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By R. Bonneville

CNES, 2 place Maurice Quentin, 75001 Paris, France
richard.bonneville@cnes.fr

Abstract

Particle physics and astrophysics tend to get closer to each other about questions associated to the position of gravitation with respect to the other fundamental interactions. That issue, which is indeed the Gordian node of today’s physics, is tightly linked with problems such as the nature of dark energy and dark matter, and the possible existence of hidden additional dimensions. In the present paper, whose goal is firstly pedagogical, we will try to present the stakes and a possible way forward, via a simple model easy to visualize and which does not require long and difficult calculations. We first consider relativistic gravitation in a 2 dimension space-time (one space-like, one time-like), then we derive the cosmological consequences, and finally we outline a field theory in a full 2+1 universe (with one extra dimension). The qualitative results obtained in this simple model of universe can easily be transposed in order to better understand our usual 4-dimension space-time.
I. Introduction

Particle physics and astrophysics tend to get closer to each other about questions associated to the position of gravitation with respect to the other fundamental interactions. That issue, which is indeed the Gordian node of today’s physics, is tightly linked with problems such as the nature of dark energy and dark matter, and the possible existence of hidden additional dimensions. Today our knowledge of the universe relies on two pillars: (i) the standard model of cosmology based on General Relativity, cold dark matter (an unknown material component only sensitive to gravitation and introduced in order to explain the dynamics of galaxies), and a cosmological constant (equivalent to a mysterious dark energy component, possibly due to the properties of the quantum vacuum); (ii) the standard model of fundamental interactions (except gravitation) based on quantum field theory and accounting for the classification of the known elementary particles and of their interactions.

In the present paper, whose goal is firstly pedagogical, we will try to present the stakes and a possible way forward, via a simple model easy to visualize and which does not require long and difficult calculations.

The next section presents the general problematics, the following one considers relativistic gravitation in a 2 dimension space-time (one space-like, one time-like), then the cosmological consequences of the former section are derived, the last section is dedicated to field theory in the full 2+1 universe (with one extra dimension).

II. Problematics

The numerous experimental confirmations of General Relativity, from the measurement of Mercury’s perihelion advance [1] to the recent evidence of gravitational waves [2], lead to acknowledge that the geometry of space-time is that of a 4-dimension curved surface $\Sigma(4)$ locally invariant under the Lorentz group; the effect of the local curvature of the surface is interpreted as resulting from a force field identified with gravitation [3].

On another hand a curved manifold of dimension $d$ can be embedded in a flat one with $d(d+1)/2$ dimensions; hence $\Sigma(4)$ can be embedded in a flat manifold with 10 dimensions. Recalling that the Minkowsky space-time of restricted relativity is a pseudo-Euclidian 4-dimension manifold we will hereafter by extension make the assumption that our universe is a pseudo-Euclidian manifold with 10 dimensions $E(10)$. 
Let $E_p(4)_M$ be the tangent vector space to $\Sigma(4)$ in any point $M$ of the surface and let $E_\perp(6)_M$ be the orthogonal set to $\Sigma(4)$ in $M$. In any point $M$ of $\Sigma(4)$ it is possible to choose a local reference frame with 4 coordinates $x^\mu$ in $E_p(4)_M$ with $\mu$ ranging from 0 to 3 and 6 coordinates $\xi^i$ in $E_\perp(6)_M$ with $i$ ranging from 1 to 6. $E_p(4)_M$ is invariant under the Lorentz group; if we define in $E_p(4)_M$ a pseudo-orthonormal reference frame, an infinite set of similar frames can be derived by a Lorentz transformation. $E_p(4)_M$ is a Minkowsky-type space-time; we will call the 4-dimension manifold $\Sigma(4)$ the “orbital” space. The 6 additional dimensions of $E_\perp(6)_M$ span a manifold whose geometrical properties are assumed to account for the bestiary of the elementary particles and of their fundamental interactions except gravitation; those fundamental interactions are described by the restriction of certain symmetry groups from a global property to a local one [4] but this is outside of the scope of the present paper\(^1\). Those additional degrees of freedom are hidden inasmuch as their perception is indirect: they manifest through what we interpret as forces acting on every point of the “orbital” space; we will call the 6-dimension manifold $E_\perp(6)_M$ the “internal” space.

In this local reference frame the 10x10 metric tensor $g^{\alpha\beta}$ has the following form

\(^1\) If $E_\perp(6)_M$ is assumed to be invariant under the orthogonal group $SO(6)$, it can be noticed here that the unitary groups $SU(3)$ and $SU(2) \times U(1)$ which respectively account for the strong and electro-weak interactions can be shown to be equivalent to two distinct invariant subgroups of $SO(6)$ [“equivalent” means that both groups have isomorphous Lie algebra].
with $g^{a\beta} = g^{\beta a}$. If the “orbital” space and the “internal” space are completely decoupled, i.e. when gravitation and the other fundamental interactions are disconnected, then $s^{ij} = \delta^{ij}$ and $g^{a\nu}$ is the metric tensor of General Relativity. If the two subspaces are not decoupled, additional terms $a^{i\nu}$ appear which are perceived in the orbital space as new massless vector-type interactions$^2$.

It is not easy to have a mental representation of a 4-dimension space-time, a fortiori of a 10-dimension one. In addition, the calculations needed by those matters are complex and tedious. In order to have a visual representation together with easy calculations but nevertheless to get a qualitative understanding of the topic, we will hereafter consider a very simple space-time with one space-like dimension and one time-like dimension, embedded in a

\[ g^{a\beta} = \begin{pmatrix}
    g^{\mu\nu} & a^{\mu j} \\
    4 \times 4 & 4 \times 6
  \end{pmatrix}
\]

\[ a^{i\nu} \\
  6 \times 4 \]

\[ s^{ij} \\
  6 \times 6 \]

(1)

\[ g^{a\beta} \]

\[ g^{a\beta} = g^{\beta a} \]

$^2$ The 10-dimension metric tensor $g^{a\beta}$ in equ.(1) presents obvious formal similarities with the metric tensor $g^{a\beta}_{KK}$ of the original 5-dimension Kaluza-Klein model where the $g^{a5}$ components account for a vector field then identified with the electromagnetic field and the $g^{55}$ component accounts for an additional scalar field:

\[ g^{a\beta}_{KK} = \begin{pmatrix}
    g^{\mu\nu} & g^{\mu5} = a^\mu \\
    4 \times 4 & g^{55} = s
  \end{pmatrix}
\]

\[ g^{a\beta}_{KK} \]

\[ g^{a\beta}_{KK} = g^{\beta a} \]
larger 3-dimension pseudo-Euclidian vector space $\mathbf{E}(3)$; the “orbital” space-time is a 2-dimension surface $\Sigma(2)$, locally invariant by the special Lorentz group. In any point $M$ of the surface there is a 3$^{rd}$ degree of freedom which is the normal to the surface in $M$, figuring an “internal” 1-dimension vector space; the only possible internal states are then scalar particles. Sections III and IV will present relativistic gravitation in a 2-dimension space-time and the cosmological consequences. Section V will present the field theory in the full 3 dimension universe when the 3$^{rd}$ dimension is taken into account.

### III. Gravitation in a 2-dimension space-time

#### III.1. Newtonian gravitation

Let us first focus onto the 2-dimension “orbital” space-time without considering any connection with the additional “internal” degree of freedom. In that simple model, gravitation is quite different from what we are used to; ith one space-like dimension Newton’s equation for the gravitation potential

$$\nabla^2 U = 4\pi G \rho(r)$$

becomes

$$\frac{d^2 U}{dx^2} = C \rho(x)$$

(2b)

The coupling constant $C$ can be linked to the usual gravitation constant $G$ by $C = 4\pi G/\sigma$ where $\sigma$ has the dimensions of a surface. For a point mass $M$ at the origin, $\rho(x) = M\delta(x)$ so that the gravitation potential is accounted for by a well with an angular point at the source i.e.

$$U(x) = CM|\mathbf{x}|.$$  

(3)

#### III.2 Relativistic gravitation

We will hereunder present “a special general relativity”, i.e. relativistic gravitation within a simple space-time with one space-like and one time-like dimension. In that 2-dimension space-time, the metric tensor has the following form:
\[ g_{\mu \nu} = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} \]  

with \( g_{01} = g_{10} \), \( x^0 = ct \), \( x^1 = x \).

The Riemann curvature tensor is then quite simple; it has only one non-zero component [5]:

\[ R_{\mu \nu \rho \sigma} = \frac{1}{2} \left( \partial_\rho g_{\mu \sigma} + \partial_\sigma g_{\mu \rho} - \partial_\mu g_{\rho \sigma} - \partial_\rho g_{\mu \sigma} \right) = -R_{\mu \nu \rho \sigma} = -R_{\rho \sigma \mu \nu} \]  \hspace{1cm} (5)

The Ricci tensor \( R_{\mu \nu} = g^{\rho \sigma} R_{\rho \sigma \mu \nu} \) is thus given by:

\[ R_{00} = g^{11} R_{1010} \]  \hspace{1cm} (6a)

\[ R_{11} = g^{00} R_{0100} \]  \hspace{1cm} (6b)

\[ R_{01} = R_{10} = -g^{01} R_{1010} \]  \hspace{1cm} (6c)

In a reference frame where \( g_{\mu \nu} \) is diagonal \( R_{\mu \nu} \) is also diagonal and the Ricci scalar

\[ R = g^{\alpha \beta} R_{\alpha \beta \mu \nu} \]  \hspace{1cm} (7)

is given by:

\[ R = g^{11} R_{11} + g^{00} R_{00} = 2g^{11} g^{00} R_{1010} \]  \hspace{1cm} (6d)

Since \( g_{00} g^{00} = g_{11} g^{11} = 1 \) we straightforwardly find that the Einstein tensor \( G_{\mu \nu} \), central in General Relativity, is null:

\[ G_{\mu \nu} = \left( R_{\mu \nu} - \frac{R}{2} g_{\mu \nu} \right) = 0 \]  \hspace{1cm} (8)

The null character of the Einstein tensor is a specificity of the 2-dimension case: let \( d \) be the space-time dimension; \( G_{\mu \nu} = 0 \) means \( R_{\mu \nu} = \frac{R}{2} g_{\mu \nu} \); taking the trace of both sides gives

\[ g^{\mu \nu} R_{\mu \nu} = \frac{R}{2} g_{\mu \nu} g_{\mu \nu} \]  \hspace{1cm} i.e. \( R \left( 1 - \frac{d}{2} \right) = 0 \).

The equality \( R_{\mu \nu} = \frac{R}{2} g_{\mu \nu} \) can be re-written as

\[ R_{\mu \nu} = \frac{1}{2} g_{\mu \nu} g^{\alpha \beta} R_{\alpha \beta} \]  \hspace{1cm} (9)

This is possible if and only if \( d = 2 \) with the Ricci tensor being proportional to the metric tensor, i.e.

\[ R_{\mu \nu} = Ag_{\mu \nu} \]  \hspace{1cm} (10)

Consequently we get

\[ R = 2A \]  \hspace{1cm} (11)
Using equ.(7) and equ.(11) we find
\[ g^{00}g^{11}R_{010} = A \tag{12} \]
or
\[ \left( \partial_0^2 g_{11} + \partial_1^2 g_{00} \right) = -2Ag_{00}g_{11} \tag{13} \]

\( A \) is determined by comparison with the classical field expression equ.(2b). The non relativistic limit of equ.(13) is obtained by neglecting the quadratic terms, i.e. by writing

\[ g_{00}g_{11} \approx -1. \]

Putting \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), with \( \eta_{\mu\nu} \) being the Minkowski metric tensor

\[ \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \text{equ.(13) is changed into} \]

\[ \left( \partial_0^2 h_{11} + \partial_1^2 h_{00} \right) \approx -2A \tag{14} \]

If the field is slowly varying with time, the time derivatives on the left hand side of equ.(14a) vanish and we find

\[ \partial_1^2 h_{00} \approx 2A \tag{15} \]

Let us consider a point particle of mass \( m \); \( u^\mu = \frac{\partial x^\mu}{\partial \tau} \) being the velocity 2-vector, the simplest way of coupling the particle to the gravitation field is to write a Lagrangian

\[ L = \frac{1}{2}mg_{\mu\nu}u^\mu u^\nu \tag{16} \]

The Lagrange equations are obtained by minimising the action integral \( S = \int Ld\tau \), \( \tau \) being the proper time:

\[ g_{\mu\nu} \frac{du^\nu}{d\tau} = \frac{1}{2} \partial g_{\mu\nu} u^\mu u^\nu \tag{17} \]

At very low velocity \( (|u| \ll c) \) and expressing \( g_{\mu\nu} \) as \( \eta_{\mu\nu} + h_{\mu\nu} \) it is straightforward to derive

\[ \frac{du^1}{dt} \approx \frac{1}{2} \partial h_{00} c^2 \tag{18} \]

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\(^3\) The linear approximation \( g_{00}g_{11} \approx -1 \) implies that \( h_{00} \approx h_{11} \); using the result equ.(19) \( U = -h_{00}c^2/2 \) we find for the quadratic form \( ds^2 = g_{00}c^2dt^2 + g_{11}dx^2 \) the approximated expression \( ds^2 \approx c^2dt^2 \left( 1 + \frac{2U}{c^2} \right) + dx^2 \left( 1 - \frac{2U}{c^2} \right). \)
hat is to say that in the quasi static limit the particle undergoes a force deriving from a potential
\[ U = -h_0 c^2 / 2 \]  
(19)

Combining equ.(2b), equ.(10) and equ.(19) we find
\[ A = - \frac{C \rho}{c^2} \]  
(20)

so that
\[ R_{\mu \nu} = - \frac{C \rho}{c^2} g_{\mu \nu} \]  
(21)

Equ.(21) is the field equation. Here \( \rho \) accounts for any type of non gravitational energy source: matter, radiation, and also the energy of the quantum vacuum. We note that the field equation does not involve the energy-momentum tensor \( T_{\mu \nu} \).

Now the Einstein tensor \( G_{\mu \nu} \) has quite generally a null divergence (it has been built from the Ricci tensor for that purpose), as well as the metric tensor and the energy-momentum tensor. The energy-momentum conservation law implies that \( R_{\mu \nu} \) as given by equ.(21) has also a null divergence.

IV. Cosmology in the 2-dimension space-time

IV.1. Cosmic evolution equation

If the universe is assumed to be homogeneous, \( g_{\mu \nu} \) is a function of time only. Assuming also isotropy i.e. in this case the symmetry \( x \rightarrow -x \) involves \( g_{0i} = 0 \). In addition, if the proper motion of the “galaxies” is neglected, \( u_1 = 0 \) and \( u_0 = c \).

\[ 4 \text{ The null character of the Einstein tensor requires the existence of a cosmological term } \lambda \text{ so that } T_{\mu \nu} = -\lambda g_{\mu \nu} \text{ with } G_{\mu \nu} = T_{\mu \nu} + \lambda g_{\mu \nu}. \text{ In the proper reference of the system } T_{\mu \nu} = \begin{pmatrix} \rho c^2 & 0 \\ 0 & -p \end{pmatrix} \text{ where } p, \text{ which in 2-dimension has the dimensions of a force, is the analogous of a pressure. We deduce } \lambda = \rho c^2 = -p, \text{ i.e. the pressure is always negative. Consequently } T_{\mu \nu} = -\rho c^2 g_{\mu \nu}; \text{ that result implies that } T_{\mu \nu} \text{ has a null divergence as required.} \]
Let us now consider on the surface the points $M$ and $M' = M + dM$ in the vicinity of $M$, $r$ the radius of curvature in $M$, $O$ the center of curvature, $d\phi$ the angle $MOM'$ and the local cartesian reference frame $\{x_0 = ct, x, \xi\}$.

At a given time $dt = 0$, $ds = dx = r d\phi$.

The general quadratic form

$$ds^2 = -c^2 dt^2 + r^2(t) d\phi^2$$

defines a system of comobile coordinates where $t$ is the proper time of an observer who would keep his space coordinate fixed along the passing of time.

In order to avoid homogeneity problems, we will henceforth put $d\tau = c dt$, $dl = r_0 d\phi$ with $r_0$ being a reference value and we introduce the scaling factor $a = r / r_0$ so that $r = r_0$ at the present time $t_0$. We get

$$ds^2 = -d\tau^2 + a^2(t) d1^2$$

The metric tensor is then given by

$$g_{00} = -1, \quad g_{11} = a^2, \quad g_{01} = 0$$

with $x^0 = \tau$, $x^1 = 1$.

From equ.(6a), equ.(6b) and equ.(7) we derive

$$R_{0101} = R_{1001} = -\frac{1}{2} \partial_0^2 a^2$$

$$R_{00} = a^2 R_{1010}$$

$$R_{11} = -R_{1001}$$

$$R = -\frac{2}{a^2} R_{0011}$$

Thanks to equ.(25) equ.(13) becomes

$$\frac{-2}{a^2} R_{0101} = 2A$$

i.e.

$$\frac{1}{a^2} \partial_0^2 (a^2) = 2A$$

Putting $\partial a = \frac{\partial a}{\partial t}, \quad \partial^2 a = \frac{\partial^2 a}{\partial t^2}$ we obtain

$$\left(\frac{\partial a}{a}\right)^2 + \frac{\partial^2 a}{a} = Ac^2$$
or, since \( A = -C \rho / c^2 \)

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} = -C \rho
\]

(29b)

In the later equation we can recognize the equivalent of the Hubble parameter \( H(t) = \frac{\dot{a}}{a} \);

equ.(29b) is in fact the 2-dimension counterpart of Friedmann’s equation [6].

The field equations are:

\[
R_{00} = -A = -a^{-2} \left( \dddot{a} + a \dot{a} \dddot{a} / c^2 \right)
\]

(30a)

\[
R_{11} = a^2 A = \left( \dddot{a} + a \dot{a} \dddot{a} / c^2 \right)
\]

(30b)

**IV.2. Consequences**

Let us now focus on to the equation of evolution equ.(29b).

Let \( M \) be the material mass of the universe, \( L \) its radius and \( \rho_{\lambda} c^2 \) the energy density of the quantum vacuum; the density of matter \( \rho_m = \frac{M}{2L} \) decreases with the expansion whereas \( \rho_{\lambda} \) is constant. With respect to a reference value at the present time \( t_0 \), the total matter-energy density \( \rho c^2 \) can be expressed as \( \left( \frac{\rho_{m0}}{a(t)} + \rho_{\lambda} \right) c^2 \), \( \rho_{m0} \) being the material density at present [7,8].

We have to solve a differential equation which can be re-written as

\[
\dddot{a} + a \dot{a} \dddot{a} = -C \left( \rho_{m0} a + \rho_{\lambda} a^2 \right)
\]

(31)

We first consider the case of a static universe, i.e. \( a \) constant. \( \rho_m \) being always positive, that is possible only if \( \rho_{\lambda} < 0 \). A negative character of \( \rho_{\lambda} \) is surprising, but we have to remember that in this special universe the gravitational energy (see equ.(3)) is not only attractive but also always positive; the need of a negative energy contribution to counterbalance the gravitational pull is thus understandable.

In the general case where \( a \) is not constant the boundary conditions are:

\[
t = 0, \ r = 0 \rightarrow a(0) = 0
\]

\[
t = t_0, \ r = r_0 \rightarrow a(t_0) = 1
\]

(32)
Two cases are to be considered: a universe with or without a material component.

IV.2. 1. A universe with no material component

In the trivial case of a completely empty universe \((\rho_\Lambda = \rho_M = 0)\) equs.(31, 32) readily lead to
\[
a(t) = \sqrt{t/t_0} \quad \text{and} \quad \frac{\&a(t)}{a} = \left(2\sqrt{t_0t}\right)^{-1}.
\]
The equivalent of the Hubble parameter is
\[
H(t) = \frac{\&a}{a} = (2t)^{-1}.
\] That empty universe is ever expanding, its expansion velocity decreasing with time.

If the universe contains only vacuum energy but no matter \((\rho_\Lambda \neq 0, \rho_M = 0)\), let us put \(y = a^2\); equ.(31) is changed into \(\&y = -2C\rho_\Lambda y\). We have to consider 2 sub-cases according to the sign of \(\rho_\Lambda\):

(i) If \(\rho_\Lambda > 0\) we get
\[
a(t) = \frac{\sin^{1/2}\left(t \sqrt{2C\rho_\Lambda}\right)}{\sin^{1/2}\left(t_0 \sqrt{2C\rho_\Lambda}\right)}
\]
(33)

If we introduce the characteristic time \(t_\Lambda = \left(\frac{C|\rho_\Lambda|}{8}\right)^{-1/2}\) (see IV.2.1 below), then
\[
a(t) = \frac{\sin^{1/2}\left(\frac{4t}{t_\Lambda}\right)}{\sin^{1/2}\left(\frac{4t_0}{t_\Lambda}\right)}
\]
(34a)

and
\[
\frac{\&a(t)}{a(t)} = 2 \frac{\cos\left(\frac{4t}{t_\Lambda}\right)}{\sin^{1/2}\left(\frac{4t}{t_\Lambda}\right) \sin^{1/2}\left(\frac{4t_0}{t_\Lambda}\right)}
\]
(34b)

The equivalent of the Hubble parameter is now
\[
H(t) = \frac{2}{t_\Lambda} \left|\cot\left(\frac{4t}{t_\Lambda}\right)\right|
\]
(35)

It describes a flat universe with a periodic succession of expansion and contraction phases.
(ii) If $\rho_\Lambda < 0$ we get

$$a(t) = \frac{\sinh^{1/2} \left( \frac{4t}{t_\Lambda} \right)}{\sinh^{1/2} \left( \frac{4t_0}{t_\Lambda} \right)}$$

(36a)

and

$$\frac{\dot{a}(t)}{a(t)} = 2 \frac{\cosh \left( \frac{4t}{t_\Lambda} \right)}{\cosh^{1/2} \left( \frac{4t}{t_\Lambda} \right) \sinh^{1/2} \left( \frac{4t_0}{t_\Lambda} \right)}$$

(36b)

The equivalent of the Hubble parameter is now

$$H(t) = \frac{2}{t_\Lambda} \coth \left( \frac{4t}{t_\Lambda} \right)$$

(37)

It describes a flat, ever expanding universe. $H(t)$ tends to infinity as $\frac{1}{2t}$ when $t \to 0$ and tends toward a constant value $\frac{2}{t_\Lambda}$ in a far future.

### IV.2. 1. A universe with a material component

In the more general situation where $\rho_M \neq 0$ a consequence of equ.(31) is that $\ddot{a}(0) = 0$. Let us put again $y = a^2$; equ.(31) becomes

$$\ddot{y} = -2C \rho_M \sqrt{y} - 2C \rho_\Lambda y$$

(38)

After multiplying both sides of equ.(38) by $\ddot{y}$ a 1st integration is possible and gives

$$\dddot{y} = -\frac{8}{3} C \rho_M y^{3/2} - 2C \rho_\Lambda y^2 + K$$

(39)

$K$ is an integration constant; at $t = 0$, $y(0) = 0$ and $\dddot{y}(0) = 0$ so that $K = 0$.

If the universe contains only matter but no vacuum energy ($\rho_\Lambda = 0$) or a positive vacuum energy ($\rho_\Lambda > 0$) the problem has no physical solution but a trivial point-like universe.

If $\rho_\Lambda < 0$ putting $y = z^2$ we can transform equ.(39) into
\[ dt = \frac{dz}{\sqrt{-\frac{2C \rho_{M0}}{3} z + \frac{C |\rho_\Lambda|}{2} z^2}} \]  

We perform a further variable change \( z = -\left(\frac{4 \rho_{M0}}{3 |\rho_\Lambda|}\right) \sinh^2(\psi) \) leading to

\[ dt = -\left(\frac{8}{C |\rho_\Lambda|}\right)^{1/2} d\psi \]  

so that

\[ t = -\left(\frac{8}{C |\rho_\Lambda|}\right)^{1/2} \text{arcsinh}\left(\frac{3 |\rho_\Lambda|}{4 \rho_{M0}} \right)^{1/2} \]  

i.e.

\[ \left(\frac{C |\rho_\Lambda|}{8}\right)^{1/2} t = \text{arsinh}\left(\frac{3 |\rho_\Lambda|}{4 \rho_{M0}} \right)^{1/2} \]  

and finally

\[ a(t) = \left(\frac{4 \rho_{M0}}{3 |\rho_\Lambda|}\right) \sinh^2\left( t \sqrt{\frac{C |\rho_\Lambda|}{8}} \right) \]  

There appear two characteristic times:

(i) a time \( t_\Lambda \) characteristic of the evolution of the universe and depending on the energy density of the vacuum, constant over the passing of time:

\[ t_\Lambda = \left(\frac{C |\rho_\Lambda|}{8}\right)^{-1/2} \]  

(ii) a time \( t_{M0} \) depending on the present value of the density of matter:

\[ t_{M0} = \left(\frac{C \rho_{M0}}{6}\right)^{-1/2} \]  

Equ.\( (42) \) can be re-written as

\[ a(t) = \left(\frac{t_\Lambda}{t_{M0}}\right)^2 \sinh^2\left( \frac{t}{t_\Lambda} \right) \]  

The speed of expansion is then given by

\[ \mathcal{\Phi} (t) = \frac{t_\Lambda}{t_{M0}} \sinh \left( \frac{2t}{t_\Lambda} \right) \]
The cosmologic parameters can be readily deduced from equ.(44) and (45). For example the equivalent of the Hubble parameter is given by:

$$ H(t) = \frac{\frac{\partial}{\partial t}}{a} = \frac{2}{t} \coth \left( \frac{t}{t_\Lambda} \right) $$

(46)

$H(t)$ tends toward a constant value $\frac{2}{t_\Lambda}$ in a far future. That far future limit is the same as in the vacuum energy-only case as expected since the material density decreases with the expansion. On the contrary we have

$$ \lim_{t \to 0} a(t) = \left( \frac{t}{t_{M0}} \right)^2, \quad \lim_{t \to 0} \partial a(t) = \frac{2t}{t_{M0}^2}, \quad \lim_{t \to 0} H(t) = \frac{2}{t}, \quad \lim_{t \to 0} H(t) = \frac{2}{t} \coth \left( \frac{t}{t_\Lambda} \right) $$

(47)

which means that the material density is predominant when $t \to 0$.

The present age of the universe $t_0$ is provided by $a(t_0) = 1$, i.e.

$$ t_0 = t_\Lambda \arcsinh \left( \frac{t_{M0}}{t_\Lambda} \right) $$

(48)

The present model of a 2-dimension relativistic space-time describes a spatially flat, ever expanding universe. It is worth noticing the formal proximity, at least at first sight, of the above formulas with those accounting for a flat 4-dimension Friedmann – Lemaître universe with pressure-free matter [9,10]; however, whereas the behaviour is the same for remote times, there are marked differences near the origin: in 4 dimensions $a(t)$ varies like $\left( \frac{t}{t_\Lambda} \right)^{2/3}$ and $\partial a(t)$ like $\left( \frac{t}{t_\Lambda} \right)^{-1/3}$ whereas in 2 dimensions $a(t)$ varies like $\left( \frac{t}{t_{M0}} \right)^2$ and $\partial a(t)$ like $\left( \frac{2t}{t_{M0}^2} \right)$, with the behaviour of the Hubble parameter being the same. Moreover $\partial a(t)$ is always positive; $a(t)$ has no inflexion point so that this 2-d universe has an ever accelerated expansion whereas the 4-d universe begins with a decelerated expansion phase followed by an accelerated one.

V. Field theory in the full 2+1 dimension universe

In the two previous sections III and IV we have considered a 2-dimension space-time. As it has been said in section II this “orbital” space-time can be viewed as a 2-dimension surface...
\( \Sigma(2) \) embedded in a larger 3-dimension pseudo-Euclidian manifold \( E(3) \). In sections III and IV there was no connection between the “orbital” space and the additional “internal” degree of freedom associated to the normal in any point of the surface, we will now consider the full 3-dimension manifold \( E(3) \).

In the same manner as we had assumed in section II that our universe is a pseudo-Euclidian manifold with 10 dimensions we now assume that \( E(3) \) is a pseudo-Euclidian manifold with 3 dimensions which preserves a pseudo norm \( \eta_{\alpha\beta}X^\alpha X^\beta \) with \( \alpha \) and \( \beta \) ranging from 0 to 2, i.e. \( \eta_{00} = -1, \eta_{0\alpha} = 1 \) if \( \alpha \neq 0 \), \( \eta_{\alpha\beta} = 0 \) otherwise. As we have already said in the section II, the existence of a single internal degree of freedom is compatible with the existence of scalar particles only. We further postulate that in a reference frame of \( E(3) \), the Lagrangian \( L \) attached to such a scalar particle \( \Theta(X^\alpha) \) simply is

\[
L = \eta^{\alpha\beta} \Theta_{\alpha} \Theta_{\beta}, \text{ with } \Theta_{\alpha} = \partial_{\alpha} \Theta \quad (49a)
\]

In the local pseudo-Euclidian reference frame \( \{ ct, x, \xi \} \) attached to the point \( \Sigma(2) \), the Lagrangian equ.(48a) is changed into

\[
L = g^{\alpha\beta} [\partial_{\alpha} \Theta]^t [\partial_{\beta} \Theta] \quad (49b)
\]

where we have introduced the 3x3 metric tensor \( g_{\alpha\beta} \).

Applying the Lagrange equations to the \( \Theta \) field, i.e.

\[
\partial_{\alpha} \frac{\partial L}{\partial \Theta_{\alpha}} = \frac{\partial L}{\partial \Theta} \quad (50)
\]

gives

\[
g^{\alpha\beta} \partial_{\alpha} \partial_{\beta} \Theta + (\partial_{\alpha} g^{\alpha\beta}) (\partial_{\beta} \Theta) = 0 \quad (51)
\]

With the gauge condition

\[
\partial_{\alpha} g^{\alpha\beta} = 0 \quad (52)
\]

the equation of evolution equ.(51) becomes

\[
g^{\alpha\beta} \partial_{\alpha} \partial_{\beta} \Theta = 0 \quad (53)
\]

That expression means that the only accessible physical states are those whose measure is null in \( E(3) \). In the local reference frame \( \{ ct, x, \xi \} \) \( g \) has the following form
In the previous sections where gravitation was disconnected from other fundamental interactions, we had \( s = 1, \ a^\mu = 0 \) and \( g^{\mu\nu} \) with \( \mu, \nu = 0 \) or 1 was the metric tensor of General Relativity. If we now consider the full 3-dimension universe, we can assume that approximately \( g^{\mu\nu} \) and \( s \) are not very different from previously but \( g^{\alpha\beta} \) also contains the additional off-diagonal terms \( a^\mu \). Let us look for \( \Theta \) special solutions which can be expressed as a product

\[ \Theta(ct, x, \xi) = \Psi(ct, x) \Phi(\xi), \]

With the expressions equ.(54) for \( g^{\alpha\beta} \) and equ.(55) for \( \Theta \), the equation of evolution equ.(53) becomes

\[ [g^{\mu\nu}\partial_\mu \partial_\nu + s\partial_\xi \partial_\xi + a^\mu \partial_\mu \partial_\xi] \Psi(ct, x) \Phi(\xi) = 0 \]

Multiplying equ.(56) by \( \Phi^\dagger \) and integrating over \( \xi \) gives

\[ g^{\mu\nu}\partial_\mu \partial_\nu \Psi = -\langle \Phi | s\partial_\xi \partial_\xi | \Phi \rangle \Psi - \langle \Phi | a^\mu \partial_\xi | \Phi \rangle \partial_\mu \Psi \]

with \( \Phi^\mu = \partial_\mu \Psi \). Using the correspondence \( p_\alpha \leftrightarrow i^\dagger h \partial_\alpha \), we equivalently get

\[ g^{\mu\nu}\partial_\mu \partial_\nu \Psi = h^{-2} \langle \Phi | s p_\xi p_\xi | \Phi \rangle \Psi - \langle \Phi | a^\mu \partial_\xi | \Phi \rangle \Phi^\mu \]

i.e.

\[ g^{\mu\nu}\partial_\mu \partial_\nu \Psi \approx h^{-2} <\Phi | s p_\xi p_\xi > \Psi - \langle a^\mu \partial_\xi \rangle \Phi^\mu \]

The second term on the right hand side of the above equation accounts for the coupling between” orbital” and “internal” degrees of freedom and thus it is presumably weak; if in a first approximation we neglect that term we get

\[ g^{\mu\nu}\partial_\mu \partial_\nu \Psi \approx h^{-2} <\Phi | s p_\xi p_\xi > \Psi \]

It can be compared with the Klein-Gordon equation in the presence of a gravitation field

\[ g^{\mu\nu}\partial_\mu \partial_\nu \Psi(ct, x) = \left(\frac{mc}{\hbar}\right)^2 \Psi(ct, x) \]

where \( m \) is the particle mass. Identifying \( m \) with the self energy term in equ.(59), i.e.

\[ mc = <\Phi | s p_\xi p_\xi >^{1/2} \]
means that the particle mass exclusively originates from its dynamics along the extra
dimension, i.e. from the local orthogonal space $E_\perp(1)_M$.

Let us now come back to the term that we have neglected on the right hand side of equ.(58b).
It can be interpreted as the coupling between $\Psi$ and an additional massless vector field

$$A^\mu = \{a^\mu \partial_\bar{s}\}$$

so that the field equation for $\Psi$ can be written as

$$g^{\mu\nu} \partial_\mu \partial_\nu \Psi = (mc/h)^2 \Psi - A^\mu \bar{\Psi} \Psi.$$

This additional vector field expresses the connection between gravitation in the “orbital” 2-
dimension space-time and the other fundamental interactions which have their origin in the
“internal” space $E_\perp(1)_M$ (a single scalar field in the present case).

In the previous section we had introduced an energy density $\rho, c^2$ of the quantum vacuum. It
has been shown [11] that the vacuum energy associated to a 5th force in a 4+1-dimension
space-time could account for the experimental value of dark energy; the vacuum energy
associated to the $A^\mu$ field is the equivalent in the present 2+1 dimension universe.

VI. Conclusion

The simple 2+1 dimension universe helps understand qualitatively the foundations of
relativistic gravitation and cosmology as it allows to visualize the physics of those topics
while avoiding complex and tedious calculations. The results presented above can easily be
transposed in our usual 4-dimension space-time; the existence of extra dimensions would
imply the existence of additional a priori massless vector fields whose vacuum energy is a
serious candidate for dark energy.
References