Boundary of the Milnor fiber of a Newton non degenerate surface singularity

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Abstract

We give in this work an explicit combinatorial algorithm for the description of the Milnor fiber of a Newton non degenerate surface singularity as a graph manifold. This is based on a previous work by the author describing a general method for the computation of the boundary of the Milnor fiber of any reduced non isolated singularity of complex surface.

1 Introduction

The study of Milnor fibers of complex-analytic functions, which began in the second half of the 20th century, gave rise to a rich interaction between algebra and topology. One of its interesting aspects is that it can be used to provide equations for sophisticated topological objects, such as exotic spheres, see [Bri00, p 46-48] and [Hir95]. See also [Mil68, Chapter 9], and [Sea19, Section 1].

This potential fabric of exotic spheres led Milnor to study further the topology of hypersurface singularities, and eventually to write his famous 1968 book [Mil68], aimed at the study of isolated singularities of hypersurfaces of $\mathbb{C}^n$, $V(f) = \{f = 0\}$. In this work he introduced two equivalent fibrations, using respectively the levels of $f/|f|$ on spheres centered at the critical points of f and the levels of f in $\mathbb{C}^n$.

The second of these two fibrations has been extended to more general contexts by Lê, see [Lê77], and is known as the **Milnor-Lê fibration**. However, it may produce singular generic fibers, due to the singularities of the ambient space. Hamm, in [Ham71], provided a setting in which the Milnor-Lê fibration is actually a **smoothing** of $V(f)$, that is, a way to put $V(f)$ in a **flat** family of analytic spaces, where the generic space is smooth. Namely, if $(X, 0)$ is a germ of equidimensional complex analytic space, and f is any holomorphic function on $(X, 0)$ such that $V(f) \supset \Sigma X$, then the function f provides a smoothing of the singularity $(V(f), 0)$, where ΣX denotes the singular locus of X.

The study of the whole Milnor fiber is very hard, even for isolated singularities, and there is only a few types of singularities where a description of the full fiber is known. It is the case for the Kleinean sigularities A, D, E, where the Milnor fiber is unique and diffeomorphic to the minimal resolution (see Brieskorn, [Bri66]), as well as for the singularities of normal

toric surfaces, with a description by surgery (see Lisca, [Lis08] and Némethi & Popescu-Pampu, [NPP10]), and for sandwich singularities (De Jong & Van Straten, [dJvS98]). As for nonisolated singularities, the only known case is that of hypersurface singularities of the form $\{f(x,y) + z \cdot g(x,y) = 0\}$, see Sigurðsson, [Sig16]. One can also refer to the survey [Gre19] for more about the topology of smoothings and deformations of singularities.

On the other hand, the study of the **boundary** ∂F of the Milnor fiber of a smoothing has been a very active area of research in the last decades. For an isolated singularity ∂F is unique, and equal to the link of the singularity. In today's words, Mumford proved in [Mum61] that the link of any isolated singularity of complex surface is a **graph manifold**, that is, a manifold describable using a decorated graph whose vertices represent fibrations in $\mathbb{S}^1$ over compact surfaces. It is Waldhausen, in [Wal67], that later introduced this vocabulary and began studying this class of manifolds in itself. Furthermore, since the work of Grauert ([Gra62]), one knows exactly which graph manifolds appear as links of isolated singularities of complex surfaces. However, this strong point is tempered by the fact that one still does not know, for example, which of these manifolds appear as links of singularities of **hypersurfaces** of $\mathbb{C}^3$.

Still, one would like to get an analogous result for boundaries of Milnor fibers associated to non isolated singularities. The first steps towards the comprehension of the topology of these manifolds were made by Randell [Ran77], then Siersma in [Sie91], [Sie01], who computed the homology of the boundary ∂F of the Milnor fiber in certain cases, and characterized the cases in which ∂F is a rational homology sphere.

Concerning the general topology of this manifold, a series of results were aimed at proving that the boundary of the Milnor fibers associated to a non isolated singularity is, again, a graph manifold, see [MP03], [MP04] and [MP16] as well as [MPW07] for the special case of Hirzebruch surface singularities, and [MPW09] for the so-called **suspensions** ($f = g(x,y) + z^n$). In [FdBMN14], the authors prove the same result for a larger context, and in [NS12], the authors give a constructive proof for the case of reduced holomorphic functions $f\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$. This is this last proof that we extend here and in [Cur19] to study the boundary of the Milnor fiber of a Newton non degenerate function $f\colon (X_\sigma, 0) \to (\mathbb{C}, 0)$, where X_σ is the 3-dimensional normal toric variety associated to a cone σ, and $V(f) \supset \mathrm{Sing}(X)$. In this setting, we adapt the general procedure of [Cur19] to produce an explicit combinatorial algorithm for the description of the boundary of the Milnor fiber of f, extending the work of Oka in [Oka87] for non degenerate isolated singularities in $\mathbb{C}^3$.

We prove the following

Theorem. *Let $(X, 0)$ be the germ at its vertex of a 3-dimensional toric variety defined by a 3-dimensional cone in a weight lattice. Let $f\colon (X, 0) \to (\mathbb{C}, 0)$ be a Newton-nondegenerate function whose zero locus contains $\mathrm{Sing}(X)$. Then the boundary ∂F of its Milnor fiber is a graph manifold determined by the local Newton polyhedron of f.*

The purely combinatorial nature of the description we make of the manifold ∂F opens the way for the computation of a great number of examples through a future implementation

in a computer program. But it also calls for more theoretic work, such as for example extending what is done in [BN07], where the authors do the opposite work, retrieving a possible Newton polyhedron of a function $f : \mathbb{C}^3 \to \mathbb{C}$ having a given graph manifold as boundary of Milnor fiber, under the hypothesis that this manifold is a rational homology sphere. Another exploitation of this method would be to answer the ambitious question of which manifolds can appear as boundaries of Milnor fibers of non degenerate surface singularities. In this direction, our algorithm already provides the following obstruction, see Proposition 5.4:

Proposition. *The normal form of the plumbing graph of the Milnor fiber of a Newton non degenerate singularity of complex surface is planar.*

The article is organized in the following way:

- In Section 2, we recall the main material of [Cur19],
- in Section 3, we give the essential definitions and results of toric geometry which are involved in our construction,
- Section 4 is dedicated to the construction of the graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$, which is one of the main ingredients for the description of the manifold ∂F in the general case,
- in Section 5, we show why the data of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ is sufficient to proceed to the rest of the computation.

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2 Reminder of the general result

We recall here some of the main material of [Cur19], one can consult this article and the references therein for more details.

2.1 Milnor fibration on a singular variety

In the sequel, a *variety* will mean a reduced and equidimensional complex analytic space.

Let $(X, 0)$ be a germ of complex variety of dimension 3, and let $f\colon (X, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function on $(X, 0)$ assumed not to divide zero, that is, not to vanish on any irreducible component of $(X, 0)$. The function f defines a germ $(V(f), 0)$ of hypersurface on $(X, 0)$, where $V(f) := \{x \in X, f(x) = 0\}$. Denote by $\Sigma V(f)$ the singular locus of $V(f)$, and by ΣX the one of X.

In the sequel, if $(X, 0)$ is a germ, X will denote a representative of this germ.

Definition 2.1. *We say that a real analytic function $\rho\colon X \to \mathbb{R}_+$ **defines** 0 in X if 0 is isolated in $\rho^{-1}(0)$, i.e. if there is another representative $X' \subset X$ of $(X,0)$ such that $\rho^{-1}_{|X'}(0) = \{0\}$.*

Theorem 2.2. (H. Hamm, [Ham71, Satz 1.6], Lê [Lê77, Theorem 1.1])
Given a real analytic function ρ defining 0 in X, and $\varepsilon > 0$, denote $X_\varepsilon := X \cap \rho^{-1}([0,\varepsilon))$, and $S_\varepsilon := X \cap \{\rho = \varepsilon\}$. Let $f\colon (X,0) \to (\mathbb{C},0)$ be a germ of holomorphic function, such that $X \setminus V(f)$ is smooth. Then there exists $\varepsilon_0 > 0$, such that $\forall\ 0 < \varepsilon \leqslant \varepsilon_0, \exists\ \delta_\varepsilon > 0$ such that $\forall\ 0 < \delta \leqslant \delta_\varepsilon$, the following two maps are diffeomorphic smooth fibrations:

- $\frac{f}{|f|}\colon S_\varepsilon \setminus V(f) \to \mathbb{S}^1$
- $f\colon \partial\left(\{|f| = \delta\} \cap X_\varepsilon\right) \to \partial\left(D_\delta\right)$, *where D_δ denotes the closed disc of radius δ around 0 in $\mathbb{C}$.*

Definition 2.3. *The first of the two fibrations above is referred to as the **Milnor fibration of** f, and the second one is called the **Milnor-Lê fibration**. The closure of the fiber of the Milnor-Lê fibration is called the **Milnor fiber** of the germ of function f.*

Remark 2.4. *The Milnor-Lê fibration is also sometimes referred to as the Milnor fibration.*

*Using transversality arguments, one may show that the diffeomorphism type of the Milnor fiber does not depend on the chosen representative, so we speak about **the Milnor fiber of the germ of function** $f \in \mathcal{O}_{X,0}$*

2.2 Graph manifolds

We explain here the definition of graph manifold (also called plumbed manifolds) that we refer to. For details, one can consult the foundational articles [Mum61] and [Wal67], as well as the article [Neu81] for a description of the so-called plumbing calculus and other topological considerations on graph manifolds.

Recall that the Euler number of a fibration in $\mathbb{S}^1$ over a surface is an integer e that characterizes entirely the total space of the fibration up to homeomorphism. See [GS99, Example 4.6.5] for more details.

Definition 2.5. *An **orientable plumbing graph** is a graph Γ with decorations of the following type:*

- *Each edge is decorated by a $\oplus$ or $\ominus$ symbol.*
- *Each vertex v is decorated by an Euler number $e_v \in \mathbb{Z}$ and a genus $[g_v]$, $g_v \in \mathbb{N}$.*

Remark 2.6. *When representing plumbing graphs, we may omit the decorations $\oplus$ and $[0]$.*

Definition 2.7. (Graph manifold)

If Γ is an orientable plumbing graph, define the oriented ***graph manifold*** *M_Γ associated to Γ in the following way: for each vertex v of Γ decorated by $([g_v], e_v)$, let M_v be the $\mathbb{S}^1$-fibration of Euler number e_v over the closed smooth surface B_v of genus g_v. Pick an orientation of the base and the fibers so that, taken together, they give the orientation of M_v.*

Now, let λ_v be the number of times the vertex v appears as endpoint of an edge. Remove from B_v disjoint open disks $(D_i)_{1\leqslant i\leqslant\lambda_v}$, consequently removing as many open solid tori from M_v. Each ∂D_i is oriented as boundary component of $B_v \setminus \bigsqcup D_i$. Denote by M_v^b the resulting circle bundle with boundary. Denote $\partial M_v^b = \bigsqcup T_i$ a disjoint union of tori.

For every edge between the vertices v and v', glue the manifolds M_v^b and $M_{v'}^b$ in the following way: pick boundary components $T = \partial D \times \mathbb{S}^1$ in M_v^b, $T' = \partial D' \times \mathbb{S}^1$ in $M_{v'}^b$, and glue T and T' according to the matrix $\epsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, ϵ being the sign on the edge.

Finally, we use the convention

$$M_{\Gamma_1 \sqcup \Gamma_2} = M_{\Gamma_1} \# M_{\Gamma_1}$$

2.3 Main aspects of the general result

We recall here the main steps of the general proof, and show how they relate to the different steps of the computation shown in Sections 4 and 5 in the setting of toric geometry.

Our computation of the boundary ∂F of the Milnor fiber of f involves a second function $g\colon (X,0) \to (\mathbb{C},0)$, called **companion of** f, which is generic with respect to f, and can be chosen to be the restriction to X of a linear form on $(\mathbb{C}^N,0) \supset (X,0)$. Using this function, one defines, for $k \in 2\mathbb{N}$, the germ

$$(\mathscr{S}_k, 0) := (\{f = |g|^k\}, 0).$$

This is a germ of 4-dimensional real analytic variety, and

Proposition 2.8. *([NS12, Proposition 11.3.3], [Cur19, Proposition 4.13])*

If k is large enough, $(\mathscr{S}_k, 0)$ is an isolated singularity, and its link $\partial\mathscr{S}_k$ is diffeomorphic to the boundary ∂F of the Milnor fiber of f.

From this point, the goal is therefore to describe the manifold $\partial\mathscr{S}_k$. This is done in analogy with the case of an isolated singularity of complex surface, by exhibiting a resolution $\Pi\colon (\widetilde{\mathscr{S}}, E) \to (\mathscr{S}_k, 0)$ of $\mathscr{S}_k$ whose exceptional locus is a transversal union of smooth orientable closed real analytic surfaces in a 4-dimensional oriented real analytic manifold $\widetilde{\mathscr{S}}$, and using Theorem A.3.

However, the existence of such a resolution is not obvious, in this category we expect the preimage of the origin to be 3-dimensional. We build an *ad hoc* resolution in several steps.

- The first step is a modification $r_X\colon (\tilde{X}, r_X^{-1}(0) \to (X,0)$ of the ambient germ which is adapted to the pair (f,g). In particular, we ask the following: denote $\mathbb{D} := r_X^{-1}(V_{f\cdot g})$, and by $\mathbb{D}_0$ the union of the irreducible components of $\mathbb{D}$ sent on the origin by r_X. Denote by $\mathbb{D}_{f,ex}$ the union of the components of $\mathbb{D}$ sent on curves in $V(f)$, and by $\widetilde{V(f)}$ and $\widetilde{V(g)}$ the strict transforms of $V(f)$ and $V(g)$ by r_X respectively. Finally, denote $\mathbb{D}_f := \mathbb{D}_{f,ex} \cup \widetilde{V(f)}$.

  Finally, denote
  $$\mathscr{C}_{tot} := (\mathbb{D}_f \cap \mathbb{D}_0) \cup (\mathbb{D}_f \cap \widetilde{V(g)}).$$
  Then we require that $\mathbb{D}$ is a simple normal crossings divisor at $\mathscr{C}_{tot}$.

  Denote
  $$\mathscr{C} := (\mathbb{D}_f \cap \mathbb{D}_0) \cup \left(\mathbb{D}_{f,ex} \cap \widetilde{V(g)}\right),$$
  the union of the compact irreducible components of $\mathscr{C}_{tot}$.

  In these conditions, define $\widetilde{\mathscr{S}_k}$ to be the strict tranform of $\mathscr{S}_k$ by r_X. We obtain a modification
  $$r_{\mathscr{S}}\colon (\widetilde{\mathscr{S}_k}, \mathscr{C}) \to (\mathscr{S}_k, 0).$$

  In this work, this first modification is the one that we build in Section 4, by refining the cone σ corresponding to the ambient variety $X = X_\sigma$.

- Then one needs to resolve the singularities of $\widetilde{\mathscr{S}_k}$, which are located along $\mathscr{C}$. This is done by normalizing $\widetilde{\mathscr{S}_k}$, and then comparing locally the normalization $\widetilde{\mathscr{S}_k}^N$ of $\widetilde{\mathscr{S}_k}$ with the normalization of a singularity of complex surface, to resolve the last singularities. The normalization step corresponds to the demultiplication of vertices and edges in Subsection 5.1, while the resolution of the remaining isolated singularities corresponds to the replacement of some edges by bamboos, according to Appendix B.

The local equations of $\widetilde{\mathscr{S}_k}$ along points of $\mathscr{C}_{tot}$ are determined by multiplicities of the pullbacks of f and g to $\tilde{X}$. If D_i is an irreducible component of $\mathbb{D}$, denote respectively m_i, n_i the multiplicities of $f \circ r_X$ and $g \circ r_X$ on D_i. The following decorated graph will contain all the necessary data for the description of the exceptional locus E of the resolution of $\mathscr{S}_k$:

Definition 2.9. *Denote $\Gamma(\mathscr{C}_{tot})$ the dual graph of the configuration of complex curves $\mathscr{C}_{tot}$. The decorated graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ is obtained from $\Gamma(\mathscr{C}_{tot})$ in the following way:*

- *If C is an irreducible component of $D_1 \cap D_2$, where $D_1 \in \mathbb{D}_f$ and $D_2 \in \mathbb{D}_0 \cup \mathbb{D}_g$, decorate v_C, the vertex corresponding to C, with the triple $(m_1; m_2, n_2)$, and with its genus $[g]$, in square brackets. If $D_1 \in \widetilde{V(f)}$ and $D_2 \in \widetilde{V(g)}$, then the vertex associated to the non-compact curve C is an arrowhead.*

- *Decorate each edge e_p of $\Gamma(\mathscr{C}_{tot})$ corresponding to a double point p of $\mathscr{C}_{tot}$ with $\oplus$ if this point is on exactly one component of $\mathbb{D}_f$, and with $\ominus$ if it is on exactly two different components of $\mathbb{D}_f$.*

3 Tools of toric geometry

We give in this Section the necessary material to proceed to Section 4 and the construction of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$. What follows is mainly based on the book [Oka97] of Oka. To go further, one can consult the books [Oda88] of Oda, or [CLS11] of Cox, Little, Schenk. See also the foundational article [Oka87].

3.1 Basic definitions

Let us start with some *ad hoc* definitions.

Definition 3.1. *In this work, a 3-dimensional **lattice** N is a free group of rank 3, that is, isomorphic to $(\mathbb{Z}^3, 0)$. Define the **integral length** of an element $u \in N$ as*

$$l(u) = max\{n \in \mathbb{N}, \exists\ v \in N \setminus \{0\} \text{ such that } u = n \cdot v\}.$$

*An element of N is called **primitive** if its integral length is equal to 1.*

*A **cone** σ in N is a subset of $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ of the form*

$$\langle v_1, \dots, v_k \rangle := \{r_1 \cdot v_1 + \dots + r_k \cdot v_k, r_1, \dots, r_k \in \mathbb{R}^+\} \subseteq N_{\mathbb{R}}.$$

*A cone σ is called **stronlgy convex** if it contains no linear subspace of $N_{\mathbb{R}}$.*

*A strongly convex cone is called **regular** if it can be generated by a family of elements of N that can be completed in a basis of the $\mathbb{Z}$-module N. We denote by $\tau \preceq \sigma$ the fact that a cone τ is a face of σ, of any dimension.*

Definition 3.2. *We will usually denote by $M := N^\vee = Hom(N, \mathbb{Z}) \simeq \mathbb{Z}^3$ the **dual lattice** of N. For σ a convex cone of dimension d in $N_{\mathbb{R}}$, let us define its **dual cone***

$$\sigma^\vee := \{m \in M_{\mathbb{R}}, \forall\ n \in \sigma, \langle m, n \rangle \geqslant 0\} \subset M_{\mathbb{R}}$$

*and its **orthogonal***

$$\sigma^\perp := \{m \in M_{\mathbb{R}}, \forall\ n \in \sigma, \langle m, n \rangle = 0\} \subset \sigma^\vee.$$

Note that if σ is of dimension 3, σ is regular if and only if $\sigma^\vee$ is regular.

Definition 3.3. *If σ is a strongly convex cone in N, define*

$$X_\sigma := Spec\left(\mathbb{C}[\sigma^\vee \cap M]\right)$$

the 3-dimensional affine variety associated to it.

Remark 3.4. *A family $m_1, \cdots, m_k \in M$ generating the semigroup $\sigma^\vee \cap M$ provides an embedding $X_\sigma \hookrightarrow Spec\,(\mathbb{C}[\chi^{m_1}, \cdots, \chi^{m_k}]) = \mathbb{C}^k{}_{x_1,\cdots,x_k}$.*

Proposition 3.5. *Closed points of X_σ correspond to semigroup morphisms $(S_\sigma, +) \to (\mathbb{C}, \cdot)$. Denote $\mathcal{T}_N$ the set of those morphisms whose image is contained in $\mathbb{C}^*$. The set $\mathcal{T}_N$ is (non canonically) isomorphic to $(\mathbb{C}^*)^3$ and dense in X_σ. This group is called the (algebraic) **torus** associated to N. The action of the group $\mathcal{T}_N$ on itself extends to a continuous action on the whole variety X_σ, making $\mathcal{T}_N$ the unique n-dimensional orbit.*

On closed points, the action is defined as on $\mathcal{T}_N$, by multiplication: if $\phi \in X_\sigma$ and $\psi \in \mathcal{T}_N$, then $\phi \cdot \psi \in X_\sigma$ is defined by

$$\forall\, m \in M, (\phi \cdot \psi)(m) = \phi(m) \cdot \psi(m).$$

Proposition 3.6. *If $\tau \subset \sigma$, then the inclusion $\mathbb{C}[\sigma^\vee] \subset \mathbb{C}[\tau^\vee]$ gives rise to a canonical birational morphism of algebraic varieties $X_\tau \to X_\sigma$. This morphism is an injection if and only if τ is a face (of any dimension) of σ.*

Definition 3.7. *A **fan** $\mathcal{F}$ in $N_\mathbb{R}$ is a finite set of strongly convex cones such that:*

1. *If $\sigma \in \mathcal{F}$, any face of σ is in $\mathcal{F}$.*
2. *The intersection of two cones of $\mathcal{F}$ is a face of each.*

*The **support** $|\mathcal{F}|$ of the fan $\mathcal{F}$ is the union $\bigcup\limits_{\sigma\in\mathcal{F}} \sigma$ of the cones composing it.*

Definition 3.8. *A fan $\mathcal{F}$ in $N_\mathbb{R}$ defines a **toric variety** $X_\mathcal{F}$ in the following way: take the disjoint union of the X_σ's, for all σ in $\mathcal{F}$, and, if σ and σ' are cones of $\mathcal{F}$, glue X_σ and $X_{\sigma'}$ along $X_{\sigma\cap\sigma'}$.*

$$X_\mathcal{F} := \left(\bigsqcup_{\sigma\in\mathcal{F}} X_\sigma\right) \Big/ \left(X_\sigma \underset{X_{\sigma\cap\sigma'}}{\sim} X_{\sigma'}\right)$$

Proposition 3.9. *The actions of the torus $\mathcal{T}_N$ on each X_σ glue together, in agreement with the gluing of the X_σ's, giving rise to a global action of the torus on $X_\mathcal{F}$, under which the torus $\mathcal{T}_N$ is the unique n-dimensional orbit. Furthermore $\mathcal{T}_N$ is open and dense in $X_\mathcal{F}$.*

Proposition 3.10. (Decomposition into orbits.)
Let σ be a strongly convex polyhedral cone of dimension d belonging to a fan $\mathcal{F}$ in $N_\mathbb{R}$, where N is n-dimensional. Let $\tau_1, \ldots, \tau_k$ be the faces of σ of codimension 1.
Then

$$O_\sigma := X_\sigma \setminus \bigcup X_{\tau_i}$$

is an orbit of $X_{\mathcal{F}}$ under the action of $\mathcal{T}_N$. It is made of all morphisms $(\sigma^\vee \cap M, +) \to (\mathbb{C}, +)$ that are different from zero exactly on $\sigma^\perp \cap M$.

In other words, a point of O_σ is a maximal ideal of $\mathbb{C}[\sigma^\vee \cap M]$ containing the monomial $\chi^m \in \mathbb{C}[\sigma^\vee \cap M]$ if and only if $m \notin \sigma^\perp \cap M$.

Furthermore, the variety O_σ can be seen as

$$O_\sigma = Spec(\mathbb{C}[\sigma^\perp \cap M])$$

providing, non-canonically, the isomorphism

$$O_\sigma \simeq (\mathbb{C}^*)^{n-d}.$$

Remark 3.11. *In particular, $O_{\{0\}} = T_N$.*

*Furthermore, if $\mathcal{F} = \sigma$, then O_σ is the unique minimal-dimensional orbit in X_σ. If $dim(\sigma) = 3$, it is called the **origin** of X_σ, and denoted 0_{X_σ}.*

*With this notation, if σ is of maximal dimension, we get the **germ of toric variety** $(X_\sigma, 0_{X_\sigma})$, or $(X_\sigma, 0)$, whose local ring of germs of holomorphic functions is*

$$\mathcal{O}_{X_\sigma,0} = \mathbb{C}\{\sigma^\vee \cap M\}.$$

Considering this for any cone of $\mathcal{F}$ will give us a decomposition of $X_{\mathcal{F}}$ into orbits under the action of the torus, a cone of $\mathcal{F}$ corresponding to an orbit:

$$X_{\mathcal{F}} = \bigsqcup_{\sigma \in \mathcal{F}} O_\sigma.$$

Proposition 3.12. (Closure of an orbit.)

The closure in $X_{\mathcal{F}}$ of an orbit corresponding to a cone $\tau \in \mathcal{F}$ is made of the union of the orbits corresponding to cones of $\mathcal{F}$ having τ as a face:

$$\overline{O_\tau} = \bigcup_{\tau \preceq \sigma} O_\sigma.$$

The following proposition is essential for the sequel and shows how one can read on a fan the singularities of the associated variety.

Proposition 3.13. *The variety $X_{\mathcal{F}}$ is smooth at a point $p \in O_\sigma$ if and only if σ is regular.*

One can encode certain modifications of a given variety $X_{\mathcal{F}}$ through a refinement of $\mathcal{F}$:

Definition 3.14. *A **refinement** of a fan $\mathcal{F}$ in $N_{\mathbb{R}}$ is another fan $\mathcal{F}'$ in $N_{\mathbb{R}}$ such that:*

$$|\mathcal{F}| = |\mathcal{F}'| \text{ and } \forall\ \sigma' \in \mathcal{F}', \exists\ \sigma \in \mathcal{F} \text{ such that } \sigma' \subset \sigma.$$

A **refinement** *of a cone is a refinement of the fan formed by its faces.*

The following definition is a matter of vocabulary and is introduced for pure reasons of convenience in the exposition made in Section 4.

Definition 3.15. *Let $\mathcal{F}'$ be a refinement of a fan $\mathcal{F}$. If τ is a cone of $\mathcal{F}'$, we call* ***minimal containing cone*** *of τ in $\mathcal{F}$ the minimal-dimensional cone of $\mathcal{F}$ containing τ.*

If τ, σ are two cones such that $\tau \subset \sigma$, the ***minimal containing face*** *of τ in σ is the minimal-dimensional face of σ containing τ. A cone τ is said to be in the interior of σ if its minimal containing face in σ is σ itself.*

Definition 3.16. *Let $\mathcal{F}$ be a fan in N, and $\mathcal{F}'$ a refinement of $\mathcal{F}$.*

The **toric morphism** *$\Pi_{\mathcal{F}',\mathcal{F}} : X_{\mathcal{F}'} \to X_{\mathcal{F}}$ associated to this refinement is obtained by gluing the morphisms given by the inclusions of cones of $\mathcal{F}'$ in the cones of $\mathcal{F}$, defined in Proposition 3.6.*

Proposition 3.17. *The morphism $\Pi_{\mathcal{F}',\mathcal{F}}$ is a modification of $X_{\mathcal{F}}$. It has the following combinatorial property: if $\sigma' \in \mathcal{F}'$, let σ be the minimal containing cone of σ' in $\mathcal{F}$. Then $\Pi_{\mathcal{F}',\mathcal{F}}(O_{\sigma'}) \subset O_\sigma$.*

The critical locus $E_{\mathcal{F}',\mathcal{F}}$ of $\Pi_{\mathcal{F}',\mathcal{F}}$ is exactly the union

$$\bigsqcup_{\tau \in \mathcal{F}', \tau \notin \mathcal{F}} O_\tau \in X_{\mathcal{F}'}$$

of orbits of $X_{\mathcal{F}'}$ corresponding to new cones, and the discriminant locus $\Delta(\Pi_{\mathcal{F}',\mathcal{F}})$ is

$$\bigsqcup_{\tau \in \mathcal{F}, \tau \notin \mathcal{F}'} O_\tau \in X_{\mathcal{F}},$$

the union of orbits of $X_{\mathcal{F}}$ corresponding to cones that have been subdivided.

3.2 modification associated to a germ of function

Let $(X_\sigma, 0)$ be the germ of affine normal variety associated to a 3-dimensional strongly convex cone.

Definition 3.18.

- *Let $f = \sum_{m_i \in \sigma^\vee \cap M} a_i \chi^{m_i} \in \mathbb{C}\{\sigma^\vee \cap M\}$. The* ***support of*** *f is defined as* $\mathrm{Supp}(f) := \bigcup_{a_i \neq 0} \{m_i\} \subset \sigma^\vee \cap M$.
- *The* ***Local Newton polyhedron of*** *f* ***at the origin of*** *X_σ is defined as*

$$\mathrm{LNP}(f) := \mathrm{Conv}\left(\mathrm{Supp}(f) + \sigma^\vee\right)$$

where "$+$" denotes the Minkowski sum in M.

This definition of Newton polyhedron for general germs may be found in [Ste83, Definition 5], or [PPS13, Definition 8.7].

The following lemma is a direct consequence of the definitions, and allows one to easily read some parts of the zero locus of a function defined on a germ of toric variety.

Lemma 3.19. *The ideal $I(\tau) \subset \mathbb{C}\{\sigma^\vee \cap M\}$ of functions cancelling on the orbit O_τ of X_σ is made of the functions $f \in \mathbb{C}\{\sigma^\vee \cap M\}$ such that* $\mathrm{Supp}(f) \cap \tau^\perp = \emptyset$.

Definition 3.20. *Let $v \in \sigma$, and $f \in \mathbb{C}\{\sigma^\vee \cap M\}$. Define the* ***height of*** $\mathrm{LNP}(f)$ ***in the direction*** *v to be*

$$h_v(f) := \min_{m \in \mathrm{LNP}(f)} \langle m, v\rangle \in \mathbb{R}_+.$$

Definition 3.21. *Let σ be a cone in N, and $f \in \mathbb{C}\{\sigma^\vee \cap M\}$. Let Δ be a face of* $\mathrm{LNP}(f)$ *of dimension d. Then the set*

$$\tau_\Delta := \{v \in \sigma, \Delta_v = \Delta\} = \{v \in \sigma / \forall\ m \in \Delta, \langle m, v\rangle = h_v(f)\}$$

is a cone of codimension d contained in σ. The set

$$\mathcal{F}_f := \{\tau_\Delta, \Delta \subset \mathrm{LNP}(f)\}$$

is the ***fan associated to*** *f.*

The morphism $\Pi_{\mathcal{F}_f} \colon X_{\mathcal{F}_f} \to X_\sigma$ coming from the refinement of σ is called the ***modification of*** *X_σ* ***associated to*** *f.*

Let us describe the behaviour of the strict tranform $\widetilde{V(f)}$ of $V(f)$ under this modification, and more generally, under any toric modification factorizing through $\Pi_{\mathcal{F}_f}$.

Let $\mathcal{F}$ be a refinement of $\mathcal{F}_f$, and $\tau \subset \sigma$ a cone of $\mathcal{F}$. Denote

$$\Pi_{\mathcal{F}} \colon (X_{\mathcal{F}}, E_{\mathcal{F}}) \to (X_\sigma, 0)$$

the modification associated to $\mathcal{F}$, and $\tilde{f} = f \circ \Pi_{\mathcal{F}}$ the pullback of f by this modification. Let

$$\Delta_\tau := \{m \in \mathrm{LNP}(f), \forall\ v \in \tau, \langle m, v\rangle = h_v(f)\}\,.$$

Note that we may have $\tau \neq \tau'$ and still $\Delta_\tau = \Delta_{\tau'}$. However, the assumption that $\mathcal{F}$ is a refinement of $\mathcal{F}_f$ ensures that this definition makes sense. In fact, if γ is the minimal containing cone of τ in $\mathcal{F}_f$, then $\Delta_\tau = \Delta_\gamma$.

Note also that $dim(\Delta_\tau) \leqslant codim(\tau)$, and that $\tau \preceq \sigma \Rightarrow \Delta_\sigma \subset \Delta_\tau$.

We have a first lemma, showing how one can read from $\mathrm{LNP}(f)$ the multiplicities of $\tilde{f}$ along some divisors in $X_{\mathcal{F}}$:

Lemma 3.22. *If $\tau = \langle v\rangle_{\mathbb{R}_+}$ for some primitive vector v in N, then the multiplicity of $\tilde{f}$ along $\overline{O_\tau}$ is equal to $h_v(f)$.*

Definition 3.23. *For a face Δ of $\mathrm{LNP}(f)$, of any dimension, we define f_Δ, f **truncated relatively to the face** Δ, as the function obtained by keeping only the terms of f corresponding to points of this face:*

$$f_\Delta := \sum_{m_i \in \Delta} a_i \chi^{m_i}.$$

In some sense, which is made more precise for example in [Oka97], f_{Δ_τ} is the equation of the intersection of $\widetilde{V(f)}$ with the orbit O_τ of $X_\mathcal{F}$.

We therefore have the following:

Lemma 3.24. *If $\mathcal{F}$ is a refinement of $\mathcal{F}_f$, and $\tau \in \mathcal{F}$, then*

$$\widetilde{V(f)} \cap O_\tau \neq \emptyset \text{ iff } \dim \Delta_\tau \geqslant 1.$$

The following definitions describe a condition in which the strict tranform $\widetilde{V(f)}$ of $V(f)$ behaves in a nice way with respect to the modification associated to $\mathcal{F}$.

Definition 3.25. *A function $f \in \mathbb{C}\{\sigma^\vee \cap M\}$ is called **suitable** if $V(f)$ does not contain any 2-dimensional orbit of X_σ, or equivalently, if $\mathrm{Supp}(f)$ has points in each 2-dimensional face of $\sigma^\vee$.*

Definition 3.26.

- *A suitable germ of function $f \in \mathbb{C}\{\sigma^\vee \cap M\}$ is said to be **nondegenerate relatively to a compact face** $\Delta \subset \mathrm{LNP}(f)$ if and only if $V(f_\Delta)$ is smooth in $(\mathbb{C}^*)^n$.*
- *A suitable germ of function is said to be **Newton-nondegenerate**, or **NND** if it is nondegenerate relatively to every compact face of its local Newton polyhedron.*

Proposition 3.27. *Let $f \in \mathbb{C}\{\sigma^\vee \cap M\}$ be a NND germ of analytic function on $(X_\sigma, 0)$, and $\mathcal{F}$ a refinement of $\mathcal{F}_f$, such that $X_\mathcal{F}$ is smooth along O_τ and Δ_τ is compact. Then the intersection $\widetilde{V(f)} \cap O_\tau$ is smooth, and at any point p of this intersection, $\left(\widetilde{V(f)}, p\right)$ is a germ of smooth hypersurface of $X_\mathcal{F}$ intersecting O_τ transversally.*

Furthermore, the following will be heavily used in Section 4:

Proposition 3.28. *Let f be a Newton-nondegenerate function on X_σ, and $\mathcal{F}$ be a refinement of $\mathcal{F}_f$. Let τ be a cone of $\mathcal{F}$, such that Δ_τ is compact. Then,*

- *If $\mathrm{codim}(\tau) = 1$, denote $l(\Delta_\tau)$ the integral length of this face in M. Then*

$$Card\left(\widetilde{V(f)} \cap O_\tau\right) = l(\Delta_\tau).$$

- *If* $\mathrm{codim}(\tau) = 2$*, denote* $i(\Delta_\tau)$ *the number of points of* M *in the* ***interior*** *of* Δ_τ*. Then*

$$C(\tau) := \widetilde{V(f)} \cap \overline{O_\tau}$$

is a smooth curve. Furthermore, if $\dim(\Delta_\tau) = 2$*, this curve is irreducible and*

$$g(C(\tau)) = i(\Delta_\tau).$$

If $\dim(\Delta_\tau) = 1$*,* $C(\tau)$ *is a disjoint union of* $l(\Delta_\tau)$ *smooth curves of genus* 0*.*

Remark 3.29. *Let us remind that, in both cases, if* $\dim(\Delta_\tau) = 0$*, then* $\widetilde{V(f)} \cap \overline{O_\tau} = \emptyset$*.*

Let us conclude this section with a lemma giving a precise meaning to the genericity of Newton-nondegenerate functions.

Lemma 3.30. *Let* f *be a holomorphic function on* X_σ*. Then in the space of coefficients of all those functions* h *such that* $\mathrm{LNP}(h) = \mathrm{LNP}(f)$*, those which are non-degenerate are Zariski-dense.*

Finally, we will need a corollary of the Bernstein-Koushnirenko-Khovanskii Theorem (see [CLO98, Theorem 7.5.4]), in dimension 2. We will need the following notations:

If $f \in \mathbb{C}[M]$ for some lattice M of dimension 2, denote $P(f) := \mathrm{Conv}(\mathrm{Supp}(f))$. Denote by $V(P)$ the lattice volume of a polyhedron P, for which a simplex is of volume 1. Finally, if P_1, P_2 are two polyhedra in M, denote

$$V(P_1, P_2) = \frac{Vol(P_1 + P_2) - Vol(P_1) - Vol(P_2)}{2}$$

called the mixed volume of P_1 and P_2.

Lemma 3.31. *Let* M *be a* 2*-dimensional lattice, and let* $f \in \mathbb{C}[M]$ *be a Newton-nondegenerate function. Denote* $P_1 := P(f)$*. Then for any convex polyhedron* P_2 *with vertices in* M*, a generic choice of coefficients for the elements of* $P_2 \cap M$ *will provide a function* g *such that* $P(g) = P_2$ *and, denoting* $P = P_1 + P_2$*, the compactifications* $\widetilde{V(f)}$ *and* $\widetilde{V(g)}$ *of* $V(f)$ *and* $V(g)$ *in* $X_{\mathcal{F}_P}$ *intersect transversally only on* O_0*, in* $V(P_1, P_2)$ *points.*

4 Construction of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$

Let $(X, 0) = (X_\sigma, 0_{X_\sigma})$ be the germ of normal 3-dimensional toric variety associated to a rational strongly convex cone $\sigma \subset M_{\mathbb{R}}$, where M is a 3-dimensional lattice. Let σ_{sing} denote the union of non regular faces of σ, i.e. the union of faces τ of σ corresponding to orbits O_τ along which X is singular.

Let $f\colon (X, 0) \to (\mathbb{C}, 0)$ be a holomorphic function such that $\mathrm{Supp}(f) \cap \sigma_{sing} = \emptyset$. This guarantees that $V(f) := \{f = 0\} \supset \mathrm{Sing}(X)$, and therefore that f is a smoothing of $V(f)$.

We also assume that f is a Newton non degenerate function, in particular it is suitable, see Definitions 3.25 and 3.26.

In this section, we explain how to compute the decorated graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ of Definition 2.9 from the cone σ and the Newton polyhedron $LNP(f)$ of f.

We can already make the following remark:

Remark 4.1. *Let τ be a face of σ of dimension 2. Then $V(f) \supset O_\tau$ iff there is a 1-dimensional cone of $\mathcal{F}_f$ whose minimal containg face in σ is τ.*

In particular, if τ is not regular, then the condition $V(f) \supset \mathrm{Sing}(X)$ implies that there is a cone of dimension 1 in $\mathcal{F}_f$ whose minimal containing cone is τ.

4.1 Universal companion polyhedron

Although there is no chance to describe a function g that will be a companion of every Newton-nondegenerate function f on X, there exists an object which will be compatible with every function f.

Definition 4.2. *For any sequence $G = (m_1, \cdots, m_k)$ generating the semigroup $\sigma^\vee \cap M$, denote*

$$P(G) := Conv\left(\bigcup_{1 \leqslant i \leqslant k} m_i + \sigma^\vee\right).$$

In the sequel, we fix such a family G.

By Remark 3.4, the sequence G leads to an embedding $(X, 0) \hookrightarrow \mathbb{C}^k{}_{x_1,\cdots,x_k}$. From this viewpoint, $P(G)$ is in fact the local Newton polyhedron of the restriction to X of a generic linear form of $\mathbb{C}^k{}_{x_1,\cdots,x_k}$.

Now, the fact that the restriction of a generic linear form is a companion of f implies:

Lemma 4.3. *For any germ of function f on $(X, 0)$, there is a function g that is a companion of f and such that*

$$LNP(g) = P(G).$$

Remark 4.4. *One can note that the Newton-nondegeneracy of f is not required for this lemma. However, for what follows, the hypothesis of Newton-nondegeneracy is central.*

From now on, fix $g\colon (X, 0) \to (\mathbb{C}, 0)$ a Newton non degenerate companion of f such that $LNP(g) = P(G)$.

Remark 4.1 has the following consequence for g:

Remark 4.5. *There is no 1-dimensional cone of $\mathcal{F}_g$ whose minimal containing face in σ is 2-dimensional.*

In the next subsections, we construct a fan $\widetilde{\mathcal{F}}$ refining σ and such that the associated modification

$$\widetilde{r} := \Pi_{\widetilde{\mathcal{F}}} \colon X_{\widetilde{\mathcal{F}}} \to X$$

is adapted to the pair (f, g). We will follow this construction on the following example:

Example 4.6. *Denote $M := \mathbb{Z}^3$, and let*

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, u_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

be vectors in $\mathbb{R}^3$, and $\sigma = \langle u_1, u_2, u_3, u_4 \rangle_{\mathbb{R}_+}$.

Let $X := X_\sigma$ be the 3-dimensional toric variety corresponding to σ. The cone σ is not simplicial, hence X is singular at the origin. Furthermore, the face $\tau_{1,2} := \langle u_1, u_2 \rangle_{\mathbb{R}_+}$ of σ is singular, hence X is singular along $0_{\tau_{1,2}}$.

The cone $\sigma^\checkmark$ is generated by the vectors $U = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, V = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, W = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, X = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$. The face of $\sigma^\checkmark$ corresponding to the face $\tau_{1,2}$ of σ is the ray generated by W. Furthermore, the semigroup $S_\sigma = \sigma^\checkmark \cap M$ is generated by U, V, W, X and $Y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. The relations between these vectors provide the description

$$X_\sigma = Spec\left(\mathbb{C}[U, V, W, X, Y] \big/ (Y^3 - XV, UX - WY^2, VW - UY)\right)$$

Figure 1 shows the local Newton polyhedron of the restriction g to X of a generic linear form in $\mathbb{C}^5_{U,V,W,X,Y}$. Full lines represent the cone σ, while dashed lines represent the axes of coordinates, left for clarity.

Consider the function $f \in \mathbb{C}[\sigma^\checkmark \cap M]$ given by

$$f = \chi^{\binom{1}{1}{2}} + \chi^{\binom{2}{0}{1}} + \chi^{\binom{0}{2}{0}} + \chi^{\binom{0}{4}{-2}} = UV + UW + Y^2 + X^2.$$

Note that $V(f) \supset \mathrm{Sing}(X)$, since f admits no multiple of W in its support. Figure 2 shows the local Newton polyhedron of f, where we kept again the coordinate axes and the cone $\sigma^\checkmark$. Full round marks represent the elements of $\mathrm{Supp}(f)$, while empty square marks represent the other points of M in the compact faces of $LNP(f)$.

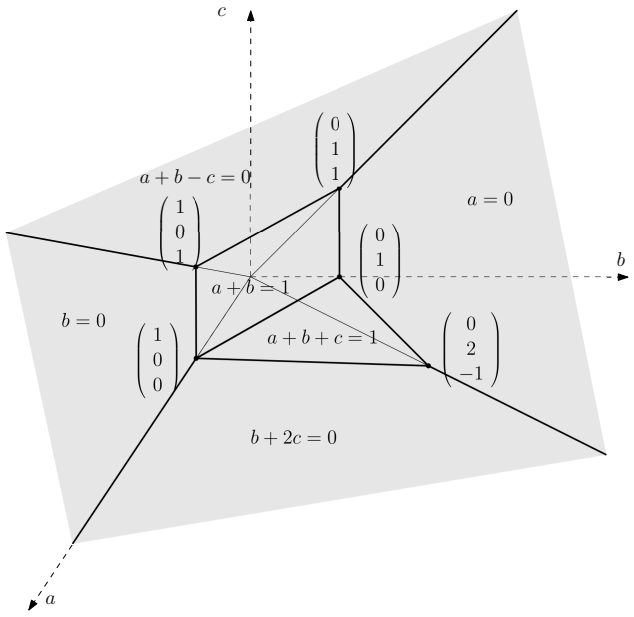


Figure 1: $LNP(g)$.

4.2 The modification associated to the polyhedra

Denote $\overset{\star}{\mathcal{F}} := \mathcal{F}_{f\cdot g}$ the fan associated to the germ $f \cdot g$. This fan is the minimal refinement of both $\mathcal{F}_f$ and $\mathcal{F}_g$. Denote $\overset{\star}{X} = X_{\overset{\star}{\mathcal{F}}}$ and

$$\overset{\star}{r} := \Pi_{\mathcal{F}_{f\cdot g}} \colon \overset{\star}{X} \to X$$

the modification associated to this refinement.

Note that we have the following commutative diagram:

$$\begin{array}{ccccc} & \Pi_{\overset{\star}{\mathcal{F}},\mathcal{F}_f} & \overset{\star}{X} & \Pi_{\overset{\star}{\mathcal{F}},\mathcal{F}_g} & \\ X_{\mathcal{F}_f} & & \downarrow \overset{\star}{r} & & X_{\mathcal{F}_g} \\ & \Pi_{\mathcal{F}_f} & X & \Pi_{\mathcal{F}_g} & \end{array}$$

Denote $\overset{\star}{\mathbb{D}} := \overset{\star}{r}^{-1}(V_{f\cdot g})$, and in the same way, $\overset{\star}{V(f)}, \overset{\star}{V(g)}$ the strict transforms of $V(f), V(g)$ by $\overset{\star}{r}$. Denote also $\overset{\star}{\mathbb{D}}_{f,ex}$ the union of components of $\overset{\star}{\mathbb{D}}$ whose images by $\overset{\star}{r}$ are curves of $V(f)$, and $\overset{\star}{\mathbb{D}}_0 := \overset{\star}{r}^{-1}(0)$.

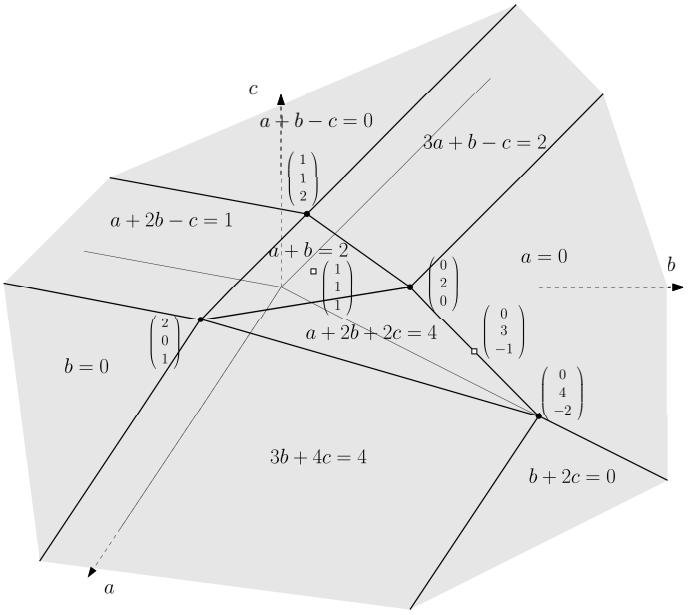


Figure 2: $LNP(f)$.

Discussion 4.7. (Reading rules.) *The irreducible components of* $\overset{\star}{\mathbb{D}}_0$ *are the closures of orbits* O_τ *of* $X_{\overset{\star}{\mathcal{F}}}$ *corresponding to* 1*-dimensional cones* τ *of* $\overset{\star}{\mathcal{F}}$ *whose minimal containing face in* σ *is* σ *itself. Such components are intersected by* $V(\overset{\star}{f})$ *(resp.* $V(\overset{\star}{g})$*) if and only if the minimal containing cone of* τ *in* F_f *(resp.* F_g*) is of dimension* 1 *or* 2*.*

The components of $\overset{\star}{\mathbb{D}}_{f,ex}$ *are the closures of orbits* O_τ *of* $X_{\overset{\star}{\mathcal{F}}}$ *corresponding to* 1*-dimensional cones* τ *of* $\overset{\star}{\mathcal{F}}$ *whose minimal containing face in* σ *is of dimension* 2*.*

Definition 4.8. *We call* ***cutting cone*** *of a fan* $\mathcal{F}$ *with support* σ *any* 2*-dimensional cone* γ *of* $\mathcal{F}$ *with* 1*-dimensional faces* τ_1, τ_2 *so that*

1. *the minimal containing face of* τ_1 *in* σ *is* σ*,*
2. *and the minimal containing cone of* τ_2 *is* 2*-dimensional.*

If γ is a cutting cone of $\overset{\star}{\mathcal{F}}$, then $\overline{O_\gamma}$ is an irreducible curve of $\overset{\star}{\mathbb{D}}_{f,ex} \cap \overset{\star}{\mathbb{D}}_0$. The denomination *cutting cone* is chosen in reference to the so called "cutting edges" introduced in [NS12, Definition 7.2.2].

Example 4.9. *The fact that the fan* $\mathcal{F}_{f\cdot g}$ *is the minimal refinement of both* $\mathcal{F}_f$ *and* $\mathcal{F}_g$ *implies that it can be computed by "superposing" the two fans* $\mathcal{F}_f$ *and* $\mathcal{F}_g$*, as in figure 3.*

This figure is to be understood as the cone over the plane figure. In this figure, thick lines indicate the 2-dimensional cones which are contained in 2-dimensional cones of $\mathcal{F}_f$, dashed lines indicate the 2-dimensional cones which are contained in 2-dimensional cones of $\mathcal{F}_g$, so the thick dashed line indicates a 2-dimensional cone which is in both.

Hollow round marks correspond to 2-dimensional orbits of $X_{\overset{\star}{\mathcal{F}}}$ whose closures are irreducible components of $\overset{\star}{\mathbb{D}}_0$ intersected by $\overset{\star}{V}(f)$.

Hollow square marks correspond to 2-dimensional orbits of $X_{\overset{\star}{\mathcal{F}}}$ whose closures are irreducible components of $\mathbb{D}_{f,ex}^{\star}$. Remarks 4.1 and 4.5 imply in fact that any 1-dimensional cone whose minimal containing face in σ is 2-dimensional correpsonds to a component of $\mathbb{D}_{f,ex}^{\star}$.

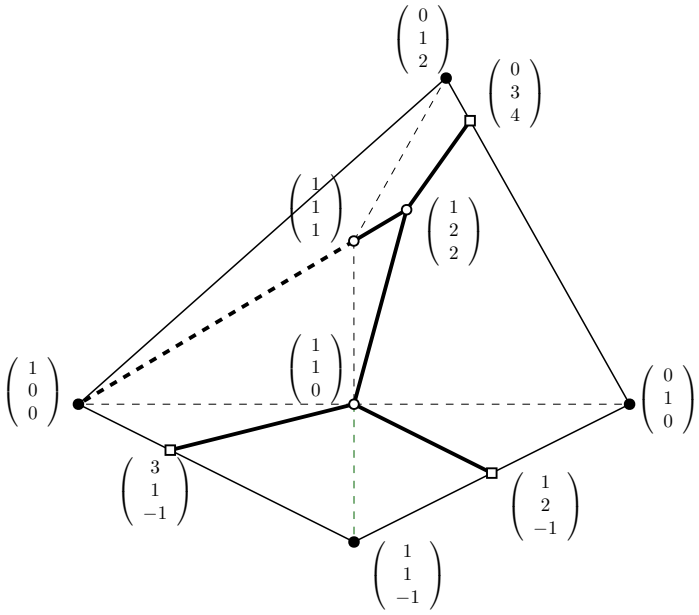


Figure 3: The fan $\overset{\star}{\mathcal{F}}$.

At this step, we do not yet have a modification of X_σ respecting all the required axioms. We still need to refine some cones of $\overset{\star}{\mathcal{F}}$.

4.3 First refinement of $\overset{\star}{\mathcal{F}}$

We obtain a new fan $\widehat{\mathcal{F}}$ by, first, refining regularly the 2-dimensional cones which are contained in 2-dimensional cones of F_f. This is done in a canonical way, see for example [Oka87] or

[PP07]. These cones correspond to 1-dimensional orbits of $X_{\overset{\star}{\mathcal{F}}}$ which are intersected by $\overset{\star}{V}(f)$. Then, one needs to add new 2-dimensional cones in a non-canonical way in order to get a collection of cones respecting the definition of a fan, see Definition 3.7. Furthermore, for convenience, we require that this last addition is made without introducing new cutting cones. This is always possible.

Denote by $\widehat{\mathcal{F}}$ the fan obtained at the end of this process.

Example 4.10. *In our example, the 2-dimensional cones of $\overset{\star}{\mathcal{F}}$ in question are already regular, so the fan $\widehat{\mathcal{F}}$ is the same as $\overset{\star}{\mathcal{F}}$.*

By analogy with the previous subsection, we introduce the notations $\widehat{\mathbb{D}_0}, \widehat{V(f)}, \widehat{V(g)}, \widehat{\mathbb{D}_{f,ex}}$, and $\widehat{r}\colon X_{\widehat{\mathcal{F}}} \to X_\sigma$.

4.4 The fan $\widetilde{\mathcal{F}}$

Here again, the components of $\widehat{\mathbb{D}_0}, \widehat{V(f)}, \widehat{V(g)}, \widehat{\mathbb{D}_{f,ex}}$ can be read on $\widehat{\mathcal{F}}$ as in Discussion 4.7, and if γ is a cutting cone of $\widehat{\mathcal{F}}$, then $\overline{O_\gamma}$ is an irreducible curve of $\widehat{\mathbb{D}_{f,ex}} \cap \widehat{\mathbb{D}_0}$. This is what motivates the final refinement of $\widehat{\mathcal{F}}$.

Denote $\widetilde{\mathcal{F}}$ a fan obtained by refining the 3-dimensional cones of $\widehat{\mathcal{F}}$ having a cutting cone as a face. These cones correspond to points of $X_{\widehat{\mathcal{F}}}$ which are on curves of $\widehat{\mathbb{D}_{f,ex}} \cap \widehat{\mathbb{D}_0}$, whence the need for $X_{\widetilde{\mathcal{F}}}$ to be smooth here.

There is no canonical way to refine regularly a 3-dimensional cone, but a first necessary step is the refining of its 2-dimensional faces. For a description of the general process, see [Oka87].

Our construction ensures the following

Lemma 4.11. *The modification $r_X := \Pi_{\widetilde{\mathcal{F}}}\colon \tilde{X} := X_{\widetilde{\mathcal{F}}} \to X$ is a modification of X adapted to the pair (f, g).*

Example 4.12. *Figure 4 shows a possible $\widetilde{\mathcal{F}}$ for our example. The new cutting cones are represented by continuous thin lines. We erased one end of the corresponding segments on the drawing to reflect the fact that the corresponding curves do not intersect, as will be explained in the next subsection. The other new 2-dimensional cone is represented with a dotted line. Other codes are the same as previously.*

4.5 Reading $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$

Let us explain now how the decorated configuration $\mathscr{C}_{tot}$ can be read from the fan $\widetilde{\mathcal{F}}$, together with $\mathrm{LNP}(f)$ and $\mathrm{LNP}(g)$.

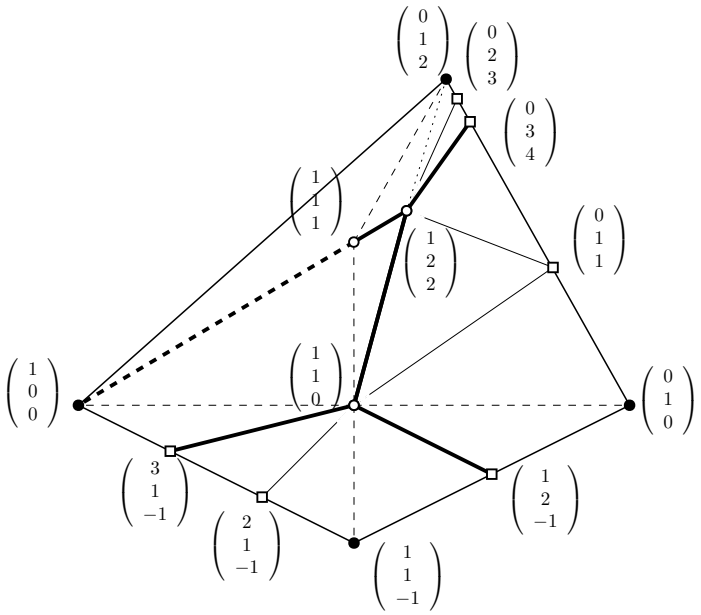


Figure 4: The fan $\widetilde{\mathcal{F}}$.

Recall that $\mathscr{C}_{tot} = (\mathbb{D}_f \cap \mathbb{D}_0) \cup \left(\mathbb{D}_f \cap \widetilde{V(g)}\right)$ can be decomposed as

$$(\mathbb{D}_{f,ex} \cap \mathbb{D}_0) \cup \left(\widetilde{V(f)} \cap \mathbb{D}_0\right) \cup \left(\widetilde{V(f)} \cap \widetilde{V(g)}\right) \cup \left(\mathbb{D}_{f,ex} \cap \widetilde{V(g)}\right).$$

Since $\mathcal{F}_g$ contains no cutting cone, the last intersection is empty. There are therefore three types of curves to be read.

We introduce the following notations:

Definition 4.13. *For $\tau \in \widetilde{\mathcal{F}}$, denote repectively $\Delta_\tau(f)$ and $\Delta_\tau(g)$ the corresponding faces of $LNP(f)$ and $LNP(g)$.*

To make short, if the dimension of those faces is 1 or less, denote

$$l_\tau(f) := l(\Delta_\tau(f)) \text{ and } l_\tau(g) := l(\Delta_\tau(g))$$

with the convention that the length of a 0-dimensional face is 0.

If $dim(\Delta_\tau(f)) \leqslant 2$, denote i_τ the number of interior points of $\Delta_\tau(f)$, with the convention that the number of interior points of a face of dimension 0 or 1 is 0.

*Finally, a cone τ in the interior of σ is called **pertinent** if $dim(\Delta_\tau(f)) \geqslant 1$, that is, if $\widetilde{V(f)} \cap O_\tau \neq \emptyset$.*

Discussion 4.14. (Two types of compact curves.)

1. ***The first type is made of curves in*** $\mathbb{D}_{f,ex} \cap \mathbb{D}_0$, *called* ***exceptional curves*** *Such curves are of the form* $\overline{O_\gamma}$, *where* γ *is a cutting cone of* $\widetilde{\mathcal{F}}$.

   Two such curves intersect if and only if the corresponding cones are faces of a same 3*-dimensional cone of* $\widetilde{\mathcal{F}}$. *The nature of their intersection point can be read on the fan: it is* $\oplus$ *if and only if they are in the same component of* $\mathbb{D}_{f,ex}$.

   The multiplicity decorations are obtained in the following way: let γ *be a cutting cone, and* $\tau_1 \subset \overset{\circ}{\sigma}, \tau_2$ *be its* 1*-dimensional faces. Then* $\overline{O_\gamma}$ *has multiplicity decoration* $(h_{\tau_1}(f); h_{\tau_2}(f), h_{\tau_2}(g))$. *Each exceptional curve is* ***rational***.

2. ***The second type is made of curves of*** $\widetilde{V(f)} \cap \mathbb{D}_0$, *called* ***strict tranform curves***. *They correspond to* 1*-dimensional pertinent cones* τ *of* $\widetilde{\mathcal{F}}$.

   Furthermore, in these conditions, the intersection $\overline{O_\tau} \cap \widetilde{V(f)}$ *is*

   - *an irreducible curve* C_τ *of genus* i_τ *if* $dim(\Delta_\tau(f)) = 2$.
   - *a disjoint union* $C_\tau = {C_\tau}^1 \bigsqcup \cdots \bigsqcup {C_\tau}^{l_\tau(f)}$ *of* $l_\tau(f)$ *irreducible rational curves if* $dim(\Delta_\tau(f)) = 1$.

   Each connected component of C_τ *has multiplicity decoration* $(1; h_\tau(f), h_\tau(g))$.

   Furthermore, let τ_1, τ_2 *be two* 1*-dimensional such cones. The possibly disconnected curves* C_{τ_1} *and* C_{τ_2} *intersect if and only if* τ_1 *and* τ_2 *are faces of the same* 2*-dimensional pertinent cone* γ. *In these conditions,*

$$Card(C_{\tau_1} \cap C_{\tau_2}) = l_\gamma(f)$$

   and these intersection points are all of type $\oplus$.

   In this situation, if $dim(\tau_1) = 1$, *then* $l_{\tau_1}(f) = l_\gamma(f)$, *and each connected component of* C_{τ_1} *intersects* C_{τ_2} *in exactly one point.*

Lemma 4.15. (Intersection of curves of the first and of the second type.)
Let $C_1 = \overline{O_\gamma}$ *be an exceptional curve, and* $C_2 = C_\tau$ *be a strict transform curve. Then*

$$C_1 \cap C_2 \neq \emptyset \Leftrightarrow \tau \prec \gamma \text{ and } \gamma \text{ is pertinent.}$$

In these conditions,

$$Card(C_1 \cap C_2) = l_\gamma(f)$$

and each intersection point is of type $\ominus$.

Now, one needs to add to the picture the non compact curves of the intersection $\widetilde{V(f)} \cap \widetilde{V(g)}$.

Pick generic coefficients of g so that for any 2-dimensional orbit O_τ of $\tilde{X}$ intersected by $\widetilde{V(f)}$, the truncations $f_{\Delta_\tau(f)}$ and $g_{\Delta_\tau(g)}$ verify the hypothesis of Lemma 3.31.

Definition 4.16. *Denote $V(\tau)$ the mixed 2-dimensional volume of $\Delta_\tau(f)$ and $\Delta_\tau(g)$.*

With this choice of g, we can access the rest of the configuration $\mathscr{C}_{tot}$. Indeed,

Lemma 4.17. (Adding non-compact curves)
Let C_τ be a strict transform curve of $\mathscr{C}$, possibly disconnected. Then

$$Card\left(C_\tau \cap \widetilde{V(g)}\right) = V(\tau),$$

each of these intersection points being an intersection point of C_τ with a curve in $\widetilde{V(f)} \cap \widetilde{V(g)}$.

Furthermore, if $dim(\Delta_\tau(f)) = 1$, $\exists\ k \in \mathbb{N}$ such that $V(\tau) = k \cdot l_\tau(f)$, and each connected component of C_τ is intersected in k points.

Each of these points is of type $\oplus$, and, in $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$, the new curves are represented by arrowheads decorated with $(1; 0, 1)$.

Example 4.18. *Figure 5 shows the graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ of our example. The representation of the graph reflects the disposition of the fan $\widetilde{\mathcal{F}}$ of figure 4. Following the classical convention, we do not indicate the decorations $\oplus$.*

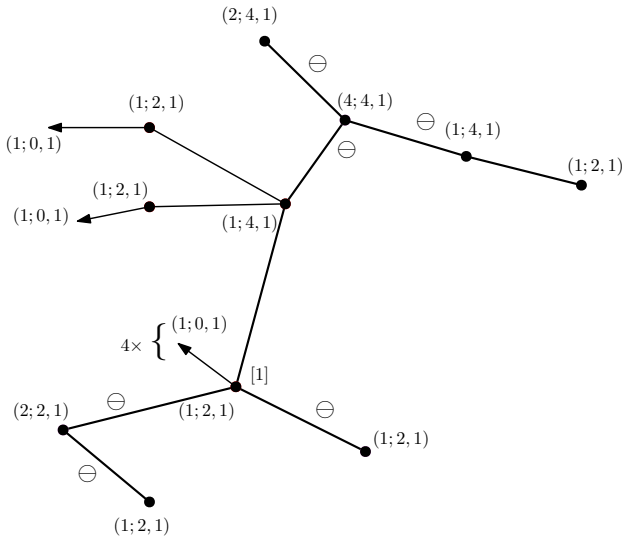


Figure 5: The graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$.

5 Final step of the computation and sufficiency of $\mathscr{C}_{tot}$

In order to get a plumbing graph for the boundary of the Milnor fiber of f, one needs to modify the graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$. Unlike in the general case, the necessary data for this operation

is entirely encoded in $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$. Indeed, one has the following direct consequences of our construction:

Lemma 5.1. *1. Every non-rational curve of $\mathscr{C}_{tot}$ has a multiplicity decoration of the form $(1; m_2, n_2)$.*

2. Each cycle in the graph $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ contains a vertex having a multiplicity decoration of the form $(1; m_2, n_2)$.

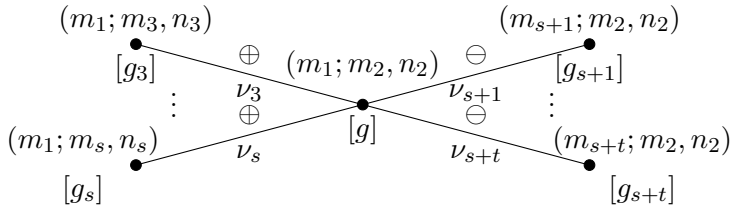
In the language developped in [Cur19], point 1 implies that the data of the so-called switches is unnecessary, and point 2 implies that a covering data of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ determines a unique graph, see [Né00, Theorem 1.20].

As in [NS12, 10.2] (see also [Cur19] for more details), a plumbing graph for the boundary ∂F of the Milnor fiber of f is obtained from $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ by the following two steps:

5.1 The multiplicity graph $\Gamma^{\mu}(E_{tot})$

Denote $\Gamma^{\mu}(E_{tot})$ the graph obtained from $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ in the following way:

1. Let v_C be a vertex of $\overset{\star}{\Gamma}(\mathscr{C}_{tot})$ corresponding to a curve C of genus g, such that the union of v_C and the adjacent vertices is of the form

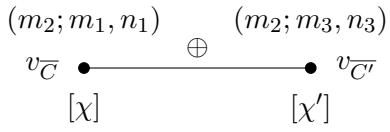


where ν_i indicates that an edge is repeated ν_i times. Then replace v_C by the union of $n_C := gcd(m_1, \cdots, m_{s+t})$ vertices, each one of them decorated by the same genus g_C verifying

$$n_C \cdot (2 - 2 \cdot g_C) = \left(2 - 2g - \sum_{i=3}^{s+t} \nu_i\right) \cdot gcd(m_1, m_2) + \sum_{i=3}^{s+t} gcd(m_1, m_2, m_i) \cdot \nu_i,$$

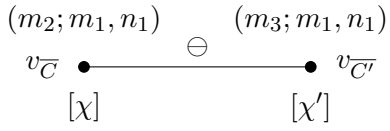
and a mutliplicity decoration given by $\mu_C := \dfrac{m_2 \cdot n_1}{gcd(m_1, m_2)}$.

2. • An edge of type $\oplus$, of the form



is replaced by $d := gcd(m_1, m_2, m_3)$ identical strings of type $Str^{\oplus}\left(\frac{m_2}{d}; \frac{m_1}{d}, \frac{m_3}{d} | 0; n_1, n_3\right)$.

- An edge of type $\ominus$, of the form



is replaced by $d := gcd(m_1, m_2, m_3)$ identical strings of type $Str^{\ominus}\left(\frac{m_1}{d}; \frac{m_2}{d}, \frac{m_3}{d} | n_1; 0, 0\right)$.

The notations Str are explained in Appendix B. In both cases, the number d of new edges is a multiple of the number of new vertices, and the edges are distributed **uniformly** on the vertices.

Example 5.2. *Figure 6 shows the graph $\Gamma^{\mu}(E_{tot})$ of our example. Here, the different bamboos are simple and do not bring any new vertices, but the upper vertex gave rise to two vertices, and the edge ending at this vertex has also produced two edges. The multiplicity decorations on $\Gamma^{\mu}(E_{tot})$ correspond to the multiplicities of the pullback of the function g on each irreducible surface of $(g \circ \Pi)^{-1}(0)$.*

5.2 The plumbing graph $\Gamma(\partial F)$

A plumbing graph $\Gamma(\partial F)$ is obtained from $\Gamma^{\mu}(E_{tot})$ by replacing the multiplicity decorations (μ_i) by the self-intersections k_i of the components of the preimage of the origin by Π in $\widetilde{\mathscr{S}}$, erasing the arrowhead vertices, and keeping the genus and edge decorations.

Let v be a vertex of $\Gamma^{\mu}(E_{tot})$, with multiplicity μ. Let $v_1, \cdots, v_n$ be the adjacent vertices (including arrowheads), with respective multiplicities $\mu_1, \cdots, \mu_n$, and denote $\epsilon_1, \cdots, \epsilon_n$ the decorations of the corresponding edges. Then, following Lemma A.8, the self-intersection decoration k of v in $\Gamma(\partial F)$ is obtained through the equality

$$k \cdot \mu = \sum_{i=1}^{n} \epsilon_i \cdot \mu_i.$$

Example 5.3. *Figure 7 shows the plumbing graph for ∂F obtained after applying the step described previously.*

Figure 8 shows the normal form of this graph. Note that, unlike in the case of isolated singularities, this graph does not have a negative definite incidence matrix.

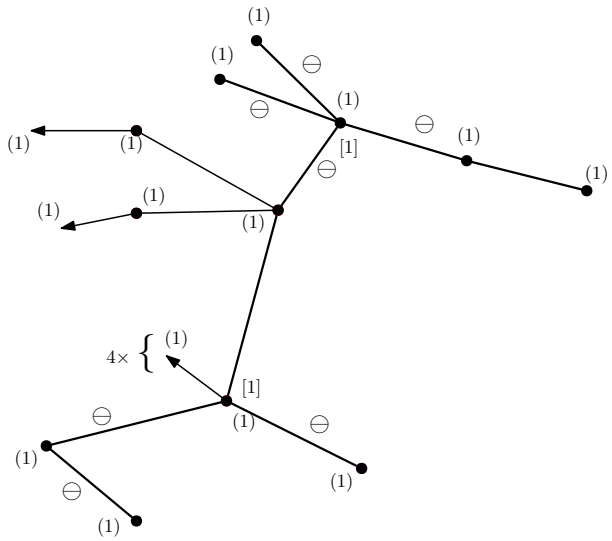


Figure 6: The graph $\Gamma^{\mu}(E_{tot})$.

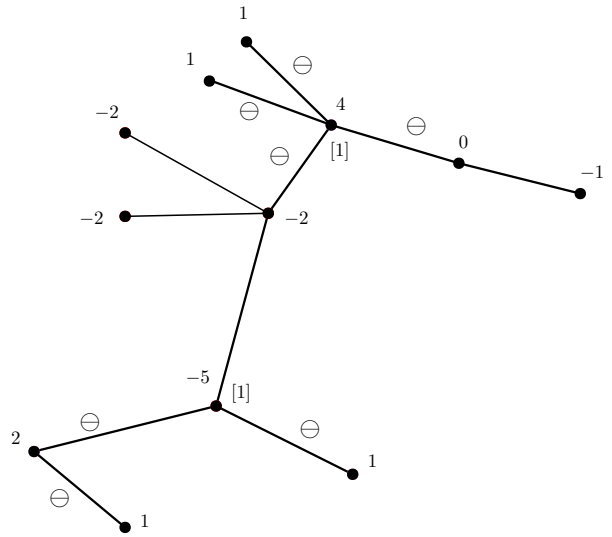


Figure 7: The graph $\Gamma(\partial F)$.

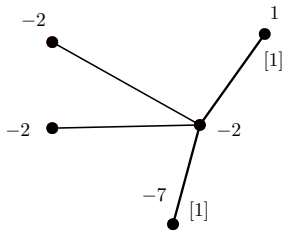


Figure 8: The normal form of $\Gamma(\partial F)$.

The plumbing calculus does not preserve planarity. However, the process of normalization of a given plumbing graph preserves this property. Therefore, a first consequence of our method of computation is the following:

Proposition 5.4. *The normal form of the plumbing graph of the Milnor fiber of a Newton non degenerate singularity of complex surface is planar.*

Appendices

A Tubular neighbourhoods and graph manifolds

Graph manifolds can appear naturally as boundaries of neigbourhoods of complex curves in smooth complex surfaces, see [Mum61]. In our context, they will appear as boundaries of neighbourhoods of real surfaces inside a smooth 4-manifold.

Definition A.1. (Simple configuration of surfaces, and its plumbing dual graphs.) *Let $\overline{\mathscr{S}}$ be a 4-dimensional oriented real analytic manifold. A **simple configuration of compact real analytic surfaces in** $\overline{\mathscr{S}}$ is a subset $E \subset \overline{\mathscr{S}}$ such that:*

1. *$E = \bigcup_{finite} E_i$, such that each E_i is an oriented closed smooth real analytic surface.*
2. *For all $i \neq j \neq k \neq i$, the intersection $E_i \cap E_j \cap E_k$ is empty.*
3. *For all $i \neq j$, the intersection $E_i \cap E_j$ is either empty or transverse. In particular, it is a finite union of points.*

*In this setting, one defines a **plumbing dual graph** $\Gamma_{\overline{\mathscr{S}}}(E)$ of E in $\overline{\mathscr{S}}$ by decorating its dual graph in the following way:*

1. *Decorate each vertex v_{E_i} by the self-intersection e_i of E_i in $\overline{\mathscr{S}}$ and by the genus $[g_i]$ of the surface E_i.*

2. *Decorate each edge of* $\Gamma(E)$ *corresponding to the intersection point* p *of* $E_i \cap E_j$*, by* $\oplus$ *if the orientation of* E_i *followed by the orientation of* E_j *is equal to the orientation of* $\overline{\mathscr{S}}$ *at* p*, and by* $\ominus$ *otherwise.*

Definition A.2. *(See [Dur83].)*

Let E *be a simple configuration of compact orientable real analytic surfaces in an oriented* 4*-dimensional real analytic manifold* $\overline{\mathscr{S}}$*. In this context, we call* ***rug function*** *any real analytic proper function* $\rho\colon \overline{\mathscr{S}} \to \mathbb{R}_+$ *such that* $\rho^{-1}(0) = E$*.*

Theorem A.3. *Let* E *be a simple configuration of compact orientable real analytic surfaces in an oriented* 4*-dimensional real analytic manifold* $\overline{\mathscr{S}}$*, that admits a rug function* ρ *as in Definition A.2. Then, for* $\varepsilon > 0$ *small enough, the boundary of the oriented* 4*-manifold* $\{\rho \leqslant \varepsilon\}$ *is orientation-preserving homeomorphic to the graph manifold associated to the graph* $\Gamma_{\overline{\mathscr{S}}}(E)$*.*

Definition A.4. *In this setting, the manifold* $\{\rho \leqslant \varepsilon\}$ *is called a* ***tubular neighbourhood*** *of* E*.*

About the proof of Theorem A.3. This theorem can be seen as an extension of what is done in [Mum61], in the case of a configuration of complex analytic curves in a smooth complex surface.

In our case, observe first that we can extend the definition of rug functions to semi-analytic functions ρ, and still have a unique homeomorphism type for the boundary of the neighbourhood $\{\rho \leqslant \varepsilon\}$ of E for $\varepsilon > 0$ small enough, following the proof of [Dur83, Proposition 3.5]. Now, one can build by hand a semi-analytic neighbourhood whose boundary is homeomorphic to the manifold $\Gamma_{\overline{\mathscr{S}}}(E)$.

This is done by building a rug function for each irreducible component E_i of E, providing a tubular neighbourhood T_i of each E_i whose boundary is an $\mathbb{S}^1$-bundle of Euler class e_i over E_i. One then plumbs those bundles using appropriate normalizations of the rug functions, building a semi-analytic neighbourhood of E which is homeomorphic to the desired graph manifold. □

Remark A.5. *Note that the decorations on the edges of the graph* $\Gamma_{\overline{\mathscr{S}}}(E)$ *depend on the orientations of the surfaces* E_i*. However, if the surfaces* E_i *are only orientable, the different possible plumbing dual graphs still encode the same graph manifold, see move* **[R0]** *of the plumbing calculus in [Neu81].*

We end this section with a tool that can be used to compute the self-intersections of the irreducible components of E:

Definition A.6. *Let* $E = \bigcup_{finite} E_i$ *be a simple configuration of compact oriented real analytic surfaces in a* 4*-dimensional real analytic manifold* $\overline{\mathscr{S}}$*. A real analytic function* $g\colon \overline{\mathscr{S}} \to \mathbb{C}$ *is called* ***adapted*** *to* E *if*

1. *$E_{tot} := g^{-1}(0)$ is a simple configuration of oriented, not necessarily compact, real analytic surfaces, such that $E_{tot} \supset E$.*

2. (a) *For any component E_i of E_{tot}, $\forall\ p \in E_i \setminus \bigcup_{j\neq i} E_j$, there is a neighbourhood U_p of p in $\overline{\mathscr{S}}$ and complex coordinates (x_p, y_p) on U_p such that $U_p \cap E_i = \{x_p = 0\}$ and $n_i \in \mathbb{N}^*$ such that*

$$g = x_p^{n_i} \cdot \varphi$$

where $\varphi\colon U_p \to \mathbb{C}$ is a unit at p.

(b) *For any components E_i of E, E_j of E_{tot}, $\forall\ P \in E_i \cap E_j$, there is a neighbourhood U_p of p in $\overline{\mathscr{S}}$ and complex coordinates (x_p, y_p) on U_p such that $U_p \cap E_i = \{x_p = 0\}$, $U_p \cap E_k = \{y_p = 0\}$, and $n_i, n_k \in \mathbb{N}^*$ such that*

$$g = x_p^{n_i} y_p^{n_j} \cdot \varphi$$

where $\varphi\colon U_p \to \mathbb{C}$ is a unit at p.

Definition A.7. *In this setting, the integer n_i of point 2a of Definition A.6, independent of the point $p \in E_i \setminus \bigcup_{j\neq i} E_j$, is called **the multiplicity of** g **on** E_i, denoted $m_{E_i}(g)$.*

Lemma A.8. (Computing self-intersections.)
Let $\overline{\mathscr{S}}, E, g, E_{tot}$ be as in Definition A.6. Let $E^{(1)}$ be an irreducible component of E. Then the self-intersection $e^{(1)}$ of the surface $E^{(1)}$ in $\overline{\mathscr{S}}$ verifies the following condition: let $p_1, \cdots, p_n$ be the intersection points of $E^{(1)}$ with other components of E_{tot}, $p_j \in E_j \cap E^{(1)}$, where the same component may appear several times. Then

$$n^{(1)} \cdot e^{(1)} = \sum_{i=1}^{n} \epsilon_i \cdot n_i$$

where $\epsilon_i \in \{-1, +1\}$ refers to the sign associated to the intersection p_i in the following sense: if $p \in E_i \cap E_j$, associate $+1$ to p if and only if the combination of the orientations of E_i and E_j at p provides the ambient orientation of $\overline{\mathscr{S}}$.

Proof. The proof follows the standard argument in the holomorphic category. The difference of the two members of the equation is the intersection number of $E^{(1)}$ with the cycle defined by $g = 0$. This cycle is homologous with that defined by a nearby level of g, which does not meet $E^{(1)}$ any more. The intersection number being invariant by homology, one gets the desired result.

In order to make this argument rigorous, one has to work in convenient tubular neighborhoods of E and to look at the cycles defined by the levels of g in the homology of the tube relative to the boundary. □

B Hirzebruch-Jung chains

We introduce here the Hirzebruch-Jung chains (or strings) that occur in the final step of our resolution of $(\widetilde{\mathscr{S}_k}, 0)$. For more details, one can consult [BPVdV84, III.5], [NS12, 4.3] or [PP07].

For $a, b, c \in \mathbb{N}$, (a, b, c) denotes $gcd(a, b, c)$. Suppose that $d := (a, b, c) = 1$. In these conditions, the normal surface $(V, p) := \left(\{x^a = y^b z^c\}, 0\right)^N$ is an isolated singularity of complex surface. We describe here a graph of resolution of this singularity, decorated with the self-intersections of the irreducible components of the exceptional divisor and the multiplicities of the pullback of the function $g(x, y, z) = x^{n_1} y^{n_2} z^{n_3}$.

Denote $\delta := \dfrac{ad}{(a, b)(a, c)}$, and by α the be the unique integer in $[0, \delta - 1]$ such that

$$ad \mid \alpha c(a, b) + b(a, c).$$

Let

$$\frac{\delta}{\alpha} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\cdots - \cfrac{1}{k_l}}}$$

be the negative fraction expansion of δ/α, $k_i \geqslant 2$, $k_i \in \mathbb{N}$.

Define $\mu_{l+1} := \dfrac{b \cdot n_1 + a \cdot n_2}{(a, b)}, \mu_0 := \dfrac{c \cdot n_1 + a \cdot n_3}{(a, c)}, \mu_1 := \dfrac{\alpha \cdot \mu_0 + \mu_{l+1}}{\delta}$ and let $\mu_2, \cdots, \mu_l$ be defined by the relation

$$\forall\ 1 \leqslant i \leqslant l, \mu_{i+1} = \mu_{i-1} - k_i \cdot \mu_i.$$

Then the graph of Figure 9 is a graph of resolution of $(V, 0)$. Left and right-hand arrows represent, respectively, the pullbacks of the z and the y-axes. The numbers between parentheses are the multplicities of the pullback of the function $g(x, y, z) = x^{n_1} y^{n_2} z^{n_3}$ on the irreducible components of its total transform.

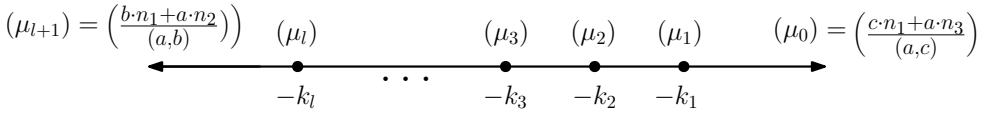


Figure 9: The string $Str(a; b, c|n_1; n_2, n_3)$.

We denote by $Str^{\oplus}(a; b, c|n_1; n_2, n_3)$, resp. $Str^{\ominus}(a; b, c|n_1; n_2, n_3)$ the chain of Figure 9 with each edge decorated by $\oplus$, resp. $\ominus$, and the self-intersection decorations removed.

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Octave Curmi, AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M, FRANCE.
E-mail address: octave.curmi@univ-amu.fr