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# Formal verification of an interior point algorithm instantiation

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**Abstract.** With the increasing power of computers, real-time algorithms tends to become more complex and therefore require better guarantees of safety. Among algorithms sustaining autonomous embedded systems, model predictive control (MPC) is now used to compute online trajectories, for example in the SpaceX rocket landing. The core components of these algorithms, such as the convex optimization function, will then have to be certified at some point. This paper focuses specifically on that problem and presents a method to formally prove a primal linear programming implementation.

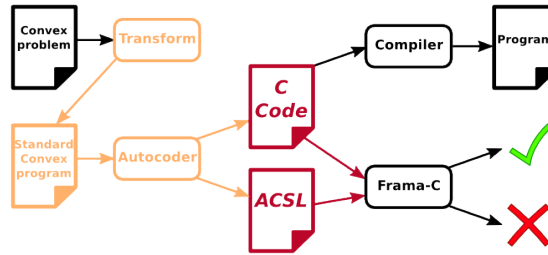
We explain how to write and annotate the code with Hoare triples in a way that eases their automatic proof. The proof process itself is performed with the WP-plugin of Frama-C and only relies on SMT solvers. Combined with a framework producing all together both the embedded code and its annotations, this work would permit to certify advanced autonomous functions relying on online optimization.

## 1 Introduction

The increasing power of computers, makes possible the use of complex numerical methods in real time within cyber-physical systems. These algorithms, despite having been studied for a long time, raise new issues when used online. Among these algorithms, we are concerned specifically with numerical optimization which is used in model predictive control (MPC) for example, by SpaceX to perform computation of trajectory for rocket landing [1].

These iterative algorithms perform complex math operations in a loop until they reach an  $\epsilon$ -close optimal value. This implies some uncertainty on the number of iterations required but also on the reliability of the computed result. We address both these issues in this paper. As a first step, we focus on linear programming, with the long-term objective of proving more general convex optimization problems. We therefore chose to study an interior point algorithm (IPM, Interior Point Method) instead of the famous simplex methods. As a matter of fact simplex is bound to linear constraints while IPMs can address more general convex cones such as the Lorentz cone problems (quadratic programming QP and second-order cone programming SOCP), or Semi-Definite Programming (SDP).

To give the best possible guarantee, we rely on formal methods to prove the algorithm soundness. More specifically we use Hoare Logic[2,3] to express what we expect from the algorithm and rely on Weakest Precondition[4] approach to prove that the code satisfy them.



**Fig. 1.** Complete toolchain we are interested in, this article focus on writing C code and annotation

Figure 1 sketches our fully automatic process which, when provided with a convex problem with some unknown values, generates the code, the associated annotation and prove it automatically. We are not going to present all the process in this paper but concentrate on how to write the embedded code, annotate it and automatize its proof.

In this first work, we focus on the algorithm itself assuming a real semantics for float variables and leave the floating point problem for future work. However the algorithm itself is expressed in C with all the associated hassle and complexity. We proved the absence of runtime error, the functionality of the code and its termination.

This paper is structured as follow. In Section 2 we present some key notions for both convex optimization and formal proof of our algorithm. In Section 3 we present the code structure supporting the later proof process. In Section 4 we introduce our code annotations. Section 5 focuses on the proof process. Section 6 presents some experimental results.

## 2 Preliminaries

In order to support the following analyses, we introduce the notions and notations used throughout the paper. First, we discuss Linear Programming (LP) problems and a primal IPM algorithm to solve it. Then, we introduce the reader to Hoare logic based reasoning.

### 2.1 Optimization

*Linear Programming* is a class of optimization problems. A linear program is defined by a matrix  $A$  of size  $m \times n$ , and two vectors  $b, c$  of respective size  $m$ , and  $n$ .

**Definition 1 (Linear program)** *Let us consider  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$  and  $c \in \mathbb{R}^n$ . We define  $P(A, b, c)$  as the linear program:*

$$\min_{x \in \mathbb{R}^n, Ax \leq b} \langle c, x \rangle \text{ with } \langle c, x \rangle = c^T x \quad (1)$$

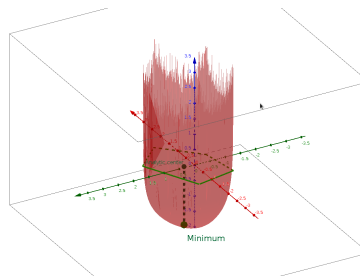
**Definition 2 (Linear program solution)** *Let us consider the problem  $P(A, b, c)$  and assume that an optimal point  $x^*$  exists and is unique. We have then the following definitions:*

$$E_f = \{x \in \mathbb{R}^n \mid Ax \leq b\} \quad (\text{feasible set of } P) \quad (2)$$

$$f(x) = \langle c, x \rangle \quad (\text{cost function}) \quad (3)$$

$$x^* = \arg \min_{x \in E_f} f \quad (\text{optimal point}) \quad (4)$$

*Primal interior point algorithm.* We decided to use interior point method (IPM) to enable the future extension of this work to more advanced convex programs. We chose a primal algorithm for its simplicity compared to primal/dual algorithms and we followed Nesterov's book [5, chap. 4] for the theory. The following definitions present the key ingredients of IPM algorithms: barrier function, central path, Newton steps and approximate centering condition.



**Fig. 2.** Barrier function

*Barrier function.* Computing the extrema, i.e. minimum or maximum, of a function without additional constraints, could be done by analyzing the zeros of its gradients. However this does not apply in presence of constraints. An approach that amounts to introducing a penalty function  $F : E_f \rightarrow \mathbb{R}$  to represent the feasible set, i.e. the acceptable region. This function must tend towards infinity when  $x$  approaches the border of  $E_f$ , cf. Figure 2 for a logarithmic barrier function encoding a set of linear constraints.

**Definition 3** *The adjusted cost function is a linear combination of the previous objective function  $f$  and the barrier function  $F$ .*

$$\tilde{f}(x, t) = t \times f(x) + F(x) \quad \text{with } t \in \mathbb{R} \quad (5)$$

The variable  $t$  balances the impact of the barrier: when  $t = 0$ ,  $\tilde{f}(x, t)$  is independent from the barrier while when  $t \rightarrow +\infty$ ,  $\tilde{f}(x, t)$  is equivalent to  $t \times f(x)$ .

*Central path.* We are interested in the values of  $x$  minimizing  $\tilde{f}$  when  $t$  varies from 0 to  $+\infty$ . These values for  $x$  characterize a path, the *central path*:

**Definition 4 (Central path and analytic center)**

$$\begin{aligned} x^* : \mathbb{R}^+ &\rightarrow E_f \\ t &\mapsto \arg \min_{x \in E_f} \tilde{f}(x, t) \end{aligned} \quad (6)$$

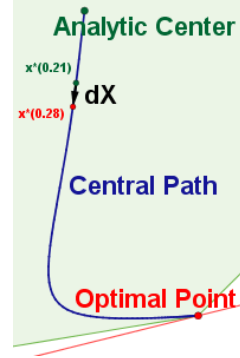
$x^*(0)$  is called the analytic center, it is independent from the cost function.

The central path has an interesting property when  $t$  increases:

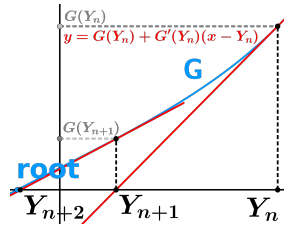
**Property 1**

$$\lim_{t \rightarrow +\infty} x^*(t) = x^* \quad (7)$$

The algorithm performs a sequence of iterations, updating a point  $X$  that follows the central path and eventually reaches the optimal point. At the beginning of an iteration, there exists a real  $t$  such that  $X = x^*(t)$ . Then  $t$  is increased by  $dt > 0$  and  $x^*(t+dt)$  is the new point  $X$ . This translation  $dX$  is performed by a Newton step as sketched in Figure 3.



**Fig. 3.** One step along the central path



**Fig. 4.** Newton step for  $k = l = 1$

*Newton step.* The Newton's method computes an approximation of a root of a function  $G : \mathbb{R}^k \rightarrow \mathbb{R}^l$ . It is a first order method, ie. it relies on the gradient of the function and, from a point in the domain of the neighbourhood of a root, performs a sequence of iterations, called Newton steps. Figure 4 illustrates one of such step.

**Definition 5** A Newton step transforms  $Y_n$  into  $Y_{n+1}$  as follows:

$$Y_{n+1} - Y_n = -(G'(Y_n))^{-1}G(Y_n) \quad (8)$$

We apply the Newton step to the gradient of  $\tilde{f}$ , computing its root which coincides with the minimum of  $\tilde{f}$ . We obtain

$$dX = -(F''(X))^{-1}((t + dt)c + F'(X)) \quad (9)$$

*Self-concordant barrier.* The convergence of the Newton method is guaranteed only in the neighbourhood to the function root. This neighbourhood is called the region of quadratic convergence; this region evolves on each iteration since  $t$  varies. To guarantee that the iterate  $X$  remains in the region after each iteration, we require the barrier function to be self-concordant:

**Definition 6 (Self-concordant barrier)** A closed convex function  $g$  is a  $\nu$ -self-concordant barrier if

$$D^3g(x)[u, u, u] \leq 2(u^T g''(x)u)^{\frac{3}{4}} \quad (10)$$

and

$$g'(x)^T g''(x) g'(x) < \nu \quad (11)$$

From now on we assume that  $F$  is a self-concordant barrier. Thus  $F''$  is non-degenerate([5, Th4.1.3]) and we can define:

**Definition 7 (Local-norm)**

$$\|y\|_x^* = \sqrt{y^T \times F''(x)^{-1} \times y} \quad (12)$$

This local-norm allows to define the Approximate Centering Condition(ACC), the crucial property which guarantees that  $X$  remains in the region of quadratic convergence:

**Definition 8 (ACC)** Let  $x \in E_f$  and  $t \in \mathbb{R}^+$ ,  $ACC(x, t, \beta)$  is a predicate defined by

$$\|\tilde{f}'(x)\|_x^* = \|tc + F'(x)\|_x^* \leq \beta \quad (13)$$

In the following, we choose a specific value for  $\beta$ , as defined in (14).

$$\beta < \frac{3 - \sqrt{5}}{2} \quad (14)$$

The only step remaining is the computation of the largest  $dt$  such that  $X$  remains in the region of quadratic convergence around  $x^*(t + dt)$ .

$$dt = \frac{\gamma}{\|c\|_x^*} \quad (15)$$

with  $\gamma$  a constant.

This choice maintains the ACC at each iteration([5, Th4.2.8]):

**Theorem 1 (ACC preserved)** If we have  $ACC(X, t, \beta)$  and  $\gamma \leq \frac{\sqrt{\beta}}{1 + \sqrt{\beta}} - \beta$  then we also have  $ACC(X + dX, t + dt, \beta)$ .

For this work, we use the classic self-concordant barrier for linear program:

$$F(x) = \sum_{i=0}^m -\log(b_i - A_i \times x) \text{ with } A_1, A_n \text{ the columns of } A.$$

*Importance of the analytic center.*  $x_F^*$  is required to initiate the algorithm. In case of offline use the value could be precomputed and validated. However in case of online use, its computation itself has to be proved. Fortunately this can be done by a similar algorithm with comparable proofs.

## 2.2 Formal methods

For the same program, different semantics can be used to specify its behavior: (i) a denotational semantics, expressing the program as a mathematical function, (ii) an operational semantics, expressing it as a sequence of basic computations, or (iii) an axiomatic semantics. In the latter case, the semantics can be defined in an incomplete way, as a set of projective statements, i.e. observations. This idea was formalized by Floyd [3], Hoare [6] as a way to specify the expected behavior, the specification, of a program through pre- and post-condition, or assume-guarantee contracts.

**Definition 9 (Hoare Triple)** Let  $C : \mathbb{M} \rightarrow \mathbb{M}$  be a program with  $\mathbb{M}$  the set of its possible memories. Let  $P$  and  $Q$ , two predicates on  $\mathbb{M}$ . We say that the Hoare triple  $\{P\} C \{Q\}$  is valid when

$$\forall m \in \mathbb{M}, P(m) \Rightarrow Q(C(m)) \quad (16)$$

- $P$  is called a precondition or *requires* and  $Q$  the postcondition or *ensures*.
- If  $C$  is a function,  $\{P\} C \{Q\}$  is called a contract.

These Hoare triples can be used to annotate programs written in C. In the following, we rely on the ANSI C Specification Language (ACSL)[7], the specification language of Frama-C, to annotate functions.

The Frama-C tool processes the annotation language, identifying each Hoare Triple and converting them into logical formulas, using the Weakest Precondition strategy.

**Definition 10 (Weakest Precondition)** The Weakest Precondition of a program  $c$  and a postcondition  $Q$  is a formula  $\mathbf{WP}(C, Q)$  such that:

1.  $\{\mathbf{WP}(C, Q)\} C \{Q\}$  is a valid Hoare triple
2. For all  $P$ ,  $\{P\} C \{Q\}$  valid implies  $P \Rightarrow \mathbf{WP}(C, Q)$

**Theorem 2 (Proving Hoare Triple)** 
$$\frac{\mathbf{WP}(C, Q) = R \quad P \Rightarrow R}{\{P\} C \{Q\}}$$

The WP property can be computed mechanically, propagating back the postcondition along the program instructions. An example of such rules is given in Figure 5. The only exception is the while loop rule which requires to be provided with an invariant, this rule is presented later in the document in Figure. 12.

$$\begin{array}{l} \text{Assignment} \frac{}{\mathbf{WP}(x = E, Q) = \forall y, y = E \Rightarrow Q[x \leftarrow y]} \\ \text{Sequence} \frac{\mathbf{WP}(S2, Q) = O \quad \mathbf{WP}(S1, O) = R}{\mathbf{WP}(S1; S2, Q) = R} \\ \text{Conditional} \frac{\mathbf{WP}(S1, Q) = P_1 \quad \mathbf{WP}(S2, Q) = P_2}{\mathbf{WP}(\text{if } (E) S1 \text{ else } S2, Q) = E \Rightarrow P_1 \wedge \neg E \Rightarrow P_2} \end{array}$$

**Fig. 5.** Examples of WP rules

*Automation* The use of SMT solvers enables the automatic proof of some programs and their annotations. This is however only feasible for properties that could be solved without the need of proof assistant. This requires to write both programs (cf. Section 3) and annotations (cf. Section 4) with some considerations for the proof process.

### 3 Writing Provable Code

In order to ease the proof process we write the algorithm code in a very specific manner. In the current section we present our modeling choices: we made all variables global, split the code into small meaningful functions for matrix operations, and transform the while loop into a for loop to address the termination issue.

*Variables.* One of the difficulties when analyzing C code are memory related issues. Two different pointers can reference the same part of the stack. A fine and complex modeling of the memory in the predicate encoding, such as separation logic[8] could address these issues. Another more pragmatic approach amounts to substitute all local variables and function arguments with global static variables. Two static arrays can't overlap since their memory is allocated at compile time.

Since we are targetting embedded system, static variables will also permit to compute and reduce memory footprints. However there are two major drawbacks: the code is a less readable and variables are accessible from any function. These two points usually lead the programmer to mistakes but could be accepted in case of code generation. We tag all variables with the function they belong to by prefixing each variable with its function name. This brings traceability.

*Function.* Proving large functions is usually hard with SMT-based reasoning since the generated goals are too complex to be discharged automatically. A more efficient approach is to associate small pieces of code with local contracts, These intermediate annotations act as cut-rules in the proof process. The Figure 6 presents the function call used in the WP algorithm.

$$\text{Function call } \frac{R \Rightarrow Q \quad \{P\} C \{R\}}{\mathbf{WP}(f(), Q) = P}$$

with void `f()` { `C` }

**Fig. 6.** WP rules used for function call

Let  $A = B[C]$  be a piece of code containing `C`. Replacing `C` by a call to `f()` `{ C }` requires either to inline the call or to write a new contract  $\{P\} f() \{Q\}$ , characterizing two smaller goals instead of a larger one. Specifically in the proof of a  $B[f()]$ , `C` has been replaced by  $P$  and  $Q$  which is simpler than a WP computation.

Therefore instead of having one large function, our code is structured into several functions: one per basic operation. Each associated contract focuses on a really specific element of the proof without interference with the others. Thereby formulas sent to SMT solvers are smaller and the code is modular.

*Matrix operation.* This is extremely useful for matrix operations. In C, a  $M \times N$  Matrix operations is written as  $M \times N$  scalar operation affecting an array representing the resulting matrix. With our methods these operations are gathered in a function annotated by the logic representation of the matrix operation, cf Figure 7. Contracts associated to this small function associate high-level matrix operation to the C low-level computation, acting as refinement contracts.



The encoding hides low-level operations to the rest of the code, leading to two kinds of goals :

- Low level operation (memory and basic matrix operation).
- High level operation (mathematics on matrices).

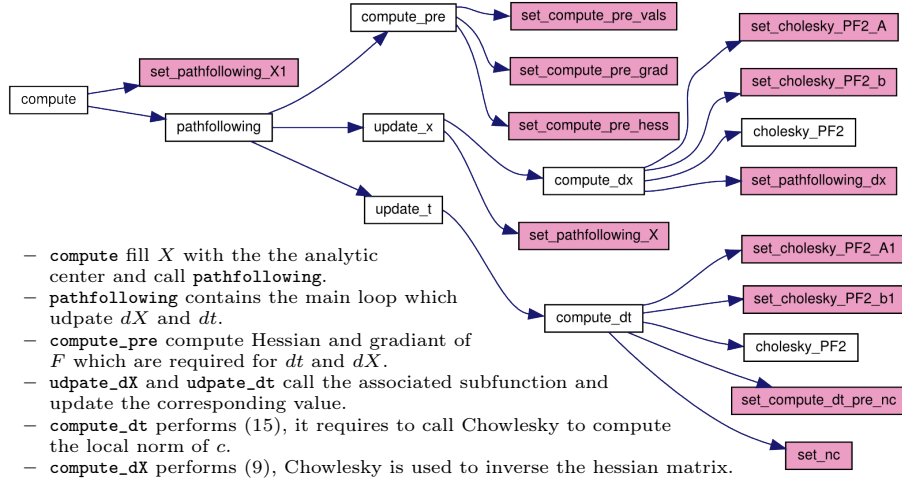
```

/*@ ensures MatVar(dx, 2, 1)
   ==\old(mat_scal(MatVar(cholesky, 2, 1), -1.0));
   @ assigns *(dx+(0..2)); */
void set_dx()
{
    dx[0] = -cholesky[0];
    dx[1] = -cholesky[1];
}

```

The structure of the final code after the split in small functions is shown in Figure 8.

**Fig. 7.** Example of matrix operation:  $dx \text{ \%} = -\text{cholesky}$  encapsulated in a function for  $dx$  and  $\text{cholesky}$  of size  $2 \times 1$



**Fig. 8.** Call tree of the implementation(rose boxes are matrix computation)

*While-loop.* The interior point algorithm is iterative: it performs the same operation until reaching the stop condition. This stopping criteria depends on the desired precision  $\epsilon$ :

$$t_k \geq t_{stop} = \frac{1}{\epsilon} \left( 1 + \frac{(1 + \beta)\beta}{1 - \beta} \right) \quad (17)$$

Proving the termination amounts to find a suitable bound guaranting the obtention of a converged value. We present here the convergence proof of [5] and use it to ensure termination. It relies on the use of the following geometric progression:

**Definition 11**

$$\text{LOWER}(k) = \frac{\gamma(1 - 2\beta)}{(1 - \beta)\|c\|_{x_F}^*} \left( 1 + \frac{\gamma}{1 + \beta} \right)^{k-1} \quad (18)$$

This sequence minimizes  $t$  for each iteration  $k$  of the algorithm([5, Th4.2.9]):

**Theorem 3** For all  $k \in \mathbb{N}^*$ ,

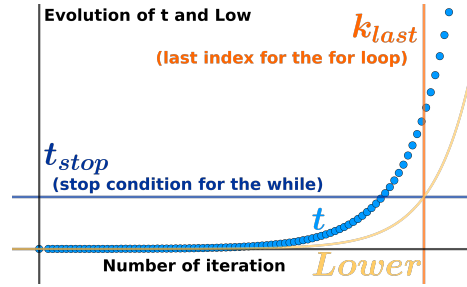
$$t_k \geq \text{LOWER}(k) \quad (19)$$

Combined with (17), a maximal number of iteration called  $k_{last}$  can be computed:

**Theorem 4 (Required number of iterations)**

$$k_{last} = 1 + \frac{\ln(1 + \frac{(\beta+1)*\beta}{1-\beta}) - \ln(\frac{\gamma*(1-2\beta)}{(1-\beta)*\|c\|_{x_F^*}^*}) - \ln(\epsilon)}{\ln(1 + \frac{\gamma}{\beta+1})} \quad (20)$$

Since we have a termination proof based on the number of iterations, the while-loop can be soundly replaced by a for-loop with  $k_{last}$  iterations. As shown in Figure 9, the number of iterations is greater than the one obtained with original while-loop with the stopping criteria but it permits to have absolute guarantee on termination.



*Analytic center*  $\|c\|_{x_F^*}^*$  is required to compute  $k_{last}$ , therefore a worst case execution time can be computed if and only if  $\|c\|_{x_F^*}^*$  has a lower bound at compilation time.

**Fig. 9.** Evolution of  $t$  and  $Lower$  with the algorithm, notice that  $t$  remains always greater than  $Lower$

## 4 Annotate the code

The code is prepared to ease its proof but the specification still remains to be formalized as function contracts, describing the computation of an  $\epsilon$ -optimal solution. This requires to enrich ACSL with some new mathematics definitions. We introduce a set of axiomatic definition to specify optimization related properties. These definitions require, in turn, additional concepts related to matrices. Similar approaches were already proposed [9] but were too specific to ellipsoid problems. We present here both the main annotation and the function local contracts, which ease the global proof.

*Matrix axiomatic.* To write the mathematics property, we need to be able to express the notion of Matrix and operations over it. Therefore we defined a new ACSL axiomatic. An ACSL axiomatic permits the specifier to extend the ACSL language with new types and operators, acting as an algebraic specification.

```
axiomatic matrix
{
  type LMat;
```

First, we defined the new type: LMat standing for Logic Matrix. This type is abstract therefore it will be defined by its operators.

```
    // Getters
  logic integer getM(LMat A);
  logic integer getN(LMat A);
  logic real mat_get(LMat A, integer i, integer j);
    // Constructors
  logic LMat MatVar(double* ar, integer m, integer n) reads ar[0..(m*n)];
  logic LMat MatCst_1_1(real x0);
  logic LMat MatCst_2_3(real x0, real x1, real x2, real x3, real x4, real x5);
```

Getters allow to extract information from the type while constructors bind new LMat object. The first constructor is followed by a read clause stating which part of the memory affects the corresponding LMat object. The Constant constructor take directly the element of the matrix as argument, it can be replaced with an ACSL array for bigger matrix sizes.

```
  logic LMat mat_add(LMat A, LMat B);
  logic LMat mat_mult(LMat A, LMat B);
  logic LMat transpose(LMat A);
  logic LMat inv(LMat A);
  ...
```

Then the theory defined the operations on the LMat type. These are defined axiomatically, with numerous axioms to cover their various behavior. We only give here a excerpt from that library.

```
  axiom getM_add: \forall LMat A, B; getM(mat_add(A, B))==getM(A);
  axiom mat_eq_def:
  \forall LMat A, B;
    (getM(A)==getM(B))=> (getN(A)==getN(A))=>
    (\forall integer i, j; 0<=i<getM(A) ==> 0<=j<getN(A) ==>
    mat_get(A,i,j)==(mat_get(B,i,j))=>
    A == B;
  ...
}
```

*Matrix operations.* As explained in previous Section, the matrix computation are encapsulated into smaller functions. Their contract states the equality between the resulting matrix and the operation computed. An extensionality axiom (mat\_eq\_def) is required to prove this kind of contract. Extensionality means that if two objects have the same external properties then they are equal.

This axiom belong to the matrix axiomatix but is too general to be used therefore lemmas specific to the matrices size are added for each matrix affectation. This lemma can be proven with the previous axioms and therefore does not introduce more assumption.

The proof remains difficult or hardly automatic for SMT solvers therefore we append additional assertions, as sketched in Figure 10, at then end of function stating all the hypothesis of the extensionality lemma. Proving these postconditions is straighfoward and smaller goals need now to be proven.

```

assert getM(MatVar(dX,2,1)) == 2;
assert getN(MatVar(dX,2,1)) == 1;
assert getM(MatVar(cholesky,2,1)) == 2;
assert getN(MatVar(cholesky,2,1)) == 1;
assert mat_get(MatVar(dX,2,1),0,0) == mat_get(\old(mat_scal(MatVar(cholesky,2,1),-1.0)),0,0);
assert mat_get(MatVar(dX, 2, 1),1,0) == mat_get(\old(mat_scal(MatVar(cholesky,2,1),-1.0)),1,0);

```

**Fig. 10.** Assertion appended to the function from figure 7

Assertions also act as cut-rules in ACSL since it introduces the property in the set of hypothesis considered (see. Figure 11).

$$\text{Assert} \frac{\text{WP}(C, P \Rightarrow Q) = R \quad \text{WP}(C, P) = S}{\text{WP}(C; \text{assert } P; , Q) \equiv R \wedge S}$$

**Fig. 11.** WP rules used for `assert`

This works for small example, when scaling each C instruction is embedded inside a correctly annotated function.

*Optimization axiomatic.* Beside generic matrix operators we also need some operators specific to our algorithm.

```

axiomatic Optim
{
  logic LMat hess(LMat x0, LMat x1, LMat x2);
  logic LMat grad(LMat x0, LMat x1, LMat x2);

```

Hessian and gradient are hard to define without real analysis which is well beyond the scope of this article. Therefore we decided to directly axiomatize some theorems relying on their definition like [5, Th4.1.14].

```

|| logic real sol(LMat x0, LMat x1, LMat x2);

```

The `sol` operator represents  $x^*$ , the exact solution which can be defined by

**Property 2 (Axiomatic characterization of Definition 2)**  $s$  is a solution of 1 if and only if

1. For all  $y \in E_f$ ,  $c^T y \geq s$
2. For all  $y \in \mathbb{R}$ ,  $\forall x \in E_f, c^T x \geq y$  implies  $s \geq y$

An ACSL equivalent definition is:

```

logic real sol(LMat A, LMat b, LMat c);
axiom sol_min: \forallall LMat A, b, c;
  \forallall LMat y; mat_gt(mat_mult(A, y), b) ==>
  dot(c, y) >= sol(A, b, c);
axiom sol_greater: \forallall LMat A, b, c;
  \forallall Real y;
  (\forallall LMat x; mat_gt(mat_mult(A, x), b) ==> dot(c, x) >= y) ==>
  sol(A, b, c) >= y;

```

Then we defined some operators representing definitions 7, 8 and 11.

```

logic real norm(LMat x0, LMat x1, LMat x2, LMat x3) =
  \sqrt(mat_get(mat_mult(transpose(x2), mat_mult(inv(hess(x0, x1, x3))), x2)), (0), (0));
logic boolean acc(LMat x0, LMat x1, LMat x2, real x3, LMat x4, real x5) =
  (norm(x0, x1, mat_add(grad(x0, x1, x4), mat_scal(x2, x3)), x4) <= (x5));
...
}

```

*Contract on pathfollowing.* A sound algorithm must produce a point in the feasible set such that its cost is  $\epsilon$ -close to  $sol$ . This is asserted by two global post-conditions:

```

ensures mat_gt(mat_mult(A, MatVar(X, N, 1)), b);
ensures dot(MatVar(X, 2, 1), c) - sol(A, b, c) < EPSILON

```

as well as two preconditions stating that  $X$  is feasible and close enough to the analytic center:

```

requires mat_gt(mat_mult(A, MatVar(X, N, 1)), b);
requires acc(A, b, c, 0, MatVar(X, N, 1), BETA);

```

Thanks to our two new theories `Matrix` and `Optim`, writing and reading this contract is straightforward and can be checked by anyone familiar with linear programming.

*Main Loop.* A loop needs to be annotated by an invariant to have its Weakest precondition computed (cf. Figure 12)

$$\text{For loop } \frac{\mathbf{WP}(E, I) = P \quad (\neg F \wedge I) \Rightarrow Q \quad \{F \wedge I\} C; G \{I\}}{\mathbf{WP}(\text{for } (E; F; G) \text{ inv } I \{C\}, Q) = P}$$

**Fig. 12.** WP rules for a loop

We need three invariants for our path following algorithms. The first one guarantees the feasibility of  $X$  while the second one states the conservation of the  $ACC$  (cf. Def. 8) The third invariant assert that  $t$  is increasing enough on each iteration, more specially that it is greater than a geometric progression (Definition 11).

```

/*@ loop-invariant mat_gt(mat_mult(A, MatVar(X, N, 1)), b);
   @ loop-invariant acc(A, b, c, t, MatVar(X, N, 1), BETA);
   @ loop-invariant t > lower(1);*/
for (int l = 0; l < NBR;l++) { ... }

```

Proving the initialization is straightforward, thanks to the main preconditions.

The first invariant preservation is stated by [5, Th4.1.5] which was translated into an ACSL lemma, the second by Theorem 1 and the third one by Theorem 3.

The last two loop invariants are combined to prove the second postcondition of `pathfollowing` thanks to Theorem 5 and  $NBR$  equals  $k_{last}(20)$ .

**Theorem 5** [5, Th4.2.7] Let  $t \geq 0$ , and  $X$  such that  $ACC(X, t, \beta)$  then

$$c^T X - c^T X^* < \frac{1}{t} \times \left(1 + \frac{(\beta + 1)\beta}{1 - \beta}\right) \quad (21)$$

*Loop body.* In the main loop there are three function calls: `update_pre` computing some common values, `update_t` and `update_x` (Figure 8). Therefore Theorem 1 is broken into several properties and the corresponding post-conditions. For example, the contract of `update_t` is:

```

/*@ requires MatVar(hess, N, N)==hess(A, b, MatVar(X, N, 1));
   @ requires acc(A, b, c, t, MatVar(X, N, 1), BETA);
   @ ensures acc(A, b, c, t, MatVar(X, N, 1), BETA + GAMMA);
   @ ensures t > \old(t)*(1 + GAMMA/(1 + BETA));*/
void update_t();

```

The first postcondition is an intermediary results stating that:

$$ACC(X, t + dt, \beta + \gamma) \quad (22)$$

This result is used as precondition for `update_x`. The second precondition corresponds to the product of  $t$  by the common ratio of the geometric progression LOWER, cf. Definition 11 which will be used to prove the second invariant of the loop. The first precondition is a postcondition from `update_pre` and the second one is the first loop invariant.

## 5 Automatic proof with SMT solvers.

For each annotated piece of code, the Frama-C WP plugin computes the Weakest precondition and generates all the first order formulas required to validate the Hoare triples.

There are two main solutions to prove goals: proving them thanks to a proof assistant – this requires to be done by a human –, or proving them with a fully automatic SMT solver. We decided to rely only on SMT solvers in order to be able to completely automatize the process. Therefore it is better to have lots of small goals instead of several larger ones. We splitted the code for this reason and we now split the proof of lemmas into several intermediate lemmas. For example, in order to prove (22) we wrote `update_t_ensures1` where  $P_1$  is  $ACC(X, t, \beta)$  and  $P_2$  is  $dt = \frac{\gamma}{\|c\|_x}$

$$\forall x, t, dt; P_1 \Rightarrow P_2 \Rightarrow ACC(X, t + dt, \beta) \quad (\text{update\_t\_ensures1})$$

which itself need `update_t_ensures1_10`

$$\forall x, t, dt; P_1 \Rightarrow P_2 \Rightarrow \|F'(X) + c(t + dt)\|_x^* \leq \beta + \gamma \quad (\text{update\_t\_ensures1\_10})$$

Equation `update_t_ensures1_10` needs 3 lemmas to be proven:

$$\forall x, t, dt; P_2 \Rightarrow \|c \times dt\|_x^* = \gamma \quad (\text{update\_t\_ensures1\_13})$$

$$\forall x, t, dt; P_1 \Rightarrow \|F'(X) + c \times t\|_x^* \leq \beta \quad (\text{update\_t\_ensures1\_12})$$

$$\forall x, t, dt; P_1 \Rightarrow P_2 \Rightarrow \|F'(X) + c(t + dt)\|_x^* \leq \|F'(X) + c \times t\|_x^* + \|c \times dt\|_x^* \quad (\text{update\_t\_ensures1\_11})$$

The proof tree for the first ensures of `update_t` can be found in Figure 13.

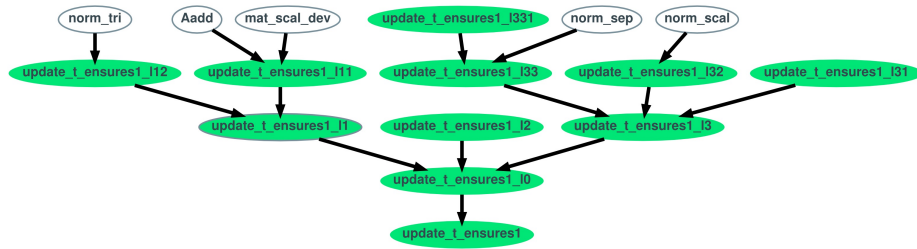


Fig. 13. Proof tree for (22)(In green proven goal, in white axioms)

## 6 Experimentations

Frama-C is a powerful tool but not always built for our specific needs therefore we had to do some tricks to make it prove our goals.

*Using multiple files* Frama-C automatically adds, for each goal, all the lemmas as hypotheses. This increases significantly the size of the goal. To avoid this issue that prevented some proofs, we wrote each function or lemma in a separate file. In this file we add as axioms all the lemmas required to prove the goal. This allows us to prove each goal independently with a minimal context.

The impact of the separation into multiple function(Section 3) and the separation into multiple files is shown in Table 1.

The annotated code can be retrieved from [https://github.com/davyg/proved\\_primal](https://github.com/davyg/proved_primal)

## 7 Related work

Related works include first activities related to the validation of numerical intensive control algorithms. This article is an extension of WANG *et al* [10] which was presenting annotations for a convex optimization algorithm, namely IMP, but the process was both manual and theoretical: the code annotated within Matlab and without machine checked proofs. An other work from the same authors [9]

Size of A	2 × 5			4 × 15			8 × 63		
Experiences	exp1	exp2	exp3	exp1	exp2	exp3	exp1	exp2	exp3
nb function	1	12	12	1	26	26	1	78	78
nb file	1	1	12	1	1	26	1	1	78
nb proven goal	21	48	48	43	97	97	12	257	264
nb goal	25	48	48	46	97	97	109	264	264

**Table 1.** Proof results for `compute_dt` with one function and one file(exp1), with multiple function(exp2) or with multiple file(exp3) and a Timeout of 30s for Alt-Ergo for random generated problem of specific sizes.

presented a similar method than ours but limited to simple control algorithms, linear controllers. The required theories in ACSL were both different and less general than the ones we are proposing here.

Concerning soundness of convex optimization, CIMINI and BEMPORAD [11] presents a termination proof for a quadratic program but without any concerns for the proof of the implementation itself. A similar criticism applies to TØNDEL, JOHANSEN and BEMPORAD [12] where another possible approach to online linear programming is proposed, moreover it is unclear how this could scale and how to extend it to other convex programs. ROUX *et al* [13,14] also presented a way to certify convex optimization algorithm, namely SDP and its sum-of-Squares (SOS) extension, but the certification is done a posteriori which is incompatible with online optimization.

A last set of works, e.g. the work of BOLDO *et al* [15], concerns the formal proof of complex numerical algorithms, relying only on theorem provers. Although code can be extracted from the proof, the code is usually not directly suitable for embedded system: too slow and require different compilation step which should also be proven to have the same guarantee than our method.

## 8 Conclusion

In this article we presented a method to guarantee the safety of numerical algorithms in a critical embedded system. This allows to embed complex algorithms in critical real-time systems with formal guarantee on both their result and termination. This method was applied to a primal algorithm solving linear program.

The implementation is first designed to be easier to prove. Then it is annotated so that in a third time Frama-C and SMT solver can prove the specification automatically. Combined to a code generator such as CVX [16] but with annotation generation it could lead to a tool taking an optimization problem and generating its code and proof automatically.

We worked with real variables to concentrate on runtime errors, termination and functionality and left floating points errors for a future work.

This proof relies on several point: the tools used, the axiomatics we wrote, the main ACSL contract and the theorems used as axioms.



There is also some unchecked code which is independent from the core proof of the algorithm and remains for further work: the Chowlesky decomposition and the Hessian and gradient computation. We also plan to extend the whole work to convex programming.

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