



**HAL**  
open science

## Remarks about the unification type of several non-symmetric non-transitive modal logics

Philippe Balbiani

► **To cite this version:**

Philippe Balbiani. Remarks about the unification type of several non-symmetric non-transitive modal logics. *Logic Journal of the IGPL*, 2018, 27 (5), pp.639-658. 10.1093/jigpal/jzy078 . hal-02378381

**HAL Id: hal-02378381**

**<https://hal.science/hal-02378381>**

Submitted on 25 Nov 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



## Open Archive Toulouse Archive Ouverte

OATAO is an open access repository that collects the work of Toulouse researchers and makes it freely available over the web where possible

This is an author's version published in:  
<http://oatao.univ-toulouse.fr/24761>

### Official URL

DOI : <https://doi.org/10.1093/jigpal/jzy078>

**To cite this version:** Balbiani, Philippe *Remarks about the unification type of several non-symmetric non-transitive modal logics*. (2018) *Logic Journal of the IGPL*, 27 (5). 639-658.  
ISSN 1367-0751

Any correspondence concerning this service should be sent to the repository administrator: [tech-oatao@listes-diff.inp-toulouse.fr](mailto:tech-oatao@listes-diff.inp-toulouse.fr)

# Remarks about the unification type of several non-symmetric non-transitive modal logics

PHILIPPE BALBIANI\*, *Institut de recherche en informatique de Toulouse, CNRS — Toulouse University, 118 route de Narbonne, 31062 Toulouse Cedex 9, France.*

## Abstract

The problem of unification in a normal modal logic  $L$  can be defined as follows: given a formula  $\varphi$ , determine whether there exists a substitution  $\sigma$  such that  $\sigma(\varphi)$  is in  $L$ . In this paper, we prove that for several non-symmetric non-transitive modal logics, there exists unifiable formulas that possess no minimal complete set of unifiers.

*Keywords:* Normal modal logics, non-symmetric modal logics, non-transitive modal logics, unification problem, types of modal logics,

## 1 Introduction

The problem of unification in a normal modal logic  $L$  can be defined as follows: given a formula  $\varphi(x_1, \dots, x_m)$  where  $x_1, \dots, x_m$  are variables, determine whether there exist formulas  $\psi_1, \dots, \psi_m$  such that  $\varphi(\psi_1, \dots, \psi_m)$  is in  $L$ . The computability of the problem of unification in transitive normal modal logics like  $S4$  and  $GL$  has been solved by Rybakov [22–24] who proved that it is decidable. With respect to its computational complexity, the problem of unification was mostly unexplored before the work of Jerábek [20] who established its membership in  $NEXPTIME$  in several normal modal logics extending  $K4$  such as  $S4$  and  $GL$ . See also [15, 17, 18, 25, 29] for a study of the problem of unification in different normal modal logics.

Within the context of the problem of unification in a normal modal logic  $L$ , an important question is the following: when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is ‘yes’, how large is this set? This question concerns the determination of the type of  $L$  for the problem of unification. Considering the type of unification in a normal modal logic  $L$  is justified from the following perspectives: deciding the unifiability of equivalences like  $\varphi \leftrightarrow \psi$  in  $L$  helps us to understand what is the overlap between the properties  $\varphi$  and  $\psi$  correspond to in  $L$  [1]; methods for deciding the unifiability of formulas in  $L$  can be used to improve the efficiency of automated theorem provers in  $L$  [2].

Ghilardi [16] has proved that the unification type of transitive normal modal logics like  $S4$  and  $GL$  is finitary. Within the context of tense logics and epistemic logics, Dzik [11–13] has studied the relationships between the unification type of a fusion of modal logics and the unification types of the modal logics composing this fusion. He has also proved that some variants of the normal modal logics studied by Jansana [19] are unitary; these variants being sound and complete with respect to

\*E-mail: philippe.balbiani@irit.fr

classes of frames satisfying conditions generalizing symmetry and transitivity. The unification type of normal modal logics such as common knowledge logics and linear temporal logics has also been studied by Babenyshev and Rybakov [3] and Rybakov [26–28].

Nevertheless, still, very little is known about the problem of unification in some of the most popular normal modal logics. For example, the types of the problem of unification in the normal modal logics  $KB$ ,  $KDB$  and  $KTB$  are still unknown [12, Chapter 5]. As well, the types of the problem of unification in the normal modal logics  $K5$ ,  $KD5$ ,  $K45$ ,  $K4.3$  and  $KD4.3$  are unknown too. In this paper, following a line of reasoning suggested by Jeřábek [21] within the context of  $K$  and furthered by Balbiani and Gencer [4] within the context of  $KD$ , we prove that for several non-symmetric non-transitive modal logics like  $KT$ , there exists unifiable formulas that possess no minimal complete set of unifiers. Such modal logics are called nullary.

The section-by-section breakdown of the paper is as follows. Sections 2–4 introduce the basic definitions about normal modal logics. In Section 5, we introduce the basic definitions about unification. In Section 6, we analyse a specific formula from the point of view of some of its unifiers in normal modal logics. In Section 7, we elaborate a sufficient condition for the nullariness of normal modal logics (the adequacy condition) and we give examples of adequate and non-adequate normal modal logics. In Section 8, we prove that if a normal modal logic is adequate then it is nullary.

## 2 Syntax

**Formulas** Let  $VAR$  be a set of *variables* (with typical members denoted  $x, y$ , etc) and  $CON$  be a set of *constants* (with typical members denoted  $p, q$ , etc). The set of all *formulas* (with typical members denoted  $\varphi, \psi$ , etc) is inductively defined as follows:

- $\varphi, \psi ::= x \mid p \mid \perp \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$ .

We adopt the standard rules for omission of the parentheses. Let  $(x_1, x_2, \dots)$  be an enumeration of  $VAR$  without repetition and  $(p_1, p_2, \dots)$  be an enumeration of  $CON$  without repetition. We write  $\varphi(x_1, \dots, x_m, p_1, \dots, p_n)$  to denote a formula whose variables form a subset of  $\{x_1, \dots, x_m\}$  and whose constants form a subset of  $\{p_1, \dots, p_n\}$ . Let  $\varphi(x_1, \dots, x_m, p_1, \dots, p_n)$  be such a formula. The result of the uniform replacement in each of their occurrences of the variables  $x_1, \dots, x_m$  by the formulas  $\psi_1, \dots, \psi_m$  and of the constants  $p_1, \dots, p_n$  by the formulas  $\chi_1, \dots, \chi_n$  is denoted  $\varphi(\psi_1, \dots, \psi_m, \chi_1, \dots, \chi_n)$ .

**Abbreviations** The Boolean connectives  $\top, \wedge, \rightarrow$  and  $\leftrightarrow$  are defined by the usual abbreviations. Let  $\Diamond$  be the modal connective defined as follows:

- $\Diamond\varphi ::= \neg\Box\neg\varphi$ .

For all  $k \geq 0$ , the modal connective  $\Box^k$  is inductively defined as follows:

- $\Box^0\varphi ::= \varphi$ ,
- $\Box^{k+1}\varphi ::= \Box\Box^k\varphi$ .

For all  $k \geq 0$ , the modal connective  $\Diamond^k$  is inductively defined as follows:

- $\Diamond^0\varphi ::= \varphi$ ,
- $\Diamond^{k+1}\varphi ::= \Diamond\Diamond^k\varphi$ .

**Example:** For all formulas  $\varphi$ ,  $\Box^2\varphi$  is  $\Box\Box\varphi$  and  $\Diamond^2\varphi$  is  $\Diamond\Diamond\varphi$ .

For all finite words  $w$  over  $CON$ , the modal connective  $[w]$  is inductively defined as follows:

- $[\epsilon]\varphi ::= \varphi$ ,
- $[pw]\varphi ::= \Box(p \rightarrow [w]\varphi)$ .

**Example:** For all constants  $p, q$  and for all formulas  $\varphi$ ,  $[pq]\varphi$  is  $\Box(p \rightarrow \Box(q \rightarrow \varphi))$ .

For all finite words  $w$  over  $CON$  and for all  $k \geq 0$ , the modal connective  $[w]^k$  is inductively defined as follows:

- $[\epsilon]^0\varphi ::= \varphi$ ,
- $[pw]^{k+1}\varphi ::= [w][w]^k\varphi$ .

**Example:** For all constants  $p, q$  and for all formulas  $\varphi$ ,  $[pq]^2\varphi$  is  $\Box(vp \rightarrow \Box(q \rightarrow \Box(p \rightarrow \Box(q \rightarrow \varphi))))$ .

For all finite words  $w$  over  $CON$  and for all  $k \geq 0$ , the modal connective  $[w]^{\leq k}$  is inductively defined as follows:

- $[w]^{\leq 0}\varphi ::= \varphi$ ,
- $[w]^{\leq k+1}\varphi ::= [w]^{\leq k}\varphi \wedge [w]^{k+1}\varphi$ .

**Example:** For all constants  $p, q$  and for all formulas  $\varphi$ ,  $[pq]^{\leq 2}\varphi$  is  $\varphi \wedge \Box(p \rightarrow \Box(q \rightarrow \varphi)) \wedge \Box(p \rightarrow \Box(q \rightarrow \Box(p \rightarrow \Box(q \rightarrow \varphi))))$ .

**Degrees** The *degree* of a formula  $\varphi$  (in symbols  $\text{deg}(\varphi)$ ) is the non-negative integer inductively defined as follows:

- $\text{deg}(x) = 0$ ,
- $\text{deg}(p) = 0$ ,
- $\text{deg}(\perp) = 0$ ,
- $\text{deg}(\neg\varphi) = \text{deg}(\varphi)$ ,
- $\text{deg}(\varphi \vee \psi) = \max\{\text{deg}(\varphi), \text{deg}(\psi)\}$ ,
- $\text{deg}(\Box\varphi) = \text{deg}(\varphi) + 1$ .

**Example:** For all constants  $p, q$  and for all formulas  $\varphi$ ,  $\text{deg}([pq]^2\varphi)$  is  $\text{deg}(\varphi) + 4$ .

**Substitutions** A *substitution* is a function  $\sigma$  associating to each variable  $x$  a formula  $\sigma(x)$ . Following the standard assumption considered in the literature about unification [1, 12, 16], we will always assume that substitutions move at most finitely many variables. For all formulas  $\varphi(x_1, \dots, x_m, p_1, \dots, p_n)$ , let  $\sigma(\varphi(x_1, \dots, x_m, p_1, \dots, p_n))$  be  $\varphi(\sigma(x_1), \dots, \sigma(x_m), p_1, \dots, p_n)$ .

**Example:** If  $\sigma$  is the substitution such that  $\sigma(x) = p$ ,  $\sigma(y) = q$  and for all variables  $z$  distinct from  $x$  and  $y$ ,  $\sigma(z) = z$  and  $\varphi$  is the formula  $(x \rightarrow p) \wedge (y \rightarrow q) \wedge (x \rightarrow [q]y) \wedge (y \rightarrow [p]x)$  then  $\sigma(\varphi)$  is  $(p \rightarrow p) \wedge (q \rightarrow q) \wedge (p \rightarrow [q]q) \wedge (q \rightarrow [p]p)$ .

The *composition*  $\sigma \circ \tau$  of the substitutions  $\sigma$  and  $\tau$  is the substitution associating to each variable  $x$  the formula  $\tau(\sigma(x))$ .

### 3 Semantics

**Frames** A *frame* is a couple  $F = (W, R)$  where  $W$  is a non-empty set of *possible worlds* and  $R$  is a binary relation on  $W$ . In a frame  $F = (W, R)$ , for all  $n \geq 0$ , the binary relation  $R^n$  on  $W$  is inductively defined as follows:

- $R^0 = Id$ , i.e. the identity relation on  $W$ ,
- $R^{n+1} = R \circ R^n$ , i.e. the composition of the binary relations  $R$  and  $R^n$  on  $W$ .

For all  $n \geq 1$ , we shall say that a frame  $F = (W, R)$  is *n-bounded* if for all  $s, t \in W$ , not  $sR^n t$ . We shall say that a frame  $F = (W, R)$  is *serial* if for all  $s \in W$ , there exists  $t \in W$  such that  $sRt$ . We shall say that a frame  $F = (W, R)$  is *reflexive* if for all  $s \in W$ ,  $sRs$ . For all  $n \geq 1$ , we shall say that a frame  $F = (W, R)$  is *n-ancestral* if for all  $s, t \in W$ , if  $sR^n t$  then there exists  $u \in W$  such that  $tRu$ . For all  $n \geq 1$ , we shall say that a frame  $F = (W, R)$  is *n-deterministic* if for all  $s \in W$  and for all  $t_1, \dots, t_{n+1} \in W$ , if  $sRt_1, \dots, sRt_{n+1}$  then there exists distinct  $i, j \geq 1$  such that  $i, j \leq n + 1$  and  $t_i = t_j$ . We shall say that a frame  $F = (W, R)$  is *symmetric* if for all  $s, t \in W$ , if  $sRt$  then  $tRs$ . We shall say that a frame  $F = (W, R)$  is *transitive* if for all  $s, t, u \in W$ , if  $sRt$  and  $tRu$  then  $sRu$ . For all  $m, n \geq 1$ , if  $(m, n) \neq (2, 1)$  then we shall say that a frame  $F = (W, R)$  is *(m, n)-compositional* if for all  $s, t \in W$ , if  $sR^m t$  then  $sR^n t$ . For all  $n \geq 1$ , let  $C_n^{bou}$  be the class of all *n-bounded* frames. Let  $C_K, C_{KD}$  and  $C_{KT}$  be, respectively, the class of all frames, the class of all serial frames and the class of all reflexive frames. Remark that  $C_{KT} \subseteq C_{KD}$ . For all  $n \geq 1$ , let  $C_n^{anc}$  be the class of all *n-ancestral* frames. Remark that for all  $n \geq 1$ ,  $C_{KD} \subseteq C_n^{anc}$ . For all  $n \geq 1$ , let  $C_n^{det}$  be the class of all *n-deterministic* frames. Let  $C_{KB}, C_{KDB}$  and  $C_{KTB}$  be, respectively, the class of all symmetric frames, the class of all symmetric serial frames and the class of all symmetric reflexive frames. Let  $C_{K4}$  be the class of all transitive frames. For all  $m, n \geq 1$ , if  $(m, n) \neq (2, 1)$  then let  $C_m^n$  be the class of all *(m, n)-compositional* frames.

**Models** A *model* based on a frame  $F = (W, R)$  is a triple  $M = (W, R, V)$  where  $V$  is a function assigning to each variable  $x$  a subset  $V(x)$  of  $W$  and to each constant  $p$  a subset  $V(p)$  of  $W$ . Given a model  $M = (W, R, V)$ , the *satisfiability* of a modal formula  $\varphi$  at  $s \in W$  (in symbols  $M, s \models \varphi$ ) is inductively defined as follows:

- $M, s \models x$  if  $s \in V(x)$ ,
- $M, s \models p$  if  $s \in V(p)$ ,
- $M, s \not\models \perp$ ,
- $M, s \models \neg\varphi$  if  $M, s \not\models \varphi$ ,
- $M, s \models \varphi \vee \psi$  if  $M, s \models \varphi$  or  $M, s \models \psi$ ,
- $M, s \models \Box\varphi$  if for all  $t \in W$ , if  $sRt$  then  $M, t \models \varphi$ .

We shall say that a formula  $\varphi$  is *true* in a model  $M = (W, R, V)$  (in symbols  $M \models \varphi$ ) if  $\varphi$  is satisfied at all  $s \in W$ .

**Validity** We shall say that a formula  $\varphi$  is *valid* in a frame  $F$  (in symbols  $F \models \varphi$ ) if  $\varphi$  is true in all models based on  $F$ . We shall say that a formula  $\varphi$  is *valid* in a class  $C$  of frames (in symbols  $C \models \varphi$ ) if  $\varphi$  is valid in all frames in  $C$ .

## 4 Normal modal logics

A *normal modal logic* is a set  $L$  of formulas such that

- $L$  contains all tautologies,
- $L$  contains all formulas of the form  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ ,
- $L$  is closed under the rule of modus ponens  $\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$ ,
- $L$  is closed under the rule of generalization  $\frac{\varphi}{\Box\varphi}$ ,
- $L$  is closed under the rule of uniform substitution  $\frac{\varphi(x_1, \dots, x_m, p_1, \dots, p_n)}{\varphi(\psi_1, \dots, \psi_m, \chi_1, \dots, \chi_n)}$ .

It is evident that for all classes  $C$  of frames, the set of all  $C$ -valid formulas is a normal modal logic.

LEMMA 4.1

Let  $L$  be a normal modal logic. For all  $k \geq 0$ , for all formulas  $\varphi$  and for all constants  $p, q$ ,

- $[qp]^k[q]\perp \rightarrow [qp]^{k+1}[q]\perp \in L$ ,
- $[qp]^{\leq k}\varphi \wedge [qp]^k[q]\perp \rightarrow [q][pq]^{\leq k}[p]\varphi \in L$ ,
- $[qp]^{\leq k}[q]\varphi \rightarrow [q][pq]^{\leq k}\varphi \in L$ ,
- $[qp]^k[q]\perp \rightarrow [q][pq]^k[p]\perp \in L$ .

PROOF. Left to the reader. □

LEMMA 4.2

Let  $L$  be a normal modal logic. For all  $k \geq 0$ , for all formulas  $\varphi, \psi$  and for all constants  $p, q$ , if  $\varphi \rightarrow [q]\psi \in L$  and  $\psi \rightarrow [p]\varphi \in L$  then  $\varphi \rightarrow [qp]^{\leq k}(\varphi \wedge [q]\psi) \in L$ .

PROOF. Left to the reader. □

It is evident that the set of all normal modal logics is closed under arbitrary intersections. For all  $n \geq 1$ , let  $K_n^{bou}$  be the least normal modal logic containing the formula  $\Box^n \perp$ . Let  $K$ ,  $KD$  and  $KT$  be, respectively, the least normal modal logic, the least normal modal logic containing all formulas of the form  $\Box\chi \rightarrow \Diamond\chi$  and the least normal modal logic containing all formulas of the form  $\Box\chi \rightarrow \chi$ . Remark that  $KD \subseteq KT$ . For all  $n \geq 1$ , let  $K_n^{anc}$  be the least normal modal logic containing the formula  $\Box^{2n-1}\Diamond\top$ . Remark that for all  $n \geq 1$ ,  $K_n^{anc} \subseteq KD$ . For all  $n \geq 1$ , let  $Alt_n$  be the least normal modal logic containing all formulas of the form  $\bigvee\{\Box(\bigwedge\{\chi_j : j \geq 1 \text{ is such that } j \leq i\} \rightarrow \chi_{i+1}) : i \geq 0 \text{ is such that } i \leq n\}$ . Let  $KB$ ,  $KDB$  and  $KTB$  be, respectively, the least normal modal logic containing all formulas of the form  $\chi \rightarrow \Box\Diamond\chi$ , the least normal modal logic containing all formulas of the form  $\chi \rightarrow \Box\Diamond\chi$  and  $\Box\chi \rightarrow \Diamond\chi$  and the least normal modal logic containing all formulas of the form  $\chi \rightarrow \Box\Diamond\chi$  and  $\Box\chi \rightarrow \chi$ . Let  $K4$  be the least normal modal logic containing all formulas of the form  $\Box\chi \rightarrow \Box\Box\chi$ . For all  $m, n \geq 1$ , if  $(m, n) \neq (2, 1)$  then let  $K_m^n$  be the least normal modal logic containing all formulas of the form  $\Diamond^m\chi \rightarrow \Diamond^n\chi$ .

PROPOSITION 4.3

For all  $n \geq 1$ ,  $K_n^{bou}$  is sound and complete with respect to  $C_n^{bou}$ .

PROOF. Let  $n \geq 1$ . It is evident that the canonical frame for  $K_n^{bou}$  is in  $C_n^{bou}$ . Hence, the result follows from the standard canonical model construction in normal modal logics as developed in [8, Chapter 4]. □

PROPOSITION 4.4

$K$ ,  $KD$  and  $KT$  are, respectively, sound and complete with respect to  $C_K$ ,  $C_{KD}$  and  $C_{KT}$ .

PROOF. See [8, Chapter 4] for a proof. □

PROPOSITION 4.5

For all  $n \geq 1$ ,  $K_n^{anc}$  is sound and complete with respect to  $C_n^{anc}$ .

PROOF. Let  $n \geq 1$ . It is evident that the canonical frame for  $K_n^{anc}$  is in  $C_n^{anc}$ . Hence, the result follows from the standard canonical model construction in normal modal logics as developed in [8, Chapter 4]. □

PROPOSITION 4.6

For all  $n \geq 1$ ,  $Alt_n$  is sound and complete with respect to  $C_n^{det}$ .

PROOF. See [9, Chapters 3 and 4] for a proof. □

PROPOSITION 4.7

$KB$ ,  $KDB$  and  $KTB$  are, respectively, sound and complete with respect to  $C_{KB}$ ,  $C_{KDB}$  and  $C_{KTB}$ .

PROOF. See [8, Chapter 4] for a proof. □

PROPOSITION 4.8

$K4$  is sound and complete with respect to  $C_{K4}$ .

PROOF. See [8, Chapter 4] for a proof. □

PROPOSITION 4.9

For all  $m, n \geq 1$ , if  $(m, n) \neq (2, 1)$  then  $K_m^n$  is sound and complete with respect to  $C_m^n$ .

PROOF. See [10, Chapter 3] for a proof. □

## 5 Unification

Let  $L$  be a normal modal logic. In this section, we shall introduce the basic definitions about unification in  $L$ .

**Unifiers** We shall say that a formula  $\varphi$  is *L-unifiable* if there exists a substitution  $\sigma$  such that  $\sigma(\varphi) \in L$ . In that case,  $\sigma$  is an *L-unifier* of  $\varphi$ .

**Example:** If  $\sigma$  is the substitution such that  $\sigma(x) = p$ ,  $\sigma(y) = q$  and for all variables  $z$  distinct from  $x$  and  $y$ ,  $\sigma(z) = z$  and  $\varphi$  is the formula  $(x \rightarrow p) \wedge (y \rightarrow q) \wedge (x \rightarrow [q]y) \wedge (y \rightarrow [p]x)$  then  $\sigma$  is a  $K$ -unifier of  $\varphi$ .

We shall say that a substitution  $\sigma$  is *L-equivalent* to a substitution  $\tau$  (in symbols  $\sigma \simeq_L \tau$ ) if for all variables  $x$ ,  $\sigma(x) \leftrightarrow \tau(x) \in L$ .

**Example:** If  $\sigma$  and  $\tau$  are the substitutions such that  $\sigma(x) = \Box p$ ,  $\tau(x) = \Box p \wedge p$ ,  $\sigma(y) = \Diamond q \vee q$ ,  $\tau(y) = \Diamond q$  and for all variables  $z$  distinct from  $x$  and  $y$ ,  $\sigma(z) = z$  and  $\tau(z) = z$  then  $\sigma \simeq_{KT} \tau$ .

LEMMA 5.1

The binary relation  $\simeq_L$  is reflexive, symmetric and transitive on the set of all substitutions.

PROOF. Left to the reader. See [1, 12] for details about the binary relation  $\simeq_L$ . □

We shall say that a substitution  $\sigma$  is more *L-general* than a substitution  $\tau$  (in symbols  $\sigma \preceq_L \tau$ ) if there exists a substitution  $\nu$  such that  $\sigma \circ \nu \simeq_L \tau$ .



LEMMA 5.2

The binary relation  $\preceq_L$  is reflexive and transitive on the set of all substitutions. Moreover, it contains  $\simeq_L$ .

PROOF. Left to the reader. See [1, 12] for details about the binary relation  $\preceq_L$ . □

We shall say that a set  $\Sigma$  of substitutions is *L-minimal* if for all  $\sigma, \tau \in \Sigma$ , if  $\sigma \preceq_L \tau$  then  $\sigma \simeq_L \tau$ . We shall say that a set  $\Sigma$  of *L-unifiers* of an *L-unifiable* formula  $\varphi$  is *L-complete* if for all *L-unifiers*  $\sigma$  of  $\varphi$ , there exists  $\tau \in \Sigma$  such that  $\tau \preceq_L \sigma$ .

**Types** An important question is the following: when a formula is *L-unifiable*, has it an *L-minimal L-complete* set of *L-unifiers*? When the answer is ‘yes’, how large is this set? We shall say that an *L-unifiable* formula

- $\varphi$  is *L-nullary* if there exists no *L-minimal L-complete* set of *L-unifiers* of  $\varphi$ ,
- $\varphi$  is *L-infinitary* if there exists an *L-minimal L-complete* set of *L-unifiers* of  $\varphi$  but there exists no finite one,
- $\varphi$  is *L-finitary* if there exists a finite *L-minimal L-complete* set of *L-unifiers* of  $\varphi$  but there exists no with cardinality 1,
- $\varphi$  is *L-unitary* if there exists an *L-minimal L-complete* set of *L-unifiers* of  $\varphi$  with cardinality 1.

We shall say that

- *L* is of unification type *nullary* if there exists an *L-nullary* formula,
- *L* is of unification type *infinitary* if every *L-unifiable* formula is *L-infinitary* or *L-finitary* or *L-unitary* and there exists an *L-infinitary* formula,
- *L* is of unification type *finitary* if every *L-unifiable* formula is *L-finitary* or *L-unitary* and there exists an *L-finitary* formula,
- *L* is of unification type *unitary* if every *L-unifiable* formula is *L-unitary*.

See [1] for a proof that *S5* is unitary, [4] for a proof that *KD* is nullary, [6] for a proof that *Alt<sub>1</sub>* is nullary, [16] for a proof that *K4* and *S4* are finitary and [21] for a proof that *K* is nullary. By the way, the proof given in [1] that *S5* is unitary can be easily adapted to a proof that  $K_1^{bou}$  is unitary (the unification type of  $K_n^{bou}$  is not known when  $n \geq 2$ ). In other respect, the proof given in [6] that *Alt<sub>1</sub>* is nullary can be easily adapted for all  $n \geq 2$  to a proof that *Alt<sub>n</sub>* is nullary. We shall say that *L* is *filtering* if for all *L-unifiable* formulas  $\varphi$  and for all *L-unifiers*  $\sigma, \tau$  of  $\varphi$ , there exists an *L-unifier*  $\mu$  of  $\varphi$  such that  $\mu \preceq_L \sigma$  and  $\mu \preceq_L \tau$ . When *L* is filtering, given two *L-unifiers* of an *L-unifiable* formula, there is always an *L-unifier* that is more *L-general* than both of them. Hence, in this case, it is known that *L* is unitary or *L* is nullary. See [17] for a proof that if  $K4 \subseteq L$  then *L* is filtering iff  $K4.2^+ \subseteq L$ . See also [5] for a proof that *K45* and *KD45* are filtering. The purpose of Sections 6–8 is to elaborate a sufficient condition for the nullariness of *L*.

**Remarks** Note that the proof of the nullariness of *K* given by Jeřábek [21] only assumed that the language contains at least one variable. As well, note that the proof of the nullariness of *KD* given by Balbiani and Gencer [4] only assumed that the language contains at least one variable and one constant. As for the nullariness of *Alt<sub>1</sub>* given by Balbiani and Tinchev [6], it only assumed that the language contains at least one variable. This means that when the language contains no constant, *K* and *Alt<sub>1</sub>* are still nullary whereas the unification type of *KD* is still unknown. In the case where the language contains infinitely many constants,

one is talking about *unification with constants* and in the case where the language contains no constant, one is talking about *elementary unification*. In Sections 6–8, we will always assume that the language of modal logic contains at least two distinct variables and two distinct constants. Hence, our results in the forthcoming sections only concern unification with constants.

## 6 Analysis of a specific formula

Let  $L$  be a normal modal logic. In this section, we shall analyse a specific formula from the point of view of some of its  $L$ -unifiers, namely the formula

$$\bullet \varphi = (x \rightarrow p) \wedge (y \rightarrow q) \wedge (x \rightarrow [q]y) \wedge (y \rightarrow [p]x)$$

in which  $x, y$  are distinct variables and  $p, q$  are distinct constants, i.e.  $\varphi$  is the conjunction of the 4 following formulas:

- $x \rightarrow p$ ,
- $y \rightarrow q$ ,
- $x \rightarrow [q]y$ ,
- $y \rightarrow [p]x$ .

Remark that Jeřábek [21] has used the formula  $x \rightarrow \Box x$  to prove that  $K$  is nullary and Balbiani and Gencer [4] have used the formula  $(x \rightarrow p) \wedge (x \rightarrow [p]x)$  to prove that  $KD$  is nullary.

Remark that in order to present our line of reasoning, we have to assume that the language of modal logic contains at least two distinct variables and two distinct constants.

Let  $\sigma_{\perp}$  be the substitution defined as follows:

- $\sigma_{\perp}(x) = \perp$ ,
- $\sigma_{\perp}(y) = \perp$ ,
- for all variables  $z$  distinct from  $x$  and  $y$ ,  $\sigma_{\perp}(z) = z$ .

LEMMA 6.1

$\sigma_{\perp}$  is an  $L$ -unifier of  $\varphi$ .

PROOF. Remark that  $\sigma_{\perp}(\varphi)$  is the conjunction of the 4 following formulas:

- $\perp \rightarrow p$ ,
- $\perp \rightarrow q$ ,
- $\perp \rightarrow [q]\perp$ ,
- $\perp \rightarrow [p]\perp$ .

Hence,  $\sigma_{\perp}(\varphi) \in L$ . Thus,  $\sigma_{\perp}$  is an  $L$ -unifier of  $\varphi$ . □

Let  $\sigma_{\top}$  be the substitution defined as follows:

- $\sigma_{\top}(x) = p$ ,
- $\sigma_{\top}(y) = q$ ,
- for all variables  $z$  distinct from  $x$  and  $y$ ,  $\sigma_{\top}(z) = z$ .

LEMMA 6.2

$\sigma_{\top}$  is an  $L$ -unifier of  $\varphi$ .

PROOF. Remark that  $\sigma_{\top}(\varphi)$  is the conjunction of the 4 following formulas:

- $p \rightarrow p$ ,
- $q \rightarrow q$ ,
- $p \rightarrow [q]q$ ,
- $q \rightarrow [p]p$ .

Hence,  $\sigma_{\top}(\varphi) \in L$ . Thus,  $\sigma_{\top}$  is an  $L$ -unifier of  $\varphi$ . □

LEMMA 6.3

Let  $\sigma$  be a substitution. The following conditions are equivalent:

1.  $\sigma_{\top} \circ \sigma \simeq_L \sigma$ ,
2.  $\sigma_{\top} \preceq_L \sigma$ ,
3.  $\sigma(x) \leftrightarrow p \in L$  and  $\sigma(y) \leftrightarrow q \in L$ .

PROOF. (1)  $\Rightarrow$  (2). By definition of  $\preceq_L$ .

(2)  $\Rightarrow$  (3). Suppose  $\sigma_{\top} \preceq_L \sigma$ . Let  $\tau$  be a substitution such that  $\sigma_{\top} \circ \tau \simeq_L \sigma$ . Hence,  $\tau(\sigma_{\top}(x)) \leftrightarrow \sigma(x) \in L$  and  $\tau(\sigma_{\top}(y)) \leftrightarrow \sigma(y) \in L$ . Since  $\tau(\sigma_{\top}(x)) = p$  and  $\tau(\sigma_{\top}(y)) = q$ , therefore  $\sigma(x) \leftrightarrow p \in L$  and  $\sigma(y) \leftrightarrow q \in L$ .

(3)  $\Rightarrow$  (1). Suppose  $\sigma(x) \leftrightarrow p \in L$  and  $\sigma(y) \leftrightarrow q \in L$ . Since  $\sigma(\sigma_{\top}(x)) = p$  and  $\sigma(\sigma_{\top}(y)) = q$ , therefore  $\sigma(\sigma_{\top}(x)) \leftrightarrow \sigma(x) \in L$  and  $\sigma(\sigma_{\top}(y)) \leftrightarrow \sigma(y) \in L$ . Moreover, since for all variables  $z$  distinct from  $x$  and  $y$ ,  $\sigma(\sigma_{\top}(z)) = \sigma(z)$ , therefore for all variables  $z$  distinct from  $x$  and  $y$ ,  $\sigma(\sigma_{\top}(z)) \leftrightarrow \sigma(z) \in L$ . Hence,  $\sigma_{\top} \circ \sigma \simeq_L \sigma$  □

For all  $k \geq 0$ , let  $\sigma_k$  be the substitution defined as follows:

- $\sigma_k(x) = p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\perp$ ,
- $\sigma_k(y) = q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\perp$ ,
- for all variables  $z$  distinct from  $x$  and  $y$ ,  $\sigma_k(z) = z$ .

LEMMA 6.4

For all  $k \geq 0$ ,  $\sigma_k$  is an  $L$ -unifier of  $\varphi$ .

PROOF. Let  $k \geq 0$ . Remark that  $\sigma_k(\varphi)$  is the conjunction of the 4 following formulas:

- $p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\perp \rightarrow p$ ,
- $q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\perp \rightarrow q$ ,
- $p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\perp \rightarrow [q](q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\perp)$ ,
- $q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\perp \rightarrow [p](p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\perp)$ .

Hence, by Lemma 4.1,  $\sigma_k(\varphi) \in L$ . Thus,  $\sigma_k$  is an  $L$ -unifier of  $\varphi$ . □

LEMMA 6.5

Let  $\sigma$  be a substitution. If  $\sigma$  is an  $L$ -unifier of  $\varphi$  then for all  $k \geq 0$ , the following conditions are equivalent:

1.  $\sigma_k \circ \sigma \simeq_L \sigma$ ,
2.  $\sigma_k \preceq_L \sigma$ ,
3.  $\sigma(x) \rightarrow [qp]^k[q]\perp \in L$  and  $\sigma(y) \rightarrow [pq]^k[p]\perp \in L$ .

PROOF. Suppose  $\sigma$  is an  $L$ -unifier of  $\varphi$ . Let  $k \geq 0$ .

(1)  $\Rightarrow$  (2). By definition of  $\preceq_L$ .

(2)  $\Rightarrow$  (3). Suppose  $\sigma_k \preceq_L \sigma$ . Let  $\tau$  be a substitution such that  $\sigma_k \circ \tau \simeq_L \sigma$ . Hence,  $\tau(\sigma_k(x)) \leftrightarrow \sigma(x) \in L$  and  $\tau(\sigma_k(y)) \leftrightarrow \sigma(y) \in L$ . Since  $\tau(\sigma_k(x)) = p \wedge [qp]^{\leq k}(\tau(x) \wedge [q]\tau(y)) \wedge [qp]^k[q]\perp$  and  $\tau(\sigma_k(y)) = q \wedge [pq]^{\leq k}(\tau(y) \wedge [p]\tau(x)) \wedge [pq]^k[p]\perp$ , therefore  $p \wedge [qp]^{\leq k}(\tau(x) \wedge [q]\tau(y)) \wedge [qp]^k[q]\perp \leftrightarrow \sigma(x) \in L$  and  $q \wedge [pq]^{\leq k}(\tau(y) \wedge [p]\tau(x)) \wedge [pq]^k[p]\perp \leftrightarrow \sigma(y) \in L$ . Thus,  $\sigma(x) \rightarrow [qp]^k[q]\perp \in L$  and  $\sigma(y) \rightarrow [pq]^k[p]\perp \in L$ .

(3)  $\Rightarrow$  (1). Suppose  $\sigma(x) \rightarrow [qp]^k[q]\perp \in L$  and  $\sigma(y) \rightarrow [pq]^k[p]\perp \in L$ . Since  $\sigma$  is an  $L$ -unifier of  $\varphi$ , therefore  $\sigma(\varphi) \in L$ . Since  $\sigma(\varphi) = (\sigma(x) \rightarrow p) \wedge (\sigma(y) \rightarrow q) \wedge (\sigma(x) \rightarrow [q]\sigma(y)) \wedge (\sigma(y) \rightarrow [p]\sigma(x))$ , therefore  $(\sigma(x) \rightarrow p) \wedge (\sigma(y) \rightarrow q) \wedge (\sigma(x) \rightarrow [q]\sigma(y)) \wedge (\sigma(y) \rightarrow [p]\sigma(x)) \in L$ . Hence, the 4 following formulas are in  $L$ :

- $\sigma(x) \rightarrow p$ ,
- $\sigma(y) \rightarrow q$ ,
- $\sigma(x) \rightarrow [q]\sigma(y)$ ,
- $\sigma(y) \rightarrow [p]\sigma(x)$ .

Since  $\sigma(x) \rightarrow [qp]^k[q]\perp \in L$  and  $\sigma(y) \rightarrow [pq]^k[p]\perp \in L$ , therefore by Lemma 4.2,  $\sigma(x) \rightarrow p \wedge [qp]^{\leq k}(\sigma(x) \wedge [q]\sigma(y)) \wedge [qp]^k[q]\perp \in L$  and  $\sigma(y) \rightarrow q \wedge [pq]^{\leq k}(\sigma(y) \wedge [p]\sigma(x)) \wedge [pq]^k[p]\perp \in L$ . In other respect,  $p \wedge [qp]^{\leq k}(\sigma(x) \wedge [q]\sigma(y)) \wedge [qp]^k[q]\perp \rightarrow \sigma(x) \in L$  and  $q \wedge [pq]^{\leq k}(\sigma(y) \wedge [p]\sigma(x)) \wedge [pq]^k[p]\perp \rightarrow \sigma(y) \in L$ . Since  $\sigma(\sigma_k(x)) = p \wedge [qp]^{\leq k}(\sigma(x) \wedge [q]\sigma(y)) \wedge [qp]^k[q]\perp$  and  $\sigma(\sigma_k(y)) = q \wedge [pq]^{\leq k}(\sigma(y) \wedge [p]\sigma(x)) \wedge [pq]^k[p]\perp$ , therefore  $\sigma(\sigma_k(x)) \leftrightarrow \sigma(x) \in L$  and  $\sigma(\sigma_k(y)) \leftrightarrow \sigma(y) \in L$ . Moreover, since for all variables  $z$  distinct from  $x$  and  $y$ ,  $\sigma(\sigma_k(z)) = \sigma(z)$ , therefore for all variables  $z$  distinct from  $x$  and  $y$ ,  $\sigma(\sigma_k(z)) \leftrightarrow \sigma(z) \in L$ . Thus,  $\sigma_k \circ \sigma \simeq_L \sigma$   $\square$

LEMMA 6.6

For all  $k, l \geq 0$ ,  $\sigma_l \preceq_L \sigma_k$  iff  $p \wedge [qp]^k[q]\perp \rightarrow [qp]^l[q]\perp \in L$  and  $q \wedge [pq]^k[p]\perp \rightarrow [pq]^l[p]\perp \in L$ .

PROOF. Let  $k, l \geq 0$ .

( $\Rightarrow$ ). Suppose  $\sigma_l \preceq_L \sigma_k$ . Hence, by Lemma 6.5,  $\sigma_k(x) \rightarrow [qp]^l[q]\perp \in L$  and  $\sigma_k(y) \rightarrow [pq]^l[p]\perp \in L$ . Since  $\sigma_k(x) = p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\perp$  and  $\sigma_k(y) = q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\perp$ , therefore  $p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\perp \rightarrow [qp]^l[q]\perp \in L$  and  $q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\perp \rightarrow [pq]^l[p]\perp \in L$ . Thus, uniformly replacing in each of their occurrences the variables  $x$  and  $y$  by, respectively, the formulas  $\top$  and  $\top$ ,  $p \wedge [qp]^k[q]\perp \rightarrow [qp]^l[q]\perp \in L$  and  $q \wedge [pq]^k[p]\perp \rightarrow [pq]^l[p]\perp \in L$ .

( $\Leftarrow$ ). Suppose  $p \wedge [qp]^k[q]\perp \rightarrow [qp]^l[q]\perp \in L$  and  $q \wedge [pq]^k[p]\perp \rightarrow [pq]^l[p]\perp \in L$ . Hence,  $p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\perp \rightarrow [qp]^l[q]\perp \in L$  and  $q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\perp \rightarrow [pq]^l[p]\perp \in L$ . Since  $\sigma_k(x) = p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\perp$  and  $\sigma_k(y) = q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\perp$ , therefore  $\sigma_k(x) \rightarrow [qp]^l[q]\perp \in L$  and  $\sigma_k(y) \rightarrow [pq]^l[p]\perp \in L$ . Thus, by Lemma 6.5,  $\sigma_l \preceq_L \sigma_k$ .  $\square$

## 7 Adequate modal logics

Let  $L$  be a normal modal logic. In this section, we shall elaborate a sufficient condition for the nullariness of  $L$ : the adequacy condition. Before elaborating it, we shall elaborate a weaker condition: the coherence condition. We shall say that  $L$  is *coherent* if for all distinct constants  $p, q$  and for all  $k \geq 0$ ,

- $p \wedge [qp]^{k+1}[q]\perp \rightarrow [qp]^k[q]\perp \notin L$ .

LEMMA 7.1

Let  $n \geq 1$ . For all distinct constants  $p, q, p \wedge [qp]^{n+1}[q]\perp \rightarrow [qp]^n[q]\perp \in K_n^{bou}$ .

PROOF. Left to the reader. □

LEMMA 7.2

For all distinct constants  $p, q$  and for all  $k \geq 0, p \wedge [qp]^{k+1}[q]\perp \rightarrow [qp]^k[q]\perp \notin KT$ .

PROOF. Let  $k \geq 0$ . Let  $F = (W, R)$  be the frame such that  $W = \mathbb{N}$  and for all  $s, t \in W, sRt$  iff  $t = s$  or  $t = s + 1$ . Obviously,  $F$  is in  $C_{KT}$ . Let  $M = (W, R, V)$  be the model based on  $F$  such that  $V(p) = \{2i : i \geq 0 \text{ is such that } i \leq k + 1\}$  and  $V(q) = \{2i + 1 : i \geq 0 \text{ is such that } i \leq k\}$ . Obviously,  $M, 0 \models p \wedge [qp]^{k+1}[q]\perp$  and  $M, 0 \not\models [qp]^k[q]\perp$ . Since  $KT$  is sound with respect to  $C_{KT}$  and  $F$  is in  $C_{KT}$ , therefore  $p \wedge [qp]^{k+1}[q]\perp \rightarrow [qp]^k[q]\perp \notin KT$ . □

LEMMA 7.3

Let  $n \geq 1$ . For all distinct constants  $p, q$  and for all  $k \geq 0, p \wedge [qp]^{k+1}[q]\perp \rightarrow [qp]^k[q]\perp \notin Alt_n$ .

PROOF. Let  $k \geq 0$ . Let  $F = (W, R)$  be the frame such that  $W = \mathbb{N}$  and for all  $s, t \in W, sRt$  iff  $t = s + 1$ . Obviously,  $F$  is in  $C_n^{det}$ . Let  $M = (W, R, V)$  be the model based on  $F$  such that  $V(p) = \{2i : i \geq 0 \text{ is such that } i \leq k + 1\}$  and  $V(q) = \{2i + 1 : i \geq 0 \text{ is such that } i \leq k\}$ . Obviously,  $M, 0 \models p \wedge [qp]^{k+1}[q]\perp$  and  $M, 0 \not\models [qp]^k[q]\perp$ . Since  $Alt_n$  is sound with respect to  $C_n^{det}$  and  $F$  is in  $C_n^{det}$ , therefore  $p \wedge [qp]^{k+1}[q]\perp \rightarrow [qp]^k[q]\perp \notin Alt_n$ . □

LEMMA 7.4

For all distinct constants  $p, q$  and for all  $k \geq 0, p \wedge [qp]^{k+1}[q]\perp \rightarrow [qp]^k[q]\perp \in KB$ .

PROOF. Left to the reader. □

LEMMA 7.5

For all distinct constants  $p, q$  and for all  $k \geq 0, p \wedge [qp]^{k+1}[q]\perp \rightarrow [qp]^k[q]\perp \notin K4$ .

PROOF. Let  $k \geq 0$ . Let  $F = (W, R)$  be the frame such that  $W = \mathbb{Q}$  and for all  $s, t \in W, sRt$  iff  $s < t$ . Obviously,  $F$  is in  $C_{K4}$ . Let  $M = (W, R, V)$  be the model based on  $F$  such that  $V(p) = \{2i : i \geq 0 \text{ is such that } i \leq k + 1\}$  and  $V(q) = \{2i + 1 : i \geq 0 \text{ is such that } i \leq k\}$ . Obviously,  $M, 0 \models p \wedge [qp]^{k+1}[q]\perp$  and  $M, 0 \not\models [qp]^k[q]\perp$ . Since  $K4$  is sound with respect to  $C_{K4}$  and  $F$  is in  $C_{K4}$ , therefore  $p \wedge [qp]^{k+1}[q]\perp \rightarrow [qp]^k[q]\perp \notin K4$ . □

LEMMA 7.6

Let  $m, n \geq 1$ . If  $(m, n) \neq (2, 1)$  then for all distinct constants  $p, q$  and for all  $k \geq 0, p \wedge [qp]^{k+1}[q]\perp \rightarrow [qp]^k[q]\perp \notin K_m^n$ .

PROOF. Similar to the proof of Lemma 7.5. □

PROPOSITION 7.7

Let  $n \geq 1$ . If  $K_n^{bou} \subseteq L$  then  $L$  is not coherent.

PROOF. By Lemma 7.1. □

PROPOSITION 7.8

If  $L \subseteq KT$  then  $L$  is coherent.

PROOF. By Lemma 7.2. □

PROPOSITION 7.9

Let  $n \geq 1$ . If  $L \subseteq K_n^{anc}$  then  $L$  is coherent.

PROOF. By Lemma 7.2. □

PROPOSITION 7.10

Let  $n \geq 1$ . If  $L \subseteq Alt_n$  then  $L$  is coherent.

PROOF. By Lemma 7.3. □

PROPOSITION 7.11

If  $KB \subseteq L$  then  $L$  is not coherent.

PROOF. By Lemma 7.4. □

PROPOSITION 7.12

If  $L \subseteq K4$  then  $L$  is coherent.

PROOF. By Lemma 7.5. □

PROPOSITION 7.13

Let  $m, n \geq 1$ . If  $(m, n) \neq (2, 1)$  and  $L \subseteq K_m^n$  then  $L$  is coherent. □

PROOF. By Lemma 7.6.

Now, we are ready to elaborate the adequacy condition. We shall say that  $L$  is *adequate* if  $L$  is coherent and for all formulas  $\varphi, \psi$ , for all distinct constants  $p, q$  and for all  $k \geq 0$ , if  $\deg(\varphi) \leq 2k$  then

- if  $\varphi \rightarrow [qp]^k[q][p]\varphi \in L$  then  $\varphi \rightarrow [qp]^k[q]\perp \in L$  or  $p \rightarrow \varphi \in L$ ,
- if  $\varphi \rightarrow [qp]^k[q]\psi \in L$  then  $\varphi \rightarrow [qp]^k\perp \in L$  or  $q \rightarrow \psi \in L$ .

PROPOSITION 7.14

Let  $n \geq 1$ . If  $K_n^{bou} \subseteq L$  then  $L$  is not adequate.

PROOF. By Proposition 7.7. □

PROPOSITION 7.15

The normal modal logic  $K$  is adequate.

PROOF. By Proposition 7.8, it suffices to prove that for all formulas  $\varphi, \psi$ , for all distinct constants  $p, q$  and for all  $k \geq 0$ , if  $\deg(\varphi) \leq 2k$  then

- if  $\varphi \rightarrow [qp]^k[q][p]\varphi \in K$  then  $\varphi \rightarrow [qp]^k[q]\perp \in K$  or  $p \rightarrow \varphi \in K$ ,
- if  $\varphi \rightarrow [qp]^k[q]\psi \in K$  then  $\varphi \rightarrow [qp]^k\perp \in K$  or  $q \rightarrow \psi \in K$ .

Let  $\varphi, \psi$  be formulas and  $k \geq 0$  be such that  $\deg(\varphi) \leq 2k$ .

Suppose  $\varphi \rightarrow [qp]^k[q]\perp \notin K$  and  $p \rightarrow \varphi \notin K$ . Since  $K$  is complete with respect to  $C_K$ , therefore let  $F = (W, R)$  be a frame in  $C_K$  such that  $F \not\models \varphi \rightarrow [qp]^k[q]\perp$  and  $F' = (W', R')$  be a frame in  $C_K$  such that  $F' \not\models p \rightarrow \varphi$ . Let  $M = (W, R, V)$  be a model based on  $F$  such that  $M \not\models \varphi \rightarrow [qp]^k[q]\perp$  and  $M' = (W', R', V')$  be a model based on  $F'$  such that  $M' \not\models p \rightarrow \varphi$ . Let  $s \in W$  be such that  $M, s \not\models \varphi \rightarrow [qp]^k[q]\perp$  and  $s' \in W'$  be such that  $M', s' \not\models p \rightarrow \varphi$ . Hence,  $M, s \models \varphi$ ,  $M, s \not\models [qp]^k[q]\perp$ ,  $M', s' \models p$  and  $M', s' \not\models \varphi$ . Let  $t_1, \dots, t_k, t_{k+1} \in V(q)$  and  $u_1, \dots, u_k \in V(p)$  be such that  $sRt_1Ru_1 \dots Rt_kRu_kRt_{k+1}$ . Let  $M_s = (W_s, R_s, V_s)$  be the unravelling of  $M$  around  $s$ . In particular,  $(s), (s, t_1), (s, t_1, u_1), \dots, (s, t_1, u_1, \dots, t_k), (s, t_1, u_1, \dots, t_k, u_k)$  and  $(s, t_1, u_1, \dots, t_k, u_k, t_{k+1})$  are in  $W_s$ . Moreover,  $(s)R_s(s, t_1)R_s(s, t_1, u_1) \dots R_s(s, t_1, u_1, \dots, t_k)R_s(s, t_1, u_1, \dots, t_k, u_k)R_s(s, t_1, u_1, \dots, t_k, u_k, t_{k+1})$ . The reader may easily verify that for all  $(s_0, \dots, s_n) \in W_s$  and for all formulas  $\chi$ , if

$\text{deg}(\chi) + n \leq 2k$  then  $M, s_n \models \chi$  iff  $M_s, (s_0, \dots, s_n) \models \chi$ . Since  $M, s \models \varphi$ , therefore  $M_s, (s) \models \varphi$ . Let  $M''$  be  $(W'', R'', V'')$  where

- $W'' = W_s \cup W'$ ,
- $R'' = R_s \cup R' \cup \{(s, t_1, u_1, \dots, t_k, u_k, t_{k+1}), s'\}$ ,
- $V'' = V_s \cup V'$ .

The reader may easily verify that for all  $(s_0, \dots, s_n) \in W_s$  and for all formulas  $\chi$ , if  $\text{deg}(\chi) + n \leq 2k$  then  $M_s, (s_0, \dots, s_n) \models \chi$  iff  $M'', (s_0, \dots, s_n) \models \chi$ . Since  $M_s, (s) \models \varphi$ , therefore  $M'', (s) \models \varphi$ . The reader may also easily verify that for all  $t' \in W'$  and for all formulas  $\chi$ ,  $M', t' \models \chi$  iff  $M'', t' \models \chi$ . Since  $M', s' \models p$  and  $M', s' \not\models \varphi$ , therefore  $M'', s' \models p$  and  $M'', s' \not\models \varphi$ . Since  $t_1, \dots, t_k, t_{k+1} \in V(q)$ ,  $u_1, \dots, u_k \in V(p)$ ,  $(s)R_s(s, t_1)R_s(s, t_1, u_1) \dots R_s(s, t_1, u_1, \dots, t_k)R_s(s, t_1, u_1, \dots, t_k, u_k)R_s(s, t_1, u_1, \dots, t_k, u_k, t_{k+1})$  and  $M'', (s) \models \varphi$ , therefore  $M'', (s) \not\models \varphi \rightarrow [qp]^k[q][p]\varphi$ . Thus,  $\varphi \rightarrow [qp]^k[q][p]\varphi \notin K$ .

Suppose  $\varphi \rightarrow [qp]^k \perp \notin K$  and  $q \rightarrow \psi \notin K$ . Since  $K$  is complete with respect to  $C_K$ , therefore let  $F = (W, R)$  be a frame in  $C_K$  such that  $F \not\models \varphi \rightarrow [qp]^k \perp$  and  $F' = (W', R')$  be a frame in  $C_K$  such that  $F' \not\models q \rightarrow \psi$ . Let  $M = (W, R, V)$  be a model based on  $F$  such that  $M \not\models \varphi \rightarrow [qp]^k \perp$  and  $M' = (W', R', V')$  be a model based on  $F'$  such that  $M' \not\models q \rightarrow \psi$ . Let  $s \in W$  be such that  $M, s \not\models \varphi \rightarrow [qp]^k \perp$  and  $s' \in W'$  be such that  $M', s' \not\models q \rightarrow \psi$ . Hence,  $M, s \models \varphi$ ,  $M, s \not\models [qp]^k \perp$ ,  $M', s' \models q$  and  $M', s' \not\models \psi$ . Let  $t_1, \dots, t_k \in V(q)$  and  $u_1, \dots, u_k \in V(p)$  be such that  $sRt_1Ru_1 \dots Rt_kRu_k$ . Let  $M_s = (W_s, R_s, V_s)$  be the unravelling of  $M$  around  $s$ . In particular,  $(s), (s, t_1), (s, t_1, u_1), \dots, (s, t_1, u_1, \dots, t_k)$  and  $(s, t_1, u_1, \dots, t_k, u_k)$  are in  $W_s$ . Moreover,  $(s)R_s(s, t_1)R_s(s, t_1, u_1) \dots R_s(s, t_1, u_1, \dots, t_k)R_s(s, t_1, u_1, \dots, t_k, u_k)$ . The reader may easily verify that for all  $(s_0, \dots, s_n) \in W_s$  and for all formulas  $\chi$ , if  $\text{deg}(\chi) + n \leq 2k$  then  $M, s_n \models \chi$  iff  $M_s, (s_0, \dots, s_n) \models \chi$ . Since  $M, s \models \varphi$ , therefore  $M_s, (s) \models \varphi$ . Let  $M''$  be  $(W'', R'', V'')$  where

- $W'' = W_s \cup W'$ ,
- $R'' = R_s \cup R' \cup \{(s, t_1, u_1, \dots, t_k, u_k), s'\}$ ,
- $V'' = V_s \cup V'$ .

The reader may easily verify that for all  $(s_0, \dots, s_n) \in W_s$  and for all formulas  $\chi$ , if  $\text{deg}(\chi) + n \leq 2k$  then  $M_s, (s_0, \dots, s_n) \models \chi$  iff  $M'', (s_0, \dots, s_n) \models \chi$ . Since  $M_s, (s) \models \varphi$ , therefore  $M'', (s) \models \varphi$ . The reader may also easily verify that for all  $t' \in W'$  and for all formulas  $\chi$ ,  $M', t' \models \chi$  iff  $M'', t' \models \chi$ . Since  $M', s' \models q$  and  $M', s' \not\models \psi$ , therefore  $M'', s' \models q$  and  $M'', s' \not\models \psi$ . Since  $t_1, \dots, t_k \in V(q)$ ,  $u_1, \dots, u_k \in V(p)$ ,  $(s)R_s(s, t_1)R_s(s, t_1, u_1) \dots R_s(s, t_1, u_1, \dots, t_k)R_s(s, t_1, u_1, \dots, t_k, u_k)$  and  $M'', (s) \models \varphi$ , therefore  $M'', (s) \not\models \varphi \rightarrow [qp]^k[q]\psi$ . Thus,  $\varphi \rightarrow [qp]^k[q]\psi \notin K$ .  $\square$

#### PROPOSITION 7.16

The normal modal logic  $KD$  is adequate.

PROOF. Similar to the proof of Proposition 7.15, seeing that the unravelling of  $M$  around  $s$  is serial when  $M$  is serial.  $\square$

#### PROPOSITION 7.17

The normal modal logic  $KT$  is adequate.

PROOF. Similar to the proof of Proposition 7.15, defining this time, in order to ensure that  $M''$  is in  $C_{KT}$ ,  $M_s = (W_s, R_s, V_s)$  to be the reflexive closure of the unravelling of  $M$  around  $s$ . Note that this reflexive closure is obtained by forcing each possible world to be related to itself.  $\square$

PROPOSITION 7.18

Let  $n \geq 1$ . The normal modal logic  $K_n^{anc}$  is not adequate.

PROOF. Suppose the normal modal logic  $K_n^{anc}$  is adequate. Let  $\varphi = \top$ ,  $\psi = \diamond\top$ . Obviously,  $\deg(\varphi) \leq 2n$ . Moreover,  $\varphi \rightarrow [qp]^n[q]\psi \in K_n^{anc}$ . Since  $K_n^{anc}$  is adequate and  $\deg(\varphi) \leq 2n$ , therefore  $\varphi \rightarrow [qp]^n\perp \in K_n^{anc}$  or  $q \rightarrow \psi \in K_n^{anc}$ . In the former case,  $[qp]^n\perp \in K_n^{anc}$ . Let  $F$  be the frame consisting of a single reflexive element. Obviously,  $F \models K_n^{anc}$  and  $F \not\models [qp]^n\perp$ . Hence,  $[qp]^n\perp \notin K_n^{anc}$ : a contradiction. In the latter case,  $q \rightarrow \diamond\top \in K_n^{anc}$ . Let  $F$  be the frame consisting of a single irreflexive element. Obviously,  $F \models K_n^{anc}$  and  $F \not\models q \rightarrow \diamond\top$ . Hence,  $q \rightarrow \diamond\top \notin K_n^{anc}$ : a contradiction.  $\square$

PROPOSITION 7.19

Let  $n \geq 1$ . The normal modal logic  $Alt_n$  is adequate.

PROOF. Similar to the proof of Proposition 7.15, defining this time, in order to ensure that  $M''$  is in  $C_n^{det}$ ,  $M_s = (W_s, R_s, V_s)$  to be the restriction of the unravelling of  $M$  around  $s$  to the paths of length at most  $2k$ .  $\square$

PROPOSITION 7.20

If  $KB \subseteq L$  then  $L$  is not adequate.

PROOF. By Proposition 7.11.  $\square$

PROPOSITION 7.21

The normal modal logic  $K4$  is not adequate.

PROOF. Suppose the normal modal logic  $K4$  is adequate. Let  $\varphi = \Box r$  and  $k = 1$ . Obviously,  $\deg(\varphi) \leq 2k$ . Moreover,  $\varphi \rightarrow [qp]^k[q][p]\varphi \in K4$ . Since  $K4$  is adequate and  $\deg(\varphi) \leq 2k$ , therefore  $\varphi \rightarrow [qp]^k[q]\perp \in K4$  or  $p \rightarrow \varphi \in K4$ . In the former case,  $\Box r \rightarrow [qp]^k[q]\perp \in K4$ . Let  $F$  be the frame consisting of a single reflexive element. Obviously,  $F \models K4$  and  $F \not\models \Box r \rightarrow [qp]^k[q]\perp$ . Hence,  $\Box r \rightarrow [qp]^k[q]\perp \notin K4$ : a contradiction. In the latter case,  $p \rightarrow \Box r \in K4$ . Let  $F$  be the frame consisting of a single reflexive element. Obviously,  $F \models K4$  and  $F \not\models p \rightarrow \Box r$ . Hence,  $p \rightarrow \Box r \notin K4$ : a contradiction.  $\square$

PROPOSITION 7.22

Let  $m, n \geq 1$ . If  $m \leq n$  then the normal modal logic  $K_m^n$  is adequate.

PROOF. Suppose  $m \leq n$ . Hence,  $(m, n) \neq (2, 1)$  and by Proposition 7.13, it suffices to prove that for all formulas  $\varphi, \psi$ , for all distinct constants  $p, q$  and for all  $k \geq 0$ , if  $\deg(\varphi) \leq 2k$  then

- if  $\varphi \rightarrow [qp]^k[q][p]\varphi \in K_m^n$  then  $\varphi \rightarrow [qp]^k[q]\perp \in K_m^n$  or  $p \rightarrow \varphi \in K_m^n$ ,
- if  $\varphi \rightarrow [qp]^k[q]\psi \in K_m^n$  then  $\varphi \rightarrow [qp]^k\perp \in K_m^n$  or  $q \rightarrow \psi \in K_m^n$ .

We will consider the case when ‘ $m = 1$  and  $n = 2$ ’, the reader being invited to adapt the following line of reasoning to the other cases. Remark that  $K_1^2$  is the least normal modal logic containing all formulas of the form  $\diamond\chi \rightarrow \diamond\diamond\chi$ . As well, remark that  $C_1^2$  is the class of all dense frames, i.e. those frames  $F = (W, R)$  such that for all  $s, t \in W$ , if  $sRt$  then there exists  $u \in W$  such that  $sRu$  and  $uRt$ . Let  $\varphi, \psi$  be formulas and  $k \geq 0$  be such that  $\deg(\varphi) \leq 2k$ .

Suppose  $\varphi \rightarrow [qp]^k[q]\perp \notin K_1^2$  and  $p \rightarrow \varphi \notin K_1^2$ . Since  $K_1^2$  is complete with respect to  $C_1^2$ , therefore let  $F = (W, R)$  be a frame in  $C_1^2$  such that  $F \not\models \varphi \rightarrow [qp]^k[q]\perp$  and  $F' = (W', R')$  be a frame in  $C_1^2$  such that  $F' \not\models p \rightarrow \varphi$ . Let  $M = (W, R, V)$  be a model based on  $F$  such that  $M \not\models \varphi \rightarrow [qp]^k[q]\perp$  and  $M' = (W', R', V')$  be a model based on



$F'$  such that and  $M' \not\models p \rightarrow \varphi$ . Let  $s \in W$  be such that  $M, s \not\models \varphi \rightarrow [qp]^k[q]\perp$  and  $s' \in W'$  be such that  $M', s' \not\models p \rightarrow \varphi$ . Hence,  $M, s \models \varphi$ ,  $M, s \not\models [qp]^k[q]\perp$ ,  $M', s' \models p$  and  $M', s' \not\models \varphi$ . Let  $t_1, \dots, t_k, t_{k+1} \in V(q)$  and  $u_1, \dots, u_k \in V(p)$  be such that  $sRt_1Ru_1 \dots Rt_kRu_kRt_{k+1}$ . Let  $M_s = (W_s, R_s, V_s)$  be the dense closure of the unravelling of  $M$  around  $s$ . Note that this dense closure is obtained as the limit of the process consisting in adding an intermediate possible world between any two related possible worlds. In particular,  $(s)$ ,  $(s, t_1)$ ,  $(s, t_1, u_1)$ ,  $\dots$ ,  $(s, t_1, u_1, \dots, t_k)$ ,  $(s, t_1, u_1, \dots, t_k, u_k)$  and  $(s, t_1, u_1, \dots, t_k, u_k, t_{k+1})$  are in  $W_s$ . Moreover,  $(s)R_s(s, t_1)R_s(s, t_1, u_1) \dots R_s(s, t_1, u_1, \dots, t_k)R_s(s, t_1, u_1, \dots, t_k, u_k)R_s(s, t_1, u_1, \dots, t_k, u_k, t_{k+1})$ . The reader may easily verify that for all  $(s_0, \dots, s_n) \in W_s$  and for all formulas  $\chi$ , if  $\deg(\chi) + n \leq 2k$  then  $M, s_n \models \chi$  iff  $M_s, (s_0, \dots, s_n) \models \chi$ . Since  $M, s \models \varphi$ , therefore  $M_s, (s) \models \varphi$ . Let  $M''$  be the dense closure of  $(W'', R'', V'')$  where

- $W'' = W_s \cup W'$ ,
- $R'' = R_s \cup R' \cup \{(s, t_1, u_1, \dots, t_k, u_k, t_{k+1}), s'\}$ ,
- $V'' = V_s \cup V'$ .

The reader may easily verify that for all  $(s_0, \dots, s_n) \in W_s$  and for all formulas  $\chi$ , if  $\deg(\chi) + n \leq 2k$  then  $M_s, (s_0, \dots, s_n) \models \chi$  iff  $M'', (s_0, \dots, s_n) \models \chi$ . Since  $M_s, (s) \models \varphi$ , therefore  $M'', (s) \models \varphi$ . The reader may also easily verify that for all  $t' \in W'$  and for all formulas  $\chi$ ,  $M', t' \models \chi$  iff  $M'', t' \models \chi$ . Since  $M', s' \models p$  and  $M', s' \not\models \varphi$ , therefore  $M'', s' \models p$  and  $M'', s' \not\models \varphi$ . Since  $t_1, \dots, t_k, t_{k+1} \in V(q)$ ,  $u_1, \dots, u_k \in V(p)$ ,  $(s)R_s(s, t_1)R_s(s, t_1, u_1) \dots R_s(s, t_1, u_1, \dots, t_k)R_s(s, t_1, u_1, \dots, t_k, u_k)R_s(s, t_1, u_1, \dots, t_k, u_k, t_{k+1})$  and  $M'', (s) \models \varphi$ , therefore  $M'', (s) \not\models \varphi \rightarrow [qp]^k[q][p]\varphi$ . Thus,  $\varphi \rightarrow [qp]^k[q][p]\varphi \notin K_1^2$ .

Suppose  $\varphi \rightarrow [qp]^k\perp \notin K_1^2$  and  $q \rightarrow \psi \notin K_1^2$ . Since  $K_1^2$  is complete with respect to  $C_1^2$ , therefore let  $F = (W, R)$  be a frame in  $C_1^2$  such that  $F \not\models \varphi \rightarrow [qp]^k\perp$  and  $F' = (W', R')$  be a frame in  $C_1^2$  such that  $F' \not\models q \rightarrow \psi$ . Let  $M = (W, R, V)$  be a model based on  $F$  such that  $M \not\models \varphi \rightarrow [qp]^k\perp$  and  $M' = (W', R', V')$  be a model based on  $F'$  such that and  $M' \not\models q \rightarrow \psi$ . Let  $s \in W$  be such that  $M, s \not\models \varphi \rightarrow [qp]^k\perp$  and  $s' \in W'$  be such that  $M', s' \not\models q \rightarrow \psi$ . Hence,  $M, s \models \varphi$ ,  $M, s \not\models [qp]^k\perp$ ,  $M', s' \models q$  and  $M', s' \not\models \psi$ . Let  $t_1, \dots, t_k \in V(q)$  and  $u_1, \dots, u_k \in V(p)$  be such that  $sRt_1Ru_1 \dots Rt_kRu_k$ . Let  $M_s = (W_s, R_s, V_s)$  be the dense closure of the unravelling of  $M$  around  $s$ . In particular,  $(s)$ ,  $(s, t_1)$ ,  $(s, t_1, u_1)$ ,  $\dots$ ,  $(s, t_1, u_1, \dots, t_k)$  and  $(s, t_1, u_1, \dots, t_k, u_k)$  are in  $W_s$ . Moreover,  $(s)R_s(s, t_1)R_s(s, t_1, u_1) \dots R_s(s, t_1, u_1, \dots, t_k)R_s(s, t_1, u_1, \dots, t_k, u_k)$ . The reader may easily verify that for all  $(s_0, \dots, s_n) \in W_s$  and for all formulas  $\chi$ , if  $\deg(\chi) + n \leq 2k$  then  $M, s_n \models \chi$  iff  $M_s, (s_0, \dots, s_n) \models \chi$ . Since  $M, s \models \varphi$ , therefore  $M_s, (s) \models \varphi$ . Let  $M''$  be the dense closure of  $(W'', R'', V'')$  where

- $W'' = W_s \cup W'$ ,
- $R'' = R_s \cup R' \cup \{(s, t_1, u_1, \dots, t_k, u_k), s'\}$ ,
- $V'' = V_s \cup V'$ .

The reader may easily verify that for all  $(s_0, \dots, s_n) \in W_s$  and for all formulas  $\chi$ , if  $\deg(\chi) + n \leq 2k$  then  $M_s, (s_0, \dots, s_n) \models \chi$  iff  $M'', (s_0, \dots, s_n) \models \chi$ . Since  $M_s, (s) \models \varphi$ , therefore  $M'', (s) \models \varphi$ . The reader may also easily verify that for all  $t' \in W'$  and for all formulas  $\chi$ ,  $M', t' \models \chi$  iff  $M'', t' \models \chi$ . Since  $M', s' \models q$  and  $M', s' \not\models \psi$ , therefore  $M'', s' \models q$  and  $M'', s' \not\models \psi$ . Since  $t_1, \dots, t_k \in V(q)$ ,  $u_1, \dots, u_k \in V(p)$ ,  $(s)R_s(s, t_1)R_s(s, t_1, u_1) \dots R_s(s, t_1, u_1, \dots, t_k)R_s(s, t_1, u_1, \dots, t_k, u_k)$  and  $M'', (s) \models \varphi$ , therefore  $M'', (s) \not\models \varphi \rightarrow [qp]^k[q]\psi$ . Thus,  $\varphi \rightarrow [qp]^k[q]\psi \notin K_1^2$ .  $\square$

#### PROPOSITION 7.23

Let  $m, n \geq 1$ . If  $m > n$  and  $(m, n) \neq (2, 1)$  then the normal modal logic  $K_m^n$  is not adequate.

PROOF. Similar to the proof of Proposition 7.21, defining this time  $\varphi = \Box^n r$  and  $k = \lceil \frac{n+2}{2(m-n)} \rceil$   $(m-n) - 1$ .  $\square$

## 8 Unification in adequate normal modal logics

Let  $L$  be a normal modal logic. In this section, we shall prove that if  $L$  is adequate then  $L$  is nullary. Let  $\varphi$  be the formula considered in Section 6. Let  $\sigma_\perp, \sigma_\top$  and for all  $k \geq 0$ ,  $\sigma_k$  be the substitutions considered in Section 6. Remind that  $\sigma_\perp, \sigma_\top$  and for all  $k \geq 0$ ,  $\sigma_k$  are  $L$ -unifiers of  $\varphi$ .

LEMMA 8.1

Let  $\sigma$  be a substitution and  $k \geq 0$ . If  $L$  is adequate,  $\sigma$  is an  $L$ -unifier of  $\varphi$ ,  $\deg(\sigma(x)) \leq 2k$  and  $\deg(\sigma(y)) \leq 2k$  then  $\sigma_\top \preceq_L \sigma$  or  $\sigma_k \preceq_L \sigma$ .

PROOF. Suppose  $L$  is adequate,  $\sigma$  is an  $L$ -unifier of  $\varphi$ ,  $\deg(\sigma(x)) \leq 2k$ ,  $\deg(\sigma(y)) \leq 2k$ ,  $\sigma_\top \not\preceq_L \sigma$  and  $\sigma_k \not\preceq_L \sigma$ . Hence, by Lemma 6.3,  $\sigma(x) \leftrightarrow p \notin L$  or  $\sigma(y) \leftrightarrow q \notin L$ . Since  $\sigma$  is an  $L$ -unifier of  $\varphi$ , therefore  $\sigma(\varphi) \in L$ . Since  $\sigma(\varphi) = (\sigma(x) \rightarrow p) \wedge (\sigma(y) \rightarrow q) \wedge (\sigma(x) \rightarrow [q]\sigma(y)) \wedge (\sigma(y) \rightarrow [p]\sigma(x))$ , therefore  $(\sigma(x) \rightarrow p) \wedge (\sigma(y) \rightarrow q) \wedge (\sigma(x) \rightarrow [q]\sigma(y)) \wedge (\sigma(y) \rightarrow [p]\sigma(x)) \in L$ . Thus,  $\sigma(x) \rightarrow p \in L$ ,  $\sigma(y) \rightarrow q \in L$ ,  $\sigma(x) \rightarrow [q]\sigma(y) \in L$  and  $\sigma(y) \rightarrow [p]\sigma(x) \in L$ . Since  $\sigma$  is an  $L$ -unifier of  $\varphi$  and  $\sigma_k \not\preceq_L \sigma$ , therefore by Lemma 6.5,  $\sigma(x) \rightarrow [qp]^k[q]\perp \notin L$  or  $\sigma(y) \rightarrow [pq]^k[p]\perp \notin L$ .

Case ' $\sigma(x) \leftrightarrow p \notin L$  and  $\sigma(x) \rightarrow [qp]^k[q]\perp \notin L$ '. Since  $\sigma(x) \rightarrow p \in L$ , therefore  $p \rightarrow \sigma(x) \notin L$ . Since  $L$  is adequate,  $\deg(\sigma(x)) \leq 2k$  and  $\sigma(x) \rightarrow [qp]^k[q]\perp \notin L$ , therefore  $\sigma(x) \rightarrow [qp]^k[q][p]\sigma(x) \notin L$ . Hence,  $\sigma(x) \rightarrow [q]\sigma(y) \notin L$  or  $\sigma(y) \rightarrow [p]\sigma(x) \notin L$ : a contradiction.

Case ' $\sigma(x) \leftrightarrow p \notin L$  and  $\sigma(y) \rightarrow [pq]^k[p]\perp \notin L$ '. Since  $\sigma(x) \rightarrow p \in L$ , therefore  $p \rightarrow \sigma(x) \notin L$ . Since  $\sigma(y) \rightarrow [pq]^k[p]\perp \notin L$ , therefore  $\sigma(y) \rightarrow [pq]^k[p]\sigma(x) \notin L$ . Since  $L$  is adequate,  $\deg(\sigma(y)) \leq 2k$  and  $p \rightarrow \sigma(x) \notin L$ , therefore  $\sigma(y) \rightarrow [pq]^k[p]\sigma(x) \notin L$ . Hence,  $\sigma(x) \rightarrow [q]\sigma(y) \notin L$  or  $\sigma(y) \rightarrow [p]\sigma(x) \notin L$ : a contradiction.

Case ' $\sigma(y) \leftrightarrow q \notin L$  and  $\sigma(x) \rightarrow [qp]^k[q]\perp \notin L$ '. Since  $\sigma(y) \rightarrow q \in L$ , therefore  $q \rightarrow \sigma(y) \notin L$ . Since  $\sigma(x) \rightarrow [qp]^k[q]\perp \notin L$ , therefore  $\sigma(x) \rightarrow [qp]^k[q]\sigma(y) \notin L$ . Since  $L$  is adequate,  $\deg(\sigma(x)) \leq 2k$  and  $q \rightarrow \sigma(y) \notin L$ , therefore  $\sigma(x) \rightarrow [qp]^k[q]\sigma(y) \notin L$ . Hence,  $\sigma(x) \rightarrow [q]\sigma(y) \notin L$  or  $\sigma(y) \rightarrow [p]\sigma(x) \notin L$ : a contradiction.

Case ' $\sigma(y) \leftrightarrow q \notin L$  and  $\sigma(y) \rightarrow [pq]^k[p]\perp \notin L$ '. Since  $\sigma(y) \rightarrow q \in L$ , therefore  $q \rightarrow \sigma(y) \notin L$ . Since  $L$  is adequate,  $\deg(\sigma(y)) \leq 2k$  and  $\sigma(y) \rightarrow [pq]^k[p]\perp \notin L$ , therefore  $\sigma(y) \rightarrow [pq]^k[p][q]\sigma(y) \notin L$ . Hence,  $\sigma(x) \rightarrow [q]\sigma(y) \notin L$  or  $\sigma(y) \rightarrow [p]\sigma(x) \notin L$ : a contradiction.  $\square$

LEMMA 8.2

If  $L$  is adequate then there exists no  $L$ -minimal  $L$ -complete set of  $L$ -unifiers of  $\varphi$ .

PROOF. Suppose  $L$  is adequate and there exists an  $L$ -minimal  $L$ -complete set of  $L$ -unifiers of  $\varphi$ . Let  $\Sigma$  be an  $L$ -minimal  $L$ -complete set of  $L$ -unifiers of  $\varphi$ . By the fact that  $\Sigma$  is an  $L$ -complete set of  $L$ -unifiers of  $\varphi$ , let  $\sigma \in \Sigma$  be such that  $\sigma \preceq_L \sigma_\perp$ . Let  $k \geq 0$  be such that  $\deg(\sigma(x)) \leq 2k$  and  $\deg(\sigma(y)) \leq 2k$ . Since  $L$  is adequate and  $\sigma \in \Sigma$ , therefore by Lemma 8.1 and the fact that  $\Sigma$  is a set of  $L$ -unifiers of  $\varphi$ ,  $\sigma_\top \preceq_L \sigma$  or  $\sigma_k \preceq_L \sigma$ .

Case ' $\sigma_\top \preceq_L \sigma$ '. Since  $\sigma \preceq_L \sigma_\perp$ , therefore by Lemma 5.2,  $\sigma_\top \preceq_L \sigma_\perp$ . Let  $\tau$  be a substitution such that  $\sigma_\top \circ \tau \preceq_L \sigma_\perp$ . Hence,  $\tau(\sigma_\top(x)) \leftrightarrow \sigma_\perp(x) \in L$  and  $\tau(\sigma_\top(y)) \leftrightarrow \sigma_\perp(y) \in L$ . Since  $\tau(\sigma_\top(x)) = p$ ,  $\sigma_\perp(x) = \perp$ ,  $\tau(\sigma_\top(y)) = q$  and  $\sigma_\perp(y) = \perp$ , therefore  $p \leftrightarrow \perp \in L$  and  $q \leftrightarrow \perp \in L$ . Thus,  $\neg p \wedge \neg q \in L$ . Consequently,  $\perp \in L$ . Hence,  $L$  is not coherent. Thus,  $L$  is not adequate: a contradiction.

Case ' $\sigma_k \preceq_L \sigma$ '. By the fact that  $\Sigma$  is an  $L$ -complete set of  $L$ -unifiers of  $\varphi$ , let  $\tau \in \Sigma$  be such that  $\tau \preceq_L \sigma_{k+1}$ . By Lemma 4.1,  $[qp]^k[q]\perp \rightarrow [qp]^{k+1}[q]\perp \in L$  and  $[pq]^k[p]\perp \rightarrow [pq]^{k+1}[p]\perp \in L$ . Hence,  $p \wedge [qp]^k[q]\perp \rightarrow [qp]^{k+1}[q]\perp \in L$  and  $q \wedge [pq]^k[p]\perp \rightarrow [pq]^{k+1}[p]\perp \in L$ . Thus, by Lemma 6.6,  $\sigma_{k+1} \preceq_L \sigma_k$ . Since  $\sigma_k \preceq_L \sigma$  and  $\tau \preceq_L \sigma_{k+1}$ , therefore by Lemma 5.2,  $\tau \preceq_L \sigma$ . Since  $\sigma, \tau \in \Sigma$ , therefore by the fact that  $\Sigma$  is an  $L$ -minimal set of substitutions,  $\tau \simeq_L \sigma$ . Since  $\sigma_k \preceq_L \sigma$  and  $\tau \preceq_L \sigma_{k+1}$ , therefore by Lemmas 5.1 and 5.2,  $\sigma_k \preceq_L \sigma_{k+1}$ . Consequently, by Lemma 6.6,  $p \wedge [qp]^{k+1}[q]\perp \rightarrow [qp]^k[q]\perp \in L$  and  $q \wedge [pq]^{k+1}[p]\perp \rightarrow [pq]^k[p]\perp \in L$ . Hence,  $L$  is not coherent. Thus,  $L$  is not adequate: a contradiction.  $\square$

PROPOSITION 8.3

If  $L$  is adequate then  $L$  is nullary.

PROOF. By Lemma 8.2.  $\square$

COROLLARY 8.4

The following normal modal logics are nullary:

- the normal modal logics  $K$ ,  $KD$  and  $KT$ ,
- for all  $n \geq 1$ , the normal modal logic  $Alt_n$ ,
- for all  $m, n \geq 1$ , if  $m \leq n$  then the normal modal logic  $K_m^n$ .

TABLE 1. Known facts and open problems in the determination of the type of unification with constants in some of the most popular normal modal logics

$L$	Type of $L$ for unif. with constants
$K$	Nullary—[21]
$KD$	Nullary—[4]
$KT$	Nullary—Corollary 8.4
$KB$	?
$KDB$	?
$KTB$	?
$K5$	?
$KD5$	?
$K45$	Unitary or nullary—[5]
$KD45$	Unitary or nullary—[5]
$S5$	Unitary—[1]
$K4$	Finitary—[16]
$S4$	Finitary—[16]
$K4.3$	?
$KD4.3$	?
$S4.3$	Unitary—[14]
$GL$	Finitary—[16]
$K_1^{bou}$	Unitary—see Section 5
$K_n^{bou}$ when $n \geq 2$	?
$Alt_1$	Nullary—[6]
$Alt_n$ when $n \geq 2$	Nullary—see Section 5
$K_m^n$ when $m \leq n$	Nullary—Corollary 8.4
$K_m^n$ when $m > n$ and $(m, n) \neq (2, 1)$	?

TABLE 2. Known facts and open problems in the determination of the type of elementary unification in some of the most popular normal modal logics

$L$	Type of $L$ for elementary unif.
$K$	Nullary—[21]
$KD$	?
$KT$	?
$KB$	?
$KDB$	?
$KTB$	?
$K5$	?
$KD5$	?
$K45$	Unitary or nullary—[5]
$KD45$	Unitary—[7]
$S5$	Unitary—[1]
$K4$	Finitary—[16]
$S4$	Finitary—[16]
$K4.3$	?
$KD4.3$	?
$S4.3$	Unitary—[14]
$GL$	Finitary—[16]
$K_1^{bou}$	Unitary—see Section 5
$K_n^{bou}$ when $n \geq 2$	?
$Alt_1$	Nullary—[6]
$Alt_n$ when $n \geq 2$	Nullary—see Section 5
$K_m^n$ when $m \leq n$	?
$K_m^n$ when $m > n$ and $(m, n) \neq (2, 1)$	?

PROOF. By Propositions 7.15–7.17, 7.19, 7.22 and 8.3. □

## 9 Conclusion and open problems

In this paper, we have proved that unification in a normal modal logic  $L$  is of nullary type when  $L$  is adequate. Remark that in order to present our proof above, we had to assume that the language of modal logic contains at least two distinct variables and two distinct constants. Note that the nullariness of  $K$  has been proved by Jeřábek [21] who only assumed that the language contains at least one variable. This shows that  $K$  is nullary both for unification with constants and for elementary unification. As well, note that the nullariness of  $KD$  has been proved by Balbiani and Gencer [4] who only assumed that the language contains at least one variable and one constant. This only shows that  $KD$  is nullary for unification with constants. As for the nullariness of  $Alt_1$ , it has also been proved by Balbiani and Tinchev [6] who only assumed that the language contains at least one variable. This shows that  $Alt_1$  is nullary both for unification with constants and for elementary unification. As mentioned in Section 5, the proof given in [6] that  $Alt_1$  is nullary can be easily adapted for all  $n \geq 2$  to a proof that  $Alt_n$  is nullary. Much remains to be done. See Tables 1 and 2 for known facts and open

problems in the determination of the type of unification with constants and the type of elementary unification in some of the most popular normal modal logics.

## Acknowledgments

We make a point of thanking the referees for their feedback: their helpful comments and their useful suggestions have been essential for improving the correctness and the readability of a preliminary version of this paper. Special acknowledgement is also heartily granted to Çiğdem Gencer (Aydın University, Turkey), Mikhail Rybakov (Tver University, Russia) and Tinko Tinchev (Sofia University, Bulgaria) for their valuable remarks.

## References

- [1] F. Baader and S. Ghilardi. Unification in modal and description logics. *Logic Journal of the IGPL*, **19**, 705–730, 2011.
- [2] S. Babenyshev, V. Rybakov, R. Schmidt and D. Tishkovsky. A tableau method for checking rule admissibility in  $S4$ . *Electronic Notes in Theoretical Computer Science*, **262**, 17–32, 2010.
- [3] S. Babenyshev and V. Rybakov. Unification in linear temporal logic  $LTL$ . *Annals of Pure and Applied Logic*, **162**, 991–1000, 2011.
- [4] P. Balbiani and Ç. Gencer.  $KD$  is nullary. *Journal of Applied Non-Classical Logics*, **27**, 196–205, 2017.
- [5] P. Balbiani and Ç. Gencer. Unification in epistemic logics. *Journal of Applied Non-Classical Logics*, **27**, 91–105, 2017.
- [6] P. Balbiani and T. Tinchev. Unification in modal logic  $Alt_1$ . *Advances in Modal Logic. College Publications*, 117–134, 2016.
- [7] P. Balbiani and T. Tinchev. Elementary unification in modal logic  $KD45$ . *Journal of Applied Logics—IFCoLog Journal of Logics and Their Applications*, **5**, 301–317, 2018.
- [8] P. Blackburn, M. de Rijke and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.
- [9] A. Chagrov and M. Zakharyashev. *Modal Logic*. Oxford University Press, 1997.
- [10] B. Chellas. *Modal Logic. An Introduction*. Cambridge University Press, 1980.
- [11] W. Dzik. Unitary unification of  $S5$  modal logics and its extensions. *Bulletin of the Section of Logic*, **32**, 19–26, 2003.
- [12] W. Dzik. *Unification Types in Logic*. Wydawnictwo Uniwersytetu Śląskiego, 2007.
- [13] W. Dzik. Remarks on projective unifiers. *Bulletin of the Section of Logic*, **40**, 37–46, 2011.
- [14] W. Dzik and P. Wojtylak. Projective unification in modal logic. *Logic Journal of the IGPL*, **20**, 121–153, 2012.
- [15] Ç. Gencer and D. de Jongh. Unifiability in extensions of  $K4$ . *Logic Journal of the IGPL*, **17**, 159–172, 2009.
- [16] S. Ghilardi. Best solving modal equations. *Annals of Pure and Applied Logic*, **102**, 183–198, 2000.
- [17] S. Ghilardi and L. Sacchetti. Filtering unification and most general unifiers in modal logic. *Journal of Symbolic Logic*, **69**, 879–906, 2004.
- [18] R. Iemhoff. On the admissible rules of intuitionistic propositional logic. *Journal of Symbolic Computation*, **66**, 281–294, 2001.
- [19] R. Jansana. Some logics related to von Wright’s logic of place. *Notre Dame Journal of Formal Logic*, **35**, 88–98, 1994.
- [20] E. Jeřábek. Complexity of admissible rules. *Archive for Mathematical Logic*, **46**, 73–92, 2007.

- [21] E. Jeřábek. Blending margins: the modal logic  $K$  has nullary unification type. *Journal of Logic and Computation*, **25**, 1231–1240, 2015.
- [22] V. Rybakov. A criterion for admissibility of rules in the model system  $S4$  and the intuitionistic logic. *Algebra and Logic*, **23**, 369–384, 1984.
- [23] V. Rybakov. Bases of admissible rules of the logics  $S4$  and  $Int$ . *Algebra and Logic*, **24**, 55–68, 1985.
- [24] V. Rybakov. *Admissibility of Logical Inference Rules*. Elsevier Science, 1997.
- [25] V. Rybakov. Construction of an explicit basis for rules admissible in modal system  $S4$ . *Mathematical Logic Quarterly*, **47**, 441–446, 2001.
- [26] V. Rybakov. Unification in common knowledge logics. *Bulletin of the Section of Logic*, **31**, 207–215, 2002.
- [27] V. Rybakov. Logical consecutions in discrete linear temporal logic. *Journal of Symbolic Logic*, **70**, 1137–1149, 2005.
- [28] V. Rybakov. Multi-modal and temporal logics with universal formula—reduction of admissibility to validity and unification. *Journal of Logic and Computation*, **18**, 509–519, 2008.
- [29] V. Rybakov, M. Terziler and Ç. Gencer. An essay on unification and inference rules for modal logics. *Bulletin of the Section of Logic*, **28**, 145–157, 1999.