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Andrei Pajitnov

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TWISTED MONODROMY HOMOMORPHISMS AND MASSEY PRODUCTS

ANDREI PAJITNOV

ABSTRACT. Let $\phi : M \to M$ be a diffeomorphism of a C^∞ compact connected manifold, and X its mapping torus. There is a natural fibration $p : X \to S^1$, denote by $\xi \in H^1(X, \mathbb{Z})$ the corresponding cohomology class. Let $\rho : \pi_1(X, x_0) \to \mathrm{GL}(n, \mathbb{C})$ be a representation (here $x_0 \in M$); denote by $H^*(X, \rho)$ the corresponding twisted cohomology of X. Denote by ρ_0 the restriction of ρ to $\pi_1(M, x_0)$, and by ρ_0^* the antirepresentation conjugate to ρ_0. We construct from these data *the twisted monodromy homomorphism* ϕ_* of the group $H_*(M, \rho_0^*)$. This homomorphism is a generalization of the homomorphism induced by ϕ in the ordinary homology of M. The aim of the present work is to establish a relation between Massey products in $H^*(X, \rho)$ and Jordan blocks of ϕ_*.

We have a natural pairing $H^*(X, \mathbb{C}) \otimes H^*(X, \rho) \to H^*(X, \rho)$; one can define Massey products of the form $\langle \xi, \dots, \xi, x \rangle$, where $x \in H^*(X, \rho)$. The Massey product containing r terms ξ will be denoted by $\langle \xi, x \rangle_r$; we say that the *length* of this product is equal to r. Denote by $M_k(\rho)$ the maximal length of a non-zero Massey product $\langle \xi, x \rangle_r$ for $x \in H^k(X, \rho)$. Given a non-zero complex number λ define a representation $\rho_\lambda : \pi_1(X, x_0) \to \mathrm{GL}(n, \mathbb{C})$ as follows: $\rho_\lambda(g) = \lambda^{\xi(g)} \cdot \rho(g)$. Denote by $J_k(\phi_*, \lambda)$ the maximal size of a Jordan block of eigenvalue λ of the automorphism ϕ_* in the homology of degree k.

The main result of the paper says that $M_k(\rho_\lambda) = J_k(\phi_*, \lambda)$. In particular, ϕ_* is diagonalizable, if a suitable formality condition holds for the manifold X. This is the case if X a compact Kähler manifold and ρ is a semisimple representation. The proof of the main theorem is based on the fact that the above Massey products can be identified with differentials in a Massey spectral sequence, which in turn can be explicitly computed in terms of the Jordan normal form of ϕ_*.

CONTENTS

1. INTRODUCTION

The relation between non-vanishing Massey products of length 2 in the cohomology of mapping tori and the Jordan blocks of size greater than 1 of the monodromy homomorphism was discovered in the work of M. Fernández, A. Gray, J.

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Morgan [7]. This relation was used by these authors to prove that certain mapping tori do not admit a structure of a Kähler manifold. In the work of G. Bazzoni, M. Fernández, V. Muñoz [1] it was proved that the existence of Jordan blocks of size 2 implies the existence of a non-zero triple Massey product of the form $\langle \xi, \xi, a\rangle$.

In the paper [14] we began a systematic treatment of this phenomena, relating the maximal length of non-zero Massey products to the maximal size of Jordan blocks of the monodromy homomorphism. The both numbers turn out to be less by a unit than the number of the sheet where the formal deformation spectral sequence degenerates. In that paper we dealt with the case of Massey products of the form $\langle \xi, \dots, \xi, x\rangle$ where ξ is the 1-dimensional cohomology class determined by the fibration of the mapping torus X over the circle, and x is an element in the cohomology of X with coefficients in a 1-dimensional local system.

In the present paper we continue the study of this phenomena, and prove a theorem relating the Jordan blocks of the monodromy homomorphisms to the Massey products of the form $\langle \xi, \dots, \xi, x\rangle$ where x is an element in the twisted cohomology of X corresponding to an arbitrary representation $\pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$. One technical problem here is that there is no immediate definition of the homomorphism induced in the twisted homology of M by the diffeomorphism $\phi : M \to M$. We construct such homomorphism in the present paper (Section 4). The construction is based on the techniques developed by P. Kirk and C. Livingston [9] in the context of twisted Alexander polynomials.

One corollary is that if X is a compact Kähler manifold, then all the Jordan blocks of this twisted monodromy homomorphism are of size 1, that is, the twisted monodromy homomorphism is diagonalizable. This result imposes new constraints on the homology of Kähler manifolds.

2. Overview of the article

Let $\phi : M \to M$ be a diffeomorphism of a C^∞ compact connected manifold, and X its mapping torus. Choose a point $x_0 \in M$, and put $H = \pi_1(M, x_0)$, $G = \pi_1(X, x_0)$. We have an exact sequence

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{p} \mathbb{Z} \longrightarrow 1 \tag{1}$$

Let V be a finite dimensional vector space over $\mathbb{C}$ and $\rho : G \to \mathrm{GL}(V)$ be a representation; denote by ρ_0 its restriction to H. Put $V^* = \mathrm{Hom}(V, \mathbb{C})$ and let $\rho^* : G \to \mathrm{GL}(V^*)$ be the antirepresentation of G conjugate to ρ. In Section 4 we construct from these data an automorphism ϕ_* of the group $H_*(M, \rho_0^*)$; we call it *the twisted monodromy homomorphism*. This homomorphism can be considered as a generalization of the map induced by ϕ in the ordinary homology. Observe however, that the homomorphism ϕ_* is not entirely determined by ϕ and ρ_0, but depends also on the values of the representation ρ on the elements of $G\backslash H$.

In the particular case when ρ_0 is the trivial 1-dimensional representation, and the representation ρ sends the positive generator u of $G/H \approx \mathbb{Z}$ to $\lambda \in \mathbb{C}^*$, the map $\lambda\phi_*$ equals the homomorphism induced by ϕ in the ordinary homology (see the details in Subsection 4.2).

The main result of the paper is the theorem A below. To state it we need some terminology. Denote by $H^*(X, \rho)$ the twisted cohomology of X with respect to the representation ρ. We have a natural pairing $H^*(X, \mathbb{C}) \otimes H^*(X, \rho) \to H^*(X, \rho)$; one can define Massey products of the form $\langle \xi, \dots, \xi, x\rangle$, where $x \in H^*(X, \rho)$.

The Massey product containing r terms ξ will be denoted by $\langle \xi, x\rangle_r$; we say that the *length* of this product is equal to r. Denote by $M_k(\rho)$ the maximal length r of a non-zero Massey product $\langle \xi, x\rangle_r$ for $x \in H^k(X, \rho)$. For a number $\lambda \in \mathbb{C}^*$ define a representation $\rho_\lambda : \pi_1(X, x_0) \to \mathrm{GL}(V)$ as follows: $\rho_\lambda(g) = \lambda^{\xi(g)} \cdot \rho(g)$. Denote by $J_k(\phi_*, \lambda)$ the maximal size of a Jordan block of eigenvalue λ of the homomorphism ϕ_* of $H_k(M, \rho_0^*)$.

Theorem A. *We have $J_k(\phi_*, \lambda) = M_k(\rho_\lambda)$ for every k and λ.*

This theorem implies that the monodromy homomorphism ϕ_* has only Jordan blocks of size 1 (that is, ϕ_* is diagonalizable) provided that the space X satisfies a suitable formality condition. Such formality conditions are discussed in details in Subsection 3.3. The main application of these ideas is the following theorem.

Theorem B. *Assume that the mapping torus X of a diffeomorphism $\phi : M \to M$ is a compact Kähler manifold. Let $\rho : \pi_1(X, x_0) \to \mathrm{GL}(n, \mathbb{C})$ be a semisimple representation. Then the twisted monodromy homomorphism ϕ_* of $H_*(M, \rho_0^*)$ is a diagonalizable linear map.*

The proofs of these theorems are given in Section 6. They are based on the construction of the twisted monodromy homomorphism ϕ_* (Section 4), and the computation of the Massey spectral sequences in terms of ϕ_* (Section 5). In Subsection 4.2 we discuss a particular case of special interest. Let us say that ϕ is π_1-*split* if the exact sequence (1) splits. For this case we give a natural geometric construction of an automorphism ϕ_*° of $H_*(M, \rho_0^*)$ which is entirely determined by ϕ and ρ_0 (see the formula (14) and Remark 4.8 of the Subsection 4.2). Theorem 6.2 states the version of Theorem A for this particular case.

Let us mention a generalization of Theorem A to the case of an arbitrary manifold Y endowed with a non-zero cohomology class $\alpha \in H^1(Y, \mathbb{Z})$ and a representation $\theta : \pi_1(Y, y_0) \to \mathrm{GL}(V)$ (the manifold Y is not assumed to be a mapping torus any more). Assume that α is indivisible and consider the corresponding infinite cyclic covering $\overline{Y}$. Let $L = \mathbb{C}[t, t^{-1}]$. Similarly to the above, denote by θ_0 the restriction of θ to $\pi_1(\overline{Y})$, and by θ_0^* its conjugate antirepresentation. The homology $H_k(\overline{Y}, \theta_0^*)$ is a finitely generated L-module; denote by $\mathcal{T}_k$ its torsion part. This is a finite dimensional vector space over $\mathbb{C}$ endowed with an action of L. In particular the element $t \in L$ represents an automorphism of $\mathcal{T}_k$.

Theorem C. *Let Y be a connected compact manifold, For every k and $\lambda \in \mathbb{C}^*$ the maximal length r of a Massey product of the form $\langle \alpha, x\rangle_r$ (where $x \in H^k(Y, \theta_\lambda)$) equals the maximal size of a Jordan block of eigenvalue λ of the automorphism $t : \mathcal{T}_k \to \mathcal{T}_k$.*

Theorem D. *Let Y be a connected compact Kähler manifold, $\theta : \pi_1(Y, y_0) \to \mathrm{GL}(V)$ a semisimple representation. Then the homomorphism $t : \mathcal{T}_k \to \mathcal{T}_k$ is diagonalizable for every k.*

2.1. **About the terminology.** We will keep the notations from Section 2 throughout the paper. Namely, X will always denote the mapping torus of a diffeomorphism ϕ of a compact connected manifold M; the corresponding cohomology class in $H^1(X, \mathbb{Z})$ will be always denoted by ξ, the ring $\mathbb{C}[t, t^{-1}]$ is denoted by L, and $\mathbb{C}[[z]]$ by Λ. There are two exceptions: in Subsection 3.3 G will denote any group,

and in Subsection 5.1 G will denote the fundamental group of a topological space Y.

2.2. **Relations with other works.** The case of the trivial representation ρ_0 was settled in the author's paper [14]. The diagonalizability of the monodromy homomorphism in the ordinary homology for Kähler manifolds was also proved by N. Budur, Y. Liu, B. Wang [2].

Another approach to the relation between the size of Jordan blocks and formality properties was developed by S. Papadima and A. Suciu [15], [16]. They prove in particular that if the monodromy homomorphism has Jordan blocks of size greater than 1, then the fundamental group of the mapping torus is not a formal group.

3. Formal deformations and Massey spectral sequences

The main aim of this section is to recall necessary definitions and results concerning the Massey spectral sequences. There are different versions of these spectral sequences in literature, see [5], [11], [12], [6], [10]. We will recall here the versions described in [10], referring to this article for details and proofs.

The only new material in the present section is the definition of *$\mathcal{F}$-formal manifold*, introduced in Subsection 3.3. This notion generalizes the classical notion of formality (D. Sullivan [18]) incorporating to it differential forms with coefficients in flat bundles.

3.1. **Formal deformations of differential graded algebras.** Let

$$\mathcal{A}^* = \{\mathcal{A}^k\}_{k\in\mathbb{N}} = \{\mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \dots\}$$

be a graded-commutative differential algebra (DGA) over $\mathbb{C}$. Let $\mathcal{N}^*$ be a graded differential module (DGM) over $\mathcal{A}^*$. We denote by $\mathcal{A}^*[[z]]$ the algebra of formal power series over $\mathcal{A}^*$ endowed with the differential extended from the differential of $\mathcal{A}^*$. Let $\theta \in \mathcal{A}^1$ be a cocycle. Consider the $\mathcal{A}^*[[z]]$-module $\mathcal{N}^*[[z]]$ and endow it with the differential

$$D_t x = dx + z\theta x.$$

Then $\mathcal{N}^*[[z]]$ is a DGM over $\mathcal{A}^*[[z]]$, and we have an exact sequence of DGMs:

$$0 \longrightarrow \mathcal{N}^*[[z]] \xrightarrow{z} \mathcal{N}^*[[z]] \xrightarrow{\pi} \mathcal{N}^* \longrightarrow 0 \tag{2}$$

where π is the natural projection $z \longmapsto 0$. The induced long exact sequence in cohomology can be considered as an exact couple

$$\begin{array}{ccc} H^*(\mathcal{N}^*[[z]]) & \xrightarrow{\quad z \quad} & H^*(\mathcal{N}^*[[z]]) \\ & {\scriptstyle\delta}\nwarrow \quad \swarrow{\scriptstyle\pi_*} & \\ & H^*(\mathcal{N}^*) & \end{array} \tag{3}$$

One can prove that the spectral sequence induced by the exact couple (3) depends only on the cohomology class of θ ([10], Prop. 2.1).

Definition 3.1. Put $\alpha = [\theta]$. The spectral sequence associated to the exact couple (3) is called *formal deformation spectral sequence* and denoted by $E_r^*(\mathcal{N}^*, \alpha)$. If the couple $(\mathcal{N}^*, \alpha)$ is clear from the context, we suppress it in the notation and write just E_r^*.

Thus $E_1^* = H^*(\mathcal{N}^*)$, and it is easy to see that $E_2^* \approx \mathrm{Ker}\, L_\alpha / \mathrm{Im}\, L_\alpha$, where L_α is the homomorphism of multiplication by α. The higher differentials in this spectral sequence can be computed in terms of special Massey products. Let $a \in H^*(\mathcal{N}^*)$. An *r-chain starting from a* is a sequence of elements $\omega_1, \dots, \omega_r \in \mathcal{N}^*$ such that

$$d\omega_1 = 0, \;\; [\omega_1] = a, \;\; d\omega_2 = \theta\omega_1, \;\; \dots, \;\; d\omega_r = \theta\omega_{r-1}.$$

Denote by $MZ^m_{(r)}$ the subspace of all $a \in H^m(\mathcal{N}^*)$ such that there exists an r-chain starting from a. Denote by $MB^m_{(r)}$ the subspace of all $\beta \in H^m(\mathcal{N}^*)$ such that there exists an $(r-1)$-chain $(\omega_1, \dots, \omega_{r-1})$ with $\theta\omega_{r-1}$ belonging to β. It is clear that $MB^m_{(i)} \subset MZ^m_{(j)}$ for every i, j. Put

$$MH^m_{(r)} = MZ^m_{(r)} \Big/ MB^m_{(r)}.$$

In the next definition we omit the upper indices and write $MH_{(r)}, MZ_{(r)}$ etc. in order to simplify the notation.

Definition 3.2. Let $a \in H^*(\mathcal{N}^*)$, and $r \geqslant 1$. We say that the r-tuple Massey product $\langle \theta, \dots, \theta, a \rangle$ is defined, if $a \in MZ_{(r)}$. In this case choose any r-chain $(\omega_1, \dots, \omega_r)$ starting from a. The cohomology class of $\theta\omega_r$ is in $MZ_{(r)}$ (actually it is in $MZ_{(N)}$ for every N) and it is not difficult to show that it is well defined modulo $MB_{(r)}$. The image of $\theta\omega_r$ in $MZ_{(r)}/MB_{(r)}$ is called the r-tuple Massey product of θ and a:

$$\langle \theta, a \rangle_r = \Big\langle \underbrace{\theta, \dots, \theta}_{r}, \; a \Big\rangle \in MZ_{(r)} \Big/ MB_{(r)}.$$

The correspondence $a \longmapsto \langle \theta, a \rangle_r$ gives rise to a well-defined homomorphism of degree 1

$$\Delta_r : MH_{(r)} \longrightarrow MH_{(r)}.$$

The following result is proved in [10], Theorem 2.5.

Theorem 3.3. 1) *For any r we have $\Delta_r^2 = 0$, and the cohomology group $H^*(MH^*_{(r)}, \Delta_r)$ is isomorphic to $MH^*_{(r+1)}$.*

2) *For any r there is an isomorphism*

$$\phi : MH^*_{(r)} \xrightarrow{\approx} E_r^*$$

commuting with differentials.

Therefore the differentials in the spectral sequence E_r^* are equal to the higher Massey products with the cohomology class of θ. Observe that these Massey products, defined above, have smaller indeterminacy than the usual Massey products.

Now let us consider some cases when the spectral sequences constructed above, degenerate in their second term. Recall that a differential graded algebra $\mathcal{A}^*$ is called *formal* if it has the same minimal model as its cohomology algebra.

Theorem 3.4. *[[10], Th. 3.14] Let $\mathcal{A}^*$ be a formal differential algebra, $\alpha \in H^1(\mathcal{A}^*)$. Then the spectral sequence $E_r^*(\mathcal{A}^*, \alpha)$ degenerates at its second term.*

Definition 3.5. A differential graded module $\mathcal{N}^*$ over a differential graded algebra $\mathcal{A}^*$ will be called *formal* if it is a direct summand of a formal differential graded algebra $\mathcal{B}^*$ over $\mathcal{A}^*$, that is,

$$\mathcal{B}^* = \mathcal{N}^* \oplus \mathcal{K}^*, \tag{4}$$

where both $\mathcal{N}^*$ and $\mathcal{K}^*$ are differential graded $\mathcal{A}^*$-submodules of $\mathcal{B}_*$.

Proposition 3.6. *[*[10]*, Th. 3.16] Let $\mathcal{N}^*$ be a formal DG-module over $\mathcal{A}^*$, and $\alpha \in H^1(\mathcal{A}^*)$. Then the spectral sequence $E_r^*(\mathcal{N}^*, \alpha)$ degenerates at its second term.*

In our applications $\mathcal{N}^*$ will be a DGM of differential forms on a manifold. Let Y be a connected C^∞ manifold, and ρ be a representation of the fundamental group of Y. Put $\mathcal{A}^* = \Omega^*(Y, \mathbb{C})$ and let $N^* = \Omega^*(Y, \rho)$ be the DGM of differential forms with coefficients in the flat bundle E_ρ, induced by ρ. Then $\mathcal{N}^*$ is a DGM over $\mathcal{A}^*$, so for any cohomology class $\alpha \in H^1(Y, \mathbb{C})$ we obtain a spectral sequence starting with the twisted cohomology $H^*(Y, \rho)$ (see the next subsection for recollections about the twisted cohomology). The differentials in this spectral sequence are the Massey products with the class α. We denote this spectral sequence by $E_r^*(Y, \rho, \alpha)$.

Corollary 3.7. *In the above notations assume that $\mathcal{N}^* = \Omega^*(Y, \rho)$ is a formal differential graded module over $\mathcal{A}^* = \Omega^*(Y, \mathbb{C})$. Then the spectral sequence $E_r^*(Y, \rho, \alpha)$ degenerates at its second term.*

3.2. **Homology with local coefficients and spectral sequences.** Let us first recall the definition of homology and cohomology with local coefficients. Let R be a commutative ring, and K a free module over R; denote by $\mathrm{GL}(K)$ the group of R-automorphisms of K. Let Y be a connected topological space, and $\rho : \pi_1(Y, y_0) \to \mathrm{GL}(K)$ a representation. Define the R-module of *ρ-twisted cochains of Y with coefficients in ρ* as follows:

$$C^*(Y, \rho) = \mathrm{Hom}_\rho\big(C_*(\widetilde{Y}), K\big). \tag{5}$$

Here $\widetilde{Y}$ denotes the universal covering of Y; we endow it with the canonical free left action of $\pi_1(Y, y_0)$. We denote by $C_*(Y)$ the group of singular chains of Y; if Y is a CW-complex we can replace it by the group of cellular chains. The group $C_*(\widetilde{Y})$ of $\widetilde{Y}$ has a natural structure of a free left module over $\mathbb{Z}\pi_1(Y, y_0)$. The cohomology $H^*(C^*(Y, \rho))$ is called *twisted cohomology of Y with coefficients in ρ*, or *cohomology of Y with local coefficients in ρ*.

Let $\beta : \pi_1(Y, y_0) \to \mathrm{GL}(K)$ be an antirepresentation (that is, $\beta(g_1 g_2) = \beta(g_2)\beta(g_1)$ for every $g_1, g_2 \in \pi_1(Y, y_0)$); it determines a right action of $\pi_1(Y, y_0)$ on K. Define the R-module of *ρ-twisted chains of Y with coefficients in K* as follows:

$$C^*(Y, \beta) = K \underset{\beta}{\otimes} C_*(\widetilde{Y}). \tag{6}$$

The homology $H_*(C_*(Y, \beta))$ is called *twisted homology of Y with coefficients in β*, or *homology of Y with local coefficients in β*.

In this paper we will be dealing with the cases when R is one of the following rings: $\mathbb{C}$, $L = \mathbb{C}[t, t^{-1}]$, $\Lambda = \mathbb{C}[[z]]$. Let V be a finite-dimensional vector space over $\mathbb{C}$, put

$$V[t^{\pm}] = V \underset{\mathbb{C}}{\otimes} \mathbb{C}[t, t^{-1}], \quad V[[z]] = V \underset{\mathbb{C}}{\otimes} \mathbb{C}[[z]].$$

If $\alpha \in H^1(Y, \mathbb{Z})$ is a non-zero cohomology class, we can define two representations:

$$[\alpha] : \pi_1(Y, y_0) \to L^\bullet; \;\; [\alpha](g) = t^{\langle \alpha, g\rangle}, \qquad \langle\alpha\rangle : \pi_1(Y, y_0) \to \Lambda^\bullet; \;\; \langle\alpha\rangle(g) = e^{z\langle \alpha, g\rangle}.$$

For a representation $\rho : \pi_1(Y, y_0) \to \mathrm{GL}(V)$ put

$$\overline{\rho} : \pi_1(Y, y_0) \to \mathrm{GL}\big(V[t^{\pm}]\big), \;\; \overline{\rho} = \rho \otimes [\alpha], \tag{7}$$

$$\widehat{\rho} : \pi_1(Y, y_0) \to \mathrm{GL}\big(V[[z]]\big), \;\; \widehat{\rho} = \rho \otimes \langle\alpha\rangle. \tag{8}$$

The representation $\overline{\rho}$ is a basic tool in the theory of twisted Alexander polynomials (see [9], Section 2.1). Observe that $\widehat{\rho} = \mathrm{Exp} \circ \overline{\rho}$, where $\mathrm{Exp} : L \to \Lambda$ is the homomorphism sending t to e^z.

Applying the functor Hom to the exact sequence $0 \to \Lambda \xrightarrow{t} \Lambda \to \mathbb{C} \to 0$ we obtain an exact sequence of groups of twisted cochains:

$$0 \longrightarrow C^*(Y, \widehat{\rho}) \xrightarrow{z} C^*(Y, \widehat{\rho}) \longrightarrow C^*(Y, \rho) \to 0.$$

The corresponding long exact sequence of twisted homomorphism modules can be considered as an exact couple

$$\begin{array}{ccc} H^*(Y, \widehat{\rho}) & \xrightarrow{\qquad z \qquad} & H^*(Y, \widehat{\rho}) \\ & \nwarrow \quad \swarrow & \\ & H^*(Y, \rho) & \end{array} \tag{9}$$

This exact couple induces a spectral sequence $D_r^*(Y, \rho, \alpha)$ starting from the module $H^*(Y, \rho)$. We have the following theorem

Theorem 3.8. *[[10], Th. 5.4] The spectral sequences $E_r^*(Y, \rho, \alpha)$ and $D_r^*(Y, \rho, \alpha)$ are isomorphic.*

In particular the differentials in the spectral sequence $D_r(Y, \rho, \alpha)$ are equal to the Massey products $\langle \alpha, \cdot \rangle_r$.

3.3. **Formality with respect to a family of representations of the fundamental group.** Let us start with a definition.

Definition 3.9. Let Y be a manifold, $y_0 \in Y$, denote $\pi_1(Y, y_0)$ by G. Let $\mathcal{F}$ be a family of complex representations of G, such that $\mathcal{F}$ is closed under tensor products, that is, if $\rho_1, \rho_2 \in \mathcal{F}$ then $\rho_1 \otimes \rho_2 \in \mathcal{F}$. Put

$$\overline{\Omega}^*(Y, \mathcal{F}) = \bigoplus_{\rho \in \mathcal{F}} \Omega^*(Y, \rho).$$

Then $\overline{\Omega}^*(Y, \mathcal{F})$ has a natural structure of a DGA. We say that Y is *$\mathcal{F}$-formal*, if this DGA is formal.

Examples.

1) If $\mathcal{F}$ is the trivial 1-dimensional representation, then the notion of $\mathcal{F}$-formality is the same as the classical notion of formality as introduced by D. Sullivan [18], [4].
2) Let $\mathcal{F}$ be the family of all 1-dimensional representations of G. Then the notion of $\mathcal{F}$-formality is the same as the notion of *strong formality* introduced in [10], see also [8]. All compact connected Kähler manifolds are strongly formal, as it follows from C. Simpson's theorem [17].

3) Let G be a fundamental group of a compact connected Kähler manifold, let ρ be a semisimple representation of G. Consider the family $\mathcal{F}$ consisting of all tensor powers of ρ (including the trivial representation of the same dimension as ρ). It follows from theorem of K. Corlette [3] that $\mathcal{F}$ is closed under tensor products, see also [17]. C. Simpson's theorem [17] implies that Y is $\mathcal{F}$-formal.

Theorem 3.10. *Assume that a manifold Y is $\mathcal{F}$-formal. Let $\rho \in \mathcal{F}$ and $\alpha \in H^1(Y, \mathbb{C})$. Then the formal deformation spectral sequence $D_r^*(Y, \rho, \alpha)$ degenerates at its second term. All Massey products $\langle \alpha, x \rangle_r$ vanish for every $x \in H^*(Y, \rho)$ and $r \geqslant 2$.*

Proof. It suffices to apply Corollary 3.7 to the module $\Omega^*(Y, E_\rho)$. □

The formality property of Example 3) above yields the following corollary.

Corollary 3.11. *Assume that Y is a connected compact Kähler manifold, and $\alpha \in H^1(Y, \mathbb{C})$ a non-zero cohomology class. Let $\rho : \pi_1(Y, y_0) \to \mathrm{GL}(n, \mathbb{C})$ be a semisimple representation. Then the spectral sequence $D_r^*(Y, \rho, \alpha)$ degenerates at its second term.* □

4. Twisted monodromy homomorphisms

Let $\phi : M \to M$ be a diffeomorphism of a C^∞ compact connected manifold, and X its mapping torus. Choose a point $x_0 \in M$, and put $H = \pi_1(M, x_0)$, $G = \pi_1(X, x_0)$. Recall the exact sequence

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{p} \mathbb{Z} \longrightarrow 1.$$

Let W be a vector space of dimension n over $\mathbb{C}$ endowed with a right action of G. Such action can be described as a map $\beta : G \to \mathrm{GL}(W) \approx \mathrm{GL}(n, \mathbb{C})$ satisfying $\beta(g_1 g_2) = \beta(g_2)\beta(g_1)$, that is, an *antirepresentation* of G. Set $\beta_0 = \beta \mid H : H \to \mathrm{GL}(W)$. In this section we associate to these data an isomorphism $\phi_* : H_*(M, \beta_0) \to H_*(M, \beta_0)$ of vector spaces that we call *twisted monodromy homomorphism* induced by ϕ. This homomorphism can be considered as a generalization of the map induced by ϕ in the ordinary homology. Observe however, that the homomorphism ϕ_* is not entirely determined by ϕ and ρ_0, but depends also on the values of ρ on the elements of $G \backslash H$ (see the details in Subsection 4.1). The constructions of this section will be applied in Section 6 to the map $\beta : G \to GL(W)$ which is conjugate to the given representation $\rho : G \to \mathrm{GL}(W)$.

4.1. **Definition of the twisted monodromy homomorphism.** Choose any path θ in M from x_0 to $\phi(x_0)$. This choice determines three more geometric objects:

A) An element $u \in G$ such that $p(u) = 1$. Namely let u be a composition of the path θ with the image of the path $\phi(x_0) \times [0, 1]$ in the mapping torus

$$X = M \times [0, 1] \Big/ (x, 0) \sim (\phi(x), 1).$$

Observe that any element u with $p(u) = 1$ can be obtained this way with a suitable choice of θ.

B) A lift of the map ϕ to a map $\widetilde{\phi} : \widetilde{M} \to \widetilde{M}$. Namely, represent a point $x \in \widetilde{M}$ by a path γ in M starting at x_0. The path $\phi(\gamma) = \phi \circ \gamma$ joins the points

$\phi(x_0)$ and $\phi(x)$. The composition of paths $\theta \cdot \phi(\gamma)$ joins x_0 and $\phi(x)$. Now put $\widetilde{\phi}(\gamma) = \theta \cdot \phi(\gamma)$.

C) A homomorphism $K_\theta : H \to H$ defined by $K_\theta(\gamma) = \theta\phi(\gamma)\theta^{-1}$ where γ is a loop starting at x_0.

These objects satisfy the following easily checked properties:

$$\widetilde{\phi}(hx) = K_\theta(h)\widetilde{\phi}(x) \text{ for every } h \in H \text{ and } x \in \widetilde{M}; \tag{10}$$

$$uhu^{-1} = K_\theta(h) \text{ for every } h \in H. \tag{11}$$

Now we can define the homomorphism ϕ_*.

Definition 4.1. To simplify the notation, we shall abbreviate $W \underset{\beta_0}{\otimes} C_*(\widetilde{M})$ to $W \otimes C_*(\widetilde{M})$ up to the end of the present subsection.

Define a map

$$\alpha : W \underset{\mathbb{C}}{\otimes} C_*(\widetilde{M}) \to W \underset{\mathbb{C}}{\otimes} C_*(\widetilde{M}); \quad \alpha(v \otimes \sigma) = vu^{-1} \otimes \widetilde{\phi}(\sigma), \tag{12}$$

where $v \in W$, and σ is a simplex in $C_*(\widetilde{M})$ (here and elsewhere we denote by vg the result of the action of $g \in G$ on the vector $v \in W$).

Lemma 4.2. *1) The map α factors to an endomorphism A of $W \otimes C_*(\widetilde{M})$.*

2) The resulting map A is a chain map, and it does not depend on the choice of the path θ.

Proof. 1) We have to check that $\alpha(vh \otimes \sigma)$ and $\alpha(v \otimes h\sigma)$ give the same element in $W \otimes C_*(\widetilde{M})$ for every $h \in H$. Observe that

$$\alpha(vh \otimes \sigma) = (vh)u^{-1} \otimes \widetilde{\phi}(\sigma) = vu^{-1} \cdot uhu^{-1} \otimes \widetilde{\phi}(\sigma) = vu^{-1} \cdot K_\theta(h) \otimes \widetilde{\phi}(\sigma)$$

and this equals $vu^{-1} \otimes K_\theta(h)\widetilde{\phi}(\sigma)$ in $W \otimes C_*(\widetilde{M})$. Apply the formula (10) and the proof of the first part of Lemma is over.

2) Let θ' be another path joining x_0 and $\phi(x_0)$, so that $\theta' = \gamma\theta$ where γ is a loop in M starting at x_0. The corresponding element $u' \in G$ satisfies $u' = \gamma u$, and $\widetilde{\phi}' = \gamma\phi$, so that $vu'^{-1} \otimes \widetilde{\phi}'(\sigma) = vu\gamma^{-1} \otimes \gamma\phi(\sigma)$ and the property 2) follows. □

Definition 4.3. The map induced by A in the homology groups $H_*(M, \beta)$ will be denoted by $\phi_*[\beta]$ and called the *twisted monodromy homomorphism* associated to ϕ and β (when the value of β is clear from the context we omit it in the notation).

Definition 4.4. For any antihomomorphism $\beta : G \to \mathrm{GL}(W)$ and $\lambda \in \mathbb{C}^*$ define an antihomomorphism $\beta_\lambda : G \to \mathrm{GL}(W)$ as follows:

$$\beta_\lambda(g) = \lambda^{\langle \xi, g \rangle} \cdot \beta(g).$$

The proof of the following proposition follows immediately from the definition of ϕ_* (see the formula (12)).

Proposition 4.5. *We have*

$$\phi_*[\beta_\lambda] = \frac{1}{\lambda} \cdot \phi_*[\beta]. \tag{13}$$

□

4.2. **The case $G = H \times \mathbb{Z}$.** The algebralically simplest case occurs when the exact sequence (1) splits. This case can be characterized by the following simple lemma (the proof will be omitted).

Lemma 4.6. *The three following properties are equivalent:*

1) *For some path θ from x_0 to $\phi(x_0)$ the homomorphism $K_\theta : H \to H$ is an inner automorphism.*
2) *For every path θ from x_0 to $\phi(x_0)$ the homomorphism $K_\theta : H \to H$ is an inner automorphism.*
3) *The extension (1) splits.* □

One can prove also that if the properties listed in the lemma hold for some choice of a base point x_0, then they hold for any other choice of the base point.

Definition 4.7. If the map ϕ satisfies the three equivalent properties of Lemma (4.6) we say that ϕ is π_1*-split*.

Assume that ϕ is π_1-split. Choose an element $u \in G$ commuting with H, and such that $p(u) = 1$. Let $\beta : H \to GL(W)$ be any antirepresentation, and let β_0 be its restriction to H. Put $B = \beta(u)$, then $B \in \mathrm{GL}(W)$. (Observe that in the split case any antirepresentation of H can be extended to an antirepresentation of G, sending u to a scalar matrix.)

1. Consider first the case when B is the identity map of W. The homomorphism ϕ_* in this case has an especially simple definition. Namely, choose a path θ from x_0 to $\phi(x_0)$ in such a way that for any $\gamma \in \pi_1(M, x_0)$ we have $\theta\phi(\gamma)\theta^{-1} = \gamma$. Then the corresponding lift $\widetilde{\phi} : \widetilde{M} \to \widetilde{M}$ has the property

$$\widetilde{\phi}(hx) = h\widetilde{\phi}(x) \quad \text{for every } h \in \pi_1(M, x_0)$$

Denote such a lift by $\widetilde{\phi}^\circ$. The automorphism of $H_*(M, \beta_0)$ corresponding to this choice will be denoted by ϕ_*°; it is defined by the following formula:

$$\phi_*^\circ(v \otimes \sigma) = v \otimes \widetilde{\phi} \circ \sigma \tag{14}$$

(where σ is a singular simplex of $\widetilde{M}$). This homomorphism ϕ_*° is entirely determined by ϕ and β_0, and does not depend on the values of β on $G\backslash H$. In the case when β_0 is the trivial representation the map ϕ_*° is just the induced map in the ordinary homology.

2. Now let $B = \lambda \cdot \mathrm{Id}$ where $\lambda \in \mathbb{C}^*$. Choosing for ϕ the same lift as in the previous case, we obtain the following formula for ϕ_*:

$$\phi_*(v \otimes \sigma) = \frac{1}{\lambda} v \otimes \widetilde{\phi}_*^\circ(\sigma) = \frac{1}{\lambda}\phi_*^\circ(v \otimes \sigma).$$

3. Now let B be an arbitrary element of $\mathrm{GL}(W)$. Since B commutes with H, it induces a well-defined linear maps $C_*(M, \beta_0) \to C_*(M, \beta_0)$ and $H_*(M, \beta_0) \to H_*(M, \beta_0)$, that will be denoted by the same letter B. We have then

$$\alpha(v \otimes \sigma) = vu^{-1} \otimes \widetilde{\phi}(\sigma) = B^{-1}(v \otimes \widetilde{\phi}(\sigma))$$

and finally

$$\phi_* = B^{-1} \circ \phi_*^\circ. \tag{15}$$

Remark 4.8. In the case $\lambda = 1$ it is possible to reformulate our definition of ϕ_*° in terms of induced representations of fundamental groups. To explain this, let us proceed to a slightly more general framework. Let $\phi : X \to Y$ be a map of connected topological spaces, $x_0 \in X$, $y_0 \in Y$. Let $\rho : \pi_1(Y, y_0) \to \mathrm{GL}(W)$ be a representation. Choose a path μ from y_0 to $\phi(x_0)$. Define a representation $\rho' : \pi_1(X, x_0) \to \mathrm{GL}(W)$ as follows $\rho'(g) = \rho(\mu\phi(g)\mu^{-1})$. It is easy to check that ϕ induces a homomorphism

$$\phi_* : H_*(X, \rho') \to H_*(Y, \rho),$$

defined on the chain level as $v \otimes \sigma \mapsto v \otimes \widetilde{\phi} \circ \sigma$. This homomorphism depends obviously on μ. In the case when $X = Y$ and ϕ is π_1-split choose a path μ in such a way that $\mu\phi(h)\mu^{-1} = h$ for every $h \in \pi_1(M, x_0)$. Then $\rho' = \rho$ and we return to the homomorphism ϕ_*° of the above definition.

4.3. **Relation wih Kirk-Livingston's setup.** Let $\overline{X}$ be the infinite cyclic covering of X corresponding to ξ. Observe that $\pi_1(\overline{X}) \approx H$, so that the twisted homology $H_*(\overline{X}, \beta_0)$ of $\overline{X}$ is defined. We have the following simple lemma; the proof follows from the observation that $\overline{X} \approx M \times \mathbb{R}$.

Lemma 4.9. *The inclusion $i : M \hookrightarrow X$ induces an isomorphism*

$$I : H_*(M, \beta_0) \xrightarrow{\approx} H_*(\overline{X}, \beta_0).$$

In the work [9] P. Kirk and C. Livingston constructed an action of the group $\mathbb{Z}$ on the space $W \underset{\beta_0}{\otimes} C_*(\overline{X})$. Namely, choose any element $u \in G$ such that $p(u) = 1$, and let the generator t of $\mathbb{Z}$ act on $W \underset{\beta_0}{\otimes} C_*(\overline{X})$ by the following formula

$$t(v \otimes \sigma) = vu^{-1} \otimes u\sigma.$$

It is shown in [9] that that this action does not depend on the particular choice of u. The next proposition follows readily from the definition of the monodromy homomorphism ϕ_*.

Proposition 4.10. *The following diagram is commutative*

$$\begin{array}{ccc} H_*(\overline{X}, \beta_0) & \xrightarrow{t} & H_*(\overline{X}, \beta_0) \\ \uparrow{\scriptstyle I} & & \uparrow{\scriptstyle I} \\ H_*(M, \beta_0) & \xrightarrow{\phi_*} & H_*(M, \beta_0) \end{array}$$

□

Remark 4.11. We worked in this section in the assumption that M is a compact C^∞ manifold; the homology groups were in coefficients in $\mathbb{C}$, in view of the applications to Kähler manifolds.

However all the constructions and results of the section generalize without any changes to the case when M is any CW-complex; the coefficient field $\mathbb{C}$ can be replaced by an arbitrary field $\mathbb{K}$.

5. Twisted monodromy maps and formal deformation spectral sequences

We begin with a discussion of a universal coefficient theorem for twisted cohomology (Subsection 5.1). The next subsection contains the computation of the formal deformation spectral sequences in terms of the monodromy maps. The proof of the main theorem in Section 6 is based on these computations.

5.1. **Universal coefficient theorem for twisted cohomology.** Let Y be a connected topological space endowed with a non-zero cohomology class $\eta \in H^1(Y, \mathbb{Z})$. Denote by G the fundamental group $\pi_1(Y, y_0)$. Let V be a finite-dimensional vector space over $\mathbb{C}$ and $\rho : G \to \mathrm{GL}(V)$ a representation. Let $L = \mathbb{Z}[t, t^{-1}]$, denote by $V[t^{\pm}]$ the free L-module $V \underset{\mathbb{C}}{\otimes} L$. Recall from Subsection 3.2 the representation

$$\overline{\rho} : G \to \mathrm{GL}(V[t^{\pm}]); \quad \overline{\rho}(g) = \rho(g) t^{\xi(g)}.$$

The representation ρ determines an action from the left of G on V; put $V^* = \mathrm{Hom}(V, \mathbb{C})$, and consider the corresponding right action ρ^* of G on V^*. Similarly we obtain a right action $\overline{\rho}^*$ of G on $V[t^{\pm}]$. If we choose a basis in V, then ρ^* is identified with an antihomomorphism $G \to \mathrm{GL}(n, \mathbb{C})$ obtained from ρ by transposition (similarly for $\overline{\rho}^*$). Associated to $\overline{\rho}^*$ there is the L-module of $\overline{\rho}^*$-twisted chains

$$V[t^{\pm}] \underset{\overline{\rho}^*}{\otimes} C_*(\widetilde{Y}),$$

and its homology $H_*(X, \overline{\rho}^*)$. Observe that for any right action χ of G on a free L-module W there is a natural isomorphism

$$\mathrm{Hom}_L\left(W \underset{\chi}{\otimes} C_*(\widetilde{Y}), L\right) \xrightarrow{\Phi} \mathrm{Hom}_\rho\big(C_*(\widetilde{Y}), W^*\big);$$

the value of Φ on an L-homomorphism $\alpha : W \underset{\chi}{\otimes} C_*(\widetilde{Y}) \to L$ is defined by the following formula:

$$\Big(\Phi(\alpha)(\sigma)\Big)(w) = \alpha(w \otimes \sigma)$$

(where $w \in W$ and $\sigma \in C_*(\widetilde{Y})$). Applying this to the right action ρ^* on $W = V^*$ we obtain the following isomorphism (see [10], Lemma 4.3)

$$H^*(Y, \overline{\rho}) \approx H^*\Big(\mathrm{Hom}_L(C_*(Y, V^*), L)\Big). \tag{16}$$

The cohomology module in the right-hand side of (16) has the advantage that we can apply to it the universal coefficient theorem:

Proposition 5.1. *For every k we have an exact sequence*

$$0 \to \mathrm{Ext}^1_L\Big(H_{k-1}(Y, \overline{\rho}^*), L\Big) \to H^k(Y, \overline{\rho}) \to \mathrm{Hom}_L\Big(H_k(Y, \overline{\rho}^*), L\Big) \to 0. \tag{17}$$ □

We will now apply these results to mapping tori. In the rest of this subsection X is the mapping torus of a homeomorphism $\phi : M \to M$ (see Subsection 2.1 for the notations). Endow the vector space $H_{k-1}(M, \rho_0^*)$ with the action of L as follows: $ta = \phi_*(a)$, where ϕ_* is the twisted monodromy map from Subsection 4.1.

Proposition 5.2. *We have an isomorphism of L-modules*

$$H^k(X, \overline{\rho}) \approx H_{k-1}(M, \rho_0^*).$$

Proof. By Theorem 2.1 of [9] we have

$$\boldsymbol{H^*(X,\overline{\rho}^*) \approx H_*(\overline{X},\rho_0^*).}$$

Lemma 4.9 says that $\boldsymbol{H_*(\overline{X},\rho_0^*) \approx H_*(M,\rho_0^*)}$, therefore this $\boldsymbol{L}$-module is a finite dimensional vector space over $\mathbb{C}$, and hence a finitely generated torsion $\boldsymbol{L}$-module. Thus we have $\mathrm{Hom}_{\boldsymbol{L}}\Big(\boldsymbol{H_k(X,\overline{\rho}^*),L}\Big) = \boldsymbol{0}$ and

$$\mathrm{Ext}^1_{\boldsymbol{L}}\Big(\boldsymbol{H_{k-1}(X,\overline{\rho}^*),L}\Big) \approx \boldsymbol{H_{k-1}(\overline{X},\rho_0^*) \approx H_{k-1}(M,\rho_0^*).}$$

The proposition follows. □

5.2. **Computation of deformation spectral sequences in terms of monodromy maps.** Our main aim here is to prove that the exact couples (18) and (19) induce isomorphic spectral sequences. We have $\widehat{\boldsymbol{\rho}} = \mathrm{Exp} \circ \overline{\boldsymbol{\rho}}$, where $\mathrm{Exp} : \boldsymbol{L} \to \boldsymbol{\Lambda}$ is the ring homomorphism sending $\boldsymbol{t}$ to $\boldsymbol{e^z}$. The exact couple $(\mathscr{D}_{\mathbf{0}})$

(18)
$$\begin{array}{ccc} \boldsymbol{H^*(X,\widehat{\rho})} & \xrightarrow{\quad\boldsymbol{z}\quad} & \boldsymbol{H^*(X,\widehat{\rho})} \\ & \nwarrow \quad \swarrow & \\ & \boldsymbol{H^*(X,\rho)} & \end{array}$$

is obviously isomorphic to the following exact couple $(\mathscr{D}_{\mathbf{1}})$

(19)
$$\begin{array}{ccc} \boldsymbol{H^*(X,\widehat{\rho})} & \xrightarrow{\quad\boldsymbol{e^z-1}\quad} & \boldsymbol{H^*(X,\widehat{\rho})} \\ & \nwarrow \quad \swarrow & \\ & \boldsymbol{H^*(X,\rho)} & \end{array}$$

The exact couple $(\mathscr{D}_{\mathbf{2}})$ below

(20)
$$\begin{array}{ccc} \boldsymbol{H^*(X,\overline{\rho})} & \xrightarrow{\quad\boldsymbol{t-1}\quad} & \boldsymbol{H^*(X,\overline{\rho})} \\ & \nwarrow \quad \swarrow & \\ & \boldsymbol{H^*(X,\rho)} & \end{array}$$

maps to $(\mathscr{D}_{\mathbf{1}})$ via the homomorphism $\boldsymbol{H_*(X,\overline{\rho}) \to H_*(X,\widehat{\rho})}$ induced by Exp, therefore the spectral sequences induced by $(\mathscr{D}_{\mathbf{2}})$ and $(\mathscr{D}_{\mathbf{0}})$ are isomorphic. Applying Proposition 5.2 we deduce that $(\mathscr{D}_{\mathbf{2}})$ is isomorphic to the exact couple $(\mathscr{D}_{\mathbf{3}})$ of the form

(21)
$$\begin{array}{ccc} \boldsymbol{H_*(M,\rho_0^*)} & \xrightarrow{\quad\boldsymbol{\phi_*-1}\quad} & \boldsymbol{H_*(M,\rho_0^*)} \\ & \nwarrow^{\boldsymbol{k}} \quad \swarrow_{\boldsymbol{j}} & \\ & \boldsymbol{H^*(X,\rho)} & \end{array}$$

where the maps $\boldsymbol{j}$ and $\boldsymbol{k}$ have the degrees respectively $\mathbf{1}$ and $\mathbf{0}$. We obtain finally a homomorphism $(\mathscr{D}_{\mathbf{3}}) \to (\mathscr{D}_{\mathbf{0}})$ of exact couples, which equals the identity map on the term $\boldsymbol{H^*(X,\rho)}$. Therefore the exact sequences derived from these exact couples are isomorphic.

Remark 5.3. Similary to the Section 4, all the constructions and results of the section generalize without any changes to the case when M is any CW-complex.

6. Proofs of the main results

Now we can complete the proofs of the main results.

6.1. **Theorem A.** Let us first prove Theorem A for the case $\lambda = 1$, that is, $\rho_\lambda = \rho$. According to the previous section the exact couples $(\mathscr{D}_0)$ and $(\mathscr{D}_3)$ induce isomorphic spectral sequences. The differentials in the spectral sequence induced by $(\mathscr{D}_0)$ are equal to Massey products: $d_r(x) = \langle \xi, x \rangle_r$, therefore the spectral sequence degenerates in degree k at the term number $M_k(\rho) + 1$. It suffices to prove that the spectral sequence induced by $(\mathscr{D}_3)$ degenerates at the term $J_k(\phi_*) + 1$ in degree k. Denote by $\phi_*^{(k)}$ the twisted monodromy homomorphism in degree k.

Let A_k be the invariant linear subspace of eigenvalue 1 of $\phi_*^{(k)}$. Let B_k be the sum of all invariant linear subspaces of $\phi_*^{(k)}$ corresponding to the eigenvalues different from 1. The restriction $(\phi_*^{(k)} - 1) \mid A_k$ is nilpotent of degree equal to $J_k(\phi_*, 1)$, and the restriction $(\phi_*^{(k)} - 1) \mid B_k$ is an isomorphism of B_k onto itself. The assertion of the theorem follows now from the following lemma ([14], Lemma 3.3).

Lemma 6.1. *Let $\mathscr{E}$ be a graded exact couple:*

(22)
$$\begin{array}{ccc} D & \xrightarrow{\ i\ } & D \\ & {\scriptstyle l}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & E & \end{array}$$

Assume that the homomorphism $i : D_k \to D_k$ decomposes as follows:

$$\delta \oplus \tau : A \oplus B \to A \oplus B$$

where δ is nilpotent of degree m and τ is injective.

1) Let $\deg i = \deg l = 0$, $\deg j = 1$. Then the spectral sequence induced by $\mathscr{E}$ degenerates at the step $m + 1$ in degree k.

2) Let $\deg i = 0$, $\deg l = 1$, $\deg j = 0$. Then the spectral sequence induced by $\mathscr{E}$ degenerates at the step $m + 1$ in degree $k - 1$.

□

Now let us consider the case when λ is an arbitrary non-zero complex number. According to the Proposition 4.5 the monodromy homomorphism $\phi_*[\rho_\lambda]$ constructed from the representation ρ_λ^* equals $\frac{1}{\lambda}\phi_*[\rho]$. Therefore the exact couple $(\mathscr{D}_3)$ for the case of the representation ρ_λ has the following form

(23)
$$\begin{array}{ccc} H_*(M, \rho_0^*) & \xrightarrow{\ \frac{1}{\lambda}\phi_* - 1\ } & H_*(M, \rho_0^*) \\ & {\scriptstyle k'}\nwarrow \quad \swarrow{\scriptstyle j'} & \\ & H^*(X, \rho_\lambda) & \end{array}$$

(where ϕ_* denotes the monodromy homomorphism corresponding to ρ). It remains to observe that $J_k(\frac{1}{\lambda}\phi_*, 1) = J_k(\phi_*, \lambda)$.

6.2. **Theorem B.** Let X be a connected compact Kähler manifold, and ρ a semisimple representation. In view of Theorem A it suffices to prove that $M_k(\rho_\lambda) = 1$ for every $\lambda \in \mathbb{C}^*$, or, equivalently, that the spectral sequence associated to the exact couple

$$\begin{array}{ccc} H^*(X, \widehat{\rho}_\lambda) & \xrightarrow{\quad z \quad} & H^*(X, \widehat{\rho}_\lambda) \\ & \nwarrow \quad \swarrow & \\ & H^*(X, \rho_\lambda) & \end{array} \tag{24}$$

degenerates at its second term. Observe that the representation ρ_λ is also semisimple; apply to it Corollary (3.11) and the proof of Theorem B is over.

6.3. **The π_1-split case.** Recall from Subsection 4.2 that in this case we have an automorphism ϕ°_* determined by ϕ and by the representation of $\pi_1(M)$. Choose an element $u \in G$ commuting with H and such that $p(u) = 1$. Let $\chi : \pi_1(M, x_0) \to \mathrm{GL}(V)$ be *any* representation of the fundamental group of M. Let $\lambda \in \mathbb{C}^*$. Extend χ to a representation $\chi_\lambda : \pi_1(X, x_0) \to \mathrm{GL}(V)$ by $\chi(u) = \lambda$ (this is possible since u commutes with H).

Theorem 6.2. *We have*

1) $J_k(\phi^\circ_*, \lambda) = M_k(\chi_\lambda)$.
2) *If moreover X is a compact Kähler manifold, and the representation χ is semisimple, then ϕ°_* is diagonalizable.*

Proof. Part 1) follows immediately from Theorem A. As for the part 2), observe that the representation χ_λ is also semisimple, so we can apply to it the Theorem B, and the proof is over. □

Remark 6.3. The particular case of the trivial representation χ corresponds to Theorems 3.1 and 5.1 of the paper [14].

6.4. **Theorem C.** We need some more terminology.

Definition 6.4. Let R be a finitely generated L-module, and $a \in L$ a polynomial of degree 1. Denote by R_a the a-primary part of R, that is, the submodule of all $x \in R$, such that $a^N x = 0$ for some N. Denote by $Nil(R, a)$ the minimal number N, such that $a^N R_a = 0$. The module R_a is a finite-dimensional vector space, and a determines a linear map of this space. The number $Nil(R, a)$ equals the maximal size of Jordan blocks of eigenvalue 0 of a.

Denote by $M_k(\alpha, \theta_\lambda)$ the maximal length of a non-zero Massey product of the form $\langle \alpha, x \rangle_r$ where $x \in H^k(Y, \theta_\lambda)$. Consider the spectral sequence associated to the exact couple

$$\begin{array}{ccc} H^*(Y, \overline{\theta}_\lambda) & \xrightarrow{\quad t-1 \quad} & H^*(Y, \overline{\theta}_\lambda) \\ & \nwarrow \quad \swarrow & \\ & H^*(Y, \theta_\lambda) & \end{array} \tag{25}$$

Applying the same argument as in the beginning of Subsection 6.1 we deduce that $M_k(\alpha, \theta_\lambda) + 1$ equals the the number r of the sheet where this spectral sequence degenerates. By Lemma 6.1 this number r equals $Nil\big(H^{k+1}(Y, \overline{\theta}_\lambda), t-1\big)$. Observe

that we have $\overline{\theta}_{\lambda} = g_{\lambda} \circ \overline{\theta}$, where $g_{\lambda} : L \to L$ is the isomorphism given by the formula $g_{\lambda}(t) = \lambda \cdot t$. Therefore the L-homomorphism $t-1 : H^*(Y, \overline{\theta}_{\lambda}) \longrightarrow H^*(Y, \overline{\theta}_{\lambda})$ is isomorphic to the L-homomorphism $\lambda^{-1}t - 1 : H^*(Y, \overline{\theta}) \longrightarrow H^*(Y, \overline{\theta})$, and we have $Nil\big(H^{k+1}(Y, \overline{\theta}_{\lambda}), t-1\big) = Nil\big(H^{k+1}(Y, \overline{\theta}), t - \lambda\big)$. The torsion submodule of $H^{k+1}(Y, \overline{\theta})$ is isomorphic to $\mathrm{Ext}\big(H_k(Y, \overline{\theta}^*), L\big)$ which is in turn isomorphic to the torsion submodule $\mathscr{T}'_k$ of $H_k(Y, \overline{\theta}^*)$. A theorem of P. Kirk and C. Livingston ([9], Th. 2.1) says that we have an isomorphism

$$H_*(Y, \overline{\theta}^*) \approx H_*(\overline{Y}, \theta_0^*).$$

Therefore the module $\mathscr{T}'_k$ is isomorphic to the torsion submodule $\mathscr{T}_k$ of $H_*(\overline{Y}, \theta_0^*)$, so, finally, $M_k(\alpha, \theta_{\lambda}) = Nil(\mathscr{T}_k, t - \lambda)$ and the proof of Theorem C is complete.

6.5. **Theorem D.** It follows readily from Theorem C; the proof is similar to the argument of Subsection 6.2.

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Laboratoire Mathématiques Jean Leray UMR 6629, Université de Nantes, Faculté des Sciences, 2, rue de la Houssinière, 44072, Nantes, Cedex
E-mail address: andrei.pajitnov@univ-nantes.fr