	N6=0
D_K=-1506599 ND=457950	H=2354 N1=94	h=1177 N2=86	N=2 N3=14	C=2041.8624465065849886 N4=0	N5=0	N6=0
D_K=-1579751 ND=480170	H=2395 N1=95	h=2395 N2=86	N=1 N3=14	C=2076.441171091851336 N4=0	N5=0	N6=0
D_K=-1614311 ND=490676	H=2421 N1=96	h=2421 N2=86	N=1 N3=14	C=2098.528720649882136 N4=0	N5=0	N6=0
D_K=-1642031 ND=499109	H=2457 N1=97	h=2457 N2=86	N=1 N3=14	C=2129.371038564755560 N4=0	N5=0	N6=0
D_K=-1658159 ND=504014	H=2468 N1=97	h=617 N2=86	N=3 N3=15	C=2138.695194658879168 N4=0	N5=0	N6=0
D_K=-1661351 ND=504988	H=2496 N1=97	h=1248 N2=87	N=2 N3=15	C=2162.917562478700930 N4=0	N5=0	N6=0
D_K=-1750271 ND=532010	H=2567 N1=98	h=2567 N2=87	N=1 N3=15	C=2223.283350513854295 N4=0	N5=0	N6=0
D_K=-1789031 ND=543800	H=2582 N1=98	h=1291 N2=88	N=2 N3=15	C=2235.785108874368914 N4=0	N5=0	N6=0
D_K=-1823951 ND=554412	H=2600 N1=98	h=1300 N2=89	N=2 N3=15	C=2250.936358940840537 N4=0	N5=0	N6=0
D_K=-1884791 ND=572912	H=2669 N1=99	h=2669 N2=89	N=1 N3=15	C=2309.914695388376059 N4=0	N5=0	N6=0
D_K=-1908191 ND=580020	H=2690 N1=99	h=1345 N2=90	N=2 N3=15	C=2327.802131753633666 N4=0	N5=0	N6=0
D_K=-1942079 ND=590318	H=2706 N1=99	h=1353 N2=91	N=2 N3=15	C=2341.235622206084445 N4=0	N5=0	N6=0
D_K=-1973039 ND=599732	H=2724 N1=99	h=1362 N2=92	N=2 N3=15	C=2356.436529443956054 N4=0	N5=0	N6=0

D_K--2040671	H=2768	h=1384	N=2	C=2393.692482410613310
ND=620289	N1=99	N2=93	N3=15	N4=0 N5=0 N6=0
D_K--2155919	H=2968	h=1484	N=2	C=2565.237453745993235
ND=655332	N1=99	N2=94	N3=15	N4=0 N5=0 N6=0
D_K--2336879	H=3036	h=1518	N=2	C=2621.895661756738780
ND=710324	N1=99	N2=95	N3=15	N4=0 N5=0 N6=0
D_K--2470079	H=3079	h=3079	N=1	C=2657.556952933138179
ND=750829	N1=100	N2=95	N3=15	N4=0 N5=0 N6=0
D_K--2508791	H=3080	h=1540	N=2	C=2658.006702093879916
ND=762606	N1=100	N2=96	N3=15	N4=0 N5=0 N6=0
D_K--2597999	H=3120	h=1560	N=2	C=2691.585649978366733
ND=789716	N1=100	N2=97	N3=15	N4=0 N5=0 N6=0
D_K--2689319	H=3150	h=1575	N=2	C=2716.527654739193685
ND=817474	N1=100	N2=98	N3=15	N4=0 N5=0 N6=0
D_K--2696711	H=3182	h=1591	N=2	C=2744.048804255267436
ND=819713	N1=100	N2=99	N3=15	N4=0 N5=0 N6=0
D_K--2704679	H=3212	h=803	N=3	C=2769.838064839484108
ND=822137	N1=100	N2=99	N3=16	N4=0 N5=0 N6=0
D_K--2798351	H=3219	h=3219	N=1	C=2774.929505870438035
ND=850609	N1=101	N2=99	N3=16	N4=0 N5=0 N6=0
D_K--2819879	H=3220	h=1610	N=2	C=2775.578833830112931
ND=857149	N1=101	N2=100	N3=16	N4=0 N5=0 N6=0
D_K--2843231	H=3290	h=1645	N=2	C=2835.683632670836421
ND=864246	N1=101	N2=101	N3=16	N4=0 N5=0 N6=0
D_K--2919479	H=3305	h=3305	N=1	C=2847.858521758496226
ND=887419	N1=102	N2=101	N3=16	N4=0 N5=0 N6=0
D_K--2954591	H=3464	h=866	N=3	C=2984.509115202034352
ND=898089	N1=102	N2=101	N3=17	N4=0 N5=0 N6=0
D_K--3068831	H=3485	h=3485	N=1	C=3001.463405484378354
ND=932809	N1=103	N2=101	N3=17	N4=0 N5=0 N6=0
D_K--3190151	H=3593	h=3593	N=1	C=3093.279082652959006
ND=969674	N1=104	N2=101	N3=17	N4=0 N5=0 N6=0
D_K--3312839	H=3632	h=1816	N=2	C=3125.675125147948027
ND=1006982	N1=104	N2=102	N3=17	N4=0 N5=0 N6=0
D_K--3406751	H=3636	h=1818	N=2	C=3128.242922219554128
ND=1035530	N1=104	N2=103	N3=17	N4=0 N5=0 N6=0
D_K--3511199	H=3640	h=1820	N=2	C=3130.738753931376549
ND=1067278	N1=104	N2=104	N3=17	N4=0 N5=0 N6=0
D_K--3524351	H=3714	h=1857	N=2	C=3194.266213425503417
ND=1071274	N1=104	N2=105	N3=17	N4=0 N5=0 N6=0
D_K--3573671	H=3720	h=1860	N=2	C=3198.981983763364178
ND=1086260	N1=104	N2=106	N3=17	N4=0 N5=0 N6=0

D_K=-3582479 ND=1088936	H=3726 h=1863 N1=104 N2=107	N=2 N3=17	C=3204.062758019106785 N4=0 N5=0 N6=0
D_K=-3654719 ND=1110901	H=3755 h=3755 N1=105 N2=107	N=1 N3=17	C=3228.355865089873748 N4=0 N5=0 N6=0
D_K=-3760511 ND=1143061	H=3807 h=3807 N1=106 N2=107	N=1 N3=17	C=3272.128934247471760 N4=0 N5=0 N6=0
D_K=-3789959 ND=1152010	H=3830 h=1915 N1=106 N2=108	N=2 N3=17	C=3291.640739899976807 N4=0 N5=0 N6=0
D_K=-3842999 ND=1168121	H=3881 h=3881 N1=107 N2=108	N=1 N3=17	C=3335.008460658430075 N4=0 N5=0 N6=0
D_K=-3983591 ND=1210873	H=3918 h=1959 N1=107 N2=109	N=2 N3=17	C=3365.593685838879712 N4=0 N5=0 N6=0
D_K=-4030991 ND=1225281	H=3920 h=1960 N1=107 N2=110	N=2 N3=17	C=3366.913420490774246 N4=0 N5=0 N6=0
D_K=-4031639 ND=1225479	H=3927 h=3927 N1=108 N2=110	N=1 N3=17	C=3372.920344194204803 N4=0 N5=0 N6=0
D_K=-4080359 ND=1240283	H=3997 h=3997 N1=109 N2=110	N=1 N3=17	C=3432.631349672983710 N4=0 N5=0 N6=0
D_K=-4145951 ND=1260219	H=4065 h=4065 N1=110 N2=110	N=1 N3=17	C=3490.473203669014561 N4=0 N5=0 N6=0
D_K=-4286831 ND=1303045	H=4068 h=2034 N1=110 N2=111	N=2 N3=17	C=3491.882171348136370 N4=0 N5=0 N6=0
D_K=-4305479 ND=1308714	H=4227 h=4227 N1=111 N2=111	N=1 N3=17	C=3628.206801079341630 N4=0 N5=0 N6=0
D_K=-4550831 ND=1383272	H=4260 h=2130 N1=111 N2=112	N=2 N3=17	C=3654.506107503567373 N4=0 N5=0 N6=0
D_K=-4704839 ND=1430106	H=4306 h=2153 N1=111 N2=113	N=2 N3=17	C=3692.738699790911247 N4=0 N5=0 N6=0
D_K=-4796039 ND=1457819	H=4355 h=4355 N1=112 N2=113	N=1 N3=17	C=3734.043150040430546 N4=0 N5=0 N6=0
D_K=-4867799 ND=1479633	H=4399 h=4399 N1=113 N2=113	N=1 N3=17	C=3771.209295058015080 N4=0 N5=0 N6=0
D_K=-4962359 ND=1508374	H=4465 h=4465 N1=114 N2=113	N=1 N3=17	C=3827.053922733881144 N4=0 N5=0 N6=0
D_K=-4972679 ND=1511510	H=4498 h=2249 N1=114 N2=114	N=2 N3=17	C=3855.258886018305127 N4=0 N5=0 N6=0
D_K=-5124479 ND=1557647	H=4505 h=4505 N1=115 N2=114	N=1 N3=17	C=3860.097710924855682 N4=0 N5=0 N6=0
D_K=-5154551 ND=1566791	H=4551 h=4551 N1=116 N2=114	N=1 N3=17	C=3899.284533343267754 N4=0 N5=0 N6=0
D_K=-5398511 ND=1640958	H=4588 h=2294 N1=116 N2=115	N=2 N3=17	C=3929.168641654753374 N4=0 N5=0 N6=0

D_K=-5416199 ND=1646328	H=4611 h=4611 N1=117 N2=115	N=1 N3=17	C=3948.736699572478907 N4=0 N5=0 N6=0
D_K=-5452919 ND=1657490	H=4612 h=2306 N1=117 N2=116	N=2 N3=17	C=3949.326216299591677 N4=0 N5=0 N6=0
D_K=-5473991 ND=1663902	H=4641 h=4641 N1=118 N2=116	N=1 N3=17	C=3974.006083741482663 N4=0 N5=0 N6=0
D_K=-5512751 ND=1675676	H=4644 h=2322 N1=118 N2=117	N=2 N3=17	C=3976.294361076295092 N4=0 N5=0 N6=0
D_K=-5645279 ND=1715966	H=4785 h=4785 N1=119 N2=117	N=1 N3=17	C=4096.048478009798967 N4=0 N5=0 N6=0
D_K=-5652071 ND=1718036	H=4802 h=2401 N1=119 N2=118	N=2 N3=17	C=4110.551366537549072 N4=0 N5=0 N6=0
D_K=-5826911 ND=1771176	H=4829 h=4829 N1=120 N2=118	N=1 N3=17	C=4132.404459586926310 N4=0 N5=0 N6=0
D_K=-5892311 ND=1791068	H=4913 h=4913 N1=121 N2=118	N=1 N3=17	C=4203.818020286060312 N4=0 N5=0 N6=0
D_K=-6077111 ND=1847238	H=5092 h=2546 N1=121 N2=119	N=2 N3=17	C=4355.634441781700724 N4=0 N5=0 N6=0
D_K=-6429671 ND=1954414	H=5097 h=5097 N1=122 N2=119	N=1 N3=17	C=4357.453347377080273 N4=0 N5=0 N6=0
D_K=-6602111 ND=2006814	H=5100 h=1275 N1=122 N2=119	N=3 N3=18	C=4358.864290883619572 N4=0 N5=0 N6=0
D_K=-6606599 ND=2008175	H=5180 h=2590 N1=122 N2=120	N=2 N3=18	C=4427.208547431550710 N4=0 N5=0 N6=0
D_K=-7032119 ND=2137511	H=5424 h=1356 N1=122 N2=120	N=3 N3=19	C=4632.856183697128412 N4=0 N5=0 N6=0
D_K=-7532279 ND=2289547	H=5454 h=2727 N1=122 N2=121	N=2 N3=19	C=4655.280665831306889 N4=0 N5=0 N6=0
D_K=-7532951 ND=2289753	H=5464 h=2732 N1=122 N2=122	N=2 N3=19	C=4663.812039921448445 N4=0 N5=0 N6=0
D_K=-7651199 ND=2325680	H=5628 h=2814 N1=122 N2=123	N=2 N3=19	C=4803.046528394405867 N4=0 N5=0 N6=0
D_K=-7741439 ND=2353117	H=5686 h=2843 N1=122 N2=124	N=2 N3=19	C=4851.975937492512519 N4=0 N5=0 N6=0
D_K=-8111039 ND=2465469	H=5712 h=1428 N1=122 N2=124	N=3 N3=20	C=4871.889549681632557 N4=0 N5=0 N6=0
D_K=-8263631 ND=2511849	H=5793 h=5793 N1=123 N2=124	N=1 N3=20	C=4940.055400147202968 N4=0 N5=0 N6=0
D_K=-8282039 ND=2517447	H=5819 h=5819 N1=124 N2=124	N=1 N3=20	C=4962.116821165968662 N4=0 N5=0 N6=0
D_K=-8442719 ND=2566266	H=5860 h=2930 N1=124 N2=125	N=2 N3=20	C=4996.119214049354727 N4=0 N5=0 N6=0

D_K=-8451959	H=5876	h=2938	N=2	C=5009.705696931064320
ND=2569076	N1=124	N2=126	N3=20	N4=0 N5=0 N6=0
D_K=-8608991	H=5958	h=2979	N=2	C=5078.681483904592450
ND=2616800	N1=124	N2=127	N3=20	N4=0 N5=0 N6=0
D_K=-8712479	H=5980	h=1495	N=3	C=5096.825519258405007
ND=2648255	N1=124	N2=127	N3=21	N4=0 N5=0 N6=0
D_K=-8737559	H=6088	h=3044	N=2	C=5188.726059132758561
ND=2655877	N1=124	N2=128	N3=21	N4=0 N5=0 N6=0
D_K=-9055199	H=6128	h=3064	N=2	C=5220.952915935371913
ND=2752425	N1=124	N2=129	N3=21	N4=0 N5=0 N6=0
D_K=-9122879	H=6144	h=3072	N=2	C=5234.194874957040551
ND=2773000	N1=124	N2=130	N3=21	N4=0 N5=0 N6=0
D_K=-9518519	H=6224	h=1556	N=3	C=5300.097881535055868
ND=2893289	N1=124	N2=130	N3=22	N4=0 N5=0 N6=0
D_K=-9679799	H=6233	h=6233	N=1	C=5306.870176633808615
ND=2942301	N1=125	N2=130	N3=22	N4=0 N5=0 N6=0
D_K=-9721799	H=6372	h=3186	N=2	C=5424.981990142689652
ND=2955071	N1=125	N2=131	N3=22	N4=0 N5=0 N6=0
D_K=-10023719	H=6446	h=3223	N=2	C=5486.305817108913837
ND=3046837	N1=125	N2=132	N3=22	N4=0 N5=0 N6=0
D_K=-10130111	H=6552	h=1638	N=3	C=5575.935568328255614
ND=3079168	N1=125	N2=132	N3=23	N4=0 N5=0 N6=0
D_K=-10289639	H=6563	h=6563	N=1	C=5584.424231397030922
ND=3127669	N1=126	N2=132	N3=23	N4=0 N5=0 N6=0
D_K=-10378079	H=6594	h=3297	N=2	C=5610.321807825583681
ND=3154550	N1=126	N2=133	N3=23	N4=0 N5=0 N6=0
D_K=-10766159	H=6720	h=3360	N=2	C=5715.426776822379577
ND=3272505	N1=126	N2=134	N3=23	N4=0 N5=0 N6=0
D_K=-10865279	H=6725	h=6725	N=1	C=5719.155170318675930
ND=3302640	N1=127	N2=134	N3=23	N4=0 N5=0 N6=0
D_K=-10984991	H=6767	h=6767	N=1	C=5754.242753006635690
ND=3339026	N1=128	N2=134	N3=23	N4=0 N5=0 N6=0
D_K=-11101151	H=6777	h=6777	N=1	C=5762.139994939395548
ND=3374344	N1=129	N2=134	N3=23	N4=0 N5=0 N6=0
D_K=-11406911	H=6896	h=3448	N=2	C=5861.726787311708545
ND=3467277	N1=129	N2=135	N3=23	N4=0 N5=0 N6=0
D_K=-11785199	H=6944	h=1736	N=3	C=5900.602282947783863
ND=3582276	N1=129	N2=135	N3=24	N4=0 N5=0 N6=0
D_K=-11915279	H=7206	h=3603	N=2	C=6122.562337412434763
ND=3621807	N1=129	N2=136	N3=24	N4=0 N5=0 N6=0
D_K=-12521759	H=7272	h=3636	N=2	C=6175.572388952824817
ND=3806170	N1=129	N2=137	N3=24	N4=0 N5=0 N6=0

D_K=-12537719 ND=3811025	H=7457 h=7457 N1=130 N2=137	N=1 N3=24	C=6332.598559043377796 N4=0 N5=0 N6=0
D_K=-13405031 ND=4074629	H=7548 h=3774 N1=130 N2=138	N=2 N3=24	C=6405.591117410744391 N4=0 N5=0 N6=0
D_K=-13518119 ND=4109018	H=7665 h=7665 N1=131 N2=138	N=1 N3=24	C=6504.336429150969353 N4=0 N5=0 N6=0
D_K=-13665791 ND=4153909	H=7697 h=7697 N1=132 N2=138	N=1 N3=24	C=6530.781274097722047 N4=0 N5=0 N6=0
D_K=-13992359 ND=4253184	H=7710 h=3855 N1=132 N2=139	N=2 N3=24	C=6540.266854300531403 N4=0 N5=0 N6=0
D_K=-13999079 ND=4255221	H=7719 h=7719 N1=133 N2=139	N=1 N3=24	C=6547.869967583099803 N4=0 N5=0 N6=0
D_K=-14351399 ND=4362323	H=7798 h=3899 N1=133 N2=140	N=2 N3=24	C=6613.240069488824306 N4=0 N5=0 N6=0
D_K=-14446871 ND=4391351	H=7966 h=3983 N1=133 N2=141	N=2 N3=24	C=6755.267700918292140 N4=0 N5=0 N6=0
D_K=-14533559 ND=4417700	H=8022 h=4011 N1=133 N2=142	N=2 N3=24	C=6802.349436505800060 N4=0 N5=0 N6=0
D_K=-14809031 ND=4501432	H=8064 h=2016 N1=133 N2=142	N=3 N3=25	C=6836.680000637058025 N4=0 N5=0 N6=0
D_K=-15570551 ND=4732906	H=8100 h=4050 N1=133 N2=143	N=2 N3=25	C=6863.758255379796392 N4=0 N5=0 N6=0
D_K=-15973151 ND=4855291	H=8303 h=8303 N1=134 N2=143	N=1 N3=25	C=7033.980045599390247 N4=0 N5=0 N6=0
D_K=-16095719 ND=4892539	H=8352 h=2088 N1=134 N2=143	N=3 N3=26	C=7074.950112024799891 N4=0 N5=0 N6=0
D_K=-16178399 ND=4917664	H=8373 h=8373 N1=135 N2=143	N=1 N3=26	C=7092.375743095949030 N4=0 N5=0 N6=0
D_K=-16368959 ND=4975593	H=8565 h=8565 N1=136 N2=143	N=1 N3=26	C=7254.160441114470195 N4=0 N5=0 N6=0
D_K=-17077199 ND=5190844	H=8588 h=4294 N1=136 N2=144	N=2 N3=26	C=7270.560110169526255 N4=0 N5=0 N6=0
D_K=-17363111 ND=5277745	H=8660 h=4330 N1=136 N2=145	N=2 N3=26	C=7330.297766141183900 N4=0 N5=0 N6=0
D_K=-17445119 ND=5302686	H=8757 h=8757 N1=137 N2=145	N=1 N3=26	C=7412.054608680323031 N4=0 N5=0 N6=0
D_K=-18031439 ND=5480889	H=8790 h=4395 N1=137 N2=146	N=2 N3=26	C=7437.527271809074609 N4=0 N5=0 N6=0
D_K=-18410831 ND=5596187	H=8931 h=8931 N1=138 N2=146	N=1 N3=26	C=7555.258975257590923 N4=0 N5=0 N6=0
D_K=-18635231 ND=5664419	H=9058 h=4529 N1=138 N2=147	N=2 N3=26	C=7661.767492376929235 N4=0 N5=0 N6=0

D_K=-18818159 ND=5720016	H=9130 h=4565 N1=138 N2=148	N=2 N3=26	C=7721.914812185435739 N4=0 N5=0 N6=0
D_K=-19090319 ND=5802757	H=9160 h=4580 N1=138 N2=149	N=2 N3=26	C=7746.175669921436377 N4=0 N5=0 N6=0
D_K=-19632791 ND=5967645	H=9361 h=9361 N1=139 N2=149	N=1 N3=26	C=7913.934013830225939 N4=0 N5=0 N6=0
D_K=-19780199 ND=6012468	H=9414 h=4707 N1=139 N2=150	N=2 N3=26	C=7958.145724461766713 N4=0 N5=0 N6=0
D_K=-20091959 ND=6107223	H=9440 h=4720 N1=139 N2=151	N=2 N3=26	C=7978.877023803495663 N4=0 N5=0 N6=0
D_K=-20357039 ND=6187803	H=9632 h=4816 N1=139 N2=152	N=2 N3=26	C=8140.092271358770220 N4=0 N5=0 N6=0
D_K=-21542351 ND=6548087	H=9779 h=9779 N1=140 N2=152	N=1 N3=26	C=8259.647537643789714 N4=0 N5=0 N6=0
D_K=-22295831 ND=6777117	H=9807 h=9807 N1=141 N2=152	N=1 N3=26	C=8280.449991561518506 N4=0 N5=0 N6=0
D_K=-22527599 ND=6847559	H=9825 h=9825 N1=142 N2=152	N=1 N3=26	C=8294.790277974804695 N4=0 N5=0 N6=0
D_K=-22993031 ND=6989014	H=9844 h=2461 N1=142 N2=152	N=3 N3=27	C=8309.131703398079277 N4=0 N5=0 N6=0
D_K=-23016551 ND=6996169	H=9849 h=9849 N1=143 N2=152	N=1 N3=27	C=8313.267112625059942 N4=0 N5=0 N6=0
D_K=-23123231 ND=7028594	H=9853 h=9853 N1=144 N2=152	N=1 N3=27	C=8316.258830656193340 N4=0 N5=0 N6=0
D_K=-23337599 ND=7093758	H=9964 h=4982 N1=144 N2=153	N=2 N3=27	C=8409.170481252799511 N4=0 N5=0 N6=0
D_K=-23489639 ND=7139966	H=10004 h=5002 N1=144 N2=154	N=2 N3=27	C=8442.380453822529996 N4=0 N5=0 N6=0
D_K=-23521559 ND=7149675	H=10008 h=2502 N1=144 N2=154	N=3 N3=28	C=8445.641365325372551 N4=0 N5=0 N6=0
D_K=-23538191 ND=7154729	H=10194 h=5097 N1=144 N2=155	N=2 N3=28	C=8602.543917071221272 N4=0 N5=0 N6=0
D_K=-23572799 ND=7165251	H=10257 h=10257 N1=145 N2=155	N=1 N3=28	C=8655.581380133549148 N4=0 N5=0 N6=0
D_K=-24339839 ND=7398395	H=10304 h=5152 N1=145 N2=156	N=2 N3=28	C=8692.459442222716579 N4=0 N5=0 N6=0
D_K=-24589319 ND=7474235	H=10494 h=5247 N1=145 N2=157	N=2 N3=28	C=8851.840805996592242 N4=0 N5=0 N6=0
D_K=-25489439 ND=7747851	H=10503 h=10503 N1=146 N2=157	N=1 N3=28	C=8856.247861295519608 N4=0 N5=0 N6=0
D_K=-25634831 ND=7792036	H=10522 h=5261 N1=146 N2=158	N=2 N3=28	C=8871.764252393603770 N4=0 N5=0 N6=0

D_K=-26060999 ND=7921584	H=10631 h=10631 N1=147 N2=158	N=1 N3=28	C=8962.191253873202892 N4=0 N5=0 N6=0
D_K=-26348999 ND=8009126	H=10896 h=5448 N1=147 N2=159	N=2 N3=28	C=9184.583215317895006 N4=0 N5=0 N6=0
D_K=-27162071 ND=8256305	H=10902 h=5451 N1=147 N2=160	N=2 N3=28	C=9186.848381363031881 N4=0 N5=0 N6=0
D_K=-27414791 ND=8333112	H=11004 h=2751 N1=147 N2=160	N=3 N3=29	C=9271.942556360555639 N4=0 N5=0 N6=0
D_K=-27939071 ND=8492486	H=11092 h=5546 N1=147 N2=161	N=2 N3=29	C=9344.320832247467688 N4=0 N5=0 N6=0
D_K=-28464719 ND=8652246	H=11105 h=11105 N1=148 N2=161	N=1 N3=29	C=9353.528930926324437 N4=0 N5=0 N6=0
D_K=-28570559 ND=8684406	H=11120 h=2780 N1=148 N2=161	N=3 N3=30	C=9365.815535272864569 N4=0 N5=0 N6=0
D_K=-28586351 ND=8689219	H=11169 h=11169 N1=149 N2=161	N=1 N3=30	C=9407.033783680700862 N4=0 N5=0 N6=0
D_K=-28684631 ND=8719096	H=11296 h=5648 N1=149 N2=162	N=2 N3=30	C=9513.672366506355691 N4=0 N5=0 N6=0
D_K=-29036999 ND=8826184	H=11421 h=11421 N1=150 N2=162	N=1 N3=30	C=9617.775029814718636 N4=0 N5=0 N6=0
D_K=-29261231 ND=8894360	H=11447 h=11447 N1=151 N2=162	N=1 N3=30	C=9638.928458613975357 N4=0 N5=0 N6=0
D_K=-29858111 ND=9075782	H=11885 h=11885 N1=152 N2=162	N=1 N3=30	C=10005.725007723046170 N4=0 N5=0 N6=0
D_K=-30706079 ND=9333539	H=12017 h=12017 N1=153 N2=162	N=1 N3=30	C=10114.020220045565043 N4=0 N5=0 N6=0
D_K=-32614559 ND=9913637	H=12084 h=6042 N1=153 N2=163	N=2 N3=30	C=10164.279561098632835 N4=0 N5=0 N6=0
D_K=-32752079 ND=9955443	H=12156 h=3039 N1=153 N2=163	N=3 N3=31	C=10224.411089027661652 N4=0 N5=0 N6=0
D_K=-33884399 ND=10299622	H=12294 h=6147 N1=153 N2=164	N=2 N3=31	C=10336.968930116852220 N4=0 N5=0 N6=0
D_K=-34084151 ND=10360340	H=12676 h=6338 N1=153 N2=165	N=2 N3=31	C=10657.533481973597802 N4=0 N5=0 N6=0
D_K=-36415991 ND=11069163	H=12719 h=12719 N1=154 N2=165	N=1 N3=31	C=10686.612105631847528 N4=0 N5=0 N6=0
D_K=-36437231 ND=11075620	H=12766 h=6383 N1=154 N2=166	N=2 N3=31	C=10726.039363180414078 N4=0 N5=0 N6=0
D_K=-36830351 ND=11195107	H=12817 h=12817 N1=155 N2=166	N=1 N3=31	C=10767.734180759178032 N4=0 N5=0 N6=0
D_K=-37492199 ND=11396290	H=12846 h=6423 N1=155 N2=167	N=2 N3=31	C=10790.175502279499082 N4=0 N5=0 N6=0

D_K=-37883039 ND=11515082	H=12933 h=12933 N1=156 N2=167	N=1 N3=31	C=10862.125829607070081 N4=0 N5=0 N6=0
D_K=-38209079 ND=11614189	H=12954 h=6477 N1=156 N2=168	N=2 N3=31	C=10878.830921710859335 N4=0 N5=0 N6=0
D_K=-38316911 ND=11646958	H=13177 h=13177 N1=157 N2=168	N=1 N3=31	C=11065.795525619164629 N4=0 N5=0 N6=0
D_K=-38833031 ND=11803839	H=13264 h=6632 N1=157 N2=169	N=2 N3=31	C=11137.366217458359875 N4=0 N5=0 N6=0
D_K=-39861839 ND=12116541	H=13322 h=6661 N1=157 N2=170	N=2 N3=31	C=11183.142426066386915 N4=0 N5=0 N6=0
D_K=-39862679 ND=12116795	H=13510 h=6755 N1=157 N2=171	N=2 N3=31	C=11340.956488427731233 N4=0 N5=0 N6=0
D_K=-40101071 ND=12189269	H=13757 h=13757 N1=158 N2=171	N=1 N3=31	C=11547.611851413558250 N4=0 N5=0 N6=0
D_K=-42099119 ND=12796600	H=13804 h=3451 N1=158 N2=171	N=3 N3=32	C=11581.430905906350555 N4=0 N5=0 N6=0
D_K=-42158639 ND=12814692	H=13996 h=6998 N1=158 N2=172	N=2 N3=32	C=11742.351267941682084 N4=0 N5=0 N6=0
D_K=-42423839 ND=12895305	H=14050 h=7025 N1=158 N2=173	N=2 N3=32	C=11786.916978647699866 N4=0 N5=0 N6=0
D_K=-42891239 ND=13037376	H=14154 h=7077 N1=158 N2=174	N=2 N3=32	C=11872.864334304536392 N4=0 N5=0 N6=0
D_K=-43787831 ND=13309927	H=14206 h=7103 N1=158 N2=175	N=2 N3=32	C=11914.018661057927337 N4=0 N5=0 N6=0
D_K=-44702759 ND=13588025	H=14384 h=7192 N1=158 N2=176	N=2 N3=32	C=12060.805980178430776 N4=0 N5=0 N6=0
D_K=-45846191 ND=13935574	H=14412 h=3603 N1=158 N2=176	N=3 N3=33	C=12081.231903891409100 N4=0 N5=0 N6=0
D_K=-45995039 ND=13980817	H=14537 h=14537 N1=159 N2=176	N=1 N3=33	C=12185.621394292240343 N4=0 N5=0 N6=0
D_K=-46381079 ND=14098162	H=14574 h=7287 N1=159 N2=177	N=2 N3=33	C=12215.615566002479214 N4=0 N5=0 N6=0
D_K=-46789679 ND=14222365	H=14587 h=14587 N1=160 N2=177	N=1 N3=33	C=12225.439542120647805 N4=0 N5=0 N6=0
D_K=-48077279 ND=14613761	H=14591 h=14591 N1=161 N2=177	N=1 N3=33	C=12225.472656815986102 N4=0 N5=0 N6=0
D_K=-48437399 ND=14723227	H=14710 h=7355 N1=161 N2=178	N=2 N3=33	C=12324.260360557409153 N4=0 N5=0 N6=0
D_K=-48537719 ND=14753723	H=15062 h=7531 N1=161 N2=179	N=2 N3=33	C=12618.910195420652593 N4=0 N5=0 N6=0
D_K=-48796439 ND=14832355	H=15195 h=15195 N1=162 N2=179	N=1 N3=33	C=12729.660556923636539 N4=0 N5=0 N6=0

D_K=-50771111 ND=15432554	H=15302 h=7651 N1=162 N2=180	N=2 N3=33	C=12814.215724172815553 N4=0 N5=0 N6=0
D_K=-50849399 ND=15456356	H=15351 h=15351 N1=163 N2=180	N=1 N3=33	C=12855.051280755096493 N4=0 N5=0 N6=0
D_K=-51579239 ND=15678204	H=15381 h=15381 N1=164 N2=180	N=1 N3=33	C=12878.338107421950026 N4=0 N5=0 N6=0
D_K=-52498991 ND=15957766	H=15622 h=7811 N1=164 N2=181	N=2 N3=33	C=13077.813024722362630 N4=0 N5=0 N6=0
D_K=-53400311 ND=16231722	H=15656 h=7828 N1=164 N2=182	N=2 N3=33	C=13104.044969129344161 N4=0 N5=0 N6=0
D_K=-54099839 ND=16444358	H=15686 h=7843 N1=164 N2=183	N=2 N3=33	C=13127.446316277320779 N4=0 N5=0 N6=0
D_K=-54694631 ND=16625151	H=16208 h=8104 N1=164 N2=184	N=2 N3=33	C=13562.819470017207533 N4=0 N5=0 N6=0
D_K=-56795639 ND=17263802	H=16330 h=8165 N1=164 N2=185	N=2 N3=33	C=13659.758927906243328 N4=0 N5=0 N6=0
D_K=-58057991 ND=17647491	H=16631 h=16631 N1=165 N2=185	N=1 N3=33	C=13908.482343424443541 N4=0 N5=0 N6=0
D_K=-62171639 ND=18897872	H=16766 h=8383 N1=165 N2=186	N=2 N3=33	C=14011.787383377556492 N4=0 N5=0 N6=0
D_K=-62702111 ND=19059122	H=17083 h=17083 N1=166 N2=186	N=1 N3=33	C=14275.499700693736568 N4=0 N5=0 N6=0
D_K=-62997479 ND=19148915	H=17167 h=17167 N1=167 N2=186	N=1 N3=33	C=14345.020573307382819 N4=0 N5=0 N6=0
D_K=-63434159 ND=19281640	H=17546 h=8773 N1=167 N2=187	N=2 N3=33	C=14660.706286697002230 N4=0 N5=0 N6=0
D_K=-67626599 ND=20555997	H=17648 h=8824 N1=167 N2=188	N=2 N3=33	C=14736.499022439230544 N4=0 N5=0 N6=0
D_K=-68049959 ND=20684699	H=18162 h=9081 N1=167 N2=189	N=2 N3=33	C=15164.754803365575434 N4=0 N5=0 N6=0
D_K=-69671159 ND=21177489	H=18455 h=18455 N1=168 N2=189	N=1 N3=33	C=15405.773880507805419 N4=0 N5=0 N6=0
D_K=-71859191 ND=21842575	H=18530 h=9265 N1=168 N2=190	N=2 N3=33	C=15463.599604729495530 N4=0 N5=0 N6=0
D_K=-71881031 ND=21849217	H=18552 h=4638 N1=168 N2=190	N=3 N3=34	C=15481.911931504111347 N4=0 N5=0 N6=0
D_K=-73396319 ND=22309821	H=18648 h=9324 N1=168 N2=191	N=2 N3=34	C=15558.779203424189371 N4=0 N5=0 N6=0
D_K=-74175191 ND=22546587	H=18672 h=2334 N1=168 N2=191	N=4 N3=34	C=15577.158967342971961 N4=1 N5=0 N6=0
D_K=-76243151 ND=23175190	H=18708 h=4677 N1=168 N2=191	N=3 N3=35	C=15602.900998341784869 N4=1 N5=0 N6=0

D_K=-76524839 ND=23260789	H=18868 h=9434 N1=168 N2=192	N=2 N3=35	C=15735.764354317781181 N4=1 N5=0 N6=0
D_K=-76642991 ND=23296718	H=18952 h=9476 N1=168 N2=193	N=2 N3=35	C=15805.575850082579888 N4=1 N5=0 N6=0
D_K=-76708271 ND=23316557	H=19191 h=19191 N1=169 N2=193	N=1 N3=35	C=16004.760643042570316 N4=1 N5=0 N6=0
D_K=-76868711 ND=23365314	H=19270 h=9635 N1=169 N2=194	N=2 N3=35	C=16070.308674732400437 N4=1 N5=0 N6=0
D_K=-77596991 ND=23586726	H=19352 h=9676 N1=169 N2=195	N=2 N3=35	C=16137.171202572712404 N4=1 N5=0 N6=0
D_K=-77852711 ND=23664456	H=19493 h=19493 N1=170 N2=195	N=1 N3=35	C=16254.212955497627534 N4=1 N5=0 N6=0
D_K=-80080439 ND=24341601	H=19521 h=19521 N1=171 N2=195	N=1 N3=35	C=16272.968995940832205 N4=1 N5=0 N6=0
D_K=-82636319 ND=25118466	H=20299 h=20299 N1=172 N2=195	N=1 N3=35	C=16916.204782081083020 N4=1 N5=0 N6=0
D_K=-87508511 ND=26599417	H=20473 h=20473 N1=173 N2=195	N=1 N3=35	C=17051.436949114886506 N4=1 N5=0 N6=0
D_K=-88713719 ND=26965758	H=20602 h=10301 N1=173 N2=196	N=2 N3=35	C=17156.530828224915471 N4=1 N5=0 N6=0
D_K=-89619911 ND=27241187	H=20715 h=20715 N1=174 N2=196	N=1 N3=35	C=17248.879671472887804 N4=1 N5=0 N6=0
D_K=-90172319 ND=27409091	H=20790 h=10395 N1=174 N2=197	N=2 N3=35	C=17310.266613260516279 N4=1 N5=0 N6=0
D_K=-90566999 ND=27529071	H=20900 h=10450 N1=174 N2=198	N=2 N3=35	C=17401.095333122603533 N4=1 N5=0 N6=0
D_K=-92525879 ND=28124492	H=21184 h=5296 N1=174 N2=198	N=3 N3=36	C=17633.776644347606017 N4=1 N5=0 N6=0
D_K=-93157679 ND=28316551	H=21507 h=21507 N1=175 N2=198	N=1 N3=36	C=17901.426862406590944 N4=1 N5=0 N6=0
D_K=-98187911 ND=29845560	H=22043 h=22043 N1=176 N2=198	N=1 N3=36	C=18337.921971024075779 N4=1 N5=0 N6=0
D_K=-98394911 ND=29908473	H=22106 h=11053 N1=176 N2=199	N=2 N3=36	C=18389.945378847540504 N4=1 N5=0 N6=0
D_K=-100682039 ND=30603686	H=22178 h=11089 N1=176 N2=200	N=2 N3=36	C=18445.603081663299435 N4=1 N5=0 N6=0
D_K=-102628679 ND=31195398	H=22235 h=22235 N1=177 N2=200	N=1 N3=36	C=18489.469322853509612 N4=1 N5=0 N6=0
D_K=-104716751 ND=31830056	H=22315 h=22315 N1=178 N2=200	N=1 N3=36	C=18552.256048921703834 N4=1 N5=0 N6=0
D_K=-105734879 ND=32139540	H=22530 h=11265 N1=178 N2=201	N=2 N3=36	C=18729.190585804667942 N4=1 N5=0 N6=0

D_K=-106105271 ND=32252144	H=23077 N1=179	h=23077 N2=201	N=1 N3=36	C=19183.240892980024564 N4=1 N5=0 N6=0
D_K=-110487599 ND=33584194	H=23275 N1=180	h=23275 N2=201	N=1 N3=36	C=19340.003764999075064 N4=1 N5=0 N6=0
D_K=-110869391 ND=33700249	H=23356 N1=180	h=5839 N2=201	N=3 N3=37	C=19406.640015194965820 N4=1 N5=0 N6=0
D_K=-111477791 ND=33885185	H=23374 N1=180	h=11687 N2=202	N=2 N3=37	C=19420.533497956226408 N4=1 N5=0 N6=0
D_K=-115066871 ND=34976149	H=23478 N1=180	h=11739 N2=203	N=2 N3=37	C=19500.762598364808532 N4=1 N5=0 N6=0
D_K=-115452431 ND=35093334	H=23501 N1=181	h=23501 N2=203	N=1 N3=37	C=19519.213378365246032 N4=1 N5=0 N6=0
D_K=-116170031 ND=35311448	H=23508 N1=181	h=11754 N2=204	N=2 N3=37	C=19523.817570980250800 N4=1 N5=0 N6=0
D_K=-116392799 ND=35379170	H=23778 N1=181	h=11889 N2=205	N=2 N3=37	C=19747.679112790713266 N4=1 N5=0 N6=0
D_K=-119504831 ND=36325122	H=24244 N1=181	h=12122 N2=206	N=2 N3=37	C=20129.381029818790830 N4=1 N5=0 N6=0
D_K=-120293879 ND=36564948	H=24503 N1=182	h=24503 N2=206	N=1 N3=37	C=20343.085515621533132 N4=1 N5=0 N6=0
D_K=-126254831 ND=38376874	H=24986 N1=182	h=12493 N2=207	N=2 N3=37	C=20734.055446965454808 N4=1 N5=0 N6=0
D_K=-131318879 ND=39916145	H=25452 N1=182	h=12726 N2=208	N=2 N3=37	C=21112.450429914118632 N4=1 N5=0 N6=0
D_K=-131486759 ND=39967180	H=25817 N1=183	h=25817 N2=208	N=1 N3=37	C=21414.944566807835268 N4=1 N5=0 N6=0
D_K=-136497191 ND=41490178	H=25891 N1=184	h=25891 N2=208	N=1 N3=37	C=21468.296640828674008 N4=1 N5=0 N6=0
D_K=-139191191 ND=42309059	H=26033 N1=185	h=26033 N2=208	N=1 N3=37	C=21581.821730147519992 N4=1 N5=0 N6=0
D_K=-139626119 ND=42441229	H=26108 N1=185	h=6527 N2=208	N=3 N3=38	C=21643.322828186284080 N4=1 N5=0 N6=0
D_K=-142867919 ND=43426607	H=26320 N1=185	h=6580 N2=208	N=3 N3=39	C=21814.061728692378016 N4=1 N5=0 N6=0
D_K=-145344239 ND=44179311	H=26748 N1=185	h=6687 N2=208	N=3 N3=40	C=22164.979619436116704 N4=1 N5=0 N6=0
D_K=-147959159 ND=44974165	H=27335 N1=186	h=27335 N2=208	N=1 N3=40	C=22647.363965396484786 N4=1 N5=0 N6=0
D_K=-153116519 ND=46541824	H=27596 N1=186	h=13798 N2=209	N=2 N3=40	C=22855.773092228261626 N4=1 N5=0 N6=0
D_K=-155194511 ND=47173486	H=27981 N1=187	h=27981 N2=209	N=1 N3=40	C=23171.517035677590206 N4=1 N5=0 N6=0

D_K=-156770591 ND=47652521	H=28032 N1=187	h=14016 N2=210	N=2 N3=40	C=23211.405490651190270 N4=1 N5=0 N6=0
D_K=-161192279 ND=48996574	H=28278 N1=187	h=14139 N2=211	N=2 N3=40	C=23408.589607970469634 N4=1 N5=0 N6=0
D_K=-165118799 ND=50190111	H=28675 N1=188	h=28675 N2=211	N=1 N3=40	C=23731.514850536918040 N4=1 N5=0 N6=0
D_K=-166353431 ND=50565418	H=28776 N1=188	h=14388 N2=212	N=2 N3=40	C=23813.328728767396488 N4=1 N5=0 N6=0
D_K=-170005391 ND=51675469	H=28978 N1=188	h=14489 N2=213	N=2 N3=40	C=23975.285149946907322 N4=1 N5=0 N6=0
D_K=-173540351 ND=52749950	H=29674 N1=188	h=14837 N2=214	N=2 N3=40	C=24546.076739362852158 N4=1 N5=0 N6=0
D_K=-179797439 ND=54651843	H=30074 N1=188	h=15037 N2=215	N=2 N3=40	C=24868.143248300853138 N4=1 N5=0 N6=0
D_K=-185836031 ND=56487367	H=30504 N1=188	h=15252 N2=216	N=2 N3=40	C=25215.378584266261906 N4=1 N5=0 N6=0
D_K=-193510511 ND=58820178	H=30831 N1=189	h=30831 N2=216	N=1 N3=40	C=25475.373830752769914 N4=1 N5=0 N6=0
D_K=-196785599 ND=59815644	H=31056 N1=189	h=15528 N2=217	N=2 N3=40	C=25656.982910318794680 N4=1 N5=0 N6=0
D_K=-197317559 ND=59977355	H=32062 N1=189	h=16031 N2=218	N=2 N3=40	C=26487.376966860446146 N4=1 N5=0 N6=0
D_K=-202665839 ND=61603030	H=32178 N1=189	h=16089 N2=219	N=2 N3=40	C=26576.099545489199192 N4=1 N5=0 N6=0
D_K=-205884839 ND=62581502	H=32529 N1=190	h=32529 N2=219	N=1 N3=40	C=26861.760265958781340 N4=1 N5=0 N6=0
D_K=-215926031 ND=65633703	H=32556 N1=190	h=8139 N2=219	N=3 N3=41	C=26871.257444877909956 N4=1 N5=0 N6=0
D_K=-216511151 ND=65811563	H=32661 N1=191	h=32661 N2=219	N=1 N3=41	C=26957.193432818245206 N4=1 N5=0 N6=0
D_K=-217860551 ND=66221716	H=33040 N1=191	h=8260 N2=219	N=3 N3=42	C=27268.311885931062050 N4=1 N5=0 N6=0
D_K=-223243151 ND=67857868	H=33852 N1=191	h=16926 N2=220	N=2 N3=42	C=27931.647384123880094 N4=1 N5=0 N6=0
D_K=-230284991 ND=69998259	H=34099 N1=192	h=34099 N2=220	N=1 N3=42	C=28126.713312157034486 N4=1 N5=0 N6=0
D_K=-236550599 ND=71902784	H=34285 N1=193	h=34285 N2=220	N=1 N3=42	C=28272.545605000215690 N4=1 N5=0 N6=0
D_K=-239041871 ND=72660049	H=34498 N1=193	h=17249 N2=221	N=2 N3=42	C=28445.212208161088348 N4=1 N5=0 N6=0
D_K=-239989559 ND=72948115	H=34504 N1=193	h=17252 N2=222	N=2 N3=42	C=28449.033826276815702 N4=1 N5=0 N6=0

D_K=-240868151 ND=73215135	H=35181 N1=194	h=35181 N2=222	N=1 N3=42	C=29006.169874263569012 N4=1 N5=0 N6=0
D_K=-242230271 ND=73629192	H=35666 N1=194	h=17833 N2=223	N=2 N3=42	C=29404.386396706681774 N4=1 N5=0 N6=0
D_K=-249936959 ND=75971745	H=35682 N1=194	h=17841 N2=224	N=2 N3=42	C=29408.365278417713108 N4=1 N5=0 N6=0
D_K=-253540031 ND=77066961	H=35720 N1=194	h=8930 N2=224	N=3 N3=43	C=29435.470690330967348 N4=1 N5=0 N6=0
D_K=-258363551 ND=78533109	H=36865 N1=195	h=36865 N2=224	N=1 N3=43	C=30373.296227365697868 N4=1 N5=0 N6=0
D_K=-269974799 ND=82062502	H=37132 N1=195	h=18566 N2=225	N=2 N3=43	C=30579.833011671226924 N4=1 N5=0 N6=0
D_K=-275070311 ND=83611344	H=37494 N1=195	h=18747 N2=226	N=2 N3=43	C=30872.182853746898098 N4=1 N5=0 N6=0
D_K=-280634591 ND=85302668	H=37638 N1=195	h=18819 N2=227	N=2 N3=43	C=30984.545205414613874 N4=1 N5=0 N6=0
D_K=-281846759 ND=85671133	H=37661 N1=196	h=37661 N2=227	N=1 N3=43	C=31002.143135995017932 N4=1 N5=0 N6=0
D_K=-282963911 ND=86010716	H=37670 N1=196	h=18835 N2=228	N=2 N3=43	C=31008.325174865081870 N4=1 N5=0 N6=0
D_K=-283768679 ND=86255335	H=38282 N1=196	h=19141 N2=229	N=2 N3=43	C=31511.202334822170740 N4=1 N5=0 N6=0
D_K=-289226999 ND=87914446	H=38977 N1=197	h=38977 N2=229	N=1 N3=43	C=32077.168152646785502 N4=1 N5=0 N6=0
D_K=-297074951 ND=90299944	H=39038 N1=197	h=19519 N2=230	N=2 N3=43	C=32118.769551139633824 N4=1 N5=0 N6=0
D_K=-299120951 ND=90921866	H=39383 N1=198	h=39383 N2=230	N=1 N3=43	C=32400.396663769135542 N4=1 N5=0 N6=0
D_K=-307210991 ND=93380922	H=39468 N1=198	h=9867 N2=230	N=3 N3=44	C=32461.662044620415180 N4=1 N5=0 N6=0
D_K=-312112919 ND=94870899	H=39840 N1=198	h=19920 N2=231	N=2 N3=44	C=32762.438015544397492 N4=1 N5=0 N6=0
D_K=-313103111 ND=95171872	H=40176 N1=198	h=10044 N2=231	N=3 N3=45	C=33037.70124065223709 N4=1 N5=0 N6=0
D_K=-323812319 ND=98427082	H=40229 N1=199	h=40229 N2=231	N=1 N3=45	C=33070.16056024920297 N4=1 N5=0 N6=0
D_K=-323886911 ND=98449772	H=40816 N1=199	h=10204 N2=231	N=3 N3=46	C=33552.62533165833114 N4=1 N5=0 N6=0
D_K=-333046271 ND=101233894	H=41134 N1=199	h=20567 N2=232	N=2 N3=46	C=33804.60751548484622 N4=1 N5=0 N6=0
D_K=-336195551 ND=102191145	H=41744 N1=199	h=20872 N2=233	N=2 N3=46	C=34302.68712492241050 N4=1 N5=0 N6=0

D_K=-348209879 ND=105843071	H=41912 h=20956 N1=199 N2=234	N=2 N3=46	C=34428.64848342800894 N4=1 N5=0 N6=0
D_K=-349879631 ND=106350632	H=42075 h=42075 N1=200 N2=234	N=1 N3=46	C=34560.89161534508842 N4=1 N5=0 N6=0
D_K=-351642191 ND=106886385	H=42298 h=21149 N1=200 N2=235	N=2 N3=46	C=34742.32055873884206 N4=1 N5=0 N6=0
D_K=-354544511 ND=107768601	H=43212 h=10803 N1=200 N2=235	N=3 N3=47	C=35490.13568691754601 N4=1 N5=0 N6=0
D_K=-368465159 ND=111999981	H=43854 h=21927 N1=200 N2=236	N=2 N3=47	C=36003.54362201812663 N4=1 N5=0 N6=0
D_K=-382592591 ND=116294180	H=43874 h=21937 N1=200 N2=237	N=2 N3=47	C=36006.41355897201070 N4=1 N5=0 N6=0
D_K=-383541479 ND=116582619	H=44736 h=11184 N1=200 N2=237	N=3 N3=48	C=36712.92831867342677 N4=1 N5=0 N6=0
D_K=-392969279 ND=119448314	H=44853 h=44853 N1=201 N2=237	N=1 N3=48	C=36800.00775604496330 N4=1 N5=0 N6=0
D_K=-393250679 ND=119533852	H=45105 h=45105 N1=202 N2=237	N=1 N3=48	C=37006.49829594266727 N4=1 N5=0 N6=0
D_K=-395177639 ND=120119599	H=45210 h=22605 N1=202 N2=238	N=2 N3=48	C=37090.83269458742499 N4=1 N5=0 N6=0
D_K=-405247079 ND=123180321	H=45381 h=45381 N1=203 N2=238	N=1 N3=48	C=37221.75641332248223 N4=1 N5=0 N6=0
D_K=-413266871 ND=125618064	H=46493 h=46493 N1=204 N2=238	N=1 N3=48	C=38126.35298530334046 N4=1 N5=0 N6=0
D_K=-422601791 ND=128455568	H=46522 h=23261 N1=204 N2=239	N=2 N3=48	C=38141.61373004616932 N4=1 N5=0 N6=0
D_K=-423536879 ND=128739777	H=46845 h=46845 N1=205 N2=239	N=1 N3=48	C=38405.58024460587069 N4=1 N5=0 N6=0
D_K=-431854799 ND=131268144	H=47063 h=47063 N1=206 N2=239	N=1 N3=48	C=38576.80271596791083 N4=1 N5=0 N6=0
D_K=-434368679 ND=132032275	H=47258 h=23629 N1=206 N2=240	N=2 N3=48	C=38734.39285188378966 N4=1 N5=0 N6=0
D_K=-438040511 ND=133148323	H=47271 h=47271 N1=207 N2=240	N=1 N3=48	C=38741.78681212866816 N4=1 N5=0 N6=0
D_K=-442212671 ND=134416542	H=47740 h=11935 N1=207 N2=240	N=3 N3=49	C=39122.45528501007771 N4=1 N5=0 N6=0
D_K=-443096111 ND=134685067	H=47760 h=23880 N1=207 N2=241	N=2 N3=49	C=39138.06396911797806 N4=1 N5=0 N6=0
D_K=-446191511 ND=135625967	H=48384 h=24192 N1=207 N2=242	N=2 N3=49	C=39646.65543872912942 N4=1 N5=0 N6=0
D_K=-458015351 ND=139220034	H=48700 h=24350 N1=207 N2=243	N=2 N3=49	C=39895.15538089343498 N4=1 N5=0 N6=0

D_K=-466912991 ND=141924556	H=49042 N1=207	h=24521 N2=244	N=2 N3=49	C=40167.59351246945375 N4=1 N5=0 N6=0
D_K=-469058399 ND=142576710	H=49913 N1=208	h=49913 N2=244	N=1 N3=49	C=40879.10741974106891 N4=1 N5=0 N6=0
D_K=-488814959 ND=148581936	H=50234 N1=208	h=25117 N2=245	N=2 N3=49	C=41125.03839029386013 N4=1 N5=0 N6=0
D_K=-490037831 ND=148953636	H=50530 N1=208	h=25265 N2=246	N=2 N3=49	C=41366.33094605944880 N4=1 N5=0 N6=0
D_K=-493193711 ND=149912933	H=50807 N1=209	h=50807 N2=246	N=1 N3=49	C=41590.42675030752527 N4=1 N5=0 N6=0
D_K=-499769591 ND=151911758	H=51733 N1=210	h=51733 N2=246	N=1 N3=49	C=42342.83826103346939 N4=1 N5=0 N6=0
D_K=-507138551 ND=154151658	H=52152 N1=210	h=26076 N2=247	N=2 N3=49	C=42679.53723116982749 N4=1 N5=0 N6=0
D_K=-538939631 ND=163818017	H=52749 N1=211	h=52749 N2=247	N=1 N3=49	C=43141.85646898487245 N4=1 N5=0 N6=0
D_K=-544915271 ND=165634421	H=52950 N1=211	h=26475 N2=248	N=2 N3=49	C=43301.47345020421439 N4=1 N5=0 N6=0
D_K=-545886431 ND=165929600	H=52990 N1=211	h=26495 N2=249	N=2 N3=49	C=43333.41305004690833 N4=1 N5=0 N6=0
D_K=-546655511 ND=166163401	H=53753 N1=212	h=53753 N2=249	N=1 N3=49	C=43956.74954353350691 N4=1 N5=0 N6=0
D_K=-548886719 ND=166841610	H=53848 N1=212	h=13462 N2=249	N=3 N3=50	C=44032.64260866141595 N4=1 N5=0 N6=0
D_K=-560493911 ND=170369788	H=54143 N1=213	h=54143 N2=249	N=1 N3=50	C=44264.60640241699044 N4=1 N5=0 N6=0
D_K=-564673919 ND=171640342	H=54565 N1=214	h=54565 N2=249	N=1 N3=50	C=44606.29808323945608 N4=1 N5=0 N6=0
D_K=-568257791 ND=172729724	H=54810 N1=214	h=27405 N2=250	N=2 N3=50	C=44803.74822899130543 N4=1 N5=0 N6=0
D_K=-578671271 ND=175895037	H=55401 N1=215	h=55401 N2=250	N=1 N3=50	C=45278.63070931557859 N4=1 N5=0 N6=0
D_K=-580284311 ND=176385362	H=55525 N1=216	h=55525 N2=250	N=1 N3=50	C=45378.71136702007682 N4=1 N5=0 N6=0
D_K=-602613959 ND=183172746	H=56826 N1=216	h=28413 N2=251	N=2 N3=50	C=46424.44228789970315 N4=1 N5=0 N6=0
D_K=-618512879 ND=188005426	H=57204 N1=216	h=14301 N2=251	N=3 N3=51	C=46721.08399905148586 N4=1 N5=0 N6=0
D_K=-625378991 ND=190092447	H=58028 N1=216	h=14507 N2=251	N=3 N3=52	C=47388.84998195933253 N4=1 N5=0 N6=0
D_K=-650467919 ND=197718535	H=58300 N1=216	h=29150 N2=252	N=2 N3=52	C=47592.25643106591283 N4=1 N5=0 N6=0

D_K=-654565271 ND=198963966	H=58982 N1=216	h=29491 N2=253	N=2 N3=52	C=48145.97269632658623 N4=1 N5=0 N6=0
D_K=-672142511 ND=204306805	H=59176 N1=216	h=29588 N2=254	N=2 N3=52	C=48291.53298751729445 N4=1 N5=0 N6=0
D_K=-679889759 ND=206661654	H=59748 N1=216	h=29874 N2=255	N=2 N3=52	C=48752.73533030156900 N4=1 N5=0 N6=0
D_K=-688333799 ND=209228365	H=60135 N1=217	h=60135 N2=255	N=1 N3=52	C=49062.46049037268961 N4=1 N5=0 N6=0
D_K=-693126671 ND=210685192	H=61506 N1=217	h=30753 N2=256	N=2 N3=52	C=50177.53907368319152 N4=1 N5=0 N6=0
D_K=-729065399 ND=221609255	H=63036 N1=217	h=31518 N2=257	N=2 N3=52	C=51399.74707836876871 N4=1 N5=0 N6=0
D_K=-745990439 ND=226753850	H=63396 N1=217	h=31698 N2=258	N=2 N3=52	C=51681.43026784882250 N4=1 N5=0 N6=0
D_K=-757836119 ND=230354496	H=63876 N1=217	h=31938 N2=259	N=2 N3=52	C=52064.53083465826248 N4=1 N5=0 N6=0
D_K=-764386559 ND=232345598	H=63963 N1=218	h=63963 N2=259	N=1 N3=52	C=52130.95662224305213 N4=1 N5=0 N6=0
D_K=-793561271 ND=241213645	H=65070 N1=218	h=32535 N2=260	N=2 N3=52	C=53013.31987621477448 N4=1 N5=0 N6=0
D_K=-803886431 ND=244352141	H=65939 N1=219	h=65939 N2=260	N=1 N3=52	C=53714.36049914315880 N4=1 N5=0 N6=0
D_K=-826388351 ND=251191968	H=66066 N1=219	h=33033 N2=261	N=2 N3=52	C=53802.96023253461807 N4=1 N5=0 N6=0
D_K=-826771391 ND=251308395	H=67332 N1=219	h=33666 N2=262	N=2 N3=52	C=54833.71372382706216 N4=1 N5=0 N6=0
D_K=-855732671 ND=260111533	H=67445 N1=220	h=67445 N2=262	N=1 N3=52	C=54906.83089380942748 N4=1 N5=0 N6=0
D_K=-863750159 ND=262548559	H=67592 N1=220	h=33796 N2=263	N=2 N3=52	C=55021.37200588626831 N4=1 N5=0 N6=0
D_K=-872441951 ND=265190540	H=68996 N1=220	h=34498 N2=264	N=2 N3=52	C=56158.63563209596284 N4=1 N5=0 N6=0
D_K=-886916351 ND=269590248	H=69024 N1=220	h=34512 N2=265	N=2 N3=52	C=56172.18232324914698 N4=1 N5=0 N6=0
D_K=-890492279 ND=270677212	H=69280 N1=220	h=17320 N2=265	N=3 N3=53	C=56378.24822539000177 N4=1 N5=0 N6=0
D_K=-913103231 ND=277550071	H=69322 N1=220	h=34661 N2=266	N=2 N3=53	C=56398.28335838298180 N4=1 N5=0 N6=0
D_K=-914001791 ND=277823219	H=69740 N1=220	h=17435 N2=266	N=3 N3=54	C=56737.79745141111156 N4=1 N5=0 N6=0
D_K=-919904831 ND=279617553	H=70368 N1=220	h=17592 N2=266	N=3 N3=55	C=57245.02886383280007 N4=1 N5=0 N6=0

D_K=-919965791 ND=279636084	H=70494 N1=220	h=35247 N2=267	N=2 N3=55	C=57347.49304553452196 N4=1	N5=0	N6=0
D_K=-921558959 ND=280120346	H=70902 N1=220	h=35451 N2=268	N=2 N3=55	C=57678.40665598553028 N4=1	N5=0	N6=0
D_K=-965839871 ND=293580100	H=71370 N1=220	h=35685 N2=269	N=2 N3=55	C=58031.88070421436389 N4=1	N5=0	N6=0
D_K=-967603199 ND=294116081	H=73018 N1=220	h=36509 N2=270	N=2 N3=55	C=59370.80811345334673 N4=1	N5=0	N6=0
D_K=-978006431 ND=297278315	H=73080 N1=220	h=36540 N2=271	N=2 N3=55	C=59414.86594395208926 N4=1	N5=0	N6=0
D_K=-984452999 ND=299237789	H=73131 N1=221	h=73131 N2=271	N=1 N3=55	C=59452.42342116160936 N4=1	N5=0	N6=0
D_K=-1001354639 ND=304375267	H=74768 N1=221	h=18692 N2=271	N=3 N3=56	C=60772.88933083096265 N4=1	N5=0	N6=0
D_K=-1017054191 ND=309147371	H=75498 N1=221	h=37749 N2=272	N=2 N3=56	C=61356.70176783838352 N4=1	N5=0	N6=0
(...)						

These two tables on imaginary quadratic fields may suggest some regularities depending on the context of definition of the successive maxima; for the non-genus part of the class number and $C_\varepsilon(D) = \frac{H}{2^{N-1}(\sqrt{D})^\varepsilon}$, the Conjecture 1.3 is verified, at least for $D \in [3, 2 \times 10^9]$, since we have not found any composite discriminant, and in the case using the whole class number $\frac{H}{(\sqrt{D})^\varepsilon}$, probabilistic densities for $\omega(D)$ seem to exist in a different manner than that predicted by the Conjecture on the non-genus part.

9.3. **Table III: Real case – Successive maxima of $\frac{H}{2^{N-1}(\sqrt{D})^\varepsilon}$.** The real quadratic fields give rise to the same results and comments as for the imaginary case.

The computations are slower and H must be computed in the restricted sense (two days of computation to obtain the last discriminant $D_K = 34574401$ of the following list).

Due to the particular case $D_K = 136$, giving a local maximum, we get $N_2 = 1$ at the end of the computations, which is an excellent verification of the conjecture:

```
{eps=0.02;bd=2;BD=10^10;ND=0;N1=0;N2=0;N3=0;
Cm=0;for(D=bD,BD,e2=valuation(D,2);d=D/2^e2;
if(e2==1||e2>3||core(d)!=d||e2==0&Mod(d,4)!=1||e2==2&Mod(d,4)!=-1),
next);ND=ND+1;P=x^2-D;K=bnfinit(P,1);H=bnfnarrow(K)[1];
N=omega(D);h=H/2^(N-1);C=h/D^(eps/2);if(C>Cm,Cm=C;
if(N==1,N1=N1+1);if(N==2,N2=N2+1);if(N>=3,N3=N3+1);print();
print("D_K=",D," H=",H," h=",h," N=",N," C=",precision(C,1));
print("ND=",ND," N1=",N1," N2=",N2," N3=",N3))}
```

```
eps=0.02
D_K=5 H=1 h=1 N=1 C=0.9840344433634576028
ND=1 N1=1 N2=0 N3=0

D_K=136 H=4 h=2 N=2 C=1.9041212797772518802
ND=40 N1=1 N2=1 N3=0

D_K=229 H=3 h=3 N=1 C=2.841338001830550898
ND=69 N1=2 N2=1 N3=0

D_K=401 H=5 h=5 N=1 C=4.709107022186433819
ND=121 N1=3 N2=1 N3=0

D_K=577 H=7 h=7 N=1 C=6.568803671814335481
ND=177 N1=4 N2=1 N3=0

D_K=1129 H=9 h=9 N=1 C=8.389103838263256518
ND=342 N1=5 N2=1 N3=0

D_K=1297 H=11 h=11 N=1 C=10.239135384664958651
ND=397 N1=6 N2=1 N3=0

D_K=7057 H=21 h=21 N=1 C=19.219102083164795610
ND=2143 N1=7 N2=1 N3=0

D_K=8761 H=27 h=27 N=1 C=24.656886007086108242
ND=2662 N1=8 N2=1 N3=0

D_K=14401 H=43 h=43 N=1 C=39.07369921920168190
ND=4379 N1=9 N2=1 N3=0

D_K=32401 H=45 h=45 N=1 C=40.56083898547825383
ND=9848 N1=10 N2=1 N3=0

D_K=41617 H=57 h=57 N=1 C=51.24861676322140264
ND=12651 N1=11 N2=1 N3=0

D_K=57601 H=63 h=63 N=1 C=56.45939878831204140
ND=17509 N1=12 N2=1 N3=0

D_K=90001 H=87 h=87 N=1 C=77.62056135842406069
ND=27358 N1=13 N2=1 N3=0
```

D_K=176401	H=153	h=153	N=1	C=135.58961275598587437
ND=53621	N1=14	N2=1	N3=0	
D_K=650281	H=207	h=207	N=1	C=181.06701198422100325
ND=197663	N1=15	N2=1	N3=0	
D_K=921601	H=235	h=235	N=1	C=204.84361566477445036
ND=280137	N1=16	N2=1	N3=0	
D_K=1299601	H=357	h=357	N=1	C=310.1202431051597330
ND=395027	N1=17	N2=1	N3=0	
D_K=2944657	H=377	h=377	N=1	C=324.8261637468532506
ND=895068	N1=18	N2=1	N3=0	
D_K=3686401	H=455	h=455	N=1	C=391.1518340310603556
ND=1120538	N1=19	N2=1	N3=0	
D_K=3920401	H=575	h=575	N=1	C=494.0086344963980478
ND=1191666	N1=20	N2=1	N3=0	
D_K=7290001	H=655	h=655	N=1	C=559.2603422239048551
ND=2215899	N1=21	N2=1	N3=0	
D_K=10497601	H=1061	h=1061	N=1	C=902.6190252870294034
ND=3190902	N1=22	N2=1	N3=0	
D_K=15210001	H=1245	h=1245	N=1	C=1055.2322581637658160
ND=4623288	N1=23	N2=1	N3=0	
D_K=34574401	H=1539	h=1539	N=1	C=1293.7521162120423479
ND=10509357	N1=24	N2=1	N3=0	
(...)				

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