Aging in Metropolis dynamics of the REM: a proof

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ABSTRACT. We study the aging behavior of the Random Energy Model (REM) evolving under Metropolis dynamics. We prove that a classical two-time correlation function converges almost surely to the arcsine law distribution function that characterizes activated aging, as predicted in the physics literature, in the optimal domain of the time-scale and temperature parameters where this result can be expected to hold. In the course of the proof we establish that a certain continuous time clock process, after proper rescaling, converges almost surely to a stable subordinator, improving upon the result of Ref. [15] where a closely related clock is shown to converge in probability only, and in a restricted region of the time-scale and temperature parameters. The random rescaling involved in this convergence is controlled at the fine level of fluctuations. As a byproduct, we refine and prove a conjecture made in Ref. [15].

1. INTRODUCTION

While there is as yet no established theory for the description of glasses, a consensus exists that this amorphous state of matter is intrinsically dynamical in nature [19], [29], [26]. Measuring suitable two-time correlation functions indeed reveals that glassy dynamics are history dependent and dominated by ever slower transients: they are *aging*. The realization in the late 80's that *mean-field* spin glass dynamics could provide a mathematical formulation for this phenomenon sparked renewed interest in models, such as Derrida's REM and p-spin SK models [16], [17], whose statics had, until then, been the main focus of attention [11]. Despite this, Bouchaud's phenomenological *trap models* first took the center stage as they succeeded in predicting the power-law decay of two-time correlation functions observed experimentally, even though they did so at the cost of an ad hoc construction and drastically simplifying assumptions [10], [12].

It was not until 2003 that a trap model dynamics was shown to result for the microscopic Glauber dynamics of a (random) mean-field spin glass Hamiltonian, namely, the REM endowed with the so-called *Random Hopping* dynamics and observed on time-scales near equilibrium [4, 5, 6]. Quite remarkably, the predicted functional form of two-time correlation functions was recovered. Rapid progress followed over the ensuing decade, beginning with [7]. The optimal domain of temperature and time-scales were this prediction applies was obtained in Ref. [22] (almost surely in the random environment except for times scales near equilibrium where the results hold in probability only) and these results were partially extended to the p-spin SK models [3], [13].

The choice of the Random Hopping dynamics, however, clearly favored the emergence of trap models. Just as in trap model constructions, its trajectories are those of a simple random walk on the underlying graph, and thus, do not depend on the random Hamiltonian. This is in sharp contrast with *Metropolis* [30] dynamics, a choice heralded in the

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physic's literature as *the* natural microscopic Glauber dynamics [27], whose trajectories are biased against increasing the energy. This dependence on the random Hamiltonian makes the analysis of the two-time correlation functions much harder. This problem was first tackled in [24] were a truncated REM is considered, and a natural two-time correlation function is proved to behave as in the Random Hopping dynamics, in the same, optimal range of time-scales and temperatures for which this result holds almost surely in the random environment. In the present paper, we free ourselves of the simplifying truncation assumption and prove that the same result holds true almost surely for the full REM. A recent paper [15], by establishing the convergence of a so-called clock process, suggested that this might be the case but failed short of proving aging: the sole clock convergence, indeed, does not suffice to deduce aging, a property of correlation functions.

1.1. **Main result.** Let us now specify the model. Denote by $\mathcal{V}_n = \{-1,1\}^n$ the n-dimensional discrete cube and by $\mathcal{E}_n$ its edge set. The Hamiltonian (or energy) of the REM is a collection of independent Gaussian random variables, $(\mathcal{H}_n(x), x \in \mathcal{V}_n)$, satisfying

$$\mathbb{E}\mathcal{H}_n(x) = 0, \quad \mathbb{E}\mathcal{H}_n^2(x) = n. \tag{1.1}$$

The sequence $(\mathcal{H}_n(x), x \in \mathcal{V}_n)$, $n > 1$, is defined on a common probability space denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. On $\mathcal{V}_n$, we consider the Markov jump process $(X_n(t), t > 0)$ with rates

$$\lambda_n(x,y) = \frac{1}{n} e^{-\beta[\mathcal{H}_n(y)-\mathcal{H}_n(x)]^+}, \quad \text{if } (x,y) \in \mathcal{E}_n, \tag{1.2}$$

and $\lambda_n(x,y) = 0$ else, were $a+ = \max\{a, 0\}$. This defines the single spin-flip continuous time Metropolis dynamics of the REM at temperature $\beta^{-1} > 0$. Note that the rates are reversible with respect to the measure that assigns to $x \in \mathcal{V}_n$ the mass

$$\tau_n(x) \equiv \exp\{-\beta \mathcal{H}_n(x)\}. \tag{1.3}$$

When studying aging the choice of the observation time-scale, c_n, is all-important. Given $0 < \varepsilon < 1$ and $0 < \beta < \infty$, we let $c_n \equiv c_n(\beta, \varepsilon)$ be the two-parameter sequence defined by

$$2^{\varepsilon n} \mathbb{P}(\tau_n(x) \geq c_n) = 1. \tag{1.4}$$

Gaussian tails estimates yield the explicit form

$$c_n = \exp\left\{ n\beta\beta_c(\varepsilon) - (1/2\alpha(\varepsilon)) \left(\log(\beta_c^2(\varepsilon) n/2) + \log 4\pi + o(1) \right) \right\} \tag{1.5}$$

where

$$\beta_c(\varepsilon) = \sqrt{\varepsilon 2 \log 2}, \tag{1.6}$$

$$\alpha(\varepsilon) = \beta_c(\varepsilon)/\beta. \tag{1.7}$$

A classical choice of two-time correlation function is the probability $\mathcal{C}_n(t,s)$ to find the process in the same state at the two endpoints of the time interval $[c_n t, c_n(t+s)]$,

$$\mathcal{C}_n(t,s) \equiv \mathcal{P}_{\mu_n}\left(X_n(c_n t) = X_n(c_n(t+s)) \right), \quad t, s > 0. \tag{1.8}$$

Here $\mathcal{P}_{\mu_n}$ denotes the law of X_n conditional on $\mathcal{F}$ (i.e. for fixed realizations of the random Hamiltonian) when the initial distribution, μ_n, is the uniform measure on $\mathcal{V}_n$.

Theorem 1.1. *For all $0 < \varepsilon < 1$ and all $\beta > \beta_c(\varepsilon)$, for all $t > 0$ and $s > 0$, $\mathbb{P}$-almost surely,*

$$\lim_{n\to\infty} \mathcal{P}_{\mu_n}\left(X_n(c_n t) = X_n(c_n(t+s)) \right) = \frac{\sin \alpha(\varepsilon)\pi}{\pi} \int_0^{t/(t+s)} u^{\alpha(\varepsilon)-1}(1-u)^{-\alpha(\varepsilon)}\, du. \tag{1.9}$$

Remark. We in fact prove the more general statement that (1.9) holds along any n-dependent sequences of the form $0 < \varepsilon_n \leq 1 - c'\beta\sqrt{n^{-1}\log n} + c''n^{-1}\log n$ where $0 < c', c'' < \infty$ are constants, that satisfy $\lim_{n\to\infty} \varepsilon_n = \varepsilon$, $0 < \varepsilon \leq 1$. Relaxation to stationarity is known to occur, to leading order, on time-scales c_n of the form (1.5) with $\varepsilon_n = 1$ [21]. At the other extremity, a behavior known as extremal aging is expected to characterize the process on times scales that are sub-exponential in the volume and defined through sequences ε_n that decay to 0 slowly enough [14], [8]. This will be the object of a follow up paper.

As in virtually all papers on aging, the proof of Theorem 1.1 relies on a two-step scheme that seeks to isolate the causes of aging by writing the process of interest, X_n, as an *exploration process* time-changed by (the inverse of) a *clock process*. Aging is then linked to the arcsine law for stable subordinators through the convergence of the suitably rescaled clock process to an α-stable subordinator, $0 < \alpha < 1$. This is provided that the two-time correlation function at hand can be brought into a suitable function of the clock. Both steps heavily depend on the properties of the exploration process.

While this scheme offers the methodological underpinnings of the analysis of aging, two distinct ways of implementing it, through *discrete* or *continuous* time objects, respectively, have emerged from the literature (we refer to the recent papers [24], [25], and [15] for in-depth bibliographies). The first arose from the study of models whose exploration process can be chosen as the simple random walk on the underlying graph. As mentioned earlier, this includes all Random Hopping dynamics and several trap models (e.g. on the complete graph or on $\mathbb{Z}^d$). In physically more realistic dynamics the discrete scheme may quickly become intractable. As shown in Ref. [24] for Metropolis dynamics of a truncated REM, the associated exploration process is itself an aging process that presents the same complexity as the original dynamics. A similar situation arises when considering asymmetric trap models on $\mathbb{Z}^d$. Initiated in that context, the continuous scheme consists in choosing a (now continuous time) exploration process that mimics the simple random walk.

Prescribing the exploration process completely determines the clock process. Clearly, having efficient tools available to prove their convergence to stable subordinators is essential. Such tools were provided in Ref. [23] and [13] for discrete time clock processes in the general setting of reversible Markov jumps processes in random environment on sequences of finite graphs and, more recently, for both discrete and continuous time clock processes of similar Markov jumps processes on infinite graphs [25]. These tools both allowed one to improve all earlier results on the Random Hopping dynamics of mean-field models [22], [13], [14], turning statements previously obtained in law into almost sure statements in the random environment, and to obtain the first aging results for several two-time correlation functions of asymmetric trap model on $\mathbb{Z}^d$ [25].

In Section 1.2 below we fill the gap left by continuous time clock processes in the case of sequences of finite graphs and, thus, extent the results of Ref. [13] to that setting. This is perhaps no more than an exercise but the results we present (Theorem 1.2 and Theorem 1.3) are the cornerstone of our approach and, hopefully, of other papers to come. We close this introduction out in Section 1.3 by stating a clock process convergence result for Metropolis dynamics of the REM (Theorem 1.4) that is at the heart of the proof of Theorem 1.1.

1.2. **Convergence of continuous time clock processes.** We now enlarge our focus to the following abstract setting. Let $G_n(\mathcal{V}_n, \mathcal{E}_n)$ be a sequence of loop-free graphs with set of vertices $\mathcal{V}_n$ and set of edges $\mathcal{E}_n$. A *random environment* is a family of possibly dependent

positive random variables, $(\tau_n(x), x \in \mathcal{V}_n)$. The sequence $(\tau_n(x), x \in \mathcal{V}_n)$, $n > 1$, is defined on a common probability space denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. On $\mathcal{V}_n$ we consider a Markov jump process, $(X_n(t), t > 0)$, with initial distribution μ_n and jump rates $(\lambda_n(x,y))_{x,y\in\mathcal{V}_n}$ satisfying $\lambda_n(y,x) = 0$ if $(x,y) \notin \mathcal{E}_n$ and

$$\tau_n(x)\lambda_n(x,y) = \tau_n(y)\lambda_n(y,x) \quad \text{if } (x,y) \in \mathcal{E}_n,\ x \neq y. \tag{1.10}$$

Thus X_n is reversible with respect to the (random measure) that assigns to $x \in \mathcal{V}_n$ the mass $\tau_n(x)$. To X_n we associate an *exploration process* Y_n. This is any Markov jump process, $(Y_n(t), t > 0)$, with state space $\mathcal{V}_n$, initial distribution μ_n, and jump rates $(\widetilde{\lambda}_n(x,y))_{x,y\in\mathcal{V}_n}$ chosen such that X_n and Y_n have the same trajectories, that is to say,

$$\frac{\lambda_n(x,y)}{\lambda_n(x)} = \frac{\widetilde{\lambda}_n(x,y)}{\widetilde{\lambda}_n(x)} \quad \forall (x,y) \in \mathcal{E}_n, \tag{1.11}$$

where $\widetilde{\lambda}_n^{-1}(x)$ and $\lambda_n^{-1}(x)$ are, respectively, the mean holding times at x of Y_n and X_n:

$$\widetilde{\lambda}_n(x) \equiv \sum_{y:(x,y)\in\mathcal{E}_n} \widetilde{\lambda}_n(x,y), \tag{1.12}$$

$$\lambda_n(x) \equiv \sum_{y:(x,y)\in\mathcal{E}_n} \lambda_n(x,y). \tag{1.13}$$

Then X_n and Y_n are related to each other through the time change

$$X_n(t) = Y_n(\widetilde{S}_n^{\leftarrow}(t)), \qquad t \geq 0, \tag{1.14}$$

where $\widetilde{S}_n^{\leftarrow}$ denotes the generalized right continuous inverse of $\widetilde{S}_n$, and $\widetilde{S}_n$, the so-called *continuous time clock process*, is given by

$$\widetilde{S}_n(t) = \int_0^t \lambda_n^{-1}(Y_n(s))\widetilde{\lambda}_n(Y_n(s))ds, \qquad t \geq 0. \tag{1.15}$$

Note that there is considerable freedom in the choice of the exploration process Y_n. We will come back to this issue at the end of this subsection and focus, for the time being, on the analysis of the asymptotic behavior of the general clock process (1.15).

For future reference, we denote by $\mathcal{F}^Y$ the σ-algebra generated by the processes Y_n. We write P for the law of the process Y_n conditional on the σ-algebra $\mathcal{F}$, i.e. for fixed realizations of the random environment. Likewise we call $\mathcal{P}$ the law of X_n conditional on $\mathcal{F}$. If the initial distribution, μ_n, has to be specified we write $\mathcal{P}_{\mu_n}$ and P_{μ_n}. Expectation with respect to $\mathbb{P}$, P_{μ_n}, and $\mathcal{P}_{\mu_n}$ are denoted by $\mathbb{E}$, E_{μ_n}, and $\mathcal{E}_{\mu_n}$, respectively.

Our main aim is to obtain simple and robust criteria for the convergence of the (suitably rescaled) clock process (1.15) to a stable subordinator. Since the clock is a doubly stochastic process, the desired convergence mode must be specified. We will ask whether there exist sequences a_n and c_n that make the rescaled clock process

$$S_n(t) = c_n^{-1}\widetilde{S}_n(a_n t)\,, \quad t \geq 0, \tag{1.16}$$

converge weakly, as $n \uparrow \infty$, as a sequence of random elements in Skorokhod's space $D((0,\infty])$, and strive to obtain $\mathbb{P}$-almost sure results in the random environment since such results (also referred to as *quenched*) contain the most useful information from the point of view of physics.

As for discrete time clock processes [23], [13], the driving force behind our approach is a powerful method developed by Durrett and Resnick [20] to prove functional limit theorems for sums of dependent variables. Clearly this method does not cover the case of our

continuous time clock processes. The simple idea (already present in [25]) is to introduce a suitable "blocking" that turns the rescaled clock process (1.16) into a partial sum process to which Durrett and Resnick method can now be applied. For this we introduce a new scale, θ_n, and set

$$k_n(t) \equiv \lfloor a_n t/\theta_n \rfloor. \tag{1.17}$$

The *blocked clock process*, $S_n^b(t)$, is defined through

$$S_n^b(t) = \sum_{i=1}^{k_n(t)} Z_{n,i} \tag{1.18}$$

where, for each $i \geq 1$,

$$Z_{n,i} \equiv c_n^{-1} \sum_{x \in \mathcal{V}_n} \left(\lambda_n^{-1}(x)\widetilde{\lambda}_n(x)\right)[\ell_n^x(\theta_n i) - \ell_n^x(\theta_n(i-1))], \tag{1.19}$$

and where, for each $x \in \mathcal{V}_n$,

$$\ell_n^x(t) = \int_0^t \mathbb{1}_{\{Y_n(s)=x\}} ds \tag{1.20}$$

is the local time at x. The next theorem gives sufficient conditions for S_n^b to converge. These conditions are expressed in terms of a small number of key quantities. For each $t > 0$, let

$$\pi_n^{Y,t}(y) = k_n^{-1}(t) \sum_{i=1}^{k_n(t)-1} \mathbb{1}_{\{Y_n(i\theta)=y\}} \tag{1.21}$$

be the empirical measure on $\mathcal{V}_n$ constructed from the sequence $(Y_n(i\theta), i \in \mathbb{N})$. For $y \in \mathcal{V}_n$ and $u > 0$, denote by

$$Q_n^u(y) \equiv P_y(Z_{n,1} > u) \tag{1.22}$$

the tail distribution of the aggregated jumps when X_n (equivalently, Y_n) starts in y. Using these quantities, define the functions

$$\nu_n^{Y,t}(u,\infty) \equiv k_n(t) \sum_{y \in \mathcal{V}_n} \pi_n^{Y,t}(y) Q_n^u(y), \tag{1.23}$$

$$\sigma_n^{Y,t}(u,\infty) \equiv k_n(t) \sum_{y \in \mathcal{V}_n} \pi_n^{Y,t}(y) \left[Q_n^u(y)\right]^2. \tag{1.24}$$

Observe that the sequence of measures $\pi_n^{Y,t}$ as well as the sequence of functions $Q_n^u(y)$, $y \in \mathcal{V}_n$, are random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the random environment. Thus, the functions $\nu_n^{Y,t}$ and $\sigma_n^{Y,t}$ also are random variables on that space.

We now formulate four conditions for the sequence S_n^b to converge to a subordinator. These conditions refer to a given sequence of initial distributions μ_n, given sequences of numbers a_n, c_n, and θ_n as well as a given realization of the random environment.

Condition (A0). For all $u > 0$,

$$\lim_{n\to\infty} P_{\mu_n}(Z_{n,1} > u) = 0. \tag{1.25}$$

Condition (A1). There exists a σ-finite measure ν on $(0,\infty)$ satisfying $\int_0^\infty (x \wedge 1)\nu(dx) < \infty$ and such that for all continuity points x of the distribution function of ν, for all $t > 0$ and all $u > 0$,

$$P_{\mu_n}\left(\left|\nu_n^{Y,t}(u,\infty) - t\nu(u,\infty)\right| < \epsilon\right) = 1 - o(1), \quad \forall \epsilon > 0. \tag{1.26}$$

Condition (A2). For all $u > 0$ and all $t > 0$,

$$P_{\mu_n}\left(\sigma_n^{Y,t}(u,\infty) < \epsilon\right) = 1 - o(1)\,, \quad \forall \epsilon > 0\,. \tag{1.27}$$

Condition (A3). For all $t > 0$,

$$\lim_{\epsilon\downarrow 0} \limsup_{n\uparrow\infty} k_n(t) \sum_{y\in\mathcal{V}_n} E_{\mu_n}(\pi_n^{Y,t}(y)) E_y(Z_{n,1}\mathbb{1}_{\{Z_{n,1}\le\epsilon\}}) = 0. \tag{1.28}$$

Theorem 1.2. *For all sequences of initial distributions μ_n and all sequences a_n, c_n, and $1 \le \theta_n \ll a_n$ for which Conditions (A0), (A1), (A2), and (A3) are verified, either $\mathbb{P}$-almost surely or in $\mathbb{P}$-probability, the following holds w.r.t. the same convergence mode:*

$$S_n^b \Rightarrow_{J_1} S_\nu, \tag{1.29}$$

where S_ν is the Lévy subordinator with Lévy measure ν and zero drift. Convergence holds weakly on the space $D([0,\infty))$ equipped with the Skorokhod J_1-topology.

Remark. Note that the theorem is stated for the *blocked* process S_n^b rather than the original process S_n (defined in (1.16)). This may falsely appear as an undesirable consequence of our techniques. We stress that for applications to correlation functions, one needs statements that are valid in the strong J_1 topology whereas forming blocks is needed in order to make sense of writing J_1 convergence statements in the setting of continuous time clocks.

Remark. Also note that convergence of S_n^b in the strong J_1 topology immediately implies the strictly weaker result that S_n converges to the same limit in the M_1 topology.

As for discrete time clocks of Ref. [13], our next step consists in reducing Conditions (A1) and (A2) of Theorem 1.2 to (i) a *mixing condition* for the chain Y_n, and (ii) a *law of large numbers* for the random variables Q_n. Again we formulate three conditions for a given sequence of initial distributions μ_n, given sequences a_n, c_n, and θ_n, and a given realization of the random environment.

Condition (B0). Denote by π_n the invariant measure of Y_n. There exists a sequence $\kappa_n \in \mathbb{N}$ and a positive decreasing sequence ρ_n, satisfying $\rho_n \downarrow 0$ as $n \uparrow \infty$, such that, for all pairs $x, y \in \mathcal{V}_n$, and all $t \ge 0$,

$$|P_x\left(Y_n(t+\kappa_n) = y\right) - \pi_n(y)| \le \rho_n \pi_n(y). \tag{1.30}$$

Condition (B1). There exists a measure ν as in Condition (A1) such that, for all $t > 0$ and all $u > 0$,

$$\nu_n^t(u,\infty) \equiv k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) Q_n^u(y) \to t\nu(u,\infty), \tag{1.31}$$

Condition (B2). For all $t > 0$ and all $u > 0$,

$$\sigma_n^t(u,\infty) \equiv k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) \left[Q_n^u(y)\right]^2 \to 0. \tag{1.32}$$

Condition (B3). For all $t > 0$,

$$\lim_{\epsilon\downarrow 0} \limsup_{n\uparrow\infty} k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) E_y(Z_{n,1}\mathbb{1}_{\{Z_{n,1}\le\epsilon\}}) = 0. \tag{1.33}$$

Theorem 1.3. *Assume that for all sequences of initial distributions μ_n and all sequences a_n, c_n, κ_n, and $\kappa_n \le \theta_n \ll a_n$, Conditions (A0), (B0), (B1), (B2), and (B3) hold $\mathbb{P}$-almost surely, respectively in $\mathbb{P}$-probability. Then, as in (1.29), $S_n^b \Rightarrow_{J_1} S_\nu$, $\mathbb{P}$-almost surely, respectively in $\mathbb{P}$-probability.*

Theorem 1.3 is our key tool for proving convergence of blocked clock processes to subordinators. It is of course essential for the success of our strategy that the convergence criteria we obtained be tractable. Going back to (1.11) we thus now ask, in this light, how best to choose the exploration process Y_n.

A tentative answer to this question is to mimic the exploration process of the Random Hopping dynamics, which means choose Y_n such that its invariant measure, π_n, is "close" to the uniform measure and its mixing time, κ_n, is short compared to that of the process X_n. The following class of jump rates, inspired from an ingenious choice made in Ref. [15], is intended to favor the emergence of these properties. Given a fresh sequence $\eta_n \geq 0$, set

$$\widetilde{\lambda}_n(x,y) = \max(\eta_n, \tau_n(x))\lambda_n(x,y). \tag{1.34}$$

One easily checks that (1.11) is verified, that Y_n is reversible with respect to the measure

$$\pi_n(x) = \frac{\min\big(\eta_n, \tau_n(x)\big)}{\sum_{x\in\mathcal{V}_n}\min\big(\eta_n, \tau_n(x)\big)}\mathbb{1}_{\{\eta_n>0\}} + |\mathcal{V}_n|^{-1}\mathbb{1}_{\{\eta_n=0\}}, \quad x\in\mathcal{V}_n, \tag{1.35}$$

and that the clock (1.15) becomes

$$\widetilde{S}_n(t) = \int_0^t \max\big(\eta_n, \tau_n(Y_n(s))\big)ds. \tag{1.36}$$

Let us discuss the role of η_n on the example of Metropolis dynamics of REM. When $\eta_n = 0$, π_n nicely reduces to the uniform measure but the mixing time, κ_n, of the resulting exploration process turns out to be of the same order as that of X_n, that is to say, of the order of $\max_{(x,y)}(\min(\tau_n(x),\tau_n(y)))^{-1} = e^{\beta n\sqrt{\log 2}(1+o(1))}$. This leaves little hope that the conditions of Theorem 1.3 can be verified. A moment's thought suffices, however, to see that such a large mixing time is a side effect of the symmetry of the Hamiltonian (1.1). By breaking this symmetry, the term $\max(\eta_n, \tau_n(x))$ in (1.34) places an η_n-dependent cap on κ_n (see Section 3.1). One is then left to choose η_n small enough so that π_n remains close to the uniform measure but large enough so that κ_n is kept as small as needed. A similar strategy should hopefully apply to more general mean-field spin glass Hamiltonians.

Remark. We stress that the sole convergence of the clock process does not suffice to deduce aging, namely, the specific power law decay of the two-time correlation function. One still has to solve the problem of reducing the behavior of the two-time correlation function, as $n\to\infty$, to the arcsine law for stable subordinators, and this requires more information on the exploration process than needed to only prove convergence of the clock. Notice also that unlike the discrete time clock process, the continuous time clock process is not a physical time. It thus has no physical meaning on its own.

1.3. **Application to Metropolis dynamics of the REM.** From that point onwards we focus on Metropolis dynamics of the REM (see (1.1)-(1.2)) started in the uniform measure on $\mathcal{V}_n$. Applying the abstract results of Section 1.2 enables us to prove $\mathbb{P}$-almost sure convergence of the blocked clock process $S_n^b(t)$, defined in (1.18), when the continuous time clock process $\widetilde{S}_n(t)$, given by (1.15), is chosen as in (1.36).

To sate this result we must specify several quantities: the parameter η_n, the time-scales, a_n and c_n, and the block length, θ_n, entering the definitions of $\widetilde{S}_n(t)$ and $S_n^b(t)$. We begin by defining a sequence, $r_n^\star$, that is ubiquitous throughout the rest of the paper: given $\beta>0$ and a constant $c_\star > 1+\log 4$, we let $r_n^\star \equiv r_n(\beta, c_\star)$ be the solution of

$$n^{c_\star}\mathbb{P}(\tau_n(x)\geq r_n^\star) = 1. \tag{1.37}$$

In explicit form

$$r_n^\star = \exp\left\{\beta\sqrt{2c_\star n\log n}\left(1-\tfrac{\log\log n}{8c_\star\log n}(1+o(1))\right)\right\}. \tag{1.38}$$

We now take $\eta_n \equiv (r_n^\star)^{-1}$ in (1.34) which, combined with (1.2), yields

$$\widetilde{\lambda}_n(x,y) = \frac{1}{nr_n^\star}\frac{\min(\tau_n(y),\tau_n(x))}{\min\left(\frac{1}{r_n^\star},\tau_n(x)\right)}, \quad \text{if } (x,y)\in\mathcal{E}_n, \tag{1.39}$$

and $\widetilde{\lambda}_n(x,y)=0$ else. The physical observation time-scale, c_n, is chosen as in (1.4). It is naturally the same as in the Random Hopping dynamics. On the contrary, the definition of the auxiliary time-scale, a_n, contrasts sharply with the simple choice $a_n = 2^{\varepsilon n}$ made in the Random Hopping dynamics. We here must take

$$a_n = 2^{\varepsilon n}/b_n \tag{1.40}$$

where the sequence b_n is defined as follows. Recalling (1.6) and (1.7), define

$$F_{\beta,\varepsilon,n}(x) \equiv x^{\alpha_n(\varepsilon)-\frac{\log x}{2n\beta^2}}\left(1-\tfrac{\log x}{n\beta\beta_c(\varepsilon)}\right)^{-1}, \quad x>0, \tag{1.41}$$

where $\alpha_n(\varepsilon)\equiv(n\beta^2)^{-1}\log c_n$, that is, in view of (1.5),

$$\alpha_n(\varepsilon) = \alpha(\varepsilon) - \tfrac{\log(\beta_c^2(\varepsilon)n/2)+\log 4\pi+o(1)}{2n\beta\beta_c(\varepsilon)}. \tag{1.42}$$

Further introduce the random set

$$T_n \equiv \left\{x\in\mathcal{V}_n \mid \tau_n(x)\geq c_n(n^2\theta_n)^{-1}\right\}. \tag{1.43}$$

Then, for ℓ_n^x as in (1.20), we set

$$b_n \equiv (\theta_n\pi_n(T_n))^{-1}\sum_{x\in T_n} E_{\pi_n}\left[F_{\beta,\varepsilon,n,}(\ell_n^x(\theta_n))\right]. \tag{1.44}$$

It now only remains to choose the block length θ_n. (The notation $x_n \ll y_n$ means that the sequences $x_n>0$ and $y_n>0$ satisfy $x_n/y_n\to 0$ as $n\to\infty$.)

Theorem 1.4. *Given $0<\varepsilon<1$ let θ_n be any sequence such that*

$$\tfrac{4}{1-\alpha(\varepsilon)}\log r_n^\star < \log\theta_n \ll n \tag{1.45}$$

and let c_n and a_n be as in (1.4) and (1.40)-(1.44), respectively. Then, for all $0<\varepsilon<1$ and all $\beta>\beta_c(\varepsilon)$, $\mathbb{P}$-almost surely,

$$S_n^b \Rightarrow_{J_1} V_{\alpha(\varepsilon)} \tag{1.46}$$

where $V_{\alpha(\varepsilon)}$ is a stable subordinator with zero drift and Lévy measure ν defined through

$$\nu(u,\infty) = u^{-\alpha(\varepsilon)}, \quad u>0, \tag{1.47}$$

and where $\Rightarrow_{J_1}$ denotes weak convergence in the space $D([0,\infty))$ of càdlàg functions equipped with the Skorokhod J_1-topology.

We again emphasize (see the remark below Theorem (1.2)) that the J_1 convergence statement of Theorem 1.4 is crucial to the control correlation functions. Of course, Theorem 1.4 implies the weaker result that the original (non blocked) clock process (1.16) converges to the same limit in the M_1 topology of Skorokhod. Such a result was proved in Ref. [15] (for the clock obtained by taking $\eta_n=1$ in (1.36)) albeit only in $\mathbb{P}$-probability and in the restricted domain of parameters $\beta>\beta_c(\varepsilon)$ and $1/2<\varepsilon<1$. As shown in

[24] (see lemma 2.1) the graph structure of the set T_n when $1/2 < \varepsilon < 1$ reduces to a collection of isolated vertices (no element of T_n has a neighbor in T_n) and this considerably simplifies the analysis.

Let us now examine the sequence b_n introduced in (1.40) and defined in (1.44). This sequence is a priori random in the random environment and depends on a sequence, θ_n, that can itself be chosen within the two widely different bounds of (1.45). The next proposition provides upper and lower bounds on b_n that are not affected by the choice of θ_n.

Proposition 1.5. *Given $0 < \varepsilon < 1$, let c_n and θ_n be as in Theorem 1.4. Then, there exists a subset $\Omega' \subseteq \Omega$ with $\mathbb{P}(\Omega') = 1$ such that on Ω', for all but a finite number of indices n*

$$\left(n^{c_-}(r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}\right)^{-1} \le b_n \le n^{c_+}(r_n^\star)^{1+\alpha_n(\varepsilon)} \tag{1.48}$$

where $0 < c_-, c_+ \le \infty$ are numerical constants. Thus $\lim_{n\to\infty} n^{-1} \log a_n = \varepsilon$ $\mathbb{P}$-a.s..

Remark. The definition (1.40)-(1.44) of a_n and that of the sequence R_N in (2.10) of Ref. [15] have an obvious family resemblance. Our control of a_n through Proposition 1.5 implies the behavior conjectured in item 4 page 4 of that paper.

Remark. One may wonder whether the lower bound of (1.45) can be improved. The main technical obstacle to doing so is the lower bound on mean hitting times of Lemma 3.5. In particular, trying to improve the bound (3.5) on the spectral gap by choosing η_n larger, say as large as 1 as in Ref. [15], can at best improve the constant $\frac{4}{1-\alpha(\varepsilon)}$ in front of $\log r_n^\star$ in (1.45).

The rest of the paper is organized as follows. Section 2 is concerned with the properties of the REM's landscape: several level sets that play an important role in our analysis are introduced and their properties collected. Section 3 gathers all needed results on the exploration process Y_n. The proof of Theorem 1.4 can then begin. Section 4, 5, and 6 are devoted, respectively, to the verification of Condition (B1), (B2), and (B3) of Theorem 1.3. The proof of Theorem 1.4 is completed in Section 7. Also in Section 7, the link between the blocked clock process of (1.46) and the two-time correlation function (1.8) is made, and the proof of Theorem 1.1 is concluded. An appendix (Section 8) contains the proof of the results of Section 1.2.

2. Level sets of the REM's landscape: the Top and other sets

Given $V \subseteq \mathcal{V}_n$ we denote by $G \equiv G(V)$ the undirected graph which has vertex set V and edge set $E(G(V)) \subseteq \mathcal{E}_n$ consisting of pairs of vertices $\{x, y\}$ in V with $\mathrm{dist}(x, y) = 1$, where $\mathrm{dist}(x, x') \equiv \frac{1}{2}\sum_{i=1}^n |x_i - x'_i|$ is the graph distance on $\mathcal{V}_n$. When $\mathrm{dist}(x, y) = 1$ we simply write $x \sim y$. We are concerned with the graph properties of level sets of the form

$$V_n(\rho) = \{x \in \mathcal{V}_n \mid \tau_n(x) \ge r_n(\rho)\} \tag{2.1}$$

where, given $\rho > 0$, the threshold level $r_n(\rho)$ is the sequence defined through

$$2^{\rho n}\mathbb{P}(\tau_n(x) \ge r_n(\rho)) = 1. \tag{2.2}$$

Observe that $V_n(\rho)$ can uniquely be decomposed into a collection of subsets

$$V_n(\rho) = \cup_{l=1}^L C_{n,l}(\rho), \quad C_{n,l}(\rho) \cap C_{n,k}(\rho) \ \ \forall 1 \le l \ne k \le L, \quad L \equiv L_n(\rho), \tag{2.3}$$

such that each graph $G(C_{n,l}(\rho))$ is connected but any two distinct graphs $G(C_{n,l}(\rho))$ and $G(C_{n,k}(\rho))$ are disconnected. With a little abuse of terminology we call the sets $C_{n,l}(\rho)$ the connected components of the graph $G(V_n(\rho))$. As ρ decreases from ∞ to 0, the set $V_n(\rho)$ grows and the graph $G(V_n(\rho))$ potentially acquires new edges. It is known [9] that

the size of the largest connected component $C_{n,l}(\rho)$ undergoes a transiton near the critical value $\rho^c \approx \frac{\log n}{n \log 2}$, with a unique "giant" component of size $\mathcal{O}(n^{-1}2^n)$ emerging slightly below this value. As ρ decreases the small components merge into the giant one, and total connectedness is achieved for ρ slightly smaller than n^{-1}. One may naturally think of the connected components $C_{n,l}(\rho)$ before criticality as containing distinct "valleys" of the REM's energy landscape, the level of emergence of the totally connected giant component then being a "ground level" connecting the local valleys.

We now introduce several sets that play key roles in our analysis.

• ***The sets*** $V_n^\star$ ***and*** $\overline{V}_n^\star$ ***(of valleys and hills).*** Let $c_\star$ be as in (1.37) and set

$$\rho_n^\star \equiv \frac{c_\star \log n}{n \log 2}. \tag{2.4}$$

Thus, taking $\rho = \rho_n^\star$ in (2.1)-(2.3), $r_n^\star \equiv r_n(\rho_n^\star)$ and the set $V_n^\star \equiv V_n(\rho_n^\star)$ decomposes into

$$V_n^\star = \cup_{l=1}^{L^\star} C_{n,l}^\star, \quad C_{n,l}^\star \cap C_{n,k}^\star \ \ \forall l \neq k, \quad L^\star \equiv L_n(\rho_n^\star), \tag{2.5}$$

where the $C_{n,l}^\star$ are the connected components of the graph $G(V_n(\rho_n^\star))$. According to our earlier picture they contain "valleys" of the landscape. Since $H_n(x)$ is symmetrical the set

$$\overline{V}_n^\star \equiv \overline{V}_n(\rho_n^\star) = \left\{x \in \mathcal{V}_n \mid \tau_n^{-1}(x) \geq r_n^\star\right\} \tag{2.6}$$

obtained from $V_n(\rho_n^\star)$ by substituting $-\mathcal{H}_n(x)$ for $\mathcal{H}_n(x)$ in (1.3) has the same random graph properties as $V_n^\star$ but now contains "hills". As in (2.5) we write

$$\overline{V}_n^\star \equiv \overline{V}_n(\rho_n^\star) = \cup_{l=1}^{L^\star} \overline{C}_{n,l}^\star, \quad \overline{C}_{n,l}^\star \cap \overline{C}_{n,k}^\star \ \ \forall l \neq k, \quad \overline{L}^\star \equiv \overline{L}_n(\rho_n^\star), \tag{2.7}$$

where $\overline{C}_{n,l}^\star$ are the connected components of the graph $G(\overline{V}_n(\rho_n^\star))$. With this definition (1.39) becomes

$$\widetilde{\lambda}_n(x,y) = \begin{cases} \frac{1}{n} e^{-\beta \max(\mathcal{H}_n(y), \mathcal{H}_n(x))}, & \text{if } x \notin \overline{V}_n^\star, \\ \frac{1}{n r_n^\star} e^{-\beta [\mathcal{H}_n(y) - \mathcal{H}_n(x)]^+}, & \text{if } x \in \overline{V}_n^\star. \end{cases} \tag{2.8}$$

Furthermore, by (1.12), denoting by $\partial A = \{x \in \mathcal{V}_n \mid \mathrm{dist}(x, A) = 1\}$ the outer boundary of $A \subset \mathcal{V}_n$, we have that for all $x \in \partial V_n^\star$,

$$\widetilde{\lambda}_n(x) = \sum_{y \in (V_n^\star)^c} \widetilde{\lambda}_n(x,y) + \left((n r_n^\star)^{-1} \mathbb{1}_{\{x \in \overline{V}_n^\star\}} + \tau_n(x) n^{-1} \mathbb{1}_{\{x \in (\overline{V}_n^\star)^c\}} \right) |\partial x \cap V_n^\star|. \tag{2.9}$$

Hence, conditional on $V_n^\star$, the mean holding time at $x \in (V_n^\star)^c$ does not depend on the variables $\{\tau_n(y), y \in V_n^\star\}$ but only depends on the variables $\{\tau_n(y), y \in (V_n^\star)^c\}$.

• ***Immersions in*** $V_n^\star$**.** Given any subset $A \subset V_n^\star$ we call the *immersion of* A *in* $V_n^\star$ and denote by $A^\star$ the set

$$A^\star \equiv \cup_{l=1}^{L^\star} A_{n,l}^\star, \quad A_{n,l}^\star = \begin{cases} C_{n,l}^\star, & \text{if } \ C_{n,l}^\star \cap A \neq \emptyset, \\ \emptyset, & \text{else.} \end{cases} \tag{2.10}$$

Thus the sets $A_{n,l}^\star$ are the valleys $C_{n,l}^\star$ that contain at least one element of A. Clearly, $\overline{V}_n^\star \cap V_n^\star = \emptyset$. Hence by (2.8), immersed sets have the property that

$$\widetilde{\lambda}_n(x,y) \leq n^{-1} r_n^\star \ \text{ for all } \ x \sim y \ \text{ such that } \ x \in A^\star, y \notin A^\star \text{ or } y \in A^\star, x \notin A^\star. \tag{2.11}$$

• ***The top,*** T_n***, and the associated sets*** $T_n^\star$***,*** T_n° ***and*** $I_n^\star$**.** Given a sequence $\delta_n \downarrow 0$ as $n \uparrow \infty$, set $\varepsilon_n \equiv \varepsilon - \delta_n$ and let the *top* be the set

$$T_n \equiv V_n(\varepsilon_n) \tag{2.12}$$

obtained by taking $\rho = \varepsilon_n$ in (2.3). (δ_n will later be chosen so that the definitions (2.12) and (1.43) of T_n coincide.) T_n contains the top of the order statistics or the $\tau_n(x)$'s, whence its name. Since $\rho_n^\star \ll \varepsilon_n$, $T_n \subset V_n^\star$, and so T_n can be immersed in $V_n^\star$. According to (2.10) we write

$$T_n^\star \equiv \cup_{l=1}^{L^\star} T_{n,l}^\star. \tag{2.13}$$

To each $x \in T_n$ corresponds a unique index $1 \le l \equiv l(x) \le L^\star$ such that $x \in T_{n,l(x)}^\star$. Of course a given valley $T_{n,l}^\star$ may contain several vertices of T_n. A set that is of special importance in the sequel is the subset T_n° of vertices of T_n that are alone in their valley,

$$T_n^\circ \equiv \left\{ x \in T_n \mid T_{n,l(x)}^\star \cap T_n = \{x\} \right\}. \tag{2.14}$$

This in particular implies that

$$T_n^\circ \subseteq \{ x \in \mathcal{V}_n \mid \tau_n(x) \ge r_n(\varepsilon_n), \forall_{y\sim x} \tau_n(y) < r_n(\varepsilon_n) \}. \tag{2.15}$$

Finally, define

$$I_n^\star \equiv \{ x \in \mathcal{V}_n \mid \tau_n(x) \ge r_n(\varepsilon_n), \forall_{y\sim x} (r_n^\star)^{-1} < \tau_n(y) < r_n^\star \} \subseteq T_n^\circ. \tag{2.16}$$

This is the largest subset of T_n° such that $\operatorname{dist}\big((V_n^\star \cup \overline{V}_n^\star), I_n^\star\big) \ge 2$.

Most of the content of the next three lemmata is taken from [24]. The first lemma gives estimates on the size of various sets.

Lemma 2.1. *There exists $\Omega^\star \subset \Omega$ with $\mathbb{P}(\Omega^\star) = 1$ such that on $\Omega^\star$, for all but a finite number of indices n,*

$$1 \le |C_{n,l}^\star| \le \{ \rho_n^\star [1 - 2c_\star^{-1}(1 + \mathcal{O}(\log n/n))] \}^{-1}, \quad 1 \le l \le L^\star. \tag{2.17}$$

The same bounds hold replacing $C_{n,l}^\star$ by $\overline{C}_{n,l}^\star$ and $L^\star$ by $\overline{L}^\star$ in (2.17). Furthermore,

$$|V_n^\star| = 2^n n^{-c_\star}(1 + o(n^{-c_\star})) \text{ and } |\overline{V}_n^\star| = 2^n n^{-c_\star}(1 + o(n^{-c_\star})), \tag{2.18}$$

$$|T_n| = 2^{n(1-\varepsilon_n)}(1 + \mathcal{O}(n 2^{-n\varepsilon_n/2})), \tag{2.19}$$

$$|T_n^\circ| = 2^{n(1-\varepsilon_n)}(1 + \mathcal{O}(n 2^{-n\varepsilon_n/2})), \tag{2.20}$$

$$|T_n \setminus T_n^\circ| \le n^4 2^{n(1-2\varepsilon_n)}(1 + o(1)), \tag{2.21}$$

$$|I_n^\star| = 2^{n(1-\varepsilon_n)}(1 - 2n^{-c_\star+1}(1 + o(1)), \tag{2.22}$$

$$|T_n^\circ \setminus I_n^\star| = 2n^{-c_\star+1} 2^{n(1-\varepsilon_n)}(1 + o(1)). \tag{2.23}$$

Finally, introducing the set

$$\mathcal{M}_n \equiv \{ x \in \mathcal{V}_n \mid \tau_n(x) > \tau_n(y) \text{ for all } y \sim x \} \tag{2.24}$$

of local minima of the Hamiltonian,

$$|\overline{V}_n^\star \cap \mathcal{M}_n| = 0. \tag{2.25}$$

Proof of Lemma 2.1. Recall that by assumption $c_\star > 1 + \log 4 > 2$. Eq. (2.17) is (2.9) of Lemma 2.2 of Ref. [24]. That the same bound holds for $|\overline{C}_{n,l}^\star|$ follows by symmetry of $\mathcal{H}_n$. The estimate (2.18) on $|V_n^\star|$ is (2.11) of Ref. [24] and the estimate on $|\overline{V}_n^\star|$ follows again by symmetry of $\mathcal{H}_n$. Eq. (2.19) and (2.22) are proved, respectively, as (2.11) of and (2.10) of Ref. [24]. The proof of (2.21) is a simple adaptation of the proof of lemma 7.1 of Ref. [24]. Clearly, (2.20) follows from (2.19) and (2.21), and (2.23) follows from (2.20)

and (2.22). It only remains to prove (2.24). For this note that $x \in \overline{V}_n^\star \cap \mathcal{M}_n$ if and only if $\tau_n(x) \le (r_n^\star)^{-1}$ and $\tau_n(y) < \tau_n(x)$ for all $y \sim x$. Thus

$$\begin{aligned} \mathbb{P}\big(|\overline{V}_n^\star \cap \mathcal{M}_n| \ge 1\big) &\le \sum_{x\in\mathcal{V}_n} \mathbb{P}\left(\tau_n(x) \le (r_n^\star)^{-1}, \forall_{y\sim x}\tau_n(y) < (r_n^\star)^{-1}\right) \quad (2.26)\\ &= 2^n n^{-c_\star}\left(n^{-c_\star}\right)^n \quad (2.27)\end{aligned}$$

which is summable. Thus, by Borel-Cantelli Lemma, there exists a set of full measure such that on that set, for all but a finite number of indices, $|\overline{V}_n^\star \cap \mathcal{M}_n| = 0$. □

The second lemma expresses the function $r_n(\rho)$ defined through (2.2).

Lemma 2.2 (Lemma 2.3 of [24])**.** *For all $\rho > 0$, possibly depending on n, and such that $\rho n \uparrow \infty$ as $n \uparrow \infty$,*

$$r_n(\rho) = \exp\left\{n\beta\beta_c(\rho) - (\beta/2\beta_c(\rho))\left[\log(\beta_c^2(\rho)n/2) + \log 4\pi\right] + o(\beta/\beta_c(\rho))\right\}. \quad (2.28)$$

Corollary 2.3. *Set $x_n = \delta_n/\varepsilon$ and assume that $x_n \downarrow 0$ as $n \uparrow \infty$ and $e^{n\beta\beta_c(\varepsilon)x} \gg r_n^\star$. Then*

$$r_n(\varepsilon_n)/r_n(\varepsilon) = \exp\left\{-n\beta\beta_c(\varepsilon)x\left[1 + \tfrac{x}{2} + \mathcal{O}(x^2)\right]\right\}. \quad (2.29)$$

The third and last lemma states needed bounds, in particular, on the maximal jump rate.

Lemma 2.4 (Lemma 2.4 of [24])**.** *There exists a subset $\Omega_0 \subseteq \Omega$ with $\mathbb{P}\big(\Omega_0\big) = 1$ such that on Ω_0, for all but a finite number of indices n the following holds:*

$$e^{-\beta\min\{\max(H_n(y),H_n(x))\,|\,(x,y)\in\mathcal{E}_n\}} \le e^{\beta n\sqrt{\log 2}(1+2\log n/n\log 2)} \equiv n\nu_n, \quad (2.30)$$

$$e^{-\beta\min\{H_n(x)\,|\,x\in\mathcal{V}_n\}} \le e^{\beta n\sqrt{2\log 2}(1+2\log n/n)}. \quad (2.31)$$

Thus, $\max_{(x,y)\in\mathcal{E}_n} \widetilde{\lambda}_n(x,y) \le \nu_n$.

3. Properties of the exploration process Y_n

In this Section we establish the properties of the exploration process needed in the rest of the paper. By (1.35) with $\eta_n \equiv (r_n^\star)^{-1}$ and (2.6), the invariant measure π_n of Y_n can be written as

$$\pi_n(x) = \frac{\mathbb{1}_{\{x\notin\overline{V}_n^\star\}} + r_n^\star\tau_n(x)\mathbb{1}_{\{x\in\overline{V}_n^\star\}}}{Z_{\beta,n}}, \quad x \in \mathcal{V}_n \quad (3.1)$$

where $Z_{\beta,n} \equiv |\mathcal{V}_n \setminus \overline{V}_n^\star| + \sum_{x\in\overline{V}_n^\star} r_n^\star\tau_n(x)$.

Lemma 3.1. *On $\Omega^\star$, for all but a finite number of indices n,*

$$2^n(1 - n^{-c_\star}(1 + o(n^{-c_\star})) \le Z_{\beta,n} \le 2^n. \quad (3.2)$$

Therefore, if A is any of the sets T_n, T_n°, $T_n \setminus T_n^\circ$, $I_n^\star$ or $T_n^\circ \setminus I_n^\star$ in (2.19)- (2.23),

$$\pi_n(A) = |A|2^{-n}(1 + o(1)) \quad (3.3)$$

whereas for any $x \in \mathcal{V}_n$,

$$\pi_n(x) \le 2^{-n}(1 + o(1)). \quad (3.4)$$

Proof. Since $\{x \in \overline{V}_n^\star\} = \{r_n^\star\tau_n(x) \le 1\}$, $|\mathcal{V}_n \setminus \overline{V}_n^\star| \le Z_{\beta,n} \le |\mathcal{V}_n \setminus \overline{V}_n^\star| + |\overline{V}_n^\star| \le 2^n$. Eq. (3.2) then follows from (2.18) of Lemma 2.1. Eq. (3.4) is then immediate and (3.3) follows from the fact that $A \cap \overline{V}_n^\star = \emptyset$ for each of the mentioned sets. □

3.1. **Spectral gap and mixing condition.** Denote by $\widetilde{L}_n$ the Markov generator matrix of Y_n (that is, the matrix with off-diagonal entries $\widetilde{\lambda}_n(x,y)$ and diagonal entries $-\widetilde{\lambda}_n(x)$), and by $0 = \vartheta_{n,0} < \vartheta_{n,1} \leq \cdots \leq \vartheta_{n,2^n-1}$ the eigenvalues of $-\widetilde{L}_n$.

Proposition 3.2. *If $c_\star > 1 + \log 4$ then for all $\beta > 0$, there exists a subset $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that, on Ω_1, for all but a finite number of indices n,*

$$1/\vartheta_{n,1} \leq \tfrac{5}{2} n^2 r_n^\star (1 + o(1)) \equiv \tilde{\kappa}_n \tag{3.5}$$

As a direct consequence on Proposition 3.2, Condition (B0) of Theorem 1.3 is satisfied $\mathbb{P}$-almost surely with e.g.

$$\kappa_n \equiv \lfloor n^4 r_n^\star (1 + o(1)) \rfloor. \tag{3.6}$$

Proposition 3.3. *Under the assumptions of Proposition 3.2, on Ω_1, for all but a finite number of indices n, for all $\beta > 0$, allpairs $x, y \in \mathcal{V}_n$, and all $t \geq 0$,*

$$|P_x(Y_n(t + \kappa_n) = y) - \pi_n(y)| \leq \rho_n \pi_n(y), \tag{3.7}$$

where κ_n is given by (3.6) and $\rho_n < e^{-n}$.

Proof of Proposition 3.2. The proof of (3.5) relies on a well known Poincaré inequality, taken from [18] (see Proposition 1' p. 38), applied to the stochastic matrix $\widetilde{P}_n = I + \nu_n^{-1} \widetilde{L}_n$ where I denotes the identity matrix and ν_n is defined in Lemma 2.4. By Lemma 2.4, on Ω_0, for all n large enough,

$$\max_{(x,y) \in \mathcal{E}_n} \widetilde{\lambda}_n(x,y) \leq \nu_n < \infty. \tag{3.8}$$

Thus, on Ω_0, for large enough n, the entries $\widetilde{p}_n(x,y)$ of $\widetilde{P}_n$ obey $0 \leq \widetilde{p}_n(x,y) \leq 1$ and $\sum_{y \in \mathcal{V}_n} \widetilde{p}_n(x,y) = 1$. The Poincaré inequality of interest now reads as follows. For each pair of distinct vertices $x, y \in \mathcal{V}_n$, choose a path $\gamma_{x,y}$ going from x to y in the graph $G(\mathcal{V}_n)$. Paths may have repeated vertices but a given edge appears at most once in a given path. Let Γ_n denote such a collection of paths (one for each pair $\{x, y\}$). Then

$$1/\vartheta_{n,1} \leq \nu_n^{-1} \max_e \rho_n^{-1}(e) \sum_{\gamma_{x,y} \ni e} |\gamma_{x,y}| \, \pi_n(x) \pi_n(y), \tag{3.9}$$

where the max is over all edges $e = \{x', y'\}$ of $G(\mathcal{V}_n)$, $\rho_n(e) \equiv \pi_{n,l}(x') \widetilde{p}_n(x', y')$, and the summation is over all paths $\gamma_{x,y}$ in Γ_n that pass through e.

The quality of the bound (3.9) now depends on making a judicious choice of the set of paths Γ_n. We adopt the following clever choice made in Ref. [21]. Given $i \in \{1, \ldots n\}$ and given two vertices x and $x' \in \mathcal{V}_n$ such that $x_i \neq x'_i$, let $\gamma^i_{x,x'}$ be the path obtained by going left to right cyclically from x to x', successively flipping the disagreeing coordinates, starting from the i-th coordinate. Set $\Gamma^i_n = \{\gamma^i_{x,x'}, x, x' \in \mathcal{V}_n\}$, $1 \leq i \leq n$. These paths are ordered in an obvious way. Given x, x' and $\gamma_{x,x'}$, let $\overline{\gamma}_{x,x'}$ be the set of vertices visited by the path $\gamma_{x,x'}$, and let $\gamma^{int}_{x,x'} = \overline{\gamma}_{x,x'} \setminus \{x, x'\}$ be the subset of "interior" vertices. We next split the set of vertices $\mathcal{V}_n$ into *good* ones and *bad* ones. Recalling (2.7), we say that a vertex is good if it does not belong to $\overline{V}^\star_n$; otherwise it is bad. We say that a path γ is good if all its interior points γ^{int} are good, and that a set of paths is good if all its elements are good.

The (random) set of path Γ_n is then constructed as follows:
(i) Consider pairs x and x' such that $\mathrm{dist}(x, x') \geq n/\log n$. If $\{\gamma^i_{x,x'}, 1 \leq i \leq n\}$ contains a good path, choose the first such for Γ_n; otherwise choose $\gamma^1_{x,x'}$.

(ii) Consider pairs x and x' such that $\mathrm{dist}(x,x') < n/\log n$. If there is a good vertex $x'' \in \mathcal{V}_n$ such that $\mathrm{dist}(x,x'') \geq n/\log n$ and $\mathrm{dist}(x'',x') \geq n/\log n$, and if there are good paths, one in $\{\gamma^i_{x,x''}, 1 \leq i \leq n\}$ and one in $\{\gamma^i_{x'',x'}, 1 \leq i \leq n\}$, such that the union of these two good paths is a self avoiding path of length less than n, select this union as the path connecting x to x' in Γ_n (notice that this is a good path); otherwise choose $\gamma^1_{x,x'}$.

It turns out that this Γ_n is almost surely good. More precisely, set $\Omega_n^{\text{GOOD}} = \{\Gamma'_n \text{ is good}\}$, $n \geq 1$, and $\Omega^{\text{GOOD}} = \liminf_{n\to\infty} \Omega_n^{\text{GOOD}}$.

Proposition 3.4 (Proposition 4.1 of [21]). *If $c_\star > 1 + \log 4$ then $\mathbb{P}\big(\Omega^{\text{GOOD}}\big) = 1$.*

From now on we assume that $\omega \in \Omega^{\text{GOOD}}$ so that, for all large enough n, Γ_n is good. Note that the paths of Γ_n have length smaller than n. Hence (3.9) yields

$$\begin{aligned} 1/\vartheta_{n,1} &\leq n \max_{e=\{x',y'\}} \big(\pi_n(x')\widetilde{\lambda}_n(x',y')\big)^{-1} \sum_{\gamma_{x,y}\ni e} \pi_n(x)\pi_n(y) \\ &= \max_{e=\{x',y'\}} \frac{n^2}{\min(\tau_n(y'),\tau_n(x'))} \sum_{\gamma_{x,y}\ni e} \frac{\min\left(1, r_n^\star\tau_n(x)\right)\min\left(1, r_n^\star\tau_n(y)\right)}{Z_{\beta,n}} \end{aligned} \tag{3.10}$$

where the final equality follows from (1.39), (1.35) (with $\eta_n \equiv (r_n^\star)^{-1}$), and (3.1). Also note that since bad vertices (i.e. vertices of $\overline{V}_n^\star$) can appear only at the ends of any path, the paths of Γ_n do not contain any edge of the graph $G_n(\overline{V}_n^\star)$. This prompts us to write $1/\vartheta_{n,1} \leq \max\{\mathcal{K}_{1,n}, \mathcal{K}_{2,n}, \mathcal{K}_{3,n}\}$ where $\mathcal{K}_{1,n}$, $\mathcal{K}_{2,n}$, and $\mathcal{K}_{3,n}$ are obtained, respectively, by restricting the maximum in (3.10) to the maximum over edges $e = \{x',y'\}$ with $x' \in \overline{V}_n^\star$ and $y' \notin \overline{V}_n^\star$, $x' \notin \overline{V}_n^\star$ and $y' \in \overline{V}_n^\star$, and $x' \notin \overline{V}_n^\star$ and $y' \notin \overline{V}_n^\star$. To bound $\mathcal{K}_{1,n}$ note that the sum over paths that contain $e = \{x',y'\}$ reduces to the sum over all paths starting in x' that contain e, so that

$$\mathcal{K}_{1,n} = \max_{e=\{x',y'\}: x'\in\overline{V}_n^\star, y'\notin\overline{V}_n^\star} \frac{n^2 \min\big(1, r_n^\star\tau_n(x')\big)}{\min(\tau_n(y'),\tau_n(x'))} \sum_{\gamma_{x',y}\ni e} \pi_n(y) \leq n^2 r_n. \tag{3.11}$$

By symmetry of the bound (3.10), $\mathcal{K}_{2,n} \leq n^2 r_n$. Finally, $\min(\tau_n(y'),\tau_n(x')) \geq 1/r_n^\star$ for all $x', y' \notin \overline{V}_n^\star$ and $\min\left(1, r_n^\star\tau_n(x)\right)\min\left(1, r_n^\star\tau_n(y)\right) \leq 1$ for all $x, y \in \mathcal{V}_n$. Thus

$$\mathcal{K}_{3,n} \leq n^2 r_n^\star Z_{\beta,n}^{-1} \max_{e\in G(\mathcal{V}_n)} |\{\gamma \in \Gamma_n \mid e \in \gamma\}| \leq n^2 r_n^\star Z_{\beta,n}^{-1}(2^{n-1} + 2^{2n/\log n}), \tag{3.12}$$

where we used that the number of paths connecting vertices at distance $n/\log n$ or more apart is at most 2^{n-1} (see e.g. Example 2.2, p. 45 in Ref. [18] for this well known bound) whereas, arguing as in Ref. [21] (see Section 4.2.2, page 934), the number of paths connecting vertices less than $n/\log n$ apart and containing e is bounded above by the volume of a hypercube of dimension at most $n/\log n$ around e, and so, is smaller than $2^{2n/\log n}$. In view of Lemma 3.1 we have that on $\Omega^\star \cap \Omega^{\text{GOOD}}$, for all but a finite number of indices n,

$$\mathcal{K}_{3,n} \leq \tfrac{1}{2} n^2 r_n^\star (1 + o(1)). \tag{3.13}$$

Collecting our bounds and taking $\Omega_1 = \Omega_0 \cap \Omega^\star \cap \Omega^{\text{GOOD}}$ yields (3.5) and ends the proof. □

Proof of Proposition 3.3. It is well know that for reversible irreducible Markov processes, bounds on spectral gaps yield bounds on their total variation distance $\|\cdot\|_{\text{var}}$ to stationarity. For instance, Proposition 3 of Ref. [18] applied to Y_n states that for all $x \in \mathcal{V}_n$ and all $t > 0$,

$$4\left\|P_x\left(Y_n(t) = \cdot\right) - \pi_n(\cdot)\right\|_{\text{var}}^2 \leq \tfrac{1-\pi_n(x)}{\pi_n(x)} e^{-2t\vartheta_{n,1}}. \tag{3.14}$$

By (3.1), Lemma 3.1, and (2.31) of Lemma 2.4, on $\Omega_0 \cap \Omega^\star$, for all but a finite number of indices n, $\sup_{z\in\mathcal{V}_n} \pi_n^{-1}(z) \le (2^n/r_n^\star)e^{\beta n\sqrt{2\log 2}(1+2\log n/n)}$. The claim of Proposition 3.3 now readily follows from this, (3.14), and Proposition 3.2, choosing κ_n as in (3.6). □

3.2. **Hitting time for the stationary chain.** Drawing heavily on Aldous and Brown's work [2], this section collects results on hitting times for the process Y_n at stationarity. Let

$$H(A) = \inf\{t \ge 0 \mid Y_n(t) \in A\} \tag{3.15}$$

be the hitting time of $A \subseteq \mathcal{V}_n$. We begin with bounds on the mean value of $H(A)$.

Lemma 3.5. *On Ω_1, for all but a finite number of indices n, for all $A \subseteq \mathcal{V}_n$,*

$$\frac{(1-n\pi_n(A))^2}{r_n^\star n\pi_n(A)(1-\pi_n(A))} \le \frac{E_{\pi_n}H(A)}{1-\pi_n(A)} \le \frac{\tilde\kappa_n}{\pi_n(A)}. \tag{3.16}$$

(If $\mathrm{dist}(V_n^\star, A) > 1$, $n\pi_n(A)$ can be replaced by $\pi_n(A)$ in the right-hand side).

The next lemma gives bounds on the density function $h_{n,A}(t)$, $t > 0$, of $H(A)$ when Y_n starts in its invariant measure, π_n.

Lemma 3.6. *On Ω_1, for all but a finite number of indices n, for all $A \subseteq \mathcal{V}_n$ and all $t > 0$,*

$$\frac{1}{E_{\pi_n}H(A)}\left(1-\frac{\tilde\kappa_n}{E_{\pi_n}H(A)}\right)^2\left(1-\frac{t}{E_{\pi_n}H(A)}\right) \le h_{n,A}(t) \le \frac{1}{E_{\pi_n}H(A)}\left(1+\frac{\tilde\kappa_n}{2t}\right).$$

The bounds of Lemma 3.6 imply that $h_{n,A}(t) \approx \frac{1}{E_{\pi_n}H(A)}$ when $\tilde\kappa_n \ll t \ll E_{\pi_n}H(A)$. Complementing this, Lemma 3.7 is well suited to dealing with "small" values of t.

Lemma 3.7. *On $\Omega^\star$, for all but a finite number of indices n, for all $A \subseteq \mathcal{V}_n$ and all $t > 0$,*

$$P_{\pi_n}(H(A) > t) \ge (1-n\pi_n(A))\exp\left(-t\frac{r_n^\star n\pi_n(A)}{1-n\pi_n(A)}\right). \tag{3.17}$$

In particular, for any A and any sequence t_n such that $t_n r_n^\star n\pi_n(A) \to 0$ as $n \to \infty$,

$$P_{\pi_n}(H(A) \le t_n) < t_n r_n^\star n\pi_n(A)(1+o(1)). \tag{3.18}$$

If $A \subset \mathcal{V}_n \setminus V_n^\star$ the factor n in front of $\pi_n(A)$ in (3.17) and (3.18) can be suppressed.

The next Corollary is stated for later convenience.

Corollary 3.8. *Under the assumptions of Lemma 3.7 the following holds: For all $0 < \varepsilon < 1$, for any sequence t_n such that $t_n r_n^\star n 2^{-n\varepsilon_n} \to 0$ as $n \to \infty$*

$$P_{\pi_n}(H(T_n \setminus T_n^\circ) \le t_n) \le t_n r_n^\star n^5 2^{-2n\varepsilon_n}(1+o(1)), \tag{3.19}$$

$$P_{\pi_n}(H(T_n^\circ) \le t_n) \le t_n r_n^\star n 2^{-n\varepsilon_n}(1+o(1)). \tag{3.20}$$

We now prove these results, beginning with Lemma 3.7.

Proof of Lemma 3.7. Write $A = B \cup B^c$ where $B = A \cap V_n^\star$ and $B^c = A \setminus B$. Let $B^\star$ be the immersion of B in $V_n^\star$ (see (2.10)). Since $A \subseteq B^\star \cup B^c$, $H(A) \ge H(B^\star \cup B^c)$, and

$$P_{\pi_n}(H(A) > t) \ge P_{\pi_n}(H(B^\star \cup B^c) > t). \tag{3.21}$$

To bound the right-hand side of (3.21), we use a well know lower bound on hitting times for stationary reversible chains taken from Ref. [2] (combine Theorem 3 and Lemma 2 therein) that states that for all $C \subseteq \mathcal{V}_n$ and all $t > 0$,

$$P_{\pi_n}(H(C) > t) \ge (1-\pi_n(C))\exp\left(-t\frac{q_n(C,C^c)}{1-\pi_n(C)}\right) \tag{3.22}$$

where, for for any two sets C and $\widetilde{C}$ such that $C \cap \widetilde{C} = \emptyset$,

$$q_n(C, \widetilde{C}) \equiv \sum_{x \in C} \sum_{y \in \widetilde{C}} \pi_n(x) \widetilde{\lambda}_n(x, y). \tag{3.23}$$

Let us thus evaluate (3.23) with $C = B^\star \cup B^c$. Clearly $q_n(B^\star \cup B^c, (B^\star \cup B^c)^c) \le q_n(B^\star, (B^\star \cup B^c)^c) + q_n(B^c, (B^\star \cup B^c)^c)$. Clearly also, by (2.8), $\widetilde{\lambda}_n(x, y) \le n^{-1} r_n^\star$ for any $x \in B^c$ and any $y \sim x$. Thus $q_n(B^c, (B^\star \cup B^c)^c) \le r_n^\star \pi_n(B^c)$. Next, by (2.11), $q_n(B^\star, (B^\star \cup B^c)^c) \le r_n^\star \pi_n(B^\star)$. Thus

$$q_n(B^\star \cup B^c, (B^\star \cup B^c)^c) \le r_n^\star [\pi_n(B^\star) + \pi_n(B^c)]. \tag{3.24}$$

Denoting by $C^\star_{n,l(x)}$ the (unique) component of $B^\star$ (see (2.10)) that contains x, we have $|B^\star| \le |\cup_{x \in B} C^\star_{n,l(x)}| \le |B| \max_{x \in B} |C^\star_{n,l(x)}|$ where by (2.17), on $\Omega^\star$, $|C^\star_{n,l(x)}| \ll n$. By this and (3.1) we get $\pi_n(B^\star) = Z_{\beta,n}^{-1} |B^\star| \le n Z_{\beta,n}^{-1} |B| = n \pi_n(B)$. Therefore,

$$\pi_n(B^\star \cup B^c) \le \pi_n(B^\star) + \pi_n(B^c) \le n\pi_n(B) + \pi_n(B^c) \le n\pi_n(B \cup B^c) = n\pi_n(A). \tag{3.25}$$

Using (3.25) in the right-and side of (3.24) and plugging the result in (3.22) finally yields (3.17). Clearly, if $A \subset \mathcal{V}_n \setminus V_n^\star$ then $B = \emptyset$ and the right-and side of (3.24) reduces to $r_n^\star [\pi_n(\emptyset) + \pi_n(B^c)] = r_n^\star \pi_n(A)$. □

Proof of Corollary 3.8. This follows from (3.3) of Lemma 3.1, (2.20), and (2.21). □

Proof of Lemma 3.6. Proceed as in Lemma 13 of Ref. [2] and use Proposition 3.2. □

Proof of Lemma 3.5. The rightmost inequality is that of Lemma 2 of Ref. [2] combined with Proposition 3.2. Lemma 2 of Ref. [2] also states that for $C \subseteq \mathcal{V}_n$ and $q_n(C, C^c)$ defined as in (3.23),

$$\frac{E_{\pi_n} H(C)}{1 - \pi_n(C)} \ge \frac{1 - \pi_n(C)}{q_n(C, C^c)}. \tag{3.26}$$

Given $A \subseteq \mathcal{V}_n$ let $B^\star$ and B^c be defined as in the first line of the proof of Lemma 3.7. Since $H(A) \ge H(B^\star \cup B^c)$, $E_{\pi_n} H(A) \ge E_{\pi_n} H(B^\star \cup B^c)$. Using (3.26) with $C = B^\star \cup B^c$, (3.16) follows from (3.24) and the bound on $\pi_n(B^\star \cup B^c)$ of (3.25). □

3.3. **On hitting the top starting in the top.** Let T_n° and $I_n^\star$ be as in (2.14) and (2.16).

Proposition 3.9. *Given $\epsilon > 0$ there exists a subset $\Omega^\circ \subset \Omega$ with $\mathbb{P}(\Omega^\circ) = 1$ such that on Ω°, for all but a finite number of indices n, for all $s > 0$*

$$|T_n^\circ|^{-1} \sum_{x \in T_n^\circ} P_x \left(H(T_n^\circ \setminus x) \le s \right) \le s n^{c_\star + 3} r_n^\star \pi_n(T_n^\circ). \tag{3.27}$$

The next proposition is a variant of Proposition 3.9 that we state for later convenience.

Proposition 3.10. *Under the assumptions and with the notation of Proposition 3.9, on Ω°, for all but a finite number of indices n, for all $s > 0$*

$$|T_n^\circ \setminus I_n^\star|^{-1} \sum_{x \in T_n^\circ \setminus I_n^\star} P_x \left(H(I_n^\star) \le s \right) \le s n^2 r_n^\star \pi_n(I_n^\star)(1 + o(1)). \tag{3.28}$$

Proof of Proposition 3.9. A key ingredient of the proof is an explicit expression of the density function $h^x_{n,A}(t)$, $t \ge 0$, of the hitting time $H(A)$ when Y_n starts in $x \in A^c \equiv \mathcal{V}_n \setminus A$. We first state this expression in full generality as given in [28] (see Section 6.2, p. 83). Consider the stochastic matrix $\widetilde{P}_n = (\widetilde{p}_n(x, y))$ defined above (3.8). Denote by $Q_n = (q_n(x, y))$ the matrix with entries $q_n(x, y) : A^c \times A^c \to \mathbb{R}$ given by $q_n(x, y) =$

$\widetilde{p}_n(x,y)$. This is the sub-matrix of $\widetilde{P}_n$ on $A^c \times A^c$. Thus Q_n is sub-stochastic. Similarly, denote by $R_n = (r_n(x,y))$ the sub-matrix of $\widetilde{P}_n$ on $A^c \times A$. Let 1_A be the vector of 1's on A and let δ_x be the vector on A^c taking value 1 at x and zero else. Then, for all $x \in A^c$,

$$h^x_{n,A}(t) = \nu_n \sum_{k=0}^{\infty} \frac{(\nu_n t)^k}{k!} e^{-\nu_n t} \left(\delta_x, Q_n^k R_n 1_A\right), \quad t \geq 0, \tag{3.29}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^{|A^c|}$. Consequently, for all $s > 0$,

$$P_x\left(H(A) \leq s\right) = \int_0^s \nu_n \sum_{k=0}^{\infty} \frac{(\nu_n t)^k}{k!} e^{-\nu_n t} \left(\delta_x, Q_n^k R_n 1_A\right) dt. \tag{3.30}$$

For later reference we also denote by $(h^x_{n,y,A}(t))_{y\in A}$ the vector whose components are, for each $y \in A$, the joint density that A is reached at time t, and that arrival to that set occurs in state y, namely, $h^x_{n,y,A}(t)$ is defined as in (3.29) substituting δ_y for 1_A therein; as a result $h^x_{n,A}(t) = \sum_{y\in A} h^x_{n,y,A}(t)$.

Returning to (3.27), a first order Tchebychev inequality yields, for all $\epsilon > 0$

$$\begin{aligned}\mathbb{P}\left[\textstyle\sum_{x\in T_n^\circ} P_x\left(H(T_n^\circ \setminus x) \leq s\right) \geq \epsilon\right] &\leq \epsilon^{-1}\mathbb{E}\left[\textstyle\sum_{x\in T_n^\circ} P_x\left(H\big(T_n^\star \setminus T^\star_{n,l(x)}\big) \leq s\right)\right] \\ &\equiv \epsilon^{-1}\mathcal{W}_n,\end{aligned} \tag{3.31}$$

where $T_n^\star \equiv \cup_{l=1}^{L^\star} T^\star_{n,l}$ is defined in (2.13) and $1 \leq l(x) \leq L^\star$ denotes the (unique) index such that $T^\star_{n,l(x)} \cap T_n^\circ = \{x\}$ in (2.14). By (3.30) with $A = T_n^\star \setminus T^\star_{n,l(x)}$,

$$\mathcal{W}_n = \sum_{x\in\mathcal{V}_n} \int_0^s dt \sum_{k=1}^{\infty} \frac{(\nu_n t)^k}{k!} e^{-\nu_n t} \mathcal{W}_{n,k}(x) \tag{3.32}$$

where

$$\mathcal{W}_{n,k}(x) \equiv \mathbb{E}\left[1_{\{x\in T_n^\circ\}} \nu_n \left(\delta_x, Q_n^k R_n 1_{T_n^\star \setminus T^\star_{n,l(x)}}\right)\right]. \tag{3.33}$$

Note that the term $k = 0$ is zero. For $k \geq 1$ the matrix term in (3.33) reads,

$$1_{\{x\in T_n^\circ\}} \nu_n \left(\delta_x, Q_n^k R_n 1_{T_n^\star \setminus T^\star_{n,l(x)}}\right) = 1_{\{x\in T_n^\circ\}} \sum_{y\in (T_n^\star \setminus T^\star_{n,l(x)})^c} q_n^{(k)}(x,y) \sum_{z\in T_n^\star \setminus T^\star_{n,l(x)}} \nu_n r_n(y,z) \tag{3.34}$$

where $q_n^{(k)}(x,y)$ denotes the entries of Q_n^k. By (2.11), for all $y \in (T_n^\star \setminus T^\star_{n,l(x)})^c$,

$$\sum_{z\in T_n^\star \setminus T^\star_{n,l(x)}} \nu_n r_n(y,z) = \sum_{z\in T_n^\star \setminus T^\star_{n,l(x)}} \widetilde{\lambda}_n(y,z) \leq n^{-1} r_n^\star \sum_{z\in T_n^\star \setminus T^\star_{n,l(x)}} 1_{\{z\sim y\}}. \tag{3.35}$$

Therefore, inserting (3.35) in (3.34), (3.33) yields

$$\mathcal{W}_{n,k}(x) \leq \frac{r_n^\star}{n} \mathbb{E}\left[\mathbb{E}\left[1_{\{x\in T_n^\circ\}} \sum_{y\in (T_n^\star \setminus T^\star_{n,l(x)})^c} q_n^{(k)}(x,y) \sum_{z\in T_n^\star \setminus T^\star_{n,l(x)}} 1_{\{z\sim y\}} \,\middle|\, V_n^\star\right]\right] \tag{3.36}$$

where $\mathbb{E}[\cdot \,|\, V_n^\star]$ denotes the conditional expectation given a realization of the set $V_n^\star = \cup_{l=1}^{L^\star} C^\star_{n,l}$ (see (2.5)), namely, expectation with respect to the measure

$$\mathbb{P}(\cdot \,|\, V_n^\star) = \frac{\mathbb{P}(\cdot \cap \{\forall_{1\leq l\leq L^\star} \forall_{x\in C^\star_{n,l}} \tau_n(x) \geq r_n^\star\} \cap \{\forall_{x\in C^\star_{n,0}} \tau_n(x) < r_n^\star\})}{\mathbb{P}(\{\forall_{1\leq l\leq L^\star} \forall_{x\in C^\star_{n,l}} \tau_n(x) \geq r_n^\star\} \cap \{\forall_{x\in C^\star_{n,0}} \tau_n(x) < r_n^\star\})}, \tag{3.37}$$

where we set $C^\star_{n,0} \equiv \mathcal{V}_n \setminus \mathcal{V}_n^\star$ for simplicity. Thus $\mathcal{V}_n = \cup_{0\leq l\leq L^\star} C^\star_{n,l}$ and $C^\star_{n,l} \cap C^\star_{n,l'} = \emptyset$ for all $0 \leq l \neq l' \leq L^\star$, so that if $\mathcal{L}_i \subset \{0,\ldots,L^\star\}$, $i = 1,\ldots,j$, is a collection of disjoint

sets, functions f_i of the variables $\{\tau_n(x), x \in \cup_{l\in\mathcal{L}_i} C^\star_{n,l}\}$, $i = 1,\dots,j$, are independent under the conditional law (3.37). Observe now that conditional on $V^\star_n$ the entries of the matrix Q_n are functions of the variables $\{\tau_n(y), y \in (T^\star_n \setminus T^\star_{n,l(x)})^c\}$ only: for off-diagonal entries, i.e. for $q_n(x,y)$ with $x \neq y$, this is an immediate consequence of (2.8); for diagonal entries, i.e. $q_n(x,x) = 1 - \nu_n^{-1}\widetilde{\lambda}_n(x)$, this claim follows from (1.12) and (2.8) if $x \notin \partial V^\star_n$ and from (2.9) and (2.8) if $x \in \partial V^\star_n$ (the boundary set ∂A of A is defined above (2.9)).

Next, observe that the sum over $y \in (T^\star_n \setminus T^\star_{n,l(x)})^c$ in (3.36) can be restricted to the sum over $y \in \partial V^\star_n \subseteq C^\star_{n,0}$ and use the definition of $T^\star_n$ (see (2.13)) to write

$$\begin{aligned}
&\sum_{y\in(T^\star_n\setminus T^\star_{n,l(x)})^c} q^{(k)}_n(x,y) \sum_{z\in T^\star_n\setminus T^\star_{n,l(x)}} \mathbb{1}_{\{z\sim y\}} \\
&= \sum_{x_1\in\mathcal{V}_n}\cdots\sum_{x_{k-1}\in\mathcal{V}_n}\sum_{y\in\partial V^\star_n}\sum_{z\sim y}\sum_{\substack{0\le l_1\le L^\star\\ C^\star_{n,l_1}\cap x_1\neq\emptyset}}\cdots\sum_{\substack{0\le l_{k-1}\le L^\star\\ C^\star_{n,l_{k-1}}\cap x_{k-1}\neq\emptyset}}\sum_{\substack{1\le l\neq l(x)\le L^\star\\ C^\star_{n,l}\cap z\neq\emptyset}} \qquad (3.38)\\
&\quad q_n(x,x_1)\dots q_n(x_{k-1},y)\mathbb{1}_{\{\forall_{x'_1\in C^\star_{n,l_1}\setminus\{x\}}\tau_n(x'_1)<r_n(\varepsilon_n)\}}\cdots\\
&\quad \dots\mathbb{1}_{\{\forall_{x'_{k-1}\in C^\star_{n,l_{k-1}}\setminus\{x\}}\tau_n(x'_{k-1})<r_n(\varepsilon_n)\}}\mathbb{1}_{\{\exists_{z'\in C^\star_{n,l}}\tau_n(z')\ge r_n(\varepsilon_n)\}}.
\end{aligned}$$

Since $\mathbb{1}_{\{\forall_{z'\in C^\star_{n,l}\setminus\{x\}}\tau_n(z')<r_n(\varepsilon_n)\}}\mathbb{1}_{\{\exists_{z'\in C^\star_{n,l}}\tau_n(z')\ge r_n(\varepsilon_n)\}} = 0$ for all $l \neq l(x)$, the sums over l in (3.38) can be restricted to $1 \le l \neq l(x), l_1,\dots,l_{k-1} \le L^\star$. We may now multiply (3.38) by $\mathbb{1}_{\{x\in T^\circ_n\}}$ and take the conditional expectation. The variables $\{\tau_n(z'), z' \in C^\star_{n,l}\}$ being independent of the variables $\{\tau_n(x'), x' \in \cup_{0\le l'\neq l\le L^\star} C^\star_{n,l'}\}$, they can be integrated out first, yielding, for all $y \in \partial V^\star_n$

$$\sum_{z\sim y}\sum_{\substack{1\le l\neq l(x),l_1,\dots,l_{k-1}\le L^\star\\ C^\star_{n,l}\cap z\neq\emptyset}} \mathbb{P}\left[\exists_{z'\in C^\star_{n,l}}\tau_n(z') \ge r_n(\varepsilon_n)\,\middle|\, V^\star_n\right] \qquad (3.39)$$

$$\le n \max_{1\le l\neq l(x),l_1,\dots,l_{k-1}\le L^\star} |C^\star_{n,l}| 2^{-(\varepsilon_n-\rho^\star_n)n} \qquad (3.40)$$

$$\le n^2 2^{-(\varepsilon_n-\rho^\star_n)n}, \qquad (3.41)$$

where we used in (3.40) that the sum over l contains at most one term while the sum over z contains at most n terms. Eq. (3.41) then follows from (2.17) and so, is valid on $\Omega^\star$ for all large enough n. This bound is uniform in $y \in \partial V^\star_n$. Therefore, using (3.41) in (3.38) and re-summing, (3.36) becomes

$$\mathcal{W}_{n,k}(x) \le \frac{r^\star_n}{n} n^2 2^{-(\varepsilon_n-\rho^\star_n)n}\mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{x\in T^\circ_n\}}\sum_{y\in\partial V^\star_n} q^{(k)}_n(x,y)\,\middle|\, V^\star_n\right]\right] \qquad (3.42)$$

$$\le \frac{r^\star_n}{n} n^2 2^{-(\varepsilon_n-\rho^\star_n)n}\mathbb{P}(x \in T^\circ_n) \qquad (3.43)$$

where we used in (3.43) that since Q_n is sub-stochastic, $\sum_{y\in\partial V^\star_n} q^{(k)}_n(x,y) \le 1$ for all x. Now, by (2.14) and (2.2), $\mathbb{P}(x \in T^\circ_n) \le \mathbb{P}(\tau_n(x) \ge r_n(\varepsilon_n)) = 2^{-\varepsilon_n n}$. Thus

$$\mathcal{W}_{n,k}(x) \le r^\star_n n 2^{-2\varepsilon_n n} 2^{\rho^\star_n n} = n^{c_\star+1} r^\star_n 2^{-2\varepsilon_n n}. \qquad (3.44)$$

The last equality is (2.4). Using this bound in (3.32) finally yields that on $\Omega^\star$, for all large enough n,

$$\mathcal{W}_n = \sum_{x\in\mathcal{V}_n} \int_0^{\theta_n} dt \sum_{k=1}^{\infty} \frac{(\nu_n t)^k}{k!} e^{-\nu_n t} S_{n,k}(x) \le \theta_n n^{c_\star+1} r_n^\star 2^n 2^{-2\varepsilon_n n}. \tag{3.45}$$

It only remains to observe that by (2.20) and (3.3) of Lemma 3.1, on $\Omega^\star$, $\pi_n(T_n^\circ) = 2^{-n\varepsilon_n}(1+o(1))$ for all but a finite number of indices n. Hence

$$\mathbb{P}\left[|T_n^\circ|^{-1}\textstyle\sum_{x\in T_n^\circ} P_x\left(H(T_n^\star \setminus T_{n,l(x)}^\star) \le s\right) \ge \epsilon\right] \le \epsilon^{-1} s n^{c_\star+1} r_n^\star \pi_n(T_n^\circ)(1+o(1)).$$

Choosing $\epsilon = n^2 n^{c_\star+1} r_n^\star \pi_n(T_n^\circ)$, the claim of the proposition follows from Borel-Cantelli Lemma. □

Proof of Proposition 3.10. This is a rerun of the proof of Proposition 3.9. □

3.4. **Rough bounds on local times.**

Lemma 3.11. *For all $0 \le \alpha \le 1$, all $x \in \mathcal{V}_n$, and all $s > 0$,*

$$E_x\left[\ell_n^x(s)\right]^\alpha \ge (\widetilde{\lambda}_n^{-1}(x))^\alpha \Gamma(1+\alpha)[1 - c_1\exp(-c_2 s\widetilde{\lambda}_n(x))] + s^\alpha \exp(-s\widetilde{\lambda}_n(x)) \tag{3.46}$$

where $0 < c_1, c_2 < \infty$ are constants, and if moreover $s r_n^\star n\pi_n(x) \to 0$ as $n \to \infty$,

$$E_x\left[\ell_n^x(s)\right]^\alpha \le (1+o(1))\left[\kappa_n^\alpha + \mathbb{1}_{\{s>\kappa_n\}} s^\alpha (s-\kappa_n) r_n^\star n\pi_n(x)\right]. \tag{3.47}$$

Proof of Lemma 3.6. The lower bound follows from the trite observation that $\ell_n^x(s)$ is at least as large as the minimum between the first jump of Y_n and s, that is,

$$\ell_n^x(s) \ge \widetilde{\lambda}_n^{-1}(x) e_1 \mathbb{1}_{s>\widetilde{\lambda}_n^{-1}(x)e_1} + s\mathbb{1}_{s\le\widetilde{\lambda}_n^{-1}(x)e_1}, \tag{3.48}$$

where e_1 is an exponential random variable of mean one. Thus

$$E_x\left[\ell_n^x(s)\right]^\alpha \ge E_x\left[\widetilde{\lambda}_n^{-1}(x) e_1 \mathbb{1}_{s>\widetilde{\lambda}_n^{-1}(x)e_1}\right]^\alpha + s^\alpha E_x\left[\mathbb{1}_{s\le\widetilde{\lambda}_n^{-1}(x)e_1}\right]^\alpha. \tag{3.49}$$

Explicit calculations yield

$$E_x\left[\widetilde{\lambda}_n^{-1}(x) e_1 \mathbb{1}_{s>\widetilde{\lambda}_n^{-1}(x)e_1}\right]^\alpha \ge (\widetilde{\lambda}_n^{-1}(x))^\alpha \Gamma(1+\alpha)[1 - c_1\exp(-c_2 s\widetilde{\lambda}_n(x))] \tag{3.50}$$

for some constants $0 < c_1, c_2 < \infty$. Eq. (3.46) now readily follows. To get an upper bound write $E_x\left[\ell_n^x(s)\right]^\alpha \le \kappa_n^\alpha$ if $s \le \kappa_n$. Otherwise write

$$\begin{aligned} E_x\left[\ell_n^x(s)\right]^\alpha &\le E_x\left[\kappa_n + \textstyle\int_{\kappa_n}^s \mathbb{1}_{\{Y_n(s)=x\}} ds\right]^\alpha &(3.51)\\ &\le (1+\rho_n) E_{\pi_n}\left[\kappa_n + \textstyle\int_0^{s-\kappa_n} \mathbb{1}_{\{Y_n(s)=x\}} ds\right]^\alpha &(3.52)\end{aligned}$$

where the last line follows from Proposition 3.3 and the Markov property. Next,

$$\begin{aligned} E_{\pi_n}\left[\kappa_n + \textstyle\int_0^{s-\kappa_n} \mathbb{1}_{\{Y_n(s)=x\}} ds\right]^\alpha &\le E_{\pi_n}\left(\kappa_n^\alpha \mathbb{1}_{\{H(x)>s-\kappa_n\}} + s^\alpha \mathbb{1}_{\{H(x)\le s-\kappa_n\}}\right)\\ &\le \kappa_n^\alpha + s^\alpha P_{\pi_n}(H(x) \le s-\kappa_n)\\ &\le \kappa_n^\alpha + s^\alpha (s-\kappa_n) r_n^\star n\pi_n(x)(1+o(1)),\end{aligned} \tag{3.53}$$

the last inequality being (3.18) of Lemma 3.7. Eq. (3.47) is proved. □

4. VERIFICATION OF CONDITION (B1)

In this section we prove a strong law of large number for the function $\nu_n^t(u,\infty)$ defined in (1.31). Recall that for $r_n^\star$ defined in (1.37), we take $\eta_n \equiv (r_n^\star)^{-1}$ in (1.34), (1.35), and (1.36). Then by (1.18)-(1.19), (1.22), and (1.34),

$$\nu_n^t(u,\infty) = k_n(t) P_{\pi_n}\left(\int_0^{\theta_n} \max\left((c_n r_n^\star)^{-1}, c_n^{-1}\tau_n(Y_n(s)) \right) ds > u \right) \tag{4.1}$$

where π_n is the invariant measure (1.35) of Y_n, θ_n is the block length of the blocked clock process (1.18), $k_n(t) = \lfloor a_n t/\theta_n \rfloor$, and, given $0<\varepsilon<1$, c_n and a_n are defined in (1.4) and (1.40)-(1.44), respectively. By Theorem 1.3, θ_n and a_n must obey

$$\lfloor n^4 r_n^\star (1+o(1)) \rfloor \equiv \kappa_n \le \theta_n \ll a_n, \tag{4.2}$$

where the left-most equality is (3.6). Further recall from Section 2 that for $\rho_n^\star$ as in (2.4),

$$\rho_n^\star \ll \varepsilon_n \equiv \varepsilon - \delta_n. \tag{4.3}$$

(Recall that $0 < x_n \ll y_n$ means that $x_n/y_n \to 0$ as $n \to \infty$.) From now on we take δ_n such that $2^{n\delta_n} = (n^2\theta_n)^{\alpha(\varepsilon)}$, i.e.

$$\delta_n \equiv \frac{1}{n\beta}\sqrt{\frac{2\varepsilon}{\log 2}}\log\left(n^2\theta_n\right). \tag{4.4}$$

Thus, given $0<\varepsilon<1$ and $\beta>0$, all sequences except θ_n are determined.

Proposition 4.1. *Given $0<\varepsilon<1$ and $\beta>0$ let the sequences c_n and a_n be defined as in (1.4) and (1.40)-(1.44), respectively, and let θ_n be such that*

$$(r_n^\star)^4 \ll \theta_n^{1-\alpha(\varepsilon)}, \tag{4.5}$$

$$n^{-1}\log\theta_n \ll 1. \tag{4.6}$$

Then, for all $0<\varepsilon<1$ and $\beta>0$, $\mathbb{P}$-almost surely,

$$\lim_{n\to\infty} \nu_n^t(u,\infty) = t u^{\alpha(\varepsilon)}, \quad \forall\, t>0, u>0. \tag{4.7}$$

Remark. Eq. (4.6) implies that $\delta_n \ll 1$ and that $\theta_n \ll c_n$ for all $\varepsilon>0$. In view of (1.38), (3.5), (4.4) and (3.6), (4.6) also implies that

$$c_0 n^{c_1}\tilde\kappa_n^{c_2}\kappa_n^{c_3}(r_n^\star)^{c_4}\theta_n^{c_5} \ll 2^{\varepsilon n} \quad\text{and}\quad c_0 n^{c_1}\tilde\kappa_n^{c_2}\kappa_n^{c_3}(r_n^\star)^{c_4}\theta_n^{c_5} \ll 2^{\varepsilon_n n} \tag{4.8}$$

for all $\varepsilon>0$ and any choice of constants $0 \le c_i < \infty$.

Remark. In order to guarantee strict equivalence of the definitions (1.43) and (2.12) of the set T_n when δ_n is given by (4.4), we should replace the term $c_n(n\theta_n)^{-1}$ in (1.43) by

$$c_n \exp\left\{ -\log(n^2\theta_n)\left[1 + (1+o(1))(2n\beta\beta_c(\varepsilon))^{-1}\log(n^2\theta_n)\right]\right\} \tag{4.9}$$

(see Corollary 2.3). We didn't state this precise formula to keep the presentation simple.

The rest of the section is organized as follows. In Section 4.1 we show that $\nu_n^t(u,\infty)$ can be reduced to the quantity $\nu_n^{\circ,t}(u,\infty)$ defined in (4.32). In Section 4.2 we prove upper et lower bounds on a sequence, b_n°, defined as b_n with T_n° substituted for T_n, and show that b_n and b_n° behave in the same way to leading order. In Section 4.3 we show that $\nu_n^{\circ,t}(u,\infty)$ concentrates around its mean value when choosing $a_n = 2^{\varepsilon n}/b_n^\circ$. The proof of Proposition 4.1 is finally completed in Section 4.2.

4.1. **Preparations.** To begin with, we bring the function $\nu_n^t(u,\infty)$ given in (4.1) into a form amenable to treatment. Let T_n be as in (2.12). For all $0<\varepsilon<1$ and δ_n as in (4.4),

$$0 \le \int_0^{\theta_n} \max\left((c_n r_n^\star)^{-1}, c_n^{-1}\tau_n(Y_n(s))\right) \mathbb{1}_{\{Y_n(s)\notin T_n\}} ds \le \theta_n \frac{r_n(\varepsilon_n)}{r_n(\varepsilon)} \le n^{-2} \tag{4.10}$$

as follows from (2.29). Hence visits of Y_n outside the set T_n only yield a negligible contribution to the event in (4.1), implying that

$$\check{\nu}_n^t(u,\infty) \le \nu_n^t(u,\infty) \le \check{\nu}_n^t\left(u-n^{-2},\infty\right) \tag{4.11}$$

where

$$\check{\nu}_n^t(u,\infty) \equiv k_n(t) P_{\pi_n}\left(\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n\}} ds > u\right). \tag{4.12}$$

Our next step consists in reducing visits to T_n in $\check{\nu}_n^t(u,\infty)$ to visits to the subset T_n° defined in (2.14). Set

$$\bar{\nu}_n^t(u,\infty) \equiv k_n(t) P_{\pi_n}\left(\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n^\circ\}} ds > u\right). \tag{4.13}$$

Lemma 4.2. *Assume that (4.6) holds. Then on $\Omega^\star$, for all but a finite number of indices n,*

$$|\check{\nu}_n^t(u,\infty) - \bar{\nu}_n^t(u,\infty)| \le 2k_n(t)\theta_n r_n^\star n^5 2^{-2n\varepsilon_n}(1+o(1)). \tag{4.14}$$

Proof of Lemma 4.2. Decomposing the event appearing in the probability in (4.12) according to whether $\{H(T_n\setminus T_n^\circ)\le\theta_n\}$ or $\{H(T_n\setminus T_n^\circ)>\theta_n\}$, (4.14) follows from (3.19) of Corollary 3.8 applied with $t_n=\theta_n$, which is licit by virtue of (4.6) (see also (4.8)). □

We next decompose (4.13) according to the hitting time, $H(T_n^\circ)$, and hitting place, $Y_n(H(T_n^\circ))$, of the set T_n°. The density of the joint distribution of $H(T_n^\circ)$ and $Y_n(H(T_n^\circ))$ is a $|T_n^\circ|$-dimensional vector, $(h_{n,x})_{x\in T_n^\circ}$, whose components are, for each $x\in T_n^\circ$, the joint density that T_n° is reached at time v, and that arrival to that set occurs in state x,

$$P_{\pi_n}(H(T_n^\circ)\le s, Y_n(H(T_n^\circ))=x) = \int_0^s h_{n,x}(v)dv. \tag{4.15}$$

For this vector of densities we have

$$\sum_{x\in T_n^\circ}\int_0^\infty h_{n,x}(v)dv = 1, \tag{4.16}$$

and, denoting by h_{n,T_n° the density of $H(T_n^\circ)$,

$$h_{n,T_n^\circ} = \sum_{x\in T_n^\circ} h_{n,x}. \tag{4.17}$$

In the notation of Section 3.3 (see the paragraph below (3.30)) $h_{n,x}=\sum_{y\in\mathcal{V}_n}\pi_n(y)h^y_{n,x,T_n^\circ}$ where, for $y\in T_n^\circ$, $h^y_{n,x,T_n^\circ}=\delta_y$. From this and the strong Markov property it follows that

$$\bar{\nu}_n^t(u,\infty) = k_n(t)\sum_{x\in T_n^\circ}\int_0^{\theta_n} h_{n,x}(v) P_x\left(\int_0^{\theta_n - v} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n^\circ\}} ds > u\right) dv. \tag{4.18}$$

Denote by $\overline{Q}_n^{u,v}(x)$ the probability appearing in (4.18). Notice that Y_n starts in $x \in T_n^\circ$ and further decompose this probability according to whether $\{H(T_n^\circ \setminus x) \le \theta_n - v\}$ or $\{H(T_n^\circ \setminus x) > \theta_n - v\}$, that is, write $\overline{Q}_n^{u,v}(x) \equiv \widetilde{Q}_n^{u,v}(x) + \widehat{Q}_n^{u,v}(x)$,

$$\widetilde{Q}_n^{u,v}(x) = P_x\left(\int_0^{\theta_n - v} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n^\circ\}}ds > u, H(T_n^\circ \setminus x) \le \theta_n - v\right), \quad (4.19)$$

$$\widehat{Q}_n^{u,v}(x) = P_x\left(\int_0^{\theta_n - v} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n^\circ\}}ds > u, H(T_n^\circ \setminus x) > \theta_n - v\right), \quad (4.20)$$

and split (4.18) accordingly. Clearly, for all $v > 0$

$$\widetilde{Q}_n^{u,v}(x) \le P_x\left(H(T_n^\circ \setminus x) \le \theta_n\right). \quad (4.21)$$

This and the bound $\int_0^{\theta_n} h_{n,x}(v)dv \le P_{\pi_n}(H(x) \le \theta_n)$ (that follows from (4.15)), yield

$$k_n(t)\sum_{x\in I_n^\circ}\int_0^{\theta_n} h_{n,x}(v)\widetilde{Q}_n^{u,v}(x)dv \quad (4.22)$$

$$\le k_n(t)\sum_{x\in T_n^\circ} P_{\pi_n}(H(T_n^\circ) \le \theta_n, Y_n(H(T_n^\circ)) = x)P_x\left(H(T_n^\circ \setminus x) \le \theta_n\right) \quad (4.23)$$

$$\le \widetilde{\nu}_n^t \quad (4.24)$$

where

$$\widetilde{\nu}_n^t \equiv k_n(t)\sum_{x\in T_n^\circ} P_{\pi_n}(H(x) \le \theta_n)P_x\left(H(T_n^\circ \setminus x) \le \theta_n\right). \quad (4.25)$$

Lemma 4.3. *Assume that (4.6) holds. Then on $\Omega^\star$, for all but a finite number of indices n,*

$$\widetilde{\nu}_n^t \le k_n(t)n^{c_\star+4}\left(\theta_n\pi_n(T_n^\circ)r_n^\star\right)^2(1+o(1)). \quad (4.26)$$

Proof of Lemma 4.3. By (3.3), (2.20), (4.3) and (4.4), on $\Omega^\star$, for all large enough n, $\theta_n n\pi_n(T_n^\circ)r_n^\star = n^{1+2\alpha(\varepsilon)}r_n^\star\theta_n^{1+\alpha(\varepsilon)}2^{-n\varepsilon}(1+o(1))$, wich decays to zero as n diverges by (4.6) (see also (4.8)). We may thus use (3.18) of Lemma 3.7 to bound the term $P_{\pi_n}(H(x) \le \theta_n)$ in (4.25), and by this and (3.3) we get that on $\Omega^\star$, for all large enough n,

$$\widetilde{\nu}_n^t \le k_n(t)\theta_n n\pi_n(T_n^\circ)r_n^\star(1+o(1))|T_n^\circ|^{-1}\textstyle\sum_{x\in T_n^\circ} P_x\left(H(T_n^\circ \setminus x) \le \theta_n\right). \quad (4.27)$$

The lemma now follows from Proposition 3.9. □

Consider now the contribution to (4.18) coming from (4.20). By definition,

$$\widehat{Q}_n^{u,v}(x) = P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u, H(T_n^\circ \setminus x) > \theta_n - v\right). \quad (4.28)$$

Thus

$$\widehat{\nu}_n^t(u,\infty) \quad (4.29)$$

$$\equiv k_n(t)\sum_{x\in T_n^\circ}\int_0^{\theta_n} h_{n,x}(v)\widehat{Q}_n^{u,v}(x)dv \quad (4.30)$$

$$= k_n(t)\sum_{x\in T_n^\circ}\int_0^{\theta_n} h_{n,x}(v)P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u, H(T_n^\circ \setminus x) > \theta_n - v\right)dv. \quad (4.31)$$

Setting

$$\nu_n^{\circ,t}(u,\infty) \equiv k_n(t)\sum_{x\in T_n^\circ}\int_0^{\theta_n} h_{n,x}(v)P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u\right)dv, \quad (4.32)$$

we have

$$\nu_n^{\circ,t}(u,\infty) - w_n^t(u,\infty) \le \widehat{\nu}_n^t(u,\infty) \le \nu_n^{\circ,t}(u,\infty) \tag{4.33}$$

where

$$w_n^t(u,\infty) \equiv k_n(t) \sum_{x\in T_n^\circ} \int_0^{\theta_n} h_{n,x}(v) P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u, H(T_n^\circ \setminus x) \le \theta_n - v\right) dv$$

$$\le k_n(t) \sum_{x\in T_n^\circ} \int_0^{\theta_n} h_{n,x}(v) P_x\left(H(T_n^\circ \setminus x) \le \theta_n - v\right) dv \le \widetilde{\nu}_n^t. \tag{4.34}$$

Inserting our bounds in (4.18), we finally get that for all $u > 0$

$$\left|\nu_n^{\circ,t}(u,\infty) - \bar{\nu}_n^t(u,\infty)\right| \le \widetilde{\nu}_n^t. \tag{4.35}$$

Our aim now is to prove almost sure convergence of $\nu_n^{\circ,t}(u,\infty)$. To do so we will need certain properties a sequence, b_n°, associated to the sequence b_n, that we now define.

4.2. **Properties of the sequences b_n and b_n°.** For $F_{\beta,\varepsilon,n}(x)$ as in (1.41) define

$$b_n^\circ \equiv (\theta_n \pi_n(T_n^\circ))^{-1} \sum_{x\in T_n^\circ} \int_0^{\theta_n} h_{n,x}(v) E_x[F_{\beta,\varepsilon,n}(\ell_n^x(\theta_n - v))] dv. \tag{4.36}$$

Thus b_n° is nothing but b_n (see (1.44)) with T_n° substituted for T_n. The next lemma collects properties of the sequences b_n and b_n° needed in the verification of both Condition (B1) and (B2). Set $\mathcal{I}_n(a,b) = (\theta_n\pi_n(T_n^\circ))^{-1} \sum_{x\in T_n^\circ} \mathcal{J}_n^x(a,b)$,

$$\mathcal{J}_n^x(a,b) = \int_a^b h_{n,x}(v) E_x[F_{\beta,\varepsilon,n}(\ell_n^x(\theta_n - v))] dv, \tag{4.37}$$

and given $0 < \zeta_n < \theta_n$ split b_n° into $b_n^\circ = \mathcal{I}_n(0,\kappa_n) + \mathcal{I}_n(\kappa_n, \theta_n - \zeta_n) + \mathcal{I}_n(\theta_n - \zeta_n, \theta_n)$.

Lemma 4.4. *Assume that (4.5) and (4.6) hold. Let $\zeta_n > 0$ be a sequence satisfying*

$$n^{-1}|\log \zeta_n| \ll 1, \ \textit{and} \ \tilde{\kappa}_n (r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} \zeta_n^{\alpha_n(\varepsilon)+o(1)} \downarrow 0 \ \textit{as} \ n \uparrow \infty. \tag{4.38}$$

Then, on $\Omega_1 \cap \Omega^\circ \cap \Omega^\star$, for all but a finite number of indices n,

$$\frac{\mathcal{I}_n(0,\kappa_n)}{\mathcal{I}_n(\kappa_n, \theta_n - \zeta_n)} \le \theta_n^{-1} \tilde{\kappa}_n \kappa_n^{1+\alpha_n(\varepsilon)} (n r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}, \tag{4.39}$$

$$0 \le (b_n - b_n^\circ)/b_n^\circ \le n (r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} \kappa_n^{1+\alpha_n(\varepsilon)} 2^{-n\varepsilon_n}, \tag{4.40}$$

and the right-hand sides of (4.39) and (4.40) decay to zero as n diverges. Furthermore

$$\kappa_n^{-1} (r_n^\star)^{-\{\alpha_n(\varepsilon)+o(1)\}} \le b_n^\circ \le (1+o(1)) n r_n^\star \kappa_n^{\alpha_n(\varepsilon)}. \tag{4.41}$$

Proof of Lemma 4.4. We first prove a lower bound on $\mathcal{I}_n(\kappa_n, \theta_n - \zeta_n)$. For this write

$$\mathcal{J}_n^x(\kappa_n, \theta_n - \zeta_n) \ge \mathcal{J}_{n,1}^x \equiv \int_{\kappa_n}^{\theta_n - \zeta_n} h_{n,x}(v) E_x[F_{\beta,\varepsilon,n}(\ell_n^x(\theta_n - v)) \mathbb{1}_{\{\zeta_n < \ell_n^x(\theta_n - v) \le \theta_n\}}] dv.$$

Since $F_{\beta,\varepsilon,n}(x) = (1+o(1)) x^{\alpha_n(\varepsilon)+o(1)}$ for all $\zeta_n < x \le \theta_n$,

$$\begin{aligned} \mathcal{J}_{n,1}^x &\ge (1+o(1)) \int_{\kappa_n}^{\theta_n - \zeta_n} h_{n,x}(v) E_x[\ell_n^x(\theta_n - v)]^{\alpha_n(\varepsilon)+o(1)} (1 - \mathbb{1}_{\{\ell_n^x(\theta_n - v) < \zeta_n\}}) dv \\ &\equiv \mathcal{J}_{n,3}^x - \mathcal{J}_{n,4}^x \end{aligned} \tag{4.42}$$

where we used the left-most inequality in (4.74) to relax the constraint $\ell_n^x(\theta_n - v) \le \theta_n$. Let us bound $\mathcal{J}_{n,3}^x$ for $x \in I_n^\star$. Note that by (2.16) and (2.8)

$$(r_n^\star)^{-1} \le \widetilde{\lambda}_n(x) \le r_n^\star, \quad \forall x \in I_n^\star. \tag{4.43}$$

Thus, setting $\zeta_n' \equiv n r_n^\star$, it follows from (3.46) of Lemma 3.11 that for all $x \in I_n^\star$,

$$\mathcal{J}_{n,3}^x \ge c_3 (\widetilde{\lambda}_n^{-1}(x))^{\alpha_n(\varepsilon)+o(1)} \int_{\kappa_n}^{\theta_n-\zeta_n'} h_{n,x}(v)dv \tag{4.44}$$

for some numerical constant $0 < c_3 < \infty$. Summing over x, wet get

$$\sum_{x\in T_n^\circ} \mathcal{J}_{n,3}^x \ge \sum_{x\in I_n^\star} \mathcal{J}_{n,3}^x \ge c_3 (r_n^\star)^{-\{\alpha_n(\varepsilon)+o(1)\}} \sum_{x\in I_n^\star} \int_{\kappa_n}^{\theta_n-\zeta_n'} h_{n,x}(v)dv \tag{4.45}$$

where the last sum in the right-hand side of (4.45) is equal to

$$P_{\pi_n}(\kappa_n < H(I_n^\star) < \theta_n - \zeta_n', H(I_n^\star) < H(T_n^\circ \setminus I_n^\star)). \tag{4.46}$$

Decomposing this probability into

$$p_1 - p_2 \equiv P_{\pi_n}(\kappa_n < H(I_n^\star) < \theta_n - \zeta_n') - P_{\pi_n}(\kappa_n < H(I_n^\star) < \theta_n - \zeta_n', H(I_n^\star) > H(T_n^\circ \setminus I_n^\star))$$

we have, by Lemma 3.6 and (3.16), whenever $\theta_n r_n^\star n \pi_n(I_n^\star) \to 0$,

$$p_1 \ge \tilde{\kappa}_n^{-1} \theta_n \pi_n(I_n^\star)(1 - \theta_n^{-1}\zeta_n')(1+o(1)) = \tilde{\kappa}_n^{-1} \theta_n \pi_n(I_n^\star)(1+o(1)) \tag{4.47}$$

where the last equality follows from (4.5). To get an upper bound on p_2, write

$$p_2 \le P_{\pi_n}(H(T_n^\circ \setminus I_n^\star) < \kappa_n) + P_{\pi_n}(H(T_n^\circ \setminus I_n^\star) < H(I_n^\star) < \theta_n) \equiv p_3 + p_4. \tag{4.48}$$

By (3.18), $p_3 \le \kappa_n r_n^\star n \pi_n(T_n^\circ \setminus I_n^\star)(1+o(1))$, whereas proceeding as in (4.22)-(4.25),

$$p_4 \le \sum_{x\in T_n^\circ \setminus I_n^\star} P_{\pi_n}(H(x) \le \theta_n) P_x\left(H(I_n^\star) \le \theta_n\right) \tag{4.49}$$

$$= n^3 (\theta_n r_n^\star)^2 \pi_n(T_n^\circ \setminus I_n^\star)\pi_n(I_n^\star)(1+o(1)) \tag{4.50}$$

where the last equality follows from (3.18) and (3.28). By (2.22), (2.23), and (3.3), on $\Omega^\star$ and for large enough n, $\pi_n(I_n^\star) = 2^{-n\varepsilon_n}(1 - n^{-c_\star}(1+o(1)))$ and $\pi_n(T_n^\circ \setminus I_n^\star) = n^{-c_\star+1} 2^{-n\varepsilon_n}(1+o(1))$ (thus in particular, $\pi_n(I_n^\star)/\pi_n(T_n^\circ) = 1 + o(1)$). In view of this, (4.5), and (4.6), one checks that $\theta_n r_n^\star n \pi_n(I_n^\star) \to 0$ (as requested above (4.47)) and that $p_2 = o(p_1)$. Thus $p_1 - p_2 = p_1(1+o(1))$ and by this, (4.47), and (4.45),

$$(\theta_n \pi_n(T_n^\circ))^{-1} \sum_{x\in T_n^\circ} \mathcal{J}_{n,3}^x \ge \tilde{\kappa}_n^{-1} (r_n^\star)^{-\{\alpha_n(\varepsilon)+o(1)\}}(1+o(1)). \tag{4.51}$$

Turning to $\mathcal{J}_{n,4}^x$ we have

$$\sum_{x\in T_n^\circ} \mathcal{J}_{n,4}^x \le (1+o(1))\zeta_n^{\alpha_n(\varepsilon)+o(1)} \sum_{x\in T_n^\circ} \int_{\kappa_n}^{\theta_n-\zeta_n} h_{n,x}(v)dv, \tag{4.52}$$

where the last sum is equal to $P_{\pi_n}(\kappa_n < H(T_n^\circ) < \theta_n - \zeta_n)$. Since by Lemma 3.6 and (3.16), $P_{\pi_n}(\kappa_n < H(T_n^\circ) < \theta_n - \zeta_n) \le (1+o(1)) r_n^\star n \theta_n \pi_n(T_n^\circ)$, we get

$$(\theta_n \pi_n(T_n^\circ))^{-1} \sum_{x\in T_n^\circ} \mathcal{J}_{n,4}^x \le (1+o(1)) n r_n^\star \zeta_n^{\alpha_n(\varepsilon)+o(1)}. \tag{4.53}$$

At this point we may observe that the right-most condition in (4.38) is tailored to guarantee that $\sum_{x\in T_n^\circ}\mathcal{J}_{n,3}^x \gg \sum_{x\in T_n^\circ}\mathcal{J}_{n,4}^x$. Hence, collecting our bounds,

$$\mathcal{I}_n(\kappa_n,\theta_n-\zeta_n) = \frac{1+o(1)}{\theta_n\pi_n(T_n^\circ)}\sum_{x\in T_n^\circ}\int_{\kappa_n}^{\theta_n-\zeta_n} h_{n,x}(v)E_x[\ell_n^x(\theta_n-v)]^{\alpha_n(\varepsilon)+o(1)} \tag{4.54}$$

$$\geq \tilde{\kappa}_n^{-1}(r_n^\star)^{-\{\alpha_n(\varepsilon)+o(1)\}}. \tag{4.55}$$

We now prove an upper bound on $\mathcal{I}_n(0,\kappa_n)$. Using that $F_{\beta,\varepsilon,n}(x)\leq(1+o(1))x^{\alpha_n(\varepsilon)}$ for all $0<x\leq\theta_n$ together with (3.47) of Lemma 3.11 (which by (4.6) and (3.4) is licit),

$$\mathcal{J}_n^x(0,\kappa_n)\leq(1+o(1))\kappa_n^{\alpha_n(\varepsilon)}\int_0^{\kappa_n}h_{n,x}(v)dv. \tag{4.56}$$

Summing over $x\in T_n^\circ$ and using (3.20) and (4.6) to bound the resulting probability,

$$\mathcal{I}_n^x(0,\kappa_n)\leq(1+o(1))nr_n^\star\theta_n^{-1}\kappa_n^{1+\alpha_n(\varepsilon)}. \tag{4.57}$$

One proves in the same way that

$$\mathcal{I}_n^x(0,\theta_n)\leq(1+o(1))nr_n^\star\kappa_n^{\alpha_n(\varepsilon)}\left[1+\theta_n^{1+\alpha_n(\varepsilon)}r_n^\star n\kappa_n^{-\alpha_n(\varepsilon)}2^{-n}\right], \tag{4.58}$$

where by (4.6) the term in square brackets (that comes from (3.47)) is equal to $1+o(1)$.

Combining (4.57) and (4.55) proves (4.39). Since $\mathcal{I}_n(\kappa_n,\theta_n-\zeta_n)\leq b_n^\circ=\mathcal{I}_n(0,\theta_n)$, (4.55) and (4.58) yield, respectively, the lower and upper bounds of (4.41). It remains to prove (4.40). By definition (see (1.44), (4.36), and the second remark below (4.7) on the definition of T_n)

$$|T_n|b_n-|T_n^\circ|b_n^\circ = 2^n\theta_n^{-1}\sum_{x\in T_n\setminus T_n^\circ}E_{\pi_n}\left[F_{\beta,\varepsilon,n}(\ell_n^x(\theta_n))\right]. \tag{4.59}$$

Conditioning on the time of the first visit to x, and proceeding as in (4.57)-(4.58) to bound the expectation starting in x, $E_{\pi_n}\left[F_{\beta,\varepsilon,n}(\ell_n^x(\theta_n))\right]\leq(1+o(1))P_{\pi_n}(H(x)\leq\theta_n)\kappa_n^{\alpha_n(\varepsilon)}$. From this and (3.18), $|T_n|b_n-|T_n^\circ|b_n^\circ\leq(1+o(1))r_n^\star n2^n\pi_n(T_n\setminus T_n^\circ)\kappa_n^{\alpha_n(\varepsilon)}$. Now by (2.19)-(2.21), $|T_n|=|T_n^\circ|(1+o(1))$ and $|T_n\setminus T_n^\circ|=|T_n^\circ|n^4 2^{-n\varepsilon_n}(1+o(1))$. Hence $b_n-b_n^\circ\leq(1+o(1))n^5r_n^\star\kappa_n^{\alpha_n(\varepsilon)}2^{-n\varepsilon_n}$. Combining this and (4.55) yields (4.40). The proof of Lemma 4.4 is now complete. □

Proof of Proposition 1.5. This is a straightforward consequence of (4.40), (4.41), the assumptions of (1.45), and (1.38). □

4.3. **Concentration of $\nu_n^{\circ,t}(u,\infty)$.** Let us now focus on the term $\nu_n^{\circ,t}(u,\infty)$ of (4.32). Recall the definitions of $k_n(t)$ and b_n° from (1.17) and (4.36), respectively.

Proposition 4.5. *Choose $a_n=2^{\varepsilon n}/b_n^\circ$ in $k_n(t)$ and assume that (4.6) holds. Let $\mathbb{P}^\circ$ denote the law of the collection $\{\tau_n(x),x\in T_n^\circ\}$ conditional on T_n°,*

$$\mathbb{P}^\circ(\cap_{x\in T_n^\circ}\{\tau_n(x)\in\cdot\})=\mathbb{P}(\cap_{x\in T_n^\circ}\{\tau_n(x)\in\cdot\}\mid T_n^\circ). \tag{4.60}$$

Then, for any sequence $u_n>0$ such that $0<u-u_n<n^{-1}$ and all $u>0$ and $t>0$,

$$\mathbb{P}^\circ\left(\left|\nu_n^{\circ,t}(u_n,\infty)-\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)\right|>n\sqrt{t\Xi_n\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)}\right)\leq n^{-2}(1+o(1)) \tag{4.61}$$

where $\Xi_n\equiv(2^{\varepsilon n}/b_n^\circ)nr_n^\star 2^{-n}$ and

$$\lim_{n\to\infty}\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)=tu^{\alpha(\varepsilon)}. \tag{4.62}$$

Proof of Proposition 4.5. We assume throughout that $\omega \in \Omega^\star$. A key ingredient of the proof is the observation that the generator $\widetilde{L}_n$ of Y_n is independent of the values of the Hamiltonian at its local minima. More precisely, recalling the definition of the set, $\mathcal{M}_n$, of local minima from (2.24), it follows from (2.8) and (2.25) that on $\Omega^\star$, for all n large enough, for all $x \in \mathcal{M}_n$, and $y \sim x$,

$$\widetilde{\lambda}_n(x,y) = n^{-1}\tau_n(y) \;\text{ and }\; \widetilde{\lambda}_n(y,x) = \begin{cases} n^{-1}\tau_n(y), & \text{if } y \notin \overline{V}_n^\star, \\ n^{-1}, & \text{if } y \in \overline{V}_n^\star, \end{cases} \tag{4.63}$$

(note that if $x \in \mathcal{M}_n$ and $y \sim x$ then $y \notin \mathcal{M}_n$). Hence the law of Y_n does not depend on the $\tau_n(x)$'s in $\mathcal{M}_n$ (but it does depend on $\mathcal{M}_n$). Now by (2.15),

$$T_n^\circ \subseteq \mathcal{M}_n \cap T_n \subseteq \mathcal{M}_n. \tag{4.64}$$

Furthermore, one easily checks that $\mathbb{P}^\circ$ in (4.60) is the product measure

$$\mathbb{P}^\circ(\cap_{x\in T_n^\circ}\{\tau_n(x) \in \cdot\}) = \prod_{x\in T_n^\circ} \frac{\mathbb{P}(\tau_n(x) \in \cdot, \tau_n(x) \geq r_n(\varepsilon_n))}{\mathbb{P}(\tau_n(x) \geq r_n(\varepsilon_n))}. \tag{4.65}$$

Consequently, for fixed T_n°, the collection $\{X_n(x), x \in T_n^\circ\}$,

$$X_n(x) \equiv \int_0^{\theta_n} h_{n,x}(v) P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u_n\right) dv, \tag{4.66}$$

viewed as a collection of r.v.'s on the sub-sigma field $\mathcal{F}_n^\circ = \sigma(\{\tau_n(x), x \in T_n^\circ\})$, forms a collection of independent random variables under $\mathbb{P}^\circ$ (that of course still depend on the variables $\tau_n(x)$ in $(T_n^\circ)^c$). The proof now hinges on a simple mean and variance argument. We deal with the variance first. By (4.32) and (4.66),

$$\mathbb{E}^\circ \nu_n^{\circ,t}(u_n,\infty) = k_n(t) \sum_{x\in T_n^\circ} \mathbb{E}^\circ X_n(x), \tag{4.67}$$

and by independence

$$\mathbb{E}^\circ(\nu_n^{\circ,t}(u_n,\infty) - \mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty))^2 \leq k_n^2(t) \sum_{x\in T_n^\circ} \mathbb{E}^\circ(X_n(x))^2. \tag{4.68}$$

Note that since

$$X_n(x) \leq \int_0^{\theta_n} h_{n,x}(v)dv \leq P_{\pi_n}(H(x) \leq \theta_n) \leq \theta_n r_n^\star n 2^{-n}(1+o(1)), \tag{4.69}$$

(the last inequality is (3.18) combined with (3.3)) then

$$k_n^2(t) \sum_{x\in T_n^\circ} \mathbb{E}^\circ(X_n(x))^2 \leq t(2^{\varepsilon n}/b_n^\circ) r_n^\star n 2^{-n}(1+o(1)) \mathbb{E}^\circ \nu_n^{\circ,t}(u_n,\infty), \tag{4.70}$$

where we used that for $a_n = 2^{\varepsilon n}/b_n^\circ$, $\theta_n k_n(t) = \theta_n \lfloor t(2^{\varepsilon n}/b_n^\circ)/\theta_n \rfloor = t(2^{\varepsilon n}/b_n^\circ)(1+o(1))$. Inserting (4.70) in (4.68), a second order Tchebychev inequality then yields (4.61).

To estimate $\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)$ in (4.67) we first use Fubini to write,

$$\mathbb{E}^\circ X_n(x) = \int_0^{\theta_n} h_{n,x}(v) E_x\mathbb{P}^\circ\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u_n\right) dv. \tag{4.71}$$

Denoting by $\mathbb{P}^x$ the law of the single variable $\tau_n(x)$,

$$\mathbb{P}^\circ\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u\right) = \frac{\mathbb{P}^x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u_n, \tau_n(x) \geq r_n(\varepsilon_n)\right)}{\mathbb{P}^x(\tau_n(x) \geq r_n(\varepsilon_n))} \tag{4.72}$$

$$= \frac{\mathbb{P}^x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u_n\right)}{\mathbb{P}^x(\tau_n(x) \geq r_n(\varepsilon_n))} \tag{4.73}$$

where (4.73) follows from the definition of c_n (see (1.4)), the a priory bound

$$\ell_n^x(\theta_n - v) \leq \theta_n - v \ll c_n, \quad 0 \leq v \leq \theta_n, \tag{4.74}$$

and the fact that δ_n in (4.4) in chosen in such a way that $\theta_n r_n(\varepsilon_n) r_n^{-1}(\varepsilon) \leq n^{-2} \downarrow 0$ as $n \uparrow \infty$ (see the last inequality in (4.10)). Using classical estimates on the asymptotics of gaussian integrals (see e.g. [1] p. 932), Lemma 2.2, and again the definition of c_n, simple calculations yield that for all $0 < u < \infty$ and $0 \leq v < \theta_n$, (4.73) is equal to

$$(1 + o(1))F_{\beta,\varepsilon,n}\left(\tfrac{\ell_n^x(\theta_n - v)}{u_n}\right)\frac{\mathbb{P}\left(\tau_n(x) > c_n\right)}{\mathbb{P}(\tau_n(x) \geq r_n(\varepsilon_n))} \tag{4.75}$$

where $F_{\beta,\varepsilon,n}(x)$ is defined in (1.41). Furthermore, by (1.4), $2^{\varepsilon n}\mathbb{P}(\tau_n(x) \geq c_n) = 1$ whereas by (2.2), (2.20), and (3.3), $\mathbb{P}(\tau_n(x) \geq r_n(\varepsilon_n)) = \pi_n(T_n^\circ)(1 + o(1))$. In view of this and (4.36) we get, combining (4.75), (4.71), (4.67), and using the a priori bound (4.74) that

$$\mathbb{E}^\circ \nu_n^{\circ,t}(u_n, \infty) = (1 + o(1))k_n(t)\theta_n(b_n^\circ/2^{\varepsilon n})\frac{I_{(0,\theta_n)}(u_n)}{I_{(0,\theta_n)}(1)} \tag{4.76}$$

where for $w > 0$

$$I_{(a,b)}(w) \;=\; \sum_{x \in T_n^\circ}\int_0^{\theta_n} h_{n,x}(v)E_x\left[F_{\beta,\varepsilon,n}\left(\tfrac{\ell_n^x(\theta_n - v)}{w}\right)\right]\mathbb{1}_{\{a \leq \ell_n^x(\theta_n - v) < b\}}dv. \tag{4.77}$$

To evaluate the ratio in (4.76) set $0 < \zeta_n \equiv e^{-n^{9/10}} \downarrow 0$ and split the integral in $I_{(0,\theta_n)}(u_n)$ into $I_{(0,\theta_n)}(u_n) \equiv I_{(0,\zeta_n)}(u_n) + I_{(\zeta_n,\theta_n)}(u_n)$. Note that $n^{-1}|\log \zeta_n| = n^{-1/10}$, $n^{-1}(\log \zeta_n)^2 = n^{4/5}$, while for all $u > 0$, $n^{-1}\log u_n \downarrow 0$, $n^{-1}(\log u_n)^2 \downarrow 0$ as $n \uparrow \infty$. Using that $F_{\beta,\varepsilon,n}(x)$ is increasing on the domain $(0, \zeta_n/u_n)$

$$I_{(0,\zeta_n)}(u_n) \;\leq\; F_{\beta,\varepsilon,n}\left(\tfrac{\zeta_n}{u_n}\right)P_{\pi_n}(H(T_n^\circ) < \theta_n) \tag{4.78}$$

where $F_{\beta,\varepsilon,n}\left(\frac{\zeta_n}{u_n}\right) = e^{o(1)\log u_n}F_{\beta,\varepsilon,n}(\zeta_n)F_{\beta,\varepsilon,n}(u_n^{-1})$ and $F_{\beta,\varepsilon,n}(\zeta_n) \leq e^{-\alpha_n(\varepsilon)n^{9/10} - n^{4/5}/2\beta^2}$. By this, (3.18), the lower bound (4.41) on b_n°, and our assumptions on u_n,

$$\frac{I_{(0,\zeta_n)}(u_n)}{I_{(0,\theta_n)}(1)} = e^{o(1)\log u_n}F_{\beta,\varepsilon,n}(u_n^{-1})F_{\beta,\varepsilon,n}(\zeta_n)n\kappa_n(r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} \to 0 \tag{4.79}$$

as $n \to \infty$. Next, since $n^{-1}\log l \downarrow 0$ as $n \uparrow \infty$ for all $\zeta_n \leq l \leq \theta_n$ we have, using (4.74),

$$\frac{I_{(\zeta_n,\theta_n)}(u_n)}{I_{(0,\theta_n)}(1)} = e^{o(1)\log u_n}F_{\beta,\varepsilon,n}(u_n^{-1})\left[1 - \tfrac{I_{(0,\zeta_n)}(u_n)}{I_{(0,\theta_n)}(1)}\right] \to u^{-\alpha(\varepsilon)} \tag{4.80}$$

as $n \to \infty$ for all $u > 0$. Inserting (4.79) and (4.80) in (4.76), choosing $a_n = 2^{\varepsilon n}/b_n^\circ$, and passing to the limit $n \to \infty$ finally gives (4.62). The proof of the lemma is done. □

4.4. **Proof of Proposition 4.1.** By (4.6), (4.3)-(4.4), and the bound $\kappa_n \le \theta_n$, (4.40) implies that on $\Omega_1 \cap \Omega^\circ \cap \Omega^\star$, for large enough n, $b_n = b_n^\circ(1+o(1))$. The assumption that $a_n = 2^{\varepsilon n}/b_n$ in (4.1) can thus be replaced by $a_n = 2^{\varepsilon n}/b_n^\circ$. Consider now (4.61) and note that by (4.41), (3.6), (1.38), and (4.6) (see also (4.8)), for all $0 < \varepsilon < 1$,

$$(2^{\varepsilon n}/b_n^\circ) r_n^\star n^3 2^{-n} \le \kappa_n (r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} n^3 2^{n\varepsilon} 2^{-n} \to 0 \tag{4.81}$$

as $n \to \infty$. Thus, by Proposition 4.5 and Borel-Cantelli Lemma we get that for all $u > 0$ and all $t > 0$,

$$\lim_{n\to\infty} \nu_n^{\circ,t}(u,\infty) = t u^{\alpha(\varepsilon)} \quad \mathbb{P} - \text{almost surely}. \tag{4.82}$$

In the same way we get that for all $u > 0$ and all $t > 0$,

$$\lim_{n\to\infty} \nu_n^{\circ,t}(u,\infty) = t u^{\alpha(\varepsilon)} \quad \mathbb{P} - \text{almost surely}. \tag{4.83}$$

Next, by Lemma 4.2, Lemma 4.3, and (4.35) we have that on $\Omega^\star$, for all but a finite number of indices n,

$$\begin{aligned} &\left|\check\nu_n^t(u,\infty) - \nu_n^{\circ,t}(u,\infty)\right| && (4.84)\\ &\le t(b_n^\circ)^{-1}[2r_n^\star n^5 \theta_n 2^{-n\varepsilon+2\delta_n n} + n^{c_\star+4} 2^{n\varepsilon} (\theta_n \pi_n(T_n^\circ) r_n^\star)^2](1+o(1)) && (4.85)\\ &\le 2t n^{c_\star+4(1+\alpha_n(\varepsilon))} (r_n^\star)^{\alpha_n(\varepsilon)+2+o(1)} \kappa_n \theta_n^{2+2\alpha(\varepsilon)} 2^{-n\varepsilon}(1+o(1)) && (4.86)\end{aligned}$$

where the last inequality follows from (4.41), (2.20), (4.3), and (4.4). Since $\kappa_n \le \theta_n$, (4.6) (see also (4.8)) implies that (4.86) decays to zero as $n \to \infty$. From this and (4.82) we get that for all $u > 0$ and all $t > 0$, $\lim_{n\to\infty} \check\nu_n^t(u,\infty) = t u^{\alpha(\varepsilon)}$ $\mathbb{P}$-almost surely. One proves in the same way that for all $u > 0$ and all $t > 0$, $\lim_{n\to\infty} \check\nu_n^t(u - n^{-2},\infty) = t u^{\alpha(\varepsilon)}$ $\mathbb{P}$-almost surely. Therefore, by (4.11), for all $u > 0$ and all $t > 0$,

$$\lim_{n\to\infty} \nu_n^t(u,\infty) = t u^{\alpha(\varepsilon)} \quad \mathbb{P} - \text{almost surely}. \tag{4.87}$$

Since ν_n^t is increasing both in t and u and since its limit continuous in those two variables, (4.87) implies that $\mathbb{P}$-almost surely,

$$\lim_{n\to\infty} \nu_n^t(u,\infty) = t u^{\alpha(\varepsilon)}, \quad \forall\, u > 0, t > 0. \tag{4.88}$$

The proof of Proposition 4.1 is done.

5. Verification of Condition (B2)

By (1.18)-(1.19), (1.22), and (1.34), Condition (B2) in (1.32) states that

$$\sigma_n^t(u,\infty) \equiv k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) \left[P_y\left(\int_0^{\theta_n} \max\left((c_n r_n^\star)^{-1}, c_n^{-1} \tau_n(Y_n(s)) \right) ds > u \right) \right]^2 \tag{5.1}$$

decays to zero as n diverges. We prove in this section that this holds true $\mathbb{P}$-almost surely.

Proposition 5.1. *Under the assumptions of Proposition 4.1, for all* $0 < \varepsilon < 1$ *and* $\beta > 0$, $\mathbb{P}$*-almost surely,*

$$\lim_{n\to\infty} \sigma_n^t(u,\infty) = 0, \quad \forall\, t > 0, u > 0. \tag{5.2}$$

As in the proof of Proposition 4.1 we first bring $\sigma_n^t(u,\infty)$ into a suitable form. Proceeding as in (4.11)-(4.12), we first write

$$\check\sigma_n^t(u,\infty) \le \sigma_n^t(u,\infty) \le \check\sigma_n^t(u - n^{-2},\infty) \tag{5.3}$$

where

$$\check{\sigma}_n^t(u,\infty) \equiv k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) \left[P_y \left(\int_0^{\theta_n} c_n^{-1} \tau_n(Y_n(s)) \mathbb{1}_{\{Y_n(s)\in T_n\}} ds > u \right) \right]^2, \tag{5.4}$$

and next reduce visits to T_n in (5.4) to visits to visits to T_n°, just as in Lemma 4.2. Set

$$\bar{\sigma}_n^t(u,\infty) \equiv k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) \left[P_y \left(\int_0^{\theta_n} c_n^{-1} \tau_n(Y_n(s)) \mathbb{1}_{\{Y_n(s)\in T_n^\circ\}} ds > u \right) \right]^2. \tag{5.5}$$

Lemma 5.2. *Assume that (4.6) holds. Then on $\Omega^\star$, for all but a finite number of indices n,*

$$|\check{\sigma}_n^t(u,\infty) - \bar{\sigma}_n^t(u,\infty)| \le 6k_n(t)\theta_n n^5 r_n^\star 2^{-2n\varepsilon_n}(1+o(1)). \tag{5.6}$$

Proof of lemma 5.2. As in the Proof of Lemma 4.2 we decompose the event appearing in the probability in (5.4) according to whether $\{H(T_n \setminus T_n^\circ) \le \theta_n\}$ or not, that is, setting

$$q_1(y) = P_y\big(\textstyle\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n\}} ds > u, H(T_n\setminus T_n^\circ) \le \theta_n\big), \tag{5.7}$$

$$q_2(y) = P_y\big(\textstyle\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n\}} ds > u, H(T_n\setminus T_n^\circ) > \theta_n\big), \tag{5.8}$$

we write $\check{\sigma}_n^t(u,\infty) = k_n(t)\sum_{y\in\mathcal{V}_n}\pi_n(y)[q_1(y)+q_2(y)]^2$. In the same way write $\bar{\sigma}_n^t(u,\infty) = k_n(t)\sum_{y\in\mathcal{V}_n}\pi_n(y)[\bar{q}_1(y)+\bar{q}_2(y)]^2$ where $\bar{q}_1(y)$ and $\bar{q}_2(y)$ are defined as in (5.7) and (5.8), respectively, substituting T_n° for T_n. Note that

$$[x_1+x_2]^2 \le 3x_1 + x_2^2, \quad 0 \le x_1, x_2 \le 1. \tag{5.9}$$

Applying (5.9) to the terms $[q_1(y)+q_2(y)]^2$ and $[\bar{q}_1(y)+\bar{q}_2(y)]^2$, and observing that $q_2^2 = \bar{q}_2^2$, we get

$$|\check{\sigma}_n^t(u,\infty) - \bar{\sigma}_n^t(u,\infty)| \le 3k_n(t)\sum_{y\in\mathcal{V}_n}\pi_n(y)(q_1(y)+\bar{q}_1(y)) \tag{5.10}$$

$$\le 6k_n(t) P_{\pi_n}(H(T_n\setminus T_n^\circ)\le\theta_n). \tag{5.11}$$

The Lemma now follows from (3.19) of Corollary 3.8. □

We continue our parallel with the proof of Proposition 4.1 and decompose (5.5) according to the hitting time and hitting place of the set T_n°. We slightly abuse the notation of Section 3 (see the paragraph below (3.30)) and denote by $h_{n,x}^y$ (instead of $h_{n,x,T_n^\circ}^y$) the joint density that T_n° is reached at time t, and that arrival to that set occurs in state x, given that the process starts in y. As already observed (see the paragraph below (4.17)), $h_{n,x} = \sum_{y\in\mathcal{V}_n}\pi_n(y)h_{n,x}^y$. Proceeding as in (4.18)-(4.20) we then get

$$\bar{\sigma}_n^t(u,\infty) = k_n(t)\sum_{y\in\mathcal{V}_n}\pi_n(y)\left[\overline{R}_n^u(y)\right]^2 \tag{5.12}$$

where, using (4.19) and (4.20),

$$\overline{R}_n^u(y) \equiv \sum_{x\in T_n^\circ}\int_0^{\theta_n} h_{n,x}^y(v)\left(\widetilde{Q}_n^{u,v}(x) + \widehat{Q}_n^{u,v}(x)\right)dv \equiv \widetilde{R}_n^u(y) + \widehat{R}_n^u(y). \tag{5.13}$$

By analogy with (4.30) we also set

$$\widehat{\sigma}_n^t(u,\infty) \equiv k_n(t)\sum_{y\in\mathcal{V}_n}\pi_n(y)\left[\widehat{R}_n^u(y)\right]^2. \tag{5.14}$$

The next lemma plays the role of Lemma 4.3.

Lemma 5.3. *Assume that (4.6) holds. Then $\Omega^\star$, for all but a finite number of indices n,*

$$0 \le \bar\sigma_n^t(u,\infty) - \widehat\sigma_n^t(u,\infty) \le 3k_n(t) n^{c_\star+4} \left(\theta_n \pi_n(T_n^\circ) r_n^\star\right)^2 (1+o(1)). \tag{5.15}$$

Proof of Lemma 5.3. As in the proof of Lemma 5.2, the proof of Lemma 5.3 relies on the observation that since $0 \le \widetilde R_n^u(y), \widehat R_n^u(y) \le 1$ in (5.13) for all $y \in \mathcal{V}_n$, then by (5.9),

$$0 < \bar\sigma_n^t(u,\infty) - \widehat\sigma_n^t(u,\infty) \le 3k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) \widetilde R_n^u(y) \tag{5.16}$$

$$= 3k_n(t) \sum_{x\in T_n^\circ} \int_0^{\theta_n} h_{n,x}(v) \widetilde Q_n^{u,v}(x) dv \le 3\widetilde\nu_n^t. \tag{5.17}$$

The equality in (5.17) follows from the identity $h_{n,x}(v) = \sum_{y\in\mathcal{V}_n} \pi_n(y) h_{n,x}^y(v)$, and the final inequality is (4.24). The claim of the lemma now follows from Lemma 4.3. □

We now need an upper bound on $\widehat\sigma_n^t(u,\infty)$. For this we proceed as in (4.31)-(4.33) and write that $0 \le \widehat\sigma_n^t(u,\infty) \le \sigma_n^{\circ,t}(u,\infty)$ where, by analogy with (4.33),

$$\sigma_n^{\circ,t}(u,\infty) = k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) \left[\sum_{x\in T_n^\circ} \int_0^{\theta_n} h_{n,x}^y(v) P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u\right) dv \right]^2 \tag{5.18}$$

Again, the quantity in between the square brackets is in $[0,1]$. Thus, splitting the integral into the sum of the integrals over $[0,\kappa_n]$ and $[\kappa_n,\theta_n]$, we get, using (5.9) and reasoning as in (5.16)-(5.17),

$$\sigma_n^{\circ,t}(u,\infty) \le 3\bar\eta_n^{\circ,t}(u,\infty) + \eta_n^{\circ,t}(u,\infty) \tag{5.19}$$

where

$$\bar\eta_n^{\circ,t}(u,\infty) \equiv k_n(t) \sum_{x\in T_n^\circ} \int_0^{\kappa_n} h_{n,x}(v) P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u\right) dv, \tag{5.20}$$

$$\eta_n^{\circ,t}(u,\infty) \equiv k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) \left[\sum_{x\in T_n^\circ} \int_{\kappa_n}^{\theta_n} h_{n,x}^y(v) P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u\right) dv \right]^2 \tag{5.21}$$

The next two propositions bound (5.20) and (5.21) in terms of the quantities $\nu_n^{\circ,t}(u_n,\infty)$ and $\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)$ defined in (4.32) and (4.67), respectively.

Proposition 5.4. *Choose $a_n = 2^{\varepsilon n}/b_n^\circ$ in (1.17). Then, for any sequence $u_n > 0$ such that $0 < u - u_n < n^{-1}$ and all $u > 0$,*

$$\mathbb{P}\left(\bar\eta_n^{\circ,t}(u_n,\infty) \ge t\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty) n^2\theta_n^{-1}\tilde\kappa_n\kappa_n^{1+\alpha_n(\varepsilon)}(nr_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}\right) \le n^{-2}. \tag{5.22}$$

Proposition 5.5. *On $\Omega^\star \cap \Omega_1$, for all but a finite number of indices n and all $u > 0$,*

$$\eta_n^{\circ,t}(u,\infty) \le \nu_n^{\circ,t}(u,\infty)\theta_n r_n^\star n 2^{-n\varepsilon_n}(1+o(1)). \tag{5.23}$$

Proof of Proposition 5.4. As in the proof of Proposition 4.5 denote by $\mathbb{P}^\circ$ the law of the collection $\{\tau_n(x), x \in T_n^\circ\}$ conditional on T_n°. By a first order Tchebychev inequality,

$$\mathbb{P}\left(\bar\eta_n^{\circ,t}(u_n,\infty) \ge \epsilon\right) \le \epsilon^{-1}\mathbb{E}\left[\mathbb{E}^\circ\bar\eta_n^{\circ,t}(u_n,\infty)\right]. \tag{5.24}$$

Note that $\mathbb{E}^\circ\bar\eta_n^{\circ,t}(u,\infty)$ only differs from the term $\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)$ of (4.67) in that the integral in (5.20) is over $[0,\kappa_n]$ instead of $[0,\theta_n]$. Taking $a_n = 2^{\varepsilon n}/b_n^\circ$, a simple adaptation of the proof of (4.62) (see (4.71)-(4.80)) yields

$$\mathbb{E}^\circ\bar\eta_n^{\circ,t}(u_n,\infty) = t(1+o(1))\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)\frac{\mathcal{I}_n(0,\kappa_n)}{\mathcal{I}_n(0,\theta_n)} \tag{5.25}$$

where $\mathcal{I}_n(a,b)$ is defined above (4.37). Eq. (4.39) of Lemma 4.4 was designed precisely to control the ratio in (5.25). Namely, on $\Omega^\circ \cap \Omega^\star$, for all but a finite number of indices n,

$$\frac{\mathcal{I}_n(0,\kappa_n)}{\mathcal{I}_n(0,\theta_n)} \le \frac{\mathcal{I}_n(0,\kappa_n)}{\mathcal{I}_n(\kappa_n,\theta_n-\zeta_n)} \le \theta_n^{-1}\tilde{\kappa}_n\kappa_n^{1+\alpha_n(\varepsilon)}(nr_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}. \tag{5.26}$$

The combination of (5.24), (5.25), and (5.26) gives (5.22). The proof is complete. □

Proof of Proposition 5.5. To prove (5.23) first observe that

$$\sum_{x\in T_n^\circ}\int_{\kappa_n}^{\theta_n} h_{n,x}^y(v)P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n-v)>u\right)dv \le P_y(\kappa_n < H(T_n^\circ)\le\theta_n) \tag{5.27}$$

$$\le (1+o(1))P_{\pi_n}(H(T_n^\circ)\le\theta_n) \tag{5.28}$$

where the last line follows from Proposition 3.3 and the Markov property, and is valid on Ω_1, for all but a finite number of indices n. Applying this bound to one of the two square brackets in (5.21) and using (4.32) to bound the remaining term, we get, under the same assumptions as above, that

$$\eta_n^{\circ,t}(u,\infty) \le (1+o(1))\nu_n^{\circ,t}(u,\infty)P_{\pi_n}(H(T_n^\circ)\le\theta_n). \tag{5.29}$$

Using Corollary (3.20) to bound the last probability yields the claim of the proposition. □

We are now ready to complete the

Proof of Proposition 5.1. Recall from the proof of Proposition 4.1 that on $\Omega_1\cap\Omega^\circ\cap\Omega^\star$ $a_n = 2^{\varepsilon n}/b_n = 2^{\varepsilon n}/b_n^\circ(1+o(1))$ for large enough n and consider (5.22). By (4.5), $n^2\theta_n^{-1}\tilde{\kappa}_n\kappa_n^{1+\alpha_n(\varepsilon)}(nr_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} \downarrow 0$ as $n\uparrow\infty$ and by (4.62), for all $u>0$ and $t>0$ $\lim_{n\to\infty}\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty) = tu^{\alpha(\varepsilon)}$. Thus, by Proposition 5.4 and Borel-Cantelli Lemma we get that for all $u>0$ and $t>0$,

$$\lim_{n\to\infty}\bar{\eta}_n^{\circ,t}(u,\infty) = 0 \quad \mathbb{P}-\text{almost surely}. \tag{5.30}$$

Turning to (5.23) and invoking (4.6) (see also (4.8)), it follows from Proposition 5.4 that for all $0<\varepsilon<1$ and for all $u>0$ and $t>0$,

$$\lim_{n\to\infty}\eta_n^{\circ,t}(u,\infty) = 0 \quad \mathbb{P}-\text{almost surely}. \tag{5.31}$$

Hence by (5.19), for all $u>0$ and $t>0$,

$$\lim_{n\to\infty}\sigma_n^{\circ,t}(u,\infty) = 0 \quad \mathbb{P}-\text{almost surely}. \tag{5.32}$$

From there on the proof is a rerun of the proof of Proposition 4.1 with Lemma 5.2 and Lemma 5.3 playing the role of Lemma 4.2 and Lemma 4.3, respectively. We omit the details. □

6. Verification of Condition (B3)

By (1.18)-(1.20), (1.22), and (1.34), Condition (B3) in (1.33) will be verified if we can establish that:

Proposition 6.1. *Under the assumptions of Proposition 4.1, for all $0<\varepsilon<1$ and all $\beta>\beta_c(\varepsilon)$, $\mathbb{P}$-almost surely,*

$$\lim_{\epsilon\downarrow 0}\limsup_{n\uparrow\infty} k_n(t)E_{\pi_n}\int_0^{\theta_n}\mathcal{M}_n(Y_n(s))\mathbb{1}_{\{\int_0^{\theta_n}\mathcal{M}_n(Y_n(s))ds\le\epsilon\}} = 0, \quad \forall t>0. \tag{6.1}$$

where $\mathcal{M}_n(Y_n(s)) = \max\left((c_n r_n^\star)^{-1}, c_n^{-1}\tau_n(Y_n(s))\right)$.

The Lemma below is central to the proof.

Lemma 6.2. *There are constants* $K, K' < \infty$ *such that for* $\alpha_n(\varepsilon)$ *as in (1.42) and any sequence* $\epsilon_n > 0$ *such that* $i\alpha_c^{-1}(\varepsilon) - 1 - \frac{\log \epsilon_n}{n\beta\beta_c(\varepsilon)} > 0$ *where* $i = 1$ *in (6.2) and* $i = 2$ *in (6.3), we have, for all large enough* n,

$$\mathbb{E}2^{\varepsilon n} c_n^{-1}\tau_n(x)\mathbb{1}_{\{c_n^{-1}\tau_n(x)\le\epsilon_n\}} \le K\frac{\epsilon_n^{1-\alpha_n(\varepsilon)-\frac{\log\epsilon_n}{2n\beta^2}}}{\alpha_c^{-1}(\varepsilon) - 1 - \frac{\log\epsilon_n}{n\beta\beta_c(\varepsilon)}}, \tag{6.2}$$

$$\mathbb{E}\left(2^{\varepsilon n} c_n^{-1}\tau_n(x)\mathbb{1}_{\{c_n^{-1}\tau_n(x)\le\epsilon_n\}}\right)^2 \le K'\frac{\epsilon_n^{2-\alpha_n(\varepsilon)-\frac{\log\epsilon_n}{2n\beta^2}}}{2\alpha_c^{-1}(\varepsilon) - 1 - \frac{\log\epsilon_n}{n\beta\beta_c(\varepsilon)}}. \tag{6.3}$$

Proof of Lemma 6.2. Using standard estimates on the asymptotics of Gaussian integrals (see e.g. [1] p. 932) the claimed result follows from straightforward computations. □

Proof of Proposition 6.1. We assume throughout that $\omega \in \Omega_1 \cap \Omega^\circ \cap \Omega^\star$ and that n is as large as desired. Note that $\mathcal{M}_n(Y_n(s)) \le (c_n r_n^\star)^{-1} + c_n^{-1}\tau_n(Y_n(s))$ and that the contribution to (6.1) coming from the term $(c_n r_n^\star)^{-1}$ if or order $o(1)$. Indeed by (1.17), (1.40), the lower bound on b_n obtained by combining (4.41) and (4.40), the expression (1.5) of c_n, the expression (3.6) of κ_n, and the fact, that follows from (1.6), that $2^n = e^{n\beta_c^2(\varepsilon)/2}$,

$$k_n(t)\theta_n(c_n r_n^\star)^{-1} \le 2tn^4(r_n^\star)^{\alpha_n(\varepsilon)+o(1)}e^{n\beta_c^2(\varepsilon)/2}e^{-n\beta\beta_c(\varepsilon)(1+o(1))} \tag{6.4}$$

and so, for all $0 < \varepsilon < 1$ and $\beta > \beta_c(\varepsilon)$, by virtue of (4.6) (see also (4.8))

$$k_n(t)\theta_n(c_n r_n^\star)^{-1} \le 2tn^4(r_n^\star)^{\alpha_n(\varepsilon)+o(1)}e^{-n\beta_c^2(\varepsilon)(1+o(1))/2} \to 0 \tag{6.5}$$

as $n \to \infty$. To prove Proposition 6.1 it thus suffices to establish that $\mathbb{P}$-almost surely,

$$\lim_{\epsilon\downarrow 0}\limsup_{n\uparrow\infty} k_n(t)E_{\pi_n}\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))ds\le\epsilon\}} = 0, \quad \forall t > 0. \tag{6.6}$$

For T_n as in (2.12) with δ_n given by (4.4), set

$$\mathcal{S}_{n,\epsilon}^{(1)}(t) \equiv k_n(t)E_{\pi_n}\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n\}}\mathbb{1}_{\{\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))ds\le\epsilon\}}ds, \tag{6.7}$$

$$\mathcal{S}_{n,\epsilon}^{(2)}(t) \equiv k_n(t)E_{\pi_n}\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\notin T_n\}}\mathbb{1}_{\{\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))ds\le\epsilon\}}ds. \tag{6.8}$$

To bound $\mathcal{S}_{n,\epsilon}^{(2)}(t)$ simply note that, using (3.4),

$$\mathcal{S}_{n,\epsilon}^{(2)}(t) \le k_n(t)E_{\pi_n}\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{\tau_n(Y_n(s))\le r_n(\varepsilon_n)\}}ds \tag{6.9}$$

$$\le k_n(t)\theta_n 2^{-n}(1+o(1))\sum_{x\in\mathcal{V}_n} c_n^{-1}\tau_n(x)\mathbb{1}_{\{\tau_n(x)\le r_n(\varepsilon_n)\}}. \tag{6.10}$$

Take $\epsilon_n = c_n^{-1}r_n(\varepsilon_n)$ and note that by (2.29), the definition of c_n, and (4.6),

$$-(n\beta\beta_c(\varepsilon))^{-1}\log\epsilon_n = o(1) \text{ and } \left(n^{2(1+c_\star 6^2/\alpha(\varepsilon))}\theta_n\right)^{-1} \le \epsilon_n \le (n^2\theta_n)^{-1}. \tag{6.11}$$

Thus, by Lemma 6.2 and a first order Tchebychev inequality, for all large enough n,

$$\mathbb{P}\left(\mathcal{S}_{n,\epsilon}^{(2)}(t) \ge n^2 t b_n^{-1}(c_n^{-1}r_n(\varepsilon_n))^{1-\alpha(\varepsilon)+o(1)}\right) \le n^{-2}K'' \tag{6.12}$$

for some constant $K'' > 0$. Using the upper bound on ϵ_n of (6.11) and the lower bound on b_n of Lemma 4.4 obtained by combining (4.41) and (4.40),

$$n^2 b_n^{-1}(c_n^{-1} r_n(\varepsilon_n))^{1-\alpha(\varepsilon)+o(1)} \le n^2 \kappa_n(r_n^\star)^{\alpha_n(\varepsilon)+o(1)} \left(n^2\theta_n\right)^{-1+\alpha(\varepsilon)+o(1)} \to 0 \tag{6.13}$$

as $n \to \infty$ by (4.5). Hence by (6.12), (6.13), and Borel-Cantelli Lemma, for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathcal{S}^{(2)}_{n,\epsilon}(t) = 0, \quad \mathbb{P} - \text{almost surely.} \tag{6.14}$$

To deal with $\mathcal{S}^{(1)}_{n,\epsilon}(t)$ we further decompose it into $\mathcal{S}^{(1)}_{n,\epsilon}(t) = \mathcal{S}^{(3)}_{n,\epsilon}(t) + \mathcal{S}^{(4)}_{n,\epsilon}(t)$, where

$$\mathcal{S}^{(3)}_{n,\epsilon}(t) \equiv k_n(t) E_{\pi_n} \int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s)) \mathbb{1}_{\{Y_n(s)\in T_n^\circ\}} \mathbb{1}_{\{\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))ds \le \epsilon\}} ds, \tag{6.15}$$

$$\mathcal{S}^{(4)}_{n,\epsilon}(t) \equiv k_n(t) E_{\pi_n} \int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s)) \mathbb{1}_{\{Y_n(s)\in T_n\setminus T_n^\circ\}} \mathbb{1}_{\{\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))ds \le \epsilon\}} ds. \tag{6.16}$$

Since $\mathcal{S}^{(4)}_{n,\epsilon}(t)$ is non zero only if the event $\{H(T_n \setminus T_n^\circ) \le \theta_n\}$ occurs,

$$\mathcal{S}^{(4)}_{n,\epsilon}(t) \le \epsilon k_n(t) E_{\pi_n} \mathbb{1}_{\{H(T_n\setminus T_n^\circ)\le\theta_n\}}. \tag{6.17}$$

Using assertion (ii) of Corollary 3.8 with $t_n = \theta_n$ as in the proof of Lemma 4.2, we get, assuming (4.6), that on $\Omega^\star$, for all but a finite number of indices n,

$$\mathcal{S}^{(4)}_{n,\epsilon}(t) \le \epsilon k_n(t) \theta_n r_n^\star n 2^{-2n\varepsilon_n}(1+o(1)), \tag{6.18}$$

Proceeding as in (6.13) to bound b_n, (4.6) (see also (4.8)) guarantees that for all $\epsilon > 0$

$$\lim_{n\to\infty} \mathcal{S}^{(4)}_{n,\epsilon}(t) = 0, \quad \mathbb{P} - \text{almost surely.} \tag{6.19}$$

Using next that $\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in A\}} = \sum_{x\in A} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n)$ for any $A \subseteq \mathcal{V}_n$,

$$\mathcal{S}^{(3)}_{n,\epsilon}(t) \le \mathcal{S}^{(5)}_{n,\epsilon}(t) \equiv k_n(t) E_{\pi_n} \sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n) \mathbb{1}_{\{\sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n)\le\epsilon\}}. \tag{6.20}$$

With the notation of (4.15)-(4.17),

$$\mathcal{S}^{(5)}_{n,\epsilon}(t) = k_n(t) \sum_{y\in T_n^\circ} \int_0^{\theta_n} dv h_{n,y}(v) E_y \sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) \mathbb{1}_{\{\sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v)\le\epsilon\}}.$$

We further split the sum over x above into $x = y$ and $x \neq y$. The latter contribution is

$$\mathcal{S}^{(6)}_{n,\epsilon}(t) \equiv k_n(t) \sum_{y\in T_n^\circ} \int_0^{\theta_n} dv h_{n,y}(v) E_y \sum_{x\in T_n^\circ\setminus y} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) \mathbb{1}_{\{\sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v)\le\epsilon\}}.$$

Observing that

$$E_y \sum_{x\in T_n^\circ\setminus y} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) \mathbb{1}_{\{\sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v)\le\epsilon\}} \le \epsilon P_y(H(T_n^\circ \setminus y) \le \theta_n), \tag{6.21}$$

yields the bound $\mathcal{S}^{(6)}_{n,\epsilon}(t) \le \epsilon\widetilde{\nu}_n^t$ where $\widetilde{\nu}_n^t$ is defined in (4.25). Thus by Lemma 4.3, reasoning as in the paragraph below (4.86), we get that for all $\epsilon > 0$

$$\lim_{n\to\infty} \mathcal{S}^{(6)}_{n,\epsilon}(t) = 0, \quad \mathbb{P} - \text{almost surely.} \tag{6.22}$$

It remains to bound $\mathcal{S}^{(5)}_{n,\epsilon}(t) - \mathcal{S}^{(6)}_{n,\epsilon}(t)$. For this we write $\mathcal{S}^{(5)}_{n,\epsilon}(t) - \mathcal{S}^{(6)}_{n,\epsilon}(t) \le \mathcal{S}^{(7)}_{n,\epsilon}(t)$ where

$$\mathcal{S}^{(7)}_{n,\epsilon}(t) \equiv k_n(t) \sum_{y\in T_n^\circ} \int_0^{\theta_n} dv h_{n,y}(v) E_y c_n^{-1}\tau_n(y)\ell_n^y(\theta_n - v) \mathbb{1}_{\{c_n^{-1}\tau_n(y)\ell_n^x(\theta_n - v)\le\epsilon\}}. \tag{6.23}$$

Let us now establish that for b_n° as in (4.36), $\mathcal{S}_{n,\epsilon}^{(7)}(t)$ obeys the following

Lemma 6.3. *Let the sequences a_n, c_n, θ_n be as in Proposition 6.1. Then, under the assumptions and with the notation of Proposition 4.5,*

$$\mathbb{P}^\circ\left(\left|\mathcal{S}_{n,\epsilon}^{(7)}(t) - \mathbb{E}^\circ\mathcal{S}_{n,\epsilon}^{(7)}(t)\right| > t\epsilon^{1/2}n2^{-n(1-\varepsilon)/2}\right) \leq n^{-2}(1+o(1)) \tag{6.24}$$

for all $\epsilon > 0$, and

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\mathbb{E}^\circ\mathcal{S}_{n,\epsilon}^{(7)}(t) = 0. \tag{6.25}$$

Proof of lemma 6.3. The proof closely follows that of Proposition 4.5. We only point out the main differences. The random variables (4.66) are now replaced by

$$X_n(y) \equiv \int_0^{\theta_n} dv h_{n,y}(v) E_y c_n^{-1}\tau_n(y)\ell_n^y(\theta_n - v)\mathbb{1}_{\{c_n^{-1}\tau_n(y)\ell_n^y(\theta_n-v)\leq\epsilon\}} \tag{6.26}$$

and

$$\mathbb{E}^\circ\mathcal{S}_{n,\epsilon}^{(7)}(t) = k_n(t)\textstyle\sum_{y\in T_n^\circ}\mathbb{E}^\circ X_n(y). \tag{6.27}$$

Proceeding as in (4.72)-(4.74) to deal with the conditional expectation and using that $\mathbb{P}(\tau_n(x)\geq r_n(\varepsilon_n)) = \pi_n(T_n^\circ)(1+o(1))$ (see the paragraph below (4.75)), we get

$$\mathbb{E}^\circ\mathcal{S}_{n,\epsilon}^{(7)}(t) = \frac{k_n(t)(1+o(1))}{\pi_n(T_n^\circ)}\sum_{y\in T_n^\circ}\int_0^{\theta_n} dv h_{n,y}(v)E_y\ell_n^y(\theta_n-v)\mathbb{E}^y c_n^{-1}\tau_n(y)\mathbb{1}_{\{c_n^{-1}\tau_n(y)\leq\epsilon_n\}}$$

where $\mathbb{P}^y$ denotes the law of $\tau_n(y)$ and where $\epsilon_n \equiv \epsilon_n(y) = \epsilon/\ell_n^y(\theta_n - v)$. Using (6.2) if $\ell_n^y(\theta_n - v) > \epsilon e^{-n\beta\beta_c(\varepsilon)(\alpha_c^{-1}(\varepsilon)-1)}$ and using that if $\ell_n^y(\theta_n - v) \leq \epsilon e^{-n\beta\beta_c(\varepsilon)(\alpha_c^{-1}(\varepsilon)-1)}$ then

$$E_y\ell_n^y(\theta_n-v)\mathbb{E}^y c_n^{-1}\tau_n(y)\mathbb{1}_{\{c_n^{-1}\tau_n(y)\leq\epsilon_n\}} \leq \epsilon e^{-n\beta\beta_c(\varepsilon)(\alpha_c^{-1}(\varepsilon)-1)}c_n^{-1}e^{n\beta^2/2}, \tag{6.28}$$

we readily see that

$$\begin{aligned}\mathbb{E}^\circ\mathcal{S}_{n,\epsilon}^{(7)}(t) \leq C_1 t\frac{\epsilon^{1-\alpha_n(\varepsilon)-\frac{\log\epsilon}{2n\beta^2}}}{b_n^\circ\theta_n\pi_n(T_n^\circ)}\sum_{y\in T_n^\circ}\int_0^{\theta_n} dv h_{n,y}(v)E_y\widetilde{F}_{\beta,\varepsilon,\epsilon,n}(\ell_n^y(\theta_n-v))\\ + C_2\epsilon n^{\alpha_n(\varepsilon)/2}e^{-n\beta^2/2}k_n(t)(\pi_n(T_n^\circ))^{-1}P_{\pi_n}(H(T_n^\circ)\leq\theta_n)\end{aligned} \tag{6.29}$$

where here and below $C_i > 0$, $i = 1, 2, \ldots$ are constants, and for $F_{\beta,\varepsilon,n}$ as in (1.41),

$$\widetilde{F}_{\beta,\varepsilon,\epsilon,n}(z) = F_{\beta,\varepsilon,n}(z)\frac{z^{\frac{\log\epsilon}{n\beta^2}}\left(1-\frac{\log z}{n\beta\beta_c(\varepsilon)}\right)}{\alpha_c^{-1}(\varepsilon)-1-\frac{\log\epsilon}{n\beta\beta_c(\varepsilon)}+\frac{\log z}{n\beta\beta_c(\varepsilon)}}\mathbb{1}_{\left\{z>\epsilon e^{-n\beta\beta_c(\varepsilon)(\alpha_c^{-1}(\varepsilon)-1)}\right\}}. \tag{6.30}$$

By the leftmost inequality of (4.74) and (4.6), $\widetilde{F}_{\beta,\varepsilon,\epsilon,n}(z) \leq C_3 F_{\beta,\varepsilon,n}(z)$. Thus, by (4.36), the first summand in (6.29) is bounded above by

$$C_4 t\epsilon^{1-\alpha_n(\varepsilon)-\frac{\log\epsilon}{2n\beta^2}}. \tag{6.31}$$

Using (3.20) and proceeding as in (6.4) to bound $k_n(t)$, the second summand is bounded above by

$$C_5 t e^{-n(\beta^2-\beta_c^2(\varepsilon))/2}\kappa_n n^{\alpha_n(\varepsilon)/2+1}(r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} \to 0 \tag{6.32}$$

as $n\to\infty$ by virtue of (3.6), (1.38), and the assumption that $\beta > \beta_c(\varepsilon)$ where $0 < \varepsilon < 1$. Note in particular that $\lim_{n\to\infty}\alpha_n(\varepsilon) = \alpha(\varepsilon) < 1$. Hence, inserting (6.31) and (6.32) in (6.29) and passing to the limit

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\mathbb{E}^\circ\mathcal{S}_{n,\epsilon}^{(7)}(t) = 0, \quad \forall t > 0. \tag{6.33}$$

This proves (6.25). Turning to the variance we have, as in (4.68), by independence, that

$$\mathbb{V}^\circ(\mathcal{S}^{(7)}_{n,\epsilon}(t)) \equiv \mathbb{E}^\circ(\mathcal{S}^{(7)}_{n,\epsilon}(t) - \mathbb{E}^\circ \mathcal{S}^{(7)}_{n,\epsilon}(t))^2 \le k_n^2(t) \sum_{y\in T_n^\circ} \mathbb{E}^\circ(X_n(y))^2. \tag{6.34}$$

Proceeding as in the proof of (6.29) but using (6.3) and the line below (6.30), we get that

$$\begin{aligned}\mathbb{V}^\circ(\mathcal{S}^{(7)}_{n,\epsilon}(t)) \le C_6 t^2 \frac{\epsilon^{2-\alpha_n(\varepsilon)-\frac{\log\epsilon}{2n\beta^2}}}{(b_n^\circ\theta_n)^2\pi_n(T_n^\circ)} \sum_{y\in T_n^\circ}\left(\int_0^{\theta_n} dv h_{n,y}(v) E_y F_{\beta,\varepsilon,\epsilon,n}(\ell_n^y(\theta_n - v))\right)^2 \\ + C_7\epsilon n^{\alpha_n(\varepsilon)/2} e^{-n\beta\beta_c(\varepsilon)} \frac{k_n^2(t)\theta}{\pi_n(T_n^\circ)} \sum_{y\in T_n^\circ}\left(\int_0^{\theta_n} dv h_{n,y}(v)\right)^2.\end{aligned}$$

From the bound $\int_0^{\theta_n} dv h_{n,y}(v) E_y F_{\beta,\varepsilon,\epsilon,n}(\ell_n^y(\theta_n - v)) \le (1+o(1))\int_0^{\theta_n} dv h_{n,y}(v)\theta_n^{\alpha_n(\varepsilon)} \le (1+o(1))\theta_n^{\alpha_n(\varepsilon)} P_{\pi_n}(H(y)\le\theta_n)$ and (3.18), (4.41), we get that on $\Omega^\star$, for all but a finite number of indices n, the first summand is bounded above by

$$C_8 t^2 \epsilon^{2-\alpha_n(\varepsilon)-\frac{\log\epsilon}{2n\beta^2}} \left(n\kappa_n\theta_n^{\alpha_n(\varepsilon)}(r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}\right)^2 2^{-n}. \tag{6.35}$$

Using the bound $\sum_{y\in T_n^\circ}\left(\int_0^{\theta_n} dv h_{n,y}(v)\right)^2 \le \sup_{y\in T_n^\circ} P_{\pi_n}(H(y)\le\theta_n) P_{\pi_n}(H(T_n^\circ)\le\theta_n)$, and proceeding as in (6.32), the second summand is bounded above by

$$C_9 t^2 \epsilon n^{\alpha_n(\varepsilon)/2}\left(n^2\kappa_n(r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}\right)^2 \theta_n e^{-n\beta_c(\varepsilon)(\beta-\beta_c(\varepsilon))} 2^{-n}. \tag{6.36}$$

Since by assumption $\beta > \beta_c(\varepsilon)$ and $0 < \varepsilon < 1$, (4.6) (see also (4.8)) enables us to conclude that on $\Omega^\star$, for all large enough n,

$$\mathbb{V}^\circ(\mathcal{S}^{(7)}_{n,\epsilon}(t)) \le C_{10} t^2 \epsilon 2^{-n(1-\varepsilon)}. \tag{6.37}$$

This yields (6.24) and concludes the proof of the Lemma. □

Arguing as in the proof of Proposition 4.1 that $b_n = b_n^\circ(1+o(1))$ on $\Omega_1\cap\Omega^\circ\cap\Omega^\star$ for all large enough n, it follows from Lemma 6.3 and Borel-Cantelli Lemma that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\left(\mathcal{S}^{(5)}_{n,\epsilon}(t) - \mathcal{S}^{(6)}_{n,\epsilon}(t)\right) = 0, \quad \mathbb{P}-\text{almost surely.} \tag{6.38}$$

Collecting (6.14), (6.19), (6.22) and (6.38) yields (6.6). The proof of Proposition 6.1 is complete. □

7. Proof of Theorem 1.1 and Theorem 1.4

Proof of Theorem 1.4. By Proposition (3.3), Proposition (4.1), Proposition (5.1) and Proposition (6.1), under the assumptions of Proposition (4.1) and Proposition (6.1), Conditions (B0), (B1), (B2), and (B3) of Theorem 1.3 are satisfied $\mathbb{P}$-a.s.. It remains to check Condition (A0), i.e. to prove that $\mathbb{P}$-a.s., for all $u > 0$,

$$\lim_{n\to\infty} P_{\mu_n}(Z_{n,1} > u) = 0 \tag{7.1}$$

where $Z_{n,1} = \int_0^{\theta_n} \max\left((c_n r_n^\star)^{-1}, c_n^{-1}\tau_n(Y_n(s))\right) ds$ and μ_n is the uniform measure on $\mathcal{V}_n$. By (3.3) and (3.4)

$$\begin{aligned} P_{\mu_n}(Z_{n,1} > u) &\le (1+o(1))P_{\pi_n}(Z_{n,1} > u) + \textstyle\sum_{x\in\overline{V}_n^\star}\mu_n(x)P_x(Z_{n,1} > u) &(7.2)\\ &\le (1+o(1))P_{\pi_n}(Z_{n,1} > u) + n^{-c_\star}(1+o(1)) &(7.3)\end{aligned}$$

where the last line is (2.18). Thus (7.1) is an immediate consequence of Proposition (4.1). One readily checks that the assumptions on a_n, c_n, and θ_n of the theorem imply that the

conditions (4.5) and (4.6) of Proposition (4.1) are verified. The proof of 1.4 is complete. □

Proof of Theorem 1.1. Reasoning as in the proof of Theorem 1.4, we may assume that the process starts in its invariant measure π_n. The main idea behind the proof is now classical. Suppose that

$$\mathcal{P}_{\pi_n}\left(A_n(t,s)\right) = P_{\pi_n}\left(\{\mathcal{R}_n \cap (t,t+s) = \emptyset\}\right) + o(1) \tag{7.4}$$

where $A_n(t,s) \equiv \{X(c_n t) = X(c_n(t+s))\}$ and where $\mathcal{R}_n$ denotes the range of the rescaled blocked clock process $S_n^b(t)$. Then Theorem 1.1 is a direct consequence of Theorem 1.4 and the arcsine law for stable subordinators. We refer to Ref. [23] for a detailed proof of this statement (see the proof of Theorem 1.6 therein) and focus on establishing (7.4). For $k \geq 1$ and $Z_{n,i}$ as in (1.19) set

$$\mathcal{B}_k = \left\{\sum_{i=1}^k Z_{n,i} < t, \sum_{i=1}^{k+1} Z_{n,i} > t+s\right\}. \tag{7.5}$$

Then by (1.18), $\{\mathcal{R}_n \cap (t,t+s) \neq \emptyset\} = \{\cup_{k\geq 1}\mathcal{B}_k\}$. Furthermore, for any $T > 0$,

$$P_{\pi_n}\left(\cup_{k>k_n(T)}\mathcal{B}_k\right) \leq P_{\pi_n}\left(S_n^b(T) < t\right) \underset{n\to\infty}{\longrightarrow} P\left(V_{\alpha(\varepsilon)}(T) < t\right) \leq \delta \tag{7.6}$$

where convergence is almost sure in the random environment as follows from Theorem 1.4, and where δ can be made as small as desired by taking T large enough. Therefore

$$0 \leq P_{\pi_n}\left(\{\mathcal{R}_n \cap (t,t+s) = \emptyset\}\right) - P_{\pi_n}\left(\cup_{1\leq k\leq k_n(T)}\mathcal{B}_k\right) \leq \delta. \tag{7.7}$$

Note that the event $\mathcal{B}_k$ is non empty if and only if the increment $Z_{n,k+1}$ straddles over the interval $(t,t+s)$. To show that (7.4) holds it now suffices to establish the following two facts:

Fact 1. Almost surely in the random environment, with overwhelming probability, non-empty events $\mathcal{B}_k$, $k \leq k_n(T)$, are produced by visits of the process Y_n to the set T_n° and, more precisely, by (many) visits of the process to one and the same element of T_n°, no other element of T_n° being visited in the time interval $(t,t+s)$. This implies that $\mathbb{P}$-a.s.

$$P_{\pi_n}\left(A_n(t,s) \cap \{\cup_{1\leq k\leq k_n(T)}\mathcal{B}_k\}\right) \geq P_{\pi_n}\left(\cup_{1\leq k\leq k_n(T)}\mathcal{B}_k\right) + o(1) \tag{7.8}$$

Fact 2. If $\mathcal{B}_k$ and $\mathcal{B}'_k$, $1 \leq k \neq k' \leq k_n(T)$, are two non-empty events then, almost surely in the random environment they are produced by visits to two distinct elements of T_n° with overwhelming probability. This implies that $\mathbb{P}$-a.s.

$$P_{\pi_n}\left(A_n(t,s) \cap (\cap_{1\leq k\leq k_n(T)}\mathcal{B}_k^c)\right) \to 0, \quad n\to\infty \tag{7.9}$$

Combining (7.7), (7.8), and (7.9) then establishes that

$$\left|P_{\pi_n}\left(A_n(t,s)\right) - P_{\pi_n}\left(\{\mathcal{R}_n \cap (t,t+s) = \emptyset\}\right)\right| \leq \delta + o(1) \tag{7.10}$$

which is tantamount to (7.4).

The proofs of Facts 1 and 2 mostly use information already obtained in the course of the verification of Conditions (B1)-(B3). We present them succinctly below, beginning with the proof of Fact 1. Fix $0 < T < \infty$ and assume that the assumption of Proposition (4.1) are satisfied. Let $H_k(A) = \inf\{t \geq \theta_n k \mid Y_n(t) \in A\}$ be the first hitting time of $A \subseteq \mathcal{V}_n$ after time $\theta_n k$. Note first that $\mathcal{B}_k = \mathcal{B}_k \cap \{Z_{n,k+1} > s\}$ and thus, by (4.10),

$$P_{\pi_n}\left(\cup_{1\leq k\leq k_n(T)}(\mathcal{B}_k \cap \{H_k(T_n) > \theta_n\})\right) = 0 \tag{7.11}$$

for all large enough n. Note next that reasoning as in (6.17)-(6.19), on $\Omega^\circ \cap \Omega^\star$,

$$P_{\pi_n}\left(\cup_{1\leq k\leq k_n(T)}(\mathcal{B}_k \cap \{H_k(T_n \setminus T_n^\circ) \leq \theta_n\})\right) \leq k_n(T) P_{\pi_n}\left(H_k(T_n \setminus T_n^\circ) \leq \theta_n\right) \to 0$$

as $n \to \infty$ by virtue of (4.6). Hence on $\Omega^\circ \cap \Omega^\star$, for all large enough n,

$$\begin{aligned} &P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}\mathcal{B}_k\right)\\ =&P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}(\mathcal{B}_k\cap\{H_k(T_n^\circ)\le\theta_n\}\cap\{H_k(T_n\setminus T_n^\circ)>\theta_n)\}\right)+o(1). \end{aligned} \tag{7.12}$$

This means that for $\mathcal{B}_k$ to be non-empty, the increment $Z_{n,k+1}$ must be produced by visits of Y_n to T_n°, and T_n° only. To prove that all these visits, if there are several of them, must be to a single vertex it suffices to show that as $n \to \infty$,

$$p_n \equiv P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}(\mathcal{B}_k\cap\{H_k(T_n^\circ)\le\theta_n\}\cap\mathcal{C}_n(Y_n(H_k(T_n^\circ))))\right)\to 0, \tag{7.13}$$

where

$$\mathcal{C}_n(Y_n(H_k(T_n^\circ)))\equiv\left\{\inf\{t>H_k(T_n^\circ)\mid Y_n(t)\in T_n^\circ\setminus Y_n(H_k(T_n^\circ))\}\le\theta_n\right\}. \tag{7.14}$$

Now,

$$\begin{aligned} p_n &= P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}\cup_{x\in T_n^\circ}\left(\mathcal{B}_k\cap\{H_k(T_n^\circ)\le\theta_n, Y_n(H_k(T_n^\circ))=x\}\cap\mathcal{C}_n(x)\right)\right.\\ &\le \widetilde{\nu}_n^T \end{aligned} \tag{7.15}$$

where $\widetilde{\nu}_n^T$ is defined in (4.25) and bounded in Lemma 4.3. Reasoning as in the paragraph below (4.86) then yields that under the assumptions (4.5) and (4.6), on $\Omega^\circ \cap \Omega^\star$, $\lim_{n\to\infty}\widetilde{\nu}_n^T = 0$. Fact 1 is now proved.

Fact 2 will be established if we can prove that as $n \to \infty$,

$$\bar{p}_n \equiv P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}\left(\{H_k(T_n^\circ)\le\theta_n\}\cap\mathcal{D}_{n,k}(Y_n(H_k(T_n^\circ)))\right)\right)\to 0, \tag{7.16}$$

where

$$\mathcal{D}_n(Y_n(H_k(T_n^\circ)))\equiv\left\{\inf\{t>(k+1)\theta_n\mid Y_n(t)=Y_n(H_k(T_n^\circ))\}\le\theta_n k_n(T)\right\}. \tag{7.17}$$

To prove this observe that the event in (7.16) can be written as

$$\cup_{x\in T_n^\circ}\cup_{y\in T_n^\circ}\left(\{H_k(T_n^\circ)\le\theta_n, Y_n(H_k(T_n^\circ))=x\}\cap\{Y_n(\theta_n(k+1))=y\}\cap\mathcal{D}_{n,k}(x)\right)$$

Thus, by the Markov property we have, using the notation of (4.15)-(4.17) and the bound $P_y\left(H(x)\le\theta_n(k_n(T)-(k+1))\right)\le P_y\left(H(x)\le\theta_n k_n(T)\right)$,

$$\bar{p}_n\le\sum_{1\le k\le k_n(T)}\sum_{x\in T_n^\circ}\sum_{y\in T_n^\circ}\int_0^{\theta_n}dv h_{n,x}(v)P_x\left(Y_n(\theta_n-v)=y\right)P_y\left(H(x)\le\theta_n k_n(T)\right).$$

To proceed, we split the domain of integration into $[0,\theta_n-\kappa_n)\cup[\theta_n-\kappa_n,\theta_n]$. Using that by Proposition 3.3, on Ω_1, for all n large enough, $P_x\left(Y_n(\theta_n-v)=y\right)=\pi_n(y)(1+o(1))$ for all $v\in[0,\theta_n-\kappa_n)$, the contribution coming from this domain is at most

$$(1+o(1))\sum_{1\le k\le k_n(T)}\sum_{x\in T_n^\circ}\int_0^{\theta_n}dv h_{n,x}(v)\sum_{y\in T_n^\circ}\pi_n(y)P_y\left(H(x)\le\theta_n k_n(T)\right) \tag{7.18}$$

$$\le(1+o(1))k_n(T)P_{\pi_n}(H(T_n^\circ)\le\theta_n)\sup_{y\in T_n^\circ}P_{\pi_n}(H(y)\le\theta_n k_n(T)) \tag{7.19}$$

$$\le(1+o(1))\left(\theta_n k_n(T)r_n^\star n2^{-n}\right)^2 2^n\pi_n(T_n^\circ) \tag{7.20}$$

where we used (3.20) with $t_n=\theta_n$ (which is licit as we many times saw) and (3.18) with $t_n=\theta_n k_n(T)$, which is licit provided that $\theta_n k_n(T)r_n^\star n2^{-n}\to 0$ as $n\to\infty$, and this is guaranteed by our assumptions on a_n. Indeed, proceeding as in the proof of Proposition 4.1 (see (4.81) and the paragraph above) we get that on $\Omega^\circ\cap\Omega^\star\cap\Omega_1$, for large enough n,

$$\theta_n k_n(T)r_n^\star n2^{-n}\le\kappa_n(r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}n2^{-(1-\varepsilon)n}\to 0 \tag{7.21}$$

as $n \to \infty$ for all $0 < \varepsilon < 1$. Since furthermore $2^n \pi_n(T_n^\circ) = (1 + o(1))2^{(1-\varepsilon)n} \left(n^2\theta_n\right)^{\alpha(\varepsilon)}$ by (4.4), (3.3), and (2.20), and we get that on $\Omega^\circ \cap \Omega^\star \cap \Omega_1$, (7.20) is bounded above by

$$(1 + o(1)) \left(\kappa_n (r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} n\right)^2 \left(n^2\theta_n\right)^{\alpha(\varepsilon)} 2^{-(1-\varepsilon)n}, \tag{7.22}$$

and by (4.6) this decays to zero as $n \to \infty$ for all $0 < \varepsilon < 1$.

Consider next the domain $[\theta_n - \kappa_n, \theta_n]$ and note that since

$$\sum_{y \in T_n^\circ} P_x \left(Y_n(\theta_n - v) = y\right) P_y \left(H(x) \le \theta_n k_n(T)\right) \le 1 \tag{7.23}$$

the corresponding contribution is bounded above by $k_n(T) P_{\pi_n} \left(\theta_n - \kappa_n \le H(T_n^\circ) \le \theta_n\right)$. By the upper bound of (3.6) and the lower bound of (3.5), on $\Omega^\star$, for all but a finite number of indices n, this is in turn bounded above by

$$n^{1+2\alpha_n(\varepsilon)} \theta_n^{-(1-\alpha(\varepsilon))} \kappa_n^2 (r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} \to 0 \tag{7.24}$$

as $n \to \infty$, where we again used that $2^{n\delta_n} = \left(n^2\theta_n\right)^{\alpha(\varepsilon)}$ by (4.4) whereas $0 < \alpha(\varepsilon) < 1$ by assumption; the final convergence then follows from (4.5). Combining the conclusions of (7.21) and (7.24) we get that on $\Omega^\circ \cap \Omega^\star \cap \Omega_1$,

$$\lim_{n \to \infty} \bar{p}_n = 0. \tag{7.25}$$

This concludes the proof of Fact 2. The proof of Theorem 1.1 is now complete. □

8. Appendix: Proof of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2. The proof closely follows that of Theorem 1.2 of Ref. [13]. Throughout we fix a realization $\omega \in \Omega$ of the random environment but do not make this explicit in the notation. We set

$$\widehat{S}_n^b(t) \equiv S_n^b(t) - Z_{n,1}. \tag{8.1}$$

Condition (A0) ensures that $S_n^b - \widehat{S}_n^b$ converges to zero, uniformly. Thus we must show that under Conditions (A1), (A2), and (A3),

$$\widehat{S}_n^b \Rightarrow_{J_1} S_\nu. \tag{8.2}$$

For this we rely on Theorem 1.1 of Ref. [13]. (This result is itself a specialized form of Theorem 4.1 of Ref. [20] suited to the present setting.) Namely, we want to show that Conditions (A1), (A2), and (A3) imply the conditions of Theorem 1.1 of Ref. [13].

To this end let $\{\mathcal{F}_{n,i}, n \ge 1, i \ge 0\}$ be the array of sub-sigma fields of $\mathcal{F}^Y$ defined (with obvious notation) through $\mathcal{F}_{n,i} = \sigma\left(Y_n(s), s \le \theta_n i\right)$, for $i \ge 0$. Note that for each n and $i \ge 1$, $Z_{n,i}$ is $\mathcal{F}_{n,i}$ measurable and $\mathcal{F}_{n,i-1} \subset \mathcal{F}_{n,i}$. Next observe that by the Markov property and the fact that, for all $i \ge 1$ and $y \in \mathcal{V}_n$, $\mathcal{P}_y(Z_{n,i} > u) = \mathcal{P}_y(Z_{n,1} > u)$,

$$\mathcal{P}_{\mu_n}\left(Z_{n,i} > u \,\middle|\, \mathcal{F}_{n,i-1}\right) = \sum_{y \in \mathcal{V}_n} \mathbb{1}_{\{Y_n((i-1)\theta) = y\}} \mathcal{P}_y(Z_{n,1} > u). \tag{8.3}$$

In view of this, (1.21), (1.22), and (1.23)

$$\sum_{i=2}^{k_n(t)} \mathcal{P}_{\mu_n}\left(Z_{n,i} > u \mid \mathcal{F}_{n,i-1}\right) = \nu_n^{Y,t}(u, \infty), \tag{8.4}$$

and in view of (1.24)

$$\sum_{i=2}^{k_n(t)} \left[\mathcal{P}_{\mu_n}\left(Z_{n,i} > u \mid \mathcal{F}_{n,i-1}\right)\right]^2 = \sigma_n^{Y,t}(u, \infty). \tag{8.5}$$

From (8.4) and (8.5) it follows that Conditions (A1) and (A2) of Theorem 1.2 are exactly the conditions of Theorem 1.1 of Ref. [13]. Similarly Condition (A3) is condition (1.9). Therefore the conditions of Theorem 1.1 of Ref. [13] are verified, and so $\widehat{S}_n^b \Rightarrow_{J_1} S_\nu$ in $D([0,\infty))$ where S_ν is a subordinator with Lévy measure ν and zero drift. □

The proof of Theorem 1.3 centers of the

Proposition 8.1. *Assume that Condition (B1) is satisfied. Then, choosing $\theta_n \geq \kappa_n$, the following holds for all initial distributions μ_n: for all $t > 0$, all $u > 0$, and all $\epsilon > 0$,*

$$P_{\mu_n}\left(\left|\nu_n^{Y,t}(u,\infty) - \nu_n^t(u,\infty)\right| \geq \epsilon\right) \leq 5\epsilon^{-2}\left[\rho_n\left(\nu_n^t(u,\infty)\right)^2 + \sigma_n^t(u,\infty)\right], \tag{8.6}$$

and

$$P_{\mu_n}\left(\sigma_n^{Y,t}(u,\infty) \geq \epsilon\right) \leq \epsilon^{-1}(1+\rho_n)\sigma_n^t(u,\infty). \tag{8.7}$$

Proof of Proposition 8.1. We assume throughout that $\theta_n \geq \kappa_n$. To prove (8.7), simply note that by a first order Tchebychev inequality

$$\begin{aligned} P_{\mu_n}\left(\sigma_n^{Y,t}(u,\infty) \geq \epsilon\right) &\leq \epsilon^{-1}k_n(t)\textstyle\sum_{y\in\mathcal{V}_n} E_{\mu_n}(\pi_n^{Y,t}(y))\left[Q_n^u(y)\right]^2 &(8.8)\\ &\leq \epsilon^{-1}(1+\rho_n)\sigma_n^t(u,\infty), &(8.9)\end{aligned}$$

where we used in the last line that by (1.30),

$$|E_{\mu_n}(\pi_n^{Y,t}(y)) - \pi_n(y)| \leq \rho_n\pi_n(y). \tag{8.10}$$

Turning to (8.6), a second order Chebychev inequality yields

$$\begin{aligned} &P_{\mu_n}\left(\left|\nu_n^{Y,t}(u,\infty) - \nu_n^t(u,\infty)\right| \geq \epsilon\right)\\ &\leq \epsilon^{-2}E_{\mu_n}\left[k_n(t)\textstyle\sum_{y\in\mathcal{V}_n}\left(\pi_n^{Y,t}(y) - \pi_n(y)\right)Q_n^u(y)\right]^2\\ &= \epsilon^{-2}\textstyle\sum_{x\in\mathcal{V}_n}\sum_{y\in\mathcal{V}_n} Q_n^u(x)Q_n^u(y)\sum_{i=1}^{k_n(t)-1}\sum_{j=1}^{k_n(t)-1}\Delta_{ij}(x,y) \end{aligned} \tag{8.11}$$

where

$$\begin{aligned} \Delta_{ij}(x,y) \equiv\ & P_{\mu_n}\left(Y_n(i\theta_n) = x, Y_n(j\theta_n) = y\right) + \pi_n(x)\pi_n(y)\\ & - \pi_n(y)P_{\mu_n}\left(Y_n(i\theta_n) = x\right) - \pi_n(x)P_{\mu_n}\left(Y_n(j\theta_n) = y\right). \end{aligned} \tag{8.12}$$

Using again (1.31) yields

$$|\Delta_{ij}(x,y)| \leq \begin{cases} \rho_n(4+\rho_n)\pi_n(x)\pi_n(y), & \text{if } i \neq j,\\ (1+\rho_n)\pi_n(x) + (1+2\rho_n)\pi_n^2(x), & \text{if } i = j \text{ and } x = y,\\ 0 & \text{else.}\end{cases} \tag{8.13}$$

Thus (8.11) is bounded above by

$$\epsilon^{-2}\rho_n(4+\rho_n)\Big[k_n(t)\sum_{y\in\mathcal{V}_n}\pi_n(y)Q_n^u(y)\Big]^2 + \epsilon^{-2}(2+3\rho_n)k_n(t)\sum_{y\in\mathcal{V}_n}\pi_n(y)\left[Q_n^u(y)\right]^2 \tag{8.14}$$

Since by assumption $\rho_n \downarrow 0$ as $n \uparrow \infty$, (8.14) is tantamount to the right-hand side of (8.6). Proposition 8.1 is proven. □

Proof of Theorem 1.3. The proof of Theorem 1.3 is now immediate: Condition (B2) combined with the conclusions of Proposition 8.1 implies both conditions (A1) and (A2), and Condition (B3) combined with (8.10) implies Condition (A3). □

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