Stochastic LQ Control Under Asymptotic Tracking for Discrete Systems over Multiple Lossy Channels

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Abstract: This paper addresses the asymptotic tracking problem subjected to linear quadratic constraints for linear discrete-time systems, where packet dropout occurs in actuating channels. In order to solve this objective control problem, the controller-coding co-design approach is adopted, i.e., the controller, encoder and decoder are designed for taking full advantage of the network resource collaboratively, thereby achieving better transmission of control signals. A stabilizability condition in the mean square sense that reveals the fundamental limitation among the \( H_2 \) norm of the plant, data arrival rates and coding matrices is first derived. Then, a solvability condition is conducted to handle the additional stochastic LQ control objective by a modified discrete-time algebraic Riccati equation, and an iterative algorithm is also given for designing the corresponding state feedback gain and coding matrices. Relied on such design, the asymptotic tracking constraint is further fulfilled through solving a Sylvester equation, and the feedforward gain related to tracking is parameterized. Finally, a simulation with the implementation of the design method on two cooperative robots is included to show the effectiveness of the current results.

1 Introduction

Unlike classical digital control systems, networked control systems (NCSs) are closed-loop systems where the plants, the sensors, the actuators and controllers are coordinated through certain form of communication network. The main features of NCSs are low cost, high reliability, ease of maintenance and expansion [1–3]. Meanwhile numerous special issues on NCSs have been concerned by many researchers, inspired by wide applications of NCSs in cooperative vehicles [4, 5], sensor networks [6, 7], multi-agent systems [8, 9] and so on. In NCSs, the imperfect communication channels can introduce various constraints and uncertainties, for instance, packet dropouts [10, 11], time delays [12, 13], fading [14, 15], limited data rates [16, 17] and quantization [18, 19], etc.

The stabilizability of NCSs with networked constraints and uncertainties is a wildly attentional research field and numerous results have been reported in the literatures. For instance, the Lyapunov function approach is applied in [20] to reach the coarsest quantization density required for the stabilization of the single-input discrete-time linear time-invariant system with logarithmic quantized state feedback. Authors in [21] consider a stabilization problem with actuating channel subjecting to packet dropout, and necessary and sufficient conditions are determined in terms of the packet dropout probability and the spectral radius of the system matrix. [22] solves the stabilization problem for single-input system with both packet dropout and logarithmic quantization and the above-mentioned problem for single-input single-output system is solved in [23] by computing two algebraic Riccati equations and an algebraic Riccati inequality, where the tradeoff between the robust stability and the robust performance is revealed. Furthermore, [24] gives the solution for the multi-input multi-output case. The output regulation problem is considered in a cooperative and distributed scheme with constraints as switching network topology in [25]. The regulation problem is alternatively tackled by using input/output weighting filters, and the classical internal model principle is extended to the co-called comprehensive admissibility in [26]. Moreover, in order to find the least total channel capacity achieving stabilization for multi-input NCSs, the technique of channel resource allocation is proposed in [27] and it is appealed to deal with the constraints as signal-to-noise ratio (SNR), logarithmic quantization and fading [28, 29]. For channel resource allocation, the designers are endowed with the freedom to allocate the capacities among different input channels with the total capacity of the communication network being given. When it comes to the case where sub-channels’ capacities are assumed to be fixed static, a coding and controller co-design method is put forward in [30] to obtain the stabilizability condition for multi-input systems under signal-to-noise ratio constraint.

Linear quadratic (LQ) control has been possessing an indispensable role in systems and control theory. Such problem for deterministic systems has been extensively investigated, and reaches a mature state in the 1970s [31]. The optimal control problem for systems subjected to stochastic perturbations and network-induced constraints has equally received considerable attention from the control community. For example, [32] addresses the LQ control problem for networked control systems subjected to data rate constraints, where a state feedback control scheme is employed in order to achieve the minimum data rate for mean square stabilization of the system. To addresses the LQ problem for Itô-type stochastic systems with input delays, Wang and Zhang introduce a forward-backward stochastic differential equation based approach for obtaining the optimal controller [33]. [34] is concerned with the stochastic LQ control for Markovian jumping systems by using the averaging approach to aggregate states according to their jump rates. [35] studies the LQ performance of systems where control signals are subject to packet dropout, and both zero-control and hold strategies are discussed. Authors in [36] study the LQ control problem for discrete-time linear systems over single packet-dropping link and a controller-coding co-design method is proposed in order to deal with the lost packet, where estimates are used to replace the missing measurements.

To the best of authors’ knowledge, the difficulties in designing NCSs mainly focus on the following aspects. First, analytical solution to controller synthesis of multi-input NCSs turns out to be an essential \( \mu \)-synthesis problem [24, 28]. Second, controller synthesis problem with multiple coupled factors becomes significantly complicated and harder to cope with [22, 26]. Finally, it is usually a considerable and difficult problem to combine the proposed results with actual application requirements. Motivated by the above...
discussion, in this paper, the stochastic LQ control under asymptotic tracking is considered for multi-input-multi-output (MIMO) discrete-time system with packet dropouts. Inspired by [30], a pair of linear encoder and decoder is introduced respectively in the transmitting side and the receiving side of the channels in order to take full advantage of the communication resource and achieve a better signal transmission. The sufficient conditions are deduced for co-designing of the controller and the coding matrices such that the tracking problem is solved and the stochastic LQ performance is guaranteed. Finally, a simulation with the implementation of the design method is solved and the stochastic LQ performance is guaranteed.

Problem formulation

The overall setup of the control problem is depicted in Fig. 1, where the controller connects to the discrete-time plant through encoder, multiple lossy channels and decoder. The plant is described by the following discrete-time space-space representation:

\[
\begin{align*}
\begin{cases}
    x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \\
    z(k) = Cx(k) + Du(k),
\end{cases}
\end{align*}
\]

(1)

where \(x(k) \in \mathbb{R}^n\) is the state, \(u(k) \in \mathbb{R}^m\) the control input, \(z(k) \in \mathbb{R}^p\) the controlled output for tracking certain reference. Assume that the matrix pair \((A, B)\) is stabilizable.

The control law \(v(k)\) is given by

\[
v(k) = F_1 r(k) + F_2 x(k),
\]

(2)

where \(F_1 \in \mathbb{R}^{m \times q}\) is the feedforward gain related to the tracking reference and \(F_2 \in \mathbb{R}^{m \times n}\) is the state feedback gain related to the LQ performance defined later, \(r(k) \in \mathbb{R}^q\) is the state of the following exo-system:

\[
\begin{align*}
\begin{cases}
    r(k+1) = A r(k), \quad r(0) = r_0, \\
    z_r(k) = C_r r(k),
\end{cases}
\end{align*}
\]

(3)

with \(z_r(k) \in \mathbb{R}^p\) being the reference. It is known that the exo-system is an autonomous system that generates the reference signal to be tracked and the disturbance to be rejected. More detailed descriptions about exo-systems may be found in [37–39]. Hence, the tracking error can be defined as:

\[
e(k) = z(k) - z_r(k).
\]

(4)

Moreover, \(l\) unreliable channels are placed in the path from the controller to the plant. In some existing results, e.g., [28, 29], the number of channels is equal to the number of control signals, and the capacities of each sub-channels subject to certain lower bounds are assumed to be flexible under total capacity constraint. Instead, in this paper the capacity of each sub-channel is fixed a priori with an appropriate coding strategy to improve the efficiency of network resource. Therefore, different from the above works, the lower bounds of sub-channels’ capacities are decreased, and relatively less expensive sub-channels may therefore be used. In this case, the number of channels is assumed to be more than the number of control signals, i.e., \(l > m\).

Then, a pair of encoder \(\mathcal{E} \in \mathbb{R}^{l \times m}\) and decoder \(\mathcal{D} \in \mathbb{R}^{m \times l}\) is introduced on each side of the \(l\) channels to give a degree of freedom for fully utilizing the resource of all the multiple channels. This transmission process is specified by the input \(v(k)\) and output \(u(k)\) as

\[
u(k) = \mathcal{D} \alpha(k) \mathcal{E} v(k),
\]

(5)

where \(\alpha(k) = \text{diag} [\alpha_1(k), \ldots, \alpha_l(k)]\) with \(\alpha_i(k), i = 0, 1, \ldots, l\) being Bernoulli variables to specify the unreliable transmission process of each channel. Here \(\alpha_i(k)\) take value in \(\{0, 1\}\) at any time instant \(k\). \(\alpha_i(k) = 1\) indicates that the transmission succeeds, otherwise \(\alpha_i(k) = 0\). The probability of successful transmission is given as \(E[\alpha_i(k)] = \beta_i\) with \(\beta_i \in (0, 1)\). As known, encoder and decoder are generally viewed as a pair of invertible operators, hence the design requirement on \(\mathcal{E}\) and \(\mathcal{D}\) is to satisfy the following constraint:

\[
\mathcal{E} \Xi \mathcal{E} = I,
\]

(6)

where \(\Xi = \text{diag} [\beta_1, \ldots, \beta_l]\).

\[\text{Fig. 1: System setup}\]
Combining (1), (3) and (5), the closed-loop system is obtained as:
\[ x(k + 1) = [A + B\varphi(x(k))E]x(k) + B\varphi(x(k))F_1r(k), \]
\[ e(k) = [C + D\varphi(x(k))E]x(k) + [D\varphi(x(k))E]F_1 - C\varphi(r(k)). \]

Furthermore, we define the cost functional \( J(x, u) \):
\[
J(x, u) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{x^T(k)Qx(k) + u^T(k)Ru(k)\},
\]
where the matrices \( Q \geq 0 \) and \( R > 0 \). And the following definition is introduced.

**Definition 1.** [14] The closed-loop system (7) is said to be mean square (MS) stable if for any bounded initial state \( x(0) \) and \( r(k) = 0 \), \( \lim_{k \to \infty} E\{x(k)E^T(k)\} = 0 \).

Now the problem under consideration is described as follows:

**Problem 1.** Find the feedforward gain \( F_1 \), the feedback gain \( F_2 \) in (2) and a pair of encoder \( \mathcal{E} \) and decoder \( \mathcal{D} \) in (5) such that the closed-loop system (7) meets the following requirements:

- \( \mathcal{R}_1 \) (MS stability): the closed-loop system (7) is MS stable;
- \( \mathcal{R}_2 \) (LQ performance): the cost functional in (8) is minimized when \( r(k) = 0 \);
- \( \mathcal{R}_3 \) (Tracking performance): for any bounded initial state \( x(0) \), \( r(0) \), the tracking error defined in (4) satisfies \( \lim_{k \to \infty} E\{e(k)\} = 0 \).

Problem 1 is defined as a multi-objective control problem with the requirements detailed in \( \mathcal{R}_1 \)-\( \mathcal{R}_3 \). In the next section, corresponding solvability conditions are successively given to fulfill the aforementioned requirements, and based on them, a complete solution to Problem 1 is further presented.

### 3 Main results

#### 3.1 MS stabilization

In this subsection, we are focused on MS stabilization. In other words, we consider Problem 1 with the requirements \( \mathcal{R}_2 \) and \( \mathcal{R}_3 \) being removed.

**Theorem 1.** \( \mathcal{R}_1 \) in Problem 1 is satisfied, if there exist \( U \in \mathbb{R}^{l \times m} \) with \( U^TU = I \), \( X > 0 \), \( W > 0 \), and \( Y \) such that the following conditions hold
\[
\begin{align*}
\{UWU^T\} & < \frac{1}{\beta_2^{-1} - 1}, \\
\begin{bmatrix} W & Y \\ * & X \end{bmatrix} & > 0, \\
\begin{bmatrix} X & AX + BY \\ * & X \end{bmatrix} & > 0.
\end{align*}
\]

Then the state feedback gain \( F_2 \) in (2), encoder \( \mathcal{E} \) and decoder \( \mathcal{D} \) in (5) are constructed respectively as
\[
F_2 = YX^{-1},
\]
\[
\mathcal{E} = \Xi^{-1/2}U, \quad \mathcal{D} = U^T\Xi^{-1/2},
\]
with \( \Xi = \text{diag} [\beta_1, \ldots, \beta_l] \).

The following lemmas are introduced here for the derivation of Theorem 1.

**Lemma 1.** [14] Given an internally stable system \( G \) with dimension \( p \times p \) and a structured random process \( \Delta(k) = \text{diag} [\Delta_1(k), \ldots, \Delta_p(k)] \), such that \( E\{\Delta_i(k)\} = 0 \), \( E\{\Delta_i^2(k)\} = \tau_i^2 \), \( i = 1, 2, \ldots, p \), then the feedback interconnection in Fig. 2 is MS stable if and only if the following condition holds \( \rho\{\phi(G)\Phi\} < 1 \), where \( \Phi = \text{diag} [\tau_1^2, \ldots, \tau_p^2] \).

[Fig. 2: System with stochastic perturbation]

**Lemma 2.** Given an internally stable system \( G \) with dimension \( p \times p \), there holds \( \rho\{\phi(G)\} = \inf_{D} \|D^{-1}GD\|_{2,1} \), where \( D \) is a diagonal matrix with all diagonal elements being positive.

**Proof:** As represented by Lemma 2.2 in [40], for nonnegative matrix \( M \), there holds
\[
\rho\{M\} = \inf_{\|D\|_{p}} \|D^{-1}MD\|_{p},
\]
for \( 1 \leq p \leq \infty \). Thus, there holds
\[
\rho\{\phi(G)\} = \inf_{D} \|D^{-1}\phi(G)D\|_{1},
\]
and by the definition of \( \|\cdot\|_{2,1} \), there holds
\[
\|D^{-1}\phi(G)D\|_{1} = \|D^{-1}GD\|_{2,1}^{2}.
\]

Then it is observed that
\[
\rho\{\phi(G)\} = \inf_{D} \|D^{-1}GD\|_{2,1}^{2}.
\]

\[\square\]

Now, we are in the position to prove Theorem 1.

**Proof of Theorem 1:**
Note that for the Bernoulli process \( \alpha_i(k) \) mentioned in (5), \( i = 1, 2, \ldots, l \), we have
\[
E\{\alpha_i(k)\} = \beta_i \neq 0, \quad E\{(\alpha_i(k) - \beta_i)^2\} = \beta_i(1 - \beta_i).
\]

Setting new random variables as \( \phi_i(k) = [\alpha_i(k) - \beta_i]/\beta_i \), yields
\[
E\{\phi_i(k)\} = 0, \quad E\{\phi_i^2(k)\} = \beta_i^{-1} - 1, \quad i = 1, 2, \ldots, l.
\]

Then the closed-loop system in (7) is rewritten as
\[
x(k + 1) = (A + BF_2 + B\mathcal{D}(\Phi(k))\mathcal{E}F_2)x(k),
\]
where \( \Xi = \text{diag} [\beta_1, \ldots, \beta_l] \), \( \Phi(k) = \text{diag} [\phi_1(k), \ldots, \phi_l(k)] \).
The above closed-loop system can be rewritten as:

\[
x(k+1) = (A + BF_2)x(k) + B\mathcal{G}u(k),
\]

\[
y(k) = \hat{\varepsilon} F_2 x(k),
\]

\[
w(k) = \Phi(k) y(k).
\]

Then it can be seen as the interconnection of two parts in Fig. 2 with

\[G = (A + BF_2, B\mathcal{G}, \varepsilon F_2, 0), \Delta(k) = \Phi(k),\]

where the system \(G\) is stable with (11) holding.

With the help of Lemma 1, it is easy to indicate that the closed-loop system (7) is MS stable if there exists a feedback gain \(F_2\), an encoder/decoder pair \(\varepsilon\) and \(\hat{\varepsilon}\) such that

\[
\rho(\bar{T}\hat{\varepsilon}^2) < 1
\]

(14)

where \(\bar{T} = \phi(T(z))\), \(T(z) = \varepsilon F_2(zI - A - BF_2)^{-1}B\mathcal{G}\), \(\bar{\varepsilon} = \text{diag} \left[\left(\beta_1^{-1} - 1\right)^\frac{1}{2}, \cdots, \left(\beta_i^{-1} - 1\right)^{\frac{1}{2}}\right]\).

Next, we will show that under the conditions (9)-(11), (14) holds with \(F_2\), \(\varepsilon\) and \(\hat{\varepsilon}\) given in (12) and (13). To this end, \(T(z)\) is first rewritten as

\[T(z) = \Xi^{-1/2} F_2 (zI - A - BF_2)^{-1} B\mathcal{G} \Xi^{1/2}.
\]

Then, it is observed by Lemma 2 that

\[
\rho(\bar{T}\hat{\varepsilon}^2) = \inf_{D} \left\| D^{-1} T(z) \bar{\varepsilon} D \right\|^2_{2,1}.
\]

Next, we will show

\[
\rho(\bar{T}\hat{\varepsilon}^2) \leq \left\| D^{-1} T(z) \bar{\varepsilon} D \right\|^2_{2,1} < 1,
\]

with \(D\) being selected as \(D = \Xi^{-1/2}\).

By the definition, \(\left\| \Xi^{1/2} T(z) \bar{\varepsilon} \Xi^{-1/2} \right\|^2_{2,1}\) can be rewritten as

\[
\left\| \Xi^{1/2} T(z) \bar{\varepsilon} \Xi^{-1/2} \right\|^2_{2,1} = \max \left\{ \beta \Xi^{-1/2} T H(e^{jw}) T(e^{jw}) \Xi^{-1/2} \beta \right\}_{ii} dw
\]

\[
= \max \left\{ \beta U \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} T H(e^{jw}) T(e^{jw}) U^T \beta \right)_{ii} \right\},
\]

where \(T(z) = F_2 (zI - A - BF_2) B\) satisfying

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} T H(e^{jw}) T(e^{jw}) dw = F_2^T W_0 F_2
\]

with \(W_0\) being the solution to equation

\[
(A + BF_2)^T W_0 (A + BF_2) - W_0 + BB^T = 0.
\]

By Schur complement, (11) and (12) indicate that

\[
(A + BF_2)^T X (A + BF_2) - X + BB^T < 0,
\]

and (10) and (12) indicate that

\[
W > F_2^T X F_2.
\]

According to III. C in [41], it can be concluded that

\[
W > F_2^T X F_2 > F_2^T W_0 F_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} T H(e^{jw}) T(e^{jw}) dw.
\]

Therefore, it can be concluded with (9) that

\[
\left\| \Xi^{1/2} T(z) \bar{\varepsilon} \Xi^{-1/2} \right\|^2_{2,1} = \max \left\{ \beta U \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} T H(e^{jw}) T(e^{jw}) dw \right) U^T \beta \right\}_{ii} < 1.
\]

Then, it is clear that

\[
\rho(\bar{T}\hat{\varepsilon}^2) \leq \left\| \Xi^{1/2} T(z) \bar{\varepsilon} \Xi^{-1/2} \right\|^2_{2,1} < 1.
\]

Consequently, (14) holds. Therefore, the closed-loop system (7) is MS stable.

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**3.2 Linear quadratic MS stabilization**

In this subsection, we are concerned with LQ performance under MS stability, that is, Problem 1 with the requirement R3 being removed.

Before proceeding, we rewrite the closed-loop system (7) with \(r(k)\) been removed as

\[
x(k+1) = Ax(k) + B\mathcal{G} [I + \Phi(k)] \varepsilon F_2 x(k),
\]

where \(\Phi(k) = \text{diag} \{\phi_1(k), \ldots, \phi_i(k)\}, \phi_i(k) = |\alpha_i(k) - \beta_i|/\beta_i, i = 1, 2, \ldots, l\). And the control law \(v(k)\) is rewritten as

\[v(k) = F_2 x(k).
\]

**Theorem 2.** R1 and R2 in Problem 1 are satisfied if there exist solution \(P > 0\) and \(U \in \mathbb{R}^{l \times n}\) with \(U^T U = I\) to the following modified discrete-time algebraic Riccati equation (MDARE)

\[
P = A^T P A + Q - A^T P B U^T \Xi^{-1/2} U B^T P A,
\]

where

\[
\Xi = [(\Xi \mathcal{H} + \Sigma) \circ (R + \Xi^{1/2} U^T B U^T \Xi^{1/2})],
\]

\[
\Sigma = \text{diag} \{\beta_1(1 - \beta_1), \cdots, \beta_l(1 - \beta_l)\},
\]

\[
\mathcal{H} = \text{diag} \{\beta_1, \cdots, \beta_l\},
\]

\(H\) denotes a constant matrix with all elements being 1. Then the feedback gain \(F_2\) is given by

\[F_2 = - U^T \Xi^{-1/2} \Sigma^{-1/2} U B^T P A,
\]

the encoder and decoder are given as

\[\varepsilon = \Xi^{-1/2} U, \ \mathcal{G} = U^T \Xi^{-1/2}.
\]
Proof. First let us consider the MS stability of the closed-loop system (7) with the feedback gain $F_2$ given in (18), encoder $\mathcal{E}$ and decoder $\mathcal{D}$ given in (19). To this end, we define

$$V[x(k)] = Tr\{E\{x(k)x^T(k)\}P\}.$$

For the closed-loop system

$$x(k + 1) = Ax(k) + B\mathcal{E}[I + \Phi(k)]\mathcal{E}F_2x(k),$$

it can be concluded that

$$V[x(k + 1)] = Tr\{E\{x(k + 1)x^T(k + 1)\}P\}$$

$$= Tr\{E\{x(k)x^T(k)[A + B\mathcal{E}F_2]^TP(A + B\mathcal{E}F_2)]\} + Tr\{E\{x(k)x^T(k)\}[B\mathcal{E}\Phi(k)\mathcal{E}F_2]^TP(B\mathcal{E}\Phi(k)\mathcal{E}F_2)]\} + Tr\{E\{x(k)x^T(k)\}^TPA\} + Tr\{E\{x(k)x^T(k)\}(\mathcal{E}F_2)^TS^{-1}(\mathcal{E}F_2)\}.$$ 

where $\bar{S} = [(\mathcal{E}H + \Sigma) \odot (\mathcal{E}^TUB^TPBU^T\mathcal{E}^T)].$

Substituting (17), (18) and (19) into the above formula, yields

$$V[x(k + 1)] = V[x(k)] - Tr\{E\{x(k)x^T(k)\}Q\} - Tr\{E\{x(k)x^T(k)\}(\mathcal{E}F_2)^T[(\mathcal{E}H + \Sigma) \odot R]^{-1}(\mathcal{E}F_2)\} < V[x(k)].$$

Therefore, $\lim_{k \to \infty} E\{x(k)x^T(k)\} = 0$, which indicates that the closed-loop system (7) is MS stable.

For further considering the functional $J(x, u)$ in (8), we define

$$V_1[x(k)] = x^T(k)P_kx(k),$$

$$z_1(k) = \left[\begin{array}{c} Q^{\frac{1}{2}}x(k) \\ R^{\frac{1}{2}}u(k) \end{array}\right].$$

Noticing

$$z_1(k) = \left[\begin{array}{c} Q^{\frac{1}{2}}x(k) \\ R^{\frac{1}{2}}\mathcal{E}(I + \Phi(k))\mathcal{E}F_2x(k) \end{array}\right],$$

there holds

$$\|z_1(k)\|^2 + V_1(k + 1) - V_1(k)$$

$$= x^T(k)Qx(k) + \bar{v}^T(k)\mathcal{E}Tz^T(k)R\mathcal{E}\mathcal{E}v(k) + \bar{v}^T(k)\Phi(k)\mathcal{E}Tz^T(k)R\mathcal{E}\mathcal{E}\mathcal{E}v(k)$$

$$+ 2\bar{v}^T(k)\mathcal{E}Tz^T(k)R\mathcal{E}\mathcal{E}\mathcal{E}v(k) + 2x^T(k)\mathcal{E}TPAx(k) + \bar{v}^T(k)\mathcal{E}Tz^T(k)B^TPB\mathcal{E}\mathcal{E}v(k)$$

$$+ \bar{v}^T(k)\mathcal{E}Tz^T(k)B^TPB\mathcal{E}\mathcal{E}\mathcal{E}v(k) + 2x^T(k)\mathcal{E}TPB\mathcal{E}\mathcal{E}v(k) + 2x^T(k)\mathcal{E}TPB\mathcal{E}\mathcal{E}\mathcal{E}v(k) + 2x^T(k)\mathcal{E}TPB\mathcal{E}\mathcal{E}\mathcal{E}v(k) + \bar{v}^T(k)\mathcal{E}Tz^T(k)B^TPB\mathcal{E}\mathcal{E}\mathcal{E}v(k) + x^T(k)Px(k).$$

Note that the random variable $\Phi(k)$ is independent of the control law $v(k)$, which is a static linear function of the state $x(k)$. Hence, taking expectations from both sides, together with (17), yields

$$E\{\|z_1(k)\|^2\} + E\{V_1(k + 1)\} - E\{V_1(k)\}$$

$$= E\{(\mathcal{E}F_2x(k))^T\bar{S}\mathcal{E}F_2x(k)\}$$

$$+ E\{x^T(k)[A^T\mathcal{E}Tz^T(k)\Sigma^{-1}\mathcal{E}^TUB^TP]Ax(k)\}$$

$$+ E\{2x^T(k)[A^T\mathcal{E}Tz^T(k)\Sigma^{-1}\mathcal{E}^TUB^TP]x(k)\}.$$ 

where $\bar{S} = [(\mathcal{E}H + \Sigma) \odot (R + \mathcal{E}^TUB^TPBU^T\mathcal{E}^T)].$ There holds

$$J(x, u) = E\{\|z_1(k)\|^2\}$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{x^T(k)Qx(k) + u^T(k)R\bar{u}(k)\}$$

$$= \|\Sigma^{\frac{1}{2}}\mathcal{E}F_2 + \Sigma^{-1}\mathcal{E}^TUB^TPAx(k)\|^2 + x^T(k)Px(k).$$

It is easy to conclude that $J(x, u)$ is minimized if

$$\mathcal{E}F_2 = -\Sigma^{-1}\mathcal{E}^TUB^TPA.$$

Noticing that $\mathcal{E}^T\mathcal{E} = I$, $\mathcal{E} = \Sigma^{-1/2}U$ and $\mathcal{D} = U^T\Sigma^{-1/2}$, multiply $\mathcal{E}^T\mathcal{E}$ at both sides of the above equation and the state feedback gain $F_2$ is given by $F_2 = -U^T\Sigma^{\frac{1}{2}}\Sigma^{-1/2}U^T\Sigma\Sigma^{-1/2}U^TPA$. 

The MDARE (17) in Theorem 2 is solvable, if the closed-loop system (7) is MS stabilizable via state feedback [29, 42]. It is well known that solving modified Riccati equation in the stochastic setting with close-form solution is still an open problem [29, 42]. To be worse, the MDARE (17) also includes the matrix $U$, which makes the corresponding solvability condition even harder. In Algorithm 1, we propose an iterative method for numerical solution to MDARE (17) with a fixed $U$ matrix.

Algorithm 1

Step 1: give the number of iteration $N_1$. Set $k = 0$ and the initial condition $P(0) = 0$.

Step 2: solve conditions (9)-(11) and obtain a feasible matrix $U$.

Step 3: update $P(k)$ as

$$P(k + 1) = A^T P(k)A + Q - A^T PBU^T \Sigma^{-1} \mathcal{E}^T U^T P A,$$

with $\Sigma = [(\mathcal{E}H + \Sigma) \odot (R + \mathcal{E}^T UB^TPBU^T\mathcal{E}^T)].$

Step 4: set $k = k + 1$, go to Steps 3. If $\|P_{k+1} - P_k\| < \epsilon$ with $\epsilon$ being a small positive number, stop. If $k > N_1$, stop.

3.3 Additional asymptotic tracking

In this subsection, a sufficient condition is given for designing the feedforward gain $F_1$ in (2) such that the tracking requirement $R^3$ defined in Problem 1 is fulfilled.
Theorem 3. R3 in Problem 1 is satisfied if there exist matrices $X_r$ and $Y_r$ such that the following Sylvester equation holds:

$$X_r A_r = AX_r + BY_r,$$

$$0 = CX_r + DY_r - C_r.$$

Moreover, the feedforward gain $F_1$ is given as:

$$F_1 = Y_r - F_2 X_r,$$

where $F_2$ is any feedback gain that ensures the MS stability of the closed-loop system (7).

Proof: Substitute (21) into (20), yields

$$X_r A_r = (A + BF_2)X_r + BF_1,$$

$$0 = (C + DF_2)X_r + DF_1 - C_r.$$

By defining $\bar{x}(k) = x(k) - X_r r(k)$ for the closed-loop system (7), it is observed that

$$x(k + 1) - X_r r(k + 1) = A x(k) + B r(k) - X_r r(k + 1)$$

$$= (A + BF_2) x(k) + BF_1 r(k) + B D \bar{x}(k) + F_2 x(k)$$

$$+ B D \bar{x}(k) F_1 r(k) - X_r A r(k),$$

where $A_r = A + BF_2 \bar{x}(k) F_2 r(k)$. Substituting (21) and (22) into the above formula, yields

$$\bar{x}(k + 1)$$

$$= x(k + 1) - X_r r(k + 1)$$

$$= (A + BF_2) x(k) + BF_1 r(k) + B D \bar{x}(k) + F_2 x(k)$$

$$+ B D \bar{x}(k) F_1 r(k)$$

By defining $\bar{x}(k) = x(k) - X_r r(k)$,

$$\lim_{k \to \infty} E[\bar{x}(k)] = 0$$

$$= \lim_{k \to \infty} E[(C + D F_2) x(k) - (C - F_1) r(k)]$$

$$= \lim_{k \to \infty} E[(C + D F_2) x(k) + (-C_r + D F_1) r(k)].$$

By defining $\bar{x}(k) = x(k) - X_r r(k)$,

$$\lim_{k \to \infty} E[\bar{x}(k)] = \lim_{k \to \infty} E[(C + D F_2) x(k) - (C + D F_2) x_r(k)$$

$$+ (C + D F_2) x_r(k) + (-C_r + D F_1) r(k)]$$

$$= \lim_{k \to \infty} E[(C + D F_2) \bar{x}(k)]$$

$$+ \lim_{k \to \infty} E[(C X_r + D F_2 X_r + D F_1 - C_r) r(k)].$$

Note that $E[(B D \bar{x}(k) F_1 r(k))] = 0$, then $\lim_{k \to \infty} E[\bar{x}(k)] = 0$. According to (6) and $e(k)$ defined in (7)

$$\lim_{k \to \infty} E[e(k)]$$

$$= \lim_{k \to \infty} E[(C + D F_2) x(k) + (-C_r + D F_1) r(k)]$$

$$= \lim_{k \to \infty} E[(C + D F_2) x(k) + (-C_r + D F_1) r(k)].$$

Note that $\lim_{k \to \infty} E[\bar{x}(k)] = 0$ and $\lim_{k \to \infty} E[e(k)] = 0$. Thus, $\lim_{k \to \infty} E[e(k)] = 0$ and the tracking requirement R3 in Problem 1 is satisfied.

Theorem 3 presents the condition for satisfying the requirement R3 in Problem 1. The corresponding feedforward gain $F_1$ given in (21) is structured with $X_r$, $Y_r$, and $F_2$, among which $F_2$ is the feedback gain related to MS stability of the closed-loop system (7) and can be previously computed by Theorem 1 or Theorem 2. So far, all the conditions needed to solve the Problem 1 have already been presented. Therefore, the following complete condition is given for solving the Problem 1.

Theorem 4. The Problem 1 is solvable if there exist matrices $U \in \mathbb{R}^{l \times m}$ satisfying $U T U = I$, $X > 0$, $W > 0$, $Y$, $P > 0$, $X_r$, $Y_r$, such that the conditions (9)-(11), (17) and (20) hold. Then the feedforward gain $F_1$ is given in (21), the feedback gain $F_2$ is given in (18), the encoder $\bar{e}$ and decoder $\mathcal{D}$ are given in (19).

Proof: The proof can be obviously obtained by combination of Theorem 1, Theorem 2 and Theorem 3.

Algorithm 2

Step 1: give the number of iteration $N_1$. Set $i = 0$, $k = 0$. Get a feasible $U$ by solving inequalities (9)-(11).

Step 2: compute MDARE (17) with $U$ in Step 1 by Algorithm 1 and structure the feedback gain $F_2$ in (18).

Step 3: calculate $W_i$ by solving equation (15) with $F_2$ in Step 2 and set $W = W_i$.

Step 4: find $U = U(0), U(1), \cdots, U(N)$ satisfying (9) with $W$ in Step 3 and set $c = 0$, $j = 0$, $P(0) = 0$. If $N = 0$, stop.

Step 5: update $P(k)$ as Step 3 in Algorithm 1 with $U = U(c)$.

Step 6: set $k = k + 1$, and go to Step 5. If $|P(k + 1) - P(k)| < \epsilon$ with $\epsilon$ being a small positive number, set $P_j = P(k + 1), U_j = U(c), j = j + 1, c = c + 1, k = 0$ and go to Step 5. If $k > N_1$, set $k = 0$, $c = c + 1$ and go to Step 5. If $c = N$, go to Step 7.

Step 7: find the matrix $P_j$, that minimizing $x^T(0) P_j x(0)$, where $j^* = 1, 2, \cdots, j$ and set $P_i = P_{j^*}, U_i = U_{j^*} , i = 1, 2, \cdots, N_1$

Step 8: if $|x^T(0) P_j - P_{j-1} x(0)| < \epsilon$ with $\epsilon$ being a small positive number, obtain the feedback gain $F_2$ with $P = P_j$, $U = U_i$ as in (18), stop.

Step 9: set $i = i + 1$, go to Step 3.

4 Application

In this section, the main results proposed in section 3 is applied to a cooperative control platform with two robots to show the specific co-design process and to verify the effectiveness of the results.
4.1 Platform description

As depicted in Fig. 3, the control platform consists of two cooperative mobile robots, one camera and a supervisor. The objective of this platform is to drive the two cooperative tank-like mobile robots in order to complete tasks such as manipulating a load too cumbersome and heavy for only one of the robots. The supervisor can measure the absolute position and velocity of each robot thanks to cameras without delays and communicate control signals to each robot through a wireless protocol.

![Diagram of the cooperative control platform](image)

**Fig. 3:** Setup of the cooperative control platform

The control objectives for the platform depicted in Fig. 3 can be formulated as follows:

- The barycenter \( G(x_G; y_G) \) of the two robots must be driven along a pre-defined path.
- A constant interdistance \( d \) between the two robots must be ensured. The interdistance \( d \) is defined here as the euclidian norm between centers of robot \( 1 (x_1; y_1) \) and robot \( 2 (x_2; y_2) \) as

\[
\begin{aligned}
x_G &= (x_1 + x_2)/2, \\
y_G &= (y_1 + y_2)/2, \\
d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}
\]  

To feet with (23), the control objectives can be defined through the 4 following signals to be controlled:

\[
\begin{aligned}
x_G &= (x_1 + x_2)/2, \\
y_G &= (y_1 + y_2)/2, \\
\Delta x &= x_1 - x_2, \\
\Delta y &= y_1 - y_2.
\end{aligned}
\]  

The reference trajectory can be generated directly by a human joystick by using some robotic trajectory generation algorithms, etc. The trajectory generator provides a vector signal \( \mathcal{X} \) as:

\[ \mathcal{X}(k) = [x_G^\text{ref}(k) \ y_G^\text{ref}(k) \ \Delta x^\text{ref}(k) \ \Delta y^\text{ref}(k)]^T \]

4.2 Control strategy formulation

The whole control implementation is depicted in Fig. 4. The two robots with their local feedback linearization law are set as the plant. The camera sends the measured absolute position and velocity of each robot to the supervisor with a sampling period tuned at 0.1 s. The controller implemented by the supervisor consists of a static feedback gain \( \mathcal{D} \) and a static feedforward gain \( \mathcal{F} \), leading to a new linearizing feedback control law is implemented \([43]\), leading to a new input-output mapping (see Fig. 3) based on two decoupled integrator chains.

\[
\text{Ref}_i(\text{lin}) = \begin{cases} 
\frac{x_i}{a_1^i} = \frac{1}{s^2}, \\
\frac{y_i}{a_2^i} = \frac{1}{s^2}.
\end{cases}
\]

The two new control inputs for each robot \( (a_1^i; a_2^i) \) are homogeneous to the acceleration of the robot. Finally, without taking into consideration the decoder \( \mathcal{D} \), embedded in each robot, the plant model, after exact discretization at \( T_{\text{sample}} = 0.1 \text{s} \), is given by

\[
\dot{x}_p = A_p x_p + B_p u_p,
\]  

with

\[
\begin{aligned}
x_p^T &= [x_1 \dot{x}_1 y_1 \dot{y}_1 x_2 \dot{x}_2 y_2 \dot{y}_2]^T, \\
u_p^T &= [u_1^p u_2^p u_3^p u_4^p]^T = [a_1^2 a_1^2 a_2^2 a_2^2]^T, \\
A_p &= \text{diag}[A_1, \ldots, A_4] , \\
B_p &= \text{diag}[b_1, \ldots, b_4], \\
A_i &= \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} , \\
b_i &= \begin{bmatrix} 0.005 \\ 0.100 \end{bmatrix} , \\
i &= 1, 2, 3, 4.
\end{aligned}
\]

Such sampling period \( T_{\text{sample}} \) leads to a good compromise between the dynamic of the robots, and the capacities of the supervisor (video tracking algorithm in particular) and of the wireless protocol (e.g. Zigbee or Bluetooth).

The standard model defined in (1) is linked to the discretize plant model (25), and the matrices for the output regulation problem are
defined as
\[ A = A_p, \quad B = B_p, \]
\[ C = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad D = 0. \]

4.3 Trajectory tracking problem formulation

The trajectory tracking problem is considered here and it is reformulated to fit with the theoretical results proposed in this paper. In this simulation, the reference trajectory \( z_r(k) \) for the gravity center \( G \) is a circle with radius 10 meters, and the robots should travel around this trajectory with a fixed interdistance of 2\( m \), at a fixed velocity of 1\( m/s \). The travel period is thus around 62.8s. This leads in discrete-time to
\[ z_r(k) = C_r r(k) = \begin{cases} x^r_G(k) = 10\sin(0.1\pi k/20), \\ y^r_G(k) = 10\cos(0.1\pi k/20), \\ \Delta^r_x(k) = 2\sin(0.1\pi k/20), \\ \Delta^r_y(k) = 2\cos(0.1\pi k/20). \end{cases} \] (26)

The 4 reference signals are clearly coupled, thus a discrete-time state-space representation based on (3) can be designed with parameters
\[ A_r = \begin{bmatrix} 0.99995 & -0.00800 \\ 0.01250 & 0.99995 \end{bmatrix}, \quad C_r = \begin{bmatrix} 0 & 10 & 0 & 2 \\ 8 & 0 & -1.6 & 0 \end{bmatrix}^T. \]

4.4 Numerical results

In this subsection, some Matlab simulations are considered for the above mentioned platform. The robots with their local feedback linearization are modeled according to (25). The MIMO channels are modeled with data arrival rates \( \beta_1 = 0.9, \beta_2 = 0.7, \beta_3 = 0.5, \beta_4 = 0.5, \beta_5 = 0.3, \beta_6 = 0.1 \). To fit with the theoretical results proposed in section 3, three different control strategies are considered for the robots platform in this subsection:

- 1. the controller set \( (F_2 = F_2^{MS}, U = U^{MS}, F_1 = F_1^{MS}) \) is obtained thanks to Theorem 1 and Theorem 3 without considering the LQ performance requirement R2;
- 2. the set \( (F_2 = F_2^{LQ1}, U = U^{LQ1}, F_1 = F_1^{LQ1}) \) and \( (F_2 = F_2^{LQ2}, U = U^{LQ2}, F_1 = F_1^{LQ2}) \) corresponding to two different weighting parameters for LQ functional (8) are obtained by Theorem 4 with a specific \( U \);
- 3. the sets \( (F_2 = F_2^{TT1}, U = U^{TT1}, F_1 = F_1^{TT1}), (F_2 = F_2^{TT2}, U = U^{TT2}, F_1 = F_1^{TT2}) \) are obtained by solving Theorem 4 with different weighting parameters for LQ functional (8), and the iterative design method proposed in Algorithm 2 is adopted to reduce the conservatism of the results.

MS stabilization

In this part, the MS stabilization feedback gain \( F_2 = F_2^{MS} \) is designed for the cooperative robots platform by Theorem 1 without considering the LQ performance, and the corresponding feedforward gain \( F_1 = F_1^{MS} \) is computed by Theorem 3. With \( F_2^{MS} \) and \( F_1^{MS} \), a simulation is introduced to show the tracking performance of the cooperative robots.

For Theorem 1, the matrix \( U^{MS} \) is given as:
\[ U^{MS} = \begin{bmatrix} -0.2823 & -0.4024 & -0.5113 & -0.2098 \\ -0.4515 & -0.1390 & 0.5950 & 0.3477 \\ -0.6033 & -0.0410 & 0.2043 & -0.0217 \\ -0.3672 & 0.7786 & -0.3076 & 0.2797 \\ -0.4164 & -0.3496 & -0.3614 & -0.0106 \\ -0.2103 & 0.2973 & 0.2327 & -0.8696 \end{bmatrix}. \]

Then the feedback gain \( F_2^{MS} \) associated to the matrix \( U^{MS} \) is obtained by solving (9)-(11) as:
\[ F_2^{MS} = \text{diag} [f_1, \ldots, f_4], \]
with
\[ f_1 = f_2 = f_3 = f_4 = [-0.3397, -1.1151]. \]

The feedforward gain \( F_1^{MS} \) corresponding to \( F_2^{MS} \) and \( U^{MS} \) is obtained by Theorem 3 as:
\[ F_1^{MS} = \begin{bmatrix} 1.2341 & 2.9099 \end{bmatrix}, \quad \begin{bmatrix} 3.6374 & -0.9873 \\ 1.0997 & 2.3808 \\ 2.9761 & -0.8078 \end{bmatrix}. \]

For the simulation, a unit step signal \( w(t) \) is added to the position of robot 1 (state \( x_1 \) in (25)) from time 30s to 42s to simulate some kind of disturbance factors. The initial positions of robot 1 and robot 2 are (0, 11) and (0, 9) respectively, and the two robots are asked to follow a circular trajectory with diameter 10\( m \), an inter-distance of 2\( m \) and a fixed velocity 1\( m/s \).

Based on \( F_2^{MS}, U^{MS} \) and \( F_1^{MS} \), Fig. 5 shows the tracking performance and the evolution of the tracking error.

As depicted in Fig. 5, the set \( (F_2^{MS}, U^{MS}, F_1^{MS}) \) can ensure an acceptable tracking performance. However, the two robots are decoupled due to the diagonal feedback gain \( F_2^{MS} \). Therefore, when the robot 1 affected by the disturbance, the robot 2 will not respond accordingly as shown in Fig. 5. This problem will be solved in the next part by further considering the LQ performance R2.

Linear quadratic MS stabilization

In this part, Theorem 4 is adopted to solve the tracking problem for the cooperative robots platform and the LQ performance is considered. To this end, two different sets of parameters are defined for the cost functional (8) as:
1. Case 1: more weighting on center error as:
\[ Q_1 = C^T \text{diag } [100, 100, 1, 1] \ C, \ R_1 = I. \]

2. Case 2: more weighting on distance error as:
\[ Q_2 = C^T \text{diag } [1, 1, 100, 100] \ C, \ R_2 = I. \]

For the specific matrix given as \( U^{LQ1} = U^{LQ2} = U^{MS} \), the set \((F^{LQ1}_1, U^{LQ1}, K^{LQ1}_1)\) associated to weighting matrices \((Q_1, R_1)\) as well as the set \((F^{LQ2}_2, U^{LQ2}, K^{LQ2}_1)\) associated to weighting matrices \((Q_2, R_2)\) are obtained by solving Theorem 4:

\[
F^{LQ1}_1 = \begin{bmatrix}
-2.7936 & 0.0344 & -1.6725 & 0.0975 \\
-2.0847 & 0.1264 & -0.6345 & 0.1643 \\
0.2188 & -3.1688 & -0.1092 & -1.2658 \\
0.1356 & -2.4344 & 0.0313 & -0.6619 \\
-1.6302 & 0.0840 & -2.8591 & -0.2314 \\
-0.6324 & 0.0440 & -2.1721 & -0.1826 \\
0.3509 & -2.1336 & -0.4160 & -1.6617 \\
0.1770 & -0.7053 & -0.1918 & -1.3351
\end{bmatrix}^T
\]

\[
F^{LQ1}_2 = \begin{bmatrix}
-2.7036 & 36.0047 \\
53.7705 & -3.6574 \\
8.2581 & 35.1215 \\
28.7722 & -0.7438
\end{bmatrix}
\]

\[
F^{LQ2}_1 = \begin{bmatrix}
-1.8522 & -0.1214 & 1.2829 & 0.5351 \\
-1.6998 & -0.1218 & 0.6285 & 0.2763 \\
-0.2653 & -1.5930 & 0.3348 & 0.0889 \\
-0.1290 & -1.5037 & 0.1945 & 0.0404 \\
1.2520 & 0.1468 & -1.9160 & -0.5303 \\
0.6270 & 0.1852 & -1.7560 & -0.2897 \\
0.3319 & 0.8771 & -0.3924 & -0.5732 \\
0.2661 & 0.0798 & -0.2818 & -1.0543
\end{bmatrix}^T
\]

\[
F^{LQ2}_2 = \begin{bmatrix}
1.2361 & 7.2705 \\
9.4986 & -1.2535 \\
0.7370 & 2.4018 \\
4.0478 & -1.4697
\end{bmatrix}
\]

The Fig. 6 and Fig. 7 show the tracking performance and the evolution of the tracking error for the simulation with \((F^{LQ1}_1, U^{LQ1}, K^{LQ1}_1)\) and \((F^{LQ2}_2, U^{LQ2}, K^{LQ2}_1)\), respectively. Compared with Fig. 5, we can see in Fig. 6 that the two robots cooperate to minimize the error between the center of the robots and the reference trajectory. Therefore, when robot 1 moves outward, robot 2 moves inward correspondingly to reduce the center error. On the contrary, we can see in Fig. 7 that the two robots cooperate to minimize the distance error. Therefore, when robot 1 moves outward, robot 2 moves outward correspondingly. For both simulations, we can see that a better cooperation can be achieved by taking the LQ performance into consideration, and the impact of different weighting can be clearly seen.

Less conservative solution

In this part, Algorithm 2 is applied to reduce the conservatism of the results obtained by Theorem 4. The less conservative solutions \(U^{IT1}\) and \(U^{IT2}\) are computed by the Algorithm 2. By solving Theorem 4,

\[
\begin{align*}
F^{IT1}_1 &= \begin{bmatrix}
-2.9862 & 0.0577 & -2.2707 & 0.3126 \\
-2.0796 & 0.0544 & -0.1571 & 0.2310 \\
0.2290 & -3.2311 & -0.0816 & -2.1212 \\
0.0630 & -2.3418 & -0.0654 & -0.8447 \\
-2.2135 & -0.0252 & -3.1356 & -0.0977 \\
-1.0488 & -0.0626 & -2.2072 & -0.0391 \\
0.3696 & -2.0695 & -0.0726 & -3.3194 \\
0.2338 & -0.8421 & -0.0476 & -2.3821
\end{bmatrix}^T \\
U^{IT1} &= \begin{bmatrix}
-0.4756 & 0.3525 & -0.2949 & 0.7469 \\
-0.3994 & -0.5220 & -0.6550 & -0.2430 \\
-0.2469 & -0.6592 & 0.3571 & 0.2601 \\
-0.4012 & 0.3929 & -0.1263 & -0.5330 \\
-0.4326 & 0.1196 & 0.4352 & -0.0938 \\
-0.4530 & -0.0127 & 0.3887 & -0.1501
\end{bmatrix} \quad \text{for Case 1}
\end{align*}
\]

\[
\begin{align*}
F^{IT2}_1 &= \begin{bmatrix}
54.0542 & -2.9927 \\
4.6939 & 42.3949 \\
52.8992 & -5.9124
\end{bmatrix}
\end{align*}
\]
The average tracking errors of one hundred sets of stochastic simulation with different choice of \((F_1, U, F_2)\) are given in Table 1. It can be seen in Table 1 that compared with the given matrix \(U^{LQ}\), the iterative solution \(U^{IT2}\) by solving Algorithm 2 greatly reduces the center error by 63.9% and also reduces the distance error by 13.6%. And compared with the given matrix \(U^{LQ2}\), the iterative solution \(U^{IT2}\) reduces the center error and distance error by 17.2% and 22%, respectively. Therefore, it is clear that Algorithm 2 can reduce the conservatism of the results by iterative computation of coding matrix \(U\). It is observed from Table 1 that the controller synthesis and coding design (equivalently channel design) are of equal importance for such network-based systems, which indeed reveals that the fusion of control theory (cybernetics) [44] and information theory [45] takes a crucial role in the analysis and synthesis of NCSs.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Average error of center (m)</th>
<th>Average error of distance (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MS stabilization</td>
<td>1.9934</td>
<td>3.9880</td>
</tr>
<tr>
<td>Case 1 (with (U^{LQ1}))</td>
<td>0.8344</td>
<td>2.0884</td>
</tr>
<tr>
<td>Case 2 (with (U^{LQ1}))</td>
<td>0.3014</td>
<td>1.8041</td>
</tr>
<tr>
<td>Case 2 (with (U^{LQ2}))</td>
<td>1.4403</td>
<td>0.5545</td>
</tr>
<tr>
<td>Case 2 (with (U^{IT2}))</td>
<td>1.9181</td>
<td>0.4523</td>
</tr>
</tbody>
</table>

For comparisons, The MS stabilization results in [30] is extended to the tracking problem by the design method proposed in this paper. In other words, a MS stabilizing feedback gain \(F_2\) is obtained by the design method proposed in [30] and the corresponding feedforward gain \(F_1\) is computed by Theorem 3. Fig. 8 shows the tracking performance and the evolution of the tracking error. As depicted in Fig. 8, the two robots are decoupled and the tracking performance is not as satisfied as the iterative solutions given by Theorem 4 in this paper.

### 6 References


