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Super-hedging a European option with a coherent risk-measure and without no-arbitrage condition

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Abstract: In this paper, we revisit the discrete-time super hedging problem of contingent claims with respect to a dynamic risk-measure defined by its acceptance sets. Without any no-arbitrage condition, we show that it is possible to characterize the prices of an European claim. Our analysis reveals a natural weak no-arbitrage condition that we study. This is a condition formulated for the prices instead of the attainable claims. Our approach is not based on a robust representation of the risk-measure and we do not suppose the existence of a risk-neutral probability measure.

Keywords and phrases: Risk-hedging prices, Dynamic risk-measures, Absence of immediate profit, Random sets, Conditional essential infimum.

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JEL Classification: C02, C61, G13.

1. Introduction

The problem of super-hedging a contingent claim ξ_T at time $T > 0$ by a self-financing portfolio process V such that $V_T \geq \xi_T$ is very classical in

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mathematical finance. For frictionless markets, we consider a no-arbitrage condition (NA) characterized by the fundamental theorem of asset pricing (FTAP), see [8, 22, 24] in discrete time and [9, 10] in continuous time. For discrete-time models, the Dalang-Morton-Willinger theorem [8] characterizes the NA condition as equivalent to the existence of a martingale probability measure under which the discounted price process is a martingale. Moreover, under NA, the set of all super-hedging prices is closed and the minimal price is the supremum of the expected discounted payoff under the equivalent probability measures, see [29] and [25, Theorem 2.1.11].

Contrarily to the classical approach, where the inequality $V_T \geq \xi_T$ holds with probability 1, Cherny [5] proposes to relax this constraint which is difficult to achieve in practice. He assumes that the portfolio manager accepts to take a (reasonable) risk for the portfolio not to super-hedge the payoff. To do so, a risk-measure ρ_T is considered and the hedging error $\epsilon_T = V_T - \xi_T$ is only supposed to be acceptable at the maturity date, i.e. $\rho_T(\epsilon_T) \leq 0$. The classical case of the literature coincides with the specific risk-measure defined by $\rho_T(X) = -\text{ess inf}(X)$, where the essential infimum $\text{ess inf}(X)$ is the minimal element of the support of X , see [2]. In the setting of coherent risk-measures, the classical notion of arbitrage opportunity is replaced by the concept of good deal, i.e. a non negative claim attainable from a negative risk. Cherny [5] formulates a version of the FTAP theorem, i.e. characterizes the absence of good deals (NGD). Moreover, he provides upper and lower bounds for the prices of super-hedging and sub-hedging strategies [4] in the case of discrete time coherent risk-measures.

It is well known that risk-measures are usually defined on L^∞ and the spaces L^p , $p \in [1, \infty)$ allow natural extensions [28]. Actually, the choice of L^∞ and, more generally L^p , is mainly motivated by the dual representations of risk-measures. However, the space L^0 is more adapted to financial and actuarial problems such as hedging, pricing, portfolio choice, equilibrium and optimal reinsurance with respect to risk-measures. Delbaen [11, 12] extends the coherent risk-measures to L^0 by enlarging their range to $\mathbf{R} \cup \{+\infty\}$ as there is no real-valued coherent risk-measure on L^0 when the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless (see [12, Theorem 5.1]).

Actually, it seems to be hopeless to axiomatize the notion of a coherent risk-measure on L^0 (see [12, Definition 5.2]) and then to deduce a robust representation. Motivated by the representation theorem in L^∞ , Delbaen constructs a support functional associated with a set of probability measures and proves that it is a coherent risk-measure on L^0 under some conditions

(see [12, Theorem 5.4]). In detail, he truncates random variables from above, i.e. only considers possible future wealth up to some threshold as the space L^0 contains non integrable random variables. It is then possible to compute a risk-measure from the robust representation in L^∞ as a supremum of expected loss with respect to some absolutely continuous probability measures (see [12, Definition 5.3]). Therefore, the robust representation on L^∞ appears to be the key point to extend coherent risk-measures to L^0 , see [3, 4, 5]. This allows to formulate a FTAP with respect to NGD and solve super-replication problems. In this approach, coherent risk-measures remain characterised through families of probability measures which are not necessarily convenient to handle in practice, see e.g. the explicit representation of this family for the Weighted VaR risk-measure [3].

Conditional risk measures have been extensively studied in the last two decades. Dual representations of convex risk measures are given in [19] in the static case and an extension to the dynamic setting is proposed. Indifferent prices are studied when there exists a risk-neutral probability measure. In the paper [16], dynamics of convex risk measures are studied for bounded random variables, see an overview in [1]. Indifferent prices are recursively computed for translation-invariant preferences defined on L^0 . [7]. The relation between risk-measures and their acceptance sets is studied in the papers [6] and [15] among others. In the paper [20], an interesting approach is to consider conditional risk measures on modules of L^p -type, which is based on the notion of conditional integrability, see [17] and, [14], [32] for the L^0 modules and [21] for a comparison between the different approaches for conditional risk measures.

The aim of this paper is not to study once more the dynamics of (conditional) risk measures but to solve the risk-hedging problem with respect to a coherent risk-measure. The novelty is that we do not use any dual representation. Precisely, we consider dynamic coherent risk-measures directly defined on the space L^0 with values in $\bar{\mathbf{R}} = [-\infty, +\infty]$. They are naturally defined from acceptable sets, i.e. a risk-measure is seen as the minimal capital requirement added to the financial position for it to be acceptable. As such a capital may be infinite, we first define a risk-measure on its *effective domain* of all positions which are acceptable when adding a finite capital and then, we extend it to the whole space L^0 .

We do not make the assumption that there exists a risk-neutral probability measure for the price process, see also the paper [18] where the classical semimartingale setting is not supposed. Actually, we are able to solve the super-

hedging problem without any no-arbitrage condition in the one step model. Precisely, we characterize the set of risk-hedging prices of a non negative contingent claim by a jointly measurable random function $g(\omega, x)$ so that it is possible to compute the minimal risk-hedging price as the ω -wise infimum of $g(\omega, \cdot)$ on \mathbf{R} .

Our analysis reveals possible immediate profits with respect to the risk-measure, as introduced in [2]. Such an arbitrage opportunity allows to super-replicate the zero contingent claim from a negative price. A version of the fundamental theorem of asset pricing is therefore formulated to characterize the absence of weak arbitrage opportunities, we call immediate profits (AIP). Actually, this weak no-arbitrage condition AIP is not formulated for the whole market as it depends on the risk-measure or, equivalently, depends on the acceptable set chosen by the portfolio manager. We also consider a slightly stronger condition under which the sets of risk-hedging prices are closed for payoff functions of linear growth. Finally, in the discrete-time setting, the minimal risk-hedging prices are defined recursively under the weak no-arbitrage condition AIP.

The paper is organized as follows. Section 2 defines the dynamic risk-measures on the space L^0 with values in $\overline{\mathbf{R}} = [-\infty, +\infty]$. Section 3 introduces the problem of risk-hedging with respect to acceptable sets. We characterize the essential infimum of the risk-hedging prices as a (deterministic) ω -wise infimum. A version of fundamental theorem of asset pricing is then formulated. Section 4 provides characterizations of minimal risk-hedging prices and lower and upper bounds are obtained.

2. Dynamic risk-measures

In discrete-time, we consider a stochastic basis $(\Omega, \mathcal{F} := (\mathcal{F}_t)_{t=0, \dots, T}, \mathbb{P})$ where the complete ¹ σ -algebra \mathcal{F}_t represents the information of the market available at time t . For any $t \leq T$, $L^0(\mathbf{R}, \mathcal{F}_t)$ is the space of all \mathbf{R} -valued random variables which are \mathcal{F}_t -measurable, and endowed with the topology of convergence in probability, so that it is a metric space. Similarly, $L^p(\mathbf{R}, \mathcal{F}_t, \mathbb{P})$, $p \in [1, \infty)$ (resp. $p = \infty$), is the normed space of all \mathbf{R} -valued random variables which are \mathcal{F}_t -measurable and admit a moment of order p under the probability measure \mathbb{P} (resp. bounded). Without any confusions, we

¹This means that the σ -algebra contains the negligible sets so that an equality between two random variables is understood up to a negligible set.

omit the notation \mathbb{P} and just denote $L^p(\mathbf{R}, \mathcal{F}_t)$. In particular, $L^p(\mathbf{R}_+, \mathcal{F}_t) = \{X \in L^p(\mathbf{R}, \mathcal{F}_t) | X \geq 0\}$ and $L^p(\mathbf{R}_-, \mathcal{F}_t) = \{X \in L^p(\mathbf{R}, \mathcal{F}_t) | X \leq 0\}$ when $p = 0$ or $p \in [1, \infty]$. All equalities and inequalities between random variables are understood to hold everywhere on Ω up to a negligible set. If A_t is set-valued mapping (i.e. a random set), we denote by $L^0(A_t, \mathcal{F}_t)$ the set of all \mathcal{F}_t -measurable random variables X_t such that $X_t \in A_t$ a.s. The topology in L^0 is defined from the convergence in probability. We say that $X_t \in L^0(A_t, \mathcal{F}_t)$ is a measurable selection of A_t . In our paper, a random set A_t is said \mathcal{F}_t -measurable if it is graph-measurable, see [31]. It is well known that $L^0(A_t, \mathcal{F}_t) \neq \emptyset$ if and only if $A_t \neq \emptyset$ a.s., see [23, Th. 4.4]. When referring to this property, we shall say "by a measurable selection argument" as it is usual to do.

2.1. Construction from the acceptance set.

The dynamic risk-measure $X \mapsto (\rho_t(X))_{t \leq T}$ considered in this work is defined on L^0 . It is constructed from its acceptance sets defined as follows:

Definition 2.1. *A dynamic acceptable set is a family $(\mathcal{A}_t)_{t \leq T}$ of non empty subsets of $L^0(\mathbf{R}, \mathcal{F}_T)$ satisfying the following conditions at time $t \leq T$:*

- 1) $X + Y \in \mathcal{A}_t$ for all $X, Y \in \mathcal{A}_t$;
- 2) $Y \in \mathcal{A}_t$ whenever $Y \geq X$ and $X \in \mathcal{A}_t$;
- 3) $\mathcal{A}_t \cap L^0(\mathbf{R}, \mathcal{F}_t) = L^0(\mathbf{R}_+, \mathcal{F}_t)$;
- 4) $k_t X \in \mathcal{A}_t$ for any $X \in \mathcal{A}_t$ and $k_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$.

Any element of \mathcal{A}_t is said acceptable at time t . Note that \mathcal{A}_t is a convex cone. For any $X \in L^0(\mathbf{R}, \mathcal{F}_T)$, we denote by \mathcal{A}_t^X the set of all $C_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ such that $X + C_t \in \mathcal{A}_t$, i.e,

$$\mathcal{A}_t^X := \{C_t \in L^0(\mathbf{R}, \mathcal{F}_t) | X + C_t \in \mathcal{A}_t\}.$$

Note that \mathcal{A}_t^X may be empty. We denote by $\text{Dom } \mathcal{A}_t$ the set of all $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ such that $\mathcal{A}_t^X \neq \emptyset$, i.e.

$$\text{Dom } \mathcal{A}_t := \{X \in L^0(\mathbf{R}, \mathcal{F}_T) | \mathcal{A}_t^X \neq \emptyset\}.$$

In the following, we use the notation $\overline{\mathbf{R}} = [-\infty, \infty]$.

Definition 2.2. Let $(\mathcal{A}_t)_{t \leq T}$ be a dynamic acceptance set. The coherent risk-measure associated to $(\mathcal{A}_t)_{t \leq T}$ is defined at any time t by the mapping $\rho_t : \text{Dom } \mathcal{A}_t \rightarrow L^0(\overline{\mathbf{R}}, \mathcal{F}_t)$ such that:

$$\rho_t(X) := \text{ess inf } \mathcal{A}_t^X, \quad X \in \text{Dom } \mathcal{A}_t. \quad (2.1)$$

Observe that $\rho_t(X)$ is the minimal \mathcal{F}_t -measurable capital requirement we add to the position X for it to be acceptable at time t . It is clear that $\rho_t(X) < \infty$ when $\mathcal{A}_t^X \neq \emptyset$.

Definition 2.3. A family $(\rho_t)_{0 \leq t \leq T}$ is called dynamic coherent risk-measure if ρ_t is a coherent risk-measure defined by (2.1) for each $t \leq T$.

In the following, we formulate some properties satisfied by the coherent risk-measures ρ_t as defined in Definition 2.2. The proofs are postponed in Appendix B.1.

Proposition 2.4. The risk-measure ρ_t defined by (2.1) satisfies the following properties on $\text{Dom } \mathcal{A}_t$:

Normalization: $\rho_t(0) = 0$;

Monotonicity: $\rho_t(X) \geq \rho_t(X')$ whatever $X, X' \in \text{Dom } \mathcal{A}_t$ s.t. $X \leq X'$;

Cash invariance: $\rho_t(X + m_t) = \rho_t(X) - m_t$ if $m_t \in L^0(\mathbf{R}, \mathcal{F}_t)$, $X \in \text{Dom } \mathcal{A}_t$;

Subadditivity: $\rho_t(X + X') \leq \rho_t(X) + \rho_t(X')$ if $X, X' \in \text{Dom } \mathcal{A}_t$;

Positive homogeneity: $\rho_t(k_t X) = k_t \rho_t(X)$ if $k_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$, $X \in \text{Dom } \mathcal{A}_t$.

Moreover, if the acceptable set \mathcal{A}_t is closed, then ρ_t is lower semi-continuous², satisfies $\rho_t(X) > -\infty$ a.s. for all $X \in \text{Dom } \mathcal{A}_t$, and \mathcal{A}_t can be represented by ρ_t as

$$\mathcal{A}_t = \{X \in \text{Dom } \mathcal{A}_t | \rho_t(X) \leq 0\}. \quad (2.2)$$

2.2. Normalizations of the sets \mathcal{A}_t^X , $X \in \text{Dom } \mathcal{A}_t$.

Recall that ρ_t is lower semi-continuous means the following: If $X_n \rightarrow X$, then $\rho_t(X) \leq \liminf_n \rho_t(X_n)$ a.s. We may suppose that the inequality holds everywhere on Ω as $\rho_t(X)$ may be modified on a negligible set since \mathcal{A}_t^X is \mathcal{F}_t -decomposable, see the proof of Proposition 2.4. Actually, this means

²The definition is recalled in the next section.

that $\rho_t(X)$ is not uniquely defined except if we consider $\rho_t(X)$ as a class of equivalent random variables. In the following, we suppose that \mathcal{A}_t is closed for every $t \leq T$ and we propose to *normalize* the sets \mathcal{A}_t^X (up to negligible sets) for $X \in \text{Dom } \mathcal{A}_t$ so that it is possible to uniquely define $\rho_t(X)$ on the whole space Ω whatever $X \in \text{Dom } \mathcal{A}_t$.

Lemma 2.5. *Suppose that \mathcal{A}_t , $t \leq T$, is closed and let $X \in \text{Dom } \mathcal{A}_t$. Then, there exists a \mathcal{F}_t -measurable closed and convex set A_t^X such that $\mathcal{A}_t^X = L^0(A_t^X, \mathcal{F}_t)$. Moreover, $A_t^X(\omega) = \text{cl} \{C_n(\omega) : n \geq 1\}$ for every $\omega \in \Omega$, where $(C_n)_{n \geq 1}$ is a countable family of \mathcal{A}_t^X ³.*

Proof. Observe that \mathcal{A}_t^X is \mathcal{F}_t -decomposable and closed by the assumption on \mathcal{A}_t . Therefore, it suffices to apply [30, Corollary 2.5 and Proposition 2.7]. \square

Let us consider for every $X \in \text{Dom } \mathcal{A}_t$, the subset $\tilde{\mathcal{A}}_t^X \subseteq \mathcal{A}_t^X$ of all elements $C \in \mathcal{A}_t^X$ such that $C(\omega) \in A_t^X(\omega)$ for every $\omega \in \Omega$. Note that $\tilde{\mathcal{A}}_t^X \neq \emptyset$ by Lemma 2.5. Moreover, each $C \in \mathcal{A}_t^X$ satisfies $C \in A_t^X$ a.s., i.e. C coincides with an element of $\tilde{\mathcal{A}}_t^X$ up to a negligible set. It is then natural to replace \mathcal{A}_t^X by $\tilde{\mathcal{A}}_t^X$ as the elements of these two sets admit the same equivalence classes, up to a negligible set. Note that $\tilde{\mathcal{A}}_t^X$ is still \mathcal{F}_t -decomposable, closed for the convergence everywhere, and it is an upper set⁴ as \mathcal{A}_t^X .

At last, for $X \in \text{Dom } \mathcal{A}_t$, it is worth noticing that $\text{ess inf } \tilde{\mathcal{A}}_t^X$ coincides with $\text{ess inf } \mathcal{A}_t^X$ up to a negligible set. Our goal is then to choose a specific element representing the equivalence class of $\text{ess inf } \tilde{\mathcal{A}}_t^X$. To do so, we use the following:

Proposition 2.6. *For all $X \in \text{Dom } \mathcal{A}_t$,*

$$\inf \tilde{\mathcal{A}}_t^X = \inf_n C_n \in A_t^X,$$

everywhere on Ω , where $(C_n)_{n \geq 1}$ is a Castaing representation of \mathcal{A}_t^X .

Proof. Since $A_t^X = \text{cl} \{C_n : n \geq 1\}$ on Ω , it is clear that $\inf_{C \in \tilde{\mathcal{A}}_t^X} C = \inf_n C_n$ is \mathcal{F}_t -measurable. The conclusion follows. \square

Corollary 2.7. *For all $X \in \text{Dom } \mathcal{A}_t$, $\inf \tilde{\mathcal{A}}_t^X$ is \mathcal{F}_t -measurable. We deduce that $\rho_t(X) := \inf \tilde{\mathcal{A}}_t^X$ is a version of the essential infimum $\text{ess inf } \mathcal{A}_t^X$. Moreover, $\mathcal{A}_t^X = L^0([\rho_t(X), \infty))$.*

³ $(C_n)_{n \geq 1}$ is called a Castaing representation of \mathcal{A}_t^X .

⁴ A set Γ is said upper if $\Gamma + \mathbf{R}_+ \subseteq \Gamma$.

Proof. As $\mathcal{A}_t^X(\omega) = \text{cl}\{C_n(\omega) : n \geq 1\}$ and $\mathcal{A}_t^X = L^0(\mathcal{A}_t^X, \mathcal{F}_t)$, it follows that $C \geq \inf \tilde{\mathcal{A}}_t^X$ a.s. for any $C \in \mathcal{A}_t^X$. Thus, $\inf \tilde{\mathcal{A}}_t^X$ is a version of $\text{ess inf } \mathcal{A}_t^X = \rho_t(X)$ and finally $\mathcal{A}_t^X = L^0([\rho_t(X), \infty))$. \square

2.3. Extension of the risk-measure to $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$.

In the following, we suppose that the acceptance sets are closed in L^0 . For any $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$, consider the set

$$\Lambda_t(X) := \{F_t \in \mathcal{F}_t : X1_{F_t} \in \text{Dom } \mathcal{A}_t\}.$$

The arguments to show the following lemma are classical, see for example [33, Section 1.2].

Lemma 2.8. *For any $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$, $\Lambda_t(X)$ is directed-upward and admits a maximal element $F_t(X)$.*

Proof. We need to show that $F_t^{(1)} \cup F_t^{(2)} \in \Lambda_t(X)$ for any $F_t^{(1)}, F_t^{(2)} \in \Lambda_t(X)$. To see it recall that, by Definition 2.1, $X_1 + X_2 \in \text{Dom } \mathcal{A}_t$ for any $X_1, X_2 \in \text{Dom } \mathcal{A}_t$ and $k_t X \in \text{Dom } \mathcal{A}_t$ for any $X \in \text{Dom } \mathcal{A}_t$ and $k_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$. Then, for any $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ and $F_t^{(1)}, F_t^{(2)} \in \Lambda_t(X)$, we get that

$$X1_{F_t^{(1)} \cup F_t^{(2)}} = X(1_{F_t^{(1)}} + 1_{F_t^{(2)}} - 1_{F_t^{(1)}}1_{F_t^{(2)}}) = X1_{F_t^{(1)}}(1 - 1_{F_t^{(2)}}) + X1_{F_t^{(2)}} \in \text{Dom } \mathcal{A}_t,$$

i.e. $F_t^{(1)} \cup F_t^{(2)} \in \Lambda_t(X)$. Consider $\gamma := \text{ess sup}\{1_{F_t} : F_t \in \Lambda_t(X)\}$. As $\Lambda_t(X)$ is directed-upward, there exists an increasing sequence $(F_t^n)_{n \geq 1} \in \Lambda_t(X)$ such that $\gamma = 1_{F_t(X)}$ where $F_t(X) = \cup_n F_t^n$. Let us define the sequence $(E_t^n)_{n \geq 1}$ by $E_t^n = F_t^n \setminus F_t^{n-1}$, $n \geq 1$, where $F_t^0 = \emptyset$. Then, any two sets of $\{E_t^n, n \geq 1\}$ are disjoint and $\cup_n F_t^n = \cup_n E_t^n$. Moreover, for every $n \geq 1$, there exists $C_t^n \in L^0(\mathbf{R}, \mathcal{F}_t)$ such that $X1_{F_t^n} + C_t^n \in \mathcal{A}_t$. We may suppose w.l.o.g. that $F_t^n = E_t^n$. Summing up, we deduce that $X1_{F_t(X)} + C_t \in \mathcal{A}_t$ where $C_t = \sum_n C_t^n 1_{E_t^n}$ belongs to \mathcal{A}_t since the later is closed and \mathcal{F}_t -decomposable. We deduce that $F_t(X)$ is the maximal element of $\Lambda_t(X)$. \square

Lemma 2.9. *Suppose that $(\zeta_n)_{n \geq 1}$ is a sequence of real-valued and bounded random variables. Suppose that $\lim_{n \rightarrow \infty} \zeta_n = X$ a.s. where $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$. Then, $\liminf_n \rho_t(\zeta_n) \in \{-\infty, \infty\}$ a.s. on the set $\Omega \setminus F_t(X)$. Moreover, if $(\zeta'_n)_{n \geq 1}$ is another sequence satisfying the same properties than $(\zeta_n)_{n \geq 1}$, then $\liminf_n \rho_t(\zeta_n) = \liminf_n \rho_t(\zeta'_n)$ on the set $\Omega \setminus F_t(X)$.*

Proof. Since ζ_n is bounded, $\zeta_n \in \text{Dom } \mathcal{A}_t$, for all $n \geq 1$, and we have $\rho_t(\zeta_n) + \zeta_n \in \mathcal{A}_t$. Suppose that $P(\Gamma_t) > 0$ where

$$\Gamma_t = \{\liminf_n \rho_t(\zeta_n) \notin \{-\infty, \infty\}\} \cap (\Omega \setminus F_t(X)).$$

Then, almost surely on Γ_t , we may construct a \mathcal{F}_t -measurable subsequence n_k such that $\rho_t(\zeta_{n_k}) \rightarrow \alpha_t \in L^0(\mathbf{R}, \mathcal{F}_t)$. Since $\rho_t(\zeta_{n_k}) + \zeta_{n_k} \in \mathcal{A}_t$ and \mathcal{A}_t is closed, we deduce that $(\alpha_t + X)1_{\Gamma_t} \in \mathcal{A}_t$, as $k \rightarrow \infty$. This contradicts the definition of $F_t(X)$ as a maximal element.

For the second statement, suppose that $\liminf_n \rho_t(\zeta_n) = +\infty$ while $\liminf_n \rho_t(\zeta'_n) = -\infty$ on a non null \mathcal{F}_t -measurable subset Γ_t of $\Omega \setminus F_t(X)$. Let us define $\gamma_n = \lambda_n \zeta_n + (1 - \lambda_n) \zeta'_n$ where $\lambda_n \in L^0([0, 1], \mathcal{F}_t)$. The sequence $(\gamma_n)_{n \geq 1}$ satisfies the same properties than $(\zeta_n)_{n \geq 1}$, i.e., $\lim_{n \rightarrow \infty} \gamma_n = X$ a.s. We now choose

$$\lambda_n =: \frac{|\rho_t(\zeta'_n)|}{|\rho_t(\zeta'_n)| + |\rho_t(\zeta_n)|}.$$

Therefore, for n large enough, on the set Γ_t , we have

$$\rho_t(\gamma_n) = \rho_t(\lambda_n \zeta_n + (1 - \lambda_n) \zeta'_n) \leq \lambda_n \rho_t(\zeta_n) + (1 - \lambda_n) \rho_t(\zeta'_n) = 0.$$

It follows that $\gamma_n 1_{\Gamma_t} \in \mathcal{A}_t$ and, taking the limit as $n \rightarrow \infty$, we deduce that $X 1_{\Gamma_t} \in \mathcal{A}_t$. This contradicts the maximality of $F_t(X)$. \square

By Lemmas 2.8 and 2.9, it is possible to extend ρ_t on the whole space $L^0(\overline{\mathbf{R}}, \mathcal{F}_t)$. Precisely, we define $f_n(x) = ((-n) \vee x) \wedge n$, $n \geq 1$, and we set

$$\begin{aligned} \rho_t(X) : &= \rho_t(X 1_{F_t(X)}), \text{ on } F_t(X), \\ &= \liminf_n \rho_t(f_n(X)) = \pm\infty, \text{ on } (\Omega \setminus F_t(X)). \end{aligned}$$

Note that by Lemma 2.9, the definition above does not depend on the sequence $(f_n(X))_{n \geq 1}$ but only on X . We claim that Proposition 2.4 may be extended to $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$. The proofs can be found in Appendix B.2.

Proposition 2.10. *Proposition 2.4 holds on $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$, with the extended acceptance set $\mathcal{A}_t = \{X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T) : \rho_t(X) \leq 0\}$ and with the conventions $0 \times (\pm\infty) = 0$, $(0, \infty) \times (\pm\infty) = \{\pm\infty\}$ and $\mathbf{R} + (\pm\infty) = \pm\infty$ and $\infty - \infty = -\infty + \infty = +\infty$. For $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$, $\rho_t(X)$ may be infinite so that $\rho_t(X) \in \mathbf{R}$ a.s. if and only if $X \in \text{Dom } \mathcal{A}_t$.*

Example 2.11. For the classical one-step super-hedging problem, a contingent claim h_T may be super-replicated at time $T - 1$ if there exist a price $P_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ and a strategy $\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ such that $P_{T-1} + \theta_{T-1}\Delta S_T - h_T \geq 0$ a.s. This means that the acceptable set \mathcal{A}_{T-1} is

$$\begin{aligned}\mathcal{A}_{T-1} &= \{X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T) \mid X \geq 0\} \\ &= \{X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T) \mid \text{ess inf}_{\mathcal{F}_{T-1}} X \geq 0\} \\ &= \{X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T) \mid -\text{ess inf}_{\mathcal{F}_{T-1}} X \leq 0\}.\end{aligned}$$

In particular, we have $\rho_{T-1}(X) = -\text{ess inf}_{\mathcal{F}_{T-1}} X$.

Remark 2.12. Since $X + \text{ess sup}_{\mathcal{F}_{T-1}}(-X) \geq 0 \in \mathcal{A}_{T-1}$ and $\text{ess sup}_{\mathcal{F}_{T-1}}(-X)$ is \mathcal{F}_{T-1} -measurable, we get that

$$\rho_{T-1}(X) \leq \text{ess sup}_{\mathcal{F}_{T-1}}(-X) = -\text{ess inf}_{\mathcal{F}_{T-1}}(X).$$

3. Risk-hedging problem and absence of immediate profit

In discrete-time, let $(S_t)_{t \leq T}$ be the discounted price process of a risky asset such that $S_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ for any $t \geq 0$. Let $(\rho_t)_{t \leq T}$ be a dynamic risk-measure as in Definition 2.3. A contingent claim with maturity T is defined by a real-valued \mathcal{F}_T -measurable random variable h_T . The goal is to find a self-financing strategy process $(\theta_t)_{t \leq T}$ to super-hedge the contingent claim h_T . Here, super-hedging needs to be understood with respect to an acceptable set.

3.1. Minimal risk-hedging prices in the one step model

Let us start with the one time-step model between time $T - 1$ and time T .

Definition 3.1. The contingent claim $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$ is said to be risk-hedged at time $T - 1$ if there exists a risk-hedging price $P_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ and a strategy $\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ such that $P_{T-1} + \theta_{T-1}\Delta S_T - h_T$ is acceptable at time $T - 1$.

Let $\mathcal{P}_{T-1}(h_T)$ be the set of all risk-hedging prices $P_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ at time $T - 1$ as in Definition 3.1. In the following, we suppose that h_T is non negative and $\mathcal{P}_{T-1}(h_T) \neq \emptyset$. This is the case if there exist $a_{T-1}, b_{T-1} \in$

$L^0(\mathbf{R}, \mathcal{F}_{T-1})$ such that $h_T \leq a_{T-1}S_T + b_{T-1}$. This property trivially holds for European call and put options.

By Proposition 2.10, we may extend the set of acceptable positions (2.2) to $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$:

$$\mathcal{A}_t = \{X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T) \mid \rho_t(X) \leq 0\}.$$

By cash invariance, we have $P_{T-1} + \theta_{T-1}\Delta S_T - h_T \in \mathcal{A}_{T-1}$ if and only if $P_{T-1} \geq \theta_{T-1}S_{T-1} + \rho_{T-1}(\theta_{T-1}S_T - h_T)$. Therefore, the set of risk-hedging prices is:

$$\begin{aligned} \mathcal{P}_{T-1}(h_T) = & \{\theta_{T-1}S_{T-1} + \rho_{T-1}(\theta_{T-1}S_T - h_T) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1}) \\ & + L^0(\mathbf{R}_+, \mathcal{F}_{T-1})\}. \end{aligned}$$

The next step is to construct a jointly measurable version of the random function

$$g_{T-1}(\omega, x) := xS_{T-1} + \rho_{T-1}(xS_T - h_T). \quad (3.3)$$

To do so, we consider the family \mathcal{G}_{T-1} of all $Z = (X, Y) \in L^0(\mathbf{R}^2, \mathcal{F}_{T-1})$ such that $Y \geq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ a.s. Since we suppose that $\mathcal{P}_{T-1}(h_T) \neq \emptyset$, \mathcal{G}_{T-1} is not empty.

Lemma 3.2. *Suppose that \mathcal{A}_{T-1} is closed. Then, \mathcal{G}_{T-1} is a closed convex subset of $L^0(\mathbf{R}^2, \mathcal{F}_{T-1})$. Moreover, \mathcal{G}_{T-1} is \mathcal{F}_{T-1} -decomposable and, for fixed $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$, the section $\{Y : (X, Y) \in \mathcal{G}_{T-1}\}$ is an upper set.*

Proof. Note that

$$\mathcal{G}_{T-1} = \{(X, Y) \in L^0(\mathbf{R}^2, \mathcal{F}_{T-1}) : X\Delta S_T - h_T + Y \in \mathcal{A}_{T-1}\}.$$

We deduce that \mathcal{G}_{T-1} is closed and convex since \mathcal{A}_{T-1} is closed by assumption and is a convex cone. Moreover, \mathcal{A}_{T-1} is \mathcal{F}_{T-1} -decomposable and so \mathcal{G}_{T-1} is. At last, for fixed $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$, since \mathcal{A}_{T-1} is an upper set, so is the X -section of \mathcal{G}_{T-1} . \square

Lemma 3.3. *Suppose that \mathcal{A}_{T-1} is closed. Then, there exists a non empty \mathcal{F}_{T-1} -measurable random closed set G_{T-1} such that $\mathcal{G}_{T-1} = L^0(G_{T-1}, \mathcal{F}_{T-1})$. Moreover, G_{T-1} is convex and $G_{T-1}(\omega) = \text{cl}\{Z^n(\omega) : n \geq 1\}$ for every $\omega \in \Omega$, where $(Z^n)_{n \geq 1}$ is a countable family of \mathcal{G}_{T-1} .*

Proof. By Lemma 3.2, we deduce that $\mathcal{G}_{T-1} = L^0(G_{T-1}, \mathcal{F}_{T-1})$ for some \mathcal{F}_{T-1} -measurable random closed set G_{T-1} , see [25, Proposition 5.4.3]. As \mathcal{G}_{T-1} is not empty, we deduce that $G_{T-1} \neq \emptyset$ a.s. Moreover, there exists a Castaing representation of G_{T-1} such that $G_{T-1}(\omega) = \text{cl}\{Z^n(\omega) : n \geq 1\}$ for every $\omega \in \Omega$, where $(Z^n)_{n \geq 1}$ is a countable family of \mathcal{G}_{T-1} , see [30, Proposition 2.7]. Then, by a contradiction argument and using a measurable selection argument, we may show that G_{T-1} is convex as \mathcal{G}_{T-1} . \square

The following propositions provide a jointly measurable function $g_{T-1}(\omega, x)$ such that $g_{T-1}(X) = XS_{T-1} + \rho_{T-1}(XS_T - h_T)$.

Proposition 3.4. *Suppose that \mathcal{A}_{T-1} is closed. There exists a $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$ -measurable function g_{T-1} such that $G_{T-1} = \{(x, y) : y \geq g_{T-1}(\omega, x)\}$ and, with $X, Y \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$, we have $Y \geq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ if and only if $Y \geq g_{T-1}(X)$. Moreover, $x \mapsto g_{T-1}(\omega, x)$ is a.s. convex and lower semi-continuous.*

Proof. Let us define the random function g_{T-1} as follows:

$$g_{T-1}(\omega, x) := \inf\{\alpha \in \mathbf{R} : (x, \alpha) \in G_{T-1}(\omega)\} \in [-\infty, \infty]. \quad (3.4)$$

We first show that g_{T-1} is $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$ -measurable. To see it, since the x -sections of G_{T-1} are upper sets, we get that

$$g_{T-1}(\omega, x) := \inf\{\alpha \in \mathbf{Q} : (x, \alpha) \in G_{T-1}(\omega)\}$$

where \mathbf{Q} is the set of all rational numbers of \mathbf{R} . Let us define the $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$ -measurable function $I(\omega, x) = 1$ if $(\omega, x) \in G_{T-1}$ and $I(\omega, x) = +\infty$ if $(\omega, x) \notin G_{T-1}$. Then, define, for each $\alpha \in \mathbf{Q}$, $\theta^\alpha(\omega, x) = \alpha I(\omega, x)$ with the convention $\mathbf{R} \times (+\infty) = +\infty$. As θ^α is $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$ -measurable, we deduce that $g_{T-1}(\omega, x) = \inf_{\alpha \in \mathbf{Q}} \theta^\alpha(\omega, x)$ is also $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$ -measurable.

Since G_{T-1} is closed, it is clear that $(x, g_{T-1}(\omega, x)) \in G_{T-1}(\omega)$ a.s. when $g_{T-1}(\omega, x) < \infty$ and, moreover, $g_{T-1}(\omega, x) > -\infty$ by Proposition 2.4. Therefore, $G_{T-1}(\omega)$ is the epigraph of the random function $x \mapsto g_{T-1}(\omega, x)$. As $Y \geq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ if and only if $(X, Y) \in \mathcal{G}_{T-1}$, or equivalently $(X, Y) \in G_{T-1}$ a.s., we deduce that it is equivalent to $Y \geq g_{T-1}(X)$.

Moreover, as G_{T-1} is convex, we deduce that $x \mapsto g_{T-1}(\omega, x)$ is a.s. convex. Let us show that $x \mapsto g_{T-1}(\omega, x)$ is a.s. lower-semi continuous. Consider a sequence $x^n \in \mathbf{R}$ which converges to $x_0 \in \mathbf{R}$. Let us denote $\beta^n := g_{T-1}(x^n)$. We have $(x^n, \beta^n) \in G_{T-1}$ from the above discussion. In the case where

$\inf_n \beta^n = -\infty$, $g_{T-1}(\omega, x) - 1 > \beta_n$ for n large enough (up to a subsequence) hence $(x^n, g_{T-1}(\omega, x) - 1) \in G_{T-1}(\omega)$ since the x^n -sections of G_{T-1} are upper sets. As $n \rightarrow \infty$, we deduce that $(x, g_{T-1}(\omega, x) - 1) \in G_{T-1}(\omega)$. This contradicts the definition of g_{T-1} . Moreover, the inequality $g_{T-1}(x) \leq \liminf_n \beta^n$ is trivial when the r.h.s. is $+\infty$. Otherwise, $\beta^\infty := \liminf_n \beta^n < \infty$ and $(x_0, \beta^\infty) \in G_{T-1}$ as G_{T-1} is closed. It follows by definition of g_{T-1} that $g_{T-1}(x_0) \leq \liminf_n g_{T-1}(x^n)$, i.e. g_{T-1} is lower-semi continuous. \square

Observe that, on the set $\{(\omega, x) : g_{T-1}(\omega, x) = +\infty\}$, G_{T-1} have empty x -sections.

Corollary 3.5. *We have $g_{T-1}(X) = XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ a.s. whatever $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$.*

Proof. Consider a measurable selection $(x_{T-1}, y_{T-1}) \in \mathcal{G}_{T-1} \neq \emptyset$. We have $y_{T-1} \geq g_{T-1}(x_{T-1})$ by definition hence $g_{T-1}(x_{T-1}) < \infty$ a.s. Let us define $X_{T-1} = x_{T-1}1_{g_{T-1}(X)=\infty} + X1_{g_{T-1}(X)<\infty}$. Since we have

$$g_{T-1}(X_{T-1}) = g_{T-1}(x_{T-1})1_{g_{T-1}(X)=\infty} + g_{T-1}(X)1_{g_{T-1}(X)<\infty},$$

is a.s. finite, $(X_{T-1}, g_{T-1}(X_{T-1})) \in G_{T-1}$ a.s. We deduce that

$$g_{T-1}(X_{T-1}) \geq X_{T-1}S_{T-1} + \rho_{T-1}(X_{T-1}S_T - h_T)$$

as $\mathcal{G}_{T-1} = L^0(G_{T-1}, \mathcal{F}_{T-1})$. Therefore, $g_{T-1}(X) \geq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ on the set $\{g_{T-1}(X) < \infty\}$. Moreover, the inequality trivially holds when $g_{T-1}(X) = +\infty$. Similarly, let us define

$$\begin{aligned} Y_{T-1} &= (XS_{T-1} + \rho_{T-1}(XS_T - h_T))1_{XS_{T-1} + \rho_{T-1}(XS_T - h_T) < \infty} \\ &+ y_{T-1}1_{XS_{T-1} + \rho_{T-1}(XS_T - h_T) = +\infty}. \end{aligned}$$

We have $(X_{T-1}, Y_{T-1}) \in \mathcal{G}_{T-1}$ a.s. hence, by definition of g , $g(X_{T-1}) \leq Y_{T-1}$. Then, $g(X) \leq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ on $\{XS_{T-1} + \rho_{T-1}(XS_T - h_T) < \infty\}$. The inequality being trivial on the complementary set, we finally conclude that the equality holds a.s. \square

From above, we rewrite $\mathcal{P}_{T-1}(h_T)$ as

$$\mathcal{P}_{T-1}(h_T) = \{g_{T-1}(\theta_{T-1}) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})\} + L^0(\mathbf{R}_+, \mathcal{F}_{T-1}). \quad (3.5)$$

Remark 3.6. When \mathcal{A}_{T-1} is closed, $\rho_{T-1}(xS_T - h_T)$ is distinct from $-\infty$ a.s. By construction of g_{T-1} , we also deduce a lower semi-continuous and jointly measurable version of

$$\hat{g}_{T-1}(\omega, x) := \rho_{T-1}(xS_T - h_T)(\omega, x) := g_{T-1}(\omega, x) - xS_{T-1}(\omega)$$

which is $\pm\infty$ if and only if $g_{T-1}(\omega, x) = \pm\infty$.

We then introduce the following random set:

$$\begin{aligned} \text{Dom } g_{T-1}(\omega) &:= \{x \in \mathbf{R} : g_{T-1}(\omega, x) < \infty\} \\ &= \{x \in \mathbf{R} : \rho_{T-1}(xS_T - h_T) < \infty\}. \end{aligned}$$

Observe that $\text{Dom } g_{T-1}$ is an upper set, i.e. an interval. Moreover, since $\mathcal{P}_{T-1}(h_T) \neq \emptyset$, there exists a strategy $a_{T-1} \in \text{Dom } g_{T-1}$ hence $\text{Dom } g_{T-1}$ contains the interval $[a_{T-1}, \infty)$. We then deduce the following:

Lemma 3.7. *Suppose that \mathcal{A}_{T-1} is closed. Then $\inf_{x \in \mathbf{R}} g_{T-1}(x)$ is \mathcal{F}_{T-1} -measurable and coincides with $\inf_{x \in \mathbf{Q}} g_{T-1}(x)$.*

Proof. The upper interval $\text{Dom } g_{T-1}$ admits a non empty interior on which g_{T-1} is convex hence continuous. It follows that

$$\inf_{x \in \mathbf{R}} g_{T-1}(x) = \inf_{x \in \text{Dom } g_{T-1}} g_{T-1}(x) = \inf_{x \in \text{int } \text{Dom } g_{T-1}} g_{T-1}(x) = \inf_{x \in \mathbf{Q} \cap \text{int } \text{Dom } g_{T-1}} g_{T-1}(x).$$

We deduce that $\inf_{x \in \mathbf{R}} g_{T-1}(x) \geq \inf_{x \in \mathbf{Q}} g_{T-1}(x)$ so that the equality holds and finally $\inf_{x \in \mathbf{R}} g_{T-1}(x)$ is \mathcal{F}_{T-1} -measurable. \square

A generalized concept of conditional essential supremum (resp. conditional essential infimum) of a family of vector-valued random variables with respect to a random partial order is introduced in [26, 27]. In the following, we use the simpler notion with respect to the natural partial order on \mathbf{R} for a family of real-valued random variables (see Appendix A).

Definition 3.8. *The minimal risk-hedging price of the contingent claim h_T at time $T - 1$ is defined as*

$$P_{T-1}^* := \text{ess inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}_{T-1}(h_T). \quad (3.6)$$

Note that the minimal risk-hedging price of h_T is not necessarily a price, i.e. an element of $\mathcal{P}_{T-1}(h_T)$ but we shall see that $P_{T-1}^* \in \mathcal{P}_{T-1}(h_T)$ under some extra conditions. If we introduce

$$\mathcal{P}'_{T-1}(h_T) := \{g_{T-1}(\theta_{T-1}) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})\},$$

we easily obtain that

$$P_{T-1}^* = \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}_{T-1}(h_T) = \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}'_{T-1}(h_T).$$

Note that the set $\mathcal{P}'_{T-1}(h_T)$ consists of all prices making it possible to hedge the contingent claim h_T up to a zero risk.

The following result concludes this section; it is possible to compute the infimum price P_{T-1}^* at time $T-1$ without any no-arbitrage condition. Precisely, P_{T-1}^* is the pointwise infimum of the random function g_{T-1} .

Lemma 3.9. *Suppose that \mathcal{A}_{T-1} is closed, then*

$$P_{T-1}^* = \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g_{T-1}(\theta_{T-1}) = \inf_{x \in \mathbf{R}} g_{T-1}(x). \quad (3.7)$$

Proof. By Lemma 3.7, we know that $\inf_{x \in \mathbf{R}} g_{T-1}(x)$ is \mathcal{F}_{T-1} -measurable. As $g_{T-1}(\theta_{T-1}) \geq \inf_{x \in \mathbf{R}} g_{T-1}(x)$ for any $\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$, we deduce that $\operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g_{T-1}(\theta_{T-1}) \geq \inf_{x \in \mathbf{R}} g_{T-1}(x)$ as $\inf_{x \in \mathbf{R}} g_{T-1}(x)$ is \mathcal{F}_{T-1} -measurable. Reciprocally, we have $g_{T-1}(x) \geq \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g_{T-1}(\theta_{T-1})$ a.s. for all $x \in \mathbf{Q}$. Therefore, we deduce that $\inf_{x \in \mathbf{Q}} g_{T-1}(x) \geq \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g_{T-1}(\theta_{T-1})$ and we get (3.7) by Lemma 3.7. \square

Note that it is unclear in general how to solve an optimization problem defined by the essential infimum. By Lemma 3.9, we have changed the problem into a deterministic one, i.e. it suffices to solve it for each fixed $\omega \in \Omega$. It is then possible to repeat this principle to time $T-2$, $T-3$, and so on. The natural condition of absence of immediate profit (IP) as introduced in [2] arises.

3.2. Fundamental theorem of asset pricing

Let us generalize the definition of acceptable set \mathcal{A}_t to $\mathcal{A}_{t,u} \subseteq L^0(\mathbf{R}, \mathcal{F}_u)$ for time between t and $u \in [t, T]$ by the same axiomatic conditions as in

Definition 2.1. In the sequel, all acceptable sets are supposed to be closed. The risk-measure $\rho_{t,u}$ is defined on $\text{Dom } \mathcal{A}_{t,u}$ and then extended to $L^0(\overline{\mathbf{R}}, \mathcal{F}_u)$ as above. The risk-measure process satisfies

$$\rho_{t,u}(X) = \text{ess inf}\{Y \in L^0(\mathbf{R}, \mathcal{F}_t) \mid X + Y \in \mathcal{A}_{t,u}\}, \quad X \in \text{Dom } \mathcal{A}_{t,u}$$

and the corresponding acceptable set can be represented as

$$\mathcal{A}_{t,u} = \{X \in \text{Dom } \mathcal{A}_{t,u} \mid \rho_{t,u}(X) \leq 0\}.$$

Consider a random variable $h_t \in L^0(\mathbf{R}, \mathcal{F}_t)$, which represents a contingent claim at time t . The general one time-step risk-hedging problem from t to $t+1$ for the contingent claim h_{t+1} aims to characterize the risk-hedging prices $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ and strategies $\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ such that $P_t + \theta_t \Delta S_{t+1} - h_{t+1}$ is acceptable with respect to the acceptable set $\mathcal{A}_{t,t+1}$. Equivalently, we may express the set of all risk-hedging prices as

$$\mathcal{P}_t(h_{t+1}) = \{\theta_t S_t + \rho_t(\theta_t S_{t+1} - h_{t+1}) : \theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)\} + L^0(\mathbf{R}_+, \mathcal{F}_t).$$

The minimal risk-hedging price at time t for this one time-step model is defined as

$$P_t^* := \text{ess inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(h_{t+1}). \quad (3.8)$$

Starting from the contingent claim h_T (see Section 3.1), we recursively define

$$P_T^* := h_T, \quad P_t^* := \text{ess inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*)$$

where P_{t+1}^* may be interpreted as a contingent claim h_{t+1} . The interesting question is whether P_t^* is actually a price, i.e. an element of $\mathcal{P}_t(P_{t+1}^*)$. However, we do not need for P_t^* to be a price to pursue the backward mechanism we propose.

An immediate profit is the possibility to super-replicate the zero contingent claim from a negative price.

Definition 3.10. *Absence of Immediate Profit (AIP) holds if, for any $t \leq T$,*

$$\mathcal{P}_t(0) \cap L^0(\mathbf{R}_-, \mathcal{F}_t) = \{0\}. \quad (3.9)$$

It is clear that AIP holds at time T since $\mathcal{P}_T(0) = L^0(\mathbf{R}_+, \mathcal{F}_T)$. Below, we characterize the AIP condition at any time.

Theorem 3.11 (Fundamental theorem of asset pricing). *The AIP condition holds if and only if, for all $t \leq T - 1$,*

$$-\rho_t(S_{t+1}) \leq S_t \leq \rho_t(-S_{t+1}). \quad (3.10)$$

Proof. Starting from $P_T^* = h_T = 0$, we obtain that the set of risk-hedging prices for the zero contingent claim at time $T - 1$ is

$$\mathcal{P}_{T-1}(0) = \{\theta_{T-1}S_{T-1} + \rho_{T-1}(\theta_{T-1}S_T) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})\} + L^0(\mathbf{R}_+, \mathcal{F}_{T-1})$$

and the minimal risk-hedging price is $P_{T-1}^* = \operatorname{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}_{T-1}(0)$. Note that AIP holds at time $T - 1$ means that $P_{T-1}^* = 0$. By Lemma 3.9, we have $P_{T-1}^* = \inf_{x \in \mathbf{R}} g_{T-1}(x)$, where $g_{T-1}(x) = xS_{T-1} + \rho_{T-1}(xS_T)$. We have

$$g_{T-1}(x) = x[S_{T-1} + \rho_{T-1}(S_T)1_{x \geq 0} - \rho_{T-1}(-S_T)1_{x < 0}].$$

With $\Lambda_{T-1} := \{-\rho_{T-1}(S_T) \leq S_{T-1} \leq \rho_{T-1}(-S_T)\}$, we can deduce that

$$P_{T-1}^* = (0)1_{\Lambda_{T-1}} + (-\infty)1_{\Omega \setminus \Lambda_{T-1}}.$$

Therefore, condition AIP holds at time $T - 1$ if and only if $\mathbb{P}(\Omega \setminus \Lambda_{T-1}) = 0$. At last, observe that, from a price for the zero claim at any instant t , it is possible to super-hedge a price at time $t + 1$. So, if the minimal price at time $t + 1$ is $P_{t+1}^* = 0$, then a price at any instant t is non negative. So, repeating the procedure backwardly at any time $t \leq T - 2$, we finally conclude. \square

Example 3.12. *In the classical problem of Example 2.11, recall that the risk-measure at time $T - 1$ is $\rho_{T-1}(X) = -\operatorname{ess\,inf}_{\mathcal{F}_{T-1}} X$, $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$. Then, by Theorem 3.11, AIP is equivalent to*

$$\operatorname{ess\,inf}_{\mathcal{F}_{t-1}} S_t \leq S_{t-1} \leq \operatorname{ess\,sup}_{\mathcal{F}_{t-1}} S_t, \quad 1 \leq t \leq T, \quad (3.11)$$

as formulated in [2, Theorem 3.4].

Remark 3.13. *The AIP condition for the risk-measure ρ_{T-1} implies the same for the risk-measure $\rho_{T-1}(X) = -\operatorname{ess\,inf}_{\mathcal{F}_{T-1}} X$. Indeed, by Remark 2.12 with S_T and $-S_T$, it holds that $\rho_{T-1}(-X) \leq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}}(X)$. Therefore,*

$$\operatorname{ess\,inf}_{\mathcal{F}_{T-1}} S_T \leq -\rho_{T-1}(S_T) \leq S_{T-1} \leq \rho_{T-1}(-S_T) \leq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} S_T.$$

Interpretation. Contrarily to classical no-arbitrage conditions, the AIP condition is not a no-arbitrage condition for the whole market, as it depends on the risk-measure or, equivalently, depends on the acceptable sets chosen by the portfolio manager. The AIP condition means that the portfolio manager evaluates the minimal price for the zero claim by 0. By Theorem 3.11, this is equivalent to say that there is no \mathcal{F}_t -measurable subset of Ω at time t on which the portfolio manager considers ΔS_{t+1} or $-\Delta S_{t+1}$ as strictly acceptable, i.e. such that $\rho_t(\pm \Delta S_{t+1}) < 0$. On the contrary case, he could make a positive profit $-\rho_t(\Delta S_{t+1})$ or $-\rho_t(-\Delta S_{t+1})$ and obtain at time t an acceptable position ΔS_{t+1} or $-\Delta S_{t+1}$ by investing in a short or long position.

Remark 3.14. *Suppose that there exists $Q \sim P$ such that S is a Q -martingale and*

$$\mathcal{A}_{T-1} \subseteq \{X \in L^1(\mathbf{R}, \mathcal{F}_T) : E_Q(X | \mathcal{F}_{T-1}) \geq 0\}.$$

Then, AIP holds.

Notice that the AIP condition is a generalization of the No Good Deal Condition introduced by Cherny, see [4, Definition 2.2].

4. Characterisation of minimal risk-hedging prices

4.1. Conditions under which the minimal risk-hedging price is a price

The main purpose of this section is to obtain the existence of optimal hedging strategies such that it is possible to hedge the contingent claim when starting from the minimal risk-hedging price, i.e. such that the minimal risk-hedging price is actually a price. In that case, the risk-measure of the hedging error vanishes.

4.1.1. One step-time model

Let us consider the one-step model between time $T - 1$ and T . We shall see that the minimal risk-hedging price P_{T-1}^* is actually a price as soon as AIP holds at time $T - 1$ with some extra conditions. To do so, we formulate the following technical lemmas. The proof of the following is postponed to Appendix B.3. Recall that g_{T-1} is defined in Section 3.1.

Lemma 4.1. *With the convention $0/0 = 0$, the following inequalities holds a.s.:*

$$S_{T-1} - \rho_{T-1}(-S_T) \leq \frac{g_{T-1}(y) - g_{T-1}(x)}{y - x} \leq S_{T-1} + \rho_{T-1}(S_T), \quad x, y \in \mathbf{R}^d.$$

Corollary 4.2. *The random function $x \mapsto g_{T-1}(x)$ is non increasing on the set $\{S_{T-1} + \rho_{T-1}(S_T) = 0\} \in \mathcal{F}_{T-1}$ and it is non decreasing on the set $\{S_{T-1} - \rho_{T-1}(-S_T) = 0\} \in \mathcal{F}_{T-1}$.*

Lemma 4.3. *If $S_{T-1} + \rho_{T-1}(S_T) > 0$, then $\lim_{x \rightarrow \infty} g_{T-1}(x) = +\infty$. If $S_{T-1} - \rho_{T-1}(-S_T) < 0$, then $\lim_{x \rightarrow -\infty} g_{T-1}(x) = +\infty$.*

Proof. For any $X \in L^0((0, \infty), \mathcal{F}_{T-1})$, we have

$$g_{T-1}(X) = XS_{T-1} + \rho_{T-1}(XS_T - h_T) = X \left(S_{T-1} + \rho_{T-1} \left(S_T - \frac{h_T}{X} \right) \right).$$

Following the proof of Lemma B.4, we get that

$$\left| \rho_{T-1} \left(S_T - \frac{h_T}{X} \right) - \rho_{T-1}(S_T) \right| \leq \frac{\max(|\rho_{T-1}(-h_T)|, |\rho_{T-1}(h_T)|)}{X}.$$

Therefore, we may choose $r_{T-1} \in L^0((0, \infty), \mathcal{F}_{T-1})$ large enough so that $X \geq r_{T-1}$ implies that $S_{T-1} + \rho_{T-1} \left(S_T - \frac{h_T}{X} \right) > 0$ (using the assumption) and finally $g_{T-1}(X) \geq M_{T-1}$ for a fixed $M_{T-1} \in L^0((0, \infty), \mathcal{F}_{T-1})$ we arbitrarily choose.

We then deduce that $\lim_{x \rightarrow \infty} g_{T-1}(x) = +\infty$. To see it, we argue by contradiction on the (non-null set) $B = \{\omega : \lim_{x \rightarrow \infty} g_{T-1}(x) \neq +\infty\}$. As g_{T-1} is continuous, the set B may be reformulated as

$$B = \{\omega : \exists M \in \mathbf{Q} \text{ s.t. } \forall r \in \mathbf{Q}, \exists x \in [r, \infty) \cap \mathbf{Q} : g(x) < M\}.$$

This proves that $B \in \mathcal{F}_{T-1}$. Moreover, the set

$$G_{T-1} = \{(\omega, M) \in \Omega \times \mathbf{R}_+ : \forall r \in \mathbf{Q}, \exists x \in [r, \infty) \cap \mathbf{Q} : g(x) < M\},$$

is jointly measurable w.r.t. $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$ and admits non empty ω -sections on B . By measurable selection arguments, there exists $M_{T-1} \in L^0(\mathbf{R}_+, \mathcal{F}_{T-1})$ such that $(\omega, M_{T-1}(\omega)) \in G_{T-1}$ on B and we set $M_{T-1} = +\infty$ otherwise.

Then, still by measurable selection arguments and using the definition of G_{T-1} , we deduce the existence of $X \geq r_{T-1}$ such that $g_{T-1}(X) < M_{T-1}$ on B . On the other hand, recall that $g_{T-1}(X)1_B \geq M_{T-1}1_B$ for every $X \geq r_{T-1}$ large enough. This yields to a contradiction hence $\lim_{x \rightarrow \infty} g_{T-1}(x) = +\infty$. Finally, we conclude for the case where $S_{T-1} - \rho_{T-1}(-S_T) < 0$ by similar arguments. \square

Since $\lim_{x \rightarrow \infty} g_{T-1}(x) = +\infty$ in the case where $S_{T-1} + \rho_{T-1}(S_T) > 0$, we deduce that $\inf_{x \in \mathbf{R}_+} g_{T-1}(x) = \min_{x \in \mathbf{R}_+} g_{T-1}(x)$ is a minimum. Otherwise, if $S_{T-1} + \rho_{T-1}(S_T) = 0$ under AIP, then g_{T-1} is non increasing and $\inf_{x \in \mathbf{R}_+} g_{T-1}(x) = g_{T-1}(\infty)$. Therefore, in the case where $S_{T-1} + \rho_{T-1}(S_T) = 0$, P_{T-1}^* is a price if and only if g_{T-1} is constant for x large enough. In the following result, we analyze the case where g_{T-1} is a constant as $x \rightarrow \pm\infty$.

Lemma 4.4. *Suppose that the random function $x \mapsto g_{T-1}(x)$ is constant for x large enough. Then, under AIP, $S_{T-1} + \rho_{T-1}(S_T) = 0$. Suppose that the random function $x \mapsto g_{T-1}(x)$ is constant for $-x$ large enough. Then, under AIP, $S_{T-1} - \rho_{T-1}(-S_T) = 0$.*

Proof. By assumption, we have $xS_{T-1} + \rho_{T-1}(xS_T - h_T) = c_{T-1}$ for all $x \geq \alpha_{T-1}$ where $\alpha_{T-1}, c_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$. We may suppose that $\alpha_{T-1} > 0$. Dividing by x and making $x \rightarrow \infty$, we deduce that $S_{T-1} + \rho_{T-1}(S_T) \leq 0$. We then conclude by Theorem 3.11. \square

Corollary 4.5. *Suppose that AIP holds. Then, the random function $x \mapsto g(x)$ is constant for x large enough and is constant for $-x$ large enough if and only if $S_{T-1} + \rho_{T-1}(S_T) = S_{T-1} - \rho_{T-1}(-S_T) = 0$. In that case, g_{T-1} is a constant function.*

Proof. It suffices to apply Lemma 4.4 and Lemma 4.1. \square

Theorem 4.6. *Suppose that AIP holds at time $T-1$ and $S_{T-1} + \rho_{T-1}(S_T) = 0$ if and only if $S_{T-1} - \rho_{T-1}(-S_T) = 0$. Then, the minimal risk-hedging price P_{T-1}^* is a price.*

Proof. When $S_{T-1} + \rho_{T-1}(S_T) = S_{T-1} - \rho_{T-1}(-S_T) = 0$, the statement is trivial as g_{T-1} is a constant function from Corollary 4.5. In particular, the strategy $\theta_{T-1}^* = 0$ is a candidate to hedge the payoff. Otherwise, we have $S_{T-1} + \rho_{T-1}(S_T) > 0$ and $S_{T-1} - \rho_{T-1}(-S_T) < 0$. By Lemma 4.3, $\lim_{x \rightarrow \infty} g_{T-1}(x) = \lim_{x \rightarrow -\infty} g_{T-1}(x) = +\infty$ so that the infimum of g is attained by

some strategy $\theta_{T-1}^* \in \mathbf{R}$. \square

The theorem above proves the existence of an optimal hedging strategy $\theta_{T-1}^* \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ such that

$$P_{T-1}^* = g_{T-1}(\theta_{T-1}^*) = \theta_{T-1}^* S_{T-1} + \rho_{T-1}(\theta_{T-1}^* S_T - h_T) \in \mathcal{P}_{T-1}(h_T).$$

4.1.2. Extension to the multi-period model

Definition 4.7. A stochastic process $(V_t)_{t \leq T}$ adapted to $(\mathcal{F}_t)_{t \leq T}$, starting from an initial endowment V_0 is a portfolio process if, for all $t \leq T-1$, there exists $\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ such that $V_t + \theta_t \Delta S_{t+1} - V_{t+1}$ is acceptable at time t . Moreover, we say that it hedges the payoff $h_T \in L^0([-\infty, \infty), \mathcal{F}_T)$ if $V_T \geq h_T$ a.s.

Note that $V_{T-1} + \theta_{T-1} \Delta S_T - V_T$ is supposed to be acceptable at time $T-1$. Therefore, $V_T \geq h_T$ implies that $V_{T-1} + \theta_{T-1} \Delta S_T - h_T$ is acceptable at time $T-1$. In the following, we actually set $V_T = h_T$ where $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$ is a European claim.

Notice that, if the infimum super-hedging price is $P_{T-1}^* = -\infty$ on some non null set, i.e. when AIP does not hold, then, the one step pricing procedure of the last section may be applied as we have extended the risk-measure on $L^0([-\infty, \infty], \mathcal{F}_T)$. This means that the following backward procedure applies without any no-arbitrage condition.

Recall that we have defined recursively $P_T^* = h_T$ and

$$\mathcal{P}_t(P_{t+1}^*) = \{\theta_t S_t + \rho_t(\theta_t S_{t+1} - P_{t+1}^*) : \theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)\} + L^0(\mathbf{R}_+, \mathcal{F}_t)$$

so that

$$P_t^* = \operatorname{ess\,inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*).$$

As in Section 3.1, in order to construct a jointly measurable version of the random function

$$g_t(\omega, x) := x S_t + \rho_t(x S_{t+1} - P_{t+1}^*), \quad (4.12)$$

we consider the family \mathcal{G}_t of all elements $Z = (X, Y) \in L^0(\mathbf{R}^2, \mathcal{F}_t)$ such that $Y \geq X S_t + \rho_t(X S_{t+1} - P_{t+1}^*)$ a.s. We need \mathcal{G}_t to be non-empty. This is why we suppose in the following that there exists a least one portfolio process $(V_t)_{t \leq T}$ such that V_T replicates h_T at time T with respect to the risk-measure

of consideration. This is the case if there exist two constants a, b such that $h_T \leq aS_T + b$. Therefore, we may express P_t^* as

$$P_t^* = \inf_{x \in \mathbf{R}} g_t(x).$$

Since we set $V_T = h_T$, $\rho_{T-1}(V_{T-1} + \theta_{T-1}\Delta S_T - h_T) \leq 0$. By definition of the one-step model, we deduce that $V_{T-1} \geq P_{T-1}^*$. By induction, we may easily show that $V_t \geq P_t^*$ for all $t \leq T$ since V_t is a risk-hedging price $V_{t+1} \geq P_{t+1}^*$ at time $t + 1$. In particular, $V_t \in \mathcal{P}_t(P_{t+1}^*) \neq \emptyset$ for all $t \in T - 1$.

So, it is possible to repeat backwardly the procedure developed in the one time step-model. In particular, if we suppose that the condition of Proposition 4.6 holds at any time $t \leq T$, we deduce by induction that each minimal risk-hedging price P_t^* is a price and we may finally obtain the minimal risk-hedging price P_0^* for the claim h_T . In any case, the computational procedure we propose allows to obtain the minimal prices at any time, which is enough for practical purposes.

4.2. Pricing under consistency in time

Definition 4.8. A dynamic risk-measure $(\rho_t)_{t \leq T}$ is said time-consistent if $\rho_{t+1}(X) = \rho_{t+1}(Y)$ implies $\rho_t(X) = \rho_t(Y)$ whatever $X, Y \in L^0(\mathbf{R}, \mathcal{F}_T)$ and $t \leq T - 1$ (see Section 5 in [13]).

The following is very well known, see [1].

Lemma 4.9. A dynamic risk-measure $(\rho_t)_{t \leq T}$ is time-consistent if and only if its family of acceptable sets $(\mathcal{A}_t)_{t \leq T}$ satisfies

$$\mathcal{A}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} \tag{4.13}$$

for any $t \leq T - 1$.

Observe that, if $(\rho_t)_{t \leq T}$ is time-consistent, we may show by induction that $\rho_t(-\rho_{t+s}(\cdot)) = \rho_t(\cdot)$ for any $t \leq T$ and $s \geq 0$ such that $s \leq T - t$. In the following, we introduce another possible definition for the risk-hedging prices in the multi-period model, where the risk is only measured at time t .

Definition 4.10. The contingent claim $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$ is said directly risk-hedged at time $t \leq T - 1$ if there exists a (direct) price $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ and a strategy $(\theta_u)_{t \leq u \leq T-1}$ such that that $P_t + \sum_{t \leq u \leq T-1} \theta_u \Delta S_{u+1} - h_T$ is acceptable at time t .

The set of all direct risk-hedging prices at time t is then given by

$$\bar{\mathcal{P}}_t(h_T) = \left\{ \rho_t \left(\sum_{t \leq u \leq T-1} \theta_u \Delta S_{u+1} - h_T \right) : (\theta_u)_{t \leq u \leq T-1} \in \Pi_{t \leq u \leq T-1} L^0(\mathbf{R}, \mathcal{F}_u) \right\} + L^0(\mathbf{R}, \mathcal{F}_t).$$

and the minimal direct risk-hedging price is

$$\bar{P}_t^*(h_T) := \operatorname{ess\,inf}_{(\theta_u)_{t \leq u \leq T-1}} \bar{\mathcal{P}}_t(h_T).$$

We propose to prove that the direct minimal risk-hedging prices coincide with the minimal ones derived from the step by step backward procedure developed before, i.e. such that $P_t^*(h_T) = \operatorname{ess\,inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*(h_T))$, where

$$P_T^*(h_T) = h_T.$$

Theorem 4.11. *Suppose that the dynamic risk-measure $(\rho_t)_{t \leq T}$ is time consistent. Then, $\bar{P}_t^*(h_T) = P_t^*(h_T)$ for any $t \leq T - 1$. Moreover, the direct minimal risk-hedging prices are direct prices if and only if the minimal prices of the backward procedure are prices.*

Proof. It is trivial to see that $\bar{P}_{T-1}^* = P_{T-1}^*$ as $\bar{\mathcal{P}}_{T-1}(h_T) = \mathcal{P}_{T-1}(h_T)$. Let us show the equality at time t by induction from the hypothesis $\bar{P}_{t+1}^* = P_{t+1}^*$. Note that this assumption is equivalent to the equality $\bar{\mathcal{P}}_{t+1}(h_T) = \mathcal{P}_{t+1}(h_T)$, since any $V_t \in \bar{\mathcal{P}}_t$ satisfies $V_t + \theta_t \Delta S_{t+1} = V_{t+1}$.

First, we show we that $\bar{P}_t^* \leq P_t^*$. Fix $\epsilon > 0$. By measurable selection argument, we may find $\theta_u^* \in L^0(\mathbf{R}, \mathcal{F}_u)$ such that $P_u^* + \epsilon \geq \rho_u(\theta_u^* \Delta S_{u+1} - P_{u+1}^*)$ for every $u = t, \dots, T-1$. Therefore, by time consistency and monotonicity, we have:

$$\begin{aligned} P_t^* &\geq \rho_t(\theta_t^* \Delta S_{t+1} - P_{t+1}^*) - \epsilon \\ P_t^* &\geq \rho_t(\theta_t^* \Delta S_{t+1} - \rho_{t+1}(\theta_{t+1}^* \Delta S_{t+2} - P_{t+2}^*)) - 2\epsilon \\ &= \rho_t(-\rho_{t+1}(\theta_t^* \Delta S_{t+1} + \theta_{t+1}^* \Delta S_{t+2} - P_{t+2}^*)) - 2\epsilon \\ &= \rho_t(\theta_t^* \Delta S_{t+1} + \theta_{t+1}^* \Delta S_{t+2} - P_{t+2}^*) - 2\epsilon \\ &\dots \\ &\geq \rho_t(\theta_t^* \Delta S_{t+1} + \theta_{t+1}^* \Delta S_{t+2} + \dots + \theta_{T-1}^* \Delta S_T - h_T) - \epsilon(T-t-1). \end{aligned}$$

Therefore, as $\epsilon \rightarrow 0$, $P_t^* \geq \bar{P}_t^*$. Similarly, if $\bar{P}_t \in \bar{\mathcal{P}}_t(h_T)$, we also get, for every

$\epsilon > 0$, inequalities as follows:

$$\begin{aligned}
\bar{P}_t^* &\geq \rho_t(\theta_t \Delta S_{t+1} + \theta_{t+1} \Delta S_{t+2} + \cdots + \theta_{T-1} \Delta S_T - h_T) - \epsilon \\
&\geq \rho_t(-\rho_{t+1}(\theta_t \Delta S_{t+1} + \theta_{t+1} \Delta S_{t+2} + \cdots + \theta_{T-1} \Delta S_T - h_T)) - \epsilon \\
&= \rho_t(\theta_t \Delta S_{t+1} - \rho_{t+1}(\theta_{t+1} \Delta S_{t+2} + \cdots + \theta_{T-1} \Delta S_T - h_T)) - \epsilon \\
&\geq \rho_t(\theta_t \Delta S_{t+1} - \bar{P}_{t+1}^*) - \epsilon.
\end{aligned}$$

As $\bar{P}_{t+1}^* = P_{t+1}^*$, we deduce, as $\epsilon \rightarrow 0$, that $\bar{P}_t^* \geq \rho_t(\theta_t \Delta S_{t+1} - P_{t+1}^*)$. Therefore, $\bar{P}_t^* \geq P_t^*$ so that the equality finally holds.

Moreover, in the case where the minimal prices are prices, we may rewrite the inequalities above as equalities so that the last statement holds. \square

Corollary 4.12. *Suppose that the dynamic risk-measure $(\rho_t)_{t \leq T}$ is time consistent. Then, $\bar{\mathcal{P}}_t(h_T) = \mathcal{P}_t(h_T)$ for all $t \leq T$.*

We now show that the minimal risk-hedging price P_t^* lies in some interval $[m_t^*, M_t^*]$ for all $t \leq T - 1$ if and only if the condition AIP holds. In this part, we assume that each asset price S_t , $1 \leq t \leq T$ is strictly positive. The proof of the following theorem is showed in the Appendix B.4.

Theorem 4.13. *The condition AIP holds at time $T - 1$ if and only if the minimal risk-hedging price of non negative contingent claim h_T satisfies*

$$m_{T-1}^* \leq P_{T-1}^* \leq M_{T-1}^*$$

where $m_{T-1}^* = S_{T-1} \left(\text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right)$ and $M_{T-1}^* = S_{T-1} \left(\text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right)$.

Corollary 4.14. *Suppose that h_T is an attainable claim. Then AIP, holds if and only if each minimal risk-hedging price P_t^* of the contingent claim h_T satisfies*

$$m_t^* \leq P_t^* \leq M_t^* \tag{4.14}$$

where $m_t^* = S_t \left(\text{ess inf}_{\mathcal{F}_t} \frac{h_T}{S_T} \right)$ and $M_t^* = S_t \left(\text{ess sup}_{\mathcal{F}_t} \frac{h_T}{S_T} \right)$.

Proof. First, observe that the inequalities (4.14) ensure that AIP holds for all $t \leq T - 1$. Indeed, $m_t^* = M_t^* = 0$ for any $t \leq T - 1$ when $h_T = 0$. So, $P_t^* = 0$ for all $t \leq T$, i.e. AIP holds.

To prove the reverse implication, we first use the tower property satisfied by the conditional essential infimum/supremum operators (see [30, Appendix

A)] and we deduce that

$$m_t^* = S_t \left(\text{ess inf}_{\mathcal{F}_t} \frac{m_{t+1}^*}{S_{t+1}} \right) \quad (4.15)$$

and

$$M_t^* = S_t \left(\text{ess sup}_{\mathcal{F}_t} \frac{M_{t+1}^*}{S_{t+1}} \right) \quad (4.16)$$

where $0 \leq t \leq T - 1$ and $m_T^* = M_T^* = h_T$. Note that (4.15) and (4.16) trivially hold when $t = T - 1$. Suppose that (4.15) and (4.16) hold for all $t + 1 \leq u \leq T - 1$, and let us prove them for the time $u = t$. Actually, it has

$$\begin{aligned} m_t^* &= S_t \left(\text{ess inf}_{\mathcal{F}_t} \frac{h_T}{S_T} \right) = S_t \left(\text{ess inf}_{\mathcal{F}_t} \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) \\ &= S_t \left(\text{ess inf}_{\mathcal{F}_t} \frac{S_{T-1} \left(\text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right)}{S_{T-1}} \right) = S_t \left(\text{ess inf}_{\mathcal{F}_t} \frac{m_{T-1}^*}{S_{T-1}} \right). \end{aligned}$$

We then conclude by induction with the tower property and, by similar arguments, we also get (4.16).

By Theorem 4.13, (4.14) holds at time $t = T - 1$. Suppose that (4.14) holds at time $t + 1$ and let us show it at time t . To do so, recall that, for any $t \leq T - 1$, $P_t^* = \text{ess inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*)$. By Theorem 4.13, we deduce that $\tilde{m}_t^* \leq P_t^* \leq \tilde{M}_t^*$ where $\tilde{m}_t^* = S_t(\text{ess inf}_{\mathcal{F}_t} \frac{P_{t+1}^*}{S_{t+1}})$ and $\tilde{M}_t^* = S_t(\text{ess sup}_{\mathcal{F}_t} \frac{P_{t+1}^*}{S_{t+1}})$. Thus $P_t^* \geq \tilde{m}_t^* \geq S_t(\text{ess inf}_{\mathcal{F}_t} \frac{m_{t+1}^*}{S_{t+1}}) = m_t^*$ and $P_t^* \leq \tilde{M}_t^* \leq S_t(\text{ess sup}_{\mathcal{F}_t} \frac{M_{t+1}^*}{S_{t+1}}) = M_t^*$. \square

Example 4.15. Consider the call option with the payoff $h_T = (S_T - K)^+$. We get by simple computations that $m_t^* = S_t \left(1 - \frac{K}{\text{ess inf}_{\mathcal{F}_t} S_T} \right) 1_{\text{ess inf}_{\mathcal{F}_t} S_T > K}$ and $M_t^* = S_t \left(1 - \frac{K}{\text{ess sup}_{\mathcal{F}_t} S_T} \right) 1_{\text{ess sup}_{\mathcal{F}_t} S_T > K}$.

Appendix A: Conditional essential supremum and random sets

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let \mathcal{H} be a sub- σ -algebra of \mathcal{F} . The concept of generalized conditional essential supremum in $L^0(\mathbf{R}^d)$ is given in [26, Definition 3.1]. Existence and uniqueness when $d = 1$ is provided by [26, Lemma 3.9].

Proposition A.1. *Let $\Gamma \neq \emptyset$ be a subset of $L^0(\mathbf{R} \cup \{+\infty\}, \mathcal{F})$. There exists a unique \mathcal{H} -measurable random variable $\hat{\gamma} \in L^0(\mathbf{R} \cup \{+\infty\}, \mathcal{H})$ denoted as $\text{ess sup}_{\mathcal{H}} \Gamma$ such that the following conditions hold:*

- (i) $\hat{\gamma} \geq \gamma$ a.s. for any $\gamma \in \Gamma$;
- (ii) if $\tilde{\gamma} \in L^0(\mathbf{R} \cup \{+\infty\}, \mathcal{H})$ satisfies $\tilde{\gamma} \geq \gamma$ for any $\gamma \in \Gamma$, then $\tilde{\gamma} \geq \hat{\gamma}$ a.s.

A random set is defined as follows. We suggest to read [30, Section 2.1] for more details.

Definition A.2. *A set-valued function $\omega \mapsto X(\omega) \subset \mathbf{R}^d$ from a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the family of all subsets of \mathbf{R}^d is called \mathcal{F} -measurable random set if its graph*

$$\text{graph } X = \{(\omega, x) \in \Omega \times \mathbf{R}^d : x \in X(\omega)\} \subset \Omega \times \mathbf{R}^d$$

belongs to the product σ -algebra $\mathcal{F} \otimes \mathcal{B}(\mathbf{R}^d)$.

The random set X is said to be closed if $X(\omega)$ is closed for almost all ω .

Appendix B: Main proofs

B.1. Proof of Proposition 2.4

First we give the following two preliminary lemmas.

Lemma B.1. *For any $X \in \text{Dom } \mathcal{A}_t$, there exists a sequence $C_n \in \mathcal{A}_t^X$ such that $\rho_t(X) = \lim_{n \rightarrow \infty} \downarrow C_n$ everywhere on Ω .*

Proof. We first observe that the set \mathcal{A}_t^X is \mathcal{F}_t -decomposable, i.e. if $\Lambda_t \in \mathcal{F}_t$ and $C_1, C_2 \in \mathcal{A}_t^X$, then $C_1 1_{\Lambda_t} + C_2 1_{\Omega \setminus \Lambda_t} \in \mathcal{A}_t^X$. To see it, we use conditions 1) and 4) of Definition 2.1. We then deduce that \mathcal{A}_t^X is directed downward, i.e. if $C_1, C_2 \in \mathcal{A}_t^X$, then $C_1 \wedge C_2 \in \mathcal{A}_t^X$. Indeed, $C_1 \wedge C_2 = C_1 1_{C_1 \leq C_2} + C_2 1_{C_1 > C_2}$ with $\{C_1 \leq C_2\} \in \mathcal{F}_t$. Therefore, there exists a sequence $C_n \in \mathcal{A}_t^X$ such that $\rho_t(X) = \lim_{n \rightarrow \infty} \downarrow C_n$ everywhere on Ω , see [25, Section 5.3.1]. \square

Lemma B.2. *Let $X \in \text{Dom } \mathcal{A}_t$ and $F_t \in \mathcal{F}_t$. Then, $X 1_{F_t} \in \text{Dom } \mathcal{A}_t$ and $\rho_t(X 1_{F_t}) = \rho_t(X) 1_{F_t}$.*

Proof. Observe that $C_t \in \mathcal{A}_t^X$ implies that $C_t 1_{F_t} \in \mathcal{A}_t^{X 1_{F_t}}$. We deduce that $X 1_{F_t} \in \text{Dom } \mathcal{A}_t$. In particular, we have $1_{F_t} \mathcal{A}_t^X \subseteq \mathcal{A}_t^{X 1_{F_t}}$. Reciprocally, consider $D_t \in \mathcal{A}_t^{X 1_{F_t}}$, then $D_t 1_{F_t} \in \mathcal{A}_t^{X 1_{F_t}}$ and finally $X + Z_t \in \mathcal{A}_t$ where $Z_t = D_t 1_{F_t} + C_t 1_{\Omega \setminus F_t}$ and $C_t \in \mathcal{A}_t^X$ is arbitrarily chosen. Therefore, $Z_t \in \mathcal{A}_t^X$

and $D_t 1_{F_t} = Z_t 1_{F_t} \in 1_{F_t} \mathcal{A}_t^X$, i.e. $1_{F_t} \mathcal{A}_t^{X1_{F_t}} \subseteq 1_{F_t} \mathcal{A}_t^X$. We finally deduce that $1_{F_t} \mathcal{A}_t^{X1_{F_t}} = 1_{F_t} \mathcal{A}_t^X$. Therefore,

$$\begin{aligned} 1_{F_t} \rho_t(X) &= 1_{F_t} \operatorname{ess\,inf} \mathcal{A}_t^X = \operatorname{ess\,inf} (1_{F_t} \mathcal{A}_t^X) \\ &= \operatorname{ess\,inf} (1_{F_t} \mathcal{A}_t^{X1_{F_t}}) = 1_{F_t} \operatorname{ess\,inf} (\mathcal{A}_t^{X1_{F_t}}) \\ &= 1_{F_t} \rho_t(X1_{F_t}). \end{aligned}$$

Since $1_{F_t} \mathcal{A}_t^{X1_{F_t}} \subseteq \mathcal{A}_t^{X1_{F_t}}$, $\rho_t(X1_{F_t}) \leq C_t 1_{F_t}$ if $C_t \in \mathcal{A}_t^{X1_{F_t}}$ hence $\rho_t(X1_{F_t}) \leq 0$ on $\Omega \setminus F_t$. At last, since $X1_{F_t} + \rho_t(X1_{F_t}) \in \mathcal{A}_t$, multiplying this property by $1_{\Omega \setminus F_t}$, we deduce that $\rho_t(X1_{F_t})1_{\Omega \setminus F_t} \in \mathcal{A}_t$ hence $\rho_t(X1_{F_t})1_{\Omega \setminus F_t} \geq 0$ and finally $\rho_t(X1_{F_t})1_{\Omega \setminus F_t} = 0$. The conclusion follows. \square

Proof of Proposition 2.4: The first four statements are directly deduced from the definition of ρ_t in (2.1). The positive homogeneity is easily seen in the case where $k_t \in L^0((0, \infty), \mathcal{F}_t)$ as $\mathcal{A}_t^{k_t X} = k_t \mathcal{A}_t^X$ when $k_t > 0$ a.s. In the general case, i.e. $k_t \geq 0$, consider $\tilde{k}_t = k_t 1_{k_t > 0} + 1_{k_t = 0} \in L^0((0, \infty), \mathcal{F}_t)$. Then, $\rho_t(\tilde{k}_t X) = \tilde{k}_t \rho_t(X)$ for any $X \in \operatorname{Dom} \mathcal{A}_t$. Replacing X by $X1_{k_t \neq 0}$, we get by Lemma B.2 that

$$\rho_t(k_t X) = \rho_t(k_t 1_{k_t \neq 0} X) = \tilde{k}_t \rho_t(X1_{k_t \neq 0}) = k_t \rho_t(X).$$

By Lemma B.1, we know that $\rho_t(X) = \lim_{n \rightarrow \infty} \downarrow C_n$ a.s. where $C_n \in \mathcal{A}_t^X$ a.s. As the set \mathcal{A}_t is closed, \mathcal{F}_t -decomposable and contains 0, we deduce that $X + \rho_t(X) \in \mathcal{A}_t$ on the set $\rho_t(X) \in \mathbf{R}$. Actually, we shall see below that $\rho_t(X) > -\infty$ holds with probability one so that $\rho_t(X) \in \mathbf{R}$ a.s. and $X + \rho_t(X) \in \mathcal{A}_t$.

Here, we suppose that $\rho_t(X) \in \mathbf{R}$ a.s. if $X \in \operatorname{Dom} \mathcal{A}_t$. Consider a sequence $X_n \in \operatorname{Dom} \mathcal{A}_t$ which converges to X_0 . With $\alpha_n := \rho_t(X_n)$, $X_n + \alpha_n \in \mathcal{A}_t$. Notice that we may assume w.l.o.g. that $\liminf \alpha_n < \infty$. Indeed, otherwise, the inequality $\rho_t(X_n) \leq \liminf \alpha_n = \infty$ trivially holds. Suppose that $\alpha_\infty := \liminf \alpha_n = -\infty$. Then, using the normalization procedure $\tilde{\alpha}_n := \frac{\alpha_n}{|\alpha_n|}$, we get that $\frac{X_n}{|\alpha_n|} + \tilde{\alpha}_n \in \mathcal{A}_t$. As $|\tilde{\alpha}_n| = 1$ and \mathcal{A}_t is closed, \mathcal{F}_t -decomposable, we may assume by [25, Lemma 2.1.2], up to some \mathcal{F}_t -measurable random subsequence, that $\tilde{\alpha}_n$ converges to some $\tilde{\alpha}$ with $|\tilde{\alpha}| = 1$. Actually, $\tilde{\alpha} = -1$ a.s. as $\tilde{\alpha}_n < 0$ for n large enough. Then $\lim_n (\frac{X_n}{|\alpha_n|} + \tilde{\alpha}_n) = \tilde{\alpha} = -1 \in \mathcal{A}_t$ if \mathcal{A}_t is closed, which contradicts the third condition in Definition 2.1. So, we may assume w.l.o.g. that $\alpha_\infty \in L^0(\mathbf{R}, \mathcal{F}_t)$ and $X_0 + \alpha_\infty \in \mathcal{A}_t$. It follows that

$\alpha_\infty \in \mathcal{A}_t^{X_0}$ and $\rho_t(X_0) \leq \alpha_\infty = \liminf \rho_t(X_n)$ a.s. Note that on the negligible set $\{\rho_t(X_0) > \alpha_\infty\}$, we may replace $\rho_t(X_0)$ by α_∞ so that we may suppose that the inequality holds everywhere, i.e. ρ_t satisfies the l.s.c. inequality on Ω .

The property $\rho_t(X) > -\infty$ for all $X \in \text{Dom } \mathcal{A}_t$ is proven similarly using Lemma B.1 and the normalization procedure above.

At last, if the set \mathcal{A}_t is closed, $X + \rho_t(X) \in \mathcal{A}_t$ on $\rho_t(X) < \infty$ for all $X \in L^0(\mathbf{R}, \mathcal{F}_T)$. Then, $\mathcal{A}_t = \{X \in \text{Dom } \mathcal{A}_t \mid \rho_t(X) \leq 0\}$. Indeed, it is clear that $\rho_t(X) \leq 0$ for all $X \in \mathcal{A}_t$. Reciprocally, if $\rho_t(X) \leq 0$ we get that $X = -\rho_t(X) + a_t$ where $a_t \in \mathcal{A}_t$. Finally, we deduce that $X \in \mathcal{A}_t$ since $0 \leq -\rho_t(X) \in \mathcal{A}_t$ and $\mathcal{A}_t + \mathcal{A}_t \subseteq \mathcal{A}_t$. \square

B.2. Proof of Proposition 2.10

Lemma B.3. *Let $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ and $F_t \in \mathcal{F}_t$. Then, $\rho_t(X1_{F_t}) = \rho_t(X)1_{F_t}$.*

Proof. First, observe that

$$\Lambda_t(X1_{F_t}) = \left\{ \tilde{F}_t \in \mathcal{F}_t : X1_{F_t}1_{\tilde{F}_t} \in \text{Dom } \mathcal{A}_t \right\} = \left\{ \tilde{F}_t \in \mathcal{F}_t : F_t \cap \tilde{F}_t \in \Lambda_t(X) \right\}.$$

It follows that $F_t(X1_{F_t}) = (F_t(X) \cap F_t) \cup (\Omega \setminus F_t)$. Therefore,

$$\begin{aligned} \rho_t(X1_{F_t}) &= \rho_t(X1_{F_t}1_{F_t(X1_{F_t})})1_{F_t(X1_{F_t})} \pm \infty 1_{\Omega \setminus F_t(X1_{F_t})} \\ &= \rho_t(X1_{F_t(X) \cap F_t})1_{F_t(X1_{F_t})} \pm \infty 1_{\Omega \setminus F_t(X1_{F_t})}. \end{aligned}$$

By Lemma B.2 applied to $X1_{F_t(X)} \in \text{Dom } \mathcal{A}_t$, we deduce that

$$\rho_t(X1_{F_t(X) \cap F_t})1_{F_t(X1_{F_t})} = \rho_t(X)1_{F_t(X) \cap F_t}.$$

On the other hand, $\Omega \setminus F_t(X1_{F_t}) = (\Omega \setminus F_t(X)) \cap F_t$ hence, since X and $X1_{F_t}$ have the same sign on F_t ,

$$\pm \infty 1_{\Omega \setminus F_t(X1_{F_t})} = \rho_t(X)1_{\Omega \setminus F_t(X)}1_{F_t}.$$

The conclusion follows. \square

Proof of Proposition 2.10: Recall that, for every $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$,

$$\rho_t(X) = \rho_t(X1_{F_t(X)})1_{F_t(X)} + (\liminf_n \rho_t(f_n(X)))1_{\Omega \setminus F_t(X)},$$

where f_n is defined by $f_n(x) = ((-n) \vee x) \wedge n$, $n \geq 1$. Let us verify the statements of Proposition 2.4. Normalization is trivial as $F_t(0) = \Omega$. If $X, X' \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ with $X \leq X'$, we deduce by Definition 2.1, Condition 2), that $\mathcal{A}_t^X \subseteq \mathcal{A}_t^{X'}$. Therefore, $\Lambda_t(X) \subseteq \Lambda_t(X')$ and, finally, $F_t(X) \subseteq F_t(X')$. We deduce that

$$\begin{aligned} \rho_t(X') &= \rho_t(X'1_{F_t(X)})1_{F_t(X)} + \rho_t(X'1_{F_t(X') \setminus F_t(X)})1_{F_t(X') \setminus F_t(X)} \\ &\quad + (\liminf_n \rho_t(f_n(X')))1_{\Omega \setminus F_t(X')} \end{aligned}$$

where $X'1_{F_t(X)} \in \text{Dom } \mathcal{A}_t$. Since $X1_{F_t(X)} \leq X'1_{F_t(X)}$, we deduce that $\rho_t(X1_{F_t(X)}) \geq \rho_t(X'1_{F_t(X)})$ so that $\rho_t(X) \geq \rho_t(X')$ on $F_t(X)$, and the inequality still holds when $\rho_t(X) = +\infty$ on $\Omega \setminus F_t(X)$. Otherwise, we have $\liminf_n \rho_t(f_n(X)) = -\infty$. As f_n is non decreasing, we have $f_n(X) \leq f_n(X')$ hence $\rho_t(f_n(X)) \geq \rho_t(f_n(X'))$ and, finally, $\liminf_n \rho_t(f_n(X')) = -\infty$. Therefore, we may conclude on Ω .

For all $m_t \in L^0(\mathbf{R}, \mathcal{F}_t)$, it has $F_t(X + m_t) = F_t(X)$ as $\mathcal{A}_t^{X+m_t} \neq \emptyset$ if and only if $\mathcal{A}_t^X \neq \emptyset$. Therefore, by Proposition 2.4,

$$\begin{aligned} \rho_t(X) - m_t &= \rho_t(X1_{F_t(X)})1_{F_t(X)} + (\pm\infty)1_{\Omega \setminus F_t(X)} - m_t1_{F_t(X)} - m_t1_{\Omega \setminus F_t(X)} \\ &= (\rho_t(X1_{F_t(X)}) - m_t1_{F_t(X)})1_{F_t(X)} + (\pm\infty)1_{\Omega \setminus F_t(X)} \\ &= \rho_t((X + m_t)1_{F_t(X)})1_{F_t(X)} + (\pm\infty)1_{\Omega \setminus F_t(X)} = \rho_t(X + m_t) \end{aligned}$$

Note that we deduce the last equality if we show that, on the set $\Omega \setminus F_t(X)$, we have $\liminf_n \rho_t(f_n(X + m_t)) = \liminf_n \rho_t(f_n(X))$. To do so, it suffices to observe that, $f_n(X) - |m_t| \leq f_n(X + m_t) \leq f_n(X) + m_t$ so that we may conclude.

For any $X, X' \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$, we have $\mathcal{A}_t^X + \mathcal{A}_t^{X'} \subseteq \mathcal{A}_t^{X+X'}$. We deduce that $F_t(X) \cap F_t(X') \subseteq F_t(X + X')$ and $\rho_t(X + X') \leq \rho_t(X) + \rho_t(X')$ by Proposition 2.4 on the set $F_t(X) \cap F_t(X')$.

On the set $(\Omega \setminus F_t(X)) \cap (\Omega \setminus F_t(X'))$, consider the sequence $h_{n,m} = f_n(X) + f_m(X')$, $n \geq 1$. Since $\lim_{n,m \rightarrow \infty} h_{n,m} = X + X'$, we deduce by Lemma 2.9 that

$$\rho_t(X + X') = \liminf_{n,m} \rho_t(f_n(X) + f_m(X')) \leq \liminf_n \rho_t(f_n(X)) + \liminf_m \rho_t(X')$$

We deduce that $\rho_t(X + X') \leq \rho_t(X) + \rho_t(X')$.

Note that, if $k_t \in L^0((0, \infty), \mathcal{F}_t)$, $\mathcal{A}_t^{k_t X} \neq \emptyset$ if and only if $\mathcal{A}_t^X \neq \emptyset$. Therefore, $\Lambda_t(X) = \Lambda_t(k_t X)$ and $F_t(k_t X) = F_t(X)$. Moreover, on $\Omega \setminus F_t(X)$, $\rho_t(k_t X) = \liminf_n \rho_t(k_t f_n(X)) = k_t \liminf_n \rho_t(f_n(X)) = k_t \rho_t(X)$ by Lemma 2.9. Therefore, we have

$$\begin{aligned} \rho_t(k_t X) &= \rho_t(k_t X 1_{F_t(X)}) 1_{F_t(X)} + (\pm\infty) 1_{\Omega \setminus F_t(X)}, \\ &= k_t \rho_t(X 1_{F_t(X)}) 1_{F_t(X)} + k_t (\pm\infty) 1_{\Omega \setminus F_t(X)} = k_t \rho_t(X). \end{aligned}$$

Let us now consider the general case $k_t \geq 0$. Consider $\tilde{k}_t = k_t 1_{k_t > 0} + 1_{k_t = 0} > 0$. By the first step, we get that $\rho_t(\tilde{k}_t X) = \tilde{k}_t \rho_t(X)$ and $\Lambda_t(X) = \Lambda_t(\tilde{k}_t X)$ whatever $X \in L^0(\mathbf{R}, \mathcal{F}_T)$. Replacing X by $X 1_{k_t \neq 0}$ and using Lemma B.3, we get that

$$\rho_t(k_t X) = \rho_t(k_t 1_{k_t \neq 0} X) = \tilde{k}_t \rho_t(X 1_{k_t \neq 0}) = k_t \rho_t(X).$$

We now show that ρ_t is l.s.c. To see it, consider $X = \lim_{n \rightarrow \infty} X^n$ a.s. On the set $F_t(X)$, $X \in \mathbf{R}$ hence $X = \lim_n X^n \vee (-n)$ a.s. where $1_{F_t(X)} X^n \vee (-n)$ belongs to $\text{Dom } \mathcal{A}_t$ as it is bounded from below and different from $+\infty$. Moreover, as $X^n \vee (-n) \geq X^n$, we deduce that $\rho_t(X^n \vee (-n)) \leq \rho_t(X^n)$ and finally, by Proposition 2.4,

$$\rho_t(X) 1_{F_t(X)} = \rho_t(X 1_{F_t(X)}) \leq \liminf_n \rho_t(X^n \vee (-n)) \leq \liminf_n \rho_t(X^n).$$

Otherwise, we only need to consider the case where $\rho_t(X) = +\infty$. On the subset $\Gamma_t = \liminf_n \rho_t(X^n) < +\infty$, we may consider a \mathcal{F}_t -measurable sequence n_k such that $\rho_t(X^{n_k}) < \infty$. We deduce that $1_{\Gamma_t} X^{n_k} \vee (-n) \in \text{Dom } \mathcal{A}_t$. As $\rho_t(X^n \vee (-n)) \leq \rho_t(X^n)$, we deduce that $\rho_t(X^{n_k}) 1_{\Gamma_t} + 1_{\Gamma_t} X^{n_k} \vee (-n) \in \mathcal{A}_t$. As $k \rightarrow \infty$, we get that $\liminf_n \rho_t(X^n) 1_{\Gamma_t} + 1_{\Gamma_t} X \in \mathcal{A}_t$ in contradiction with the maximality of $F_t(X)$. \square

B.3. Proof of Lemma 4.1

Lemma B.4. For all $X, Y \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ such that $X \neq Y$ a.s., we have

$$S_{T-1} - \rho_{T-1}(-S_T) \leq \frac{g_{T-1}(Y) - g_{T-1}(X)}{Y - X} \leq S_{T-1} + \rho_{T-1}(S_T).$$

Proof. Suppose that $X, Y \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$. On the set $\{Y > X\} \in \mathcal{F}_{T-1}$, we have:

$$\begin{aligned} \rho_{T-1}(Y S_T - h_T) &= \rho_{T-1}(X S_T - h_T + (Y - X) S_T) \\ &\leq \rho_{T-1}(X S_T - h_T) + \rho_{T-1}((Y - X) S_T) \\ &\leq \rho_{T-1}(X S_T - h_T) + (Y - X) \rho_{T-1}(S_T). \end{aligned}$$

We deduce that $g_{T-1}(Y) \leq g_{T-1}(X) + (Y - X)(S_{T-1} + \rho_{T-1}(S_T))$ and finally

$$\frac{g_{T-1}(Y) - g_{T-1}(X)}{Y - X} \leq S_{T-1} + \rho_{T-1}(S_T). \quad (\text{B.17})$$

By symmetry, we deduce that (B.17) holds also on $\{Y < X\}$. Similarly, on the set $\{Y < X\}$,

$$\begin{aligned} \rho_{T-1}(YS_T - h_T) &\leq \rho_{T-1}(XS_T - h_T) + \rho_{T-1}((Y - X)S_T) \\ &\leq \rho_{T-1}(XS_T - h_T) + (X - Y)\rho_{T-1}(-S_T). \end{aligned}$$

Therefore,

$$\frac{g_{T-1}(Y) - g_{T-1}(X)}{Y - X} \geq S_{T-1} - \rho_{T-1}(-S_T). \quad (\text{B.18})$$

By symmetry, we deduce that (B.18) holds also on $\{Y > X\}$. \square

Corollary B.5. *We have $g_{T-1}(x) \in \mathbf{R}$, for all $x \in \mathbf{R}$ a.s. In particular, g_{T-1} is a.s. continuous.*

Proof. By Lemma B.4, we have $|g_{T-1}(X)| \leq k_{T-1}|X| + |\rho_{T-1}(-h_T)|$ for all $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ so that $g_{T-1}(X) \in \mathbf{R}$, where

$$k_{T-1} := \max(|S_{T-1} + \rho_{T-1}(S_T)|, |S_{T-1} - \rho_{T-1}(-S_T)|).$$

Consider the set

$$A := \{\omega \in \Omega : \forall x \in \mathbf{R}, g_{T-1}(\omega, x) < \infty\} = \{\omega \in \Omega : \forall x \in \mathbf{Q}, g_{T-1}(\omega, x) < \infty\}.$$

The equality above holds because g_{T-1} is convex and any $x \in \mathbf{R}$ is a convex combination of two rational numbers. We deduce that $B = \Omega \setminus A \in \mathcal{F}_{T-1}$. If $B \neq \emptyset$, as the jointly measurable set $\{(\omega, x) \in \Omega \times \mathbf{R} : g_{T-1}(\omega, x) = \infty\}$ admit non empty ω -sections, we may use a measurable selection argument. We then deduce $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ such that $g_{T-1}(X) = +\infty$ on B , which yields a contradiction. \square

Proof of Lemma 4.1: Consider the set $B = B_1 \cup B_2$ where

$$B_1 = \bigcup_{y>x} \{\omega \in \Omega : g_{T-1}(y) - g_{T-1}(x) > (S_{T-1} + \rho_{T-1}(S_T))(y - x)\},$$

$$B_2 = \bigcup_{y>x} \{\omega \in \Omega : g_{T-1}(y) - g_{T-1}(x) < (S_{T-1} - \rho_{T-1}(-S_T))(y - x)\},$$

By Corollary B.5, g is a.s. continuous hence B_1 and B_2 may be rewritten as countable unions. Therefore, B is \mathcal{F}_{T-1} -measurable. Suppose that $P(B) > 0$ and let us introduce the jointly measurable set $G_{T-1} = G_{T-1}^1 \cup G_{T-1}^2$ where

$$\begin{aligned} G_{T-1}^1 &= \{(\omega, x, y) \in \Omega \times \mathbf{R}^2 : g_{T-1}(y) - g_{T-1}(x) > (S_{T-1} + \rho_{T-1}(S_T))(y - x)\}, \\ G_{T-1}^2 &= \{(\omega, x, y) \in \Omega \times \mathbf{R}^2 : g_{T-1}(y) - g_{T-1}(x) < (S_{T-1} - \rho_{T-1}(-S_T))(y - x)\}. \end{aligned}$$

As the ω -sections of G_{T-1} are not empty on B , we deduce by a measurable selection argument the existence of $X, Y \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ such that $(\omega, X(\omega), Y(\omega)) \in G_{T-1}$ a.s. on B and we set $X = Y = 0$ otherwise. This leads to a contradiction by Lemma B.4. \square

B.4. Proof of theorem 4.13

Notice that $m_{T-1}^* \geq 0$ when $h_T \geq 0$ hence AIP holds with $h_T = 0$. Let us show the reverse implication. To do so, we observe that

$$g_{T-1}(x) = xS_{T-1} + \rho_{T-1} \left(\left(x - \frac{h_T}{S_T} \right) S_T \right).$$

Since $\text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \leq \frac{h_T}{S_T} \leq \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}$, we deduce that

$$\left(x - \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) S_T \leq \left(x - \frac{h_T}{S_T} \right) S_T \leq \left(x - \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) S_T.$$

Therefore,

$$\rho_{T-1} \left(\left(x - \frac{h_T}{S_T} \right) S_T \right) \geq \rho_{T-1} \left(\left(x - \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) S_T \right)$$

and

$$\rho_{T-1} \left(\left(x - \frac{h_T}{S_T} \right) S_T \right) \leq \rho_{T-1} \left(\left(x - \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) S_T \right).$$

This implies that

$$g_{T-1}(x) \geq xS_{T-1} + \rho_{T-1} \left(\left(x - \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) S_T \right) := m_{T-1}(x) \quad (\text{B.19})$$

and

$$g_{T-1}(x) \leq xS_{T-1} + \rho_{T-1} \left(\left(x - \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) S_T \right) := M_{T-1}(x), \quad (\text{B.20})$$

where the r.h.s. $m_{T-1}(x)$ and $M_{T-1}(x)$ of the inequalities (B.19) and (B.20) need to be understood as jointly measurable and l.s.c. random functions w.r.t. (ω, x) . To see it, it suffices to repeat the construction of g_{T-1} . As we have $m_{T-1}(x) \leq g_{T-1}(x) \leq M_{T-1}(x)$ for all x , we get by Lemma 3.9 that

$$m_{T-1}^* := \inf_{x \in \mathbf{R}} m_{T-1}(x) \leq P_{T-1}^* \leq \inf_{x \in \mathbf{R}} M_{T-1}(x) =: M_{T-1}^*.$$

Notice that

$$m_{T-1}(x) = \begin{cases} x(S_{T-1} + \rho_{T-1}(S_T)) - \left(\text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) \rho_{T-1}(S_T), \\ \quad \text{if } x \geq \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \\ x(S_{T-1} - \rho_{T-1}(-S_T)) + \left(\text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) \rho_{T-1}(-S_T), \\ \quad \text{if } x < \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}. \end{cases}$$

By Theorem 3.11, $S_{T-1} + \rho_{T-1}(S_T) \geq 0$ and $S_{T-1} - \rho_{T-1}(-S_T) \leq 0$ under AIP. Thus, the infimum of $m_{T-1}(x)$ is reached at point $\underline{x}^* := \text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}$ and

$$m_{T-1}^* = S_{T-1} \left(\text{ess inf}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right).$$

Similarly, the upper bound function M_{T-1} may be expressed as

$$M_{T-1}(x) = \begin{cases} x(S_{T-1} + \rho_{T-1}(S_T)) - \left(\text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) \rho_{T-1}(S_T), \\ \quad \text{if } x \geq \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \\ x(S_{T-1} - \rho_{T-1}(-S_T)) + \left(\text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right) \rho_{T-1}(-S_T), \\ \quad \text{if } x < \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}. \end{cases}$$

Similarly, under AIP, the infimum of $M_{T-1}(x)$ is reached at $\bar{x}^* := \text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T}$ and

$$M_{T-1}^* = S_{T-1} \left(\text{ess sup}_{\mathcal{F}_{T-1}} \frac{h_T}{S_T} \right).$$

The conclusion follows. \square

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