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# POSSIBILISTIC REASONING FROM PARTIALLY ORDERED BELIEF BASES WITH THE SURE THING PRINCIPLE

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## Abstract

We consider the problem of reasoning from logical bases equipped with a partial order expressing relative certainty, with a view to construct a partially ordered deductive closure via syntactic inference. At the syntactic level we use a language expressing pairs of related formulas and axioms describing the properties of the order. Reasoning about uncertainty using possibility theory relies on the idea that if an agent believes each among two propositions to some extent, then this agent should believe their conjunction to the same extent. This principle is known as adjunction. Adjunction is often accepted in epistemic logic but fails with probabilistic reasoning. In the latter, another principle prevails, namely the sure thing principle, that claims that the certainty ordering between propositions should be invariant to the addition or deletion of possible worlds common to both sets of models of these propositions. Pursuing our work on relative certainty logic based on possibility theory, we propose a qualitative likelihood logic that respects the sure thing principle, albeit using a likelihood relation that preserves adjunction.

**Keywords** : partially ordered bases, possibility theory, adjunction rule, comparative probability

## 1 Introduction

The representation of partial belief often uses a numerical setting, prominently the one of probability theory, but also weaker non-additive settings such as belief functions or imprecise probabilities (see [18] for a survey). However, this kind of approach requires the use of

elicitation procedures so as to force agents to provide degrees of belief through a given protocol (for instance, using the betting metaphor, assigning prices to gambles or risky events, or using analogy with the frequentist setting of drawing balls from a known urn). Inevitably, the resulting numbers will not have infinite precision, which leads either to consider precise figures as suitable idealization, or to take into account the imprecision of assessments, which may lead to more complex computations.

Reasoning with uncertain knowledge often consists of attaching belief weights to propositions of interest and computing belief weights of other propositions of interest, using some appropriate inference methods. This approach was early considered by De Finetti [10] (see [29] for a translation), and then taken over by many other scholars (Adams and Levine[1], Coletti and Scozzafava [9], Nilsson [34], etc.).

In this paper, we deliberately give up assigning belief weights to propositions. We assume that uncertain knowledge is based on stating that some propositions are more believed than others. This is the least we can expect from agents expressing their beliefs. In the case of probability theory, it comes down to studying properties of the relation “more probable than” first introduced by De Finetti [10], and later by Ramsey, and Savage, among others (see Fishburn[21] for an early survey). Comparative probabilities are total orders on propositions, that obey a special case of the so-called sure thing principle of Savage [35], stating that the fact that a proposition is more probable than another one is not affected by the probabilities of their common models. We call this property *preadditivity*, to highlight the known fact that on finite settings this property is not sufficient to ensure the existence of a probability measure representing the ordering between propositions [26]. There is not a long tradition on logics for comparative probability that do not refer to a numerical underpinning. This point is discussed in detail by Walley and Fine [37] who provide an early overview on modal, conditional and comparative probability logics.

Another kind of uncertainty relation, originally introduced by Lewis [32], are comparative possibility relations, independently introduced by Dubois [11] along with their dual called necessity relations. While Lewis introduced these concepts in connection with the logical representation of counterfactuals, Dubois viewed possibility relations as the ordinal counterpart of Zadeh’s possibility measures [39]. These relations are weak orders that do not obey the sure thing principle, but they are instrumental in non-monotonic reasoning and belief revision [17, 2] (where necessity relations are called epistemic entrenchments). This setting also captures the notion of accepted beliefs [15]: the agent reasons with such beliefs as if they were true ones, so that the condition that the conjunction of accepted beliefs is an accepted belief is adopted, like in epistemic logic. For the sake of clarity, we call qualitative plausibility and certainty relations the generalisation of possibility and necessity relations to the partially ordered setting. The key property for such relations is called *qualitativeness* [22], which encodes the idea that a possible world is always more likely than the disjunction of less likely worlds.

Logics for reasoning with totally ordered comparative possibility statements have been first studied by Lewis [31]. Possibilistic logic [19] is an alternative setting where a total order on a subset of propositions is encoded by means of weighted formulas, where weights attached to formulas are taken from a totally ordered symbolic scale. In this paper we focus on partial orders, as we consider that agents may only have a lacunary knowledge of the relative beliefs of propositions. Approaches to reasoning from logical bases equipped with a partial order expressing relative certainty have been proposed by Halpern [25] using a modal logic framework inspired by Lewis works, which means a very rich language. A simpler framework, called relative certainty logic and focusing on strict partial orders, yet adopting similar axioms as Halpern, is presented in [36], where the purpose is to construct a partially ordered deductive closure. The idea is to interpret a partially ordered base as a partial necessity ordering. At the syntactic level the language expresses pairs of related formulas; axioms and inference rules describe the properties of the partial certainty order. The semantics consists in assuming that the partial order on formulas stems from a partial order between the corresponding sets of models (and not between models as in possibilistic logic).

Moving from the totally ordered to the partially ordered setting is non-trivial. The difficult points are twofold: (i) equivalent definitions in the totally ordered case are no longer equivalent in the partially ordered one, and (ii) a partial possibility order on subsets of a set cannot be represented by a partial order between elements of this set. This point is especially explained in [36].

In this paper, we pursue the work initiated in [36] with a view to study how the preadditivity of comparative probability can be used to refine the relative certainty logic. In the totally ordered case, qualitiveness is almost incompatible with preadditivity [15]. In the partially ordered setting, we get a qualitative likelihood logic that is adjunctive, but respects the sure thing principle, that we compare with the qualitative certainty logic of [36]. Moreover we show that the latter logic can be used to facilitate inference in the former.

The paper is structured as follows: in the next section we provide an overview of confidence relations between sets of states, including comparative possibility and probability. Then we provide characteristic properties of qualitative plausibility, certainty and (preadditive) likelihood relations, in the partially ordered setting. We show that there is a bijection between qualitative plausibility and qualitative likelihood relations. Based on these new results we propose in section 3 a general setting for reasoning about uncertainty using confidence relations, which extends the methodology introduced in [36] for qualitative certainty logic. Then, in sections 4 and 5 we respectively focus on the qualitative likelihood logic and on its connection with relative certainty logic.

## 2 Qualitative confidence relations comparing subsets

In a non-numerical setting, it is natural to represent confidence in propositions by means of a partial preorder  $\succeq$  on subsets  $A, B, C, \dots$  of a set of states of affairs  $S$ . This idea goes back to De Finetti's [10] comparative probabilities, and is presented in more details in Fine's book [20]. Other proposals are comparative possibilities of Lewis [32] later independently proposed, along with their dual necessity relations by one of the authors [11] in contrast with comparative probabilities. These are examples of complete preorders (reflexive, complete and transitive relations) on the power set  $\wp(S)$ . Various examples of confidence relations have been discussed by Halpern [23, 22] in connection with non-monotonic reasoning. They are called acceptance relations in [15]. In some cases, confidence orderings stem from a total or partial plausibility ordering on  $S$ . This is the case for comparative possibility relations and their refinements [14, 12], and also for relations built from a partial order on elements, studied by Halpern [25]. In this section we review such relations and their properties.

Given a reflexive relation  $\succeq$  on  $\wp(S)$  we can derive three companion relations:

- The strict part of  $\succeq$ :  $A \succ B$  iff  $A \succeq B$ , but not  $B \succeq A$
- The indifference relation  $A \sim B$  iff  $A \succeq B$  and  $B \succeq A$
- The incomparability relation:  $A \pm B$  iff neither  $A \succeq B$  nor  $B \succeq A$

Moreover, we can also define the *dual*  $\succeq^d$  of a relation  $\succeq$  on  $\wp(S)$  as:

$$A \succeq^d B \text{ iff } \overline{B} \succeq \overline{A}$$

There are minimal requirements a confidence relation should satisfy in order to justify this name.

### 1. Compatibility with Inclusion (CI) If $B \subseteq A$ then $A \succeq B$

Indeed if  $B$  implies  $A$  there is no point for  $B$  to be more likely than  $A$ .<sup>1</sup>

### 2. Orderliness (O) If $A \succ B$ , $A \subseteq A'$ , and $B' \subseteq B$ , then $A' \succ B'$

This property, already mentioned by Walley and Fine [37], and also used by Friedman and Halpern [22], is a variant of, but not equivalent to, the former. It also reflects compatibility with logical deduction.

### 3. Quasi-Transitivity (QT) If $A \succ B$ , and $B \succ C$ , then $A \succ C$

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<sup>1</sup>Friedman and Halpern [23] call "Plausibility measure" a partial relation that satisfies (CI); however this name may be judged misleading, since plausibility is a notion dual to belief, as often used in evidence theory.

Should this property be false for an agent, one may question her rationality. These are the three minimal properties we can expect from a partial confidence relation.

**Definition 1.** A relation on  $\wp(S)$  is called a confidence relation if it satisfies (CI, O, QT). Its strict part is called a strict confidence relation.

This terminology was proposed in [12]. It is clear that a confidence relation is reflexive and consistent in the sense that  $S \succeq A \succeq \emptyset$  for all subsets  $A$  of  $S$ . Note that we can do away with the two monotonicity conditions (CI) and (O) if we modify the latter by requiring it for  $\succeq$  instead of  $\succ$ . Moreover, a strict confidence relation is a strict partial order satisfying (O). Finally, it can be easily verified that the dual of a confidence relation is again a confidence relation.

## 2.1 Complete and transitive confidence relations

It is quite often the case that partial belief is represented numerically via a set-function  $f : \wp(S) \rightarrow [0, 1]$ , for instance a probability measure. A set-function  $f$  is said to *represent* a confidence relation  $\succeq$  provided that for all subsets  $A, B$  of  $S$ ,  $A \succeq B$  if and only if  $f(A) \geq f(B)$ ,  $f(\emptyset) = 0$ ,  $f(S) = 1$ .

Of course, if this is so, the confidence relation  $\succeq$  should be transitive and complete (hence reflexive):

- *Transitivity:* If  $A \succeq B$ , and  $B \succeq C$ , then  $A \succeq C$
- *Completeness:*  $A \succeq B$  or  $B \succeq A$

It is easy to see that complete and transitive confidence relations are represented by *capacities*, which are monotonic set-functions, such that if  $A \subseteq B$  then  $f(A) \leq f(B)$ , which expresses (CI) (for instance, [18]). In fact, for transitive and complete relations, (CI) implies (O). Important examples of complete and transitive confidence relations are

- *Comparative probabilities* [10, 20]: They are complete and transitive confidence relations that obey the preadditivity property:

**Preadditivity (P)** If  $A \cap (B \cup C) = \emptyset$  then  $(B \succeq C \text{ iff } A \cup B \succeq A \cup C)$

- *Comparative possibilities* [32, 11]: They are complete and transitive confidence relations that satisfy a property that is a variant of the former:

**Stability for Union (SU)** If  $A \succeq B$  then  $A \cup C \succeq B \cup C$

Comparative possibility relations, denoted by  $\succeq_{\Pi}$ , can be represented by and only by possibility measures [11]. They are set-functions  $\Pi : \wp(S) \rightarrow [0, 1]$  such that  $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$  [39, 16]. This is because the (SU) axiom for complete and transitive confidence relations is equivalent to: If  $A \succeq_{\Pi} B$  then  $A \sim_{\Pi} A \cup B$ . Comparative possibility relations on finite sets are completely characterised by the restriction  $\succeq_{\pi}$  of  $\succeq_{\Pi}$  to singletons on  $S$ . Namely [11]:

$$A \succeq_{\Pi} B \iff \forall s_2 \in B, \exists s_1 \in A : s_1 \succeq_{\pi} s_2 \quad (1)$$

$$\iff \exists s_1 \in A, \forall s_2 \in B : s_1 \succeq_{\pi} s_2 \quad (2)$$

This property, which shows the simplicity of this approach, reflects the fact that a possibility measure  $\Pi$  derives from a possibility distribution  $\pi : S \rightarrow [0, 1]$ , in the sense that  $\Pi(A) = \max_{s \in A} \pi(s)$ . In the scope of uncertainty modeling,  $\pi(s)$  can be viewed as a degree of plausibility of  $s$ , and the condition  $\max_{s \in S} \pi(s) = 1$  must be satisfied. The possibility degree  $\Pi(A)$  can be interpreted as a degree of unsurprisingness of  $A$ , i.e., the degree to which there is no reason not to believe  $A$  (which does not imply a reason for believing it).

The conjugate functions  $N(A) = 1 - \Pi(\overline{A})$ , called necessity measures [16], express the idea that  $A$  is certain to some extent, that is,  $A$  is true in all situations that are plausible enough. The corresponding necessity relations  $\succeq_N$  have a characteristic axiom called

**Stability for intersection (SI):** If  $A \succeq_N B$  then  $A \cap C \succeq_N B \cap C$

It is easy to check [11] that necessity relations can be defined from possibility relations by duality:  $A \succeq_N B$  if and only if  $\overline{B} \succeq_{\Pi} \overline{A}$ , so that

$$A \succeq_N B \iff \forall s_2 \in \overline{A}, \exists s_1 \in \overline{B} : s_1 \succeq_{\pi} s_2 \quad (3)$$

$$\iff \exists s_1 \in \overline{B}, \forall s_2 \in \overline{A} : s_1 \succeq_{\pi} s_2 \quad (4)$$

Comparative possibility relations satisfy properties that indicate their qualitative nature:

**Qualitiveness (Q)** If  $A \cup B \succ_{\Pi} C$  and  $A \cup C \succ_{\Pi} B$ , then  $A \succ_{\Pi} B \cup C$

**Negligibility (N)** If  $A \succ_{\Pi} B$  and  $A \succ_{\Pi} C$ , then  $A \succ_{\Pi} B \cup C$

the second one being a consequence of the first. Negligibility expresses the non-compensatory nature of possibility measures, according to which the union of unlikely singletons cannot override a very plausible one.

Necessity relations obey counterparts of (Q) and (N):

**Dual qualitiveness (Q<sup>d</sup>):** If  $A \succ_N B \cap C$  and  $B \succ_N A \cap C$  then  $A \cap B \succ_N C$

**Adjunction (A):** If  $A \succ_N C$  and  $B \succ_N C$  then  $A \cap B \succ_N C$

These properties make it clear that the family of sets  $\{A : A \succ_N C\}$  is a filter (closed under inclusion and intersection) or in terms of propositions, deductively closed. This closure property for confidence measures is also characteristic of necessity relations for complete and transitive confidence relations [15].

Preadditive complete and transitive confidence relations  $\succeq_P$ , called comparative probabilities, behave very differently. Given a probability measure on  $S$ , the relation  $A \succeq B$  if and only if  $P(A) \geq P(B)$  for some probability measure on a finite set is indeed preadditive, complete and transitive. However the converse is false, namely it has been known since the 1950's [26] that there are comparative probability relations that are not representable by a probability measure; see also [33]. Nevertheless comparative probability relations are self dual, in the sense of the following property:

$$\text{Self-duality (D)} \quad A \succeq_P B \text{ iff } \overline{B} \succeq_P \overline{A}$$

However the fact that comparative probability relations are more general than confidence relations induced by probabilities highlights the fact that, contrary to comparative possibility and necessity relations, they cannot be defined by a complete preorder on  $S$ : the restriction of  $\succeq_P$  on singletons is not enough to reconstruct it. In fact comparative probabilities can be represented by special kinds of belief functions inducing a self-dual order [38].

Interestingly, there are comparative probability relations that satisfy the qualitiveness properties. It is proved in [3] that they correspond to so-called big-stepped probabilities on  $S$ : there is a probability distribution  $p$  such that  $p(s_1) > p(s_2) > \dots > p(s_{n-1}) > p(s_n)$ , with  $\forall i = 1, \dots, n-1, p(s_i) > \sum_{j=i+1}^n p(s_j)$ , and then  $A \succeq_P B$  if and only if  $P(A) \geq P(B)$ . The probabilities of singletons form a super-increasing sequence. Moreover if we consider the possibility ordering  $s_1 >_\pi s_2 >_\pi \dots >_\pi s_{n-1} >_\pi s_n$ , then, for non-elementary events  $A, B$  we have that  $A \succ_\Pi B$  implies  $A \succ_P B$ . In other words, the comparative probability relation induced by a big-stepped probability refines the possibility relation (see also [12]).

In this paper, we generalize possibility relations and necessity relations to partial orders on  $S$ , and consider their preadditive refinements.

## 2.2 Partial qualitative confidence relations

In this section we consider partial confidence relations satisfying property (Q). The four properties (CI), (O), (QT) and (Q) are not independent [25, 5].

**Proposition 1.** *If a relation on  $\wp(S)$  satisfies (Q) and (O), this relation and its dual are transitive.*

**Proof of Proposition 1:**

We use a relation denoted by  $\triangleright$  that can stand for  $\succeq$  or its strict part. Suppose  $A \triangleright B$  and  $B \triangleright C$ . Then, from (O),  $A \cup C \triangleright B$  and  $A \cup B \triangleright C$ , and from (Q):  $A \triangleright B \cup C$ , then by (O),  $A \triangleright C$ . A similar proof holds for the dual relation.  $\square$

Partial confidence relations satisfying property (Q) generalize comparative possibilities. However, in the following we consider asymmetric relations of this kind, to which (CI) does not apply:

**Definition 2.** A qualitative plausibility relation is an asymmetric relation  $\succ_{pl}$  on  $\wp(S)$  that satisfies (Q) and (O).

Due to Proposition 1, a qualitative plausibility relation is indeed a strict partial order on  $\wp(S)$  since it is transitive. Moreover,

**Proposition 2.** A qualitative plausibility relation satisfies (N), and (SU) in contrapositive form: If  $A \cup C \succ_{pl} B \cup C$  then  $A \succ_{pl} B$ .

**Proof of Proposition 2:**

(N) is an obvious consequence of (Q) and (O). For (SU), suppose  $A \cup C \succ_{pl} B \cup C$ . By (O), we infer that  $A \cup (B \cup C) \succ_{pl} C$ . Applying (Q) yields  $A \succ_{pl} B \cup C$ , which by (O), results in  $A \succ_{pl} B$  [7].  $\square$

Another useful property related to (SU) is:

**Proposition 3.** A qualitative plausibility relation is such that: If  $A \succ_{pl} B$  and  $C \succ_{pl} D$  then  $A \cup C \succ_{pl} B \cup D$ .

**Proof of Proposition 3:**

Due to (O),  $A \succ_{pl} B$  and  $C \succ_{pl} D$  imply  $A \cup C \cup D \succ_{pl} B$  and  $A \cup C \cup B \succ_{pl} D$ , and then by (Q),  $A \cup C \succ_{pl} B \cup D$  follows.  $\square$

Now we introduce another partial order on a set of events, called a qualitative certainty relation:

**Definition 3.** A qualitative certainty relation, denoted by  $\succ_{cr}$ , is an asymmetric relation on  $\wp(S)$  that satisfies  $Q^d$  and O.

It is clear that  $\succ_{cr}$  is a qualitative certainty relation if and only if its dual relation is a qualitative plausibility relation. In particular, from the above results, it easily follows that a qualitative certainty relation is transitive, satisfies adjunction, and (SI) under the form: If  $A \cap C \succ_{cr} B \cap C$  then  $A \succ_{cr} B$ . Moreover, if  $A \succ_{cr} B$  and  $C \succ_{cr} D$  then  $A \cap C \succ_{cr} B \cap D$ .

Contrary to the terminology used in [23], the use of *plausibility* vs. *certainty* to name confidence relations satisfying (Q) vs. its dual property (Q<sup>d</sup>) makes the point that such relations are dual to each other, and reflect the dual pairs (possibility, necessity), (plausibility, belief) in other uncertainty theories, where the second concept in each pair is more committing than the first one.

Qualitative plausibility and certainty relations are instrumental for defining a semantics for non-monotonic reasoning (as explained in [22, 15]). Namely consider the following properties for a partial order on  $\wp(S)$ , inspired from [27]:

- **Conditional Closure by Implication (CCI)** If  $A \subseteq B$  and  $A \cap C \succ \overline{A} \cap C$  then  $B \cap C \succ \overline{B} \cap C$
- **Conditional Closure by Conjunction (CCC)** If  $C \cap A \succ C \cap \overline{A}$  and  $C \cap B \succ C \cap \overline{B}$  then  $C \cap (A \cap B) \succ C \cap \overline{(A \cap B)}$
- **Left Disjunction (OR)** If  $A \cap C \succ A \cap \overline{C}$  and  $B \cap C \succ B \cap \overline{C}$  then  $(A \cup B) \cap C \succ (A \cup B) \cap \overline{C}$
- **Cut (CUT)** If  $A \cap B \succ A \cap \overline{B}$  and  $A \cap B \cap C \succ A \cap B \cap \overline{C}$  then  $A \cap C \succ A \cap \overline{C}$
- **Cautious Monotony (CM)** If  $A \cap B \succ A \cap \overline{B}$  and  $A \cap C \succ A \cap \overline{C}$  then  $A \cap B \cap C \succ A \cap B \cap \overline{C}$

These properties are intuitive when  $A \succ \overline{A}$  is interpreted as “A is an accepted belief”, and  $A \cap C \succ \overline{A} \cap C$  as “A is an accepted belief in the context C”, hence the name “acceptance relations” for qualitative plausibility relations in [15]. In that work, it has been proved that:

**Proposition 4.**

- (O) implies (CCI).
- If a relation between subsets of S satisfies (Q) and (O), then it satisfies (CCI), (CCC), (OR), (CUT), (CM).
- For any relation that satisfies (O), (CCC) is equivalent to (Q).

See also [6] for the two first results.

It is clear that qualitative plausibility relations satisfy all these properties and are ideally fit for non-monotonic reasoning with conditional assertions of the form  $A \mid \sim B$ , which stand for  $A \cap B \succ_{pl} A \cap \overline{B}$  [27]. Note that properties (CCI), (CCC), (OR), (CUT), (CM) only involve the comparison of disjoint subsets. It is proved in [15], and follows from Proposition 4 that if the restriction of a confidence relation to disjoint subsets satisfies (CCI), (CCC), (OR), (CUT), (CM) then it is the restriction of a qualitative plausibility relation.

One way to construct a qualitative plausibility relation is to proceed as suggested by Halpern [25]. Let  $(S, \triangleright)$  be a partially ordered set, where  $\triangleright$  is an asymmetric and transitive relation. Various possible definitions for extending the comparative possibility to qualitative plausibility relations have been reviewed in [6] and arguments have been given for selecting one of them. Here, like in our previous paper [36] we consider the extensions (1) and (2) of the strict part of  $\geq_\pi$  to build a partial order between subsets. It turns out they are no longer equivalent, and the one possessing the greatest number of properties is:

**Definition 4** (Weak optimistic strict dominance). *Let  $\triangleright$  be an asymmetric and transitive relation on  $S$ . Then  $A \succ_{wos}^\triangleright B$  iff  $A \neq \emptyset$  and  $\forall b \in B, \exists a \in A, a \triangleright b$ .*

It is clear that if  $\triangleright$  is the strict part of a complete preorder on  $S$  encoded by a possibility distribution  $\pi$ ,  $A \succ_{wos}^\triangleright B$  if and only if  $\Pi(A) > \Pi(B)$ . In the partially ordered setting, the following properties have been established [25, 6, 36]:

**Proposition 5.** *The weak optimistic strict dominance  $\succ_{wos}^\triangleright$  is a strict partial order that satisfies Qualitativeness (Q) and Orderliness (O).*

Unfortunately, contrary to the totally ordered case, not all qualitative plausibility relations can be generated from a partial order on  $S$ . This is because knowing only the restriction to the singletons of  $S$  of a qualitative plausibility relation  $\succ_{pl}$  on  $\wp(S)$  is insufficient to reconstruct  $\succ_{pl}$ . Namely, let a partial order on  $S$  be defined by  $s_1 \triangleright_{pos} s_2$  if and only if  $\{s_1\} \succ_{pl} \{s_2\}$ , where  $\succ_{pl}$  satisfies (Q) and (O). Consider the relation  $\succ_{wos}^{\triangleright_{pos}}$  induced by  $\triangleright_{pos}$  via Definition 4. Then  $A \succ_{wos}^{\triangleright_{pos}} B$  implies  $A \succ_{pl} B$ , but generally the converse does not hold [6].

**Example 1** (due to Halpern). *Let  $S = \{a, b, c\}, A = \{a\}, B = \{b\}, C = \{c\}$ . Suppose relation  $\succ$  is the smallest asymmetric partial order relation including constraints  $B \cup C \succ A, A \succ \emptyset, B \succ \emptyset, C \succ \emptyset$ , and that is closed for (O) and (T). It obviously satisfies (Q). It is a qualitative plausibility relation. Define the partial order on  $S$  as  $s_1 \triangleright_{pos} s_2$  if and only if  $\{s_1\} \succ \{s_2\}$ . Then elements  $a, b, c$  are not comparable. So we do not have  $\{b, c\} \succ_{wos}^{\triangleright_{pos}} \{a\}$  and we cannot retrieve  $B \cup C \succ A$ .*

**Remark** A result due to Halpern [25] says that qualitative plausibility relations on  $\wp(S)$  can be generated from a partial order on a set larger than  $S$ , which stands as a refinement of it. Namely, for any qualitative plausibility relation  $\succ_{pl}$  on  $S$ , there is a set  $\Omega$ , a surjective map  $f : \Omega \rightarrow S$ , and a partial order  $\triangleright$  on  $\Omega$  such that, if  $A, B$  are subsets of  $S$ ,  $A \succ_{pl} B$  if and only if  $f^{-1}(A) \succ_{wos}^\triangleright f^{-1}(B)$ . This is in fact the semantics adopted by Lehmann and colleagues [27] for non-monotonic relations from conditional assertions.

Another way to generate qualitative plausibility relations is to start from a family  $\mathcal{L}$  of linear orders  $>_\sigma$  on  $S$  defined by permutations  $\sigma$  of elements  $(s_{\sigma(1)} >_\sigma s_{\sigma(2)} >_\sigma \cdots >_\sigma s_{\sigma(n)})$  and let the relation  $\succ_{\mathcal{L}}$  on  $\wp(S)$  be defined as follows:

$$A \succ_{\mathcal{L}} B \iff \forall >_\sigma \in \mathcal{L}, A \succ_{\Pi}^\sigma B$$

where  $\succ_{\Pi}^\sigma$  is the strict part of the comparative possibility relation induced by  $>_\sigma$  on  $S$  [3]. It is easy to check that the relation  $\succ_{\mathcal{L}}$  is a qualitative plausibility relation, i.e., it satisfies the properties (Q) and (O). An interesting question addressed below is whether any qualitative plausibility relation can be generated in this way. To this end, we introduce two more properties of relations between sets:

$$\text{Non-Dogmaticism (NoD)} \quad \forall A \neq \emptyset, A \succ \emptyset$$

$$\text{Semi-Cancellativity (SC)} \quad A \succ B \text{ if and only if } A \setminus B \succ B$$

We can establish the following proposition:

**Proposition 6.** *A qualitative plausibility relation is semi-cancellative.*

**Proof of Proposition 6:**

It is clear that by (O),  $A \setminus B \succ_{pl} B$  implies  $A \succ_{pl} B$ . The less obvious part is the converse: suppose  $A \succ_{pl} B$ . It can be written as

- $(A \setminus B) \cup (A \cap B) \succ_{pl} B$
- and also as  $(A \setminus B) \cup B \succ_{pl} B$  which implies  $(A \setminus B) \cup B \succ_{pl} A \cap B$ .

Now applying (Q) yields  $A \setminus B \succ_{pl} B \cup (A \cap B) = B$ . □

It is proved in [15] that for any non-dogmatic, semi-cancellative qualitative plausibility relation  $\succ_{pl}$ , there exists a family  $\mathcal{L}$  of linear orders on  $S$ , such that  $\succ_{pl}$  coincides with the relation  $\succ_{\mathcal{L}}$  on disjoint subsets.

Using this result, we get the representation theorem for qualitative plausibility relations as follows:

**Corollary 1.** *A non-dogmatic relation  $\succ$  between sets is a qualitative plausibility relation if and only if there is a family  $\mathcal{L}$  of linear orders  $>_\sigma$  on  $S$ , such that  $A \succ B$  if and only if  $A \succ_{\mathcal{L}} B$ .*

**Proof of Corollary 1:**

Let  $\succ_{pl}$  be a non-dogmatic qualitative plausibility relation. From [15], there exists a family  $\mathcal{L}$  of linear orders  $>_\sigma$  on  $S$ , such that  $\succ_{pl}$  coincides with the relation  $\succ_{\mathcal{L}}$  on disjoint subsets.  $A \succ_{pl} B$

if and only if  $A \setminus B \succ_{pl} B$  (by semi-cancellativity). So  $A \succ_{pl} B$  if and only if  $A \setminus B \succ_{\mathcal{L}} B$  if and only if  $\forall \succ_{\sigma} \in \mathcal{L}, A \setminus B \succ_{\Pi}^{\sigma} B$ , if and only if  $\forall \succ_{\sigma} \in \mathcal{L}, A \succ_{\Pi}^{\sigma} B$  if and only if  $A \succ_{\mathcal{L}} B$ . For the converse it has been already said that the relation  $\succ_{\mathcal{L}}$  built from a family of linear orders is a qualitative plausibility relation.  $\square$

This is the answer to the question of whether any qualitative plausibility relation can be constructed from a family of possibility orderings.

### 2.3 Preadditive substitutes of confidence relations

The property of preadditivity considers that the common part of two sets should play no role in their comparison. This is the idea behind Savage sure thing principle [35], which applies to the comparison of more general functions than characteristic functions of sets. One may say that preadditivity is precisely an instance of this principle. Preadditivity is a sufficient condition to make a relation between subsets self-dual:

**Proposition 7.** *For any relation  $\succ$  on  $\wp(S)$ , (P) implies (D).*

**Proof of Proposition 7:**

Let  $A \succ B$ .  $A = (A \setminus B) \cup (A \cap B)$  and similarly  $B = (B \setminus A) \cup (A \cap B)$ . Applying (P) produces  $(A \setminus B) \succ (B \setminus A)$ . Applying (P) again yields  $(A \setminus B) \cup \overline{(A \cup B)} \succ (B \setminus A) \cup \overline{(A \cup B)}$ . That is  $\overline{B} \succ \overline{A}$ .  $\square$

As a direct consequence, we have an equivalent form of (P), which is to (P) what (SI) is to (SU):

$$(P) \Leftrightarrow \text{If } A \cup (B \cap C) = S \text{ then } (B \succ C \text{ iff } A \cap B \succ A \cap C)$$

Moreover, the two following properties are direct consequences of (P):

- $B \succ C$  iff  $B \setminus C \succ C \setminus B$  (a stronger property than semi-cancellativity)
- $B \succ C$  iff  $B \cup \overline{C} \succ C \cup \overline{B}$

A preadditive approach for comparing two sets  $A$  and  $B$  then consists in eliminating the common part and then comparing  $A \setminus B$  and  $B \setminus A$ . This is not a new idea (see [24], [25]). Given a partial order  $\succ$  on  $\wp(S)$  one can define a preadditive ordering  $\succ^+$ , called *preadditive substitute* of  $\succ$  as follows:

$$A \succ^+ B \text{ if and only if } A \setminus B \succ B \setminus A$$

Clearly  $\succ^+$  and  $\succ$  coincide on pairs of disjoint subsets, and it is obvious that  $\succ^+$  is preadditive, which implies it is self-dual, due to Proposition 7.

Consider a confidence relation  $\succeq$  in the sense of Definition 1 and its preadditive substitute  $\succeq^+$ . It is obvious that, as soon as the strict part of this relation is non-dogmatic (which means that all elements in  $S$  are in some sense useful or possible), the latter satisfies a strong form of compatibility with inclusion:

**Strict Compatibility with Inclusion (SCI)** If  $B \subset A$  then  $A \succ B$

**Proposition 8.** *If a relation  $\succ$  between subsets satisfies Preadditivity, then SCI is equivalent to its weak form: If  $A \neq \emptyset$  then  $A \succ \emptyset$  (NoD)*

**Proof of Proposition 8:**

Assume that  $\succ$  satisfies (P) and (NoD). Let  $B \subset A$ . We have  $B \setminus A = \emptyset$  and  $A \setminus B \neq \emptyset$ . By (NoD) we obtain  $(A \setminus B) \succ (B \setminus A)$ . By (P), we add  $A \cap B = B$  to each side and we obtain  $((A \setminus B) \cup B) = A \succ ((B \setminus A) \cup B) = B$ .  $\square$

The following relaxed versions of properties (Q) and (N) are appropriate for preadditive relations.

- **Qualitativeness for disjoint sets (QD)** If  $A \cup C \succ B$  and  $A \cup B \succ C$  then  $A \succ B \cup C$ , provided that  $A \cap B = A \cap C = B \cap C = \emptyset$
- **Negligibility for disjoint sets (ND)** If  $A \succ B$  and  $A \succ C$  then  $A \succ B \cup C$ , provided that  $A \cap B = A \cap C = \emptyset$

It is easy to verify:

**Proposition 9.** *The properties (Q) and (QD) are equivalent when  $\succ$  is applied to disjoint sets.*

**Proof of Proposition 9:**

Obviously, (Q) implies (QD).

Conversely, let us assume that  $\succ$  satisfies (QD) and consider that  $A \cup C \succ B$  and  $A \cup B \succ C$ , with  $(A \cup C) \cap B = (A \cup B) \cap C = \emptyset$ . As  $(A \cup C) \cap B = \emptyset$ , we have that  $A \cap B = C \cap B = \emptyset$ . Similarly, we have  $A \cap C = \emptyset$ . So (QD) can be applied, producing  $A \succ B \cup C$ .  $\square$

**Proposition 10.** *For any relation  $\succ$  on  $\wp(S)$ , if  $\succ$  satisfies:*

- *transitivity (T) and (SCI), then it satisfies (O);*
- *(QD) and (O), then it satisfies (ND);*

- (QD) and (O), then it satisfies (CCI), (CCC), (OR), (CUT), (CM);
- (CCC), then it satisfies (QD).

**Proof of Proposition 10:**

**T, SCI  $\Rightarrow$  O:** Assume that  $A \succ B$ ,  $A \subseteq A'$ , and  $B' \subseteq B$ . We have to prove that  $A' \succ B'$ .

If  $A = A'$  we have  $A' \succ B$ . If  $A \subset A'$  we obtain  $A' \succ A$  by (SCI) and then  $A' \succ B$  by transitivity (T).

Now, if  $B = B'$  we obtain  $A' \succ B'$ . Otherwise  $B' \subset B$ , so  $B \succ B'$  by (SCI) and by transitivity we obtain  $A' \succ B'$ .

**O, QD  $\Rightarrow$  ND:** Assume that  $A \cap B = A \cap C = \emptyset$ ,  $A \succ B$  and  $A \succ C$ . We have to prove that  $A \succ (B \cup C)$ . From  $A \succ B$  and (O):  $(A \cup C) \succ (B \setminus C)$  (1). From  $A \succ C$  and (O), we obtain  $A \cup (B \setminus C) \succ C$  (2).

Due to the assumptions, we have  $A \cap (B \setminus C) = A \cap C = \emptyset$  and obviously  $C \cap (B \setminus C) = \emptyset$ . Applying (QD) from (1) and (2) yields  $A \succ (C \cup (B \setminus C))$  that is  $A \succ (B \cup C)$ .

**O, QD  $\Rightarrow$  CCI, CCC, OR, CUT, CM:** As  $A \cap C$  and  $\overline{A} \cap C$  (resp.  $B \cap C$  and  $\overline{B} \cap C$ ) are disjoint sets, the proof of Proposition 4 can be used.

(CCC)  $\Rightarrow$  (QD) This is Theorem 1 in [15]. □

As (P) implies Self-duality, it follows that the property (QD) possesses a dual property (QD<sup>d</sup>) equivalent to the former for preadditive relations:

$$\text{QD}^d: \text{ If } A \cup B = A \cup C = B \cup C = S, \text{ then if } C \succ A \cap B \text{ and } B \succ A \cap C, \text{ then } B \cap C \succ A$$

So, for preadditive substitutes, we can use the dual property (QD<sup>d</sup>), in place of (QD).

An important question is whether a strict confidence relation  $\succ$  is refined or not by its preadditive substitute. We can prove this property for confidence relations that obey the following weak form of both preadditivity and stability for disjunction (first proposed in [11] for weak – transitive and complete – orders):

$$\text{Stability for Disjoint Union (SDU) If } A \cap (B \cup C) = \emptyset \text{ then } A \cup B \succ A \cup C \text{ implies } B \succ C$$

**Proposition 11.** *If an asymmetric relation  $\succ$  satisfies (SDU), then its preadditive substitute  $\succ^+$  is a self-dual refinement of  $\succ$  and of its dual.*

**Proof of Proposition 11:**

The result is obvious from  $\succ$  to  $\succ^+$  since by (SDU), if  $A \succ B$  then  $A \setminus B \succ B \setminus A$  which is  $A \succ^+ B$ . For the dual relation  $\succ^d$ ,  $A \succ^d B$  means  $\overline{B} \succ \overline{A}$  which also reads  $(A \setminus B) \cup (\overline{A} \cap \overline{B}) \succ (B \setminus A) \cup (\overline{A} \cap \overline{B})$ , which by (SDU) implies  $A \succ^+ B$ . □

## 2.4 From qualitative plausibility to qualitative likelihood and back

Consider now qualitative plausibility relations  $\succ_{pl}$  and their preadditive substitutes  $\succ_{pl}^+$ . It is obvious that  $\succ_{pl}^+$  satisfies (O), (QD) and (P). Moreover, due to Propositions 7 and 10, the preadditive relation  $\succ_{pl}^+$  also satisfies the properties of Self-duality (D), Negligibility for disjoint sets (ND), and also Conditional Closure by Implication (CCI), Conditional Closure by Conjunction (CCC), Left Disjunction (OR), (CUT), (CM). The use of  $\succ_{pl}$  or  $\succ_{pl}^+$  for non-monotonic inference is immaterial as it only involves disjoint subsets.

We can prove that the preadditive substitute of a qualitative plausibility relation is transitive.

**Proposition 12.** *If  $A \succ_{pl}^+ B$  and  $B \succ_{pl}^+ C$ , then  $A \succ_{pl}^+ C$ .*

**Proof of Proposition 12:**

We can write the two assumptions as (we omit the intersection symbol for simplicity):  $A\bar{B}\bar{C} \cup \bar{A}B\bar{C} \succ_{pl} \bar{A}B\bar{C} \cup \bar{A}B\bar{C}$  and  $AB\bar{C} \cup \bar{A}B\bar{C} \succ_{pl} \bar{A}B\bar{C} \cup \bar{A}B\bar{C}$ . We must prove that  $AB\bar{C} \cup \bar{A}B\bar{C} \succ_{pl} \bar{A}B\bar{C} \cup \bar{A}B\bar{C}$ . Taking the union on both sides it yields, using Proposition 3:

$$A\bar{B}\bar{C} \cup \bar{A}B\bar{C} \cup AB\bar{C} \cup \bar{A}B\bar{C} \succ_{pl} \bar{A}B\bar{C} \cup \bar{A}B\bar{C} \cup \bar{A}B\bar{C} \cup \bar{A}B\bar{C}$$

Due to property (SU) contraposed (Prop. 2) we can cancel  $\bar{A}B\bar{C}$  and  $\bar{A}B\bar{C}$  which yields  $AB\bar{C} \cup \bar{A}B\bar{C} \succ_{pl} \bar{A}B\bar{C} \cup \bar{A}B\bar{C}$ .  $\square$

**Remark:** Due to the representation result in Corollary 1, there is an alternative proof that goes as follows: there exists a family  $\mathcal{L}$  of linear orders  $>_{\pi}$  on  $S$  that generates  $\succ_{pl}$  in the sense that  $A \succ_{pl} B$  if and only if  $A \succ_{\Pi} B, \forall >_{\pi} \in \mathcal{L}$ . Then suppose  $A \succ_{pl}^+ B$ , which means  $A \setminus B \succ_{pl} B \setminus A$ , which means  $A \setminus B \succ_{\Pi} B \setminus A, \forall >_{\pi} \in \mathcal{L}$ . Likewise with  $B \succ_{pl}^+ C$ . Using transitivity of  $\succ_{\Pi}^+$  (claimed in [14]), we conclude that  $A \setminus C \succ_{\Pi} C \setminus A, \forall >_{\pi} \in \mathcal{L}$ , which is  $A \succ_{pl}^+ B$ . However the use of linear orders on  $S$  presupposes a non-dogmatic qualitative plausibility relation.

Since a qualitative plausibility relation  $\succ_{pl}$  satisfies a strong form of axiom (SDU) (without the condition  $A \cap (B \cup C) = \emptyset$ ), we get the following result, which is a direct consequence of Proposition 11:

**Corollary 2.** *The preadditive relation  $\succ_{pl}^+$  is a self-dual refinement of  $\succ_{pl}$  and of its dual:*

- *If  $A \succ_{pl} B$  then  $A \succ_{pl}^+ B$ .*
- *If  $\bar{B} \succ_{pl} \bar{A}$  then  $A \succ_{pl}^+ B$ .*

This fact has already been known for a long time for comparative possibility and necessity relations [14]. But it is not valid for any kind of confidence relation. For instance

it is easy to find capacities for which  $f(A) > f(B)$  but  $f(A \cup C) < f(B \cup C)$ , for disjoint  $A, B, C$ . So using the order  $\succ_f$  induced by  $f$ , one would have  $B \cup C \succ_f A \cup C$  but  $A \cup C \succ_f^+ B \cup C$ .

These results can be applied to special cases of qualitative plausibility relations  $\succ_{pl}$ :

- Comparative possibility relations  $\succ_{\Pi}$
- Weak optimistic strict dominance relations  $\succ_{\triangleright_{wos}}$  (renamed as  $\succ_{wos}$  for short in the following)

In particular, we can consider the preadditive substitute of a comparative possibility relation. It is a special case of the discrimax relation for comparing vectors of values in a totally ordered scale [13]. It is defined equivalently as follows in terms of a possibility distribution:  $A \succ_{\Pi}^+ B$  if and only if  $\max_{s \in A \setminus B} \pi(s) > \max_{s \in B \setminus A} \pi(s)$  [14]. It is a transitive refinement of the comparative possibility relation (as pointed out, in [13, 14], but not proved for transitivity).

The preadditive substitute of a weak optimistic strict dominance relation is as follows:

**Definition 5** (Weak preadditive strict dominance).  $A \succ_{wos}^+ B$  if and only if  $A \neq B$  and  $A \setminus B \succ_{wos} B \setminus A$ .

This relation has been thoroughly studied in [6]<sup>2</sup>. It coincides with  $\succ_{wos}$  on disjoint sets. The above results can also be applied to the weak preadditive strict dominance.

**Proposition 13.** *The weak preadditive strict dominance  $\succ_{wos}^+$  is a strict partial order that satisfies Preadditivity (P), Strict Compatibility with Inclusion (SCI) and Qualitativeness for disjoint sets (QD).*

**Proof of Proposition 13:**

(T), (P) and (QD) hold due to the above results about  $\succ_{pl}^+$ . Transitivity has already been proved in [8] (see also [7], Proposition 30, p. 35). (SCI) follows from the fact that  $C \succ_{wos}^+ \emptyset$  when  $C \neq \emptyset$ .  $\square$

As a consequence of Corollary 2, the weak optimistic dominance is also refined by its preadditive substitute.

**Corollary 3.**  $\succ_{wos}^+$  refines  $\succ_{wos}$  and its dual variant:

- If  $A \succ_{wos} B$  then  $A \succ_{wos}^+ B$ .
- If  $\overline{B} \succ_{wos} \overline{A}$  then  $A \succ_{wos}^+ B$ .

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<sup>2</sup>A loose preadditive dominance has also been studied in [7].

These results lead us to define a qualitative likelihood relation as follows:

**Definition 6.** A qualitative likelihood relation is an asymmetric relation  $\succ^+$  on  $\wp(S)$  that satisfies (O), (P) and (QD).

Due to the above results, the preadditive substitute of a qualitative plausibility relation is a qualitative likelihood relation. More importantly, we can get a representation theorem for qualitative likelihood relations as follows:

**Proposition 14.** Any qualitative likelihood relation is the preadditive substitute of a qualitative plausibility relation.

**Proof of Proposition 14:**

Let  $\succ$  be a qualitative likelihood relation. Let us define  $\triangleright$  as  $A \triangleright B$  whenever  $A \setminus B \succ B$ . We have to prove that  $\triangleright$  satisfies (Q) and (O) and that the preadditive substitute of  $\triangleright$  is  $\succ$ .

- First we show that  $\triangleright$  satisfies (Q). That is: if  $A \cup C \triangleright B$  and  $B \cup C \triangleright A$ , then  $C \triangleright A \cup B$ . Due to the definition of  $\triangleright$ , we must prove that if  $(A \cup C) \setminus B \succ B$  and  $(B \cup C) \setminus A \succ A$ , then  $C \setminus (A \cup B) \succ A \cup B$ . Let  $C'$  denote  $C \setminus (A \cup B)$  and  $AB$  denote  $A \cap B$ . The hypothesis can be written as  $(A \setminus B) \cup C' \succ (B \setminus A) \cup AB$  (1) and  $(B \setminus A) \cup C' \succ (A \setminus B) \cup AB$  (2). The conclusion can be written as  $C' \succ AB \cup A\Delta B$ . Applying (O) to (1) and (2) produces  $(A \setminus B) \cup C' \succ (B \setminus A)$  and  $(B \setminus A) \cup C' \succ (A \setminus B)$ . Now using (QD) we obtain  $C' \succ (A \setminus B) \cup (B \setminus A)$  or equivalently  $C' \succ A\Delta B$  (3). Applying (O) to (3) produces  $AB \cup C' \succ A\Delta B$ . Using (O) once again from (1) yields  $(A\Delta B) \cup C' \succ AB$ . From (QD) we obtain  $C' \succ (A\Delta B) \cup AB$  which is exactly the expected conclusion.
- $\triangleright$  satisfies (O). Assume that  $A \triangleright B$ ,  $A \subseteq A'$  and  $B' \subseteq B$ . Due to the definition of  $\triangleright$ , we have  $A \setminus B \succ B$ . Obviously,  $A \setminus B \subseteq A' \setminus B'$ . As  $\succ$  satisfies (O), we conclude that  $A' \setminus B' \succ B'$  which is exactly  $A' \triangleright B'$ .
- It remains to prove that the preadditive substitute of  $\triangleright$ , say  $\triangleright^+$ , is  $\succ$ . By definition,  $A \triangleright^+ B$  iff  $A \setminus B \triangleright B \setminus A$  iff  $A \setminus B \succ B \setminus A$  since  $A \setminus B$  and  $B \setminus A$  are disjoint. As  $\succ$  satisfies (P),  $A \setminus B \succ B \setminus A$  is equivalent to  $A \succ B$ . So we have proved that  $A \triangleright^+ B$  iff  $A \succ B$ .

□

As a corollary of Propositions 12 and 14, we conclude that any qualitative likelihood relation is transitive, which was not obvious from its definition. In fact what this result shows is that the application  $\rho : \succ_{pl} \mapsto \succ_{pl}^+$  that assigns to each qualitative plausibility relation its preadditive refinement is a bijection between the set of qualitative plausibility relations  $\succ_{pl}$  and the set of qualitative likelihood relations  $\succ^+$ , namely:

- $A \succ^+ B$  such that  $\succ^+ = \rho(\succ_{pl})$  is defined as  $A \setminus B \succ_{pl} B \setminus A$ .
- $A \succ_{pl} B$  such that  $\succ_{pl} = \mu(\succ^+)$  is defined as  $A \setminus B \succ^+ B$ .

Then, relation  $\rho(\succ_{pl})$  is a qualitative likelihood relation, and relation  $\mu(\succ^+)$  is a qualitative plausibility relation. Moreover:  $\mu(\rho(\succ_{pl})) = \succ_{pl}$  and  $\rho(\mu(\succ^+)) = \succ^+$ .

### 3 Relative confidence and certainty logics

In [36], a logic for reasoning about partially ordered bases has been proposed, with inference rules inspired from the properties of a qualitative certainty relation.

In the following, we define a logical language capable of expressing relative confidence between logical propositions, and a semantics based on confidence relations between sets of interpretations. An example of such a logic is the one in [36]. After recalling this logic, we consider a logic for qualitative likelihood, for which the preadditivity axiom holds. The results in the previous section indicate that the relative certainty logic and qualitative likelihood logic are closely related due to the bijection between the two notions. Especially they will coincide for pairs of formulas whose disjunction is a tautology.

#### 3.1 A logical framework for confidence relations

We consider a propositional language  $\mathcal{L}$  where formulas are denoted by  $\phi, \psi$  etc., and  $\Omega$  is the set of its interpretations.  $[\phi]$  denotes the set of models of  $\phi$ , a subset of  $\Omega$ . We denote by  $\models$  the classical semantic inference. We also denote by  $\vdash_X$  the syntactic inference in the proof system  $X$ .

Let  $\mathcal{K} \subseteq \mathcal{L}$  be a finite set of formulas equipped with a relation  $>$ . The idea is that this relation should represent a fragment of a strict partial ordering. We call  $(\mathcal{K}, >)$  a partially ordered belief base (po-base, for short) where  $\phi > \psi$  is supposed to express that  $\phi$  is more prone to being true than  $\psi$ , for an agent. The standard language  $\mathcal{L}$  is encapsulated inside a language equipped with a binary connective  $>$  (interpreted as a partial order relation). Formally, an atom  $\Phi \in \mathcal{L}_{>}$  is of the form  $\phi > \psi$  where  $\phi$  and  $\psi$  are formulas of  $\mathcal{L}$ . A formula of  $\mathcal{L}_{>}$  is either an atom  $\Phi$  of  $\mathcal{L}_{>}$ , or a conjunction of formulas, that is,  $\Psi \wedge \Phi \in \mathcal{L}_{>}$  if  $\Psi, \Phi \in \mathcal{L}_{>}$ . We also have the formulas  $\perp$  and  $\top$  in  $\mathcal{L}_{>}$ . In contrast with Halpern [25], we exclude negations and disjunctions of atomic formulas just like in basic possibilistic logic, where we do not use negations nor disjunctions of weighted formulas.

A relative confidence base  $\mathcal{B}$  is a finite subset of  $\mathcal{L}_{>}$ . We associate to a po-base  $(\mathcal{K}, >)$  the set of formulas of the form  $\phi > \psi$  and forming a base  $\mathcal{B}_{(\mathcal{K}, >)} \subset \mathcal{L}_{>}$ . In the following, we shall often write  $(\mathcal{K}, >)$  instead of  $\mathcal{B}_{(\mathcal{K}, >)}$  for simplicity.

We consider a semantics defined by a strict confidence relation between sets of interpretations. The idea is to interpret the formula  $\phi > \psi$  on  $2^\Omega$  by  $[\phi] \succ [\psi]$  for a strict confidence relation  $\succ$  (Definition 1). A relative confidence model  $\mathcal{M}$  is a structure  $(2^\Omega, \succ)$  where  $\succ$  is a strict confidence relation on  $2^\Omega$  (that is a strict partial order on  $2^\Omega$  satisfying the properties O and T).

We define the satisfiability of a formula  $\phi > \psi \in \mathcal{L}_{>}$  in  $\mathcal{M}$  as  $\mathcal{M} \models \phi > \psi$  iff  $[\phi] \succ [\psi]$ . The satisfiability of the set of formulas  $\mathcal{B}_{(\mathcal{K}, >)}$  is defined by  $\mathcal{M} \models \mathcal{B}_{(\mathcal{K}, >)}$  iff  $\mathcal{M} \models \phi_i > \psi_i, \forall \phi_i > \psi_i \in \mathcal{B}_{(\mathcal{K}, >)}$ . Note that there is not always a relative confidence model of a po-base  $(\mathcal{K}, >)$ . For instance, if  $\phi > \psi \in \mathcal{B}_{(\mathcal{K}, >)}$  such that  $\phi \models \psi$ , it is impossible to find a confidence relation  $\succ$  such that  $[\phi] \succ [\psi]$  since  $\succ$  should satisfy property O. This comes down to saying that no model of this formula in  $\mathcal{L}_{>}$  exists for the semantics of relative confidence.

We say that  $(\mathcal{K}, >)$  is *inconsistent with respect to the relative confidence semantics*, in short *rc-inconsistent*, iff there is no relative confidence model for  $\mathcal{B}_{(\mathcal{K}, >)}$ .

A logic for relative confidence, denoted by  $CO$ , can be defined as follows: It directly interprets the atoms  $\phi > \psi$  in  $\mathcal{L}_{>}$  by means of the strict confidence relation  $\succ$  having properties (O) and (T) for comparing the sets of models  $[\phi]$  and  $[\psi]$ . The idea behind the proof system is to use the characteristic properties of the confidence relation  $\succ$ , expressed in terms of inference rules that define the syntactic entailment  $\vdash_{CO}$ . We need one axiom and three inference rules:

### Axiom

$$ax_{NT}: \top > \perp$$

### Inference rules

$$RI_O : \text{If } \phi \models \phi' \text{ and } \psi' \models \psi \text{ then } \phi > \psi \vdash \phi' > \psi' \quad (O)$$

$$RI_T : \{\phi > \psi, \psi > \chi\} \vdash \phi > \chi \quad (T)$$

$$RI_{AS} : \{\phi > \psi, \psi > \phi\} \vdash \perp \quad (AS)$$

The axiom says that the order relation is not trivial<sup>3</sup>. Rules  $RI_O$  and  $RI_T$  correspond to the properties of Orderliness and Transitivity. Rule  $RI_{AS}$  expresses the asymmetry of the relation  $>$ . The proof system of the logic of relative confidence is composed of the axiom  $ax_{NT}$  and the three inference rules  $RI_O - RI_{AS}$ .

**Remark 1.** *The order relation  $>$  does not contradict classical inference. Indeed, if we have  $\psi \models \phi$  and  $\psi > \phi \in \mathcal{B}_{(\mathcal{K}, >)}$ , we prove that  $\phi > \phi$  by  $RI_O$  and the contradiction by  $RI_{AS}$ .*

The associated semantic consequence  $\models_{CO}$  can then be defined in the usual way:

$$(\mathcal{K}, >) \models_{CO} \phi > \psi \text{ iff } \forall \mathcal{M}, \text{ if } \mathcal{M} \models \mathcal{B}_{(\mathcal{K}, >)} \text{ then } \mathcal{M} \models \phi > \psi. \quad (5)$$

The proof system of the logic of relative confidence is sound and complete for the relative confidence semantics:

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<sup>3</sup>This axiom could be replaced by  $\phi \vee \neg\phi > \psi \wedge \neg\psi$ , in the presence of the inference rule  $RI_T$ .

**Proposition 15.** *Let  $(\mathcal{K}, >)$  be a partially ordered base and  $\phi, \psi \in \mathcal{L}$ .*

- **Soundness:**

*If  $(\mathcal{K}, >) \vdash_{CO} \phi > \psi$  then  $(\mathcal{K}, >) \models_{CO} \phi > \psi$*

- **Completeness:**

*If  $(\mathcal{K}, >)$  is rc-consistent and  $(\mathcal{K}, >) \models_{CO} \phi > \psi$  then  $(\mathcal{K}, >) \vdash_{CO} \phi > \psi$*

*If  $(\mathcal{K}, >)$  is rc-inconsistent then  $(\mathcal{K}, >) \vdash_{CO} \perp$*

**Proof of Proposition 15:**

Let  $\mathcal{B}_{(\mathcal{K}, >)} = \{(\phi_i > \psi_i), i = 1 \dots n\}$ .

- **Soundness:**

Let  $\succ$  be a strict partial order on  $2^\Omega$  satisfying O. We must show that if  $\forall i = 1 \dots n, [\phi_i] \succ [\psi_i]$  then  $[\phi] \succ [\psi]$ . We assume that  $\phi > \psi$  was obtained from  $(\phi_i > \psi_i)$  by inference rules  $RI_O, RI_T, RI_{AS}$  and the axiom. So we just have to show that each of the rules is sound and that the axiom  $ax_{NT}$  is valid.

$ax_{NT}$ : It holds because  $S \succ \emptyset$  for a confidence relation.

$RI_O$ : It holds because  $\succ$  satisfies (O)

$RI_T$ : It holds because  $\succ$  is transitive.

$RI_{AS}$ : The presence of both  $\phi > \psi$  and  $\psi > \phi$  leads to a semantic contradiction because the relation  $\succ$  being asymmetric, we can not have both  $[\psi] \succ [\phi]$  and  $[\phi] \succ [\psi]$ .

- **Completeness:**

We assume that  $(\mathcal{K}, >)$  is rc-consistent. We suppose that for each strict partial order  $\succ$  on  $2^\Omega$  satisfying O, if  $\forall i = 1 \dots n, \phi_i \succ \psi_i$  then  $[\phi] \succ [\psi]$ . We must show that  $(\mathcal{K}, >) \vdash_{CO} \phi > \psi$ . If  $\phi > \psi$  appears in  $\mathcal{B}_{(\mathcal{K}, >)}$ , it is proven.

Otherwise, consider the strict partial order  $\succ$  defined on  $2^\Omega$  as the smallest order containing pairs  $[\phi_i] \succ [\psi_i]$  and closed for the properties O, T.

This relation exists because  $(\mathcal{K}, >)$  is rc-consistent. According to the hypothesis, we have  $[\phi] \succ [\psi]$ . And, by definition of  $\succ$ , the pair  $([\phi], [\psi])$  is obtained by successive applications of the properties O, T. This amounts to getting  $\phi > \psi$  by successive applications of inference rules  $RI_O, RI_T$ .

It remains to prove that if  $(\mathcal{K}, >)$  is rc-inconsistent, then  $(\mathcal{K}, >) \vdash_{CO} \perp$ .

Note that, as  $\mathcal{L}_>$  contains only atomic comparison constraints and their conjunctions, the only form of syntactic inconsistency is the presence of both  $\phi > \psi$  and  $\psi > \phi$  derived from  $(\mathcal{K}, >)$ . This is the only way to get  $(\mathcal{K}, >) \vdash_{CO} \perp$ . In this case, we know that  $\mathcal{B}_{(\mathcal{K}, >)}$  does not have a model of relative confidence. So if  $(\mathcal{K}, >) \vdash_{CO} \perp$  does not hold, then the relation  $>$  obtained on  $\mathcal{L}_>$  by the syntactic closure is asymmetric and transitive, and so is the relation  $\succ$  on  $2^\Omega$  defined by  $[\phi] \succ [\psi]$  if and only if  $(\mathcal{K}, >) \vdash_{CO} \phi > \psi$ . In addition,  $\succ$  will be the smallest relation containing the pairs  $([\phi_i], [\psi_i])$  with  $\phi_i > \psi_i$  in  $(\mathcal{K}, >)$ , and closed for the properties O, T. It is a model of  $\mathcal{B}_{(\mathcal{K}, >)}$ , which is rc-consistent.

□

**Example 2.**  $\mathcal{K}_1 = \{\phi \wedge \psi, \phi \wedge \neg\psi, \neg\phi\}$  with  $\phi \wedge \psi > \phi \wedge \neg\psi > \neg\phi$ .

Then we let  $\mathcal{B}_{(\mathcal{K}_1, >)} = \{\phi \wedge \psi > \phi \wedge \neg\psi, \phi \wedge \neg\psi > \neg\phi\}$ . With the proof system of CO, by  $RI_T$ , we deduce  $\phi \wedge \psi > \neg\phi$  and by  $RI_O$ ,  $\psi > \neg\phi$ .

Next is a case where inconsistency can be detected.

**Example 3.**  $\mathcal{K}_2 = \{\phi, \phi \wedge \psi\}$  with  $\mathcal{B}_{(\mathcal{K}_2, >)} = \{\phi \wedge \psi > \phi\}$ . With the proof system of CO, we obtain a contradiction by  $RI_O$  (we have  $(\mathcal{K}_2, >) \vdash_{CO} \phi \wedge \psi > \phi \wedge \psi$ ) and  $RI_{AS}$ .

### 3.2 Axioms and inference rules for relative certainty logic

The logic for relative certainty described in [36], here denoted by  $\mathcal{C}$ , directly interprets the atoms  $\phi > \psi$  in  $\mathcal{L}_{>}$  by means of a qualitative certainty relation  $\succ_{cr}$  having properties (O) and (Q<sup>d</sup>) for comparing the sets of models  $[\phi]$  and  $[\psi]$ . A relative certainty model is a structure  $(2^\Omega, \succ_{cr})$  where  $\succ_{cr}$  is a qualitative certainty relation on  $2^\Omega$ .

The idea behind the proof system is again to use the characteristic properties of the relation  $\succ_{cr}$ , expressed in terms of inference rules. We need again one axiom and three inference rules in the language  $\mathcal{L}_{>}$ : the same axiom as for the relative confidence logic above, and we can add the following inference rule to the inference rules  $RI_O$  and  $RI_{AS}$  of the confidence relation logic:

$$RI_{Q^d} : \{\chi > \phi \wedge \psi, \psi > \phi \wedge \chi\} \vdash \psi \wedge \chi > \phi \quad (Q^d)$$

This rule corresponds to the properties of dual Qualitativeness. So the relative certainty logic proof system is made of axiom  $ax_{NT}$ , and rules  $RI_O$ ,  $RI_{AS}$  and  $RI_{Q^d}$ .

The inference rule  $RI_T$  can be derived in this system (see also Proposition 1), as well as the following inference rules, some of which are established in [36]:

$$RI_A : \{\psi > \phi, \chi > \phi\} \vdash \psi \wedge \chi > \phi \quad (A)$$

$$RI_{OR^d} : \{\phi \rightarrow \chi > \phi \rightarrow \neg\chi, \psi \rightarrow \chi > \psi \rightarrow \neg\chi\} \vdash (\phi \vee \psi) \rightarrow \chi > (\phi \vee \psi) \rightarrow \neg\chi \quad (OR^d)$$

$$RI_{CCC^d} : \{\chi \rightarrow \phi > \chi \rightarrow \neg\phi, \chi \rightarrow \psi > \chi \rightarrow \neg\psi\} \vdash \chi \rightarrow (\phi \wedge \psi) > \chi \rightarrow \neg(\phi \wedge \psi) \quad (CCC^d)$$

$$RI_{CUT^d} : \{\phi \rightarrow \psi > \phi \rightarrow \neg\psi, (\phi \wedge \psi) \rightarrow \chi > (\phi \wedge \psi) \rightarrow \neg\chi\} \vdash \phi \rightarrow \chi > \phi \rightarrow \neg\chi \quad (CUT^d)$$

$$RI_{CM^d} : \{\phi \rightarrow \psi > \phi \rightarrow \neg\psi, \phi \rightarrow \chi > \phi \rightarrow \neg\chi\} \vdash (\phi \wedge \psi) \rightarrow \chi > (\phi \wedge \psi) \rightarrow \neg\chi \quad (CM^d)$$

$$RI_{Nec} : \phi > \perp \vdash \phi > \neg\phi$$

$$RI_{SC^d} : \phi > \psi \vdash \phi > \neg\phi \vee \psi \text{ (semi-cancellativity).}$$

The first derived rule expresses adjunction and ensures that formulae that are more certain than another one will form a deductively closed set. The next four rules are key inference properties in non-monotonic logic of the KLM type [27]. Rule  $RI_{Nec}$  results from applying  $RI_{Q^d}$  to  $\phi > \phi \wedge \neg\phi$ , and reminds of the property  $\min(N(A), N(\bar{A})) = 0$  of necessity measures  $N$  in possibility theory. The last rule can be proved by implementing the proof of Proposition 6 in  $\mathcal{C}$ .

**Example 4.** Let  $\mathcal{K}_3 = \{\phi, \neg\phi, \psi, \neg\psi\}$  with  $\mathcal{B}_{(\mathcal{K}_3, >)} = \{\phi > \neg\phi, \psi > \neg\psi\}$ . Using  $RI_{CC^d}$  by considering  $\phi > \neg\phi$  as  $\top \rightarrow \phi > \top \rightarrow \neg\phi$  and  $\psi > \neg\psi$  as  $\top \rightarrow \psi > \top \rightarrow \neg\psi$ , we have  $\phi \wedge \psi > \neg\phi \vee \neg\psi$ . Then by  $RI_O$  we obtain  $\psi > \neg\phi$ . And similarly we obtain  $\phi > \neg\psi$ .

The proof system of the relative certainty logic  $\mathcal{C}$  has been proved sound and complete [36] for the semantics of relative certainty. Namely, define  $(\mathcal{K}, >) \models_{\mathcal{C}} \phi > \psi$  to mean: for each qualitative certainty relation  $\succ_{cr}$ , if  $[\phi_i] \succ_{cr} [\psi_i], \forall i$  s.t.  $\phi_i > \psi_i \in \mathcal{B}_{(\mathcal{K}, >)}$ , then  $[\phi] \succ_{cr} [\psi]$ . Moreover  $(\mathcal{K}, >)$  is said to be rcr-consistent if it has a relative certainty model  $(2^\Omega, \succ_{cr})$ . Then we have proved in [36]:

**Proposition 16.** Let  $(\mathcal{K}, >)$  be a partially ordered base and  $\phi, \psi \in \mathcal{L}$ .

- **Soundness:**

If  $(\mathcal{K}, >) \vdash_{\mathcal{C}} \phi > \psi$  then  $(\mathcal{K}, >) \models_{\mathcal{C}} \phi > \psi$ .

- **Completeness:**

If  $(\mathcal{K}, >)$  is rcr-consistent and  $(\mathcal{K}, >) \models_{\mathcal{C}} \phi > \psi$  then  $(\mathcal{K}, >) \vdash_{\mathcal{C}} \phi > \psi$ .

If  $(\mathcal{K}, >)$  is rcr-inconsistent then  $(\mathcal{K}, >) \vdash_{\mathcal{C}} \perp$ .

## 4 Qualitative likelihood logic

In this section, we will present the preadditive version of the relative certainty logic. As done for relative certainty, we propose an inference system for qualitative likelihood relations, which is preadditive, with a semantics defined by a relation between sets of interpretations. As before, we interpret a partially ordered base as a fragment of a qualitative likelihood ordering. We propose a logic system for reasoning with comparative statements interpreted by such a relation. We keep the syntax as defined in the previous section.

## 4.1 Semantics of qualitative likelihood

Let us interpret the formula  $\phi > \psi$  on  $2^\Omega$  by  $[\phi] \succ^+ [\psi]$  for a qualitative likelihood relation  $\succ^+$  in the sense of Definition 6. We assume it is non-dogmatic. A qualitative likelihood model  $\mathcal{M}^+$  is a structure  $(2^\Omega, \succ^+)$  where  $\succ^+$  is a non-dogmatic qualitative likelihood relation on  $2^\Omega$ . We define the satisfiability of a formula  $\phi > \psi \in \mathcal{L}_>$  in  $\mathcal{M}^+$  as  $\mathcal{M}^+ \models \phi > \psi$  iff  $[\phi] \succ^+ [\psi]$ . The satisfiability of the set of formulas  $\mathcal{B}_{(\mathcal{K}, >)}$  is defined by  $\mathcal{M}^+ \models \mathcal{B}_{(\mathcal{K}, >)}$  iff  $\mathcal{M}^+ \models (\phi_i > \psi_i), \forall \phi_i > \psi_i \in \mathcal{B}_{(\mathcal{K}, >)}$ . The associated semantic consequence  $\models_+$  can then be defined in the usual way:

**Definition 7.**  $(\mathcal{K}, >) \models_+ \phi > \psi$  iff  $\forall \mathcal{M}^+,$  if  $\mathcal{M}^+ \models \mathcal{B}_{(\mathcal{K}, >)}$  then  $\mathcal{M}^+ \models \phi > \psi$ .

In other words,  $(\mathcal{K}, >) \models_+ \phi > \psi$  iff for every strict partial order  $\succ^+$  on  $2^\Omega$  verifying O, P, QD, NoD, if  $\forall i = 1 \dots n, [\phi_i] \succ^+ [\psi_i]$  then  $[\phi] \succ^+ [\psi]$ .

We say that  $(\mathcal{K}, >)$  is inconsistent with respect to the qualitative likelihood semantics, in short *ql-inconsistent*, iff there is no qualitative likelihood model for  $\mathcal{B}_{(\mathcal{K}, >)}$ .

## 4.2 Proof system

The logic for qualitative likelihood directly interprets the atoms  $\phi > \psi$  in  $\mathcal{L}_>$  by means of a qualitative likelihood relation  $\succ^+$  for comparing the sets of models  $[\phi]$  and  $[\psi]$ . The idea behind the proof system is again to use the characteristic properties of the relation  $\succ^+$ , expressed in terms of formulas, as inference rules. Indeed, we need one axiom and four inference rules in the language  $\mathcal{L}_>$ , owing to Proposition 8, that indicates that SCI is a derived property in this setting.

### Axiom

$ax_{NoD}$ : If  $\phi \not\equiv \perp$  then  $\phi > \perp$  (NoD)

### Inference rules: $RI_O$ , $RI_{AS}$ and

$RI_{QD^d}$  : If  $\models \phi \vee \psi, \models \phi \vee \chi$  and  $\models \psi \vee \chi$ , then  $\{\chi > \phi \wedge \psi, \psi > \phi \wedge \chi\} \vdash \psi \wedge \chi > \phi$  (QD<sup>d</sup>)

$RIP_1$  : If  $\neg\chi \models \phi \wedge \psi$  then  $\phi > \psi \vdash \phi \wedge \chi > \psi \wedge \chi$  ( $\Rightarrow$  P)

$RIP_2$  : If  $\neg\chi \models \phi \wedge \psi$  then  $\phi \wedge \chi > \psi \wedge \chi \vdash \phi > \psi$  ( $\Leftarrow$  P)

We denote by  $QL$  this logic and by  $\vdash_+$  the associated syntactic inference. Note that the axiom  $ax_{NoD}$  encodes property (NoD). Besides,  $(RI_{QD^d})$  could be replaced

by ( $RI_{QD}$ ) of the form

$RI_{QD}$ : If  $\phi \wedge \psi \models \perp$  and  $\phi \wedge \chi \models \perp$  and  $\psi \wedge \chi \models \perp$ , then  $\{\phi \vee \psi > \chi, \phi \vee \chi > \psi\} \vdash \phi > \psi \vee \chi$

since, in the presence of (P), the properties (QD) and (QD<sup>d</sup>) are equivalent.

Due to Propositions 8 and 10, it can be proved that other rules can be derived from the rules of the proof system of  $QL$ . Some of these derived rules are theorems of the proof system of  $\mathcal{C}$ :  $RI_T$ ,  $RI_{OR^d}$ ,  $RI_{CCC^d}$ ,  $RI_{CUT^d}$ ,  $RI_{CM^d}$ .

Other derived rules are new:

$RI_{ND^d}$  : If  $\models \phi \vee \psi$  and  $\models \phi \vee \chi$ , then  $\{\psi > \phi, \chi > \phi\} \vdash \psi \wedge \chi > \phi$  (ND<sup>d</sup>)

$RI_D$  :  $\{\phi > \psi\} \vdash \neg\psi > \neg\phi$  (Self-duality D)

$RI_{SCI}$  : If  $\psi \models \phi$  and not  $\phi \models \psi$  then  $\phi > \psi$  (SCI)

$RI_{SD}$  : If  $\phi \wedge \psi = \chi \wedge \xi = \perp$ ,  $\{\phi > \chi, \psi > \xi\} \vdash \phi \vee \psi > \chi \vee \xi$  (SD)

$RI_{STP}$  :  $\phi > \psi$  iff  $\phi \wedge \neg\psi > \psi \wedge \neg\phi$  (direct consequence of P)

The last rule is a consequence of  $RIP_1$  and  $RIP_2$  (taking  $\neg\chi = \phi \wedge \psi$ ), that expresses the sure thing principle for events.

**Example 5.** Let  $\mathcal{K}_4 = \{\neg\phi \vee \neg\psi, \neg\phi, \phi \wedge \psi, \phi\}$  with  $\neg\phi \vee \neg\psi > \phi \wedge \psi > \neg\phi$  and  $\phi > \neg\phi$ . So  $\mathcal{B}_{(\mathcal{K}_4, >)} = \{\neg\phi \vee \neg\psi > \phi \wedge \psi, \phi \wedge \psi > \neg\phi, \phi > \neg\phi\}$ . Using  $RI_D$  (Self-duality) we obtain  $\phi > \neg\phi \vee \neg\psi$  and so we get the chain  $\phi > \neg\phi \vee \neg\psi > \phi \wedge \psi > \neg\phi$ .

Of interest is to prove rule  $RI_{SD}$  and  $RI_T$ . The derivation of  $RI_{SD}$  follows from the following lemma.

**Lemma 1.** If  $A, B, C, D$  satisfy  $A \cap B = A \cap C = B \cap D = C \cap D = \emptyset$  then, for any qualitative likelihood relation  $\succ^+$ , it holds that whenever  $A \succ^+ B$  and  $C \succ^+ D$  then  $A \cup C \succ^+ B \cup D$ .

**Proof of Lemma 1:**

Let  $\succ^+$  be a qualitative likelihood relation. Let  $\succ_{pl}$  denote the plausibility relation which is refined by  $\succ^+$ , as defined in Proposition 14. We have  $\succ^+ = \rho(\succ_{pl})$  and  $\succ_{pl} = \mu(\succ^+)$ .

So,  $A \succ^+ B$  and  $C \succ^+ D$  can be written as  $A \setminus B \succ_{pl} B \setminus A$  and  $C \setminus D \succ_{pl} D \setminus C$ . Moreover as  $A \cap B = C \cap D = \emptyset$ , we obtain  $A \succ_{pl} B$  and  $C \succ_{pl} D$ .

From Proposition 3, it follows that  $(A \cup C) \succ_{pl} (B \cup D)$ . Then from Proposition 2, we obtain  $(A \setminus D) \cup (C \setminus B) \succ_{pl} (B \setminus C) \cup (D \setminus A)$  (deleting  $(A \cap D) \cup (B \cap C)$  on both sides). As  $\succ_{pl} = \mu(\succ^+)$  and the sets are disjoint, we also have  $(A \setminus D) \cup (C \setminus B) \succ^+ (B \setminus C) \cup (D \setminus A)$ . Then

applying (P) we get  $A \cup C \succ^+ B \cup D$  (adding  $(A \cap D) \cup (B \cap C)$  to both sides).  $\square$

Then the derivation of  $RI_T$  goes as follows:

- $\{\phi > \psi, \psi > \chi\} \vdash_{QL} \{\phi \wedge \neg\psi > \psi \wedge \neg\phi, \psi \wedge \neg\chi > \chi \wedge \neg\psi\}$  (using  $RI_{STP}$ )
- $\{\phi \wedge \neg\psi > \psi \wedge \neg\phi, \psi \wedge \neg\chi > \chi \wedge \neg\psi\} \vdash_{QL} (\phi \wedge \neg\psi) \vee (\psi \wedge \neg\chi) > (\psi \wedge \neg\phi) \vee (\chi \wedge \neg\psi)$  (using  $RI_{SD}$ )
- $(\phi \wedge \neg\psi) \vee (\psi \wedge \neg\chi) > (\psi \wedge \neg\phi) \vee (\chi \wedge \neg\psi) \vdash_{QL} \phi \wedge \neg\chi > \neg\phi \wedge \chi$  (using  $RI_{STP}$ )
- $\phi \wedge \neg\chi > \neg\phi \wedge \chi \vdash_{QL} \phi > \chi$  (using  $RI_{STP}$ ).

The proof system of  $QL$  is sound and complete for the semantics of qualitative likelihood.

**Proposition 17.** *Let  $(\mathcal{K}, >)$  be a partially ordered base and  $\phi, \psi \in \mathcal{L}$ .*

- **Soundness:**  
If  $(\mathcal{K}, >) \vdash_+ \phi > \psi$  then  $(\mathcal{K}, >) \models_+ \phi > \psi$
- **Completeness:**  
If  $(\mathcal{K}, >)$  is *ql-consistent* and  $(\mathcal{K}, >) \models_+ \phi > \psi$  then  $(\mathcal{K}, >) \vdash_+ \phi > \psi$   
If  $(\mathcal{K}, >)$  is *ql-inconsistent* then  $(\mathcal{K}, >) \vdash_+ \perp$

**Proof of Proposition 17:**

The proof follows the same pattern as for the soundness and completeness of the relative confidence proof system (proof of Proposition 15).

- **Soundness:**  
Let  $\succ^+$  be a strict partial order on  $2^\Omega$  satisfying O, P,  $QD^d$  and NoD. We must show that if  $\forall i = 1 \dots n, [\phi_i] \succ^+ [\psi_i]$  then  $[\phi] \succ^+ [\psi]$ . We do it for axioms and rules not previously encountered.
  - ax<sub>NoD</sub>* We must show that  $\forall \mathcal{M}^+$ , if  $\phi \not\equiv \perp$  then  $\mathcal{M}^+ \models \phi > \perp$ . Or equivalently, for any strict relation  $\succ^+$  on  $2^\Omega$  that satisfies the properties O, P,  $QD^d$  and NoD, if  $\phi \not\equiv \perp$  then  $[\phi] \succ^+ [\perp]$ . It follows from Proposition 8 since  $[\phi] \neq \emptyset$  when  $\phi \not\equiv \perp$ .
  - RIP<sub>1</sub>* We must show that if  $[\phi] \succ^+ [\psi]$  and  $\neg\chi \models \phi \wedge \psi$  then  $[\phi \wedge \chi] \succ^+ [\psi \wedge \chi]$ . This is true since the relation  $\succ^+$  is preadditive.
  - RIP<sub>2</sub>* We must show that if  $[\phi \wedge \chi] \succ^+ [\psi \wedge \chi]$  and  $\neg\chi \models \phi \wedge \psi$  then  $[\phi] \succ^+ [\psi]$ . This is true since the relation  $\succ^+$  is preadditive.

- **Completeness:**

The proof is exactly the same as for the relative confidence logic CO, (Proposition 15) replacing (O) by (O), (P), (QD<sup>d</sup>) and (NoD). Note that the only possible form of syntactic inconsistency that can be detected in  $(\mathcal{K}, >)$  is again when  $(\mathcal{K}, >) \vdash_+ \phi > \psi$  and  $(\mathcal{K}, >) \vdash_+ \psi > \phi$ .

□

## 5 Comparison between proof systems of $QL$ and $\mathcal{C}$

Based on results from the previous sections, it is interesting to compare relative certainty and qualitative likelihood logics  $\mathcal{C}$  and  $QL$  in terms of strength of their proof systems.

### 5.1 Is one system more productive than the other?

Recall that for any relative certainty relation, there is a qualitative likelihood relation that refines it. Due to Proposition 2 and Proposition 14, we have: if  $\mathcal{M}$  is a relative certainty model and  $\mathcal{M}^+$  its associated qualitative likelihood model,  $\mathcal{M} \models \phi > \psi$  implies  $\mathcal{M}^+ \models \phi > \psi$  and so  $\mathcal{M} \models \mathcal{B}_{(\mathcal{K}, >)}$  implies  $\mathcal{M}^+ \models \mathcal{B}_{(\mathcal{K}, >)}$ . However it does not imply that, applied to a set of constraints in the form of a partially ordered set of formulas, the system  $QL$  will produce more comparative statements than  $\mathcal{C}$ .

Indeed, the following points must be noticed:

- Inference rules  $RI_O$  and  $RI_{AS}$  belong to both systems.
- Axiom  $ax_{NoD}$  is stronger than axiom  $ax_{NT}$ .<sup>4</sup>
- The proof system of  $QL$  adds two preadditivity rules that are not part of  $\mathcal{C}$ .
- $\mathcal{C}$  uses rule  $RI_{Q^d}$ , but the qualitativeness rule  $RI_{QD^d}$  used in  $QL$  is weaker than  $RI_{Q^d}$  as it only applies to relative confidence statements when the disjunction of the two compared formulas forms a tautology.

Note that due to semi-cancellativity,  $\phi > \psi$  in  $QL$  is equivalent to  $\phi > \neg\phi \vee \psi$  in  $\mathcal{C}$ , and the latter statement obeys the condition that the disjunction of the two formulas forms a tautology, which enables the use of  $RI_{QD^d}$ . But the form of the obtained statements does not allow to apply it directly. So it seems that the two logics are not comparable. Moreover, if a partially ordered base is inconsistent for relative certainty semantics, it may be consistent for qualitative likelihood semantics. The following example illustrates this point.

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<sup>4</sup>but non-dogmaticism is not compulsory: one can specialize system  $\mathcal{C}$  adding it in the form  $\top > \phi$  if  $\top \not\vdash \phi$ , or weaken system  $QL$  by using  $ax_{NT}$  in place of  $ax_{NoD}$ .

**Example 6.**  $\mathcal{K}_4 = \{\neg\phi \vee \neg\psi, \neg\phi, \phi \wedge \psi, \phi\}$  with  $\neg\phi \vee \neg\psi > \phi \wedge \psi > \neg\phi$  and  $\phi > \neg\phi$ , (the same case as in Example 5).

With system  $\mathcal{C}$ , we obtain a contradiction: By  $RI_T$  we obtain  $\neg\phi \vee \neg\psi > \neg\phi$ . Then by  $RI_A$  we obtain  $(\neg\phi \vee \neg\psi) \wedge (\phi \wedge \psi) > \neg\phi$ . Applying  $RI_O$  produces  $\perp > \neg\phi$  and applying  $RI_O$  again produces  $\neg\phi > \neg\phi$ , which by  $RI_{AS}$  yields a contradiction.

With system  $QL$ , we obtain  $\phi > \neg\phi \vee \neg\psi > \phi \wedge \psi > \neg\phi$  (see Example 5).

Note that the reason why we get a contradiction in  $\mathcal{C}$  is because we have  $\neg\phi \vee \neg\psi > \phi \wedge \psi > \perp$ , of the form  $\neg\varphi > \varphi > \perp$  which is forbidden in  $\mathcal{C}$  due to the inference  $\{\varphi > \perp, \neg\varphi > \perp\} \vdash_{\mathcal{C}} \perp$  valid in  $\mathcal{C}$  (just use  $RI_A$ ).

However, nothing prevents  $\neg\varphi > \varphi > \perp$  in  $QL$  (e.g.,  $\varphi > \perp$  is axiom  $ax_{NoD}$ ). Note that we cannot apply rule  $RI_A$  to  $\{\neg\varphi > \perp, \varphi > \perp\}$  in  $QL$ .

In this example it can be shown that the resulting total order in the case of the system  $QL$  is the refinement of a relative certainty ordering that differs from the set of constraints given in the original  $(\mathcal{K}_4, >)$ . Namely, consider a big-stepped probability that represents the linear order  $\phi > \neg\phi \vee \neg\psi > \phi \wedge \psi > \neg\phi$ , letting

$$p_1 = P([\phi \wedge \neg\psi]) \propto 8; p_2 = P([\phi \wedge \psi]) \propto 4; p_3 = P([\neg\phi \wedge \psi]) \propto 2; p_4 = P([\neg\phi \wedge \neg\psi]) \propto 1.$$

Then, the reader can check that  $P([\phi]) > P([\neg\phi \vee \neg\psi]) > P([\phi \wedge \psi]) > P([\neg\phi])$ . It ensures  $ql$ -consistency of the linear order. The big-stepped probability assignment viewed as a possibility ordering corresponds to the strict constraints (using the max instead of the sum)

$$\Pi([\neg\phi \vee \neg\psi]) > \Pi([\phi \wedge \psi]) > \Pi([\neg\phi]) \text{ and } \Pi([\phi]) > \Pi([\phi \wedge \psi]).$$

Indeed,  $\Pi([\phi]) = \Pi([\neg\phi \vee \neg\psi])$ . The corresponding plausibility ordering, expressed in terms of a partial certainty relation, leads to the new set of constraints obtained by duality from the possibility constraints:

$$\mathbb{C}: \phi \gg \neg\phi \vee \neg\psi \gg \phi \wedge \psi \text{ and } \neg\phi \vee \neg\psi \gg \neg\phi.$$

This new partially ordered base is no longer  $\mathcal{C}$ -inconsistent and is refined by means of the  $QL$  logic. In  $\mathcal{C}$ , we cannot prove that  $\mathbb{C}$  implies  $\phi \wedge \psi \gg \neg\phi$ , while this is obtained in  $QL$  logic using self-duality rule  $RI_D$  applied to  $\phi \gg \neg\phi \vee \neg\psi$ .

However, even rcr-consistent bases do not necessarily produce less inferences using system  $\mathcal{C}$  than using system  $QL$ , as shown now.

**Example 7.**  $\mathcal{K}_5 = \{\phi, \phi \wedge \psi\}$  with the constraint  $\phi > \phi \wedge \psi$ .

With system  $\mathcal{C}$ , we obtain  $\phi > \psi$ , using  $RI_{Qd}$  but we do not have that  $\psi > \phi \wedge \psi$ .

With system  $QL$ , the partially ordered base  $(\mathcal{K}_5, >)$  gives no information. Indeed from  $RI_{SCI}$  and  $ax_{NoD}$ ,  $\phi > \phi \wedge \psi$  and  $\psi > \phi \wedge \psi$  are theorems of  $QL$ . But we cannot infer  $\phi > \psi$  in  $QL$ .

## 5.2 Using system $\mathcal{C}$ to compute inference in $QL$

As seen above, we do not have that  $(\mathcal{K}, >) \models_{\mathcal{C}} \phi > \psi$  implies  $(\mathcal{K}, >) \models_+ \phi > \psi$ . Indeed, while the proof system of  $QL$  contains inference rules that are not in the proof system of  $\mathcal{C}$  (the preadditivity property which is translated into the inference rules  $RIP_1$  and  $RIP_2$ ), it contains one less powerful inference rule (qualitativeness for disjoint sets, which is translated into  $RI_{QD^d}$ ) than  $RI_{Q^d}$  for system  $\mathcal{C}$ . Examples above have shown that if applied to a bunch of comparative confidence statements, one system is, strictly speaking, not more powerful than the other. Nevertheless we can try to use  $\mathcal{C}$  to compute inferences in  $QL$ , provided that we modify the original base in a suitable way.

The idea is to exploit Proposition 14 that says that qualitative likelihood orderings are in bijection with qualitative plausibility ones. Due to Proposition 9, the properties (Q) and (QD) are equivalent when considering disjoint sets. By duality, it follows that the properties  $(Q^d)$  and  $(QD^d)$  are equivalent on pairs  $(A, B)$  such that  $\overline{A} \cap \overline{B} = \emptyset$ . As a consequence, the rules  $RI_{Q^d}$  and  $RI_{QD^d}$  are equivalent for bases consisting of  $\phi > \psi$  such that  $\overline{[\phi]} \cap \overline{[\psi]} = \emptyset$ , or equivalently such that  $\models \phi \vee \psi$  (two such formulas are said to be subcontraries). So, the first step is to transform a partially ordered base understood as a fragment of a qualitative likelihood ordering, into a partially ordered base with comparative propositions involving only subcontraries.

More precisely, applying the transformation in Proposition 14, if  $>$  is interpreted as a qualitative likelihood ordering, we consider  $>_{pl}$  the qualitative plausibility ordering that is refined by  $>$ , and denote by  $>_{cr}$  the certainty ordering dual of  $>_{pl}$ . We have  $> = \rho(>_{pl})$  and  $>_{pl} = \mu(>)$ . So the formula  $\phi > \psi$  stands for  $\phi \wedge \neg\psi >_{pl} \psi \wedge \neg\phi$  and can be written as  $\phi \vee \neg\psi >_{cr} \psi \vee \neg\phi$ .

Once we obtain such a relative certainty base, the inference rules of  $\mathcal{C}$  can be applied. Then, applying the converse transformation in Proposition 14, we obtain formulas belonging to the  $QL$ -closure  $QL(\mathcal{B}_{(\mathcal{K}, >)})$ . More precisely, if the formula  $\phi' >_{cr} \psi'$  is produced using  $\mathcal{C}$ , as  $>_{cr}$  is the certainty ordering dual of  $>_{pl}$  and  $>_{pl} = \mu(>)$ ,  $\phi' >_{cr} \psi'$  stands for  $\phi' \wedge \neg\psi' > \neg\phi'$ .

The strategy is summarized as follows. Starting from a partially ordered  $QL$ -base  $(\mathcal{K}, >)$ :

1. Turn  $(\mathcal{K}, >)$  into a new partially ordered  $\mathcal{C}$ -base  $\mu(\mathcal{K}, >) = (\mathcal{K}', \gg)$ , replacing each  $\phi > \psi$  by  $\phi \vee \neg\psi \gg \psi \vee \neg\phi$ .
2. Apply the rules of  $(\mathcal{C})$  to the base  $\mathcal{B}_{(\mathcal{K}', \gg)}$ , thus obtaining the closure  $\mathcal{C}(\mathcal{B}_{(\mathcal{K}', \gg)})$ .
3. Turn  $\mathcal{C}(\mathcal{B}_{(\mathcal{K}', \gg)})$  into a new base  $\rho(\mathcal{C}(\mathcal{B}_{(\mathcal{K}', \gg)}))$  by replacing each  $\phi' \gg \psi'$  by  $\phi' \wedge \neg\psi' > \neg\phi'$ . (Note that this is equivalent to applying rule  $RI_{SC^d}$  (semi-cancellativity), a rule of system  $\mathcal{C}$ ).

Obviously, following the above strategy we obtain only a subset of  $QL(\mathcal{B}_{(\mathcal{K}, >)})$ .

**Example 8.** Let  $\mathcal{K} = \{\phi, \psi\}$  with  $\phi > \psi$ .

As  $> = \rho(>_{pl})$  the formula  $\phi > \psi$  stands for  $\phi \wedge \neg\psi >_{pl} \psi \wedge \neg\phi$ . Conversely, as  $>_{pl} = \mu(>)$ , we obtain the formula  $\phi \wedge \neg\psi > \psi \wedge \neg\phi$ . So we do not recover the initial formula  $\phi > \psi$ . The rule  $RI_{STP}$  must be used for that purpose.

**Example 9.** Let  $\mathcal{K} = \{\phi, \neg\phi \vee \psi, \neg\psi\}$  with  $\mathcal{B}_{(\mathcal{K}, >)} = \{\phi > \neg\psi, \neg\phi \vee \psi > \neg\psi\}$  interpreting  $>$  as qualitative likelihood.

First, we transform the  $QL$ -base into a  $\mathcal{C}$ -base:

- $\phi > \neg\psi$  will be turned into  $\phi \vee \psi \gg \neg\psi \vee \neg\phi$
- $\neg\phi \vee \psi > \neg\psi$  into  $\neg\phi \vee \psi \gg \neg\psi$ , which remains unchanged

Then we use  $\mathcal{C}$ . By  $RI_O$  on  $\phi \vee \psi \gg \neg\psi \vee \neg\phi$  we obtain  $\phi \vee \psi \gg \neg\psi$ . Then by  $RI_A$  and  $RI_O$  again we obtain  $\psi \gg \neg\psi$ . Finally, applying  $RI_{SC^d}$  produces no other formula. As  $\phi \vee \psi, \neg\psi$  are subcontraries, and so are  $\psi, \neg\psi$ , we do get  $\phi \vee \psi > \neg\psi$  and  $\psi > \neg\psi$ . By  $QL$  we directly compute the partial preadditive deductive closure.

- By  $RI_O$  we obtain  $\phi \vee \psi > \neg\psi$
- By  $RI_{ND^d}$  we obtain  $\psi > \neg\psi$ .

To conclude, applying the proof system of  $QL$  to a comparative base does not give the same results as applying system  $\mathcal{C}$  first and then  $QL$  (see Example 7). However, by changing a  $QL$ -base into a  $\mathcal{C}$ -base, applying the transformation in Proposition 14 enables us to derive  $QL$  consequences using inference rules of system  $\mathcal{C}$ . It is yet to be proved whether adding axiom  $ax_{NoD}$  and using preadditivity rules (or just the sure thing principle rule  $RI_{STP}$ ) to a  $QL$  base, completed by its consequences obtained applying system  $\mathcal{C}$  to the transformed original base, will generate the whole  $QL$  closure of the latter.

That it can be conjectured relies on the following reasoning. If we consider a qualitative likelihood relation  $\succ^+$  and its associated plausibility relation  $\succ_{pl} = \mu(\succ^+)$ , these relations coincide on pairs of disjoint sets. Consider a relation  $\succ$  relating only  $A, B$  such that  $A \cap B = \emptyset$ ; it is clear that

- $\succ^+$  can be obtained from  $\succ$  using  $C \succ^+ D$  if and only if  $C \setminus D \succ D \setminus C$ , for  $C, D$  not disjoint.
- $\succ_{pl}$  can be obtained from  $\succ$  using  $C \succ_{pl} D$  if and only if  $C \setminus D \succ D$ .

So if a  $QL$  base  $(\mathcal{K}, >)$  is changed into a  $\mathcal{C}$ -base using the transformation  $\mu$  (and taking the certainty relation dual of  $\mu(>)$ ), we can extract from the  $\mathcal{C}$ -closure of the transformed base

all statements  $\phi > \psi$  where  $\phi, \psi$  are subcontraries. Call this set of comparative statements  $SC(K, >)$ . All statements in  $SC(K, >)$  are in the  $QL$ -closure of  $(K, >)$  and we can argue that the  $\mathcal{C}$ -closure of the transformed base contains all  $QL$ -consequences of  $(K, >)$  involving subcontraries. So if we apply the sure thing principle rule  $RI_{STP}$  to  $SC(K, >)$ , we can hope to recover the  $QL$ -closure  $(K, >)$ .

### 5.3 Case of a flat base

One interesting issue is whether classical propositional logic is a special case of the logics of relative certainty and of qualitative likelihood. To see it, we can encode a propositional knowledge base in the syntax of these logics, and show that the standard closure of the original propositional knowledge base can be recovered respectively from the  $\mathcal{C}$ -closure, and the  $QL$ -closure, of the set of comparative statements obtained by such encodings.

Consider a flat propositional base of the form  $\mathcal{K} = \{\phi_1, \dots, \phi_n\}$ , where each formula  $\phi_i$  expresses a piece of information given by an agent. We thus suppose that each formula is certain. In consequence a natural encoding of  $\mathcal{K}$  in terms of comparative statements consists in translating each formula  $\phi_i$  into  $\phi_i > \neg\phi_i$ . Let  $\mathcal{B}_{\mathcal{K}} = \{\phi_1 > \neg\phi_1, \dots, \phi_n > \neg\phi_n\}$ .

We try to show that introducing the comparative statement  $\phi_i > \neg\phi_i$  for each formula  $\phi_i$  of the flat base  $\mathcal{K}$ , we can recover a classical consequence  $\psi$  of  $\mathcal{K}$  as the consequence  $\psi > \neg\psi$  of  $\mathcal{B}_{\mathcal{K}}$ . We will successively study the deductive closures of  $\mathcal{B}_{\mathcal{K}}$  in the sense of relative certainty and qualitative likelihood logics.

**Example 10.** Let  $\mathcal{K} = \{\phi, \neg\phi \vee \psi\}$  be a classical base. So,  $\mathcal{B}_{\mathcal{K}} = \{\phi > \neg\phi, \neg\phi \vee \psi > \phi \wedge \neg\psi\}$ .

By *modus ponens* on  $\mathcal{K}$ ,  $\psi$  can be derived. So, we would like to obtain  $\psi > \neg\psi$  from  $\mathcal{B}_{\mathcal{K}}$ .

- We compute the  $\mathcal{C}$ -closure: by  $RI_{CCCa}$  on  $\phi > \neg\phi$  and  $\neg\phi \vee \psi > \phi \wedge \neg\psi$  we obtain  $\phi \wedge \psi > \neg\psi \vee \neg\phi$ . Then by  $RI_O$  we obtain  $\psi > \neg\psi$ . The  $\mathcal{C}$ -closure also contains:
  - $\phi \wedge \psi > \neg\psi, \phi \wedge \psi > \neg\phi, \psi > \neg\psi, \phi > \neg\psi$  and  $\psi > \neg\phi$ .
  - $\phi > \neg\phi \vee \neg\psi$  and  $\psi > \neg\phi \vee \neg\psi$ .
- We compute the  $QL$ -closure. Each formula is of the form  $\phi_i > \neg\phi_i$ , so  $\mathcal{B}_{\mathcal{K}}$  contains only pairs of disjoint formulas, that are also subcontraries. Inference rule  $RI_{CCCa}$  can still be applied and so the same conclusion  $\psi > \neg\psi$  can be inferred. Other comparative formulas can be inferred such as
  - By axiom  $ax_{NoD}$ , we obtain  $\phi \wedge \neg\psi > \perp$  and  $\psi \wedge \neg\phi > \perp$  if  $\phi$  and  $\psi$  are not equivalent. So we have  $\phi \wedge \neg(\phi \wedge \psi) > \neg\phi \wedge (\phi \wedge \psi)$ .
  - By  $RI_{STP}$ , we obtain  $\phi > \phi \wedge \psi$  and similarly  $\psi > \phi \wedge \psi$ .

With both systems, we obtain  $\psi > \neg\psi$ . Moreover the  $QL$ -closure contains  $\phi > \phi \wedge \psi$  and  $\psi > \phi \wedge \psi$  (when  $\phi$  and  $\psi$  are not equivalent).

What the above example suggests holds more generally:

**Proposition 18.** *If  $\{\phi_1, \dots, \phi_n\} \vdash \phi$  then  $\{\phi_1 > \neg\phi_1, \dots, \phi_n > \neg\phi_n\} \vdash_X \phi > \neg\phi$  for  $X \in \{\mathcal{C}, QL\}$ .*

**Proof of Proposition 18:**

In both systems  $\mathcal{C}$  and  $QL$ , we can apply inference rule  $RI_{CCC^d}$  to  $\{\phi_1 > \neg\phi_1, \dots, \phi_n > \neg\phi_n\}$  and get the consequence  $\phi_1 \wedge \dots \wedge \phi_n > \neg\phi_1 \vee \dots \vee \neg\phi_n$ . And indeed,  $\{\phi_1, \dots, \phi_n\} \vdash \phi_1 \wedge \dots \wedge \phi_n$ .

Now it is well-known that  $\{\phi_1, \dots, \phi_n\} \vdash \phi$  if and only if  $\phi_1 \wedge \dots \wedge \phi_n \vdash \phi$ . In this case  $\{\phi_1 > \neg\phi_1, \dots, \phi_n > \neg\phi_n\} \vdash_X \phi > \neg\phi$  also holds using  $RI_O$ , valid for  $X \in \{\mathcal{C}, QL\}$ .  $\square$

For the converse proposition, the situation is different between  $\mathcal{C}$  and  $QL$ . Note that

**Lemma 2.** *In  $\mathcal{C}$ ,  $\phi_i > \neg\phi_i$  is equivalent to  $\phi_i > \perp$ .*

**Proof of Lemma 2:**

Rule  $RI_{Nec}$  expresses that  $\phi_i > \perp$  implies  $\phi_i > \neg\phi_i$ , and for the converse, apply  $RI_O$ .  $\square$

So we can prove:

**Proposition 19.** *If  $\{\phi_1 > \neg\phi_1, \dots, \phi_n > \neg\phi_n\} \vdash_{\mathcal{C}} \phi > \neg\phi$  then  $\{\phi_1, \dots, \phi_n\} \vdash \phi$ .*

**Proof of Proposition 19:**

In  $\mathcal{C}$ , the knowledge base  $\{\phi_1 > \neg\phi_1, \dots, \phi_n > \neg\phi_n\}$  is equivalent to  $\{\phi_1 > \perp, \dots, \phi_n > \perp\}$ . Only rules  $RI_A$  and  $RI_O$  can be used to the latter base, which ensures that  $\{\phi_1 > \perp, \dots, \phi_n > \perp\} \vdash_{\mathcal{C}} \phi > \perp$  only when  $\{\phi_1, \dots, \phi_n\} \vdash \phi$ , so that  $\{\phi_1 > \neg\phi_1, \dots, \phi_n > \neg\phi_n\} \vdash_{\mathcal{C}} \phi > \neg\phi$  implies  $\{\phi_1, \dots, \phi_n\} \vdash \phi$ .  $\square$

In  $QL$ , the base  $\{\phi_1 > \perp, \dots, \phi_n > \perp\}$  brings no information as it follows from non-dogmaticism axiom  $ax_{NoD}$ , so it is not equivalent to  $\{\phi_1 > \neg\phi_1, \dots, \phi_n > \neg\phi_n\}$ . Moreover, we cannot apply the  $QL$  rule  $RI_{Q^d}$  to the knowledge base  $\{\phi_1 > \neg\phi_1, \dots, \phi_n > \neg\phi_n\}$ . We can only apply inference rules  $RI_{CCC^d}$  and  $RI_O$ . But then what we get is again the  $\mathcal{C}$ -closure. The inference rules we can use on top are  $RIP_1$  and  $RIP_2$ , or better the sure thing principle  $R_{STP}$ . However they would only deduce statements of the form  $\phi \vee \psi > \phi$  whenever  $\psi \not\equiv \phi$  from axiom  $ax_{NoD}$ . But note that we cannot apply  $R_{STP}$  to statements of the form  $\phi_i > \neg\phi_i$ . So inference from such statements in  $QL$  is again equivalent to inference in classical logic.

## 6 Conclusion

In their early survey on qualitative approaches to probabilistic reasoning, Walley and Fine [37] pointed out in 1979 that

there is a uniform disregard for the formal analysis of probability concepts that cannot be reduced in some fashion to numerical probability.

Due to the assumption that probability is intrinsically numerical, most logical approaches to reasoning with absolute or comparative probability statements in a symbolic framework still reject the adjunction principle according to which the conjunction of two beliefs is still a belief (see for instance the logic of risky knowledge [30], or yet Burgess comparative probability logic [4]). In this paper we have tried to reconcile two uncertain reasoning traditions in a symbolic framework, namely the non-monotonic reasoning approach of the Kraus, Lehman and Magidor style [27] as captured in the possibility theory setting, and the probabilistic reasoning approach as captured via the sure thing principle. There is a clash of intuitions between the two frameworks as the first one respects deductive closure for beliefs, while the latter often rejects it, for instance on the basis of the lottery paradox, originally introduced by Kyburg [28]. In this example, a conjunction of strong beliefs may turn out to be inconsistent. As explained in [15], the lottery paradox is less convincing in situations where some possible worlds are much more frequent than other ones, and probabilities tend to be big-stepped on a suitable partition, which brings probability orderings much closer to possibilistic orderings. However, if the considered probability ordering is total, a certain trivialization results from adopting the adjunction principle, as it enforces a linear order of possible worlds ([15] again).

In this paper, we restrict to partial orders expressing relative likelihood, giving up the reference to numerical probabilities, thus avoiding this trivialization. We show that strict partial comparative plausibility and qualitative likelihood relations coincide on pairs of disjoint sets and are in bijection with one another, and we provide a logic for relative likelihood that is both adjunctive and respects the sure thing principle.

A possible extension of this work would be to consider similar notions dropping the asymmetry property, so as to capture equal likelihoods between propositions as distinct from incomparability due to incompleteness, as studied in [12]. However it is clear that such a logic should then allow for negation and disjunction of comparative statements, in order to express relations between strict and weak preference, which would make the language more complex. Another line of further research would be to extend  $QL$  to comparative conditional statements.

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