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Michele Ancona, Thomas Letendre

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Michele Ancona, Thomas Letendre. Roots of Kostlan polynomials: moments, strong Law of Large Numbers and Central Limit Theorem. 2019. hal-02380440v1

HAL Id: hal-02380440

<https://hal.science/hal-02380440v1>

Preprint submitted on 26 Nov 2019 (v1), last revised 6 Dec 2021 (v3)

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Roots of Kostlan polynomials: moments, strong Law of Large Numbers and Central Limit Theorem

Michele Ancona* Thomas Letendre†

November 26, 2019

Abstract

We study the number of real roots of a Kostlan (or elliptic) random polynomial of degree d in one variable. More generally, we are interested in the distribution of the counting measure of the set of real roots of such a polynomial. We compute the asymptotics of the central moments of any order of these random variables, in the large degree limit. As a consequence, we prove that these quantities satisfy a strong Law of Large Numbers and a Central Limit Theorem. In particular, the real roots of a Kostlan polynomial almost surely equidistribute as the degree diverges. Moreover, the fluctuations of the counting measure of this random set around its mean converge in distribution to the Gaussian White Noise. We also prove that the random variables we study concentrate in probability around their mean faster than any negative power of d . More generally, our results hold for the real zeros of a random real section of a line bundle of degree d over a real projective curve, in the complex Fubini–Study model.

Keywords: Kostlan polynomials, Elliptic polynomials, Complex Fubini–Study model, Kac–Rice formula, Law of Large Numbers, Central Limit Theorem, Method of moments.

Mathematics Subject Classification 2010: 14P99, 32L05, 60F05, 60F15, 60G15, 60G57.

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*Michele Ancona, Tel Aviv University, School of Mathematical Sciences; e-mail: michi.ancona@gmail.com. Michele Ancona is supported by the Israeli Science Foundation through the ISF Grants 382/15 and 501/18.

†Thomas Letendre, Sorbonne Université, Institut de Mathématiques de Jussieu – Paris Rive Gauche; e-mail: letendre@math.cnrs.fr. Thomas Letendre is supported by the French National Research Agency through the ANR grants SpInQS (ANR-17-CE40-0011) and UniRaNDom (ANR-17-CE40-0008).

1 Introduction

Kostlan polynomials. A real *Kostlan polynomial* of degree d is a univariate random polynomial of the form $\sum_{k=0}^d a_k \sqrt{\binom{d}{k}} X^k$, where the coefficients $(a_k)_{0 \leq k \leq d}$ are independent real Gaussian variables with zero mean and unit variance. These random polynomials are also known as *elliptic polynomials* in the literature (see [28] for example). The roots of such a polynomial form a random subset of \mathbb{R} that we denote by Z_d . Kostlan proved (cf. [16]) that for all $d \in \mathbb{N}$, the average number of roots of this random polynomial is $\mathbb{E}[\text{Card}(Z_d)] = d^{\frac{1}{2}}$, where $\text{Card}(Z_d)$ is the cardinality of Z_d . It was later proved by Dalmao (see [10]) that $\text{Var}(\text{Card}(Z_d)) \sim \sigma^2 d^{\frac{1}{2}}$ as $d \rightarrow +\infty$, where σ is some explicit positive constant. Dalmao also proved that $\text{Card}(Z_d)$ satisfies a Central Limit Theorem as $d \rightarrow +\infty$.

In this paper, we study the higher moments of $\text{Card}(Z_d)$ in the large degree limit. Let $p \in \mathbb{N}$, we denote by $m_p(\text{Card}(Z_d))$ the p -th central moments of $\text{Card}(Z_d)$. A consequence of our main result (Theorem 1.12) is that, as $d \rightarrow +\infty$, we have:

$$m_p(\text{Card}(Z_d)) = \mu_p \sigma^p d^{\frac{p}{2}} + o(d^{\frac{p}{2}}),$$

where σ is the constant appearing in Dalmao's variance estimate, and $(\mu_p)_{p \in \mathbb{N}}$ is the sequence of moments of the standard real Gaussian distribution $\mathcal{N}(0, 1)$. This result allows us to prove a strong Law of Large Numbers: $d^{-\frac{1}{2}} \text{Card}(Z_d) \rightarrow 1$ almost surely. We also prove that $d^{-\frac{1}{2}} \text{Card}(Z_d)$ concentrates around 1 in probability, faster than any negative power of d . Finally, we recover Dalmao's Central Limit Theorem by the method of moments. The original proof used the Wiener–Itô expansion of the number of roots. In fact, we improve this result by proving a Central Limit Theorem for the counting measure of Z_d (see Theorem 1.9 below).

Equivalently, one can define Z_d as the set of zeros, on the real projective line $\mathbb{R}P^1$, of the homogeneous Kostlan polynomial $\sum_{k=0}^d a_k \sqrt{\binom{d}{k}} X^k Y^{d-k}$, where $(a_k)_{0 \leq k \leq d}$ are i.i.d. (independent identically distributed) $\mathcal{N}(0, 1)$ variables. In this setting, an homogeneous Kostlan polynomial of degree d is a standard Gaussian vector in the space of real global holomorphic sections of the line bundle $\mathcal{O}(d) \rightarrow \mathbb{C}P^1$, equipped with the natural L^2 -inner product (see Section 2.1). Recall that $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$, where $\mathcal{O}(1)$ is the hyperplane line bundle over $\mathbb{C}P^1$. Then, Z_d is the real zero set of this random section. In this paper, we study more generally the real zeros of random real sections of positive Hermitian line bundles over real algebraic curves.

Framework and background. Let us introduce our general framework. More details are given in Section 2.1 below. Let \mathcal{X} be a smooth complex projective manifold of dimension 1. Let \mathcal{E} and \mathcal{L} be holomorphic line bundles over \mathcal{X} . We assume that \mathcal{X} , \mathcal{E} and \mathcal{L} are endowed with compatible real structures and that the real locus of \mathcal{X} is not empty. We denote by M this real locus, which is then a smooth closed (i.e. compact without boundary) submanifold of \mathcal{X} of real dimension 1.

Let $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$ denote Hermitian metrics on \mathcal{E} and \mathcal{L} respectively, that are compatible with the real structures. We assume that $(\mathcal{L}, h_{\mathcal{L}})$ has positive curvature ω , so that \mathcal{L} is ample and ω is a Kähler form on \mathcal{X} . The form ω induces a Riemannian metric g on \mathcal{X} , hence on M . Let us denote by $|dV_M|$ the arc-length measure on M defined by g .

For all $d \in \mathbb{N}$, we denote by $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ the space of global real holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d \rightarrow \mathcal{X}$. Let $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we denote by $Z_s = s^{-1}(0) \cap M$ the real zero set of s . Since s is holomorphic, if $s \neq 0$ its zeros are isolated and Z_s is finite. In this case, we denote by ν_s the counting measure of Z_s , that is $\nu_s = \sum_{x \in Z_s} \delta_x$, where δ_x stands for the unit Dirac mass at x . For any $\phi \in C^0(M)$, we denote by $\langle \nu_s, \phi \rangle = \sum_{x \in Z_s} \phi(x)$. Quantities of the form $\langle \nu_s, \phi \rangle$ are called the *linear statistics* of ν_s . Note that $\langle \nu_s, \mathbf{1} \rangle = \text{Card}(Z_s)$, where $\mathbf{1}$ is the constant unit function.

For any $d \in \mathbb{N}$, the space $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is finite-dimensional, and its dimension can be computed by the Riemann–Roch Theorem. Moreover, the measure $|dV_M|$ and the metrics $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$ induce a Euclidean L^2 -inner product on this space (see Equation (2.1)). Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, see Section 2.3. Then, ν_{s_d} is an almost surely well-defined random Radon measure on M . We denote by $Z_d = Z_{s_d}$ and by $\nu_d = \nu_{s_d}$ for simplicity. In this setting, the linear statistics of ν_d were studied in [2, 13, 18, 20], among others. In particular, the exact asymptotics of their expected values and their variances are known.

Theorem 1.1 (Gayet–Welschinger). *For every $d \in \mathbb{N}$, let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then the following holds as $d \rightarrow +\infty$:*

$$\forall \phi \in \mathcal{C}^0(M), \quad \mathbb{E}[\langle \nu_d, \phi \rangle] = d^{\frac{1}{2}} \left(\frac{1}{\pi} \int_M \phi |dV_M| \right) + \|\phi\|_\infty O(d^{-\frac{1}{2}}),$$

where the error term $O(1)$ does not depend on ϕ . That is: $d^{-\frac{1}{2}} \mathbb{E}[\nu_d] = \frac{1}{\pi} |dV_M| + O(d^{-1})$.

Theorem 1.1 is Theorem 1.2 in [13] with $n = 1$ and $i = 0$. See also [18, Theorem 1.3] when \mathcal{E} is not trivial. In [18], the case the linear statistics is discussed in Section 5.3.

We use the following notation for the central moments of ν_d .

Definition 1.2. Let $d \in \mathbb{N}$ and let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Let ν_d denote the counting measure of the real zero set of s_d . For all $p \in \mathbb{N}^*$, for all $\phi_1, \dots, \phi_p \in \mathcal{C}^0(M)$ we denote:

$$m_p(\nu_d)(\phi_1, \dots, \phi_p) = \mathbb{E} \left[\prod_{i=1}^p \left(\langle \nu_d, \phi_i \rangle - \mathbb{E}[\langle \nu_d, \phi_i \rangle] \right) \right].$$

For all $\phi \in \mathcal{C}^0(M)$, we denote by $m_p(\langle \nu_d, \phi \rangle) = m_p(\nu_d)(\phi, \dots, \phi)$ the p -th central moment of $\langle \nu_d, \phi \rangle$. As above, we denote the p -th central moment of $\text{Card}(Z_d)$ by $m_p(\text{Card}(Z_d)) = m_p(\nu_d)(\mathbf{1}, \dots, \mathbf{1})$.

Of course, $m_p(\nu_d)$ is only interesting for $p \geq 2$. For $p = 2$, the bilinear form $m_2(\nu_d)$ encodes the covariance structure of the linear statistics of ν_d . In particular, $m_2(\nu_d)(\phi, \phi) = \text{Var}(\langle \nu_d, \phi \rangle)$ for all $\phi \in \mathcal{C}^0(M)$. The large degree asymptotics of $m_2(\nu_d)$ have been studied in [20, Theorem 1.6].

Theorem 1.3 (Letendre–Puchol). *For all $d \in \mathbb{N}$, let $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a standard Gaussian vector. There exists $\sigma > 0$ such that, for all ϕ_1 and $\phi_2 \in \mathcal{C}^0(M)$, the following holds as $d \rightarrow +\infty$:*

$$m_2(\nu_d)(\phi_1, \phi_2) = d^{\frac{1}{2}} \sigma^2 \int_M \phi_1 \phi_2 |dV_M| + o(d^{\frac{1}{2}}).$$

Remarks 1.4.

- In fact, $\sigma = \left(\frac{1 + \mathcal{I}_{1,1}}{\pi} \right)^{\frac{1}{2}}$, where $\mathcal{I}_{1,1}$ is the constant appearing in [20, Theorem 1.6]. Since σ is universal, it is the same as the one appearing in Dalmao’s variance estimate [10, Theorem 1.1]. An integral expression of σ is given by [10, Proposition 3.1]. The positivity of σ is non-trivial and is given by [10, Corollary 1.2]. See also [20, Theorem 1.8].

- The error term $o(d^{\frac{1}{2}})$ in Theorem 1.3 can be improved (see [20, Theorem 1.6]).

In [2], Ancona derived a two terms asymptotic expansion of the non-central moments of the linear statistics of ν_d . As a consequence, he proved the following (cf. [2, Theorem 1.5]).

Theorem 1.5 (Ancona). *For all $d \in \mathbb{N}$, let $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a standard Gaussian vector. Let $p \geq 2$, for all $\phi_1, \dots, \phi_p \in \mathcal{C}^0(M)$ we have: $m_p(\nu_d)(\phi_1, \dots, \phi_p) = o(d^{\frac{p-1}{2}})$.*

Remark 1.6. In Ancona’s paper the line bundle \mathcal{E} is trivial. The results of [2] rely on precise estimates for the Bergman kernel of \mathcal{L}^d . These estimates still hold if we replace \mathcal{L}^d by $\mathcal{E} \otimes \mathcal{L}^d$, where \mathcal{E} is any fixed real Hermitian line bundle (see [21, Theorem 4.2.1]). Thus, all the results in [2] are still valid for random real sections of $\mathcal{E} \otimes \mathcal{L}^d \rightarrow \mathcal{X}$.

Main results. In this paper, we prove a strong Law of Large Numbers (Theorem 1.7) and a Central Limit Theorem (Theorem 1.9) for the linear statistics of the random measures $(\nu_d)_{d \in \mathbb{N}}$ defined above. These results are deduced from our main result (Theorem 1.12), which gives the precise asymptotics of the central moments $(m_p(\nu_d))_{p \geq 3}$ (cf. Definition 1.2), as $d \rightarrow +\infty$.

Recall that all the results of the present paper apply when Z_d is the set of real roots of a Kostlan polynomial of degree d . If one considers homogeneous Kostlan polynomials in two variables, then $Z_d \subset M = \mathbb{R}P^1$. Since $\mathbb{R}P^1$ is obtained from the Euclidean unit circle \mathbb{S}^1 by identifying antipodal points, it is a circle of length π . In this case, $|dV_M|$ is the Lebesgue measure on this circle, normalized so that $\text{Vol}(M) = \pi$. If one wants to consider the original Kostlan polynomials in one variable, then $Z_d \subset M = \mathbb{R}$, where \mathbb{R} is seen as a standard affine chart in $\mathbb{R}P^1$. In this case, the measure $|dV_M|$ admits the density $t \mapsto (1 + t^2)^{-1}$ with respect to the Lebesgue of \mathbb{R} , and $\text{Vol}(M) = \pi$ once again.

Theorem 1.7 (Strong Law of Large Numbers). *Let \mathcal{X} be a real projective curve whose real locus M is non-empty. Let $\mathcal{E} \rightarrow \mathcal{X}$ and $\mathcal{L} \rightarrow \mathcal{X}$ be real Hermitian line bundles such that \mathcal{L} is positive. Let $(s_d)_{d \geq 1}$ be a sequence of independent standard Gaussian vectors in $\prod_{d \geq 1} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. For any $d \geq 1$, let Z_d denote the real zero set of s_d and let ν_d denote the counting measure of Z_d .*

Then, almost surely, $d^{-\frac{1}{2}}\nu_d \xrightarrow{d \rightarrow +\infty} \frac{1}{\pi} |dV_M|$ in the weak- sense. That is, almost surely:*

$$\forall \phi \in \mathcal{C}^0(M), \quad d^{-\frac{1}{2}} \langle \nu_d, \phi \rangle \xrightarrow{d \rightarrow +\infty} \frac{1}{\pi} \int_M \phi |dV_M|.$$

In particular, almost surely, $d^{-\frac{1}{2}} \text{Card}(Z_d) \xrightarrow{d \rightarrow +\infty} \frac{\text{Vol}(M)}{\pi}$ and $\text{Card}(Z_d)^{-1} \nu_d \xrightarrow{d \rightarrow +\infty} \frac{|dV_M|}{\text{Vol}(M)}$.

Definition 1.8. The *Gaussian White Noise* on M is a random Gaussian generalized function $W \in \mathcal{D}'(M)$, whose distribution is characterized by the following facts:

- for any $\phi \in \mathcal{C}^\infty(M)$, $\langle W, \phi \rangle_{(\mathcal{D}', \mathcal{C}^\infty)}$ is a real centered Gaussian variable;
- for all $\phi_1, \phi_2 \in \mathcal{C}^\infty(M)$, we have: $\mathbb{E} \left[\langle W, \phi_1 \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \langle W, \phi_2 \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \right] = \int_M \phi_1 \phi_2 |dV_M|$.

Here, $\langle \cdot, \cdot \rangle_{(\mathcal{D}', \mathcal{C}^\infty)}$ denotes the usual duality pairing between $\mathcal{D}'(M)$ and $\mathcal{C}^\infty(M)$.

Here and in the following, we avoid using the term ‘‘distribution’’ when talking about elements of $\mathcal{D}'(M)$ and rather use the term ‘‘generalized function’’. This is to avoid any possible confusion with the distribution of a random variable.

Theorem 1.9 (Central Limit Theorem). *Let \mathcal{X} be a real projective curve whose real locus M is non-empty. Let $\mathcal{E} \rightarrow \mathcal{X}$ and $\mathcal{L} \rightarrow \mathcal{X}$ be real Hermitian line bundles such that \mathcal{L} is positive. Let $(s_d)_{d \geq 1}$ be a sequence of independent standard Gaussian vectors in $\prod_{d \geq 1} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. For any $d \geq 1$, let Z_d denote the real zero set of s_d and let ν_d denote the counting measure of Z_d .*

Then, the following holds in distribution, in the space $\mathcal{D}'(M)$ of generalized functions on M , and in the Sobolev space $H^{-\frac{1}{2}-\varepsilon}(M)$ for any $\varepsilon > 0$:

$$\frac{1}{d^{\frac{1}{4}}\sigma} \left(\nu_d - \frac{d^{\frac{1}{2}}}{\pi} |dV_M| \right) \xrightarrow{d \rightarrow +\infty} W,$$

where W denotes the Gaussian White Noise on M (see Definition 1.8).

In particular, for any test-functions $\phi_1, \dots, \phi_k \in \mathcal{C}^\infty(M)$, the random vector:

$$\frac{1}{d^{\frac{1}{4}}\sigma} \left(\left(\langle \nu_d, \phi_1 \rangle - \frac{d^{\frac{1}{2}}}{\pi} \int_M \phi_1 |dV_M| \right), \dots, \left(\langle \nu_d, \phi_k \rangle - \frac{d^{\frac{1}{2}}}{\pi} \int_M \phi_k |dV_M| \right) \right)$$

converges in distribution to a centered Gaussian vector of variance matrix $\left(\int_M \phi_i \phi_j |dV_M| \right)_{1 \leq i, j \leq k}$.

Moreover, for all $\phi \in \mathcal{C}^0(M)$,

$$\frac{1}{d^{\frac{1}{4}}\sigma} \left(\langle \nu_d, \phi \rangle - \frac{d^{\frac{1}{2}}}{\pi} \int_M \phi |dV_M| \right) \xrightarrow{d \rightarrow +\infty} \mathcal{N} \left(0, \int_M \phi^2 |dV_M| \right).$$

In particular, $\left(d^{\frac{1}{4}}\sigma \text{Vol}(M)^{\frac{1}{2}} \right)^{-1} \left(\text{Card}(Z_d) - \frac{d^{\frac{1}{2}}}{\pi} \text{Vol}(M) \right) \xrightarrow{d \rightarrow +\infty} \mathcal{N}(0, 1)$ in distribution.

Before stating our main result, we need to introduce some additional notations.

Definition 1.10. For all $p \in \mathbb{N}$, we denote by μ_p the p -th moment of the standard real Gaussian distribution. Recall that, for all $p \in \mathbb{N}$, we have $\mu_{2p} = \frac{(2p)!}{2^p p!}$ and $\mu_{2p+1} = 0$.

Definition 1.11. Let A be a non-empty finite set, a *partition* of A is a family $\mathcal{I} = \{I_1, \dots, I_m\}$ of non-empty disjoint subsets of A such that $\bigsqcup_{i=1}^m I_i = A$. We denote by \mathcal{P}_A (resp. \mathcal{P}_k) the set of partitions of A (resp. $\{1, \dots, k\}$). A *partition into pairs* of A is a partition $\mathcal{I} \in \mathcal{P}_A$ such that $\text{Card}(I) = 2$ for all $I \in \mathcal{I}$. We denote by \mathcal{PP}_A , (resp. \mathcal{PP}_k) the set of partitions into pairs of A (resp. $\{1, \dots, k\}$). We also use the convention that $\mathcal{P}_\emptyset = \{\emptyset\} = \mathcal{PP}_\emptyset$.

Theorem 1.12. Let \mathcal{X} be a real projective curve whose real locus M is non-empty. Let $\mathcal{E} \rightarrow \mathcal{X}$ and $\mathcal{L} \rightarrow \mathcal{X}$ be real Hermitian line bundles such that \mathcal{L} is positive. For all $d \in \mathbb{N}$, let $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a standard Gaussian vector, let Z_d denote the real zero set of s_d and let ν_d denote its counting measure. For all $p \geq 3$, for all $\phi_1, \dots, \phi_p \in C^0(M)$, the following holds as $d \rightarrow +\infty$:

$$\begin{aligned} m_p(\nu_d)(\phi_1, \dots, \phi_p) &= \sum_{\{\{a_i, b_i\} | 1 \leq i \leq \frac{p}{2}\} \in \mathcal{PP}_p} \prod_{i=1}^{\frac{p}{2}} m_2(\nu_d)(\phi_{a_i}, \phi_{b_i}) + \left(\prod_{i=1}^p \|\phi_i\|_\infty \right) O\left(d^{\frac{1}{2} \lfloor \frac{p-1}{2} \rfloor} (\ln d)^p\right) \\ &= d^{\frac{p}{4}} \sigma^p \sum_{\{\{a_i, b_i\} | 1 \leq i \leq \frac{p}{2}\} \in \mathcal{PP}_p} \prod_{i=1}^{\frac{p}{2}} \left(\int_M \phi_{a_i} \phi_{b_i} |dV_M| \right) + o(d^{\frac{p}{4}}), \end{aligned}$$

where $\lfloor \cdot \rfloor$ denotes the integer part, $\|\cdot\|_\infty$ denotes the sup-norm, σ is the same positive constant as in Theorem 1.3, and the error term $O\left(d^{\frac{1}{2} \lfloor \frac{p-1}{2} \rfloor} (\ln d)^p\right)$ does not depend on (ϕ_1, \dots, ϕ_p) .

In particular, for all $\phi \in C^0(M)$, we have:

$$\begin{aligned} m_p(\langle \nu_d, \phi \rangle) &= \mu_p \text{Var}(\langle \nu_d, \phi \rangle)^{\frac{p}{2}} + \|\phi\|_\infty^p O\left(d^{\frac{1}{2} \lfloor \frac{p-1}{2} \rfloor} (\ln d)^p\right) \\ &= \mu_p d^{\frac{p}{4}} \sigma^p \left(\int_M \phi^2 |dV_M| \right)^{\frac{p}{2}} + o(d^{\frac{p}{4}}). \end{aligned}$$

Remark 1.13. If p is odd, then the first term vanishes in the asymptotic expansions of Theorem 1.12. Indeed, in this case $\mathcal{PP}_p = \emptyset$ and $\mu_p = 0$. Hence, if p is odd, for all $\phi_1, \dots, \phi_p \in C^0(M)$, we have $m_p(\nu_d)(\phi_1, \dots, \phi_p) = O(d^{\frac{p-1}{4}} (\ln d)^p)$. If p is even, we have $m_p(\nu_d)(\phi_1, \dots, \phi_p) = O(d^{\frac{p}{4}})$ for all $\phi_1, \dots, \phi_p \in C^0(M)$.

Other interesting corollaries of Theorem 1.12 include the following.

Corollary 1.14 (Concentration in probability). *In the setting of Theorem 1.12, let $(\varepsilon_d)_{d \geq 1}$ denote a sequence of positive numbers and let $\phi \in C^0(M)$. Then, for all $p \in \mathbb{N}^*$, as $d \rightarrow +\infty$, we have:*

$$\mathbb{P}\left(d^{-\frac{1}{2}} \left| \langle \nu_d, \phi \rangle - \mathbb{E}[\langle \nu_d, \phi \rangle] \right| > \varepsilon_d\right) = O\left(\left(d^{\frac{1}{4}} \varepsilon_d\right)^{-2p}\right).$$

In particular, for all $p \in \mathbb{N}^*$, as $d \rightarrow +\infty$, we have:

$$\mathbb{P}\left(d^{-\frac{1}{2}} \left| \text{Card}(Z_d) - \mathbb{E}[\text{Card}(Z_d)] \right| > \varepsilon_d\right) = O\left(\left(d^{\frac{1}{4}} \varepsilon_d\right)^{-2p}\right).$$

Corollary 1.15 (Hole probability). *In the setting of Theorem 1.12, let U be a non-empty open subset of M . Then, for all $p \in \mathbb{N}^*$, as $d \rightarrow +\infty$, we have:*

$$\mathbb{P}(Z_d \cap U = \emptyset) = O(d^{-\frac{p}{2}}).$$

About the proofs. The proof of Theorem 1.12 relies on several ingredients. Some of them are classical, such as Kac–Rice formulas and estimates for the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$, other are new, such as the key combinatorial argument that we develop in Sections 3.2 to 3.6.

Kac–Rice formulas are a classical tool in the study of the number of real roots of random polynomials (see [1, 5] for example). More generally, they allow to express the moments of local quantities associated with the level sets of a Gaussian process, such as their volume or their Euler characteristic, only in terms of the correlation function of the process. Even if these formulas are well-known, it is the first time, to the best of our knowledge, that they are used to compute the

exact asymptotics of central moments of order greater than 3. The Kac–Rice formulas we use in this paper were proved in [2]. We recall them in Proposition 2.24. They allow us to write the p -th central moment $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ as the integral over M^p of $\phi : (x_1, \dots, x_p) \mapsto \prod_{i=1}^p \phi_i(x_i)$ times some density function \mathcal{D}_d^p . Here we are cheating a bit. What we just said would be true if the diagonal part of Z_d^p was negligible in Z_d^p , which is not the case since Z_d is almost surely a discrete set. The correct statement (see Lemma 3.1) is more complicated because of additional terms accounting for the contribution of the diagonal part of Z_d^p . For the purpose of this sketch of proof, let us pretend that we have indeed:

$$m_p(\nu_d)(\phi_1, \dots, \phi_p) = \int_{M^p} \phi \mathcal{D}_d^p |dV_M|^p.$$

This simplified situation is enough to understand the main ideas of the proof.

The density \mathcal{D}_d^p is a polynomial in the Kac–Rice densities $(\mathcal{R}_d^k)_{1 \leq k \leq p}$ appearing in Definition 2.22. As such, it only depends on the correlation function of the Gaussian process $(s_d(x))_{x \in M}$, which is the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. This kernel admits a universal local scaling limit at scale $d^{-\frac{1}{2}}$, which is exponentially decreasing (cf. [21, 22]). In [2], the author used these Bergman kernel asymptotics and Olver multispaces (see [24]) to prove estimates for the $(\mathcal{R}_d^k)_{1 \leq k \leq p}$ in the large d limit. These key estimates are recalled in Propositions 2.25 and 2.27 below. They allow us to show that $\mathcal{D}_d^p(x) = O(d^{\frac{p}{2}})$, uniformly in $x \in M^p$. Moreover, we show that $\mathcal{D}_d^p(x) = O(d^{\frac{p}{4}-1})$, uniformly in $x \in M^p$ such that one of the components of x is far from the others. By this we mean that $x = (x_1, \dots, x_p)$ and there exists $i \in \{1, \dots, p\}$ such that, for all $j \neq i$, the distance from x_i to x_j is larger than $b_p \frac{\ln d}{\sqrt{d}}$, where $b_p > 0$ is some well-chosen constant.

In order to understand the integral of $\phi \mathcal{D}_d^p$, we split M^p as follows. For $x = (x_1, \dots, x_p) \in M^p$, we define a graph (see Definition 3.2) whose vertices are the integers $\{1, \dots, p\}$, with an edge between i and j if and only if $i \neq j$ and the distance from x_i to x_j is less than $b_p \frac{\ln d}{\sqrt{d}}$. The connected components of this graph yield a partition $\mathcal{I}(x) \in \mathcal{P}_p$ (see Definition 3.3) encoding how the $(x_i)_{1 \leq i \leq p}$ are clustered in M , at scale $d^{-\frac{1}{2}}$. Denoting by $M_{\mathcal{I}}^p = \{x \in M^p \mid \mathcal{I}(x) = \mathcal{I}\}$, we have:

$$m_p(\nu_d)(\phi_1, \dots, \phi_p) = \sum_{\mathcal{I} \in \mathcal{P}_p} \int_{M_{\mathcal{I}}^p} \phi \mathcal{D}_d^p |dV_M|^p.$$

Thanks to our estimates on \mathcal{D}_d^p , we show that if \mathcal{I} contains a singleton then the integral over $M_{\mathcal{I}}^p$ is $O(d^{\frac{p}{4}-1})$. Hence it contributes only an error term in Theorem 1.12. Moreover, denoting by $|\mathcal{I}|$ the cardinality of \mathcal{I} (i.e. the number of clusters), the volume of $M_{\mathcal{I}}^p$ is $O(d^{\frac{|\mathcal{I}|p}{2}} (\ln d)^p)$. Hence, the integral over $M_{\mathcal{I}}^p$ is $O(d^{\frac{|\mathcal{I}|p}{2}} (\ln d)^p)$. If $|\mathcal{I}| < \frac{p}{2}$, this is also an error term in Theorem 1.12. Thus, the main contribution in $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ comes from the integral of $\phi \mathcal{D}_d^p$ over the pieces $M_{\mathcal{I}}^p$ indexed by partitions $\mathcal{I} \in \mathcal{P}_p$ without singletons and such that $|\mathcal{I}| \geq \frac{p}{2}$. These are exactly the partitions into pairs of $\{1, \dots, p\}$. Finally, if $\mathcal{I} = \{\{a_i, b_i\} \mid 1 \leq i \leq \frac{p}{2}\} \in \mathcal{PP}_p$, we prove the contribution of the integral over $M_{\mathcal{I}}^p$ to be equal to the product of covariances $\prod_{i=1}^{\frac{p}{2}} m_2(\nu_d)(\phi_{a_i}, \phi_{b_i})$, up to an error term. When all the test-functions $(\phi_i)_{1 \leq i \leq p}$ are equal, the p -th moment μ_p of the standard Gaussian distribution appears as the cardinality of the set of partitions of $\{1, \dots, p\}$ into pairs.

Concerning the corollaries, Corollaries 1.14 and 1.15 follow from Theorem 1.12 and Markov’s Inequality for the $2p$ -th moment. The strong Law of Large Number (Theorem 1.7) is deduced from Theorem 1.12 for $p = 6$ by a Borel–Cantelli type argument. The Central Limit Theorem (Theorem 1.9) for the linear statistics is obtained by the method of moments. The functional version of this Central Limit Theorem is then obtained by the Lévy–Fernique Theorem (cf. [11]), which is an extension of Lévy’s Continuity Theorem adapted to generalized random processes.

Higher dimension. In this paper, we are concerned with the real roots of a random polynomial (or a random section) in an ambient space of dimension 1. There is a natural higher dimensional analogue of this problem. Namely, one can consider the common zero set $Z_d \subset \mathbb{R}P^n$ of r independent real Kostlan polynomials in $n+1$ variables, where $r \in \{1, \dots, n\}$. More generally, we consider the real zero set Z_d of a random real section of $\mathcal{E} \otimes \mathcal{L}^d \rightarrow \mathcal{X}$ in the complex Fubini–Study model, where \mathcal{X} is a real projective manifold of complex dimension n whose real locus M is non-empty,

\mathcal{L} is a positive line bundle as above, and \mathcal{E} is a rank r real Hermitian bundle with $1 \leq r \leq n$. Then, for d large enough, Z_d is almost surely a smooth closed submanifold of codimension r in the smooth closed n -dimensional manifold M . In this setting, M is equipped with a natural Riemannian metric that induces a volume measure $|dV_M|$ on M and a volume measure ν_d on Z_d . As in the 1-dimensional case, ν_d is an almost surely well-defined random Radon measure on M . In this higher dimensional setting, we have the following analogues of Theorem 1.1 and 1.3 (see [18, 19, 20]):

$$\begin{aligned} \forall \phi \in \mathcal{C}^0(M), \quad \mathbb{E}[\langle \nu_d, \phi \rangle] &= d^{\frac{r}{2}} \frac{\text{Vol}(\mathbb{S}^{n-r})}{\text{Vol}(\mathbb{S}^n)} \int_M \phi |dV_M| + \|\phi\|_\infty O(d^{\frac{r}{2}-1}), \\ \forall \phi_1, \phi_2 \in \mathcal{C}^0(M), \quad m_2(\nu_d)(\phi_1, \phi_2) &= d^{r-\frac{n}{2}} \sigma_{n,r}^2 \int_M \phi_1 \phi_2 |dV_M| + o(d^{r-\frac{n}{2}}), \end{aligned}$$

where $\sigma_{n,r} > 0$ is a universal constant depending only on n and r . In [20], Letendre and Puchol proved some analogues of Corollaries 1.14 and 1.15 for any n and r . They also showed that the strong Law of Large Numbers (Theorem 1.7) holds if $n \geq 3$.

Most of the proof of Theorem 1.12 is valid in any dimension and codimension. In fact, the combinatorics are simpler when $r < n$. The only things we are missing, in order to prove the analogue of Theorem 1.12 for any n and r , are higher dimensional versions of Propositions 2.25 and 2.27. The proofs of these propositions (see [2]) rely on the compactness of Olver multispaces, which holds when $n = 1$ but fails for $n > 1$. This seems to be only a technical obstacle and the authors are currently working toward the following.

Conjecture. *Let s_d be a random section in the complex Fubini–Study model in dimension n and codimension $r \in \{1, \dots, n\}$. Let ν_d denote the volume measure of the real zero set of s_d . For all $p \geq 3$, for all $\phi_1, \dots, \phi_p \in \mathcal{C}^0(M)$, the following holds as $d \rightarrow +\infty$:*

$$m_p(\nu_d)(\phi_1, \dots, \phi_p) = d^{\frac{p}{2}(r-\frac{n}{2})} \sigma_{n,r}^p \sum_{\{a_i, b_i\}_{1 \leq i \leq \frac{p}{2}} \in \mathcal{P}\mathcal{P}_p} \prod_{i=1}^{\frac{p}{2}} \left(\int_M \phi_{a_i} \phi_{b_i} |dV_M| \right) + o(d^{\frac{p}{2}(r-\frac{n}{2})}).$$

In particular, for all $\phi \in \mathcal{C}^0(M)$, $m_p(\langle \nu_d, \phi \rangle) = \mu_p d^{\frac{p}{2}(r-\frac{n}{2})} \sigma_{n,r}^p \left(\int_M \phi^2 |dV_M| \right)^{\frac{p}{2}} + o(d^{\frac{p}{2}(r-\frac{n}{2})}$.

Proving this conjecture for $n = 2$ and $p = 4$ is enough to prove that the strong Law of Large Numbers (Theorem 1.7) holds for $n = 2$, which is the only missing case. This conjecture also implies the Central Limit Theorem (Theorem 1.9) in dimension n and codimension r , with the same proof as the one given in Section 4.2. Note that a Central Limit Theorem for the volume of the common zero set of r Kostlan polynomials in $\mathbb{R}P^n$ was proved in [3, 4].

Other related works. The complex roots of complex Kostlan polynomials have been studied in relation with Physics in [9]. More generally, complex zeros of random holomorphic sections of positive bundles over projective manifolds were studied in [26] and some subsequent papers by the same authors. In [26], they computed the asymptotics of the expected current of integration over the complex zeros of such a random section, and proved a Law of Large Numbers similar to Theorem 1.7. In [27], they obtained a variance estimate for this random current, and proved that it satisfies a Central Limit Theorem. This last paper extends the results of [28] for the complex roots of a family of random polynomials, including elliptic ones. In [8], Bleher, Shiffman and Zelditch studied the p -points zero correlation function associated with random holomorphic sections. These functions are the Kac–Rice densities for the non-central p -th moment of the linear statistics in the complex case. The results of [2] are also valid in the 1-dimensional complex case, see [2, Section 6]. Thus, Theorem 1.12 can be extended to the complex case, with the same proof.

In Corollaries 1.14 and 1.15, we deduce from Theorem 1.12 some concentration in probability, faster than any negative power of d . However, our results are not precise enough to prove that this concentration is exponentially fast in d . In order to obtain such a large deviation estimate, one would need to investigate how the constants involved in the error terms in Theorem 1.12 grow with p . Some large deviations estimates are known for the complex roots of random polynomials. As far as real roots are concerned, the only result of this kind we are aware of is [6].

The Central Limit Theorem (Theorem 1.9) was already known for the roots of Kostlan polynomials [10]. In the wider context of random real geometry, Central Limit Theorems are known in several settings, see [3, 4, 25] and the references therein. The proofs of all these results rely on Wiener chaos techniques developed by Kratz–Leòn [17]. So far, these techniques have only been applied in highly symmetric ambient spaces (spheres, projective spaces and tori), where one considers the zero set of a random process satisfying some stationarity assumption. Namely, one needs the distribution of the random process to be invariant under an isometry group that acts transitively on the unit tangent bundle of the ambient space. Our proof of Theorem 1.9 follows a different path, that allows us to consider non-stationary processes.

For real zeros in ambient dimension $n = 1$, Nazarov and Sodin [23] proved a strong Law of Large Numbers, as $R \rightarrow +\infty$, for the number of zeros of a Gaussian process lying in the interval $[-R, R]$. Finally, to the best of our knowledge, Theorem 1.12 gives the first precise estimate for the central moments of the number of real zeros of a family of random processes.

Organization of the paper. This paper is organized as follows. In Section 2 we introduce the object of our study and recall some useful previous results. More precisely, we introduce our geometric framework in Section 2.1. In Section 2.2 we introduce various notations that will allow us to make sense of the combinatorics involved in our problem. The random measures we study are defined in Section 2.3. Finally, we state the Kac–Rice formulas for higher moments in Section 2.4 and we recall several results from [2] concerning the density functions appearing in these formulas. In Section 3, we prove our main result, that is the moments estimates of Theorem 1.12. Section 4 is concerned with the proofs of the corollaries of Theorem 1.12. We prove the Law of Large Numbers (Theorem 1.7) in Section 4.1, the Central Limit Theorem (Theorem 1.9) in Section 4.2 and the remaining corollaries (Corollaries 1.14 and 1.15) in Section 4.3.

Acknowledgments. Thomas Letendre thanks Hugo Vanneville for a useful discussion about the method of moments and for finding the reference [7]. He is also grateful to Julien Fageot for inspiring conversations about generalized random processes. Finally, the authors thank Damien Gayet and Jean-Yves Welschinger for their unwavering support.

2 Framework and background

2.1 Geometric setting

In this section, we introduce our geometric framework, which is the same as that of [2, 13, 18, 19, 20].

- Let $(\mathcal{X}, c_{\mathcal{X}})$ be a smooth real projective curve, that is a smooth complex projective manifold \mathcal{X} of complex dimension 1, equipped with an anti-holomorphic involution $c_{\mathcal{X}}$. We denote by M the real locus of \mathcal{X} , that is the set of fixed points of $c_{\mathcal{X}}$. Throughout the paper, we assume that M is not empty. In this case, M is a smooth compact submanifold of \mathcal{X} of real dimension 1, without boundary.
- Let $(\mathcal{E}, c_{\mathcal{E}})$ and $(\mathcal{L}, c_{\mathcal{L}})$ be two real holomorphic line bundles over $(\mathcal{X}, c_{\mathcal{X}})$. Denoting by $\pi_{\mathcal{E}}$ (resp. $\pi_{\mathcal{L}}$) the bundle projection, this means that $\mathcal{E} \rightarrow \mathcal{X}$ (resp. $\mathcal{L} \rightarrow \mathcal{X}$) is an holomorphic line bundle such that $\pi_{\mathcal{E}} \circ c_{\mathcal{E}} = c_{\mathcal{X}} \circ \pi_{\mathcal{E}}$ (resp. $\pi_{\mathcal{L}} \circ c_{\mathcal{L}} = c_{\mathcal{X}} \circ \pi_{\mathcal{L}}$) and that $c_{\mathcal{E}}$ (resp. $c_{\mathcal{L}}$) is anti-holomorphic and fiberwise \mathbb{C} -anti-linear. For any $d \in \mathbb{N}$, we denote by $c_d = c_{\mathcal{E}} \otimes c_{\mathcal{L}}^d$. Then, $(\mathcal{E} \otimes \mathcal{L}^d, c_d)$ is a real holomorphic line bundle over $(\mathcal{X}, c_{\mathcal{X}})$.
- We equip \mathcal{E} with a real Hermitian metric $h_{\mathcal{E}}$. That is $(\mathcal{E}, h_{\mathcal{E}})$ is an Hermitian line bundle, and moreover $c_{\mathcal{E}}^* h_{\mathcal{E}} = \overline{h_{\mathcal{E}}}$. Similarly, let $h_{\mathcal{L}}$ denote a real Hermitian metric on \mathcal{L} . For all $d \in \mathbb{N}$, we denote by $h_d = h_{\mathcal{E}} \otimes h_{\mathcal{L}}^d$, which defines a real Hermitian metric on $\mathcal{E} \otimes \mathcal{L}^d \rightarrow \mathcal{X}$. We assume that $(\mathcal{L}, h_{\mathcal{L}})$ is positive, in the sense that its curvature form ω is a Kähler form. Recall that ω is locally defined as $\frac{1}{2i} \partial \bar{\partial} \ln(h_{\mathcal{L}}(e, e))$, where e is any local holomorphic frame of \mathcal{L} . The Kähler structure defines a Riemannian metric $g = \omega(\cdot, i\cdot)$ on \mathcal{X} , hence on M . The Riemannian volume form on \mathcal{X} associated with g is simply ω . We denote by $|dV_M|$ the arc-length measure on M associated with g . For all $k \in \mathbb{N}^*$, we denote by $|dV_M|^k$ the product measure on M^k .

- For any $d \in \mathbb{N}$, we denote by $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ the space of global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$. This is a complex vector space of complex dimension N_d . By the Riemann–Roch Theorem, N_d is finite and diverges to infinity as $d \rightarrow +\infty$. We denote by:

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \{s \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \mid s \circ c_{\mathcal{X}} = c_d \circ s\}$$

the space of global real holomorphic sections of $\mathcal{E} \rightarrow \mathcal{L}^d$, which is a real vector space of real dimension N_d . Let $x \in M$, the fiber $(\mathcal{E} \otimes \mathcal{L}^d)_x$ is a complex line equipped with a \mathbb{C} -anti-linear involution $c_d(x)$. We denote by $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ the set of fixed points of $c_d(x)$, which is a real line. Then, $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \rightarrow M$ is a real line bundle and, for any $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, the restriction of s to M is a smooth section of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \rightarrow M$.

- The volume form ω and the Hermitian metric h_d induce an Hermitian L^2 -inner product $\langle \cdot, \cdot \rangle$ on $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. It is defined by:

$$\forall s_1, s_2 \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d), \quad \langle s_1, s_2 \rangle = \int_{\mathcal{X}} h_d(s_1, s_2) \omega. \quad (2.1)$$

The restriction of this inner product to $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is a Euclidean inner product.

- For any section $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we denote by $Z_s = s^{-1}(0) \cap M$ its real zero set. Since s is holomorphic, if $s \neq 0$ its zeros are isolated. In this case, Z_s is finite by compactness of M , and we denote by $\nu_s = \sum_{x \in Z_s} \delta_x$, where δ_x stands for the unit Dirac mass at $x \in M$. The measure ν_s is called the *counting measure* of Z_s . It is a Radon measure, that is a continuous linear form on the space $(\mathcal{C}^0(M), \|\cdot\|_{\infty})$ of continuous functions equipped with the sup-norm. It acts on continuous functions by: for all $\phi \in \mathcal{C}^0(M)$, $\langle \nu_s, \phi \rangle = \sum_{x \in Z_s} \phi(x)$.

Example 2.1 (Kostlan scalar product). We conclude this section by giving an example of our geometric setting. We consider $\mathcal{X} = \mathbb{C}P^1$, equipped with the conjugation induced by the one in \mathbb{C}^2 . Its real locus is $M = \mathbb{R}P^1$. We take \mathcal{E} to be trivial and $\mathcal{L} = \mathcal{O}(1)$, the dual of the tautological line bundle $\{(v, x) \in \mathbb{C}^2 \times \mathbb{C}P^1 \mid v \in x\}$. Both \mathcal{E} and \mathcal{L} are canonically real Hermitian line bundle over $\mathbb{C}P^1$, and the curvature of \mathcal{L} is the Fubini–Study form, normalized so that $\text{Vol}(\mathbb{C}P^1) = \pi$. The corresponding Riemannian metric on $\mathbb{R}P^1$ is the quotient of the metric on the Euclidean unit circle, so that the length of $\mathbb{R}P^1$ is π . In this setting, $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ (resp. $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$) is the space of homogeneous polynomials of degree d in two variables with complex (resp. real) coefficients. If $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is such a polynomial, then Z_s is the set of its roots in $\mathbb{R}P^1$. Finally, the inner product defined by Equation (2.1) is the one that makes the family $\left\{ \sqrt{\binom{d}{k}} X^k Y^{d-k} \mid 0 \leq k \leq d \right\}$ an orthonormal basis.

2.2 Partitions, products and diagonal inclusions

In this section, we introduce some notations that will be useful throughout the paper, in particular to sort out the combinatorics involved in the proof of Theorem 1.12 (see Section 3). Recall that we already defined the set \mathcal{P}_A (resp. \mathcal{P}_p) of partitions of a finite set A (resp. of $\{1, \dots, p\}$) and the set \mathcal{PP}_A (resp. \mathcal{PP}_p) of its partitions into pairs (see Definition 1.11). In all this section, M denotes a smooth manifold.

Definition 2.2. Let B be a finite set. For all $\mathcal{I} \in \mathcal{P}_B$, we denote by $\mathcal{S}_{\mathcal{I}}$ the set of subsets of B adapted to \mathcal{I} , that is:

$$\mathcal{S}_{\mathcal{I}} = \{A \subset B \mid \forall I \in \mathcal{I}, \text{ if } \text{Card}(I) \geq 2, \text{ then } I \subset A\}.$$

Moreover, for all $A \subset B$, we define $f_A : \mathcal{I} \mapsto \mathcal{I} \setminus \{\{i\} \mid i \notin A\}$, from $\{\mathcal{I} \in \mathcal{P}_B \mid A \in \mathcal{S}_{\mathcal{I}}\}$ to \mathcal{P}_A . Finally, we define

$$F_B : \{(A, \mathcal{I}) \mid \mathcal{I} \in \mathcal{P}_B, A \in \mathcal{S}_{\mathcal{I}}\} \mapsto \{(A, \mathcal{I}) \mid A \subset B, \mathcal{I} \in \mathcal{P}_A\}$$

by $F_B(A, \mathcal{I}) = (A, f_A(\mathcal{I}))$ for all (A, \mathcal{I}) .

Lemma 2.3. For all finite set B , the map F_B is a bijection.

Proof. Let $\mathcal{I} \in \mathcal{P}_B$. By definition, if $A \in \mathcal{S}_{\mathcal{I}}$, then \mathcal{I} contains all the singletons $(\{i\})_{i \notin A}$. Hence f_A is well-defined. Moreover, for all $A \subset B$, the map f_A is a bijection and $f_A^{-1} : \mathcal{I} \mapsto \mathcal{I} \sqcup \{\{i\} \mid i \notin A\}$ from \mathcal{P}_A to $\{\mathcal{I} \in \mathcal{P}_B \mid A \in \mathcal{S}_{\mathcal{I}}\}$. Thus, F_B is also well-defined and is a bijection whose inverse is $(A, \mathcal{I}) \mapsto (A, f_A^{-1}(\mathcal{I}))$. \square

Notations 2.4. Let A be a finite set.

- We denote by $\text{Card}(A)$ or by $|A|$ the cardinality of A .
- We denote by M^A the Cartesian product of $|A|$ copies of M , indexed by the elements of A .
- A generic element of M^A is denoted by $\underline{x}_A = (x_a)_{a \in A}$. If $B \subset A$ we denote by $\underline{x}_B = (x_a)_{a \in B}$.
- Let $(\phi_a)_{a \in A}$ be continuous functions on M , we denote by $\phi_A = \boxtimes_{a \in A} \phi_a$ the function on M^A defined by: $\phi_A(\underline{x}_A) = \prod_{a \in A} \phi_a(x_a)$, for all $\underline{x}_A = (x_a)_{a \in A} \in M^A$.

If A is of the form $\{1, \dots, k\}$ with $k \in \mathbb{N}^*$, we use the simpler notations x for \underline{x}_A and ϕ for ϕ_A .

Definition 2.5. Let A be a finite set, we denote by Δ_A the *large diagonal* of M^A , that is:

$$\Delta_A = \left\{ (x_a)_{a \in A} \in M^A \mid \exists a, b \in A \text{ such that } a \neq b \text{ and } x_a = x_b \right\}.$$

Moreover, for all $\mathcal{I} \in \mathcal{P}_A$, we denote by

$$\Delta_{A, \mathcal{I}} = \left\{ (x_a)_{a \in A} \in M^A \mid \forall a, b \in A, (x_a = x_b \iff \exists I \in \mathcal{I} \text{ such that } a \in I \text{ and } b \in I) \right\}.$$

If $A = \{1, \dots, k\}$, we use the simpler notations Δ_k for Δ_A , and $\Delta_{k, \mathcal{I}}$ for $\Delta_{A, \mathcal{I}}$.

Definition 2.6 (Diagonal inclusions). Let A be a finite set and let $\mathcal{I} \in \mathcal{P}_A$. The *diagonal inclusion* $\iota_{\mathcal{I}} : M^{\mathcal{I}} \rightarrow M^A$ is the function defined, for all $\underline{y}_{\mathcal{I}} = (y_I)_{I \in \mathcal{I}} \in M^{\mathcal{I}}$, by $\iota_{\mathcal{I}}(\underline{y}_{\mathcal{I}}) = (x_a)_{a \in A}$, where for all $I \in \mathcal{I}$, for all $a \in I$, we set $x_a = y_I$.

Remark 2.7. With these definitions, we have $M^A = \bigsqcup_{\mathcal{I} \in \mathcal{P}_A} \Delta_{A, \mathcal{I}}$ and $\Delta_A = \bigsqcup_{\mathcal{I} \in \mathcal{P}_A \setminus \{\mathcal{I}_0\}} \Delta_{A, \mathcal{I}}$, where we denoted by $\mathcal{I}_0 = \{\{a\} \mid a \in A\}$. Moreover, $\iota_{\mathcal{I}}$ is a smooth diffeomorphism from $M^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}$ onto $\Delta_{A, \mathcal{I}} \subset M^A$. Note that $\Delta_{A, \mathcal{I}_0}$ is the configuration space $M^A \setminus \Delta_A$ of $|A|$ distinct points in M . In the following, we avoid using the notation $\Delta_{A, \mathcal{I}_0}$ and use $M^A \setminus \Delta_A$ instead.

Let us now go back to the setting of Section 2.1, in which M is the real locus of the projective manifold \mathcal{X} . Let $d \in \mathbb{N}$ and let $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \setminus \{0\}$. In Section 2.1, we defined the counting measure ν_s of the real zero set Z_s of s . More generally, for any finite set A , we can define the counting measure of $Z_s^A \subset M^A$.

Definition 2.8. Let $d \in \mathbb{N}$ and let A be a finite set. For all $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \setminus \{0\}$, we denote by:

$$\nu_s^A = \sum_{x \in Z_s^A} \delta_x \quad \text{and} \quad \tilde{\nu}_s^A = \sum_{x \in Z_s^A \setminus \Delta_A} \delta_x,$$

where δ_x is the unit Dirac mass at $x \in M^A$ and Δ_A is defined by Definition 2.5. Both ν_s^A and $\tilde{\nu}_s^A$ are Radon measure on M^A . They act on $\mathcal{C}^0(M^A)$ as follows: for any $\phi \in \mathcal{C}^0(M^A)$,

$$\langle \nu_s^A, \phi \rangle = \sum_{x \in Z_s^A} \phi(x) \quad \text{and} \quad \langle \tilde{\nu}_s^A, \phi \rangle = \sum_{x \in Z_s^A \setminus \Delta_A} \phi(x).$$

As usual, if $A = \{1, \dots, k\}$, we denote ν_s^k for ν_s^A and $\tilde{\nu}_s^k$ for $\tilde{\nu}_s^A$.

Lemma 2.9. Let $d \in \mathbb{N}$ and let A be a finite set. For any $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \setminus \{0\}$, we have:

$$\nu_s^A = \sum_{\mathcal{I} \in \mathcal{P}_A} (\iota_{\mathcal{I}})_* (\tilde{\nu}_s^{\mathcal{I}}).$$

Proof. Recall that $M^A = \bigsqcup_{\mathcal{I} \in \mathcal{P}_A} \Delta_{A, \mathcal{I}}$. For all $\phi \in \mathcal{C}^0(M^A)$, we have:

$$\langle \nu_s^A, \phi \rangle = \sum_{x \in Z_s^A} \phi(x) = \sum_{\mathcal{I} \in \mathcal{P}_A} \left(\sum_{x \in Z_s^A \cap \Delta_{A, \mathcal{I}}} \phi(x) \right).$$

Let $\mathcal{I} \in \mathcal{P}_A$, recall that $\iota_{\mathcal{I}}$ defines a smooth diffeomorphism from $M^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}$ onto $\Delta_{A, \mathcal{I}}$. Moreover, $\iota_{\mathcal{I}}(Z_s^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}) = Z_s^A \cap \Delta_{A, \mathcal{I}}$ (see Definition 2.5 and 2.6). Hence,

$$\sum_{x \in Z_s^A \cap \Delta_{A, \mathcal{I}}} \phi(x) = \sum_{y \in Z_s^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}} \phi(\iota_{\mathcal{I}}(y)) = \langle \tilde{\nu}_s^{\mathcal{I}}, \phi \circ \iota_{\mathcal{I}} \rangle = \langle (\iota_{\mathcal{I}})_* (\tilde{\nu}_s^{\mathcal{I}}), \phi \rangle.$$

Thus, for all $\phi \in \mathcal{C}^0(M^A)$, we have: $\langle \nu_s^A, \phi \rangle = \langle \sum_{\mathcal{I} \in \mathcal{P}_A} (\iota_{\mathcal{I}})_* (\tilde{\nu}_s^{\mathcal{I}}), \phi \rangle$. \square

2.3 Zeros of random real sections

Let us now introduce the main object of our study: the random Radon measure ν_d encoding the real zeros of a random real section $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. The model of random real sections we study is often referred to as the complex Fubini–Study model. It was introduced in this generality by Gayet and Welschinger in [12]. This model is the real counterpart of the model of random holomorphic sections studied by Shiffman and Zelditch in [26] and subsequent articles.

Definition 2.10. Let $(V, \langle \cdot, \cdot \rangle)$ be a Euclidean space of dimension N , and let Λ denote a positive self-adjoint operator on V . Recall that a random vector X in V is said to be a *centered Gaussian* with *variance operator* Λ if its distribution admits the following density with respect to the normalized Lebesgue measure:

$$v \mapsto \frac{1}{(2\pi)^{\frac{N}{2}} \det(\Lambda)^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \langle v, \Lambda^{-1}v \rangle\right).$$

We denote by $X \sim \mathcal{N}(0, \Lambda)$ the fact that X follows this distribution. If $X \sim \mathcal{N}(0, \text{Id})$, where Id is the identity of V , we say that X is a *standard Gaussian vector* in V .

Remark 2.11. Recall that if (e_1, \dots, e_N) is an orthonormal basis of V , a random vector $X \in V$ is a standard Gaussian vector if and only if $X = \sum_{i=1}^N a_i e_i$ where the $(a_i)_{1 \leq i \leq N}$ are independent identically distributed $\mathcal{N}(0, 1)$ real random variables.

In the setting of Section 2.1, for any $d \in \mathbb{N}$, the space $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is endowed with the Euclidean inner product $\langle \cdot, \cdot \rangle$ defined by Equation (2.1). We denote by s_d a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Almost surely, $s_d \neq 0$, hence its real zero set and the associated counting measure are well-defined. For simplicity, we denote by $Z_d = Z_{s_d}$ and by $\nu_d = \nu_{s_d}$. Similarly, for any finite set A , we denote by $\nu_d^A = \nu_{s_d^A}$ and by $\tilde{\nu}_d^A = \tilde{\nu}_{s_d^A}$ (see Definition 2.8).

Example 2.12 (Kostlan polynomials). In the case described in Example 2.1, we have $M = \mathbb{R}P^1$, and $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is the set of real homogeneous polynomials of degree d in two variables, endowed with the inner product for which $\left\{ \sqrt{\binom{d}{k}} X^k Y^{d-k} \mid 0 \leq k \leq d \right\}$ is an orthonormal basis.

In this case, the standard Gaussian vector $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is an homogeneous Kostlan polynomial of degree d .

Lemma 2.13. *Let $d \geq 1$ and let $s_d \sim \mathcal{N}(0, \text{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. For all $\phi \in \mathcal{C}^0(M)$, the random variable $\langle Z_d, \phi \rangle$ is bounded. In particular, it admits finite moments of any order.*

Proof. We have $|\langle \nu_d, \phi \rangle| = \left| \sum_{x \in Z_d} \phi(x) \right| \leq \|\phi\|_{\infty} \text{Card}(Z_d)$. Hence it is enough to prove that $\text{Card}(Z_d)$ is a bounded random variable.

The cardinality of Z_d is the number of zeros of s_d in M , which is smaller than the number of zeros of s_d in \mathcal{X} . Now, almost surely, $s_d \neq 0$, and the complex zero set of s_d defines a divisor which is Poincaré-dual to the first Chern class of $\mathcal{E} \otimes \mathcal{L}^d$ (see [14, pp. 136 and 141]). Hence, almost surely:

$$\text{Card}(Z_d) \leq \text{Card} \{x \in \mathcal{X} \mid s_d(x) = 0\} \leq \int_{\mathcal{X}} c_1(\mathcal{E} \otimes \mathcal{L}^d) = d \cdot \deg(\mathcal{L}) + \deg(\mathcal{E}). \quad \square$$

Remark 2.14. In the case of polynomials, the proof is clearer: the number of real roots of a non-zero polynomial of degree d is bounded by the number of its complex roots, which is at most d .

2.4 Kac-Rice formulas and density functions

In this section we recall some important facts about Kac-Rice formulas. These formulas are classical tools in the study of moments of local quantities such as the cardinality, or more generally the volume, of the zero set of a smooth Gaussian process. Classical references for this material are [1, 5]. With a more geometric point of view, the following formulas were proved and used in [2, 13, 20], see also [18].

Remark 2.15. In some of the papers we refer to in this section, the line bundle \mathcal{E} is taken to be trivial. That is the authors considers random real sections of \mathcal{L}^d instead of $\mathcal{E} \otimes \mathcal{L}^d$. As we already explained (see Remark 1.6), the proofs of the results we cite below rely on asymptotics for the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$, as $d \rightarrow +\infty$. These asymptotics do not depend on \mathcal{E} . Hence the results established in the case of a trivial line bundle \mathcal{E} can be extended to the case of a general \mathcal{E} without modifying the proofs.

Definition 2.16. Let V and V' be two Euclidean spaces of dimension N and N' respectively. Let $L : V \rightarrow V'$ be a linear map and let L^* denote its adjoint. We denote by $|\det^\perp(L)| = \det(LL^*)^{\frac{1}{2}}$ the *Jacobian* of L .

Remark 2.17. If $N' \leq N$, an equivalent definition of the Jacobian is the following: $|\det^\perp(L)|$ is the absolute value of the determinant of the matrix of L , restricted to $\ker(L)^\perp$, in orthonormal bases. Note that, in any case, $|\det^\perp(L)| > 0$ if and only if L is surjective.

Let us now consider the geometric setting of Section 2.1.

Definition 2.18. Let $k \in \mathbb{N}^*$, for all $d \in \mathbb{N}$, for all $x = (x_1, \dots, x_k) \in M^k$, we denote by:

$$\text{ev}_x^d : \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \longrightarrow \bigoplus_{i=1}^k \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_{x_i}$$

the *evaluation map* at x , defined by $s \mapsto (s(x_1), \dots, s(x_k))$.

Lemma 2.19 ([2, Proposition 3.11]). *Let $k \in \mathbb{N}^*$, there exists $d_k \in \mathbb{N}$ such that, for all $d \geq d_k$, for all $x \in M^k \setminus \Delta_k$, the evaluation map ev_x^d is surjective (i.e. $|\det^\perp(\text{ev}_x^d)| > 0$).*

Remark 2.20. The dimension N_d of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ does not depend on k , while the dimension of the target space of ev_x^d equals k . In particular, for any d , the linear map ev_x^d can only be surjective if $k \leq N_d$. This shows that $d_k \xrightarrow[k \rightarrow +\infty]{} +\infty$. Moreover, $(d_k)_{k \geq 1}$ is non-decreasing.

Remark 2.21. If $s_d \sim \mathcal{N}(0, \text{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, then $\text{ev}_x^d(s_d) = (s_d(x_1), \dots, s_d(x_k))$ is a centered Gaussian variable in $\bigoplus_{i=1}^k \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_{x_i}$ whose variance operator is $\text{ev}_x^d(\text{ev}_x^d)^*$. If $d \geq d_k$, then for all $x \in M^k \setminus \Delta_k$, this variance operator is positive and $\text{ev}_x^d(s_d)$ is non-degenerate.

Definition 2.22. Let $k \in \mathbb{N}^*$, for all $d \geq d_k$ (cf. Lemma 2.19), we define the *density function* $\mathcal{R}_d^k : M^k \setminus \Delta_k \rightarrow \mathbb{R}$ as follows:

$$\forall x = (x_1, \dots, x_k) \in M^k \setminus \Delta_k, \quad \mathcal{R}_d^k(x) = (2\pi)^{-\frac{k}{2}} \frac{\mathbb{E} \left[\prod_{i=1}^k \|\nabla_{x_i} s_d\| \middle| \text{ev}_x^d(s_d) = 0 \right]}{|\det^\perp(\text{ev}_x^d)|}.$$

Here, ∇ is any connection on $\mathcal{E} \otimes \mathcal{L}^d$ and $\mathbb{E} \left[\prod_{i=1}^k \|\nabla_{x_i} s_d\| \middle| \text{ev}_x^d(s_d) = 0 \right]$ stands for the conditional expectation of $\prod_{i=1}^k \|\nabla_{x_i} s_d\|$ given that $\text{ev}_x^d(s_d) = 0$.

Remark 2.23. Recall that if s is a section and $s(x) = 0$, then the derivative $\nabla_x s$ does not depend on the choice of the connection ∇ . In the conditional expectation appearing at the numerator in Definition 2.22, we only consider derivatives of s_d at the points x_1, \dots, x_k , under the condition that s_d vanishes at these points. Hence \mathcal{R}_d^k does not depend on the choice of ∇ .

We can now state the Kac-Rice formula we are interested in, see [2, Propositions 3.5 and 3.9]. See also [20, Theorem 5.5] in the case $k = 2$. Recall that the counting measure $\tilde{\nu}_d^k$ was defined by Definition 2.8.

Proposition 2.24 (Kac–Rice formula). *Let $k \in \mathbb{N}^*$ and let $d \geq d_k$. Let $s_d \sim \mathcal{N}(0, \text{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, for any $\phi \in C^0(M^k)$, we have:*

$$\mathbb{E}[\langle \tilde{\mathcal{V}}_d^k, \phi \rangle] = \int_{x \in M^k} \phi(x) \mathcal{R}_d^k(x) |dV_M|^k.$$

where \mathcal{R}_d^k is the density function defined by Definition 2.22.

Let $k \geq 2$ and $d \in \mathbb{N}$. If $x \in \Delta_k$, then the evaluation map ev_x^d can not be surjective. Hence, the continuous map $x \mapsto |\det^\perp(\text{ev}_x^d)|$ from M^k to \mathbb{R} vanishes on Δ_k , and one would expect \mathcal{R}_d^k to be singular along the diagonal. Yet, Ancona showed that one can extend continuously \mathcal{R}_d^k to the whole of M^k and that the extension vanishes on Δ_k , see [2, Theorem 5.7]. Moreover, he showed that $d^{-\frac{k}{2}} \mathcal{R}_d^k$ is bounded.

Proposition 2.25 ([2, Theorem 4.1]). *For any $k \in \mathbb{N}^*$, there exists a constant $C_k > 0$ such that, for all d large enough, for all $x \in M^k \setminus \Delta_k$, we have $d^{-\frac{k}{2}} \mathcal{R}_d^k(x) \leq C_k$.*

Let $d \geq 1$ and let $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a standard Gaussian. A fundamental idea in our problem is that the values (and more generally the k -jets) of s_d at two points x and $y \in M$ are “quasi-independent” if x and y are far from one another, at scale $d^{-\frac{1}{2}}$. More precisely, $(s_d(x))_{x \in M}$ defines a Gaussian process with values in $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)$ whose correlation kernel is the Bergman kernel e_d of $\mathcal{E} \otimes \mathcal{L}^d$. This kernel admits a universal local scaling limit at scale $d^{-\frac{1}{2}}$ around any point of \mathcal{X} (cf. [21, Theorem 4.2.1]). Moreover, the correlations are exponentially decreasing at scale $d^{-\frac{1}{2}}$, in the sense that there exists $C > 0$ such that $\|e_d(x, y)\| = O\left(\exp\left(-C\sqrt{d}\rho_g(x, y)\right)\right)$ as $d \rightarrow +\infty$, uniformly in (x, y) , where ρ_g is for the geodesic distance. Similar estimates hold for the derivatives of e_d with the same constant C . We refer to [22] for a precise statement. These facts were extensively used in [2, 19, 20] and we refer to these papers for a more detailed discussion of how estimates for the Bergman kernel are used in the context of random real algebraic geometry. An important consequence of these estimates that we use in the present paper is Proposition 2.27 below.

Definition 2.26. Let $p \in \mathbb{N}$, we denote by $b_p = \frac{1}{C} \left(1 + \frac{p}{4}\right)$, where $C > 0$ is the same as above. That is C is the constant appearing in the exponential in [22, Theorem 1, Equation (1.3)].

Proposition 2.27 ([2, Proposition 4.2]). *Let $p \geq 2$. Recall that ρ_g denotes the geodesic distance and that b_p is defined by Definition 2.26. The following holds uniformly for all $k \in \{2, \dots, p\}$, for all A and $B \subset \{1, \dots, k\}$ disjoint such that $A \sqcup B = \{1, \dots, k\}$, for all $x \in M^k \setminus \Delta_k$ such that for all $a \in A$ and $b \in B$ we have $\rho_g(x_a, x_b) \geq b_p \frac{\ln d}{\sqrt{d}}$:*

$$\mathcal{R}_d^k(x) = \mathcal{R}_d^{|A|}(\underline{x}_A) \mathcal{R}_d^{|B|}(\underline{x}_B) + O(d^{\frac{k}{2} - \frac{p}{4} - 1}).$$

Here we used the notations defined in Section 2.2 (see Notation 2.4), and by convention $\mathcal{R}_d^0 = 1$ for all d .

Proof. In the case $p = 2$, this is precisely what is used to prove [20, Proposition 5.25], see also the proof of [19, Proposition 4.12]. For $p \geq 2$, this was proved in [2, Proposition 4.2] with b_p replaced by b_0 and the error term replaced by $O(d^{\frac{k}{2} - 1})$. The same proof shows that the error term is $O(d^{\frac{k}{2} - \frac{p}{4} - 1})$ in our setting (cf. the estimates for the Bergman kernel discussed previously).

Note that in [2, Proposition 4.2], the proof is only given for $k = 2$ and $|A| = 1 = |B|$, for clarity of exposition. The proof of the general case follows the same lines, but there is a small subtlety. Indeed, we need to compute the asymptotics as $d \rightarrow +\infty$ of the numerator and the denominator of $\mathcal{R}_d^k(x)$ (see Definition 2.22), uniformly for x in some subdomain of $M^k \setminus \Delta_k$. For $k \geq 3$, it can happen that what we want to be the leading terms in these asymptotics vanish along the diagonal Δ_k . This leads to some difficulties as we need to understand the asymptotics of the ratio $\mathcal{R}_d^k(x)$ in the double limit $d \rightarrow +\infty$ and $x \rightarrow x_0 \in \Delta_k$. In order to avoid this problem, we need to use another expression of $\mathcal{R}_d^k(x)$, which is obtained using the Olver multispaces introduced in [2, Section 5]. This new expression is derived in [2, Proposition 5.21]. It is still a ratio of two functions depending on d and $x \in M^k \setminus \Delta_k$, but the leading terms in the asymptotics as $d \rightarrow +\infty$ of the numerator and the denominator are not singular along Δ_k (cf. [2, Propositions 5.22 and 5.23]). Using this new expression, the proof of [2, Proposition 4.2] is the same in the general case as in the case $k = 2$. \square

3 Asymptotics of the central moments

The goal of this section is to prove Theorem 1.12. In Section 3.1 we derive an integral expression of the central moments we want to estimate, see Lemma 3.1 below. Then, in Section 3.2, we define a decomposition of the manifolds M^A , where M is as in Section 2.1 and A is a finite set. In Sections 3.3, 3.4 and 3.5, we compute the contributions of the various pieces of the decomposition defined in Section 3.2 to the asymptotics of the integrals appearing in Lemma 3.1. Finally, we conclude the proof of Theorem 1.12 in Section 3.6.

3.1 An integral expression of the central moments

In this section, we derive an integral expression for the central moments $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ (see Definition 1.2). This expression follows from the Kac–Rice formulas of Section 2.4 and some combinatorics. Recall that $(d_k)_{k \geq 1}$ was defined in Lemma 2.19. In the following we use the notations introduced in Definition 2.2 and Notation 2.4.

Lemma 3.1. *Let $p \geq 2$ and let $\phi_1, \dots, \phi_p \in \mathcal{C}^0(M)$. For all $d \geq d_p$, we have:*

$$m_p(\nu_d)(\phi_1, \dots, \phi_p) = \sum_{\mathcal{I} \in \mathcal{P}_p} \int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}} (\iota_{\mathcal{I}}^* \phi)(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) |dV_M|^{|\mathcal{I}|},$$

where, for any finite set B , any $\mathcal{I} \in \mathcal{P}_B$ and any $\underline{x}_{\mathcal{I}} = (x_I)_{I \in \mathcal{I}} \in M^{\mathcal{I}}$,

$$\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \sum_{A \in \mathcal{S}_{\mathcal{I}}} (-1)^{|B| - |A|} \mathcal{R}_d^{f_A(\mathcal{I})}(\underline{x}_{f_A(\mathcal{I})}) \prod_{i \notin A} \mathcal{R}_d^1(x_{\{i\}}).$$

Proof. Developing the product, we get:

$$\begin{aligned} m_p(\nu_d)(\phi_1, \dots, \phi_p) &= \sum_{A \subset \{1, \dots, p\}} (-1)^{p - |A|} \mathbb{E} \left[\prod_{i \in A} \langle \nu_d, \phi_i \rangle \right] \prod_{i \notin A} \mathbb{E}[\langle \nu_d, \phi_i \rangle] \\ &= \sum_{A \subset \{1, \dots, p\}} (-1)^{p - |A|} \mathbb{E}[\langle \nu_d^A, \phi_A \rangle] \prod_{i \notin A} \mathbb{E}[\langle \nu_d, \phi_i \rangle] \\ &= \sum_{A \subset \{1, \dots, p\}} \sum_{\mathcal{I} \in \mathcal{P}_A} (-1)^{p - |A|} \mathbb{E}[\langle \tilde{\nu}_d^{\mathcal{I}}, \iota_{\mathcal{I}}^* \phi_A \rangle] \prod_{i \notin A} \mathbb{E}[\langle \nu_d, \phi_i \rangle], \end{aligned}$$

where the last equality comes from Lemma 2.9. By the Kac–Rice formulas of Proposition 2.24, this equals:

$$\sum_{A \subset \{1, \dots, p\}} \sum_{\mathcal{I} \in \mathcal{P}_A} (-1)^{p - |A|} \left(\int_{M^{\mathcal{I}}} (\iota_{\mathcal{I}}^* \phi_A) \mathcal{R}_d^{|\mathcal{I}|} |dV_M|^{|\mathcal{I}|} \right) \prod_{i \notin A} \left(\int_M \phi_i \mathcal{R}_d^1 |dV_M| \right).$$

By Lemma 2.3, we can exchange the two sums and obtain the following:

$$\begin{aligned} &\sum_{\mathcal{I} \in \mathcal{P}_p} \sum_{A \in \mathcal{S}_{\mathcal{I}}} (-1)^{p - |A|} \left(\int_{M^{f_A(\mathcal{I})}} (\iota_{f_A(\mathcal{I})}^* \phi_A) \mathcal{R}_d^{f_A(\mathcal{I})} |dV_M|^{f_A(\mathcal{I})} \right) \prod_{i \notin A} \left(\int_M \phi_i \mathcal{R}_d^1 |dV_M| \right) \\ &= \sum_{\mathcal{I} \in \mathcal{P}_p} \sum_{A \in \mathcal{S}_{\mathcal{I}}} (-1)^{p - |A|} \int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}} (\iota_{\mathcal{I}}^* \phi)(\underline{x}_{\mathcal{I}}) \mathcal{R}_d^{f_A(\mathcal{I})}(\underline{x}_{f_A(\mathcal{I})}) \prod_{i \notin A} \mathcal{R}_d^1(x_{\{i\}}) |dV_M|^{|\mathcal{I}|} \\ &= \sum_{\mathcal{I} \in \mathcal{P}_p} \int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}} (\iota_{\mathcal{I}}^* \phi)(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) |dV_M|^{|\mathcal{I}|}, \end{aligned}$$

which concludes the proof. \square

3.2 Cutting M^A into pieces

In this section, we define a way to cut the Cartesian product M^A into disjoint pieces, for any finite subset A . The upshot is to use this decomposition of M^A with $A = \mathcal{I} \in \mathcal{P}_p$ in order to get an estimate of the integral of $\mathcal{D}_d^{\mathcal{I}}$ over $M^{\mathcal{I}}$ (see Lemma 3.1).

Definition 3.2. Let A be a finite set and let $p \in \mathbb{N}$. For all $d \geq 1$, for all $x = (x_a)_{a \in A} \in M^A$, we define a graph $G_d^p(x)$ as follows.

- The vertices of $G_d^p(x)$ are the elements of A .
- Two vertices a and b are joined by an edge of $G_d^p(x)$ if and only if $a \neq b$ and $\rho_g(x_a, x_b) \leq b_p \frac{\ln d}{\sqrt{d}}$, where ρ_g is the geodesic distance in M and b_p is the constant defined by Definition 2.26.

Definition 3.3. Let A be a finite set and let $p \in \mathbb{N}$. For all $d \geq 1$ we define a map $\mathcal{I}_d^p : M^A \rightarrow \mathcal{P}_A$ as follows: for all $x \in M^A$, $\mathcal{I}_d^p(x)$ is the partition of A given by the connected components of $G_d^p(x)$. That is a and $b \in A$ belong to the same element of $\mathcal{I}_d^p(x)$ if and only if they are in the same connected component of $G_d^p(x)$.

Definition 3.4. Let A be a finite set, let $\mathcal{I} \in \mathcal{P}_A$, let $p \in \mathbb{N}$ and let $d \geq 1$, we define:

$$M_{\mathcal{I},d}^{A,p} = \{x \in M^A \mid \mathcal{I}_d^p(x) = \mathcal{I}\}.$$

Remarks 3.5. • For any $d \geq 1$ and any $p \in \mathbb{N}$, we have: $M^A = \bigsqcup_{\mathcal{I} \in \mathcal{P}_A} M_{\mathcal{I},d}^{A,p}$.

- Let $\mathcal{I} \in \mathcal{P}_A$ and let $(x_a)_{a \in A} \in M_{\mathcal{I},d}^{A,p}$. For all $I \in \mathcal{I}$, for any a and $b \in I$, the geodesic distance from x_a to x_b satisfies: $\rho_g(x_a, x_b) \leq (|I| - 1)b_p \frac{\ln d}{\sqrt{d}} \leq |A| b_p \frac{\ln d}{\sqrt{d}}$.

We conclude this section by bounding from above the volume of $M_{\mathcal{I},d}^{A,p}$.

Lemma 3.6. Let A be a finite set, let $\mathcal{I} \in \mathcal{P}_A$ and let $p \in \mathbb{N}$. For all $d \geq 1$, we have:

$$\text{Vol}(M_{\mathcal{I},d}^{A,p}) = O\left(\left(\frac{\ln d}{\sqrt{d}}\right)^{|A| - |\mathcal{I}|}\right).$$

Proof. For all $I \in \mathcal{I}$, we fix a preferred element a_I of I . Let $x = (x_a)_{a \in A} \in M_{\mathcal{I},d}^{A,p}$. For all $I \in \mathcal{I}$, for all $a \in I \setminus \{a_I\}$, the point x_a belongs to the geodesic ball of center x_{a_I} and radius $|A| b_p \frac{\ln d}{\sqrt{d}}$. Since M is 1-dimensional, this ball has volume $2|A| b_p \frac{\ln d}{\sqrt{d}}$. Hence,

$$\text{Vol}(M_{\mathcal{I},d}^{A,p}) \leq \text{Vol}(M)^{|\mathcal{I}|} \prod_{I \in \mathcal{I}} \left(2|A| b_p \frac{\ln d}{\sqrt{d}}\right)^{|I| - 1} = O\left(\prod_{I \in \mathcal{I}} \left(\frac{\ln d}{\sqrt{d}}\right)^{|I| - 1}\right).$$

The result follows from $\sum_{I \in \mathcal{I}} (|I| - 1) = |A| - |\mathcal{I}|$. \square

3.3 An upper bound on the contribution of each piece

Recalling Lemma 3.1, we want to estimate the integral of $\mathcal{D}_d^{\mathcal{I}}$ over $M^{\mathcal{I}}$, where $p \geq 2$ and $\mathcal{I} \in \mathcal{P}_p$. This is done by decomposing $M^{\mathcal{I}}$ as in Section 3.2 and estimating the contribution of each piece. One difficulty is that $\mathcal{I} \in \mathcal{P}_p$, and the pieces of our decomposition of $M^{\mathcal{I}}$ are indexed by partitions of \mathcal{I} , seen as a finite set. Hence we need to consider partitions of partitions.

In this section, we compute an upper bound for the contribution of each piece in the decomposition of $M^{\mathcal{I}}$ to the integral over $M^{\mathcal{I}}$ appearing in Lemma 3.1.

Lemma 3.7. Let $p \geq 2$ and let $\phi_1, \dots, \phi_p \in \mathcal{C}^0(M)$. Let $\mathcal{I} \in \mathcal{P}_p$, for all $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$, we have:

$$\int_{\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}} (\iota_{\mathcal{I}}^* \phi)(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) |\text{dV}_M|^{|\mathcal{I}|} = \left(\prod_{i=1}^p \|\phi_i\|_{\infty}\right) O\left(d^{\frac{|\mathcal{I}|}{2}} (\ln d)^{|\mathcal{I}| - |\mathcal{J}|}\right),$$

where the error term does not depend on (ϕ_1, \dots, ϕ_p) .

Proof. We have $\|\iota_{\mathcal{I}}^* \phi\|_{\infty} \leq \|\phi\|_{\infty} \leq \prod_{i=1}^p \|\phi_i\|_{\infty}$. Besides, by the definition of $\mathcal{D}_d^{\mathcal{I}}$ (see Lemma 3.1) and Proposition 2.25, the density $\mathcal{D}_d^{\mathcal{I}}$ is bounded on $M^{\mathcal{I}}$ by $O(d^{\frac{|\mathcal{I}|}{2}})$. Integrating over $M_{\mathcal{J},d}^{\mathcal{I},p}$ yields:

$$\int_{\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}} (\iota_{\mathcal{I}}^* \phi)(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) |dV_M|^{|\mathcal{I}|} \leq \left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) O\left(d^{\frac{|\mathcal{I}|}{2}}\right) \text{Vol}\left(M_{\mathcal{J},d}^{\mathcal{I},p}\right).$$

Then the result follows from the volume estimate of Lemma 3.6. \square

3.4 Contribution of the partitions with an isolated point

In this section, we still work with a fixed $p \geq 2$ and some fixed $\mathcal{I} \in \mathcal{P}_p$. Lemma 3.7 seems to say that the major contribution to the integral:

$$\int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}} (\iota_{\mathcal{I}}^* \phi)(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) |dV_M|^{|\mathcal{I}|}$$

is given by the pieces of the form $M_{\mathcal{J},d}^{\mathcal{I},p}$, where $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$ is such that $|\mathcal{J}|$ is large. However, by looking more carefully at $\mathcal{D}_d^{\mathcal{I}}$, we can prove that the contribution of $M_{\mathcal{J},d}^{\mathcal{I},p}$ is also small if \mathcal{J} contains a singleton. In particular, if $|\mathcal{J}|$ is too large, the corresponding piece will only contribute an error term.

Lemma 3.8. *Let $p \geq 2$ and let $\mathcal{I} \in \mathcal{P}_p$. We assume that there exists $j \in \{1, \dots, p\}$ such that $\{j\} \in \mathcal{I}$. Then, for all $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$ such that $\{\{j\}\} \in \mathcal{J}$, we have:*

$$\forall \underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}, \quad \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = O(d^{\frac{p}{4}-1}),$$

as $d \rightarrow +\infty$, uniformly in $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$.

Proof. Recall that we defined $\mathcal{S}_{\mathcal{I}}$ in Definition 2.2. Since $\{j\} \in \mathcal{I}$, we can split $\mathcal{S}_{\mathcal{I}}$ into the subsets $A \in \mathcal{S}_{\mathcal{I}}$ that contain j , and those that do not. Moreover, $A \mapsto A \sqcup \{j\}$ is a bijection from $\{A \in \mathcal{S}_{\mathcal{I}} \mid j \notin A\}$ to $\{A \in \mathcal{S}_{\mathcal{I}} \mid j \in A\}$.

Let $A \in \mathcal{S}_{\mathcal{I}}$ be such that $j \notin A$. Regrouping the terms corresponding to A and $A \sqcup \{j\}$ in the sum defining $\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}})$ (see Lemma 3.1), we obtain:

$$\begin{aligned} & (-1)^{p-|A|} \mathcal{R}_d^{|f_A(\mathcal{I})|}(\underline{x}_{f_A(\mathcal{I})}) \prod_{i \notin A} \mathcal{R}_d^1(x_{\{i\}}) + (-1)^{p-|A|-1} \mathcal{R}_d^{|f_A(\mathcal{I})|+1}(\underline{x}_{f_A(\mathcal{I}) \sqcup \{j\}}) \prod_{i \notin A \sqcup \{j\}} \mathcal{R}_d^1(x_{\{i\}}) \\ &= (-1)^{p-|A|} \prod_{i \notin A \sqcup \{j\}} \mathcal{R}_d^1(x_{\{i\}}) \left(\mathcal{R}_d^{|f_A(\mathcal{I})|}(\underline{x}_{f_A(\mathcal{I})}) \mathcal{R}_d^1(x_{\{j\}}) - \mathcal{R}_d^{|f_A(\mathcal{I})|+1}(\underline{x}_{f_A(\mathcal{I}) \sqcup \{j\}}) \right), \end{aligned} \quad (3.1)$$

for all $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}$. Here we use once again the convention that $\mathcal{R}_d^0 = 1$ for all $d \geq 1$.

Let $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$. Since $\{\{j\}\} \in \mathcal{J}$, by definition of $M_{\mathcal{J},d}^{\mathcal{I},p}$ (see Definition 3.4), the point $x_{\{j\}}$ is far from the other coordinates of $\underline{x}_{\mathcal{I}}$. More precisely, for all $I \in \mathcal{I} \setminus \{\{j\}\}$ we have $\rho_g(x_I, x_{\{j\}}) \geq b_p \frac{\ln d}{\sqrt{d}}$. Then, by Proposition 2.27, the term on the right-hand side of Equation (3.1) is $O(d^{\frac{1}{2}(|f_A(\mathcal{I})|-|A|)+\frac{p}{4}-1})$, uniformly in $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$. Since $f_A(\mathcal{I}) \in \mathcal{P}_A$, we have $|f_A(\mathcal{I})| \leq |A|$. Hence, the previous term is $O(d^{\frac{p}{4}-1})$. We conclude the proof by summing this estimates over $\{A \in \mathcal{S}_{\mathcal{I}} \mid j \notin A\}$. \square

Corollary 3.9. *Let $p \geq 2$ and $\mathcal{I} \in \mathcal{P}_p$. We assume that there exists $j \in \{1, \dots, p\}$ such that $\{j\} \in \mathcal{I}$. Then, for all $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$ such that $\{\{j\}\} \in \mathcal{J}$, we have:*

$$\forall \phi_1, \dots, \phi_p \in \mathcal{C}^0(M), \quad \int_{\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}} (\iota_{\mathcal{I}}^* \phi)(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) |dV_M|^{|\mathcal{I}|} = \left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) O(d^{\frac{p}{4}-1}),$$

where the error term does not depend on (ϕ_1, \dots, ϕ_p) .

Proof. We obtain this corollary by integrating the estimate of Lemma 3.8 over $M_{\mathcal{J},d}^{\mathcal{I},p}$, using the fact that $\|\iota_{\mathcal{I}}^* \phi\|_{\infty} \leq \prod_{i=1}^p \|\phi_i\|_{\infty}$. \square

3.5 Contribution of the partitions into pairs

Here, as in the previous two sections, we are interested in computing the contributions of some pieces of the decomposition $M^{\mathcal{I}} = \bigsqcup_{\mathcal{J} \in \mathcal{P}_{\mathcal{I}}} M_{\mathcal{J},d}^{\mathcal{I},p}$ to the integrals appearing in Lemma 3.1. More precisely, we consider the contributions coming from partitions into pairs.

Lemma 3.10. *Let $p \geq 2$ and let $\mathcal{I} \in \mathcal{P}_p$. Let $S \subset \{1, \dots, p\}$ be such that $|S|$ is even and $\{\{s\} \mid s \in S\} \subset \mathcal{I}$. We denote by $\mathcal{I}'' = \{\{s\} \mid s \in S\}$ and by $\mathcal{I}' = \mathcal{I} \setminus \mathcal{I}''$. Let $\mathcal{J}' \in \mathcal{PP}_{\mathcal{I}'}$ and $\mathcal{J}'' \in \mathcal{P}_{\mathcal{I}''}$, we denote by $\mathcal{J} = \mathcal{J}' \sqcup \mathcal{J}'' \in \mathcal{P}_{\mathcal{I}}$. Then, the following holds uniformly for all $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$:*

$$\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \mathcal{D}_d^{\mathcal{I}'}(\underline{x}_{\mathcal{I}'}) \left(\prod_{J \in \mathcal{J}'} \mathcal{D}_d^J(\underline{x}_J) \right) + O(d^{\frac{p}{4}-1}).$$

Proof. If S is empty, then $\mathcal{I}'' = \emptyset$ and $\mathcal{J}' = \emptyset$ by convention. Moreover, $\mathcal{I} = \mathcal{I}'$. Hence, the result holds in this case. Let us now assume that S is not empty. Then \mathcal{I}'' is non-empty and contains an even number of elements, so that $\mathcal{PP}_{\mathcal{I}''}$ is non-empty. Let $\mathcal{J}' \in \mathcal{PP}_{\mathcal{I}'}$, let $\mathcal{J}'' \in \mathcal{P}_{\mathcal{I}''}$ and let $\mathcal{J} = \mathcal{J}' \sqcup \mathcal{J}''$.

Let $J \in \mathcal{J}'$, then there exists s and $t \in S$ such that $s \neq t$ and $J = \{\{s\}, \{t\}\}$. Recall that $\mathcal{D}_d^{\mathcal{I}}$ is defined as a sum indexed by $A \subset \{1, \dots, p\}$ (see Lemma 3.1). Let $A \subset \{1, \dots, p\} \setminus \{s, t\}$, we regroup the four terms in this sum corresponding to A , $A \sqcup \{s\}$, $A \sqcup \{t\}$ and $A \sqcup \{s, t\}$. For all $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}$, we obtain:

$$\begin{aligned} (-1)^{p-|A|} \left(\prod_{i \notin A \sqcup \{s, t\}} \mathcal{R}_d^1(x_{\{i\}}) \right) & \left(\mathcal{R}_d^{|f_A(\mathcal{I})|+2}(\underline{x}_{f_A(\mathcal{I}) \sqcup \{s, t\}}) - \mathcal{R}_d^{|f_A(\mathcal{I})|+1}(\underline{x}_{f_A(\mathcal{I}) \sqcup \{\{s\}\}}) \mathcal{R}_d^1(x_{\{t\}}) \right. \\ & \left. - \mathcal{R}_d^{|f_A(\mathcal{I})|+1}(\underline{x}_{f_A(\mathcal{I}) \sqcup \{\{t\}\}}) \mathcal{R}_d^1(x_{\{s\}}) + \mathcal{R}_d^{|f_A(\mathcal{I})|}(\underline{x}_{f_A(\mathcal{I})}) \mathcal{R}_d^1(x_{\{s\}}) \mathcal{R}_d^1(x_{\{t\}}) \right) \end{aligned} \quad (3.2)$$

If $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$, then for any $I \in \mathcal{I} \setminus \{\{s\}, \{t\}\}$ we have $\rho_g(x_I, x_{\{s\}}) \geq b_p \frac{\ln d}{\sqrt{d}}$ and $\rho_g(x_I, x_{\{t\}}) \geq b_p \frac{\ln d}{\sqrt{d}}$, by definition. Applying Proposition 2.27 in Equation (3.2), we obtain:

$$(-1)^{p-|A|} \left(\prod_{i \notin A \sqcup \{s, t\}} \mathcal{R}_d^1(x_{\{i\}}) \right) \mathcal{R}_d^{|f_A(\mathcal{I})|}(\underline{x}_{f_A(\mathcal{I})}) \left(\mathcal{D}_d^J(\underline{x}_J) + O(d^{-\frac{p}{4}}) \right).$$

Summing these terms over $A \subset \{1, \dots, p\} \setminus \{\{s\}, \{t\}\}$, we get that, for all $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$:

$$\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \mathcal{D}_d^{\mathcal{I}' \setminus J}(\underline{x}_{\mathcal{I}' \setminus J}) \left(\mathcal{D}_d^J(\underline{x}_J) + O(d^{-\frac{p}{4}}) \right). \quad (3.3)$$

We can repeat the argument for $\mathcal{D}_d^{\mathcal{I}' \setminus J}(\underline{x}_{\mathcal{I}' \setminus J})$, with \mathcal{J}' replaced by $\mathcal{J}' \setminus \{J\}$. More formally, we prove by induction on the cardinality of \mathcal{J}' that, for all $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$ the following holds uniformly:

$$\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \mathcal{D}_d^{\mathcal{I}'}(\underline{x}_{\mathcal{I}'}) \prod_{J \in \mathcal{J}'} \left(\mathcal{D}_d^J(\underline{x}_J) + O(d^{-\frac{p}{4}}) \right).$$

The result is true if $\mathcal{J}' = \emptyset$, and the inductive step is given by Equation (3.3).

To conclude the proof, recall that by Proposition 2.25, for any finite set B , any $\mathcal{I} \in \mathcal{P}_B$ and any $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}$, we have $\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = O(d^{\frac{|\mathcal{I}|}{2}})$, uniformly in $\underline{x}_{\mathcal{I}}$. Here $\frac{1}{2}|\mathcal{I}'| + |\mathcal{J}'| - 1 \leq \frac{1}{2}|\mathcal{I}| - 1 \leq \frac{p}{2} - 1$, using the fact that $|\mathcal{J}'| \leq \frac{1}{2}|\mathcal{I}''|$ since $\mathcal{J}' \in \mathcal{PP}_{\mathcal{I}''}$. Hence, all the error terms are of the form $O(d^{\frac{p}{4}-1})$. \square

Definition 3.11. Let $p \geq 2$. For all $\mathcal{I} \in \mathcal{P}_p$, we denote by $\mathcal{I}' = \{I \in \mathcal{I} \mid |I| \geq 2\}$ and by $\mathcal{I}'' = \{I \in \mathcal{I} \mid |I| = 1\}$. We denote by \mathfrak{C}_p the set of couples $(\mathcal{I}, \mathcal{J})$ such that $\mathcal{I} \in \mathcal{P}_p$, $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$ and the following two conditions are satisfied:

1. for all $I \in \mathcal{I}'$, we have $|I| = 2$;
2. there exists $\mathcal{J}' \in \mathcal{PP}_{\mathcal{I}'}$ such that $\mathcal{J} = \mathcal{J}' \sqcup \mathcal{J}''$, where $\mathcal{J}'' = \{\{I\} \mid I \in \mathcal{I}''\}$.

Remark 3.12. If $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$, then \mathcal{I}'' admits a partition into pairs, hence $|\mathcal{I}''|$ is even. Moreover, $p = 2|\mathcal{I}'| + |\mathcal{I}''|$ is also even.

Lemma 3.13. *Let $p \geq 2$ and let $\mathcal{I} \in \mathcal{P}_p$. We denote by $\mathcal{I}' = \{I \in \mathcal{I} \mid |I| \geq 2\}$ and by $\mathcal{I}'' = \{I \in \mathcal{I} \mid |I| = 1\}$. Let $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$ be such that $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$, then the following holds uniformly for all $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J}, d}^{\mathcal{I}, p}$:*

$$\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \left(\prod_{I \in \mathcal{I}'} \mathcal{R}_d^1(x_I) \right) \left(\prod_{J \in \mathcal{J}'} \mathcal{D}_d^J(\underline{x}_J) \right) + O(d^{\frac{p}{4}-1}),$$

where $\mathcal{J}' \in \mathcal{PP}_{\mathcal{I}''}$ is as in Definition 3.11 and \mathcal{R}_d^1 is defined in Definition 2.22.

Proof. Let us denote by $S = \bigsqcup_{I \in \mathcal{I}''} I$. We have $|S| = |\mathcal{I}''|$, and by Remark 3.12, this cardinality is even. By Lemma 3.10, we have the following uniform estimate: for all $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J}, d}^{\mathcal{I}, p}$,

$$\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \mathcal{D}_d^{\mathcal{I}'}(\underline{x}_{\mathcal{I}'}) \left(\prod_{J \in \mathcal{J}'} \mathcal{D}_d^J(\underline{x}_J) \right) + O(d^{\frac{p}{4}-1}).$$

Let us denote by $B = \bigsqcup_{I \in \mathcal{I}'} I$. Then $\mathcal{I}' \in \mathcal{P}_B$ and, recalling Definition 2.2, we have $\mathcal{S}_{\mathcal{I}'} = \{B\}$ because of Condition 1 in Definition 3.11. Hence, $\mathcal{D}_d^{\mathcal{I}'} = \mathcal{R}_d^{|\mathcal{I}'|}$. Let $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J}, d}^{\mathcal{I}, p}$, by Condition 2 in Definition 3.11, for all $I, I' \in \mathcal{I}'$ such that $I \neq I'$, we have $\rho_g(x_I, x_{I'}) \geq b_p \frac{\ln d}{\sqrt{d}}$. We apply Proposition 2.27 several times, and we obtain:

$$\forall \underline{x}_{\mathcal{I}} \in M_{\mathcal{J}, d}^{\mathcal{I}, p}, \quad \mathcal{D}_d^{\mathcal{I}'}(\underline{x}_{\mathcal{I}'}) = \mathcal{R}_d^{|\mathcal{I}'|}(\underline{x}_{\mathcal{I}'}) = \prod_{I \in \mathcal{I}'} \mathcal{R}_d^1(x_I) + O(d^{\frac{|\mathcal{I}'|}{2} - \frac{p}{4} - 1}),$$

where the error term is obtained by using Proposition 2.25.

By Proposition 2.25, for any $J \in \mathcal{J}'$, we have $\mathcal{D}_d^J(\underline{x}_J) = O(d)$ uniformly in $\underline{x}_J \in M^J$. Hence, the result follows from the inequality:

$$\frac{1}{2} |\mathcal{I}'| + |\mathcal{J}'| = \frac{1}{2} (|\mathcal{I}'| + |\mathcal{I}''|) = \frac{|\mathcal{I}|}{2} \leq \frac{p}{2}, \quad (3.4)$$

where we used the fact that \mathcal{J}' is partition into pairs of \mathcal{I}'' . \square

3.6 Proof of Theorem 1.12

We now prove our main result, that is the moments estimates of Theorem 1.12. The key argument is that the leading term in the asymptotics of the $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ comes from the partitions into pairs studied in Section 3.5, while all the other terms are negligible. We use the same notations as in the previous sections.

Lemma 3.14. *Let $p \geq 2$ and $\phi_1, \dots, \phi_p \in C^0(M)$. Let $\mathcal{I} \in \mathcal{P}_p$, we denote by $\mathcal{I}' = \{I \in \mathcal{I} \mid |I| \geq 2\}$ and by $\mathcal{I}'' = \{I \in \mathcal{I} \mid |I| = 1\}$. Let $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$ be such that $(\mathcal{I}, \mathcal{J}) \notin \mathfrak{C}_p$, where \mathfrak{C}_p is defined by Definition 3.11. Then, we have:*

$$\int_{\underline{x}_{\mathcal{I}} \in M_{\mathcal{J}, d}^{\mathcal{I}, p}} (\iota_{\mathcal{I}}^* \phi)(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) |dV_M|^{|\mathcal{I}|} = \left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) O\left(d^{\frac{1}{2} \lfloor \frac{p-1}{2} \rfloor} (\ln d)^p\right),$$

where the error term does not depend on $(\phi_i)_{1 \leq i \leq p}$.

Proof. Let $p \geq 2$ and let $\mathcal{I} \in \mathcal{P}_p$. Denoting by $\mathcal{I}' = \{I \in \mathcal{I} \mid |I| \geq 2\}$ and by $\mathcal{I}'' = \{I \in \mathcal{I} \mid |I| = 1\}$, we have $\mathcal{I} = \mathcal{I}' \sqcup \mathcal{I}''$. Then, we have $\{1, \dots, p\} = (\bigsqcup_{I \in \mathcal{I}'} I) \sqcup (\bigsqcup_{I \in \mathcal{I}''} I)$, so that:

$$2|\mathcal{I}'| + |\mathcal{I}''| \leq p. \quad (3.5)$$

Moreover, equality holds in Equation (3.5) if and only if, for all $I \in \mathcal{I}'$, we have $|I| = 2$, that is if and only if Condition 1 of Definition 3.11 is satisfied.

Let $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$. Let $\mathcal{J}' = \{J \in \mathcal{J} \mid |J| \geq 2\}$ and $\mathcal{J}'' = \{J \in \mathcal{J} \mid |J| = 1\}$, so that $\mathcal{J} = \mathcal{J}' \sqcup \mathcal{J}''$. For all $J \in \mathcal{J}''$, there exists $I_J \in \mathcal{I}$ such that $J = \{I_J\}$. By Corollary 3.9, if $I_J \in \mathcal{I}'$, that is if I_J is a singleton, we have:

$$\forall \phi_1, \dots, \phi_p \in \mathcal{C}^0(M), \quad \int_{\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}} (\iota_{\mathcal{I}}^* \phi)(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) |dV_M|^{|\mathcal{I}|} = \left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) O(d^{\frac{p}{4}-1}),$$

where the error term does not depend on (ϕ_1, \dots, ϕ_p) . Since $\frac{p}{4} - 1 = \frac{1}{2} \left(\frac{p-2}{2} - 1 \right) < \frac{1}{2} \lfloor \frac{p-1}{2} \rfloor$, this proves the result if there exists $J \in \mathcal{J}''$ such that $I_J \in \mathcal{I}'$.

Let us now assume that this is not the case. That is, for all $J \in \mathcal{J}''$, there exists $I_J \in \mathcal{I}'$ such that $J = \{I_J\}$. Thanks to this assumption, we have:

$$|\mathcal{J}''| \leq |\mathcal{I}'|, \quad (3.6)$$

and equality holds in Equation (3.6) if and only if, $\mathcal{J}'' = \{\{I\} \mid I \in \mathcal{I}'\}$. Besides, by definition of \mathcal{J}' and \mathcal{J}'' , we have:

$$|\mathcal{J}'| \leq \frac{1}{2} (|\mathcal{I}| - |\mathcal{J}''|), \quad (3.7)$$

and equality holds in Equation (3.7) if and only if $|J| = 2$ for all $J \in \mathcal{J}'$. In particular, equality holds in both Equations (3.6) and (3.7) if and only if Condition 2 of Definition 3.11 holds.

By Equations (3.5), (3.6) and (3.7), we have:

$$|\mathcal{J}| = |\mathcal{J}'| + |\mathcal{J}''| \leq \frac{1}{2} (|\mathcal{I}| + |\mathcal{J}''|) \leq \frac{1}{2} (|\mathcal{I}| + |\mathcal{I}'|) = \frac{1}{2} (2|\mathcal{I}'| + |\mathcal{I}''|) \leq \frac{p}{2}, \quad (3.8)$$

and $|\mathcal{J}| = \frac{p}{2}$ if and only if $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$. Since we excluded this case, we have $|\mathcal{J}| < \frac{p}{2}$. Since $|\mathcal{J}|$ is an integer, we have in fact $|\mathcal{J}| \leq \lfloor \frac{p-1}{2} \rfloor$. Then, the result follows from Lemma 3.7, since $|\mathcal{I}| \leq p$. \square

Lemma 3.15. *Let $p \geq 2$ and let $\mathcal{I} \in \mathcal{P}_p$. Let us denote by $\mathcal{I}' = \{I \in \mathcal{I} \mid |I| \geq 2\}$ and by $\mathcal{I}'' = \{I \in \mathcal{I} \mid |I| = 1\}$. Let $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$ be such that $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$, where \mathfrak{C}_p is defined by Definition 3.11. Then, for any $\phi_1, \dots, \phi_p \in \mathcal{C}^0(M)$, we have:*

$$\int_{\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}} \iota_{\mathcal{I}}^* \phi(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) |dV_M|^{|\mathcal{I}|} = \left(\prod_{\{i,j\} \in \mathcal{I}'} \int_M \phi_i \phi_j \mathcal{R}_d^1 |dV_M| \right) \left(\prod_{J \in \mathcal{J}'} \int_{M^J} \phi_J \mathcal{D}_d^J |dV_M|^2 \right) + \left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) O\left(d^{\frac{1}{2} \lfloor \frac{p-1}{2} \rfloor} (\ln d)^p\right),$$

where the constant involved in the $O\left(d^{\frac{1}{2} \lfloor \frac{p-1}{2} \rfloor} (\ln d)^p\right)$ is independent of (ϕ_1, \dots, ϕ_p) .

Proof. Let $(\phi_i)_{1 \leq i \leq p}$ and let $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}$, we have:

$$\iota_{\mathcal{I}}^* \phi(\underline{x}_{\mathcal{I}}) = \phi(\iota_{\mathcal{I}}(\underline{x}_{\mathcal{I}})) = \prod_{I \in \mathcal{I}} \prod_{i \in I} \phi_i(x_I) = \left(\prod_{\{i,j\} \in \mathcal{I}'} \phi_i(x_I) \phi_j(x_I) \right) \left(\prod_{\{i\} \in \mathcal{I}''} \phi_i(x_{\{i\}}) \right).$$

Hence, by Lemma 3.13, for all $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$, we have that $\iota_{\mathcal{I}}^* \phi(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}})$ equals:

$$\left(\prod_{\{i,j\} \in \mathcal{I}'} \phi_i(x_I) \phi_j(x_I) \mathcal{R}_d^1(x_I) \right) \left(\prod_{\{\{i\}, \{j\}\} = J \in \mathcal{J}'} \phi_i(x_{\{i\}}) \phi_j(x_{\{j\}}) \mathcal{D}_d^J(\underline{x}_J) \right) + \left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) O(d^{\frac{p}{4}-1}), \quad (3.9)$$

where the error term does not depend on $\underline{x}_{\mathcal{I}}$ or $(\phi_i)_{1 \leq i \leq p}$. The quantity we want to compute is the integral of the function (3.9) over $M_{\mathcal{J},d}^{\mathcal{I},p}$. Since $\frac{p}{4} - 1 < \frac{1}{2} \lfloor \frac{p-1}{2} \rfloor$, we only need consider the leading term of Equation (3.9) in the following.

Let $\mathcal{K} \in \mathcal{P}_{\mathcal{I}}$, we denote by $\mathcal{K} \geq \mathcal{J}$ if \mathcal{J} is finer than \mathcal{K} . This means that every element of \mathcal{K} is obtained as the disjoint union of some elements of \mathcal{J} . Equivalently, for all $J \in \mathcal{J}$ there exists

$K \in \mathcal{K}$ such that $J \subset K$. We denote by $\mathcal{K} > \mathcal{J}$ if $\mathcal{K} \geq \mathcal{J}$ and $\mathcal{K} \neq \mathcal{J}$. With this definition in mind, we claim that:

$$M_{\mathcal{J},d}^{\mathcal{I},p} \subset \left(\prod_{I \in \mathcal{I}'} M \right) \times \left(\prod_{J \in \mathcal{J}''} M_{\{J\},d}^{J,p} \right) \subset \bigsqcup_{\{\mathcal{K} \in \mathcal{P}_{\mathcal{I}} | \mathcal{K} \geq \mathcal{J}\}} M_{\mathcal{K},d}^{\mathcal{I},p}. \quad (3.10)$$

First, note that these three sets are subsets of $M^{\mathcal{I}}$. In order to prove the first inclusion in Equation (3.10), let us fix a point $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$. Since \mathcal{J} satisfies Condition 2 of Definition 3.11, the associated graph $G_d^p(\underline{x}_{\mathcal{I}})$ (see Definition 3.2) is formed of $|\mathcal{I}'|$ isolated vertices $\{I \mid I \in \mathcal{I}'\}$ and $|\mathcal{J}'|$ pairs of vertices of the form $\{I, I'\} \in \mathcal{J}'$ with an edge between I and I' . Hence, for all $J = \{I, I'\} \in \mathcal{J}'$, we have $\rho_g(x_I, x_{I'}) \leq b_p \frac{\ln d}{\sqrt{d}}$, and $\underline{x}_J \in M_{\{J\},d}^{J,p}$. Let us now prove the second inclusion in Equation (3.10). Let $\underline{x}_{\mathcal{I}} = (x_I)_{I \in \mathcal{I}} \in M^{\mathcal{I}}$ and let us assume that $\underline{x}_{\mathcal{I}}$ belongs to the middle set in Equation (3.10). By Definition 3.2, for all $J = \{I, I'\} \in \mathcal{J}'$, the graph $G_d^p(\underline{x}_{\mathcal{I}})$ associated with $\underline{x}_{\mathcal{I}}$ has an edge between I and I' . Hence the associated partition $\mathcal{K} = \mathcal{I}_d^p(\underline{x}_{\mathcal{I}})$ (see Definition 3.3) is such that, for all $J \in \mathcal{J}'$ there exists $K \in \mathcal{K}$ such that $J \subset K$. Since \mathcal{J} satisfies Condition 2 in Definition 3.11, this is enough to ensure that $\mathcal{K} \geq \mathcal{J}$. Note that, in general, both inclusion in Equation (3.10) are strict.

Let $\mathcal{K} \in \mathcal{P}_{\mathcal{I}}$ be such that $\mathcal{K} > \mathcal{J}$. The same proof as that of Lemma 3.7 (using the uniform upper bounds of Proposition 2.25 and the volume estimate of Lemma 3.6) shows that the integral of the leading term in Equation (3.9) over $\left(\left(\prod_{I \in \mathcal{I}'} M \right) \times \left(\prod_{J \in \mathcal{J}''} M_{\{J\},d}^{J,p} \right) \right) \cap M_{\mathcal{K},d}^{\mathcal{I},p}$ is bounded by:

$$\left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) \text{Vol}\left(M_{\mathcal{K},d}^{\mathcal{I},p}\right) = \left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) O\left(d^{\frac{|\mathcal{K}|}{2}} (\ln d)^{|\mathcal{I}|-|\mathcal{K}|}\right). \quad (3.11)$$

We have $|\mathcal{I}| - |\mathcal{K}| \leq |\mathcal{I}| \leq p$. Besides, since $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$ we have equality in Equation (3.8), that is $|\mathcal{J}| = \frac{p}{2}$. Since $|\mathcal{K}| < |\mathcal{J}|$, we have $|\mathcal{K}| \leq \frac{p}{2} - 1 = \lfloor \frac{p-1}{2} \rfloor$. Hence, the error term in Equation (3.11) can be replaced by $O\left(d^{\frac{1}{2} \lfloor \frac{p-1}{2} \rfloor} (\ln d)^p\right)$. This proves that:

$$\int_{\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}} \iota_{\mathcal{I}}^* \phi(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) |dV_M|^{|\mathcal{I}|} = \left(\prod_{\{i,j\} \in \mathcal{I}'} \int_M \phi_i \phi_j \mathcal{R}_d^1 |dV_M| \right) \left(\prod_{J \in \mathcal{J}'} \int_{M_{\{J\},d}^{J,p}} \phi_J \mathcal{D}_d^J |dV_M|^2 \right) + \left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) O\left(d^{\frac{1}{2} \lfloor \frac{p-1}{2} \rfloor} (\ln d)^p\right). \quad (3.12)$$

Note that, using Notation 2.4, for all $J = \{\{i\}, \{j\}\} \in \mathcal{J}'$ we have:

$$\phi_J = \phi_{\{\{i\}, \{j\}\}} = \phi_{\{i\}} \boxtimes \phi_{\{j\}} = \phi_i \boxtimes \phi_j.$$

In order to conclude the proof, we need to replace the integral over $M_{\{J\},d}^{J,p}$ by an integral over M^J in Equation (3.12), for all $J \in \mathcal{J}'$. Let $J \in \mathcal{J}'$, there exists I and $I' \in \mathcal{I}''$ such that $J = \{I, I'\}$. Then, \mathcal{P}_J has exactly two elements: $\{J\}$ and $\{\{I\}, \{I'\}\}$. By Proposition 2.27 and the definition of \mathcal{D}_d^J (see Lemma 3.1), we have the following uniform estimate:

$$\forall \underline{x}_J \in M_{\{\{I\}, \{I'\}\},d}^{J,p}, \quad \mathcal{D}_d^J(\underline{x}_J) = O(d^{-\frac{p}{4}}).$$

Hence, for all $J \in \mathcal{J}'$, we have:

$$\int_{M_{\{J\},d}^{J,p}} \phi_J \mathcal{D}_d^J |dV_M|^2 = \int_{M^J} \phi_J \mathcal{D}_d^J |dV_M|^2 + \|\phi_J\|_{\infty} O(d^{-\frac{p}{4}}),$$

since $M^J = M_{\{J\},d}^{J,p} \sqcup M_{\{\{I\}, \{I'\}\},d}^{J,p}$. Then, by Proposition 2.25, we have:

$$\begin{aligned} & \left(\prod_{\{i,j\} \in \mathcal{I}'} \int_M \phi_i \phi_j \mathcal{R}_d^1 |dV_M| \right) \left(\prod_{J \in \mathcal{J}'} \int_{M_{\{J\},d}^{J,p}} \phi_J \mathcal{D}_d^J |dV_M|^2 \right) = \\ & \left(\prod_{\{i,j\} \in \mathcal{I}'} \int_M \phi_i \phi_j \mathcal{R}_d^1 |dV_M| \right) \left(\prod_{J \in \mathcal{J}'} \int_{M^J} \phi_J \mathcal{D}_d^J |dV_M|^2 \right) + \left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) O\left(d^{\frac{|\mathcal{I}'|}{2} + |\mathcal{J}'| - \frac{p}{4} - 1}\right), \end{aligned}$$

and the conclusion follows from $\frac{1}{2}|\mathcal{I}'| + |\mathcal{J}'| \leq \frac{p}{2}$ (see Equation (3.4)) and $\frac{p}{4} - 1 < \frac{1}{2} \lfloor \frac{p-1}{2} \rfloor$. \square

We conclude this section with the proof of Theorem 1.12.

Proof of Theorem 1.12. Let $p \geq 3$ and $\phi_1, \dots, \phi_p \in \mathcal{C}^0(M)$. Let $d \in \mathbb{N}$ be large enough, let $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a standard Gaussian vector and let ν_d denote the counting measure on the real zero set of s_d .

By Lemma 3.1 and the fact that $M^{\mathcal{I}} = \bigsqcup_{\mathcal{J} \in \mathcal{P}_{\mathcal{I}}} M_{\mathcal{J},d}^{\mathcal{I},p}$ for all $\mathcal{I} \in \mathcal{P}_p$, we have:

$$m_p(\nu_d)(\phi_1, \dots, \phi_p) = \sum_{\mathcal{I} \in \mathcal{P}_p} \sum_{\mathcal{J} \in \mathcal{P}_{\mathcal{I}}} \int_{\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}} \iota_{\mathcal{I}}^* \phi(\underline{x}_{\mathcal{I}}) \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) |dV_M|^{|\mathcal{I}|}.$$

By Lemma 3.14, up to an error term of the form $(\prod_{i=1}^p \|\phi_i\|_{\infty}) O(d^{\frac{1}{2} \lfloor \frac{p-1}{2} \rfloor} (\ln d)^p)$ we need only consider the terms in this double sum indexed by $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$, where \mathfrak{C}_p is defined by Definition 3.11. The expression of these terms is given by Lemma 3.15. Thus, we have:

$$m_p(\nu_d)(\phi_1, \dots, \phi_p) = \sum_{(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p} \left(\prod_{\{i,j\} \in \mathcal{I}'} \int_M \phi_i \phi_j \mathcal{R}_d^1 |dV_M| \right) \left(\prod_{J \in \mathcal{J}', \int_{M^J}} \phi_J \mathcal{D}_d^J |dV_M|^2 \right) + \left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) O(d^{\frac{1}{2} \lfloor \frac{p-1}{2} \rfloor} (\ln d)^p),$$

where we used the same notations as in Definition 3.11. Recall that $\mathcal{I}' = \{I \in \mathcal{I} \mid |I| = 2\}$ and $\mathcal{I}'' = \{I \in \mathcal{I} \mid |I| = 1\}$, so that $\mathcal{I} = \mathcal{I}' \sqcup \mathcal{I}''$ thanks to Condition 1 in Definition 3.11. Recall also that $\mathcal{J}'' = \{\{I\} \mid I \in \mathcal{I}'\}$ and $\mathcal{J} = \mathcal{J}' \sqcup \mathcal{J}''$ for some $\mathcal{J}' \in \mathcal{PP}_{\mathcal{I}''}$, thanks to Condition 2 in Definition 3.11.

We just wrote $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ as a sum of terms indexed by \mathfrak{C}_p . In the following, we define a bijection $\Phi: \mathfrak{C}_p \rightarrow \{(\Pi, \mathcal{S}) \mid \Pi \in \mathcal{PP}_p, \mathcal{S} \subset \Pi\}$, which will allow us to rewrite $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ as sum over $\{(\Pi, \mathcal{S}) \mid \Pi \in \mathcal{PP}_p, \mathcal{S} \subset \Pi\}$, by a change of variable.

- Let $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$ and let us denote by $S = \{i \mid \{i\} \in \mathcal{I}''\}$ and by $\mathcal{S} = \{\{i, j\} \mid \{\{i\}, \{j\}\} \in \mathcal{J}'\}$. Since $\mathcal{J}' \in \mathcal{PP}_{\mathcal{I}''}$, we have $\mathcal{S} \in \mathcal{PP}_{\mathcal{S}}$. Since \mathcal{I}' is a partition of $\bigsqcup_{I \in \mathcal{I}'} I = \{1, \dots, p\} \setminus S$ into pairs, we define a partition into pairs $\Pi \in \mathcal{PP}_p$ by $\Pi = \mathcal{I}' \sqcup \mathcal{S}$. We obtain a couple (Π, \mathcal{S}) where $\Pi \in \mathcal{PP}_p$ and $\mathcal{S} \subset \Pi$. Let us denote this couple by $\Phi(\mathcal{I}, \mathcal{J})$.
- Conversely, let $\Pi \in \mathcal{PP}_p$ and let $\mathcal{S} \subset \Pi$. We set $S = \bigsqcup_{I \in \mathcal{S}} I$ and $\mathcal{I}' = \Pi \setminus \mathcal{S}$, so that \mathcal{I}' is a partition into pairs of $\{1, \dots, p\} \setminus S$. Let us denote by $\mathcal{I}'' = \{\{i\} \mid i \in S\}$ and by $\mathcal{J}' = \{\{\{i\}, \{j\}\} \mid \{i, j\} \in \mathcal{S}\}$, so that $\mathcal{J}' \in \mathcal{PP}_{\mathcal{I}''}$. Finally, let $\mathcal{I} = \mathcal{I}' \sqcup \mathcal{I}'' \in \mathcal{P}_p$, and let $\mathcal{J}'' = \{\{I\} \mid I \in \mathcal{I}'\}$ so that $\mathcal{J} = \mathcal{J}' \sqcup \mathcal{J}'' \in \mathcal{P}_{\mathcal{I}}$. We just defined a couple $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$ that we denote by $\Psi(\Pi, \mathcal{S})$.

By construction, Φ is a bijection from \mathfrak{C}_p to $\{(\Pi, \mathcal{S}) \mid \Pi \in \mathcal{PP}_p, \mathcal{S} \subset \Pi\}$ such that $\Psi = \Phi^{-1}$. Moreover, for all $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$, denoting by $(\Pi, \mathcal{S}) = \Phi(\mathcal{I}, \mathcal{J})$, we have:

$$\left(\prod_{\{i,j\} \in \mathcal{I}'} \int_M \phi_i \phi_j \mathcal{R}_d^1 |dV_M| \right) \left(\prod_{J \in \mathcal{J}', \int_{M^J}} \phi_J \mathcal{D}_d^J |dV_M|^2 \right) = \left(\prod_{\{i,j\} \in \Pi \setminus \mathcal{S}} \int_M \phi_i \phi_j \mathcal{R}_d^1 |dV_M| \right) \left(\prod_{J \in \mathcal{S}} \int_{M^J} \phi_J \mathcal{D}_d^J |dV_M|^2 \right).$$

Hence,

$$m_p(\nu_d)(\phi_1, \dots, \phi_p) = \sum_{\Pi \in \mathcal{PP}_p} \sum_{\mathcal{S} \subset \Pi} \left(\prod_{\{i,j\} \in \Pi \setminus \mathcal{S}} \int_M \phi_i \phi_j \mathcal{R}_d^1 |dV_M| \right) \left(\prod_{J \in \mathcal{S}} \int_{M^J} \phi_J \mathcal{D}_d^J |dV_M|^2 \right) + \left(\prod_{i=1}^p \|\phi_i\|_{\infty} \right) O(d^{\frac{1}{2} \lfloor \frac{p-1}{2} \rfloor} (\ln d)^p). \quad (3.13)$$

Note that if p is odd, then $\mathcal{PP}_p = \emptyset$ and this is enough to prove Theorem 1.12 in this case.

On the other hand, we have:

$$\sum_{\{\{a_i, b_i\} | 1 \leq i \leq \frac{p}{2}\} \in \mathcal{PP}_p} \prod_{i=1}^{\frac{p}{2}} m_2(\nu_d)(\phi_{a_i}, \phi_{b_i}) = \sum_{\Pi \in \mathcal{PP}_p} \prod_{\{i, j\} \in \Pi} m_2(\nu_d)(\phi_i, \phi_j).$$

By Lemma 3.1 applied with $p = 2$, for any i and $j \in \{1, \dots, p\}$ we have:

$$m_2(\nu_d)(\phi_i, \phi_j) = \int_{M^{\{i, j\}}} \phi_{\{i, j\}} \mathcal{D}_d^{\{i, j\}} |dV_M|^2 + \int_M \phi_i \phi_j \mathcal{R}_d^1 |dV_M|.$$

Thus,

$$\sum_{\Pi \in \mathcal{PP}_p} \prod_{\{i, j\} \in \Pi} m_2(\nu_d)(\phi_i, \phi_j) = \sum_{\Pi \in \mathcal{PP}_p} \sum_{S \subset \Pi} \left(\prod_{\{i, j\} \in \Pi \setminus S} \int_M \phi_i \phi_j \mathcal{R}_d^1 |dV_M| \right) \left(\prod_{J \in S} \int_{M^J} \phi_J \mathcal{D}_d^J |dV_M|^2 \right),$$

which is the leading term in Equation (3.13). This proves the first claim in Theorem 1.12. We obtain the second expression of $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ in Theorem 1.12 by replacing $m_2(\nu_d)$ by its asymptotics, computed in Theorem 1.3.

Let $\phi \in \mathcal{C}^0(M)$. Using what we just proved with $\phi_i = \phi$ for all $i \in \{1, \dots, p\}$, we have:

$$m_p(\langle \nu_d, \phi \rangle) = m_p(\nu_d)(\phi, \dots, \phi) = \text{Card}(\mathcal{PP}_p) (m_2(\nu_d)(\phi, \phi))^{\frac{p}{2}} + \|\phi\|_\infty^p O\left(d^{\frac{1}{2}} \lfloor \frac{p-1}{2} \rfloor (\ln d)^p\right).$$

By definition, $m_2(\nu_d)(\phi, \phi) = \text{Var}(\langle \nu_d, \phi \rangle)$. Besides, the cardinality of \mathcal{PP}_p is equal to 0 if p is odd, and to $2^{-\frac{p}{2}} \left(\frac{p!}{2}\right)^{-1} p!$ if p is even. In both cases we have $\text{Card}(\mathcal{PP}_p) = \mu_p$ (see Definition 1.10). This proves the first expression of $m_p(\langle \nu_d, \phi \rangle)$ stated in Theorem 1.12. We obtain the second expression of $m_p(\langle \nu_d, \phi \rangle)$ by Theorem 1.3, which states that: $\text{Var}(\langle \nu_d, \phi \rangle) = d^{\frac{1}{2}} \sigma^2 \int_M \phi^2 |dV_M| + o(d^{\frac{p}{2}})$. \square

4 Proof of the corollaries of Theorem 1.12

In this section, we prove the corollaries of Theorem 1.12. The strong Law of Large Numbers (Theorem 1.7) is proved in Section 4.1. The Central Limit Theorem (Theorem 1.9) is proved in Section 4.2. Finally, we prove Corollaries 1.14 and 1.15 in Section 4.3.

4.1 Proof of the strong Law of Large Numbers (Theorem 1.7)

The purpose of this section is to prove the strong Law of Large Numbers (Theorem 1.7). This result follows from the moments estimates of Theorem 1.12 by a Borel–Cantelli type argument, using the separability of $\mathcal{C}^0(M)$. This method was already used in [19, 20, 26], for example.

We follow the notation of Section 2. In particular, recall that $(s_d)_{d \geq 0}$ is a sequence independent random vectors such, for all $d \geq 0$, $s_d \sim \mathcal{N}(0, \text{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, Z_d is the real zero set of s_d and ν_d is the counting measure of Z_d .

Proof of Theorem 1.7. Let us first consider the case of one test-function $\phi \in \mathcal{C}^0(M)$. We have:

$$\mathbb{E} \left[\sum_{d \geq 1} \left(\frac{\langle \nu_d, \phi \rangle - \mathbb{E}[\langle \nu_d, \phi \rangle]}{d^{\frac{1}{2}}} \right)^6 \right] = \sum_{d \geq 1} d^{-3} m_6(\nu_d)(\phi, \dots, \phi).$$

By Theorem 1.12 this sum is finite. Indeed, $m_6(\nu_d)(\phi, \dots, \phi) = O(d^{\frac{3}{2}})$. Then, almost surely, we have:

$$\sum_{d \geq 1} \left(\frac{\langle \nu_d, \phi \rangle - \mathbb{E}[\langle \nu_d, \phi \rangle]}{d^{\frac{1}{2}}} \right)^6 < +\infty,$$

hence $d^{-\frac{1}{2}} \left(\langle \nu_d, \phi \rangle - \mathbb{E}[\langle \nu_d, \phi \rangle] \right) \xrightarrow{d \rightarrow +\infty} 0$. Thus, for all $\phi \in \mathcal{C}^0(M)$, we have almost surely:

$$d^{-\frac{1}{2}} \langle \nu_d, \phi \rangle = d^{-\frac{1}{2}} \mathbb{E}[\langle \nu_d, \phi \rangle] + o(1) = \frac{1}{\pi} \int_M \phi |dV_M| + o(1).$$

Applying this to $\phi = \mathbf{1}$, the constant unit function, we obtain: $d^{-\frac{1}{2}} \text{Card}(Z_d) \xrightarrow{d \rightarrow +\infty} \frac{1}{\pi} \text{Vol}(M)$ almost surely.

Now, recall that $(\mathcal{C}^0(M), \|\cdot\|_\infty)$ is separable. Let $(\phi_k)_{k \geq 0}$ denote a dense sequence in this space such that $\phi_0 = \mathbf{1}$. Almost surely, for all $k \geq 0$, we have $d^{-\frac{1}{2}} \langle \nu_d, \phi_k \rangle \xrightarrow{d \rightarrow +\infty} \frac{1}{\pi} \int_M \phi_k |dV_M|$. Let $\underline{s} = (s_d)_{d \geq 1} \in \prod_{d \geq 1} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ denote a fixed sequence belonging to the probability 1 event on which this condition holds. For every $\phi \in \mathcal{C}^0(M)$ and $k \in \mathbb{N}$, we have:

$$\begin{aligned} & \left| d^{-\frac{1}{2}} \langle \nu_d, \phi \rangle - \frac{1}{\pi} \int_M \phi |dV_M| \right| \\ & \leq d^{-\frac{1}{2}} |\langle \nu_d, \phi - \phi_k \rangle| + \left| d^{-\frac{1}{2}} \langle \nu_d, \phi_k \rangle - \frac{1}{\pi} \int_M \phi_k |dV_M| \right| + \frac{1}{\pi} \int_M |\phi - \phi_k| |dV_M| \\ & \leq \|\phi - \phi_k\|_\infty \left(d^{-\frac{1}{2}} \text{Card}(Z_d) + \frac{\text{Vol}(M)}{\pi} \right) + \left| d^{-\frac{1}{2}} \langle \nu_d, \phi_k \rangle - \frac{1}{\pi} \int_M \phi_k |dV_M| \right|. \end{aligned}$$

Since $\phi_0 = \mathbf{1}$, the sequence $\left(d^{-\frac{1}{2}} \text{Card}(Z_d) \right)_{d \geq 1}$ converges, hence is bounded by some $K > 0$. Let $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|\phi - \phi_k\|_\infty < \varepsilon$. Then, for every d large enough, we have:

$$\left| d^{-\frac{1}{2}} \langle \nu_d, \phi_k \rangle - \frac{1}{\pi} \int_M \phi_k |dV_M| \right| < \varepsilon,$$

hence

$$\left| d^{-\frac{1}{2}} \langle \nu_d, \phi \rangle - \frac{1}{\pi} \int_M \phi |dV_M| \right| < \varepsilon \left(1 + K + \frac{\text{Vol}(M)}{\pi} \right).$$

Thus, for all $\phi \in \mathcal{C}^0(M)$, we have $d^{-\frac{1}{2}} \langle \nu_d, \phi \rangle \xrightarrow{d \rightarrow +\infty} \frac{1}{\pi} \int_M \phi |dV_M|$, which concludes the proof. \square

4.2 Proof of the Central Limit Theorem (Theorem 1.9)

In this section, we prove the Central Limit Theorem (Theorem 1.9). The result follows from Theorem 1.12 by the method of moments, see [7, Chap. 30]. This method allows to prove the Central Limit Theorem for any fixed continuous test-function. Then, we apply the Lévy–Fernique Theorem [11, Theorem III.6.5], which is the analogue of Lévy’s Continuity Theorem for random generalized functions. Using this theorem, we prove the convergence in distribution to the Gaussian White Noise in $\mathcal{D}'(M)$, the space of generalized functions on M .

Concerning the convergence in distribution towards the Gaussian White Noise in Sobolev spaces, let us recall some facts and definitions. With these definitions, we prove the last part of Theorem 1.9, that is the convergence in distribution in $H^{-\frac{1}{2}-\varepsilon}(M)$ for all $\varepsilon > 0$, by applying the Lévy–Fernique Theorem once again.

Let us denote by $L^2(M)$ the space of square integrable functions on M . There exists a Hilbert basis $(\phi_k)_{k \geq 0}$ of $L^2(M)$ whose elements are eigenfunctions of the Laplace–Beltrami operator Δ . Let $(\lambda_k)_{k \geq 0}$ denote the sequence of associated eigenvalues. We assume the basis to be ordered in such a way that $(\lambda_k)_{k \geq 0}$ is non-decreasing. Note that we use the convention that Δ is a positive operator. Recall that $\langle \cdot, \cdot \rangle_{(\mathcal{D}', \mathcal{C}^\infty)}$ denotes the duality pairing between $\mathcal{D}'(M)$ and $\mathcal{C}^\infty(M)$.

Definition 4.1. For all $\alpha \in \mathbb{R}$, $H^\alpha(M)$ is the space:

$$H^\alpha(M) = \left\{ T \in \mathcal{D}'(M) \left| \sum_{k \geq 0} (1 + \lambda_k)^\alpha \left| \langle T, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \right|^2 < +\infty \right. \right\}.$$

This space is a Hilbert space for the inner product defined by:

$$\forall T, T' \in H^\alpha(M), \quad \langle T, T' \rangle_{H^\alpha} = \sum_{k \geq 0} (1 + \lambda_k)^\alpha \langle T, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \langle T', \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)},$$

and we denote by $\|\cdot\|_{H^\alpha}$ the associated norm. For $\alpha = 0$, $H^0(M) = L^2(M)$ and we simply denote by $\|\cdot\|_2$ the associated norm.

Note that the linear map $(\text{Id} + \Delta)^{-\frac{\alpha}{2}}$ defined by: for all $k \geq 0$, $(\text{Id} + \Delta)^{-\frac{\alpha}{2}}(\phi_k) = (1 + \lambda_k)^{-\frac{\alpha}{2}} \phi_k$ is an isometry from $L^2(M)$ to $H^\alpha(M)$. Moreover, if $\alpha \leq \beta$ then $H^\beta(M) \subset H^\alpha(M)$, and $H^{-\alpha}(M)$ is the topological dual of $H^\alpha(M)$ with the duality pairing defined by:

$$\forall T \in H^{-\alpha}(M), \forall \phi \in H^\alpha(M), \quad \langle T, \phi \rangle_{(H^{-\alpha}, H^\alpha)} = \sum_{k \geq 0} \langle T, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \langle \phi, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)}.$$

In the course of the proof of Theorem 1.9, we will show that the random generalized functions we consider belong to $H^{-\frac{1}{2}-\varepsilon}(M)$ for any $\varepsilon > 0$. The proof of this fact relies on a good understanding of the Laplace eigenfunctions and eigenvalues on M . Namely we use the Weyl Law and the Sobolev Embedding Theorem. The advantage of this proof is that it can be adapted to a higher dimensional setting. However, let us note that in the framework of this paper, M is a 1-dimensional smooth closed manifold. Hence it is isometric to the disjoint union of a finite number of circles. In this case, the eigenfunctions of Δ are explicit and are given by complex exponentials whose periodicities depend on the lengths of the connected components of M . Thus, it is possible to replace our proof by a more elementary one, involving only considerations about Fourier series on a circle.

Proof of Theorem 1.9. CLT for a fixed test-function. Let $\phi \in \mathcal{C}^0(M) \setminus \{0\}$. We define a sequence (X_d) of centered and normalized real random variables by:

$$X_d = \frac{\langle \nu_d, \phi \rangle - \mathbb{E}[\langle \nu_d, \phi \rangle]}{\text{Var}(\langle \nu_d, \phi \rangle)^{\frac{1}{2}}}.$$

Note that since $\phi \neq 0$, by Theorem 1.3, $\text{Var}(\langle \nu_d, \phi \rangle) = m_2(\nu_d)(\phi, \phi)$ is positive for d large enough. Hence X_d is well-defined for d large enough, and we want to prove that $X_d \xrightarrow{d \rightarrow +\infty} \mathcal{N}(0, 1)$ in distribution. By Theorem 1.3 and 1.12, for any integer $p \geq 3$, we have:

$$\mathbb{E}[X_d^p] = \frac{m_p(\nu_d)(\phi, \dots, \phi)}{\text{Var}(\langle \nu_d, \phi \rangle)^{\frac{p}{2}}} \xrightarrow{d \rightarrow +\infty} \mu_p,$$

where μ_p is the p -th moment of an $\mathcal{N}(0, 1)$ variable (recall Definition 1.10). By the Theorem of Moments (cf. [7, Theorem 30.2]), this implies that $X_d \xrightarrow{d \rightarrow +\infty} \mathcal{N}(0, 1)$ in distribution. Replacing the expectation and the variance of $\langle \nu_d, \phi \rangle$ by their asymptotics (Theorem 1.1 and Theorem 1.3), we get:

$$X_d = \frac{1}{d^{\frac{1}{4}}\sigma} \left(\int_M \phi^2 |dV_M| \right)^{-\frac{1}{2}} \left(\langle \nu_d, \phi \rangle - \frac{d^{\frac{1}{2}}}{\pi} \int_M \phi |dV_M| \right) + o(1).$$

Hence,

$$\frac{1}{d^{\frac{1}{4}}\sigma} \left(\langle \nu_d, \phi \rangle - \frac{d^{\frac{1}{2}}}{\pi} \int_M \phi |dV_M| \right) \xrightarrow{d \rightarrow +\infty} \mathcal{N}(0, \|\phi\|_2^2),$$

in distribution, where $\|\phi\|_2^2 = \int_M \phi^2 |dV_M|$. Of course, this also holds for $\phi = 0$. Taking $\phi = \mathbf{1}$ yields the Central Limit Theorem for the cardinality of Z_d .

CLT in $\mathcal{D}'(M)$. Let us now consider the random measure $T_d = \left(d^{\frac{1}{4}}\sigma\right)^{-1} \left(\nu_d - d^{\frac{1}{2}} |dV_M|\right)$. For all $d \geq 1$, T_d defines a random generalized function. Recall that the characteristic functional of this random generalized function is the map $\chi_d : \phi \mapsto \mathbb{E} \left[\exp \left(i \langle T_d, \phi \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \right) \right]$ from $\mathcal{C}^\infty(M)$ to \mathbb{C} . Let $\phi \in \mathcal{C}^\infty(M)$, we just proved that $\langle T_d, \phi \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} = \langle T_d, \phi \rangle \xrightarrow{d \rightarrow +\infty} \mathcal{N}(0, \|\phi\|_2^2)$. Hence,

$$\chi_d(\phi) = \mathbb{E} \left[e^{i \langle T_d, \phi \rangle} \right] \xrightarrow{d \rightarrow +\infty} e^{-\frac{1}{2} \|\phi\|_2^2},$$

where we recognized the characteristic function of $\langle T_d, \phi \rangle$ evaluated at 1. Thus, the sequence (χ_d) converges pointwise to $\chi : \phi \mapsto e^{-\frac{1}{2}\|\phi\|_2^2}$, which is the characteristic functional of the Gaussian White Noise W (recall Definition 1.8). Then, by the Lévy–Fernique Theorem (cf. [11, Theorem III.6.5]), we have $T_d \xrightarrow{d \rightarrow +\infty} W$ in distribution, as random variables in $\mathcal{D}'(M)$.

CLT for a family of smooth test-functions. Let $\phi_1, \dots, \phi_k \in \mathcal{C}^\infty(M)$, by the Continuous Mapping Theorem, we have:

$$\left(\langle T_d, \phi_1 \rangle_{(\mathcal{D}', \mathcal{C}^\infty)}, \dots, \langle T_d, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \right) \xrightarrow{d \rightarrow +\infty} \left(\langle W, \phi_1 \rangle_{(\mathcal{D}', \mathcal{C}^\infty)}, \dots, \langle W, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \right)$$

in distribution, as random vectors in \mathbb{R}^k . This yields the second claim.

CLT in Sobolev spaces. Let $\varepsilon > 0$, the last point we need to prove is the convergence of (T_d) to W in the Sobolev space $H^{-\frac{1}{2}-\varepsilon}(M)$, where (T_d) denotes the same sequence of random Radon measures as before.

Let $\phi \in H^{\frac{1}{2}+\varepsilon}(M)$. Since $\left((1 + \lambda_k)^{-\frac{1}{4}-\frac{\varepsilon}{2}} \phi_k \right)_{k \geq 0}$ is a Hilbert basis of $H^{\frac{1}{2}+\varepsilon}(M)$, we have:

$$\phi = \sum_{k \geq 0} (1 + \lambda_k)^{-\frac{1}{2}-\varepsilon} \langle \phi, \phi_k \rangle_{H^{\frac{1}{2}+\varepsilon}} \phi_k = \sum_{k \geq 0} \langle \phi, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \phi_k, \quad (4.1)$$

in $(H^{\frac{1}{2}+\varepsilon}(M), \|\cdot\|_{H^{\frac{1}{2}+\varepsilon}})$. By the Sobolev Embedding Theorem, we have $H^{\frac{1}{2}+\varepsilon}(M) \subset \mathcal{C}^0(M)$ and the inclusion is continuous, cf. [29, Chap. 7]. More precisely, with the notations of [29], we have: $H^{\frac{1}{2}+\varepsilon}(M) = F_{2,2}^{\frac{1}{2}+\varepsilon}(M)$ (see Equation (5) p. 320) and $F_{2,2}^{\frac{1}{2}+\varepsilon}(M) \subset B_{2,2}^{\frac{1}{2}+\varepsilon}(M) \subset \mathcal{C}^\varepsilon(M) \subset \mathcal{C}^0(M)$ (see Equations (2) and (6) p. 315), where we used the fact that M is 1-dimensional. Thus, $\phi \in \mathcal{C}^0(M)$ and Equation (4.1) also holds in $(\mathcal{C}^0(M), \|\cdot\|_\infty)$.

For all d large enough, T_d is almost surely a continuous linear form on $(\mathcal{C}^0(M), \|\cdot\|_\infty)$. Using the Sobolev Embedding $H^{\frac{1}{2}+\varepsilon}(M) \subset \mathcal{C}^0(M)$ once again, T_d almost surely defines a continuous linear form on $(H^{\frac{1}{2}+\varepsilon}(M), \|\cdot\|_{H^{\frac{1}{2}+\varepsilon}})$. That is $T_d \in H^{-\frac{1}{2}-\varepsilon}(M)$ almost surely. Then, using Equation (4.1) in $\mathcal{C}^0(M)$, for any d large enough the following holds for all $\phi \in H^{\frac{1}{2}+\varepsilon}(M)$:

$$\langle T_d, \phi \rangle_{(H^{-\frac{1}{2}-\varepsilon}, H^{\frac{1}{2}+\varepsilon})} = \sum_{k \geq 0} \langle T_d, \phi_k \rangle \langle \phi, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} = \left\langle T_d, \sum_{k \geq 0} \langle \phi, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \phi_k \right\rangle = \langle T_d, \phi \rangle. \quad (4.2)$$

Let us denote by $\tilde{\chi}_d : H^{\frac{1}{2}+\varepsilon}(M) \rightarrow \mathbb{C}$ the characteristic functional of T_d , seen as a random variable in $H^{-\frac{1}{2}-\varepsilon}(M)$. By Equation (4.2) and the definition of the characteristic functional, for all $\phi \in H^{\frac{1}{2}+\varepsilon}(M)$, we have:

$$\tilde{\chi}_d(\phi) = \mathbb{E} \left[\exp \left(i \langle T_d, \phi \rangle_{(H^{-\frac{1}{2}-\varepsilon}, H^{\frac{1}{2}+\varepsilon})} \right) \right] = \mathbb{E} \left[e^{i \langle T_d, \phi \rangle} \right] \xrightarrow{d \rightarrow +\infty} e^{-\frac{1}{2}\|\phi\|_2^2},$$

where we used once again that $\langle T_d, \phi \rangle \xrightarrow{d \rightarrow +\infty} \mathcal{N}(0, \|\phi\|_2^2)$ in distribution.

By Definition 1.8, we have:

$$\mathbb{E} \left[\sum_{k \geq 0} (1 + \lambda_k)^{-\frac{1}{2}-\varepsilon} \left| \langle W, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \right|^2 \right] = \sum_{k \geq 0} \frac{\text{Var} \left(\langle W, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \right)}{(1 + \lambda_k)^{\frac{1}{2}+\varepsilon}} = \sum_{k \geq 0} \frac{1}{(1 + \lambda_k)^{\frac{1}{2}+\varepsilon}}.$$

Since M is 1-dimensional, the Weyl Law (cf. [15]) yields $\lambda_k \sim \frac{\pi}{\text{Vol}(M)} k^2$. Hence the previous sum is finite, which implies that $W \in H^{-\frac{1}{2}-\varepsilon}(M)$ almost surely. For all $\phi \in H^{\frac{1}{2}+\varepsilon}(M)$, we have:

$$\langle W, \phi \rangle_{(H^{-\frac{1}{2}-\varepsilon}, H^{\frac{1}{2}+\varepsilon})} = \sum_{k \geq 0} \langle W, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \langle \phi, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)}, \quad (4.3)$$

and by definition of W , the $\left(\langle W, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \right)_{k \geq 0}$ are i.i.d. real standard Gaussian variables. Since

$$\sum_{k \geq 0} \left| \langle \phi, \phi_k \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \right|^2 = \sum_{k \geq 0} |\langle \phi, \phi_k \rangle_{L^2}|^2 = \|\phi\|_2^2 < +\infty$$

the series on the right-hand side of Equation (4.3) converges absolutely in the space of square integrable random variables, hence in distribution. Moreover, the characteristic function of the partial sums converges pointwise to $t \mapsto e^{-\frac{t}{2}\|\phi\|_2^2}$. Thus, for all $\phi \in H^{\frac{1}{2}+\varepsilon}(M)$, the random variable $\langle W, \phi \rangle_{(H^{-\frac{1}{2}-\varepsilon}, H^{\frac{1}{2}+\varepsilon})}$ is distributed as $\mathcal{N}(0, \|\phi\|_2^2)$. In particular, the characteristic functional of W , seen as a random variable in $H^{-\frac{1}{2}-\varepsilon}(M)$, is $\tilde{\chi} : \phi \mapsto e^{-\frac{1}{2}\|\phi\|_2^2}$.

Finally, for all $\phi \in H^{-\frac{1}{2}-\varepsilon}(M)$, $\tilde{\chi}_d(\phi) \xrightarrow{d \rightarrow +\infty} \tilde{\chi}(\phi)$. Hence, by the Lévy–Fénelix Theorem we have $T_d \xrightarrow{d \rightarrow +\infty} W$ in distribution in $H^{-\frac{1}{2}-\varepsilon}(M)$. \square

4.3 Proof of Corollaries 1.14 and 1.15

We first prove Corollary 1.14, that is the concentration in probability. This is just an application of Markov's Inequality.

Proof of Corollary 1.14. Let $(\varepsilon_d)_{d \geq 1}$ be a positive sequence and let $p \in \mathbb{N}^*$. By Markov's inequality for the $2p$ -th moment, we have:

$$\mathbb{P}\left(d^{-\frac{1}{2}} |\langle \nu_d, \phi \rangle - \mathbb{E}[\langle \nu_d, \phi \rangle]| > \varepsilon_d\right) \leq \varepsilon_d^{-2p} d^{-p} m_{2p}(\nu_d)(\phi, \dots, \phi).$$

By Theorem 1.12, the term on the right-hand side is $O\left(\left(d^{\frac{1}{4}}\varepsilon_d\right)^{-2p}\right)$. As usual, one obtains the statement about $\text{Card}(Z_d)$ by specializing the result for $\phi = \mathbf{1}$. \square

The proof of Corollary 1.15, that is of the Hole probability, uses Corollary 1.14 with the right test-function and a constant sequence $(\varepsilon_d)_{d \geq 1}$. It is similar to the proof of [19, Corollary 1.10].

Proof of Corollary 1.15. Let U be a non-empty open subset of M , and let $\phi_U : M \rightarrow \mathbb{R}$ be a continuous function such that for all $x \in M$, $\phi_U(x) > 0$ if $x \in U$, and $\phi_U(x) = 0$ otherwise. Let $\varepsilon > 0$ be such that $\varepsilon < \frac{1}{\pi} \int_M \phi_U |dV_M|$. By Theorem 1.1, for all d large enough we have $d^{-\frac{1}{2}} \mathbb{E}[\langle Z_d, \phi_U \rangle] > \varepsilon$. For a degree d such that this condition holds, we have:

$$\mathbb{P}(Z_d \cap U = \emptyset) = \mathbb{P}(\langle Z_d, \phi_U \rangle = 0) \leq \mathbb{P}\left(d^{-\frac{1}{2}} \left| \langle Z_d, \phi_U \rangle - \mathbb{E}[\langle Z_d, \phi_U \rangle] \right| > \varepsilon\right).$$

By Corollary 1.14, the right-hand side is $O(d^{-\frac{p}{2}})$ for all $p \in \mathbb{N}^*$. \square

References

- [1] R. J. Adler and J. E. Taylor, *Random fields and geometry*, 1st ed., Monographs in Mathematics, Springer, New York, 2007.
- [2] M. Ancona, *Random sections of line bundles over real Riemann surfaces*, arXiv: 1806.10481 (2018).
- [3] D. Armentano, J.-M. Azaïs, F. Dalmao, and J. R. Leõn, *Central Limit Theorem for the number of real roots of Kostlan Shub Smale random polynomial systems*, arXiv: 1801.06331 (2018).
- [4] ———, *Central Limit Theorem for the volume of the zero set of Kostlan Shub Smale random polynomial systems*, arXiv: 1808.02967 (2018).
- [5] J.-M. Azaïs and M. Wschebor, *Level sets and extrema of random processes and fields*, 1st ed., John Wiley & Sons, Hoboken, NJ, 2009.
- [6] R. Basu, A. Dembo, N. Feldheim, and O. Zeitouni, *Exponential concentration of zeroes of stationary Gaussian processes*, arXiv: 1709.06760 (2017).
- [7] P. Billingsley, *Probability and measure*, 3rd ed., Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, New York, 1995.

- [8] P. Bleher, B. Shiffman, and S. Zelditch, *Universality and scaling of correlations between zeros on complex manifolds*, *Invent. Math.* **142** (2000), no. 2, 351–395.
- [9] E. Bogomolny, O. Bohigas, and P. Leboeuf, *Distribution of roots of random polynomials*, *Phys. Rev. Lett.* **68** (1992), no. 18, 2726–2729.
- [10] F. Dalmao, *Asymptotic variance and CLT for the number of zeros of Kostlan–Shub–Smale random polynomials*, *C. R. Math. Acad. Sci. Paris* **353** (2015), no. 12, 1141–1145.
- [11] X. Fernique, *Processus linéaires, processus généralisés*, *Ann. Inst. Fourier* **17** (1967), 1–92.
- [12] D. Gayet and J.-Y. Welschinger, *Exponential rarefaction of real curves with many components*, *Publ. Math. Inst. Hautes Études Sci.* (2011), no. 113, 69–96.
- [13] ———, *Betti numbers of random real hypersurfaces and determinants of random symmetric matrices*, *J. of Eur. Math. Soc.* **18** (2016), no. 4, 733–772.
- [14] Ph. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, New York, 1994, Reprint of the 1978 original.
- [15] L. Hörmander, *The spectral function of an elliptic operator*, *Acta Math.* **121** (1968), 193–218.
- [16] E. Kostlan, *On the distribution of roots of random polynomials*, *From topology to computation: proceedings of the Smalefest (Berkeley, CA, 1990)*, Springer, New York, 1993, pp. 419–431.
- [17] M. F. Kratz and J. R. León, *Central limit theorems for level functionals of stationary Gaussian processes and fields*, *J. Theoret. Probab.* **14** (2001), no. 3, 639–672.
- [18] T. Letendre, *Expected volume and Euler characteristic of random submanifolds*, *J. Funct. Anal.* **270** (2016), no. 8, 3047–3110.
- [19] ———, *Variance of the volume of random real algebraic submanifolds*, *Trans. Amer. Math. Soc.* **371** (2019), no. 6, 4129–4192.
- [20] T. Letendre and M. Puchol, *Variance of the volume of random real algebraic submanifolds II*, *Indiana Univ. Math. J.* (2019), to be published.
- [21] X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, 1st ed., *Progress in Mathematics*, vol. 254, Birkhäuser, Basel, 2007.
- [22] ———, *Exponential estimate for the asymptotics of Bergman kernels*, *Math. Ann.* **362** (2015), no. 3–4, 1327–1347.
- [23] F. Nazarov and M. Sodin, *Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions*, *Zh. Mat. Fiz. Anal. Geom.* **12** (2016), no. 3, 205–278.
- [24] P. Olver, *Geometric foundations of numerical algorithms and symmetry*, *Appl. Algebra Engrg. Comm. Comput.* **11** (2001), no. 5, 417–436.
- [25] M. Rossi, *Random nodal lengths and Wiener chaos*, arxiv: 1803.09716 (2018).
- [26] B. Shiffman and S. Zelditch, *Distribution of zeros of random and quantum chaotic sections of positive line bundles*, *Comm. Math. Phys.* **200** (1999), no. 3, 661–683.
- [27] ———, *Number variance of random zeros on complex manifolds II: smooth statistics*, *Pure Appl. Math. Q.* **6** (2010), no. 4, 1145–1167.
- [28] M. Sodin and B. Tsirelson, *Random complex zeroes I: asymptotic normality*, *Israel J. Math.* **144** (2004), 125–149.
- [29] H. Triebel, *Theory of function spaces II*, *Monographs in Mathematics*, vol. 84, Birkhäuser Verlag, Basel, 1992.