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# A finite-volume scheme for a cross-diffusion model arising from interacting many-particle population systems

Ansgar Jüngel and Antoine Zurek

**Abstract** A finite-volume scheme for a cross-diffusion model arising from the mean-field limit of an interacting particle system for multiple population species is studied. The existence of discrete solutions and a discrete entropy production inequality is proved. The proof is based on a weighted quadratic entropy that is not the sum of the entropies of the population species.

**Key words:** Finite volume scheme, cross-diffusion system, entropy method.

**MSC (2010):** 35K51, 35K55, 35Q92, 65M08

## 1 Introduction

### 1.1 Presentation of the model

We consider the following cross-diffusion system:

$$\partial_t u_i + \operatorname{div}(-\delta \nabla u_i - u_i \nabla p_i(u)) = 0, \quad p_i(u) = \sum_{j=1}^n a_{ij} u_j \quad \text{in } \Omega, t > 0, \quad (1)$$

where  $i = 1, \dots, n$  with  $n \geq 2$ ,  $\Omega \subset \mathbb{R}^2$  is an open bounded polygonal domain, and  $\delta > 0$ ,  $a_{ij} > 0$ . We impose the initial and no-flux boundary conditions

$$u_i(0) = u_i^0 \geq 0 \quad \text{in } \Omega, \quad \nabla u_i \cdot \nu = 0 \quad \text{on } \partial\Omega, t > 0, i = 1, \dots, n, \quad (2)$$

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where  $\nu$  is the exterior unit normal vector on  $\partial\Omega$ . We write  $u := (u_1, \dots, u_n)$  and  $u^0 := (u_1^0, \dots, u_n^0)$ . Equations (1) are derived from a weakly interacting stochastic many-particle system in the mean-field limit [7]. It can be seen as a simplification of the Shigesada-Kawasaki-Teramoto (SKT) population model [12], where the diffusion is reduced to  $\delta\nabla u_i$ . The two-species system was analyzed first in [3], but up to now, no analytical or numerical results are available for the  $n$ -species system. The diffusion matrix associated to (1) is neither symmetric nor positive definite but we show below that system (1) possesses an entropy structure [10] yielding gradient estimates that are the basis for the numerical analysis.

We assume that  $A := (a_{ij}) \in \mathbb{R}^{n \times n}$  is positively stable (i.e., all eigenvalues of  $A$  have positive real part) and that the detailed-balance condition holds, i.e., there exist numbers  $\pi_1, \dots, \pi_n > 0$  such that

$$\pi_i a_{ij} = \pi_j a_{ji} \quad \text{for all } i, j = 1, \dots, n. \quad (3)$$

We refer to [6] for an interpretation of this condition and its connection to Markov chains. Note that for the two-species model this condition is always satisfied, just set  $\pi_1 = a_{21}$  and  $\pi_2 = a_{12}$ . Since  $A_1 := \text{diag}(\pi_i^{-1})$  is symmetric, positive definite and  $A_2 := (\pi_i a_{ij})$  is symmetric, by [11, Prop. 6.1], the number of positive eigenvalues of  $A = A_1 A_2$  equals that for  $A_2$ . Thus,  $A_2$  has only positive eigenvalues, which together with the symmetry means that  $A_2$  is symmetric, positive definite.

Our (numerical) analysis is based on the observation that system (1) possesses an entropy structure with a weighted quadratic entropy that has not been observed before in cross-diffusion systems:

$$H[u] = \int_{\Omega} h(u) dx, \quad \text{where } h(u) := \frac{1}{2\delta} \sum_{i,j=1}^n \pi_i a_{ij} u_i u_j = \frac{1}{2\delta} u^T A_2 u,$$

where  $(A_2)_{ij} = \pi_i a_{ij}$ . Interestingly, this entropy is not of the form  $\sum_{i=1}^n h_i(u_i)$ , but it mixes the species. A formal computation shows that

$$\frac{dH}{dt} + \sum_{i,j=1}^n \pi_i a_{ij} \int_{\Omega} \nabla u_i \cdot \nabla u_j dx + \frac{1}{\delta} \sum_{i=1}^n \pi_i \int_{\Omega} u_i |\nabla p_i(u)|^2 dx = 0.$$

With  $\lambda > 0$  being the smallest eigenvalue of  $A_2$ , we conclude the following entropy production inequality:

$$\frac{dH}{dt} + \lambda \sum_{i=1}^n \int_{\Omega} |\nabla u_i|^2 dx + \frac{1}{\delta} \sum_{i=1}^n \pi_i \int_{\Omega} u_i |\nabla p_i(u)|^2 dx \leq 0.$$

Our aim is to prove this inequality for the finite-volume solutions.

## 1.2 The numerical scheme

A mesh of  $\Omega$  is given by a set  $\mathcal{T}$  of open polygonal control volumes, a set  $\mathcal{E}$  of edges, and a set  $\mathcal{P}$  of points  $(x_K)_{K \in \mathcal{T}}$ . We assume that the mesh is admissible in the sense of Definition 9.1 in [9]. We distinguish in  $\mathcal{E}$  the interior edges  $\sigma = K|L$  and the exterior edges such that  $\mathcal{E} = \mathcal{E}_{\text{int}} \cup \mathcal{E}_{\text{ext}}$ . For a given control volume  $K \in \mathcal{T}$ , we denote by  $\mathcal{E}_K$  the set of its edges. This set splits into  $\mathcal{E}_K = \mathcal{E}_{\text{int},K} \cup \mathcal{E}_{\text{ext},K}$ . For any  $\sigma \in \mathcal{E}$ , there exists at least one cell  $K \in \mathcal{T}$  such that  $\sigma \in \mathcal{E}_K$  and we denote this cell by  $K_\sigma$ . When  $\sigma$  is an interior edge,  $\sigma = K|L$ ,  $K_\sigma$  can be either  $K$  or  $L$ . For all  $\sigma \in \mathcal{E}$ , we define  $d_\sigma = d(x_K, x_L)$  if  $\sigma = K|L \in \mathcal{E}_{\text{int}}$  and  $d_\sigma = d(x_K, \sigma)$  if  $\sigma \in \mathcal{E}_{\text{ext},K}$ . Then the transmissibility coefficient is defined by  $\tau_\sigma = m(\sigma)/d_\sigma$  for all  $\sigma \in \mathcal{E}$ . We assume that the mesh satisfies the following regularity constraint:

$$\exists \xi > 0, \forall K \in \mathcal{T}, \forall \sigma \in \mathcal{E}_K : d(x_K, \sigma) \geq \xi d_\sigma. \quad (4)$$

The size of the mesh is denoted by  $\Delta x = \max_{K \in \mathcal{T}} \text{diam}(K)$ . Let  $N_T \in \mathbb{N}$  be the number of time steps,  $\Delta t = T/N_T$  be the time step size, and  $t_k = k\Delta t$  for  $k = 0, \dots, N_T$ .

Let  $\mathcal{H}_{\mathcal{T}}$  be the linear space of functions  $\Omega \rightarrow \mathbb{R}$  which are constant on each  $K \in \mathcal{T}$ . For  $v \in \mathcal{H}_{\mathcal{T}}$ , we introduce

$$D_{K,\sigma} v = v_{K,\sigma} - v_K, \quad D_\sigma v = |D_{K,\sigma} v| \quad \text{for all } K \in \mathcal{T}, \sigma \in \mathcal{E}_K,$$

where  $v_{K,\sigma}$  is either  $v_L$  ( $\sigma = K|L$ ) or  $v_K$  ( $\sigma \in \mathcal{E}_{\text{ext},K}$ ). Finally, we define the (squared) discrete  $H^1$  norm

$$\|v\|_{1,2,\mathcal{T}}^2 = \sum_{\sigma \in \mathcal{E}} \tau_\sigma (D_\sigma v)^2 + \sum_{K \in \mathcal{T}} m(K) v_K^2.$$

For all  $K \in \mathcal{T}$  and  $i = 1, \dots, n$ ,  $u_{i,K}^0$  denotes the mean value of  $u_i^0$  over  $K$ . The finite-volume scheme for (1) reads as

$$\frac{m(K)}{\Delta t} (u_{i,K}^k - u_{i,K}^{k-1}) + \sum_{\sigma \in \mathcal{E}_K} \mathcal{F}_{i,K,\sigma}^k = 0, \quad i = 1, \dots, n, \quad (5)$$

$$\mathcal{F}_{i,K,\sigma}^k = -\tau_\sigma (\delta D_{K,\sigma} u_i^k + u_{i,\sigma}^k D_{K,\sigma} p_i(u^k)) \quad \text{for all } K \in \mathcal{T}, \sigma \in \mathcal{E}_K, \quad (6)$$

with  $u^k = (u_1^k, \dots, u_n^k)$  and  $u_{i,\sigma}^k := \min\{u_{i,K}^k, u_{i,K,\sigma}^k\}$ . As in [1], this definition of  $u_{i,\sigma}^k$  allows us to prove the nonnegativity of  $u_{i,K}^k$ . This property can be also obtained by an upwind approximation of  $u_i \nabla p_i(u)$  in (1).

## 1.3 Main result

The main result of this work is the existence of nonnegative solutions to scheme (5)-(6), which preserve the entropy production inequality.

**Theorem 1 (Existence of discrete solutions).** *Assume that  $u^0 \in L^2(\Omega)^n$  with  $u_i^0 \geq 0$ ,  $\delta > 0$ ,  $a_{ij} > 0$ ,  $A$  is positively stable, and (3) holds. Then there exists a solution  $(u_K^k)_{K \in \mathcal{T}, k=0, \dots, N_T}$  with  $u_K^k = (u_{1,K}^k, \dots, u_{n,K}^k)$  to scheme (5)-(6) satisfying  $u_{i,K}^k \geq 0$  for all  $K \in \mathcal{T}$ ,  $i = 1, \dots, n$ , and  $k = 0, \dots, N_T$ . Moreover, the following discrete entropy production inequality holds:*

$$\begin{aligned} \sum_{K \in \mathcal{T}} m(K)h(u_K^k) + \Delta t \lambda \sum_{i=1}^n \sum_{\sigma \in \mathcal{E}} \tau_\sigma (D_\sigma u_i^k)^2 \\ + \frac{\Delta t}{\delta} \sum_{i=1}^n \sum_{\sigma \in \mathcal{E}} \tau_\sigma \pi_i u_{i,\sigma}^k (D_\sigma p_i(u^k))^2 \leq \sum_{K \in \mathcal{T}} m(K)h(u_K^{k-1}), \end{aligned} \quad (7)$$

where  $\lambda$  denotes the smallest eigenvalue of  $A_2$ .

We expect that the detailed-balance condition (3) can be replaced by a weak cross-diffusion condition as in [6]. The positive stability of  $A$  implies the parabolicity of (1) in the sense of Petrovskii. Indeed,  $A_2$ , defined by  $(A_2)_{ij} = \pi_i a_{ij}$ , and  $A_3 = \text{diag}(u_i/\pi_i)$  are symmetric, positive definite matrices for  $u \in (0, \infty)^n$ . Thus, its product  $(u_i a_{ij})$  has only positive eigenvalues [4, Theorem 7] which proves the claim. The assumption that the diffusion coefficient  $\delta$  is the same for all species is a simplification needed to conclude that  $h(u)$  is coercive,  $h(u) \geq (\lambda/2\delta)|u|^2$  for  $u \in \mathbb{R}^n$ . It can be removed by exploiting the Shannon entropy to show first that  $u_i$  is nonnegative, but this requires more technical effort which will be detailed in a future work.

## 2 Proof of Theorem 1

We proceed by induction. For  $k = 0$ , we have  $u_i^0 \geq 0$  by assumption. Assume that there exists a solution  $u^{k-1}$  for some  $k \in \{1, \dots, N_T\}$  such that  $u_i^{k-1} \geq 0$  in  $\Omega$ ,  $i = 1, \dots, n$ . The construction of a solution  $u^k$  is split in several steps.

*Step 1: Definition of a linearized problem.* Let  $R > 0$ , we set

$$Z_R := \{w = (w_1, \dots, w_n) \in (\mathcal{H}_{\mathcal{T}})^n : \|w_i\|_{1,2,\mathcal{T}} < R \text{ for } i = 1, \dots, n\},$$

and let  $\varepsilon > 0$  be given. We define the mapping  $F_\varepsilon : Z_R \rightarrow \mathbb{R}^{\theta n}$  by  $F_\varepsilon(w) = w^\varepsilon$ , with  $\theta = \#\mathcal{T}$ , where  $w^\varepsilon = (w_1^\varepsilon, \dots, w_n^\varepsilon)$  is the solution to the linear problem

$$\varepsilon \left( - \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma D_{K,\sigma} (w_i^\varepsilon) + m(K) w_{i,K}^\varepsilon \right) = - \left( \frac{m(K)}{\Delta t} (u_{i,K} - u_{i,K}^{k-1}) + \sum_{\sigma \in \mathcal{E}_K} \mathcal{F}_{i,K,\sigma}^+ \right), \quad (8)$$

for  $K \in \mathcal{T}$ ,  $i = 1, \dots, n$ , and  $\mathcal{F}_{i,K,\sigma}^+$  is defined in (6) with  $u_{i,\sigma}$  replaced by  $\bar{u}_{i,\sigma} = \min\{u_{i,K}^+, u_{i,K,\sigma}^+\}$ , where  $z^+ = \max\{0, z\}$ . Here,  $u_{i,K}$  is a function of  $w_{1,K}, \dots, w_{n,K}$ , defined by the entropy variables

$$w_{i,K} = \frac{\pi_i}{\delta} p_i(u_K) = \sum_{j=1}^n \frac{\pi_i a_{ij}}{\delta} u_j \quad \text{for all } K \in \mathcal{T}, i = 1, \dots, n. \quad (9)$$

This is a linear system with the invertible coefficient matrix  $A_2/\delta$ , and so, the function  $u_K = u(w_K)$  is well-defined. The existence of a unique solution  $w_i^\varepsilon$  to the linear scheme (8)-(9) is now a consequence of [9, Lemma 3.2].

*Step 2: Continuity of  $F_\varepsilon$ .* We fix  $i \in \{1, \dots, n\}$ . Multiplying (8) by  $w_{i,K}^\varepsilon$  and summing over  $K \in \mathcal{T}$ , we obtain, after discrete integration by parts,

$$\varepsilon \|w_i^\varepsilon\|_{1,2,\mathcal{T}}^2 = - \sum_{K \in \mathcal{T}} \frac{m(K)}{\Delta t} (u_{i,K} - u_{i,K}^{k-1}) w_{i,K}^\varepsilon + \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \mathcal{F}_{i,K,\sigma}^+ D_{K,\sigma} w_i^\varepsilon =: J_1 + J_2.$$

By the Cauchy-Schwarz inequality and the definition of  $\mathcal{F}_{i,K,\sigma}^+$ , we find that

$$\begin{aligned} |J_1| &\leq \frac{1}{\Delta t} \left( \sum_{K \in \mathcal{T}} m(K) (u_{i,K} - u_{i,K}^{k-1})^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}} m(K) (w_{i,K}^\varepsilon)^2 \right)^{1/2} \\ |J_2| &\leq \left( \sum_{\sigma \in \mathcal{E}} \tau_\sigma (\delta D_\sigma u_i + \bar{u}_{i,\sigma} D_\sigma p_i(u))^2 \right)^{1/2} \left( \sum_{\sigma \in \mathcal{E}} \tau_\sigma (D_\sigma w_i^\varepsilon)^2 \right)^{1/2}. \end{aligned}$$

Hence, since  $u_i$  is a linear combination of  $(w_1, \dots, w_n) \in Z_R$ , there exists a constant  $C(R) > 0$  which is independent of  $w^\varepsilon$  such that  $|J_1| + |J_2| \leq C(R) \|w_i^\varepsilon\|_{1,2,\mathcal{T}}$ . Inserting these estimations, it follows that  $\varepsilon \|w_i^\varepsilon\|_{1,2,\mathcal{T}} \leq C(R)$ .

We turn to the proof of the continuity of  $F_\varepsilon$ . Let  $(w^m)_{m \in \mathbb{N}} \subset Z_R$  be such that  $w^m \rightarrow w$  as  $m \rightarrow \infty$ . The previous estimate shows that  $w^{\varepsilon,m} := F_\varepsilon(w^m)$  is bounded uniformly in  $m \in \mathbb{N}$ . Thus, there exists a subsequence of  $(w^{\varepsilon,m})$ , which is not relabeled, such that  $w^{\varepsilon,m} \rightarrow w^\varepsilon$  as  $m \rightarrow \infty$ . Passing to the limit  $m \rightarrow \infty$  in scheme (8)-(9) and taking into account the continuity of the nonlinear functions, we see that  $w_i^\varepsilon$  is a solution to (8)-(9) for  $i = 1, \dots, n$  and  $w^\varepsilon = F_\varepsilon(w)$ . Because of the uniqueness of the limit function, the whole sequence converges, which proves the continuity.

*Step 3: Existence of a fixed point.* We claim that the map  $F_\varepsilon$  admits a fixed point. We use a topological degree argument [8], i.e., we prove that  $\deg(I - F_\varepsilon, Z_R, 0) = 1$ , where  $\deg$  is the Brouwer topological degree. Since  $\deg$  is invariant by homotopy, it is sufficient to prove that any solution  $(w^\varepsilon, \rho) \in \bar{Z}_R \times [0, 1]$  to the fixed-point equation  $w^\varepsilon = \rho F_\varepsilon(w^\varepsilon)$  satisfies  $(w^\varepsilon, \rho) \notin \partial Z_R \times [0, 1]$  for sufficiently large values of  $R > 0$ . Let  $(w^\varepsilon, \rho)$  be a fixed point and  $\rho \neq 0$ , the case  $\rho = 0$  being clear. Then  $w_i^\varepsilon$  solves

$$\varepsilon \left( - \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma D_{K,\sigma} (w_i^\varepsilon) + m(K) w_{i,K}^\varepsilon \right) = -\rho \left( \frac{m(K)}{\Delta t} (u_{i,K}^\varepsilon - u_{i,K}^{k-1}) + \sum_{\sigma \in \mathcal{E}_K} \mathcal{F}_{i,K,\sigma}^{+,\varepsilon} \right), \quad (10)$$

for all  $K \in \mathcal{T}$ ,  $i = 1, \dots, n$ , and  $\mathcal{F}_{i,K,\sigma}^{+,\varepsilon}$  is defined as in (6) with  $u$  replaced by  $w^\varepsilon$  which is related to  $w^\varepsilon$  by (9). The following discrete entropy production inequality is the key argument.

**Lemma 1 (Discrete entropy production inequality).** *Let the assumptions of Theorem 1 hold. Then, for any  $\rho \in (0, 1]$  and  $\varepsilon > 0$ ,*

$$\begin{aligned} & \rho \sum_{K \in \mathcal{T}} m(K) h(u_K^\varepsilon) + \varepsilon \Delta t \sum_{i=1}^n \|w_i^\varepsilon\|_{1,2,\mathcal{T}}^2 + \rho \Delta t \lambda \sum_{i=1}^n \sum_{\sigma \in \mathcal{E}} \tau_\sigma (D_\sigma u_i^\varepsilon)^2 \\ & + \rho \frac{\Delta t}{\delta} \sum_{i=1}^n \sum_{\sigma \in \mathcal{E}} \tau_\sigma \pi_i \bar{u}_{i,\sigma}^\varepsilon (D_\sigma p_i(u^\varepsilon))^2 \leq \rho \sum_{K \in \mathcal{T}} m(K) h(u_K^{k-1}), \end{aligned} \quad (11)$$

with  $\lambda > 0$  being the smallest eigenvalue of  $A_2$  and obvious notations for  $\bar{u}_{i,\sigma}^\varepsilon$ .

*Proof.* We multiply (10) by  $\Delta t w_{i,K}^\varepsilon$  and sum over  $i$  and  $K \in \mathcal{T}$ . This gives, after discrete integration by parts,  $\varepsilon \Delta t \sum_{i=1}^n \|w_i^\varepsilon\|_{1,2,\mathcal{T}}^2 + J_3 + J_4 + J_5 = 0$ , where

$$\begin{aligned} J_3 &= \rho \sum_{i=1}^n \sum_{K \in \mathcal{T}} m(K) (u_{i,K}^\varepsilon - u_{i,K}^{k-1}) w_{i,K}^\varepsilon, \\ J_4 &= -\rho \Delta t \sum_{i=1}^n \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma \delta D_{K,\sigma} u_i^\varepsilon w_{i,K}^\varepsilon, \\ J_5 &= \rho \Delta t \sum_{i=1}^n \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma \bar{u}_{i,\sigma}^\varepsilon D_{K,\sigma} p_i(u^\varepsilon) D_{K,\sigma} w_{i,K}^\varepsilon. \end{aligned}$$

To estimate  $J_3$ , we use the convexity of  $h$ ; for  $J_4$ , we take into account the symmetry of  $\tau_\sigma$  with respect to  $\sigma = K|L$ , definition (9) of  $w_i^\varepsilon$  and the positive definiteness of  $A_2$ ; and for  $J_5$ , we employ definition (9) of  $w_i^\varepsilon$ :

$$\begin{aligned} J_3 &\geq \rho \sum_{K \in \mathcal{T}} m(K) (h(u_K^\varepsilon) - h(u_K^{k-1})), \\ J_4 &= \rho \Delta t \sum_{i,j=1}^n \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma \pi_i a_{ij} D_{K,\sigma} u_i^\varepsilon D_{K,\sigma} u_j^\varepsilon \geq \rho \Delta t \lambda \sum_{i=1}^n \sum_{\sigma \in \mathcal{E}} \tau_\sigma (D_\sigma u_i^\varepsilon)^2, \\ J_5 &= \rho \frac{\Delta t}{\delta} \sum_{i=1}^n \sum_{\sigma \in \mathcal{E}} \tau_\sigma \pi_i \bar{u}_{i,\sigma}^\varepsilon (D_\sigma p_i(u^\varepsilon))^2. \end{aligned}$$

Putting all the estimations together completes the proof.  $\square$

We proceed with the topological degree argument. Lemma 1 implies that

$$\varepsilon \Delta t \sum_{i=1}^n \|w_i^\varepsilon\|_{1,2,\mathcal{T}}^2 \leq \rho \sum_{K \in \mathcal{T}} m(K) h(u_K^{k-1}) \leq \sum_{K \in \mathcal{T}} m(K) h(u_K^{k-1}).$$

Then, if we define  $R := (\varepsilon \Delta t)^{-1/2} (\sum_{K \in \mathcal{T}} m(K) h(u_K^{k-1}))^{1/2} + 1$ , we conclude that  $w^\varepsilon \notin \partial Z_R$  and  $\deg(I - F_\varepsilon, Z_R, 0) = 1$ . Thus,  $F_\varepsilon$  admits a fixed point.

*Step 4: Limit  $\varepsilon \rightarrow 0$ .* Recall that  $h(u_K) \geq \lambda / (2\delta) |u_K|^2$  (note that  $u_{i,K} \in \mathbb{R}$  at this point). Thus, by Lemma 1, there exists a constant  $C > 0$  depending only on the mesh

but not on  $\varepsilon$  such that for all  $K \in \mathcal{T}$  and  $i = 1, \dots, n$ ,

$$|u_{i,K}^\varepsilon| \leq C(\lambda) \left( \sum_{K \in \mathcal{T}} m(K) h(u_K^{k-1}) \right)^{1/2}.$$

Thus, up to a subsequence, for  $i = 1, \dots, n$  and for all  $K \in \mathcal{T}$ , we infer the existence of  $u_{i,K} \in \mathbb{R}$  such that  $u_{i,K}^\varepsilon \rightarrow u_{i,K}$  as  $\varepsilon \rightarrow 0$ . We deduce from (11) that there exists a subsequence (not relabeled) such that  $\varepsilon w_{i,K}^\varepsilon \rightarrow 0$  for any  $K \in \mathcal{T}$  and  $i = 1, \dots, n$ . Hence, the limit  $\varepsilon \rightarrow 0$  in (8) yields the existence of a solution to (8) with  $\varepsilon = 0$ .

Let  $i \in \{1, \dots, n\}$  and  $K \in \mathcal{T}$  such that  $u_{i,K} = \min_{L \in \mathcal{T}} u_{i,L}$ . We multiply (8) with  $\varepsilon = 0$  by  $\Delta t u_{i,K}^-$  with  $z^- = \min\{0, z\}$  and use the induction hypothesis:

$$\begin{aligned} m(K)(u_{i,K}^-)^2 - \Delta t \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma (\delta + a_{ii} \bar{u}_{i,\sigma}) D_{K,\sigma}(u_i) u_{i,K}^- \\ - \Delta t \sum_{j \neq i} \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma a_{ij} \bar{u}_{i,\sigma} D_{K,\sigma}(u_j) u_{i,K}^- = 0. \end{aligned}$$

The second term is nonpositive since  $\bar{u}_{i,\sigma} \geq 0$  and  $D_{K,\sigma}(u_i) \geq 0$ , by the choice of  $K$ . The last term vanishes since  $\bar{u}_{i,\sigma} u_{i,K}^- = u_{i,K}^+ u_{i,K}^- = 0$ , by the definition of  $\bar{u}_{i,\sigma}$ . This shows that  $u_{i,L} \geq u_{i,K} \geq 0$  for all  $L \in \mathcal{T}$  and  $i = 1, \dots, n$ . Passing to the limit  $\varepsilon \rightarrow 0$  in (11) yields inequality (7), which completes the proof of Theorem 1.

### 3 Convergence analysis and perspectives

In this section, we sketch the proof of the convergence of the scheme and possible extensions of the method presented in this paper.

- Let us give the main features of the proof of convergence. First, thanks to the a priori estimates given by (7) and assumption (4), we prove the existence of a constant  $C > 0$  independent of  $\Delta x$  and  $\Delta t$  such that for all  $i = 1, \dots, n$  and  $\phi \in C_0^\infty(Q_T)$ , where  $Q_T := \Omega \times (0, T)$ ,

$$\sum_{k=1}^{N_T} \sum_{K \in \mathcal{T}} m(K) (u_{i,K}^k - u_{i,K}^{k-1}) \phi(x_K, t_k) \leq C \|\nabla \phi\|_{L^\infty(Q_T)}. \quad (12)$$

Next, we consider a sequence of admissible meshes  $(\mathcal{T}_\eta, \Delta t_\eta)_{\eta > 0}$  of  $Q_T$ , indexed by the size  $\eta = \{\Delta x, \Delta t\}$ , satisfying (4) uniformly in  $\eta$ . For any  $\eta > 0$ , we denote by  $u_\eta = (u_{1,\eta}, \dots, u_{n,\eta})$  the piecewise constant (in time and space) finite-volume solution constructed in Theorem 1. We deduce, thanks to [2, Theorem 3.9] and (12), that there exist nonnegative functions  $u_1, \dots, u_n$  such that, up to a subsequence,

$$u_{i,\eta} \rightarrow u_i \quad \text{a.e. in } Q_T \text{ as } \eta \rightarrow 0, \quad i = 1, \dots, n.$$



Moreover, we conclude from (7) that  $u_{i,\eta}$  is uniformly bounded in  $L^\infty(0, T; L^2(\Omega))$  and  $L^2(0, T; L^p(\Omega))$  for  $p < \infty$  thanks to (7) and Sobolev embedding. We deduce from the Riesz–Thorin theorem that  $(u_{i,\eta})$  is bounded in  $L^r(Q_T)$  for some  $2 < r < 4$  and thus, it is equi-integrable. Thus, applying the Vitali convergence theorem, we infer that, up to a subsequence,  $u_{i,\eta} \rightarrow u_i$  strongly in  $L^r(Q_T)$  for all  $r < 4$  as  $\eta \rightarrow 0$ ,  $i = 1, \dots, n$ . The discrete entropy production inequality yields a uniform bound of the discrete gradient  $\nabla^\eta$  of  $u_{i,\eta}$  in  $L^2(Q_T)$ ; see [5] for a definition of  $\nabla^\eta$ . It follows from [5, Lemma 4.4] that, up to a subsequence,

$$\nabla^\eta u_{i,\eta} \rightharpoonup \nabla u_i \quad \text{weakly in } L^2(Q_T) \text{ as } \eta \rightarrow 0, \quad i = 1, \dots, n.$$

Finally, following the method developed in [5], we prove that the limit function  $u = (u_1, \dots, u_n)$  is a weak solution to (1)-(2).

- We already mentioned that system (1) can be interpreted as a simplification of the SKT model. In a future work, we will analyze a structure-preserving finite-volume approximation of the full SKT model. Such a discretization was analyzed in [1], but only for positive definite diffusion matrices associated to (1). We will extend the analysis of [1] without this assumption.

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