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Possibilistic Logic: From Certainty-Qualified Statements to Two-Tiered Logics – A Prospective Survey

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Abstract. Possibilistic logic (PL) is more than thirty years old. The paper proposes a survey of its main developments and applications in artificial intelligence, together with a short presentation of works in progress. PL amounts to a classical logic handling of certainty-qualified statements. Certainty is estimated in the setting of possibility theory as a lower bound of a necessity set-function. An elementary possibilistic formula is a pair made of a classical logic formula, and a certainty level belonging to a bounded scale. Basic PL handles only conjunctions of such formulas, and PL bases can be viewed as classical logic bases layered in terms of certainty. Semantics is in terms of epistemic states represented by fuzzy sets of interpretations. A PL base is associated with an inconsistency level above which formulas are safe from inconsistency. Applications include reasoning with default rules, belief revision, Bayesian possibilistic networks, information fusion, and preference modeling (in this latter case, certainty is turned into priority). Different extensions of basic PL are briefly reviewed, where levels take values in lattices, are replaced by vectors of levels, or are handled in a purely symbolic manner (without being instantiated). This latter extension may be of interest for explanation purposes. A paraconsistent treatment of inconsistency is also discussed. Still another extension allows for associating possibilistic formulas with sets of agents or sources that support them. In generalized possibilistic logic (GPL), negation and disjunction can be applied as well as conjunction, to possibilistic formulas. It may be viewed as a fragment of modal logic (such as KD45) where modalities cannot be nested. GPL can be still extended to a logic involving both objective and non-nested multimodal formulas. Applications of GPL to the modeling of ignorance, to the representation of answer set programs, to reasoning about other agents' beliefs, and to a logic of argumentation are outlined. Generally speaking, the interest and the strength of PL relies on a sound alliance between classical logic and possibility theory which offers a rich representation setting allowing an accurate modeling of partial ignorance. The paper focuses more on ideas than on technicalities and provides references for details (Invited talk presented by the second author).

1 Introduction

An important part of the pieces of information one has to deal with are pervaded with uncertainty. In other words, we have to handle statements that are not all fully certain. This does not mean that we are always able to quantify the certainty of a given piece of information in a precise manner. This calls for a setting that may be qualitative. Moreover, the fact that one has no certainty at all in favor of some statement should not entail that we have some certainty about the opposite statement, since one may be fully ignorant about a situation. This rules out probabilities ($\text{Prob}(A) = 0$ entails $\text{Prob}(\text{not}A) = 1$).

Possibility theory has not this drawback since uncertainty about A is assessed in terms of two dual set functions, called possibility and necessity measures, by $\Pi(A)$ and $N(A) = 1 - \Pi(\text{not}A)$, and $N(A) = 0 = N(\text{not}A)$ in case of total ignorance about A . Moreover, possibility theory may be numerical or qualitative [47]. In the first case, possibility measures and the dual necessity measures can be regarded respectively as upper bounds and lower bounds of ill-known probabilities; they are also particular cases of plausibility and belief functions respectively [41, 72]. In fact, possibility measures and necessity measures constitute the simplest, non trivial, imprecise probabilities system [73]. Second, when qualitative, possibility theory provides a natural approach to the grading of possibility and necessity modalities on finite scales in an ordinal way.

Possibility theory has a logical counterpart, called possibilistic logic (PL) [35, 37, 48, 50], which remains close to classical logic, and propagates certainty in a qualitative manner, retaining the weakest link in a chain of inferences. As this survey is going to show, PL logic turns to be an unexpectedly versatile tool. The paper is organized as follows. We give a short background on possibility theory in Sect. 2. Section 3 is devoted to basic PL, which handles conjunctions of PL formulas, made of a classical logic formula associated with a lower bound of necessity measure. Section 4 provides an overview of various applications of basic PL to default reasoning, belief revision, information fusion, decision under uncertainty, and uncertainty handling in databases; some other applications are briefly mentioned, as well as the close relationship of PL bases with possibilistic networks. Section 5 covers different extensions of PL where classical logic formulas may be associated with values in lattices, with vectors of certainty levels, with purely symbolic (non instantiated) values, with paraconsistent valuations, or still with sets of agents or sources that support them. Section 6 deals with generalized possibilistic logic, a two-tiered logic having a powerful representation ability for modeling uncertain epistemic states, which can capture answer set programming. Section 7 points out potential applications to multiple agent reasoning and to argumentative reasoning.

2 Short Refresher on Possibility Theory

Possibility theory originates in Zadeh's paper [75] and has been more extensively investigated in [40, 41]. Zadeh starts from the idea of a possibility distribution to which he associates a possibility measure. A possibility distribution is a mapping

π from a set of states, or universe of discourse, U (it may be an attribute domain, the set of interpretation of a propositional language, etc) to a totally ordered scale \mathcal{S} , with top denoted by 1 and bottom by 0. It is assumed that \mathcal{S} is equipped with an order-reversing map denoted by $\lambda \in \mathcal{S} \mapsto 1 - \lambda$. Different types of scales may be used from a finite scale $\mathcal{S} = \{1 = \lambda_1 > \dots \lambda_n > \lambda_{n+1} = 0\}$ in the qualitative case, to the unit interval in the quantitative case, see [52] for other options. $\pi(u) = 0$ means that state u is rejected as impossible; $\pi(u) = 1$ means that state u is totally possible (= plausible). The larger $\pi(u)$, the more possible the state u is. If the universe U is exhaustive, at least one of the elements in S should be the actual world, so that $\exists u, \pi(u) = 1$ (normalization). This condition expresses the consistency of the epistemic state described by π .

A possibility distribution π is said to be at least as specific as another π' if and only if for each state of affairs u : $\pi(u) \leq \pi'(u)$ [74]. Then, π is at least as restrictive and informative as π' , since it rules out at least as many states with at least as much strength. In this setting, extreme forms of partial knowledge can be captured, namely: (i) complete knowledge: for some $u_0, \pi(u_0) = 1$ and $\pi(u) = 0, \forall u \neq u_0$ (only u_0 is possible); (ii) complete ignorance: $\pi(u) = 1, \forall u \in U$ (all states are possible).

Two dual set functions are associated with a possibility distribution, namely a possibility measure Π and a necessity measure N : $\forall A \subseteq U$,

$$\Pi(A) = \sup_{u \in A} \pi(u); \quad N(A) = 1 - \Pi(A^c) = \inf_{s \notin A} 1 - \pi(u),$$

with $A^c = U \setminus A$. $\Pi(A)$ (resp. $N(A)$) evaluates to what extent A is consistent with π (resp. A is certainly implied by π). Generally, $\Pi(U) = N(U) = 1$ and $\Pi(\emptyset) = N(\emptyset) = 0$ (since π is normalized to 1). In the Boolean case, the possibility distribution reduces to a disjunctive (epistemic) set $E \subseteq U$, and possibility and necessity are s.t.: (i) $\Pi(A) = 1$ if $A \cap E \neq \emptyset$, and 0 otherwise; (ii) $N(A) = 1$ if $E \subseteq A$, and 0 otherwise.

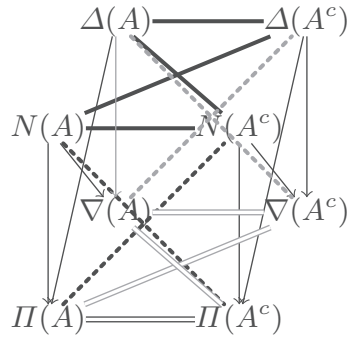
Possibility measures satisfy a characteristic “maxitivity” property $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$, and necessity measures a “minitivity” property $N(A \cap B) = \min(N(A), N(B))$. On infinite spaces, these axioms must hold for infinite families of sets. As a consequence, of the normalization of π , $\min(N(A), N(A^c)) = 0$ and $\max(\Pi(A), \Pi(A^c)) = 1$, or equivalently $\Pi(A) = 1$ whenever $N(A) > 0$, namely that something somewhat certain should be first fully possible, i.e. consistent with the available information. Moreover, one cannot be somewhat certain of both A and A^c , without being inconsistent. But we only have $N(A \cup B) \geq \max(N(A), N(B))$. This goes well with the idea that one may be certain about the event $A \cup B$, without being really certain about more specific events such as A and B .

Human knowledge is often expressed by statements to which belief degrees are attached. Certainty-qualified pieces of uncertain information of the form “ A is certain to degree α ” can then be modeled by the constraint $N(A) \geq \alpha$. It represents a family of possible epistemic states π that obey this constraint. The least specific possibility distribution among them exists and is defined by [41]: $\pi_{(A, \alpha)}(u) = 1$ if $u \in A$, $\pi_{(A, \alpha)}(u) = 1 - \alpha$ otherwise. If $\alpha = 1$ we get the characteristic function of A . If $\alpha = 0$, we get total ignorance. It is a key building-block

to construct possibility distributions from several pieces of uncertain knowledge. It is instrumental in PL semantics.

There are two other set functions: (i) a measure of *guaranteed possibility* or *strong possibility* [31,47]: $\Delta(A) = \inf_{u \in A} \pi(u)$ which estimates to what extent *all* states in A are possible according to evidence. $\Delta(A)$ can be used as a degree of evidential support for A , and its dual conjugate ∇ such that $\nabla(A) = 1 - \Delta(A^c) = \sup_{u \notin A} 1 - \pi(u)$. $\nabla(A)$ evaluates the degree of potential or *weak necessity* of A , as it is 1 only if some state u out of A is impossible. Thus, the functions Δ and ∇ are *decreasing* wrt set inclusion (in full contrast with Π and N which are increasing). They satisfy the characteristic properties $\Delta(A \cup B) = \min(\Delta(A), \Delta(B))$ and $\nabla(A \cap B) = \max(\nabla(A), \nabla(B))$.

Interestingly enough, the four evaluations of an event A and the four evaluations of its opposite A^c can be organized in a cube of opposition [49] (see below), whose front and back facets are graded extension of the traditional square of opposition [67]. Counterparts of the characteristic properties of the square of opposition do hold. First, the diagonals (in dotted lines) of these facets link dual measures through the involutive order-reversing function $1 - (\cdot)$. The vertical edges of the cube, as well as the diagonals of the side facets, which are bottom-oriented arrows, correspond to entailments here expressed by inequalities. Indeed, provided that π and $1 - \pi$ are both normalized, we have for all A , $\max(N(A), \Delta(A)) \leq \min(\Pi(A), \nabla(A))$. The thick black lines of the top facets express mutual exclusiveness under the form $\min(N(A), N(A^c)) = \min(\Delta(A), \Delta(A^c)) = \min(N(A), \Delta(A^c)) = \min(\Delta(A), N(A^c)) = 0$. Dually, the double lines of the bottom facet correspond to $\max(\Pi(A), \Pi(A^c)) = \max(\nabla(A), \nabla(A^c)) = \max(\Pi(A), \nabla(A^c)) = \max(\nabla(A), \Pi(A^c)) = 1$. Thus, the following cube summarizes the interplay between the different set functions in possibility theory.



3 Basic Possibilistic Logic

A basic PL formula is a pair (a, α) made of a classical logic formula a associated with a certainty level $\alpha \in (0, 1]$, viewed as a lower bound of a *necessity measure*, i.e., (a, α) is semantically understood as $N(a) \geq \alpha$. Formulas of the form $(a, 0)$, contain no information ($N(a) \geq 0$ always holds), and are not considered. Thanks to the minitivity property of necessity measures for conjunction, i.e., $N(a \wedge b) = \min(N(a), N(b))$, a PL base, i.e., a set of PL formulas, can be always put in an equivalent clausal form.

Syntactic Aspects. Here we only consider the case of (basic) possibilistic *propositional* logic, i.e., PL formulas (a, α) are s.t. a is a formula in a propositional language; for (basic) possibilistic *first order* logic, the reader is referred to [37].

Axioms and Inference Rules. The PL axioms [37] are those of propositional logic, where each axiom schema is now supposed to hold with certainty 1. Its inference rules are:

- if $\beta \leq \alpha$ then $(a, \alpha) \vdash (a, \beta)$ (certainty weakening).
- $(\neg a \vee b, \alpha), (a, \alpha) \vdash (b, \alpha), \forall \alpha \in (0, 1]$ (modus ponens).

We may also use the certainty weakening rule with the following *PL*-resolution rule:

- $(\neg a \vee b, \alpha), (a \vee c, \alpha) \vdash (b \vee c, \alpha), \forall \alpha \in (0, 1]$ (resolution).

Using certainty weakening, it is easy to see that the following inference rule is valid:

- $(\neg a \vee b, \alpha), (a \vee c, \beta) \vdash (b \vee c, \min(\alpha, \beta))$ (weakest link resolution).

So in a reasoning chain, the certainty level of the conclusion is the smallest of the certainty levels of the formulas involved in the premises. The following inference rule, we call *formula weakening* holds also as a consequence of α - β -resolution.

- if $a \vdash b$ then $(a, \alpha) \vdash (b, \alpha), \forall \alpha \in (0, 1]$ (formula weakening).

Inference and Consistency. Let $\Gamma = \{(a_i, \alpha_i), i = 1, \dots, m\}$ be a set of possibilistic formulas. In a way quite similar to propositional logic, proving $\Gamma \vdash (a, \alpha)$ amounts to proving $\Gamma, (\neg a, 1) \vdash (\perp, \alpha)$ by repeated application of the weakest link-resolution rule. Moreover, note that $\Gamma \vdash (a, \alpha)$ if and only if $\Gamma_\alpha \vdash (a, \alpha)$ if and only if $(\Gamma_\alpha)^* \vdash a$, where $\Gamma_\alpha = \{(a_i, \alpha_i) \in \Gamma, \alpha_i \geq \alpha\}$ and $\Gamma^* = \{a_i \mid (a_i, \alpha_i) \in \Gamma\}$. The certainty levels stratify the knowledge base Γ into nested level cuts Γ_α , i.e. $\Gamma_\alpha \subseteq \Gamma_\beta$ if $\beta \leq \alpha$. A consequence (a, α) from Γ can only be obtained from formulas having a certainty level at least equal to α , so from formulas in Γ_α ; then a is a classical consequence from the PL knowledge base $(\Gamma_\alpha)^*$, and $\alpha = \max\{\beta \mid (\Gamma_\beta)^* \vdash a\}$.

The *inconsistency level* of Γ is defined by $inc(\Gamma) = \max\{\alpha \mid \Gamma \vdash (\perp, \alpha)\}$. The possibilistic formulas in Γ whose level is strictly above $inc(\Gamma)$ are safe from inconsistency, namely $inc(\{(a_i, \alpha_i) \mid (a_i, \alpha_i) \in \Gamma \text{ and } \alpha_i > inc(\Gamma)\}) = 0$. Indeed, if $\alpha > inc(\Gamma)$, $(\Gamma_\alpha)^*$ is consistent. The classical consistency of Γ^* is equivalent to $inc(\Gamma) = 0$.

Semantic Aspects. The semantics of PL [37] is expressed in terms of possibility distributions, and necessity measures on the set Ω of interpretations of the language. A PL formula (a, α) encodes the statement $N(a) \geq \alpha$. Its semantics is given by the following possibility distribution $\pi_{(a, \alpha)}$ defined in agreement with certainty qualification:

$$\forall \omega \in \Omega, \pi_{(a, \alpha)}(\omega) = 1 \text{ if } \omega \models a \text{ and } \pi_{(a, \alpha)}(\omega) = 1 - \alpha \text{ if } \omega \models \neg a$$

where ω is any interpretation of the considered propositional language. Intuitively, this means that any interpretation that is a counter-model of a , is all the less possible as a is more certain, i.e. as α is higher. It can be easily checked that the associated necessity measure is such that $N_{(a,\alpha)}(a) = \alpha$, and $\pi_{(a,\alpha)}$ is the *least informative* possibility distribution (i.e. maximizing possibility degrees) such that this constraint holds. In fact, any possibility distribution π such that $\forall \omega, \pi(\omega) \leq \pi_{(a,\alpha)}(\omega)$ is such that its associated necessity measure N satisfies $N(a) \geq N_{(a,\alpha)}(a) = \alpha$ (hence is more committed).

The base Γ is semantically associated with the possibility distribution:

$$\pi_{\Gamma}(\omega) = \min_{i=1,\dots,m} \pi_{(a_i,\alpha_i)}(\omega) = \min_{i=1,\dots,m} \max([a_i](\omega), 1 - \alpha_i)$$

where $[a_i]$ is the characteristic function of the models of a_i , namely $[a_i](\omega) = 1$ if $\omega \models a_i$ and $[a_i](\omega) = 0$ otherwise. Thus, the least informative induced possibility distribution π_{Γ} is obtained as the min-based conjunction of the fuzzy sets of interpretations (with membership functions $\pi_{(a_i,\alpha_i)}$), representing each formula. It can be checked that $N_{\Gamma}(a_i) \geq \alpha_i$ for $i=1,\dots,m$, where N_{Γ} is the necessity measure defined from π_{Γ} . Note that we may only have an inequality here since Γ may, for instance, include two formulas associated to equivalent propositions, but with distinct certainty levels. The semantic entailment is defined by $\Gamma \models (a, \alpha)$ if and only if $\forall \omega, \pi_{\Gamma}(\omega) \leq \pi_{\{(a,\alpha)\}}(\omega)$. PL is sound and complete [37] wrt this semantics: $\Gamma \vdash (a, \alpha)$ if and only if $\Gamma \models (a, \alpha)$.

Moreover, we have $inc(\Gamma) = 1 - \max_{\omega \in \Omega} \pi_{\Gamma}(\omega)$, which acknowledges the fact that the normalization of π_{Γ} is equivalent to the classical consistency of Γ^* . Thus, an important feature of PL is its ability to deal with inconsistency.

4 Applications of Basic Possibilistic Logic

Before briefly surveying different uses of basic PL, we mention possibilistic networks as another compact representation setting that can be related to PL.

Possibilistic Networks. We first need to define conditioning in possibility theory. Conditional possibility can be defined similarly to probability theory using a Bayesian-like equation of the form [42] $\Pi(B \cap A) = \Pi(B \mid A) \star \Pi(A)$ where $\Pi(A) > 0$ and \star may be the minimum or the product; moreover $N(B \mid A) = 1 - \Pi(B^c \mid A)$. If operation \star is the minimum, the equation $\Pi(B \cap A) = \min(\Pi(B \mid A), \Pi(A))$ fails to characterize $\Pi(B \mid A)$, and we must resort to the minimal specificity principle to define a qualitative conditioning [41]: $\Pi(B \mid A) = 1$ if $\Pi(B \cap A) = \Pi(A) > 0$, $\Pi(B \mid A) = \Pi(B \cap A)$ otherwise. It is clear that $N(B \mid A) > 0$ if and only if $\Pi(B \cap A) > \Pi(B^c \cap A)$. Note also that $N(B \mid A) = N(A^c \cup B)$ if $N(B \mid A) > 0$. In the numerical setting, we must choose $\star =$ product that preserves continuity, so that $\Pi(B \mid A) = \frac{\Pi(B \cap A)}{\Pi(A)}$ which makes possibilistic and probabilistic conditionings very similar, and then gradual positive reinforcement of possibility is allowed.

There are several notions of possibilistic independence between events. Let us just mention two main directions (see [9] for details, discussions and reference):

- *Unrelatedness*: $\Pi(A \cap B) = \min(\Pi(A), \Pi(B))$. When it does not hold, it indicates an epistemic form of mutual exclusion between A and B . It is symmetric but sensitive to negation. When it holds for all pairs made of A, B and their complements, it is an epistemic version of logical independence, useful in default reasoning.
- *Causal independence*: $N(B | A) = N(B)$. This notion is different from the former one and stronger. It is a form of directed epistemic independence whereby learning A does not affect the certainty of B . It is neither symmetric nor insensitive to negation. A weaker qualitative version is $N(B | A) > 0$ and $N(B) > 0$.

Graphical Structures. Like joint probability distributions, joint possibility distributions can be decomposed into a conjunction of conditional possibility distributions (using $\star = \text{minimum, or product}$), once an ordering of the variables is chosen, in a way similar to Bayes nets [13]. A joint possibility distribution associated with ordered variables X_1, \dots, X_n , can be decomposed by the chain rule

$$\pi(X_1, \dots, X_n) = \pi(X_n | X_1, \dots, X_{n-1}) \star \dots \star \pi(X_2 | X_1) \star \pi(X_1).$$

Since possibilistic nets and PL bases are compact representations of possibility distributions, it should not come as a surprise that possibilistic nets can be directly translated into PL bases and vice-versa, both when conditioning is based on minimum or on product [13]. Hybrid representations formats have been introduced where local PL bases are associated to the nodes of a graphical structure rather than conditional possibility tables [25]. An important feature of the PL setting is the existence of equivalent representation formats: set of prioritized logical formulas, preorders on interpretations (possibility distributions) at the semantical level, possibilistic nets, but also set of conditionals of the form $\Pi(a \wedge b) > \Pi(a \wedge \neg b)$. There are algorithms for translating one format in another [13].

Default Reasoning. PL can be used for describing the normal course of things. A default rule “if a then b , generally” is understood formally as the constraint $\Pi(a \wedge b) > \Pi(a \wedge \neg b)$ on a possibility measure Π describing the semantics of the available knowledge. It expresses that in the context where a is true, there exists situations where having b true is strictly more plausible than any situations where b is false in the same context. Any finite consistent set of constraints of the above form, representing a set of defaults $\mathcal{D} = \{a_i \rightsquigarrow b_i, i = 1, \dots, n\}$, is compatible with a non-empty family of possibility measures Π , and induces a partially defined ranking on the set of interpretations, that can be completed according to the principle of minimal specificity, e.g. [18]. This principle assigns to each world ω the highest possibility level without violating the constraints. This defines a unique complete preorder. The method then consists in turning each default $a_i \rightsquigarrow b_i$ into a possibilistic clause $(\neg a_i \vee b_i, N(\neg a_i \vee b_i))$, where N is computed from the greatest possibility distribution π induced by the set of constraints corresponding to the default knowledge base, as already explained. We thus obtain a PL base K . This encodes the generic knowledge embedded in

the default rules. Then we apply the possibilistic inference for reasoning with the formulas in K encoding the defaults together with the available factual knowledge encoded as fully certain possibilistic formulas in a base F . However, the conclusions that can be obtained from $K \cup F$ with a certainty level strictly greater than the level of inconsistency of this base are safe. Roughly speaking, it turns out that in this approach, the most specific rules w.r.t. a given context remain above the level of inconsistency. Such an approach has been proved to be in full agreement with the Kraus-Lehmann-Magidor postulates-based approach to nonmonotonic reasoning [60]. More precisely, two nonmonotonic entailments can be defined in the possibilistic setting, the one presented above, based on the least specific possibility distribution compatible with the constraints encoding the set of defaults, and another one more cautious, where one considers that b can be deduced in the situation where all we know is $F = \{a\}$ iff the inequality $\Pi(a \wedge b) > \Pi(a \wedge \neg b)$ holds true for *all* the Π compatible with the constraints encoding the set of defaults. The first entailment coincides with the rational closure inference [63], while the later corresponds to the (cautious) preferential entailment [60]; see [15, 45].

PL can be also applied to inductive logic programming (ILP). Indeed having a stratified set of first-order logic rules as an hypothesis in ILP is of interest for learning both rules covering normal cases and more specific rules for exceptional cases [71].

Belief Revision. Since nonmonotonic reasoning and belief revision can be closely related, PL finds application also in belief revision. In fact, comparative necessity relations (which can be encoded by necessity measures) [28] are nothing but the epistemic entrenchment relations [43] that underly well-behaved belief revision processes [57]. This enables the PL setting to provide syntactic revision operators that apply to possibilistic knowledge bases, including the case of uncertain inputs [21, 46, 70]. Note that in PL, where formulas (a, α) are pieces of belief with certain levels, the epistemic entrenchment of formulas is made explicit through these levels. Besides, in a revision process it is expected that all formulas independent of the validity of the input information should remain in the revised state of belief; this idea may receive a precise meaning using a definition of possibilistic causal independence between events [29].

Updating in a dynamic world obeys other principles than the revision of a belief state by an input information in a static world. It is linked to the idea of Lewis' *imaging*, whose a possibilistic counterpart is proposed in [44]. A PL transposition of Kalman filtering that combines the ideas of updating and revision can be found in [19].

In contrast with static beliefs, expected to be closed under conjunctions, (positive) desires are such that endorsing $a \vee b$ as a desire means to desire a *and* to desire b . However, desiring both a and $\neg a$ does not sound rational; so when a new desire is added to the set of desires of an agent, a revision process is necessary. Just as belief revision relies on an epistemic entrenchment relation (and thus on a necessity measure), well-behaved desire revision relies on a guaranteed possibility function Δ [39].

Information Fusion. The combination of possibility distributions can be equivalently performed in terms of PL bases: The syntactic counterpart of the point-wise combination of two possibility distributions π_1 and π_2 into a distribution $\pi_1 \otimes \pi_2$ by any monotonic combination operator \otimes such that $1 \otimes 1 = 1$, can be computed, following an idea first proposed in [26]. Namely, if the PL base Γ_1 is associated with π_1 and the base Γ_2 with π_2 , a PL base $\Gamma_{1 \otimes 2}$ semantically equivalent to $\pi_1 \otimes \pi_2$ is given by [16]:

$$\{(a_i, 1 - (1 - \alpha_i) \otimes 1) \text{ s.t. } (a_i, \alpha_i) \in \Gamma_1\} \cup \{(b_j, 1 - 1 \otimes (1 - \beta_j)) \text{ s.t. } (b_j, \beta_j) \in \Gamma_2\} \\ \cup \{(a_i \vee b_j, 1 - (1 - \alpha_i) \otimes (1 - \beta_j)) \text{ s.t. } (a_i, \alpha_i) \in \Gamma_1, (b_j, \beta_j) \in \Gamma_2\}.$$

For $\otimes = \min$, we get $\Gamma_{1 \oplus 2} = \Gamma_1 \cup \Gamma_2$ with $\pi_{\Gamma_1 \cup \Gamma_2} = \min(\pi_1, \pi_2)$ as expected (conjunctive combination). For $\otimes = \max$ (disjunctive combination), we get $\Gamma_{1 \oplus 2} = \{(a_i \vee b_j, \min(\alpha_i, \beta_j)) \text{ s.t. } (a_i, \alpha_i) \in \Gamma_1, \text{ and } (b_j, \beta_j) \in \Gamma_2\}$. With non idempotent \oplus operators, some reinforcement effects may be obtained. See [20, 58, 69] for further studies on possibilistic logic merging operators. Besides, this approach can be also applied to the syntactic encoding of the merging of *classical* logic bases based on Hamming distance (where distances are computed between each interpretation and the different classical logic bases, thus giving birth to counterparts of possibility distributions) [14].

Decision Under Uncertainty. Possibility theory provides a valuable setting for qualitative decision under uncertainty where a pessimistic and an optimistic decision criterion have been axiomatized [53]. The counterpart of these criteria, when knowledge and preferences are under the form of *two distinct* PL bases, is given by the definitions [38]:

- the pessimistic utility $u_*(d)$ of decision d is the maximal $\alpha \in \mathcal{S}$ s.t. $K_\alpha \wedge d \vdash_{PL} P_{\underline{\nu(\alpha)}}$,
- the optimistic utility $u^*(d)$ of d is the maximal $\nu(\alpha) \in \mathcal{S}$ s.t. $K_\alpha \wedge d \wedge P_\alpha \not\equiv \perp$,

where \mathcal{S} is a finite bounded totally ordered scale, ν the ordered reversing map of this scale; K_α is a set of classical logic formulas gathering the pieces of knowledge that are certain at a level at least α , and where P_β is a set of classical logic formulas made of a set of goals (modeling preferences) whose priority level is *strictly* greater than β . An optimal pessimistic decision leads for sure to the satisfaction of all goals in $P_{\underline{\nu(\alpha)}}$ with a priority as low as possible, using only a part K_α of knowledge which has high certainty. An optimal optimistic decision maximizes the consistency of all the more or less important goals with all the more or less certain pieces of knowledge.

Other Applications. In a computational perspective, possibilistic logic has also impacted logic programming [1, 6, 65, 66]. Besides, the possibilistic handling of uncertainty in description logic [68, 76] has also computational advantages, in particular in the case of the *possibilistic DL-Lite* family [11, 12]. Another application is the encoding of control access policies [22]. Lastly, PL has been recently shown to be of interest in database design where the presence of tuples in the database is possible only to some extent, and where functional dependencies are certain only to some extent [59, 64].

5 Extensions of Basic Possibilistic Logic

This section surveys various extensions of basic PL where logical formulas are no longer associated with simple levels valued in a linearly ordered scale.

Lattices. A first idea is to use lattices of values instead of a scale. Examples are: (i) a timed PL where logical formulas are associated with *fuzzy* sets of time instants where the formula is known as being certain to some extent. Semantically, it leads to define necessity measures valued in a distributive lattice; (ii) a *logic of supporters* [61], where formulas a are associated with sets of logical arguments in their favor, (iii) an *interval-based PL* [23] where levels are replaced by intervals, modeling imprecise certainty.

Vectors of Certainty Levels. An obvious consequence of the PL resolution rule is that only the smallest weight of the formulas used in a proof is retained. Thus no difference is made between, e.g., getting (b, β) from $(\neg a \vee b, 1)$ and (a, β) , or getting it from $(\neg c \vee d, 1)$, $(\neg d \vee b, \alpha)$ and (c, β) assuming $\alpha \geq \beta$, although we may find the first proof stronger. This idea can be captured by a new resolution rule $(\neg a \vee b, \alpha); (a \vee c, \beta) \vdash (b \vee c, \alpha\beta)$ where α and β are lists of weights, and $\alpha\beta$ is the list obtained as the concatenation of α and β . In the above example, the first proof yields $(b, (1, \beta))$, while the second one leads to $(b, (1, \alpha, \beta))$. Assuming a finite scale, we have then to rank-order the proofs according to their strength. This can be done by a refinement of min-based ordering, called *leximin* [48] which amounts to a lexicographic ordering of the vectors once they have been completed with 1's for making them of equal length, and increasingly reordered. This can be equivalently encoded by treating the vectors as multisets, replacing $\alpha\beta$ by the union \otimes of the corresponding multisets, and defining an associative operation \oplus that selects the multiset containing the least possible value with a number of occurrences lower than its occurrence in the other multiset (after discarding the values common in equal number in the two multi sets). See [30] for the semiring structure based on \oplus and \otimes . On this basis an extended PL could be fully developed.

Purely Symbolic Levels. Another extension of interest is to consider that the values of certainty levels associated to formulas (still assumed to belong to a totally ordered scale) may be unknown, while the relative ordering between some of them may be known. In such a case, we have to process these levels in a purely symbolic manner, i.e., computing the level from a derived formula as a symbolic expression. For instance, $\Gamma = \{(a, \alpha), (\neg a \vee b, \beta), (b, \gamma)\} \vdash (b, \max(\min(\alpha, \beta), \gamma))$. There still exists a partial order between formulas based on the partial order between symbolic levels (e.g., $\max(\min(\alpha, \beta), \alpha, \gamma) \geq \min(\alpha, \delta)$ for any values of $\alpha, \beta, \gamma, \delta$). See [24] for details and [27] for the proof of completeness.

The use of symbolic levels may serve *explanation* purposes by providing a tool for tracing the impact of the certainty of some pieces of information on a conclusion of interest, as early suggested in [56]. Possibilistic logic formulas with symbolic weights have been used in preference modeling [10]. Then, interpretations (corresponding to the different alternatives) are compared in terms of

symbolic vectors acknowledging the satisfaction or the violation of the formulas associated with the different (conditional) preferences, using suitable order relations. Thus, partial orderings of interpretations can be obtained, and may be refined in case some additional information on the relative priority of the preferences is given. Another use may concern access rights: The different contexts of an ontology, like the access rights of a user, the trust level or the level of detail requested by the user, may be expressed by elements of a lattice, leading to a calculus similar to PL with symbolic weights [3].

Paraconsistent Valuations. An extension of the possibilistic inference has been proposed for handling paraconsistent information [17]. The idea is the following. Given a PL base Γ , we build a set Γ° of bi-weighted formulas: for each formula (a, α) in Γ , we compute triples (a, β, γ) where β (resp. γ) is the highest degree with which a (resp. $\neg a$) is supported in Γ (a is said to be *supported* in Γ *at least at degree* β if there is a *consistent* sub-base of $(\Gamma_\beta)^*$ that entails a , with $\Gamma_\beta^* = \{a_i \mid (a_i, \alpha_i) \in \Gamma, \alpha_i \geq \beta\}$).

Clearly the set of formulas of the form $(a, \beta, 0)$ in Γ° is not paraconsistent, and thus leads to safe conclusions. However, one may obtain a larger set of consistent conclusions from Γ° as explained now. This requires two evaluations: (i) the *undefeasibility* degree of a consistent set A of formulas: $UD(A) = \min\{\beta \mid (a, \beta, \gamma) \in \Gamma^\circ \text{ and } a \in A\}$; (ii) the *unsafeness* degree of a consistent set A of formulas: $US(A) = \max\{\gamma \mid (a, \beta, \gamma) \in \Gamma^\circ \text{ and } a \in A\}$. Then an entailment \vdash_{SS} , named *safely supported* consequence relation, is defined by $\Gamma^\circ \vdash_{SS} b$ if and only if \exists a minimal consistent subset A that classically entails b such that $UD(A) > US(A)$. It can be shown that the set $\{b \mid \Gamma^\circ \vdash_{SS} b\}$ is classically consistent [17]. See [32,51] for details, discussions and other approaches to the handling of inconsistency in the PL setting.

Subsets of Agents. Another early proposed idea, in an information fusion perspective, is to associate each formula with a set of distinct explicit sources that support its truth [36]. Then formulas are associated with sets of sources. This has led to the proposal of a “multiple agent” logic (*ma-L*) where formulas are of the form (a, A) , where A denotes a subset of agents and the formula means that *at least all* the agents in A believe that a is true. In spite of an obvious formal parallel with PL, (a, A) should not only be seen as another way of expressing the strength of the support in favor of a (the larger A , the stronger the support) [33], but also as a piece of information linking a proposition with a particular subset of agents. *ma-L* has two inference rules: (i) if $B \subseteq A$ then $(a, A) \vdash (a, B)$ (subset weakening); (ii) $(\neg a \vee b, A), (a, A) \vdash (b, A), \forall A \in 2^{ALL} \setminus \emptyset$ (modus ponens). As a consequence, we also have the resolution rule if $A \cap B \neq \emptyset$, then $(\neg a \vee b, A), (a \vee c, B) \vdash (b \vee c, A \cap B)$. If $A \cap B = \emptyset$, the resulting information is trivial: (a, \emptyset) is a formula of no use. An *inconsistent subset* of agents for Γ can be defined as

$$inc-s(\Gamma) = \bigcup \{A \subseteq All \mid \Gamma \vdash (\perp, A)\} \text{ and } inc-s(\Gamma) = \emptyset \text{ if } \nexists A \text{ s.t. } \Gamma \vdash (\perp, A).$$

Clearly, $inc-s(\Gamma) = \emptyset$ does not imply the consistency of $\Gamma^\circ = \{a_i \mid (a_i, A_i) \in \Gamma, i = 1, m\}$. It contrasts with possibilistic logic. Just consider the example

$\Gamma = \{(a, A), (\neg a, \overline{A})\}$, then $inc-s(\Gamma) = A \cap \overline{A} = \emptyset$ while Γ° is inconsistent. This is compatible with situations where agents contradict each other. Yet, the consistency of Γ° does entail $inc-s(\Gamma) = \emptyset$. What matters in *ma-L* is the collective consistency of *subsets* of agents (while the collection of the beliefs held by the whole set of agents may be inconsistent).

The semantics of *ma-L* is expressed in terms of set-valued possibility distributions, and set-valued possibility and necessity measures. Namely, the semantics of formula (a, A) is given by set-valued distribution $\pi_{\{(a,A)\}}: \forall \omega \in \Omega, \pi_{\{(a,A)\}}(\omega) = All$ if $\omega \models a, \pi_{\{(a,A)\}}(\omega) = A^c$ if $\omega \models \neg a$ where $A^c = All \setminus A$, and the formula (a, A) is understood as expressing the constraint $\mathbf{N}(a) \supseteq A$ where \mathbf{N} is a set-valued necessity measure. Soundness and completeness results can be established with respect to this semantics [7].

Basic PL and *ma-L* may be combined in a possibilistic multiple agent logic (P*ma-L*). Formulas are pairs (a, F) where F is a fuzzy subset of *All*. One may in particular consider the fuzzy sets $F = (\alpha/A)$ s.t. $(\alpha/A)(k) = \alpha$ if $k \in A$, and $(\alpha/A)(k) = 0$ if $k \in A^c$; it encodes the piece of information “at least all agents in A believe a at least at level α ”. Interpretations are then associated with fuzzy sets of agents. Soundness and completeness of P*ma-L* has been established [8].

6 Generalized Possibilistic Logic

In basic possibilistic logic, only conjunctions of possibilistic logic formulas are allowed. But since (a, α) is semantically interpreted as $N(a) \geq \alpha$, a possibilistic formula can be manipulated as a propositional formula that is true (if $N(a) \geq \alpha$) or false (if $N(a) < \alpha$). Then possibilistic formulas can be combined with all propositional connectives, including disjunction and negation. This is *generalized possibilistic logic* (GPL) [34, 55]. GPL is a two-tiered propositional logic, in which propositional formulas are encapsulated by weighted modal operators interpreted in terms of uncertainty measures from possibility theory. Let $\Lambda_k = \{0, \frac{1}{k}, \frac{2}{k}, \dots, 1\}$ with $k \in \mathbb{N} \setminus \{0\}$ be a finite set of certainty degrees, and let $\Lambda_k^+ = \Lambda_k \setminus \{0\}$. The language of GPL, $\mathcal{L}_{\mathbf{N}}^k$, with $k + 1$ certainty levels is built on top of the propositional language \mathcal{L} as follows: (i) If $a \in \mathcal{L}, \alpha \in \Lambda_k^+$, then $\mathbf{N}_\alpha(a) \in \mathcal{L}_{\mathbf{N}}^k$; (ii) if $\varphi \in \mathcal{L}_{\mathbf{N}}^k, \psi \in \mathcal{L}_{\mathbf{N}}^k$, then $\neg\varphi$ and $\varphi \wedge \psi$ are also in $\mathcal{L}_{\mathbf{N}}^k$.

Here we use the notation $\mathbf{N}_\alpha(a)$, instead of (a, α) , emphasizing the closeness with modal logic. So, an agent asserting $\mathbf{N}_\alpha(a)$ has an epistemic state π such that $N(a) \geq \alpha > 0$. Hence $\neg\mathbf{N}_\alpha(a)$ stands for $N(a) < \alpha$, which, given the finiteness of the set of considered certainty degrees, means $N(a) \leq \alpha - \frac{1}{k}$ and thus $\Pi(\neg a) \geq 1 - \alpha + \frac{1}{k}$. Let $\nu(\alpha) = 1 - \alpha + \frac{1}{k}$. Then, $\nu(\alpha) \in \Lambda_k^+$ iff $\alpha \in \Lambda_k^+$, and $\nu(\nu(\alpha)) = \alpha, \forall \alpha \in \Lambda_k^+$. Thus, we can write $\mathbf{\Pi}_\alpha(p) \equiv \neg\mathbf{N}_{\nu(\alpha)}(\neg p)$. In particular, $\mathbf{\Pi}_1(a) \equiv \neg\mathbf{N}_{\frac{1}{k}}(\neg a)$ if $k > 1$. So, in GPL, one can distinguish between the absence of sufficient certainty that a is true ($\neg\mathbf{N}_\alpha(a)$) and the stronger statement that a is somewhat certainly false ($\mathbf{N}_\alpha(\neg a)$).

The semantics of GPL is as in PL defined in terms of normalized possibility distributions over propositional interpretations, where possibility degrees are

limited to Λ_k . A model of a GPL formula $\mathbf{N}_\alpha(a)$ is any Λ_k -valued possibility distribution π such that $N(a) \geq \alpha$, where N is the necessity measure induced by π , and then the standard definition for $\pi \models \varphi_1 \wedge \varphi_2$ $\pi \models \neg\varphi$. As usual, π is called a model of a set of GPL formulas Γ , written $\pi \models \Gamma$, if π is a model of each formula in Γ . We write $\Gamma \models \phi$, for Γ a set of GPL formulas and ϕ a GPL formula, iff every model of Γ is also a model of ϕ . Note that a formula in GPL will not always have a least specific possibility distribution that satisfies it. For instance, the set of possibility distributions satisfying the disjunction ‘ $\mathbf{N}_\alpha(a) \vee \mathbf{N}_\alpha(b)$ ’ no longer has a unique least informative model as it is the case for conjunction in PL. The soundness and completeness of the following axiomatization of GPL holds with respect to the above semantics [55]:

(**PL**) The Hilbert axioms of classical logic; (**K**) $\mathbf{N}_\alpha(a \rightarrow b) \rightarrow (\mathbf{N}_\alpha(a) \rightarrow \mathbf{N}_\alpha(b))$;
 (**N**) $\mathbf{N}_1(\top)$; (**D**) $\mathbf{N}_\alpha(a) \rightarrow \mathbf{\Pi}_1(a)$; (**W**) $\mathbf{N}_{\alpha_1}(a) \rightarrow \mathbf{N}_{\alpha_2}(a)$, if $\alpha_1 \geq \alpha_2$.

with modus ponens as the only inference rule.

Note that when α is fixed we get a fragment of the modal logic KD. See [5] for a survey of studies of the links between modal logics and possibility theory, and extensions of GPL with objective and non-nested multimodal formulas, as in KD45 and S5.

7 Applications of Generalized Possibilistic Logic

Nonmonotonic Logic Programming. A remarkable application of GPL is its capability to encode answer set programs, using a 3-valued scale $\Lambda_2 = \{0, 1/2, 1\}$. Then, we can discriminate between propositions we are fully certain of and propositions we consider only more plausible than not. It is enough to encode nonmonotonic ASP rules (with negation as failure) within GPL and lay bare their epistemic semantics. For instance, the ASP rule $a \leftarrow b \wedge \text{not } c$ is encoded as $\mathbf{N}_1(b) \wedge \mathbf{\Pi}_1(\neg c) \rightarrow \mathbf{N}_1(a)$ in GPL. See [55] for theoretical results, and [54] for the GPL encoding of Pearce equilibrium logic.

Multiple Agent Reasoning. We consider the muddy children problem: Two children come home from garden. The father sees their muddy foreheads. They sit by him. Father declares at least one of them has a muddy forehead. Then he asks them whoever has mud on the forehead to stand up. None does. Then the question is asked again. Both stand up. Why? Informally, the children did not stand up in the first place because they do not see their own forehead and they see the other is muddy, but as the latter did not stand up it means he did not know either it was muddy because he sees the former is muddy.

Since there is no uncertainty, we use a particular case of GPL, where $k = 1$ with value scale Λ_1 , known as Meta-Epistemic Logic (MEL) [4]; $\Box a$ is identified with $\mathbf{N}_1(a)$ and $\Diamond a$ with $\mathbf{\Pi}_1(a)$. Moreover $\mathbf{\Pi}_1(a) \equiv \neg\mathbf{N}_1(\neg a)$. For the example, we consider the point of view of one child (say child 1): we say “I”. The other is “You”. We use a standard propositional language \mathcal{L} with variables

$V = \{m_I, m_Y, s_Y\}$ (for I am muddy, you are muddy, you stand up). We use one MEL modality \Box_Y standing for “I know that You know” Each pure propositional formula $a \in \mathcal{L}$ not in the scope of \Box_Y is interpreted as “I know a ” (modality \Box_I) is not explicitly used because we exclude for the reference agent (“I”) the possibility of declaring ignorance). So “My” knowledge is the following:

1. Father’s public announcement: I know one of us is muddy and I know You know it too: $m_I \vee m_Y; \Box_Y(m_I \vee m_Y)$;
2. I know that You know whether I am muddy or not (You see me): $\Box_Y m_I \vee \Box_Y \neg m_I$;
3. I know that you ignore if You are muddy or not (like me): $\neg \Box_Y m_Y \wedge \neg \Box_Y \neg m_Y$;
4. I know that if You knew I was not muddy and that one of us is muddy anyway, then You would know that you are muddy: $\Box_Y \neg m_I \wedge \Box_Y(m_I \vee m_Y) \rightarrow \Box_Y(m_Y)$;
5. I know that if You knew you were muddy You would stand up when invited too: $\Box_Y(m_Y) \rightarrow s_Y$; we also have $\Box_Y(m_Y) \rightarrow m_Y$.

I see You did not stand up, so I add $\neg s_Y$ to my knowledge base, and I reason as follows: $\frac{\neg s_Y, \Box_Y(m_Y) \rightarrow s_Y}{\neg \Box_Y(m_Y)}$. You did not stand up because you did not know if you were muddy (and now I know it). $\frac{\neg \Box_Y(m_Y), \Box_Y(m_I \vee m_Y), \Box_Y \neg m_I \wedge \Box_Y(m_I \vee m_Y) \rightarrow \Box_Y(m_Y)}{\neg \Box_Y \neg m_I}$. As one of us at least is muddy and you have no knowledge to claim you are muddy, it means that I know You cannot claim I am not muddy. $\frac{\neg \Box_Y \neg m_I, \Box_Y m_I \vee \Box_Y \neg m_I}{\Box_Y m_I}$. Since it is wrong that you know I am not muddy, while you know if I am muddy or not, it is because you know I am muddy. $\frac{\Box_Y m_I, \Box_Y m_I \rightarrow m_I}{m_I}$. Now I know that You know I am muddy, so, I know I am muddy, and I will stand up next time I am invited to.

Argumentative Reasoning. A logic of arguments similar to GPL has been outlined in [2]. The basic formulas are pairs (x, y) (“ y is a reason for x ”), which are manipulated as propositional formulas using \wedge, \vee, \neg connectives. Thus, e.g., we allow the use of negation in three places: $(x, \neg y)$, $(\neg x, y)$, $\neg(x, y)$, making also difference between (x, y) and $(\neg y \vee x, \top)$, and $(\neg x \vee x', y)$, $(x \vee z, y') \vdash (x' \vee z, y \wedge y')$ is a valid inference rule.

8 Conclusion

This survey has covered old results and works in progress about PL, a simple, but quite powerful approach to the handling of uncertainty that remains as close as possible to classical logic. For complexity issues of PL and GPL the reader is referred to [62] and [55].

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