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Geometrization of Piecewise homeomorphisms of the circle.

Jérôme Los and Natalia A. Viana Bedoya

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Abstract

In this work we study the following realization problem: given a piecewise homeomorphism $\Phi : S^1 \rightarrow S^1$, which geometrical and dynamical conditions on ϕ are sufficient to realize it as a Bowen-Series-like map associated to a surface group, seen as a discrete subgroup of $\text{Homeo}^+(S^1)$?

1 Introduction

In a paper published in 1979, R. Bowen and C. Series [BS] introduced a surprising object:

a dynamical system associated to the geometry of a Fuchsian group G , i.e. a discrete subgroup of $\text{PSL}(2, \mathbb{R})$.

This group is also understood as a discrete subgroup of $\text{Diff}^+(S^1)$ by elements satisfying strong additional properties like the Möbius property. The dynamical system they obtained from G is a piecewise diffeomorphism with two strong properties:

- the map is Markov and,
- it is orbit equivalent to the action of the group on S^1 .

More recently the construction has been revisited in [L] by considering less structures: the group is a topological surface group $G_0 = \pi_1(S)$ given by a presentation in a particular class of presentations called “geometric”, meaning that the Cayley two complex is homeomorphic to a plane. The dynamical system obtained has less structures, it is a piecewise homeomorphism with the same two features: the Markov property and the orbit equivalence to the group action on S^1 .

An additional property has been obtained, very much in the spirit of R. Bowen: the topological entropy of the map is equal to the volume entropy of the group presentation.

In the present work we consider the reverse question: starting with a dynamical system given by a piecewise homeomorphism $\Phi : S^1 \rightarrow S^1$ we want to find sufficient dynamical conditions that allow to construct a group G_Φ so that Φ is a Bowen-Series-like (BSL) map for G_Φ . Here a piecewise homeomorphism of S^1 is called BSL with a group G , acting on S^1 , if the G -action and the map action are orbit equivalent.

The map, being a piecewise homeomorphism, is given by a partition of S^1 into a finite number of intervals. For reasons that will be clear soon, the number of partition intervals is supposed to be even. A point at the boundary of two partition intervals is called a *cutting point*.

We impose four kind of conditions on the map Φ (see section 2):

- A Strong Expansivity (SE) condition on each interval of the partition.
- An Eventual Coincidence (EC) condition for the left and right orbits of the cutting points.
- A control for the left and right orbits of the cutting points before the coincidence (E+) and (E-). This is a refinement of condition (SE).

Finally we do not require the map Φ to satisfy a Markov property (as in [BS] and in [L]), we replace it by a weaker condition:

- The map is conjugated to a piecewise affine map with constant slope (CS).

Under this set of conditions we obtain:

Theorem 1. *Let $\Phi : S^1 \rightarrow S^1$ be a piecewise orientation preserving homeomorphism satisfying the conditions: (SE), (EC), (E+), (E-), (CS). Then there exists a discrete subgroup G_Φ of $\text{Homeo}^+(S^1)$ such that:*

- G_Φ is a surface group.
- G_Φ and Φ are orbit equivalent.

The set of maps satisfying the above properties is not empty, they are satisfied in particular by the maps obtained by the R. Bowen and C. Series construction [BS] and by many of the maps obtained in [L], namely from the geometric presentations where all relations have even length. The strategy to prove this result has several steps. The first one is to analyse the dynamical properties of the map Φ . Then we construct a group G_Φ as a subgroup of $\text{Homeo}^+(S^1)$ by producing a generating set X_Φ from the map Φ . This step depends on many choices for the set X_Φ . The next step is to prove that the group obtained, G_Φ is a hyperbolic group in the sense of M. Gromov (see [Gr] or [GdlH]). We obtain this result by showing

that G_Φ acts geometrically on a hyperbolic metric space. This is the main technical step. It requires to define a “nice” hyperbolic space, to construct an action on it and prove that this action is geometric. The hyperbolic space is obtained via a dynamical construction inspired by one due to P. Haissinsky and K. Pilgrim [HP]. We adapt to our situation their ideas using the specific properties of the map Φ . The main difficulty is to define an action and prove it is geometric. This is the step where all the properties of the map are used.

At this point we have the ingredients to conclude that the group G_Φ is hyperbolic with boundary S^1 . Then, by a result of E. Freden [F], the group is a discrete convergence group, as defined by F. Gehring and G. Martin [GM]. Then using the famous geometrization result of P. Tukia [T], D. Gabai [G] and A. Casson-D. Jungreis [CJ], we conclude that G_Φ is virtually Fuchsian. Proving that the group G_Φ and the map Φ are orbit equivalent is similar than in [BS] or in [L]. The last point is to extract more properties on the group G_Φ , in order to prove that in fact it is a surface group. We use a result of H. Zieschang [Zi] to that end.

As it is clear from this introduction, there are many more to understand between group presentations for groups acting on the S^1 and dynamics of piecewise homeomorphisms of the circle. Even for surface groups, the conditions described here are far from being optimal.

It is interesting to notice that some of the conditions we use, in particular the Eventual Coincidence condition, seems to appear for the first time here. The class of maps satisfying this condition (EC) seems to be much larger and covers much more than this application to surface groups. The relation between the growth properties of the map and of the group have not been considered in this work. This is clearly a direction for further works.

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2 A class of piecewise homeomorphisms on the circle

We first give the definitions and properties of the map Φ that is the main object of this paper.

The map $\Phi : S^1 \rightarrow S^1$ is a *piecewise homeomorphism of the circle* S^1 , more precisely, there is a partition of S^1 with an even number of intervals such that:

$$(I) \quad (a) \quad S^1 = \bigcup_{j=1}^{2N} I_j, \text{ and}$$

(b) $\Phi_j := \Phi|_{I_j}$ is an orientation preserving homeomorphism.

In order to state the next properties we introduce some notations. Let $\zeta, \iota, \delta, \gamma$ be permutations of $\{1, 2, \dots, 2N\}$, such that:

- ζ is a cyclic permutation of order $2N$,
- ι is a fixed point free involution, i.e. $\iota(j) \neq j$ and $\iota^2 = \text{id}$, such that:
 $\iota(j) \neq \zeta^{\pm 1}(j)$.
 This property implies that N is large enough (larger than 3) and to avoid special cases we assume for the rest of the paper that $N \geq 4$.
- $\gamma := \zeta^{-1}\iota$
- $\delta := \zeta \iota$.

Geometrically we consider ζ to be the permutation that realizes the adjacency permutation of the intervals $\{I_1, \dots, I_{2N}\}$ along a given positive orientation of S^1 . By convention $I_{\zeta(j)}$ is the interval that is adjacent to I_j in the positive direction. The interval $I_{\iota(j)}$ is an interval that is not I_j and is not adjacent to I_j . The two intervals $I_{\gamma(j)}$ and $I_{\delta(j)}$ are the adjacent intervals to $I_{\iota(j)}$ (see Figure 1).

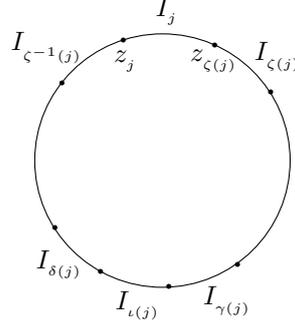


Figure 1: The permutations and their realizations

Using the permutations above we consider the *Strong Expansivity* condition:

(SE) $\forall j \in \{1, \dots, 2N\}$, the map Φ satisfies: $\Phi(I_j) \cap I_k \neq \emptyset, \forall k \neq \iota(j)$.

This condition has some immediate consequences:

(II) $\Phi(I_j) \cap I_k = I_k, \forall k \neq \iota(j), \gamma(j), \delta(j)$,

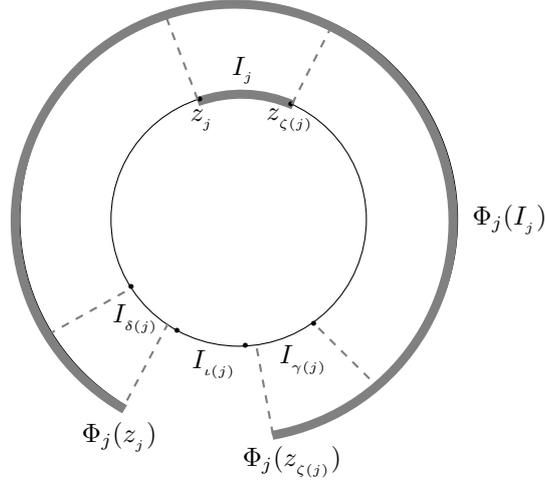


Figure 2: Condition (SE)

(III) the map Φ has an expanding fixed point in the interior of each I_j .

This property is immediate by the definition of ι and the property (II), since $I_j \subset \Phi(I_j)$.

Observe that if the map Φ satisfies the condition (SE) then all the permutations above are given by the map. Indeed, the permutation ζ is simply the cyclic ordering on S^1 and the involution ι comes from condition (SE). The two other permutations are obtained from ζ and ι .

In order to simplify some notations we write each interval as $I_j := [z_j, z_{\zeta(j)})$. The points $z_j \in S^1$ are called the *cutting points* of Φ .

The next condition is the one that makes the map Φ really particular. It is called the *Eventual coincidence* condition:

$$\begin{aligned}
 \text{(EC)} \quad & \forall j \in \{1, \dots, 2N\}, \exists k(j) \in \mathbb{N}^*, k(j) \geq 2, \text{ such that:} \\
 & \forall n \geq k(j) - 1, \quad \Phi^n(\Phi_{\zeta^{-1}(j)}(z_j)) = \Phi^n(\Phi_j(z_j)).
 \end{aligned}$$

In other words, each cutting point has a priori two different orbits, one from the positive side and one from the negative side of the point. The condition (EC) says that after $k(j)$ iterates these two orbits coincide.

The next set of conditions on the map are some control of the first $k(j) - 1$ iterates of the cutting points z_j , namely:

For all $j \in \{1, \dots, 2N\}$ and all $0 \leq m \leq k(j) - 2$:

$$\text{(E+)} \quad \Phi^m(\Phi_j(z_j)) \in I_{\delta^{m+1}(j)},$$

$$(E-) \quad \Phi^m(\Phi_{\zeta^{-1}(j)}(z_j)) \in I_{\gamma^{m+1}(\zeta^{-1}(j))}.$$

Observe that the two conditions (E+), (E-) imply condition (SE), by using $m = 0$ (see Figure 2).

The last condition on the map is perhaps not absolutely necessary but simplifies many arguments, it is called the *Constant Slope* condition:

(CS) Φ is conjugated to a piecewise affine map with constant slope $\lambda > 1$.

Remark 1. *We do not require that Φ satisfies a Markov property as in the Bowen-Series constructions in [BS] and in [L]. The constant slope condition (CS) is weaker since the Markov property implies (CS), by an easy Perron-Frobenius argument.*

Remark 2. *The piecewise homeomorphisms constructed by Bowen and Series [BS] from a Fuschian group action on \mathbb{H}^2 and those constructed in [L] from a geometric presentation of a surface group satisfies all the above conditions when the presentation has relations with even length. In other words the set of maps satisfying the above conditions is not empty.*

3 The combinatorics

Recall that $\zeta, \iota, \delta, \gamma$ are permutations of $\{1, 2, \dots, 2N\}$, for $N \geq 4$, such that:

- ζ is a cyclic permutation of order $2N$,
- ι is a fixed point free involution such that $\iota(j) \neq \zeta^{\pm 1}(j)$,
- $\gamma := \zeta^{-1}\iota$,
- $\delta := \zeta \iota$.

In this section we will point out some elementary properties of these permutations that are fundamental for the rest of the work.

Lemma 1. *The permutations γ and δ are conjugated, more precisely $\gamma = \iota^{-1}\delta^{-1}\iota$.*

Proof. Since δ and δ^{-1} are conjugated and

$$\iota^{-1}\delta^{-1}\iota = \iota(\iota\zeta^{-1})\iota = \zeta^{-1}\iota = \gamma,$$

then δ and γ are conjugated. □

To simplify the notation we put :

$$\bar{j} := \iota(j). \tag{1}$$

Remark 3. From basic permutation group theory and Lemma 1 we have that γ and δ have the same cycle structure and we obtain γ from δ^{-1} by changing j to \bar{j} on its cycles. Hence the cycle of γ that contains j and the cycle of δ that contains \bar{j} have the same length. Denote this number by $\ell[j]$.

Lemma 2. The integers $\zeta^{-1}(j)$, \bar{j} and $\overline{\delta^{m-1}(j)}$ are in the same cycle of γ of length $\ell[j]$, for all $j \in \{1, \dots, 2N\}$ and $0 < m \leq \ell[j]$.

Proof. Notice that the equalities

$$\gamma(\bar{j}) = \zeta^{-1}\iota(\iota(j)) = \zeta^{-1}(j)$$

$$\overline{\delta^{m-1}(j)} = \iota(\delta^m \delta^{-1}(j)) = (\iota \delta^m \iota^{-1})\zeta^{-1}(j) = \gamma^{-m}(\zeta^{-1}(j))$$

follow from (1), the definitions of ι , γ , δ and Lemma 1. \square

Lemma 3. If $1 \leq m \leq \ell[j]$, then $\zeta(\gamma^m(j)) = \overline{\gamma^{m-1}(j)}$. In particular if $\ell[j]$ is even and $k(j) = \ell[j]/2$ then

$$\zeta(\overline{\delta^{k(j)-1}\zeta(j)}) = \overline{\gamma^{k(j)-1}(j)}.$$

Proof. Notice that $\zeta(\gamma^m(j)) = \zeta\gamma(\gamma^{m-1}(j)) = \zeta(\zeta^{-1}\iota)(\gamma^{m-1}(j)) = \overline{\gamma^{m-1}(j)}$.

Now suppose that $\ell[j]$ is even and $k(j) = \ell[j]/2$. From the first part of this lemma, to obtain $\zeta(\overline{\delta^{k(j)-1}\zeta(j)}) = \overline{\gamma^{k(j)-1}(j)}$, it is enough to show that $\overline{\delta^{k(j)-1}\zeta(j)} = \gamma^{k(j)}(j)$. In fact, by Lemma 1 and the definition of δ we have:

$$\gamma^{k(j)}(j) = \iota^{-1}\delta^{-k(j)}\iota(j) = \overline{\delta^{-k(j)-1}\delta\iota(j)} = \overline{\delta^{-k(j)-1}\zeta(j)} = \overline{\delta^{k(j)-1}\zeta(j)}. \quad \square$$

Lemma 4. For $m = 1, \dots, \ell[j]$ we have:

$$\gamma(\overline{\delta^m(j)}) = \overline{\delta^{m-1}(j)} \text{ and } \delta(\overline{\gamma^m(\zeta^{-1}(j))}) = \overline{\gamma^{m-1}(\zeta^{-1}(j))}.$$

Proof. In fact, by (1) and the definitions of ι , γ , δ :

$$\gamma(\overline{\delta^m(j)}) = \zeta^{-1}\iota(\iota\delta^m(j)) = \zeta^{-1}\delta(\delta^{m-1}(j)) = \zeta^{-1}\zeta(\overline{\delta^{m-1}(j)}) = \overline{\delta^{m-1}(j)},$$

and

$$\delta(\overline{\gamma^m(\zeta^{-1}(j))}) = \zeta\iota(\iota\gamma^m(\zeta^{-1}(j))) = \zeta\gamma(\gamma^{m-1}(\zeta^{-1}(j))) = \overline{\gamma^{m-1}(\zeta^{-1}(j))}.$$

\square

From now on we assume that the permutations above satisfies : $\ell[j]$ is even, for all $j \in \{1, \dots, 2N\}$.

4 A group from the map Φ

From a particular map Φ , as defined above, we construct in this section a subgroup of $\text{Homeo}^+(S^1)$ by giving a set of generators.

Lemma 5. *Assume $\Phi : S^1 \rightarrow S^1$ is a piecewise homeomorphism of S^1 satisfying the conditions (SE) and (CS) then, for each $j \in \{1, \dots, 2N\}$, there is a diffeomorphism $\varphi_j \in \text{Diff}^+(S^1)$ such that:*

$$(1) \ (\varphi_j)|_{I_j} = \tilde{\Phi}|_{I_j} \text{ and } (\varphi_j)^{-1}|_{I_{\iota(j)}} = \tilde{\Phi}|_{I_{\iota(j)}},$$

(2) φ_j is a hyperbolic Möbius like diffeomorphism, i.e. with one attractive and one repelling fixed point and one pair of neutral points (with derivative one).

$$(3) \ (\varphi_j)^{-1} = \varphi_{\iota(j)},$$

where $\tilde{\Phi}$ is the piecewise affine map conjugated to Φ by condition (CS).

Proof. Using condition (CS) we replace our initial piecewise homeomorphism Φ by the piecewise affine map $\tilde{\Phi}$ with constant slope $\lambda > 1$, where $\tilde{\Phi} = g^{-1} \circ \Phi \circ g$, with $g \in \text{Homeo}^+(S^1)$. The piecewise affine map $\tilde{\Phi}$ is defined by a partition: $S^1 = \bigcup_{j=1}^{2N} \tilde{I}_j$, where $\tilde{I}_j = g^{-1}(I_j)$ for all $j \in \{1, \dots, 2N\}$. We will often abuse notation by identifying the intervals \tilde{I}_j with the original I_j .

The condition (1) of Lemma 5 has no constraints since the two intervals I_j and $I_{\iota(j)}$ are disjoint and satisfies the property: $\Phi(I_j) \cap I_{\iota(j)} = \emptyset = \Phi(I_{\iota(j)}) \cap I_j$ by the condition (SE). The same conditions are satisfied for the intervals \tilde{I}_j and $\tilde{I}_{\iota(j)}$.

In addition, the slopes of $\tilde{\Phi}$ being the same in \tilde{I}_j and $\tilde{I}_{\iota(j)}$, by imposing condition in (1) we obtain that $(\varphi_j)|_{\tilde{I}_{\iota(j)}}$ is linear with slope λ^{-1} .

Hence, the map φ_j is defined when restricted to $\tilde{I}_j \cup \tilde{I}_{\iota(j)}$, it remains to define it on the complementary intervals:

$$S^1 - (\tilde{I}_j \cup \tilde{I}_{\iota(j)}) = L_j \cup R_j,$$

where $L_j := [z_{\delta(j)}, z_j]$ and $R_j := [z_{\zeta(j)}, z_{\bar{j}}]$. The existence is a simple “differentiable connect-the-dots” construction where the constraints are the images of the extreme points:

$$\varphi_j(\partial \tilde{I}_j) = \tilde{\Phi}_j(\partial \tilde{I}_j) \text{ and } \varphi_j(\partial \tilde{I}_{\iota(j)}) = (\tilde{\Phi}_{\iota(j)})^{-1}(\partial \tilde{I}_{\iota(j)}),$$

together with the derivatives at these points being respectively λ and λ^{-1} (see Figure 3).

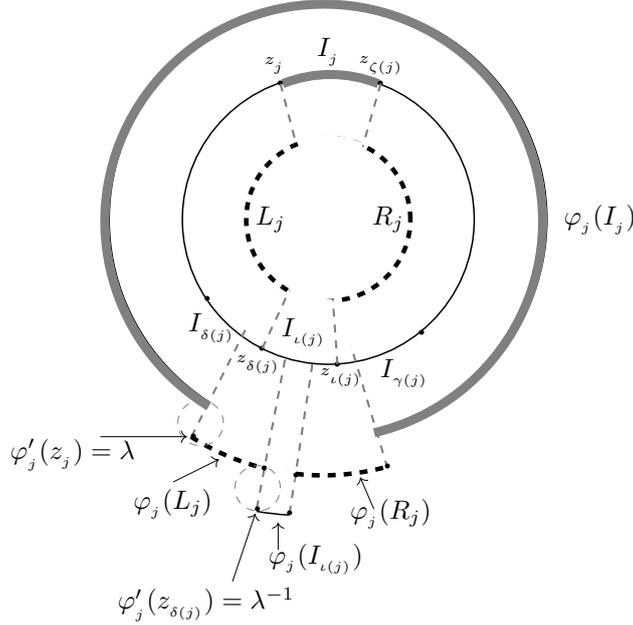


Figure 3: The connect-the-dot construction of $\varphi_j \in \text{Diff}^+(S^1)$

This connect the dot construction is simple enough not to be more explicit. We define the map φ_j on the two intervals L_j and R_j with the image intervals given by the two components of:

$$S^1 - [\tilde{\Phi}_j(\tilde{I}_j) \cup (\tilde{\Phi}_{\iota(j)})^{-1}(\tilde{I}_{\iota(j)})].$$

We have to control the derivative of φ_j on R_j and L_j which varies continuously from $\lambda > 1$ to $\lambda^{-1} < 1$ since φ_j is required to be a diffeomorphism. The construction implies there is at least one neutral point, i.e. with derivative one, in each interval L_j and R_j , by the intermediate value theorem. Clearly this construction of φ_j is non unique.

Condition (2) requires the existence of exactly one neutral point in each R_j and L_j . This is the simplest situation which is realized by imposing the derivative to vary monotonically in R_j and L_j .

By condition (SE), see (III), the map φ_j has exactly two fixed points, one expanding in \tilde{I}_j and one contracting in $\tilde{I}_{\iota(j)}$. Therefore, with the above choices, condition (2) is satisfied for φ_j and $\varphi_{\iota(j)}$.

Let us denote by $\{\varphi_j\}$ the subset of $\text{Diff}^+(S^1)$ satisfying conditions (1) and (2). Fixing $\varphi_j \in \{\varphi_j\}$, by construction we have $\varphi_j^{-1} \in \{\varphi_{i(j)}\}$. Therefore the pair $\varphi_j, \varphi_j^{-1}$ satisfies the condition (3) of Lemma 5. \square

Let us denote by $[\varphi_j]$ the subset of $\text{Diff}^+(S^1)$ satisfying the conditions (1), (2), (3) of Lemma 5mark. By the above observation, this subset is non empty.

Definition 1. Given a piecewise homeomorphism $\Phi : S^1 \rightarrow S^1$ satisfying the conditions (SE), (E+), (E-), (EC), (CS), we define a set $X_\Phi := \{\varphi_1, \dots, \varphi_{2N}\}$ of elements in $\text{Diff}^+(S^1)$, where for each $j \in \{1, \dots, 2N\}$, $\varphi_j \in [\varphi_j]$ is a diffeomorphism given by Lemma 5. We define, for each choice of X_Φ , the group G_{X_Φ} as a subgroup of $\text{Homeo}^+(S^1)$ generated by X_Φ .

This definition depends on the choice of the set X_Φ obtained from Lemma 5. One goal in the following is to prove that the group G_{X_Φ} does not depends on all these choices. The ultimate goal is to show that, under our assumptions on Φ , the group is in fact a surface group.

From now on we assume that:

$$k(j) = \ell[j]/2, \text{ for } j = 1, \dots, 2N, \quad (2)$$

see Section §3. This means that all the cycles of γ , (and δ by Lemma 1) in its cycle decomposition, have length that are even and larger than 2 by (EC).

Lemma 6. Given a set X_Φ for a piecewise homeomorphism $\Phi : S^1 \rightarrow S^1$ as in Definition 1, there exists an open neighborhood U_j of the cutting point z_j , for all $j \in \{1, \dots, 2N\}$, such that $\tilde{\Phi}^{k(j)}|_{U_j}$ is an affine diffeomorphism with slope $\lambda^{k(j)}$, where $k(j)$ is the integer given by condition (EC) on Φ .

Proof. Let $Z_j^{k(j)}$ be the point defined by condition (EC) for the cutting point z_j , i.e.

$$Z_j^{k(j)} = \tilde{\Phi}^{k(j)-1}(\tilde{\Phi}_{\zeta^{-1}(j)}(z_j)) = \tilde{\Phi}^{k(j)-1}(\tilde{\Phi}_j(z_j)),$$

and suppose that $Z_j^{k(j)} \in I_{a_j}$, for some $I_{a_j} \in \{I_1, \dots, I_{2N}\}$ satisfying:

$$a_j \neq \overline{\gamma^{k(j)-1}(\zeta^{-1}(j))}, \overline{\delta^{k(j)-1}(j)},$$

by the condition (SE), (see (7)).

- If $Z_j^{k(j)} \in \text{Int}(I_{a_j})$ then we write:

$$I_{a_j} = [z_{a_j}, Z_j^{k(j)}] \cup [Z_j^{k(j)}, z_{\zeta(a_j)}].$$

By the conditions (E+), (E-) and the continuity of $\tilde{\Phi}$ at the right and the left of the cutting point z_j there is a right and a left neighborhood of z_j , denoted $U_j^- \subset I_{\zeta^{-1}(j)}$ and $U_j^+ \subset I_j$, such that for all $1 \leq m \leq k(j) - 1$:

$$\forall z \in U_j^- \text{ then } \tilde{\Phi}^m(z) \in I_{\gamma^m(\zeta^{-1}(j))} \text{ and, } \forall z \in U_j^+ \text{ then } \tilde{\Phi}^m(z) \in I_{\delta^m(j)}. \quad (3)$$

The two half intervals U_j^- and U_j^+ are defined to satisfy:

$$\tilde{\Phi}^{k(j)}(U_j^-) = (z_{a_j}, Z_j^{k(j)}) \text{ and } \tilde{\Phi}^{k(j)}(U_j^+) = [Z_j^{k(j)}, z_{\zeta(a_j)}].$$

In other words, U_j^+ and U_j^- are the right and the left $\tilde{\Phi}^{k(j)}$ pre-image of I_{a_j} along the right and left orbits of the cutting point z_j .

By condition (EC) then $\tilde{\Phi}^{k(j)}$ is continuous at the cutting point z_j and is an affine diffeomorphism of slope $\lambda^{k(j)}$ on the neighborhood $U_j := U_j^- \cup U_j^+$ of the cutting point z_j . Indeed, it is a composition of $k(j)$ affine diffeomorphisms, each of slope λ .

In addition, by definition of the diffeomorphisms $\varphi_j \in X_{\Phi}$ of Lemma 5 and the condition (3) above we obtain:

$$\begin{aligned} \forall z \in U_j^+ \text{ then : } \tilde{\Phi}^{k(j)}(z) &= \varphi_{\delta^{k(j)-1}(j)} \circ \cdots \circ \varphi_{\delta(j)} \circ \varphi_j(z) \text{ and ,} \\ \forall z \in U_j^- \text{ then : } \tilde{\Phi}^{k(j)}(z) &= \varphi_{\gamma^{k(j)-1}(\zeta^{-1}(j))} \circ \cdots \circ \varphi_{\gamma(\zeta^{-1}(j))} \circ \varphi_{\zeta^{-1}(j)}(z). \end{aligned} \quad (4)$$

This proves the Lemma in this case.

- If $Z_j^{k(j)} \in \partial(I_{a_j})$, the construction above applies if we replace I_{a_j} either by $I_{\zeta^{-1}(a_j)} \cup I_{a_j}$ or by $I_{a_j} \cup I_{\zeta(a_j)}$, depending on whether $Z_j^{k(j)} = z_{a_j}$ or $Z_j^{k(j)} = z_{\zeta(a_j)}$. \square

Lemma 7. *Given a set X_{Φ} for a piecewise homeomorphism $\Phi : S^1 \rightarrow S^1$ as in Definition 1, then there is an extension of $\tilde{\Phi}^{k(j)}|_{U_j}$, where U_j is given in Lemma 6, to a diffeomorphism of S^1 that is topologically conjugated to a hyperbolic Möbius diffeomorphism and is an element of the group G_{Φ} .*

Proof. From Lemma 6 we obtain a neighborhood of the cutting point z_j so that $\tilde{\Phi}^{k(j)}|_{U_j}$ is an affine diffeomorphism with slope $\lambda^{k(j)}$. We now want to find a maximal interval $V_j \supset U_j$ with the same property. For that we consider the pre-image of the point $Z_j^{k(j)}$ from the left and the right, along the orbits of the cutting point z_j , namely we consider the points:

$$\begin{aligned} Z_{\delta^{k(j)-1}(j)} &= \tilde{\Phi}^{k(j)-2}(\tilde{\Phi}_j(z_j)) \in I_{\delta^{k(j)-1}(j)} \text{ and} \\ Z_{\gamma^{k(j)-1}(\zeta^{-1}(j))} &= \tilde{\Phi}^{k(j)-2}(\tilde{\Phi}_{\zeta^{-1}(j)}(z_j)) \in I_{\gamma^{k(j)-1}(\zeta^{-1}(j))}. \end{aligned}$$

In order to simplify the notations let us define:

$$\begin{aligned} c_j &:= \gamma^{k(j)-1}(\zeta^{-1}(j)), \quad d_j := \delta^{k(j)-1}(j), \\ J_{c_j} &:= [z_{c_j}, Z_{\gamma^{k(j)-1}(\zeta^{-1}(j))}] \subset I_{c_j}, \quad J_{d_j} := [Z_{\delta^{k(j)-1}(j)}, z_{\zeta(d_j)}] \subset I_{d_j}, \end{aligned} \quad (5)$$

(see Figure 4).

From the condition (EC) and the definitions above we obtain that:

$$\tilde{\Phi}_{c_j}(J_{c_j}) \cup \tilde{\Phi}_{d_j}(J_{d_j}) = \varphi_{c_j}(J_{c_j}) \cup \varphi_{d_j}(J_{d_j}) \text{ is connected, and}$$

$$\tilde{\Phi}_{c_j}(J_{c_j}) \cap \tilde{\Phi}_{d_j}(J_{d_j}) = Z_j^{k(j)}.$$

As in the proof of Lemma 6, we define the $(\tilde{\Phi}^{k(j)-1})$ - pre-images of J_{c_j} and J_{d_j} along the orbit of z_j , respectively in the intervals I_j and $I_{\zeta^{-1}(j)}$ and we obtain:

$$V_j^{c_j} = [\tilde{\Phi}^{-k(j)+1}(z_{c_j}), z_j] \subset I_{\zeta^{-1}(j)} \text{ and } V_j^{d_j} = [z_j, \tilde{\Phi}^{-k(j)+1}(z_{\zeta(d_j)})] \subset I_j.$$

From the definition of the generators φ_j via $\tilde{\Phi}$, we can rewrite these two intervals as:

$$\begin{aligned} V_j^{c_j} &= \varphi_{\zeta^{-1}(j)}^{-1} \circ \dots \circ \varphi_{\gamma^{k(j)-2}(\zeta^{-1}(j))}^{-1}(J_{c_j}) \\ V_j^{d_j} &= \varphi_j^{-1} \circ \varphi_{\delta(j)}^{-1} \circ \dots \circ \varphi_{\delta^{k(j)-2}(j)}^{-1}(J_{d_j}). \end{aligned} \quad (6)$$

We obtain thus that: $V_j := \text{Int}(V_j^{c_j} \cup V_j^{d_j})$ is an open neighborhood of z_j (see Figure 4) which satisfies: $U_j \subsetneq V_j$, where U_j is given by Lemma 6.

Moreover we observe that:

$$\tilde{\Phi}^{k(j)}(V_j) = \tilde{\Phi}_{c_j}(J_{c_j}) \cup \tilde{\Phi}_{d_j}(J_{d_j}) = \varphi_{c_j}(J_{c_j}) \cup \varphi_{d_j}(J_{d_j}) = [\tilde{\Phi}_{c_j}(z_{c_j}), \tilde{\Phi}_{d_j}(z_{\zeta(d_j)})],$$

and satisfies, by condition (SE) the following (see Figure 4):

$$\tilde{\Phi}^{k(j)}(V_j) \cap I_k \neq \emptyset, \forall k \neq \bar{c}_j, \bar{d}_j. \quad (7)$$

The property (7) implies, in particular, that: $V_j \subset \tilde{\Phi}^{k(j)}(V_j)$ and thus $\tilde{\Phi}^{k(j)}|_{V_j}$ has a unique fixed point in V_j . And, as in Lemma 6, the map: $\tilde{\Phi}^{k(j)}|_{V_j}$ is an affine diffeomorphism of slope $\lambda^{k(j)}$.

Notice that, by Lemma 3, \bar{c}_j and \bar{d}_j are adjacent with $\zeta(\bar{d}_j) = \bar{c}_j$ and, from Remark 3 the length of the cycles containing j and \bar{c}_j are the same and thus $k(\bar{c}_j) = k(j)$.

Therefore we can make the same construction for $\tilde{\Phi}^{k(j)}$ around the cutting point $z_{\bar{c}_j}$ and we obtain a neighborhood $V_{\bar{c}_j}$ of $z_{\bar{c}_j}$ so that $\tilde{\Phi}^{k(j)}|_{V_{\bar{c}_j}}$ is an affine diffeomorphism

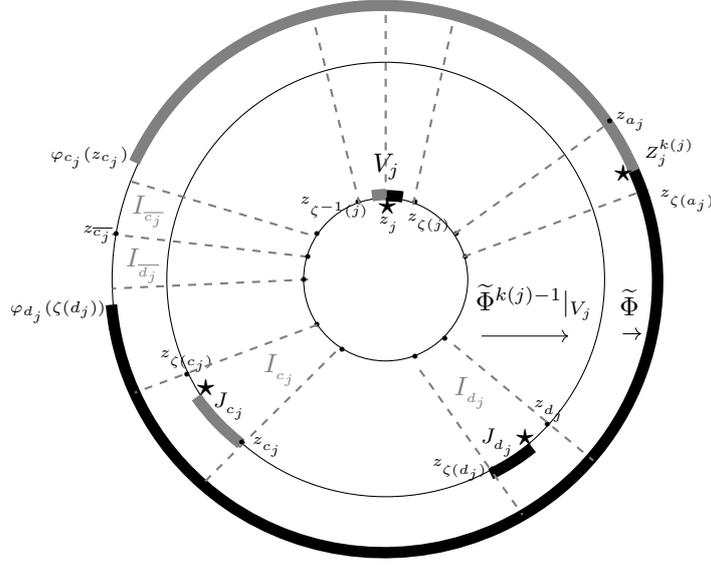


Figure 4: $\tilde{\Phi}^{k(j)}|_{V_j}$

of slope $\lambda^{k(j)}$ and with a unique fixed point in $V_{\bar{c}_j}$ (see Figure 5). By the same arguments than in (4) we obtain:

$$(\tilde{\Phi}^{k(j)}|_{V_{\bar{c}_j}})(z) = \begin{cases} \varphi_{\gamma^{k(j)-1}(\bar{d}_j)} \circ \varphi_{\gamma^{k(j)-2}(\bar{d}_j)} \circ \cdots \circ \varphi_{\gamma(\bar{d}_j)} \circ \varphi_{\bar{d}_j}(z); & z \in I_{\bar{d}_j} \cap V_{\bar{c}_j} \\ \varphi_{\delta^{k(j)-1}(\bar{c}_j)} \circ \varphi_{\delta^{k(j)-2}(\bar{c}_j)} \circ \cdots \circ \varphi_{\delta(\bar{c}_j)} \circ \varphi_{\bar{c}_j}(z); & z \in I_{\bar{c}_j} \cap V_{\bar{c}_j}. \end{cases} \quad (8)$$

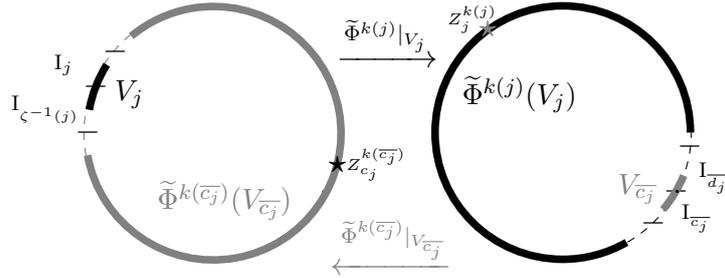


Figure 5

Let us define

$$\mathcal{G}_j : S^1 \rightarrow S^1 \quad \text{by}$$

$$\mathcal{G}_j(z) := \begin{cases} \varphi_{d_j} \circ \varphi_{\delta^{k(j)-2(j)}} \circ \cdots \circ \varphi_{\delta(j)} \circ \varphi_j(z), & \text{if } z \in [z_j, z_{\bar{c}_j}] \\ \varphi_{c_j} \circ \varphi_{\gamma^{k(j)-2(\zeta^{-1}(j))}} \circ \cdots \circ \varphi_{\gamma(\zeta^{-1}(j))} \circ \varphi_{\zeta^{-1}(j)}(z), & \text{if } z \in (z_{\bar{c}_j}, z_j]. \end{cases}$$

Observe that the two writings of \mathcal{G}_j agrees at the cutting point z_j by (4) and we have:

$$\mathcal{G}_j|_{V_j} = \tilde{\Phi}^{k(j)}|_{V_j}.$$

Recall that by (5), $c_j := \gamma^{k(j)-1}(\zeta^{-1}(j))$ and $d_j := \delta^{k(j)-1}(j)$ and, by Lemma 4 we have: $\gamma^{i-1}(\bar{d}_j) = \delta^{k(j)-i}(j)$ and $\delta^{i-1}(\bar{c}_j) = \gamma^{k(j)-i}(\zeta^{-1})$ for $i = 2, \dots, k(j)$.

The inverse of \mathcal{G}_j is given by:

$$(\mathcal{G}_j)^{-1}(z) = \begin{cases} \varphi_j^{-1} \circ \varphi_{\delta(j)}^{-1} \circ \cdots \circ \varphi_{d_j}^{-1}, & \text{if } z \in [z_j, z_{\bar{c}_j}] \\ \varphi_{\zeta^{-1}(j)}^{-1} \circ \varphi_{\gamma(\zeta^{-1}(j))} \circ \cdots \circ \varphi_{c_j}^{-1}(z), & \text{if } z \in (z_{\bar{c}_j}, z_j]. \end{cases} \quad (9)$$

Recall also that the definition of the diffeomorphisms φ_j given by Lemma 5 implies that: $\forall j \in \{1, \dots, 2N\}$ then $\varphi_j^{-1} = \varphi_{\bar{j}}$. Therefore by comparing (8) and (9) we obtain that:

$$\mathcal{G}_j|_{V_{\bar{c}_j}} = (\tilde{\Phi}^{k(j)}|_{V_{\bar{c}_j}})^{-1}.$$

This implies, in particular, that \mathcal{G}_j is well defined and continuous, at each extreme point z_j and $z_{\bar{c}_j}$ and thus is a diffeomorphism. We also obtain that \mathcal{G}_j is an affine diffeomorphism of constant slope $\lambda^{k(j)}$, when restricted to V_j and of slope $1/\lambda^{k(j)}$ in $V_{\bar{c}_j}$. This imply that \mathcal{G}_j has one repelling fixed point in V_j and one attracting fixed point in $V_{\bar{c}_j}$.

Claim: \mathcal{G}_j has exactly two fixed points.

Proof of the Claim. The proof is the same than in Lemma 5. Indeed, by construction the interval V_j is a strict sub-interval of $I_j \cup I_{\zeta^{-1}(j)}$ and $V_{\bar{c}_j}$ is a strict sub-interval of $I_{\bar{c}_j} \cup I_{\bar{d}_j}$. In addition the length of the cycle containing j satisfies $k(j) \geq 2$ and by Lemma 3 the two intervals $I_j \cup I_{\zeta^{-1}(j)}$ and $I_{\bar{c}_j} \cup I_{\bar{d}_j}$ have disjoint interior and thus $V_j \cap V_{\bar{c}_j} = \emptyset$. The proof that \mathcal{G}_j has exactly two fixed points is thus the same than for Lemma 5. This completes the proof of the Claim.

Then, by [T] Theorem 2A, \mathcal{G}_j is topologically conjugated to a hyperbolic Möbius diffeomorphism. By definition of \mathcal{G}_j , it is a composition of generators in X_{Φ} and is thus an element of G_{Φ} . □

Proposition 2. *For each cutting point z_j , $j \in \{1, \dots, 2N\}$ the following equality holds in S^1 :*

$$\varphi_{\gamma^{k(j)-1}(\zeta^{-1}(j))} \circ \cdots \circ \varphi_{\gamma(\zeta^{-1}(j))} \circ \varphi_{\zeta^{-1}(j)} = \varphi_{\delta^{k(j)-1}(j)} \circ \cdots \circ \varphi_{\delta(j)} \circ \varphi_j.$$

Proof. By Lemma 7, $\tilde{\Phi}^{k(j)}|_{U_j}$ is the restriction to the neighborhood U_j , given by Lemma 6, of the diffeomorphism \mathcal{G}_j which is conjugated to a hyperbolic Möbius diffeomorphism h_j by $f \in \text{Homeo}^+(S^1)$. By [T] this Möbius diffeomorphism is unique by analytic properties.

Therefore we have: $\tilde{\Phi}^{k(j)}|_{U_j} = \mathcal{G}_j|_{U_j}$ and $\mathcal{G}_j = f \circ h_j \circ f^{-1}$. On the other hand, by definition of $\tilde{\Phi}^{k(j)}|_{U_j}$ we have that:

$$\mathcal{G}_j|_{U_j}(z) = \varphi_{\delta^{k(j)-1(j)}} \circ \cdots \circ \varphi_{\delta(j)} \circ \varphi_j(z) \text{ for } z \in U_j \cap I_j, \text{ and}$$

$$\mathcal{G}_j|_{U_j}(z) = \varphi_{\gamma^{k(j)-1(\zeta^{-1(j)})}} \circ \cdots \circ \varphi_{\gamma(\zeta^{-1(j)})} \circ \varphi_{\zeta^{-1(j)}}(z) \text{ for } z \in U_j \cap I_{\zeta^{-1(j)}}.$$

Therefore the Möbius diffeomorphism h_j has two writings in the neighborhood $f^{-1}(U_j)$ and, by uniqueness, these two writings are the same.

This completes the proof. \square

The relation given by Proposition 2 will be called a *cutting point relation* for z_j .

5 Some geometries associated to a piecewise homeomorphism of S^1

The group G_{X_Φ} is defined via a generating set obtained from the map Φ . This group depends on many choices made during the construction in Lemma 5. We have to prove that the group does not depends on these choices. The idea is to construct a *geometric action* of the group on a metric space, i.e. an action by isometries that is co-compact and properly discontinuous.

In this section we associate, to the dynamics of Φ , two geometrical spaces with the aim of defining a geometric action. In the following we will not distinguish between Φ and $\tilde{\Phi}$ as well as between the partition intervals I_j and \tilde{I}_j .

5.1 A first dynamical graph: Γ_Φ^0

The first space we consider is directly inspired by one given by P. Haissinsky and K. Pilgrim [HP] in the context of coarse expanding conformal maps. In [HP] the authors uses the dynamics of a map F on a compact metric space Y . They construct a graph out of a sequence of coverings of the space Y by open sets obtained from one covering and by the sequence of pre-image coverings. A related construction can be found in [H18].

We use here the same idea where the space is S^1 and the dynamics is given by our map Φ . We replace their coverings by our partition and their sequence of pre-image coverings by the sequence of pre-image partitions. In order to fit with

this description we use a partition by closed intervals, in order for the different pieces to have very simple intersections (points). With our previous description we consider the initial partition:

$$S^1 = \bigcup_{j=1}^{2N} I_j, \text{ with } I_j = [z_j, z_{\zeta(j)}].$$

Thus, each interval I_j intersects the two adjacent intervals $I_{\zeta^{\pm 1}(j)}$ exactly at a cutting points.

We define the graph Γ_{Φ}^0 by an iterative process (see Figure 6):

Level 0: A base vertex v_0 is defined.

Level 1:

- a) To each interval I_j of the partition is associated a vertex v_j .
- b) v_0 is connected to v_j by an edge.
- c) v_j is connected to v_k if $I_j \neq I_k$ and $I_j \cap I_k \neq \emptyset$.

Level 2:

- a) A vertex v_{j_1, j_2} is defined for each non empty connected component of

$$I_{j_1, j_2} := I_{j_1} \cap \Phi^{-1}(I_{j_2}).$$

This notation is unambiguous since $\Phi^{-1}(I_{j_2})$ has several connected components but only one in I_{j_1} .

- b) v_{j_1} is connected to v_{j_1, j_2} by an edge.
- c) v_{j_1, j_2} is connected to $v_{j'_1, j'_2}$ if $I_{j_1, j_2} \neq I_{j'_1, j'_2}$ and $I_{j_1, j_2} \cap I_{j'_1, j'_2} \neq \emptyset$.

Level k:

- a) We repeat level 2 by iteration, i.e. we consider a sequence of intervals $\{I_{j_1}; I_{j_1, j_2}; \dots; I_{j_1, j_2, \dots, j_k}; \dots\}$ such that:

$$I_{j_1, j_2, \dots, j_k} := I_{j_1, j_2, \dots, j_{k-1}} \cap \Phi^{-k+1}(I_{j_k}) \neq \emptyset,$$

It is important to note that when the sequence j_1, j_2, \dots, j_k defines an interval of level k then $j_{i+1} \neq \overline{j_i}$, for $1 \leq i \leq k-1$, because of the condition (SE).

- b) A vertex v_{j_1, j_2, \dots, j_k} is associated to the interval I_{j_1, j_2, \dots, j_k} ,

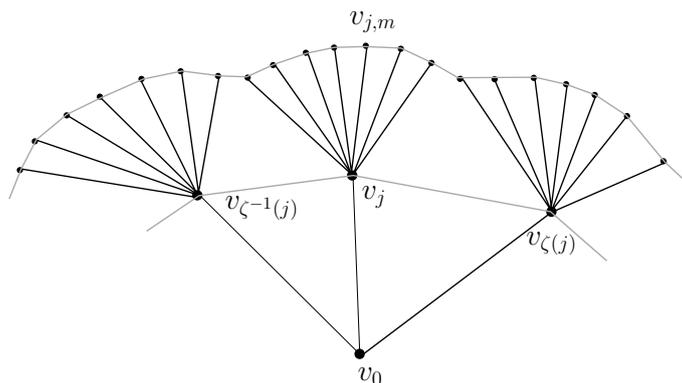


Figure 6: The first levels of the graph Γ_{Φ}^0 .

- c) v_{j_1, j_2, \dots, j_k} is connected to $v_{j_1, j_2, \dots, j_{k-1}}$ by an edge,
- d) v_{j_1, j_2, \dots, j_k} is connected to $v_{j'_1, j'_2, \dots, j'_k}$ if $I_{j'_1, j'_2, \dots, j'_k} \neq I_{j_1, j_2, \dots, j_k}$ and $I_{j_1, j_2, \dots, j_k} \cap I_{j'_1, j'_2, \dots, j'_k} \neq \emptyset$.

Lemma 8. *If Φ is a piecewise homeomorphism of S^1 satisfying the condition (SE) then the graph Γ_{Φ}^0 , endowed with the combinatorial metric (each edge has length one), is Gromov hyperbolic with boundary S^1 .*

Proof. We adapt word for word the proof in [HP]. Indeed, the essential ingredients in the proof in [HP] are the facts that each vertex is associated to a connected component of the pre-image cover with two properties:

- each component has a uniformly bounded number of pre-images,
- the size of each connected component goes to zero when the level goes to infinity.

These two properties are verified here:

- each interval has at most $2N - 1$ pre-images,
- the size of the intervals I_{j_1, j_2, \dots, j_k} in the sequence of pre-images goes to zero when k goes to infinity by the expansivity property.

In fact a much weaker expansivity property than our condition (SE) would be enough to obtain the conclusion that the graph is hyperbolic. Observe that the distance of any vertex to the base vertex is simply the level k and the edges connecting v_{j_1, j_2, \dots, j_k} to $v_{j'_1, j'_2, \dots, j'_k}$ belongs to the sphere of radius k centered at the based vertex. By this observation and our definition of the edges, each sphere of radius k centered at the based vertex is homeomorphic to S^1 . Therefore the limit space when k goes to infinity is homeomorphic to S^1 and the Gromov boundary $\partial\Gamma_{\Phi}^0$ is homeomorphic to S^1 . \square

5.2 Dynamical graph Γ_Φ

In this part we define another graph with a similar iterative process from the map Φ . The main difference is that we do not consider edges on the spheres. We replace them by an equivalence relation that identifies some vertices on some of the spheres using the specific properties of the map Φ , namely condition (EC).

We use the definitions in Level 0, Level 1: a), b), Level 2: a), b) and Level k : a), b), c) of Γ_Φ^0 , in §5.1.

Labeling the edges: we label the edge connecting $v_{j_1, j_2, \dots, j_{k-1}}$ to v_{j_1, j_2, \dots, j_k} by a symbol Ψ_{j_k} .

The graph obtained at this stage has a tree structure and is denoted by T_Φ .

We define now a quotient map: two vertices of T_Φ are identified if they belongs to the same level $k > 1$, i.e. we write them v_{j_1, j_2, \dots, j_k} and v_{l_1, l_2, \dots, l_k} , and:

- (a) There is an integer $0 \leq r < k - 1$ such that:
 - (a1) $I_{j_1} = I_{l_1}; \dots; I_{j_1, \dots, j_r} = I_{l_1, \dots, l_r}$ as intervals in S^1 (if $r = 0$ the vertex is v_0),
 - (a2) $I_{j_1, \dots, j_r, j_{r+1}}$ and $I_{l_1, \dots, l_r, l_{r+1}}$ are adjacent in the cyclique ordering of S^1 , and $I_{j_1, \dots, j_{r+p}} \neq I_{l_1, \dots, l_{r+p}}$ for all $1 \leq p < k - r$.
- (b) At level k :
 - (b1) $\Phi^k(I_{j_1, \dots, j_k}) \cap \Phi^k(I_{l_1, \dots, l_k}) = \text{one point}$,
 - (b2) $\Phi^k(I_{j_1, \dots, j_k}) \cup \Phi^k(I_{l_1, \dots, l_k}) = \text{a non degenerate interval}$.

Definition 2. *The dynamical graph is defined by $\Gamma_\Phi := T_\Phi / \sim_\Phi$, where \sim_Φ is the following relation:*

- (i) *Two vertices v_{j_1, j_2, \dots, j_k} and v_{l_1, l_2, \dots, l_k} of T_Φ are identified if the conditions (a) and (b) above are satisfied.*
- (ii) *Two edges with the same label and starting from an identified vertex are identified.*

Remark 4. *It could happens that the point in the condition (b1) is a cutting point, i.e. a boundary point of an interval. In this case the edges starting from the identified vertex have all different labels and the identification (ii) is vacuous in this case.*

It is interesting to notice that the identification operation is essentially a Stallings folding [Sta].

Lemma 9. *If Φ is a piecewise homeomorphism of S^1 satisfying the conditions (SE) and (EC) then the dynamical graph Γ_Φ is well defined.*

Proof. Let us start the study of the relation \sim_Φ when $r = 0$ in condition (a). Condition (a2) means that the two intervals I_{j_1} and I_{l_1} are adjacent, then they have a cutting point z in common. By condition (a2) the $k - 1$ first intervals in the sequence, up to $I_{j_1, \dots, j_{k-1}}$ and $I_{l_1, \dots, l_{k-1}}$ are different. Condition (b1) says that the Φ^k image of I_{j_1, \dots, j_k} and I_{l_1, \dots, l_k} have one point in common. This point has to be the Φ^k image of a cutting point z . By the Eventual coincidence condition (EC) on Φ , there is indeed an integer $k(z)$ for each cutting point, so that the two orbits of z coincides after $k(z)$ iterates. Therefore condition (b1) is satisfied for this iterate $k(z)$ and such a condition is satisfied for each cutting point and therefore for each pair of adjacent intervals. Since Φ is a piecewise orientation preserving homeomorphism then condition (b2) is also satisfied for the same iterate $k(z)$.

When $r > 0$, for each pair of adjacent intervals $I_{j_1, \dots, j_r, j_{r+1}}$ and $I_{l_1, \dots, l_r, l_{r+1}}$ as in condition (a2), the Φ^r image of these intervals is as above, i.e. adjacent of level 1, and thus there is an integer k for which conditions (b1) and (b2) are satisfied. The identification in Definition 2 (i) is well defined and occurs at each level after some minimal level $k_0 = \min\{k(j) | j = 1, \dots, 2N\}$, where the $k(j)$'s are the integers given by condition (EC).

If the point $w = \Phi^k(I_{j_1, \dots, j_k}) \cap \Phi^k(I_{l_1, \dots, l_k})$ in condition (b1) belongs to the interior of a sub-interval I_α then there is a sub-interval $I_{j_1, \dots, j_k, \alpha}$ of I_{j_1, \dots, j_k} and an edge labeled Ψ_α connecting v_{j_1, \dots, j_k} to $v_{j_1, \dots, j_k, \alpha}$ and similarly an edge, labeled Ψ_α , connecting v_{l_1, \dots, l_k} to $v_{l_1, \dots, l_k, \alpha}$ in T_Φ . The identification of these two vertices by \sim_Φ implies that two edges labelled Ψ_α starts from the new vertex \tilde{v} . The identification in Definition 2 (ii) is simply an identification of these two edges with the label Ψ_α . This identification is well defined and the dynamical graph Γ_Φ is well defined. \square

Observe that condition (EC) on the dynamics of Φ is essential for the definition of the dynamical graph Γ_Φ .

Lemma 10. *Every vertex $w \neq v_0$ in the dynamical graph Γ_Φ of Definition 2 is identified with an interval I_w of S^1 , this interval could be of the following types:*

- (i) $I_w = I_{j_1, \dots, j_k}$,
- (ii) $I_w = I_{j_1, \dots, j_k} \cup I_{l_1, \dots, l_k}$, where I_{j_1, \dots, j_k} and I_{l_1, \dots, l_k} are of the same level and adjacent on S^1 .

Proof. If w is a vertex of Γ_Φ that comes from a single vertex in T_Φ then the result is clear, the associated interval is of the form I_{j_1, \dots, j_k} , this is an interval of type (i). If w comes from two vertices v_{j_1, j_2, \dots, j_k} and v_{l_1, l_2, \dots, l_k} satisfying conditions (a) and (b) in T_Φ then the associated interval is of the form $I_w := I_{j_1, \dots, j_k} \cup I_{l_1, \dots, l_k}$. This is

indeed an interval since, by the condition (EC) and the proof above, I_{j_1, \dots, j_k} and I_{l_1, \dots, l_k} are adjacent on S^1 , this is an interval of type (ii). \square

Lemma 11. *The two graphs Γ_Φ and Γ_Φ^0 , endowed with the combinatorial metric (every edge has length one), are quasi-isometric.*

Proof. We observe that the two sets of vertices $V(\Gamma_\Phi^0)$ and $V(\Gamma_\Phi)$ are related by a map $\mathcal{V} : V(\Gamma_\Phi^0) \rightarrow V(\Gamma_\Phi)$ which is induced by the equivalence relation \sim_Φ of Definition 2. Indeed each vertex $v \in V(\Gamma_\Phi^0) \setminus \{v_0\}$ is identified with an interval $I_v := I_{j_1, \dots, j_k}$ and thus with a vertex of the tree T_Φ . The map \mathcal{V} is either one to one or two to one. Two vertices of Γ_Φ^0 with the same \mathcal{V} -image corresponds to adjacent intervals at the same level k and are thus at distance one in Γ_Φ^0 .

Let us denote by $d_{\Gamma_\Phi^0}$ and d_{Γ_Φ} the combinatorial distances in Γ_Φ^0 and Γ_Φ , respectively. Two vertices connected by an edge along a sphere S_p^0 centered at the base vertex v_0 and of radius p of Γ_Φ^0 are mapped either to a single vertex in the sphere S_p of radius p and centered at v_0 of Γ_Φ or to two distinct vertices. These two vertices also belongs to S_p of Γ_Φ and are connected in Γ_Φ by a path of length at most $k(j)$, for some $j \in \{1, \dots, 2N\}$, given by the relation \sim_Φ via the condition (EC) on the map Φ .

If we let $K_\Phi := \max\{k(j) | j = 1, \dots, 2N\}$ then we obtain that:

$$d_{\Gamma_\Phi}(\mathcal{V}(v_\alpha^0), \mathcal{V}(v_\beta^0)) \leq K_\Phi \cdot d_{\Gamma_\Phi^0}(v_\alpha^0, v_\beta^0),$$

for any two vertices (v_α^0, v_β^0) in $V(\Gamma_\Phi^0) \times V(\Gamma_\Phi^0)$. Indeed a minimal length path between v_α^0 and v_β^0 is a concatenation of some paths along the spheres centered at v_0 and some paths along rays starting at v_0 . The length of the paths along the rays are preserved by the map \mathcal{V} and the length of the paths along the spheres are expanded by a factor bounded by K_Φ . On the other direction, the same observation implies that:

$$d_{\Gamma_\Phi}(\mathcal{V}(v_\alpha^0), \mathcal{V}(v_\beta^0)) \geq \frac{1}{K_\Phi} d_{\Gamma_\Phi^0}(v_\alpha^0, v_\beta^0) + 1.$$

\square

Since a metric space which is quasi-isometric to a Gromov hyperbolic space is Gromov hyperbolic with the same boundary [GdlH], the following is an immediate consequence of Lemma 8 and Lemma 11:

Corollary 3. *The graph Γ_Φ , with the combinatorial distance, is hyperbolic with boundary homeomorphic to S^1 .* \square

6 An action of G_{X_Φ} on Γ_Φ

One way to understand the group G_{X_Φ} is to define a “geometric” action on a “nice metric space”. Recall that a geometric action of a group on a metric space is an action by isometries that is co-compact and properly discontinuous. Our candidate to be such a nice space is the dynamical graph Γ_Φ of Definition 2. It is defined only by the dynamics of Φ , just as the group G_{X_Φ} .

Before defining an action on Γ_Φ , let us describe how the generators $\varphi_j \in X_\Phi$ of the group, given by Lemma 5, do act on the partition intervals I_m for all $m \in \{1, \dots, 2N\}$, i.e. on the intervals of level 1.

Lemma 12. *If Φ is piecewise homeomorphism of S^1 satisfying the conditions (SE), (E-), (E+), (EC) and (CS), let $\varphi_j \in X_\Phi$ be a generator of the group G_{X_Φ} given by Lemma 5. If I_m is a partition interval, $m \in \{1, \dots, 2N\}$ then $\varphi_j(I_m)$ satisfies one of the following conditions:*

- (a) if $m = j$ then: $\varphi_j(I_j) \cap I_k \neq \emptyset$ for all $k \neq \iota(j)$.
- (b) For all $m \neq j, \zeta^{\pm 1}(j)$ then: $\varphi_j(I_m) = I_{\iota(j), m}$.
- (c) $\varphi_j(I_{\zeta(j)}) = I_{\iota(j), \zeta(j)} \cup A_{\gamma(j)}$ (resp. $\varphi_j(I_{\zeta^{-1}(j)}) = I_{\iota(j), \zeta^{-1}(j)} \cup A_{\delta(j)}$), where $A_{\gamma(j)}$ (resp. $A_{\delta(j)}$) is a sub-interval of $I_{\gamma(j)}$ (resp. $I_{\delta(j)}$) which does not contains any sub-interval $I_{\gamma(j), r}$, (resp. $I_{\delta(j), p}$) of level 2.
- (c') More precisely the interval $A_{\gamma(j)}$ (resp. $A_{\delta(j)}$) satisfies:
 $A_{\gamma(j)} \subsetneq I_{\gamma(j), \gamma^2(j), \dots, \gamma^{k(\zeta(j))-1}(j)}$ (resp. $A_{\delta(j)} \subsetneq I_{\delta(j), \delta^2(j), \dots, \delta^{k(j)-1}(j)}$),
 where the integers $k(j)$ and $k(\zeta(j))$ are given by the condition (EC).

Proof. The proof is a case by case study.

- (a) This is simply the condition (SE) on the map Φ .
- (b) By Lemma 5 we have that $(\varphi_j)_{|I_{\iota(j)}}^{-1} = \tilde{\Phi}_{\iota(j)}$. The condition (SE) implies that $\tilde{\Phi}_{\iota(j)}(I_{\iota(j)}) \cap I_m = I_m$ for all $m \neq j, \zeta^{\pm 1}(j)$. Therefore we obtain that: $I_{\iota(j)} \cap \tilde{\Phi}_{\iota(j)}^{-1}(I_m) = \tilde{\Phi}_{\iota(j)}^{-1}(I_m)$ which reads: $I_{\iota(j), m} = \varphi_j(I_m)$ (see Figure 7).
- (c) The two situations in this case requires more work and are symmetric, we restrict to one of them, for instance $\varphi_j(I_{\zeta(j)})$. By condition (SE), applied to I_j and $I_{\iota(j)}$, we have:
 - (i) $\varphi_j(I_j) \cap I_{\gamma(j)} \subsetneq I_{\gamma(j)}$,
 - (ii) $\varphi_{\iota(j)}(I_{\iota(j)}) \cap I_{\zeta(j)} \subsetneq I_{\zeta(j)}$.

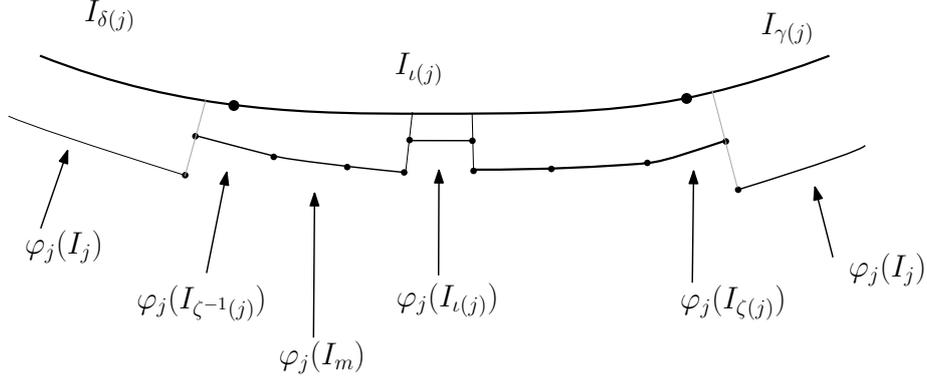


Figure 7: Image of the intervals I_m under φ_j .

These two inclusions imply, by the definition of φ_j given by Lemma 5, that:

$$I_{l(j)} \cap \varphi_{l(j)}^{-1}(I_{\zeta(j)}) = I_{l(j),\zeta(j)} = I_{l(j)} \cap \varphi_j(I_{\zeta(j)}) \text{ by (ii)}$$

and

$$\varphi_j(I_{\zeta(j)}) \cap I_{\gamma(j)} = A_{\gamma(j)} \neq \emptyset \text{ by (i) and the continuity of } \varphi_j,$$

see Figures 7 and 8.

In order to complete the proof we need to verify that the interval $A_{\gamma(j)}$ is strictly contained in the “last” sub-interval $I_{\gamma(j),l}$ of level 2 of $I_{\gamma(j)}$, where last means with respect to the orientation of S^1 . With our notations of the cutting points, this interval is written as:

$$A_{\gamma(j)} = [\varphi_j(z_{\zeta(j)}), z_{l(j)}] \text{ (see Figure 8).}$$

Recall that condition (E+), at the cutting point $z_{\zeta(j)}$, is the following:

$$\forall i, 0 \leq i \leq k(\zeta(j)) - 2 : \Phi^i(\Phi_j(z_{\zeta(j)})) \in I_{\gamma^{i+1}(j)}.$$

Hence, for $i = 0 : \Phi_j(z_{\zeta(j)}) = \varphi_j(z_{\zeta(j)}) \in I_{\gamma(j)}$, for $i = 1 : \Phi(\Phi_j(z_{\zeta(j)})) \in I_{\gamma^2(j)}$ which reads: $\varphi_{\gamma(j)}(\varphi_j(z_{\zeta(j)})) \in I_{\gamma^2(j)}$.

This last condition means that the $\varphi_{\gamma(j)}$ image of the point $\varphi_j(z_{\zeta(j)}) \in I_{\gamma(j)}$ belongs to the same partition interval than the $\varphi_{\gamma(j)}$ image of the cutting point $z_{l(j)}$, because of the condition (SE), applied to the interval $I_{\gamma(j)}$. Therefore the point $\varphi_j(z_{\zeta(j)})$ belongs to the interior of the last sub-interval of level 2 of the partition interval $I_{\gamma(j)}$ which is $I_{\gamma(j),\gamma^2(j)}$. This completes the proof of statement (c) in this case since $\varphi_j(z_{\zeta(j)})$ is the extreme point of the interval $A_{\gamma(j)}$ (see Figure 8).

For the symmetric situation, in the case (c), the arguments are exactly the same where ζ is replaced by ζ^{-1} , γ is replaced by δ and the condition (E+) is replaced by (E-).

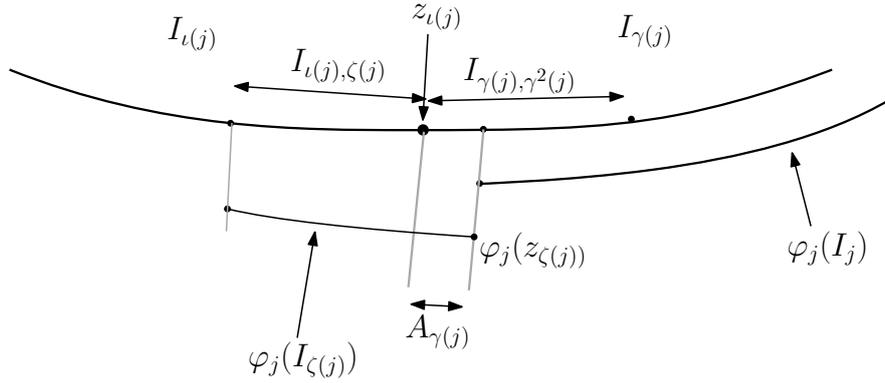


Figure 8: Details of the image around the cutting point $z_{\iota(j)}$.

(c') The above argument implies that: $A_{\gamma(j)} \subsetneq I_{\gamma(j), \gamma^2(j)}$ which is part of statement (c'). At this point only the first iterate $i = 1$ in the conditions (E+) has been used. In order to complete the proof of (c') we apply exactly the same arguments for all iterates: $i \leq k(\zeta(j)) - 2$ given by the condition (E+) and we obtain:

$$A_{\gamma(j)} \subsetneq I_{\gamma(j), \gamma^2(j), \dots, \gamma^{k(\zeta(j))-1}(j)}.$$

The symmetric situation is obtained, as above, by replacing ζ by ζ^{-1} , γ by δ and the condition (E+) by (E-).

□

Lemma 12 is a guide line for the definition of an action of $G_{X_{\Phi}}$ on the graph Γ_{Φ} . It describes how the generators act on the first level intervals.

Our goal is to define a map $\mathcal{A}_g : \Gamma_{\Phi} \rightarrow \Gamma_{\Phi}$ for all $g \in G_{X_{\Phi}}$. From Lemma 10, each vertex $v \neq v_0$ of Γ_{Φ} is identified with an interval I_v of S^1 . Each element $g \in G_{X_{\Phi}}$ is, in particular, a homeomorphism of S^1 and $g(I_v)$ is an interval of S^1 . All we have to understand is how the intervals $g(I_v)$ are related to some interval I_w , for some vertex w of Γ_{Φ} .

Definition 3. Let $G_{X_{\Phi}}$ be the group of Definition 1 and Γ_{Φ} be the dynamical graph of Definition 2. Each vertex $v \in V(\Gamma_{\Phi})$, $v \neq v_0$ is identified with an interval I_v of S^1 which could be of type (i) or (ii) according to Lemma 10. For any $g \in G_{X_{\Phi}}$, let $\mathcal{A}_g : \Gamma_{\Phi} \rightarrow \Gamma_{\Phi}$ be a map defined on the vertices $v \in V(\Gamma_{\Phi})$, $v \neq v_0$ as follows:

- (1) If $g(I_v)$ intersects more than 3 partition intervals I_j (first level), then we set $\mathcal{A}_g(v) = v_0$, where v_0 is the base vertex of Γ_{Φ} .
- (2) If there is $w \in V(\Gamma_{\Phi})$ such that $g(I_v) \subseteq I_w$ and $g(I_v)$ is not contained in any $I_{w'} \subset I_w$ of higher level than w , then we set $\mathcal{A}_g(v) = w$.

- (3) (i) If there is $w \in V(\Gamma_\Phi)$ a vertex of type (i) such that $I_w \subset g(I_v)$ and no other $I_{w'}$, for w' of the same level as w is contained in $g(I_v)$ then we set $\mathcal{A}_g(v) = w$.
- (ii) If there is $w \in V(\Gamma_\Phi)$ a vertex of type (ii) such that $I_w \subset g(I_v)$ and $g(I_v)$ does not contain $I_{w'}$ for w' of level one less than w then we set $\mathcal{A}_g(v) = w$.

Remark 5. This definition does not covers the image of the based vertex v_0 , this image will be given in Proposition 6.

The definition of the action \mathcal{A} is tricky and, as a warm up, let us check it is well defined for the generators φ_j on each vertex of level 1. We consider $\varphi_j(I_m)$ for all $m \in \{1, \dots, 2N\}$ and Lemma 12 gives all the possibilities:

- If $m = j$ then, by case (a) in Lemma 12 and case (1) of Definition 3, $\mathcal{A}_{\varphi_j}(v_j) = v_0$.
- If $m \neq j, \zeta^{\pm 1}(j)$ then, by case (b) of Lemma 12 and case (2) of Definition 3, $\mathcal{A}_{\varphi_j}(v_m) = v_{i(j),m}$.
- If $m = \zeta^{\pm 1}(j)$ then, by case (c) of Lemma 12 and case (3-i) of Definition 3, $\mathcal{A}_{\varphi_j}(v_m) = v_{i(j),m}$.

The goal is to show that, with the above notations and definitions, for any $g \in G_{X_\Phi}$, the map $\mathcal{A}_g : \Gamma_\Phi \rightarrow \Gamma_\Phi$ is well defined and is an isometry.

That the map \mathcal{A}_g is well defined means that for any $g \in G_{X_\Phi}$, it sends a vertex to a vertex. In order to prove that \mathcal{A} is an isometry we have to check that it has a well defined extension to the edges and it sends an edge to an edge and the link at a vertex to the link at the image vertex.

By Lemma 10 each vertex $v \neq v_0 \in V(\Gamma_\Phi)$ is identified with a sub-interval I_v which is of the following form:

$$(i) I_{j_1, \dots, j_k} \quad \text{or} \quad (ii) I_{j_1, \dots, j_k} \cup I_{l_1, \dots, l_k}.$$

Since the group G_{X_Φ} is generated by the $\varphi_j \in X_\Phi$ we only have to check that the action is well defined for each φ_j .

For the intervals of type (i), we consider the three situations:

- (a) $I_v \subset I_j$,
- (b) $I_v \subset I_m$ for $m \neq j, \zeta^{\pm 1}(j)$,
- (c) $I_v \subset I_m$ for $m = \zeta^{\pm 1}(j)$.

By the proof of Lemma 12 the difficult situations, for the action of φ_j , are when the interval I_v contains the cutting points z_j or $z_{\zeta(j)}$. The cutting point could be in the boundary of the interval in case (i) or in the interior in case (ii).

Proposition 4. *With the above definitions and notations the image, under φ_j , of the intervals of level less than $k(j) - 1$ or $k(\zeta(j)) - 1$ of condition (EC), are given by the following cases:*

1) *If the interval I_{j_1, \dots, j_r} does not contains a cutting point z_j (resp. $z_{\zeta(j)}$) in its boundary and:*

(a) *if $j_1 = j$ then $\varphi_j(I_{j, j_2, \dots, j_r}) = I_{j_2, \dots, j_r}$.*

(b) *if $j_1 \neq j$ then $\varphi_j(I_{j_1, j_2, \dots, j_r}) = I_{\iota(j), j_1, j_2, \dots, j_r}$.*

2) *If the interval I_{j_1, \dots, j_r} contains a cutting point, say z_j , in its boundary, then it has the form:*

$$I_{j, \delta(j), \delta^2(j), \dots, \delta^m(j)} \text{ or } I_{\zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \gamma^2(\zeta^{-1}(j)), \dots, \gamma^m(\zeta^{-1}(j))},$$

for $1 \leq m \leq k(j) - 2$. These two sets of intervals have the following image, under φ_j :

(a) *$\varphi_j(I_{j, \delta(j), \dots, \delta^m(j)}) \subseteq I_{\delta(j), \dots, \delta^m(j)}$ and $\varphi_j(I_{j, \delta(j), \dots, \delta^m(j)}) \cap I_{\delta(j), \dots, \delta^m(j), \alpha} \neq \emptyset$, for all possible such $\alpha \in \{1, 2, \dots, 2N\}$.*

(b) *$\varphi_j(I_{\zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^m(\zeta^{-1}(j))}) = I_{\iota(j), \zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^m(\zeta^{-1}(j))} \cup A_{\delta(j)}$, where $A_{\delta(j)}$ is a sub-interval of $I_{\delta(j)}$ such that $A_{\delta(j)} \subsetneq I_{\delta(j), \delta^2(j), \dots, \delta^{k(j)-1}(j)}$.*

Proof. 1. We start by assuming that the cutting points z_j and $z_{\zeta(j)}$ are not boundary points of I_{j_1, \dots, j_k} .

(a) *If $j_1 = j$ then $I_{j_1, j_2, \dots, j_k} = I_{j, j_2, \dots, j_k}$. Thus by the definition of φ_j , and Lemma 12(a) we obtain that: $\varphi_j(I_{j, j_2, \dots, j_k}) = I_{j_2, \dots, j_k}$.*

(b) *If $j_1 \neq j$, by Lemma 12(b), we have that: $\varphi_j(I_{j_1, j_2, \dots, j_k}) = I_{\iota(j), j_1, j_2, \dots, j_k}$.*

2. When a cutting point of the interval $I_j = [z_j, z_{\zeta(j)}]$ belongs to the boundary of an interval I_v then, with our notations, I_v corresponds to one of the following intervals (for the first levels):

- $I_{j, \delta(j), \delta^2(j), \dots, \delta^m(j)}$ or $I_{\zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \gamma^2(\zeta^{-1}(j)), \dots, \gamma^m(\zeta^{-1}(j))}$, if $z_j \in I_v$,
- $I_{j, \gamma(j), \gamma^2(j), \dots, \gamma^m(j)}$ or $I_{\zeta(j), \delta(\zeta(j)), \delta^2(\zeta(j)), \dots, \delta^m(\zeta(j))}$, if $z_{\zeta(j)} \in I_v$.

Let us consider the sequence of intervals around the cutting point z_j .

Recall that condition (E+), at the cutting point z_j is:

$$\Phi^m(\varphi_j(z_j)) \in I_{\delta^{m+1}(j)} \text{ for all } 0 \leq m \leq k(j) - 2.$$

For $m = 0$, the argument used in the proof of Lemma 12 implies that $\varphi_j(I_{j,\delta(j)}) \subset I_{\delta(j)}$ and $\varphi_j(I_{j,\delta(j)}) \cap I_{\delta(j),\alpha} \neq \emptyset$ for all such possible α .

The same argument applies for all $1 \leq m \leq k(j) - 2$, and it implies the statement 2(a) of the proposition.

In order to prove statement 2(b), we apply condition (E-) at the cutting point z_j :

$$\Phi^m(\varphi_{\zeta^{-1}(j)}(z_j)) \in I_{\gamma^{m+1}(\zeta^{-1}(j))} \text{ for all } 0 \leq m \leq k(j) - 2.$$

Around the cutting point z_j we consider the image of the following intervals:

$$\varphi_j(I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\dots,\gamma^m(\zeta^{-1}(j))}),$$

this is a situation where the case (c) in Lemma 12 applies. From Lemma 12(c) we have that:

$$\varphi_j(I_{\zeta^{-1}(j)}) = I_{\iota(j),\zeta^{-1}(j)} \cup A_{\delta(j)},$$

where $A_{\delta(j)}$ satisfies, by Lemma 12(c'):

$$A_{\delta(j)} \subsetneq I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j)}.$$

With the same arguments and for all $1 \leq m \leq k(j) - 2$ we obtain:

$$\varphi_j(I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\gamma^2(\zeta^{-1}(j)),\dots,\gamma^m(\zeta^{-1}(j))}) = I_{\iota(j),\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\dots,\gamma^m(\zeta^{-1}(j))} \cup A_{\delta(j)},$$

where $A_{\delta(j)}$ is as above.

This completes the proof of the proposition. \square

Proposition 4 describes the possible action of a generator on the intervals I_v of level less than $k(j) - 1$ and of type (i) in Lemma 10, i.e. corresponding to the intervals where the equivalence relation \sim_{Φ} of Definition 2 is one to one. We now consider the action on the intervals of type (ii) according to Lemma 10. These intervals are of the form: $I_{j_1,\dots,j_k} \cup I_{l_1,\dots,l_k}$. The first such non trivial equivalence arises, around the cutting point z_j , at level $k(j)$ as observed during the proof of Lemma 10.

Remark 6. *With the notations above the following properties are satisfied:*

1. By the item 2 of Proposition 4 and using the conditions (E-) and (E+), the intervals of level $k(j)$ around the cutting point z_j in the tree T_Φ are:

$$I_{j,\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j)} \text{ and } I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\gamma^2(\zeta^{-1}(j)),\dots,\gamma^{k(j)-1}(\zeta^{-1}(j))}.$$

2. The conditions (EC), (E-) and Lemma 12(c'), applied to the cutting point z_j , imply:

$$\varphi_j(z_j) \in I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j)}.$$

3. The Eventual Coincidence condition (EC) on Φ at z_j is:

$$\Phi^{k(j)-1}(\Phi_{\zeta^{-1}(j)}(z_j)) = \Phi^{k(j)-1}(\Phi_j(z_j)) = Z_j^{k(j)}.$$

Assume that $Z_j^{k(j)} \in I_\alpha$, for some interval $I_\alpha \in \{I_1, \dots, I_{2N}\}$ with $\alpha \neq \gamma^{k(j)-1}(\zeta^{-1}(j)), \delta^{k(j)-1}(j)$, by (SE). Then the two intervals of level $k(j) + 1$ in T_Φ around z_j are:

$$I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\dots,\gamma^{k(j)-1}(\zeta^{-1}(j)),\alpha} \text{ and } I_{j,\delta(j),\dots,\delta^{k(j)-1}(j),\alpha},$$

the extreme (“right”) interval of level $k(j) + 1$ of $I_{\zeta^{-1}(j)}$ and the extreme (“left”) interval of level $k(j) + 1$ of I_j , respectively. These two intervals have the cutting point z_j in common (see Figure 9).

4. The interval $I_{\tilde{v}}$ of Lemma 10, corresponding to the vertex \tilde{v} obtained by the equivalence relation of Definition 2 at level $k(j)$ is:

$$I_{\tilde{v}} = I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\dots,\gamma^{k(j)-1}(\zeta^{-1}(j))} \cup I_{j,\delta(j),\dots,\delta^{k(j)-1}(j)}.$$

This interval contains, as a sub-interval of level $k(j) + 1$, the interval:

$$I_{\tilde{v},\alpha} = I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\dots,\gamma^{k(j)-1}(\zeta^{-1}(j)),\alpha} \cup I_{j,\delta(j),\dots,\delta^{k(j)-1}(j),\alpha},$$

which contains the cutting point z_j in its interior (see Figure 9).

Proposition 5. *With the notations above, we have:*

1. $\varphi_j(I_{\tilde{v},\alpha}) = I_{\delta(j),\dots,\delta^{k(j)-1}(j),\alpha}$.
2. $\varphi_j(I_{\tilde{v}}) \subset I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j)}$ and is not contained in any sub-interval of higher level.

- 2'. More precisely:

$$\varphi_j(I_{\tilde{v}}) \cap I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j),m} \neq \emptyset$$

for all $m \neq \delta^{k(j)}(j)$.

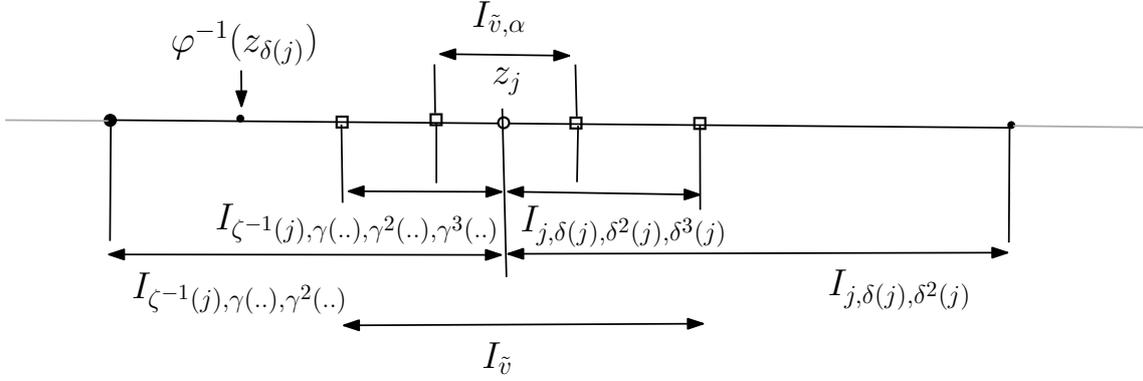


Figure 9: The intervals around the cutting point z_j , $k(j) = 4$.

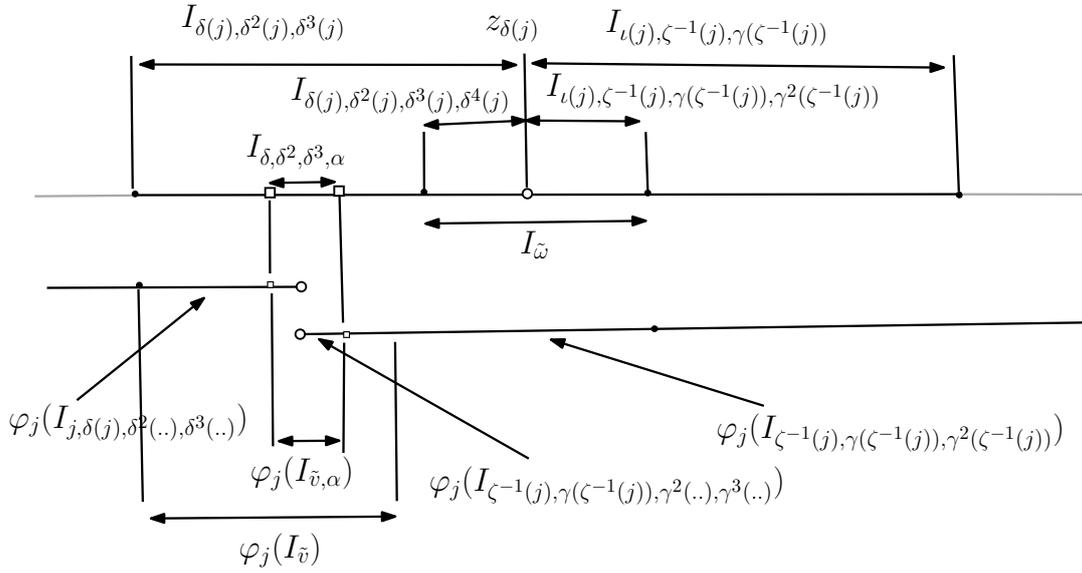


Figure 10: Image of the intervals around the cutting point $z_{\delta(j)}$

Proof. 1. Notice that $\varphi_j(I_{j,\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j),\alpha)}) \subset I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j),\alpha}$ up to the point $\varphi_j(z_j)$. By continuity of φ_j at z_j , the interval $\varphi_j(I_{\zeta^{-1}(j)})$ starts at $\varphi_j(z_j)$ up to the extreme point of $I_{\iota(j),\zeta^{-1}(j)}$. The very beginning of this interval thus covers the part of $I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j),\alpha}$ starting at $\varphi_j(z_j)$ (see Figure 10). This implies that $\varphi_j(I_{\bar{v},\alpha}) = I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j),\alpha}$.

2. Observe that for the cyclic ordering of S^1 , along the subinterval: $I_{\delta(j),\dots,\delta^{k(j)-1}(j)}$, of level $k(j) - 1$ the following subintervals of level $k(j)$ are ordered as follows, with an obvious abuse of notations (see Figure 10):

$$I_{\delta(j),\dots,\delta^{k(j)-1}(j),\alpha} \geq I_{\delta(j),\dots,\delta^{k(j)-1}(j),\beta} \geq I_{\delta(j),\dots,\delta^{k(j)-1}(j),\delta^{k(j)}}.$$

Where $I_{\delta(j),\dots,\delta^{k(j)-1}(j),\alpha}$ is the subinterval described above and containing the point $\varphi_j(z_j)$ and $I_{\delta(j),\dots,\delta^{k(j)-1}(j),\beta}$ is the subinterval containing the neutral point $N_{\iota(j)}^+$ of $\varphi_{\iota(j)} = \varphi_j^{-1}$. Indeed, by Lemma 5 and Definition 1, φ_j is expanding at z_j and is neutral at N_j^- and thus: $N_j^- < z_j$. For the image under φ_j we have: $N_{\iota(j)}^+ = \varphi_j(N_j^-)$ since $\varphi_{\iota(j)} = \varphi_j^{-1}$ and by the chain rule. Since φ_j is orientation preserving then $\varphi_j(z_j) > \varphi_j(N_j^-)$. By the same arguments we obtain (see Figure 10):

$$I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\dots,\gamma^{k(j)-2}(\zeta^{-1}(j)),\alpha'} \leq I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\dots,\gamma^{k(j)-2}(\zeta^{-1}(j)),\beta'} \leq I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\dots,\gamma^{k(j)-2}(\zeta^{-1}(j)),\gamma^{k(j)-1}(\zeta^{-1}(j))}.$$

Where $I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\dots,\gamma^{k(j)-2}(\zeta^{-1}(j)),\alpha'}$ is the subinterval containing the point $\varphi_{\iota(j)}(z_{\delta(j)})$ and $I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\dots,\gamma^{k(j)-2}(\zeta^{-1}(j)),\beta'}$ is the subinterval containing the neutral point N_j^- .

By the above argument we obtain:

$$\varphi_j(I_{\zeta^{-1}(j),\gamma(\zeta^{-1}(j)),\dots,\gamma^{k(j)-1}(\zeta^{-1}(j))}) \subset I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j)}$$

and does not intersects $I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)}(j)}$. We also obtain that: $\varphi_j(I_{j,\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j)}) \subset I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j)}$ from the point $\varphi_j(z_j)$ up to the extreme point of $I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j)}$ by a direct application of Lemma 12.

With these observations we have:

$$\varphi_j(I_{\bar{v}}) \subsetneq I_{\delta(j),\delta^2(j),\dots,\delta^{k(j)-1}(j)},$$

and, by item 1, $\varphi_j(I_{\bar{v}})$ is not contained in any sub-interval of level $k(j) + 1$ since the subinterval $I_{\delta(j),\dots,\delta^{k(j)-1}(j),\alpha}$ is contained in $\varphi_j(I_{\bar{v}})$.

2'. Recall that the cutting point z_j defines, via Proposition 2 the relation:

$$\varphi_{\delta^{k(j)-1(j)}} \circ \cdots \circ \varphi_{\delta(j)} \circ \varphi_j = \varphi_{\gamma^{k(j)-1(\zeta^{-1}(j))}} \circ \cdots \circ \varphi_{\gamma(\zeta^{-1}(j))} \circ \varphi_{\zeta^{-1}(j)}.$$

Applying this relation to the interval $I_{\bar{v}}$ gives:

$$\varphi_{\delta^{k(j)-1(j)}} \circ \cdots \circ \varphi_{\delta(j)}[\varphi_j(I_{\bar{v}})] = \varphi_{\gamma^{k(j)-1(\zeta^{-1}(j))}} \circ \cdots \circ \varphi_{\gamma(\zeta^{-1}(j))}[\varphi_{\zeta^{-1}(j)}(I_{\bar{v}})].$$

Condition 2 above, for φ_j and $\varphi_{\zeta^{-1}(j)}$, in the last equality implies that:

$$\tilde{\Phi}^{k(j)-1}[\varphi_j(I_{\bar{v}})] = \tilde{\Phi}^{k(j)-1}[\varphi_{\zeta^{-1}(j)}(I_{\bar{v}})]. \quad (10)$$

Each side of this equality is a union of two intervals with the $\tilde{\Phi}^k$ image of the cutting point in common, i.e. the point $Z_j^{k(j)} \in I_\alpha$. The union of these intervals intersects all possible intervals I_m except two (see the proof of Proposition 2). The equality (10) thus implies that the interval $\varphi_j(I_{\bar{v}})$ intersects all sub-intervals of level $k(j)$ of the interval $I_{\delta(j), \dots, \delta^{k(j)-1}(j)}$, except the “first” one, i.e. $I_{\delta(j), \dots, \delta^{k(j)-1}(j), \delta^{k(j)}(j)}$.

This completes the proof of the proposition. \square

We can now check how the action given by Definition 3 applies to the first level vertices.

Proposition 6. *The action \mathcal{A}_{φ_j} of Definition 3, for all vertices $v \neq v_0$, together with the setting: $\mathcal{A}_{\varphi_j}(v_0) := v_{i(j)}$, is well defined for all vertices at distance less than $k(j)$ or $k(\zeta(j))$ of the based vertex v_0 for all $j \in \{1, \dots, 2N\}$.*

Proof. By Lemma 12 and Definition 3, we checked that all vertices at distance one of v_0 are mapped by \mathcal{A}_{φ_j} to a vertex at distance one from $v_{i(j)}$. By adding the definition above : $\mathcal{A}_{\varphi_j}(v_0) := v_{i(j)}$ then the action \mathcal{A}_{φ_j} is well defined for all vertices in the ball of radius one centered at v_0 and it is mapped one to one to the ball of radius one centered at $v_{i(j)}$.

For intervals of level less than $k(j) - 1$ or $k(\zeta(j)) - 1$, Proposition 4 gives the following cases:

- (1) $\mathcal{A}_{\varphi_j}(v_{j_1, j_2, \dots, j_r}) = v_{j_2, \dots, j_r}$, by Proposition 4 case 1) and Definition 3 case 2).
- (2) $\mathcal{A}_{\varphi_j}(v_{j_1, j_2, \dots, j_r}) = v_{i(j), j_1, \dots, j_r}$, if $j_1 \neq j, \zeta^{\pm 1}(j)$, by Proposition 4 case 2) and Definition 3 case 2).
- (3) $\mathcal{A}_{\varphi_j}(v_{j, \delta(j), \dots, \delta^m(j)}) = v_{\delta(j), \dots, \delta^m(j)}$, for all $1 \leq m \leq k(j) - 2$ by Proposition 4 case 2(a) and Definition 3 case 2).
- (4) $\mathcal{A}_{\varphi_j}(v_{\zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^m(\zeta^{-1}(j))}) = v_{i(j), \zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^m(\zeta^{-1}(j))}$ for all $0 \leq m \leq k(j) - 3$, by Proposition 4 case 2(b) and Definition 3 case 3(i).

The cases (3) and (4) correspond to the vertices associated to intervals around the cutting point z_j . The same arguments apply, by symmetry, to the corresponding intervals around the cutting point $z_{\zeta(j)}$.

The cases above correspond to vertices of type (i) according to Lemma 10, they are mapped to vertices of type (i).

We have to check the cases where $\tilde{v} \in V(\Gamma_\Phi)$ is of type (ii). This is covered by Proposition 5 case (2) and it gives:

- (5) $\mathcal{A}_{\varphi_j}(\tilde{v}) = v_{\delta(j), \dots, \delta^{k(j)-1}(j)}$, where \tilde{v} is given by Remark 6 case 4. This situation, for the action, is given by Definition 3 case (2).

The last cases are when an interval of type (i) is mapped to an interval of type (ii). This happens for the following interval of level $k(j) - 1$ around z_j :

$$I_{\zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^{k(j)-2}(\zeta^{-1}(j))}.$$

Indeed, by Proposition 4 case 2(b), the φ_j image is:

$$\varphi_j(I_{\zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^{k(j)-2}(\zeta^{-1}(j))}) = I_{\iota(j), \zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^{k(j)-2}(\zeta^{-1}(j))} \cup A_{\delta(j)},$$

where $A_{\delta(j)} \subsetneq I_{\delta(j), \dots, \delta^{k(j)-1}(j)}$. By Remark 6 case 4, the interval of type (ii) containing the cutting point $z_{\delta(j)}$ is:

$$I_{\tilde{w}} = I_{\iota(j), \zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^{k(j)-2}(\zeta^{-1}(j))} \cup I_{\delta(j), \dots, \delta^{k(j)}(j)}.$$

By the property of the interval $A_{\delta(j)}$ given above we observe that:

$$I_{\delta(j), \dots, \delta^{k(j)}(j)} \subset A_{\delta(j)},$$

see the proof of case (2) in Proposition 5.

Hence, we obtain that:

$$I_{\tilde{w}} \subset \varphi_j(I_{\zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^{k(j)-2}(\zeta^{-1}(j))}),$$

and is not contained in any interval of level $k(j) - 1$. Therefore, by Definition 3 case 3(ii), we obtain that:

$$(6) \quad \mathcal{A}_{\varphi_j}(v_{\zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^{k(j)-2}(\zeta^{-1}(j))}) = I_{\tilde{w}}.$$

Thus the action is well defined for all vertices at distance less or equal to $k(j)$ or $k(\zeta(j))$ if the cutting point z_j is replaced by $z_{\zeta(j)}$. \square

Before going further observe that the map Φ do act on the intervals I_v defining the graph Γ_Φ . The proof bellow makes this action explicit.

Proposition 7. *For any vertex $v \in V(\Gamma_\Phi) \setminus \{v_0\}$ let I_v be the corresponding interval given by Lemma 10, then:*

There is an integer $n(v) \in \mathbb{N}$ so that $\Phi^{n(v)}(I_v)$ is an interval of level at most $K = \max\{k(j) | j = 1, \dots, 2N\}$, where $k(j)$ is given by the condition (EC).

Proof. This statement is a consequence of the strong expansion condition (SE). More precisely, if $I_v = I_{j_1, j_2, \dots, j_m}$ is an interval of type (i) then $\Phi(I_v) = \varphi_{j_1}(I_v)$. By the argument of Proposition 4 case 1(a), it gives : $\varphi_{j_1}(I_v) = I_{j_2, \dots, j_m}$ and the level has decreased by one. By iterating L times for $m - L \leq K$ the resulting interval has level less than K .

For a vertex of type (ii), the Definition 2 of the equivalence relation \sim_Φ (cases (a1), (a2)) implies that iterating r times, $0 \leq r < k(j) - 1$, the map Φ on this interval gives the same conclusion: $\Phi^r(I_v)$ is an interval of level less or equal to some $k(j)$. \square

Corollary 8. *For any vertex $v \in V(\Gamma_\Phi)$ there is a group element $g \in G_\Phi$ so that the action $\mathcal{A}_g : \Gamma_\Phi \rightarrow \Gamma_\Phi$ satisfies: $\mathcal{A}_g(v) = w \in \text{Ball}(v_0, K)$, for some integer K .*

Proof. For any vertex v , Proposition 7 gives an iterate $\Phi^{n(v)}$ applied to the interval I_v so that $\Phi^{n(v)}(I_v)$ is of level less than K . This iterate is written as a composition: $\Phi^{n(v)}(I_v) = \varphi_{j_n} \circ \varphi_{j_{n-1}} \circ \dots \circ \varphi_{j_1}(I_v)$, by definition of the map Φ and the generators of the group G_Φ . There is thus a group element $g \in G_\Phi$ so that $g(I_v)$ is an interval I_w of level less than K and thus $\mathcal{A}_g(v) = w$ so that w is at a distance less than K to the based vertex v_0 . \square

Corollary 9. *The action $\mathcal{A}_g : \Gamma_\Phi \rightarrow \Gamma_\Phi$ is well defined for any vertex and any $g \in G_\Phi$ and is co-compact.*

Proof. By Corollary 8, any vertex of Γ_Φ is mapped, by some $g \in G_\Phi$, to a vertex in the ball of radius K centered at v_0 and by Proposition 6 the action is well defined in this ball. Thus the action is well defined on Γ_Φ and is co-compact. \square

Proposition 10. *The action $\mathcal{A}_g : \Gamma_\Phi \rightarrow \Gamma_\Phi$ extends to a well defined map on edges and satisfies: $\mathcal{A}_g(\text{one edge}) = \text{one edge}$. The action \mathcal{A}_g is an isometry of Γ_Φ for the combinatorial metric.*

Proof. Since the action \mathcal{A} is well defined and co-compact on the vertices, we can restrict the study to the compact ball of radius K given by Corollary 8, and to the action by the generators φ_j of the group G_Φ . So we can use the Propositions 4 and 5. Observe that the action being an isometry is obtained by Proposition 6 for the ball of radius one. A vertex $v \neq v_0$ of type (i), according to Lemma 10, is identified with an interval of the form: $I_v = I_{j_1, \dots, j_m}$. This vertex is connected by an edge to w if $I_w = I_{j_1, \dots, j_m, j_{m+1}}$. In all cases of Proposition 4 if $\mathcal{A}_{\varphi_j}(v) = v'$ and

$\mathcal{A}_{\varphi_j}(w) = w'$ then Proposition 4 implies that $I_{w'}$ is a sub-interval of $I_{v'}$ of level one more than $I_{v'}$. For an interval of type (ii) in Lemma 10, we apply Proposition 5, with the same conclusion. This proves that the action \mathcal{A}_{φ_j} extends to an action on edges and sends one edge to one edge. In addition no two edges are mapped to the same one. \square

Lemma 13. *The action \mathcal{A}_g of Definition 3 of the group G_Φ on the graph Γ_Φ is geometric.*

Proof. At this point there is one missing property for the action to be geometric. We need to prove that the action \mathcal{A}_g of G_Φ on Γ_Φ is properly discontinuous. Since we already know, from the above results that the action is by isometries (Proposition 10) and is co-compact (Corollary 9).

The graph Γ_Φ is locally finite so a compact C in Γ_Φ is contained in a finite ball. We consider two compact sets K_1 and K_2 . By the action of the group we can assume for instance that K_1 is contained in a ball of radius R centered at the base vertex v_0 .

By Corollary 8 there are elements $g \in G_\Phi$ so that $\mathcal{A}_g(K_2) \cap K_1 \neq \emptyset$. These elements g have finite length (in the generators X_Φ) which is bounded by the maximum distance between K_1 and K_2 . Therefore the number of elements in: $\{g \in G_\Phi \text{ so that } \mathcal{A}_g(K_2) \cap K_1 \neq \emptyset\}$ is finite. This completes the proof that the action is properly discontinuous.

There is an alternative way to prove the same fact. The set of vertices $v \neq v_0$ in K_1 (say) is identified with a set of intervals in the circle. If we normalise S^1 to have length 1. The set of intervals $\{I_v; v \in K_2\}$ is finite and the set of lengths $\{l(I_v); v \in K_2\}$ is uniformly bounded, above and below by say L_2 and L'_2 . On the group side, each element of G_Φ have bounded expansion and contraction rates (slopes) between λ_g and λ_g^{-1} . There are thus finitely many elements in G_Φ so that the set of lengths in $\{l(g(I_w)); w \in K_1\}$ intersect the interval $[L'_2, L_2]$. \square

As a consequence of the above properties we obtain the following result.

Theorem 11. *Let Φ be a piecewise homeomorphism on the circle satisfying the properties: (SE), (E+), (E-), (EC), (CS). Let G_Φ be the group given in Definition 1 for one choice of generating set X_Φ then:*

- (1) *The group G_Φ is discrete and does not depends on the choices in the generating set X_Φ from Definition 1.*
- (2) *The group G_Φ is a Gromov-hyperbolic group with boundary S^1 .*
- (3) *The group G_Φ is a surface group.*

Proof. (1) The group acts geometrically on a discrete metric space by Lemma 13 and so it is a discrete group. The graph Γ_Φ and the action of Definition 3 on it does not depends on the particular generating set obtained from Lemma 5.

(2) By Lemma 13 and Corollary 3 the group acts geometrically on a Gromov hyperbolic space with boundary S^1 . Therefore the group is Gromov hyperbolic with boundary S^1 by the Milnor-Swarc Lemma (see for instance §3 [GdlH]).

(3) The group is a convergence group by a result of E. Freden [F]. Therefore by [G], [T] and [CJ] the group G_Φ is virtually a surface group.

In order to complete the proof of the Theorem it suffices to check:

Claim. The group G_Φ is torsion free.

Proof of the Claim. We already observed that each $g \in G_\Phi$ has bounded expansion and contraction factors. This implies, in particular, that the action \mathcal{A}_g is free. Indeed any vertex $v \neq v_0$ of Γ_Φ is associated to an interval I_v . We observe that for any $g \in G_\Phi$ and any v , $g(I_v)$ satisfies either: (a) $g(I_v) \cap I_v = \emptyset$, or (b) $I_v \subset g(I_v)$.

In the last case $g(I_v)$ intersects at least 3 intervals of the same level than I_v by Lemma 12. The Definition 3 of the action imply it is free.

We also obtain, by this observation, that each element $g \in G_\Phi$ is associated to an interval I_v as above on which g is expanding. This expansion property also implies that $g^n \neq id$ for all $g \in G_\Phi$ and all n . \square

To complete the proof we observe that a virtual surface group which is torsion free is a surface group by [Zi]. \square

7 Orbit equivalence

In this section we complete the proof of the main theorem. Let us first recall the definition of orbit equivalence:

Definition 4. A map $\Phi : S^1 \rightarrow S^1$ and a group G acting on the circle S^1 are orbit equivalent if, except for a finite number of pairs of points $(x, y) \in S^1 \times S^1$: $\exists \gamma \in G$ so that $y = \gamma(x)$ if and only if $\exists (m, n) \in \mathbb{N} \times \mathbb{N}$ so that $\Phi^n(x) = \Phi^m(y)$.

Lemma 14. If $\Phi : S^1 \rightarrow S^1$ is a piecewise homeomorphism of the circle satisfying the conditions (SE), (EC), (E-), (E+) and (CS) and the group G_Φ of Definition 1 are orbit equivalent.

Proof. One direction of the orbit equivalence is direct from the definition of the map and the group.

The first step is, as in the previous paragraphs, to replace the map Φ by the piecewise affine map $\tilde{\Phi}$ to which Φ is conjugated by the condition (CS).

Indeed, if $\tilde{\Phi}^n(x) = \tilde{\Phi}^m(y)$ then there are two sequences of integers $\{j_1, \dots, j_n\}$ and $\{l_1, \dots, l_m\}$ such that:

$$\varphi_{j_n} \circ \dots \circ \varphi_{j_1}(x) = \varphi_{l_m} \circ \dots \circ \varphi_{l_1}(y).$$

This implies that $x = \gamma(y)$ with $\gamma = (\varphi_{j_n} \circ \dots \circ \varphi_{j_1})^{-1} \circ \varphi_{l_m} \circ \dots \circ \varphi_{l_1}(y) \in G_\Phi$.

For the other direction we assume $y = \gamma(x)$ and, since $X_\Phi = \{\varphi_1, \dots, \varphi_{2N}\}$ is generating G_Φ , it is sufficient to restrict to $\gamma = \varphi_j \in X_\Phi$.

Recall that each generator $\varphi_j \in X_\Phi$ is, by Lemma 5, a Möbius like diffeomorphism with exactly two neutral points, N_j^+ and N_j^- , i.e. two points with derivative one. By construction, each interval of the partition satisfies:

$I_j = [z_j, z_{\zeta(j)}] \subset (N_j^-, N_j^+)$ since $(\varphi_j)|_{I_j}$ is expanding. By definition of φ_j and by the chain rule we have:

$$\varphi_{\iota(j)}(N_{\iota(j)}^-) = N_j^+ \text{ and } \varphi_{\iota(j)}(N_{\iota(j)}^+) = N_j^- \quad (11)$$

Let us assume that x is not a neutral point for φ_j and so either:

$$\varphi'_j(x) > 1 \text{ or } \varphi'_j(x) < 1.$$

In the second case we have that $x = \varphi_j^{-1}(y)$ and $(\varphi_j^{-1})'(y) > 1$. By this symmetry we can assume that $y = \varphi_j(x)$ and $\varphi'_j(x) > 1$. In other words we assume that $x \in (N_j^-, N_j^+)$. Two cases can arise:

$$(a) \ x \in I_j \quad \text{or} \quad (b) \ x \in (N_j^-, N_j^+) - I_j.$$

In case (a) $\varphi_j(x) = \tilde{\Phi}(x)$ and thus $y = \tilde{\Phi}(x)$ and (x, y) are in the same $\tilde{\Phi}$ -orbit and thus in the same Φ -orbit.

In case (b) there is another symmetry: $x \in (N_j^-, z_j)$ or $x \in (z_{\zeta(j)}, N_j^+)$. We assume that $x \in (N_j^-, z_j)$. By definition of φ_j the derivative of $\varphi_{\iota(j)} = \varphi_j^{-1}$ is larger than one at any points in $I_{\iota(j)}$. In particular $\varphi'_{\iota(j)}(z_{\delta(j)}) > 1$ and by the chain rule we have $\varphi'_j(\varphi_{\iota(j)}(z_{\delta(j)})) < 1$. This implies:

$$N_j^- \in (\varphi_{\iota(j)}(z_{\delta(j)}), z_j). \quad (12)$$

With the notations of Lemma 12 we obtain:

$$N_j^- \in A_{\zeta^{-1}(j)} \text{ and, by symmetry } N_j^+ \in A_{\zeta(j)}. \quad (13)$$

By Lemma 12 (*c'*) we obtain a precise localization of the interval $A_{\zeta^{-1}(j)}$, namely:

$$A_{\zeta^{-1}(j)} \subset I_{\zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^{k(j)-2}(\zeta^{-1}(j))}.$$

By definition of these intervals we obtain the following property:

$$\forall u \in I_{\zeta^{-1}(j), \gamma(\zeta^{-1}(j)), \dots, \gamma^{k(j)-2}(\zeta^{-1}(j))} : \tilde{\Phi}^i(u) \in I_{\gamma^i(\zeta^{-1}(j))}, \forall i \in \{0, \dots, k(j) - 2\}.$$

In particular, condition (b) above implies:

$$\tilde{\Phi}^i(x) \in I_{\gamma^i(\zeta^{-1}(j))}, \forall i \in \{0, \dots, k(j) - 2\}. \quad (14)$$

With a similar argument we obtain, by property (11) that

$$y = \varphi_j(x) \in (N_{\iota(j)}^+, \varphi_j(z_j)) \subset A_{\delta(j)}$$

with $A_{\delta(j)} \subset I_{\delta(j), \dots, \delta^{k(j)-1}(j)}$ which implies:

$$\tilde{\Phi}^i(y) \in I_{\delta^{i+1}(j)}, \forall i \in \{1, \dots, k(j) - 2\}. \quad (15)$$

These two orbits are written as:

$$\tilde{\Phi}^{k(j)-1}(y) = \varphi_{\delta^{k(j)-1}(j)} \circ \dots \circ \varphi_{\delta(j)}(y) \quad (16)$$

and

$$\tilde{\Phi}^{k(j)-1}(x) = \varphi_{\gamma^{k(j)-2}(\zeta^{-1}(j))} \circ \dots \circ \varphi_{\gamma(\zeta^{-1}(j))} \circ \varphi_{\zeta^{-1}(j)}(x). \quad (17)$$

Recall that each cutting point z_j defines a relation R_{z_j} in the group G_{Φ} , by Proposition 2, this relation has the form:

$$\varphi_{\delta^{k(j)-1}(j)} \circ \dots \circ \varphi_{\delta(j)} \circ \varphi_j = \varphi_{\gamma^{k(j)-1}(\zeta^{-1}(j))} \circ \dots \circ \varphi_{\gamma(\zeta^{-1}(j))} \circ \varphi_{\zeta^{-1}(j)}. \quad (18)$$

If the relation R_{z_j} is applied to the point x we obtain:

$$\tilde{\Phi}^{k(j)-1}(y) = \varphi_{\gamma^{k(j)-1}(\zeta^{-1}(j))}[\tilde{\Phi}^{k(j)-1}(x)]. \quad (19)$$

Indeed, when we replace $y = \varphi_j(x)$ in the left hand side of (18) we obtain the equality (16) which is the left hand side of (19). The right hand side of (19) is obtained by replacing, in the right hand side of (18) the equality (17).

Let $j_1 := \gamma^{k(j)-1}(\zeta^{-1}(j)) \in \{1, \dots, 2N\}$, $x_1 := \tilde{\Phi}^{k(j)-1}(x)$ and $y_1 := \varphi_{j_1}(x_1)$.

The equality (19) implies that an alternative, similar to (a) or (b) above applies again, more precisely:

$$(a_1) \quad x_1 \in I_{j_1} \quad \text{or} \quad (b_1) \quad x_1 \notin I_{j_1}.$$

In case (a₁) the relation (19) gives:

$$\tilde{\Phi}^{k(j)-1}(y) = \tilde{\Phi}[\tilde{\Phi}^{k(j)-1}(x)] = \tilde{\Phi}^{k(j)}(x),$$

and the orbit equivalence is proved in this case.

In case (b₁), by derivation, the relation (19) gives:

$$[\tilde{\Phi}^{k(j)-1}(\varphi_j(x))]' = [\tilde{\Phi}^{k(j)-1}]'(\varphi_j(x)).[\varphi_j]'(x) = [\varphi_{j_1}]'(\tilde{\Phi}^{k(j)-1}(x)).[\tilde{\Phi}^{k(j)-1}]'(x) \quad (20)$$

Recall that $\tilde{\Phi}$ is an affine map of constant slope $\lambda > 1$ and therefore the equality (20) implies:

$$\lambda \geq (\varphi_j)'(x) = (\varphi_{j_1})'(x_1) \geq 1.$$

This means that the alternative (b₁) is exactly the same at the point x_1 than (b) at the point x . This implies, in particular that: $x_1 \in I_{\zeta^{-1}(j_1)}$ and, more precisely:

$$x_1 \in I_{\zeta^{-1}(j_1), \gamma(\zeta^{-1}(j_1)), \dots, \gamma^{k(j_1)-2}(\zeta^{-1}(j_1))} \quad \text{and} \quad y_1 = \varphi_{j_1}(x_1) \in I_{\delta(j_1), \dots, \delta^{k(j_1)-1}},$$

by the same arguments than for the points x and y .

The previous arguments then produce:

- a sequence of integers: $\{j, j_1, \dots, j_n, \dots\}$ where each $j_m \in \{1, \dots, 2N\}$,
- a sequence of points: $x_n := \tilde{\Phi}^{k(j_{n-1})-1}(x_{n-1})$ and $y_n := \varphi_{j_n}(x_n)$, with the following alternative:

$$(a_n) \quad x_n \in I_{j_n} \quad \text{or} \quad (b_n) \quad x_n \notin I_{j_n}.$$

Lemma 15. *With the above notations, if $x \in [N_j^-, z_j]$ is such that there is an integer n_0 so that $x_{n_0} \in I_{j_{n_0}}$ then $y_{n_0} = \varphi_{n_0}(x_{n_0}) = \tilde{\Phi}(x_{n_0})$ and there is an integer $K(n_0)$ such that $\tilde{\Phi}^{K(n_0)}(y) = \tilde{\Phi}^{K(n_0)+1}(x)$.*

Proof. The situation is the alternative (a_{n₀}), similar to the initial alternative (a), for the iterate $K(n_0) = (k(j_{n_0}) - 1) + (k(j_{n_0-1}) - 1) + \dots + (k(j) - 1)$. \square

It remains to consider the situation where the alternative (b_n) occurs for all $n \in \mathbb{N}$. Let us observe the following:

Lemma 16. *Assume that x and \tilde{x} are two points in $[N_j^-, z_j]$ such that there is $n_0 \geq 1$ so that x_{n_0} satisfies (a_{n₀}) and \tilde{x}_{n_0} satisfies (b_{n₀}) but \tilde{x}_{n_0+1} satisfies (a_{n₀+1}) then the following inequalities are satisfied:*

$$1 < (\varphi_j)'(\tilde{x}) < (\varphi_j)'(x) < \lambda.$$

Proof. By the definition of φ_j given by Lemma 5, $(\varphi_j)'$ is strictly increasing in $[N_j^-, z_j]$ between 1 and $\lambda > 1$. It remains to prove that:

$$N_j^- < \tilde{x} < x < z_j.$$

From the hypothesis on \tilde{x} and x , we have:

$$x_{n_0} = \tilde{\Phi}^{k(j_{n_0-1})-1}(x_{n_0-1}) \in I_{j_{n_0}} \text{ by } (a_{n_0}), \text{ and}$$

$$\tilde{x}_{n_0} = \tilde{\Phi}^{k(j_{n_0-1})-1}(\tilde{x}_{n_0-1}) \in I_{\zeta^{-1}(j_{n_0})} \text{ by } (b_{n_0}).$$

Therefore $\tilde{x}_{n_0} < x_{n_0}$ since $I_{\zeta^{-1}(j_{n_0})}$ occurs before $I_{j_{n_0}}$ along the cyclic ordering of S^1 and thus $\tilde{x} < x$ since the map $\tilde{\Phi}$ is orientation preserving. \square

Lemma 17. *If the point $x \in [N_j^-, z_j]$ is such that the alternative (b_n) occurs for all $n \in \mathbb{N}$ then x is a neutral point, i.e. $x = N_j^-$.*

Proof. The point x so that the alternative (b_n) is satisfied for all n is an accumulation of the point \tilde{x}_{n_0} given by Lemma 16 when n_0 goes to infinity. This sequence is decreasing by Lemma 16 and the derivative is strictly decreasing in $[1, \lambda]$. The accumulation point of this sequence is such that the derivative of φ_j is 1 and thus $x = N_j^-$. \square

This completes the proof of Lemma 14 and thus of the main Theorem 1. \square

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