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Emeric Bouin, Jean Dolbeault, Laurent Lafleche, Christian Schmeiser

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# FRACTIONAL HYPOCOERCIVITY

by

Emeric Bouin, Jean Dolbeault, Laurent Laffleche & Christian Schmeiser

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**Abstract.** — This research report is devoted to kinetic equations without confinement. We investigate the large time behaviour induced by collision operators with fat tailed local equilibria. Such operators have an anomalous diffusion limit. In the appropriate scaling, the macroscopic equation involves a fractional diffusion operator so that the optimal decay rate is determined by a fractional Nash inequality. At kinetic level we develop an  $L^2$  hypocoercivity approach and establish a rate of decay compatible with the anomalous diffusion limit.

## Table of Contents

1. Introduction .....	2
1.1. The model .....	2
1.2. Main results: decay rates .....	3
1.3. The hypocoercivity strategy .....	7
1.4. A brief review of the literature .....	8
2. Mode by mode hypocoercivity .....	9
3. Estimates in weighted $L^2$ spaces .....	16
3.1. The boundedness in $L^\infty(dx dv)$ .....	17
3.2. The boundedness in $L^1(F \langle v \rangle^k dx dv)$ .....	17
3.3. A factorization method .....	18
4. Interpolation inequalities of Poincaré type .....	19
5. Proof of the theorems .....	22
5.1. Proof of Theorem 1 in the case $\gamma + \beta > 0$ .....	22
5.2. Proof of Theorems 1–3 in the case $\gamma + \beta \leq 0$ .....	24
5.3. Proof of Theorem 2: the critical case $\gamma + \beta = 2$ . ..	26
6. Quantitative estimates of $\mu_L$ and $\lambda_L$ . .....	28
6.1. Fokker-Planck operators .....	28

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6.2. Scattering collision operators .....	29
6.3. Fractional Fokker-Planck operator .....	30
Appendix A. Steady states and force field for the fractional Laplacian with drift .....	34
Acknowledgments .....	37
References .....	37

## 1. Introduction

**1.1. The model.** — We consider the Cauchy problem

$$(1) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f = \mathbf{L}f \\ f(0, x, v) = f^{\text{in}}(x, v), \end{cases}$$

for a distribution function  $f(t, x, v)$  depending on a *position* variable in the whole space,  $x \in \mathbb{R}^d$ , on a *velocity* variable  $v \in \mathbb{R}^d$ , and on *time*  $t \geq 0$ . All collision operators share the properties that the null space of  $\mathbf{L}$  is spanned by the *local equilibrium*  $F$ , and that  $\mathbf{L}$  only acts on the velocity variable.

The aim of this work is to handle the situation where  $F$  has a polynomial decay:

$$(2) \quad \forall v \in \mathbb{R}^d, \quad F(v) = \frac{c_\gamma}{\langle v \rangle^{d+\gamma}},$$

with  $\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}$ ,  $\gamma > 0$  and  $c_\gamma$  a normalization constant ensuring that  $F$  is a probability density. We shall also consider the measure  $d\mu = F^{-1}(v) dv$  and define a scalar product and a norm by

$$\langle f, g \rangle = \int_{\mathbb{R}^d} \bar{f} g d\mu \quad \text{and} \quad \|f\| = \int_{\mathbb{R}^d} |f|^2 d\mu,$$

and the orthogonal projection  $\Pi$  on the subspace generated by  $F$  given by

$$\Pi g = \rho_g F \quad \text{where} \quad \rho_g := \int_{\mathbb{R}^d} g dv.$$

We shall cover three cases of linear collision operators: the *Fokker-Planck* operator

$$(a) \quad \mathbf{L}_1 f = \nabla_v \cdot \left( F \nabla_v (F^{-1} f) \right),$$

the *linear Boltzmann* operator, or *scattering* collision operator

$$(b) \quad \mathbf{L}_2 f = \int_{\mathbb{R}^d} b(\cdot, v') \left( f(v') F(\cdot) - f(\cdot) F(v') \right) dv',$$

and the *fractional Fokker-Planck* operator

$$(c) \quad \mathbf{L}_3 f = \Delta_v^{\frac{\alpha}{2}} f + \nabla_v \cdot (E f),$$

with  $\alpha \in (0, 2)$ . In this latter case, we shall simply assume that the *friction force*  $E$  is radial and solves the equation

$$(3) \quad \Delta_v^{\frac{\alpha}{2}} F + \nabla_v \cdot (E F) = 0.$$

As in [20], for the linear Boltzmann operator, we have in mind a *collision kernel*  $b$  which take the form  $b(v, v') = \langle v' \rangle^\beta \langle v \rangle^\beta$  or  $b(v, v') = |v' - v|^\beta$  and in particular we shall assume that the *collision frequency*  $\nu$  is positive, locally bounded and verifies

$$(H0) \quad \nu(v) := \int_{\mathbb{R}^d} b(v, v') F(v') dv' \underset{|v| \rightarrow \infty}{\sim} |v|^\beta,$$

for a given  $\beta \in \mathbb{R}$ . Inspired by the fractional diffusion limit proved in [20] and remarking that the above operators can formally be written on the form  $B(f) - \nu(v)f$ , we define more generally  $\beta$  as the exponent of polynomial behavior at infinity of the formal function  $\nu$ . We thus define  $\beta$  through

$$\beta = -2 \text{ in case (a), and } \beta = \gamma - \mathbf{a} \text{ in case (c).}$$

The definition of  $\beta$  in the last case comes from the fact that under the condition (3),  $E$  verifies

$$E(v) = G(v) \langle v \rangle^\beta v,$$

where  $G \in L^\infty(\mathbb{R}^d)$  is a positive function such that  $G^{-1} \in L^\infty(B_0^c(1))$ , which is proved in Proposition 20 in appendix.

The collision kernel is also supposed to conserve the mass, which is equivalent to

$$(H1) \quad \int_{\mathbb{R}^d} (b(v, v') - b(v', v)) F(v') dv' = 0.$$

We also need an assumption which will be used to guarantee the microscopic coercivity or a weak version of it, saying that there exists  $\tilde{\gamma} < \gamma$  such that

$$(H2) \quad \forall R > 0, \quad \inf_{(v', v) \in B_R^2} \frac{b(v, v') \langle v' \rangle^{\tilde{\gamma}}}{\langle v \rangle^\eta} \geq C \langle R \rangle^{-(\eta - \beta)_+}.$$

Moreover, for technical reasons we assume additionally that

$$(H3) \quad \iint_{\mathbb{R}^{2d}} \frac{b(v', v)^2}{\nu(v')\nu(v)} F F' dv dv' < \infty,$$

and when  $\beta < 0$ ,

$$(H4) \quad \forall k < \gamma - \beta, \quad \int_{\mathbb{R}^d} \frac{b(v', v) \langle v' \rangle^k}{\langle v \rangle^\beta} F' dv' < \infty.$$

All these assumptions are verified for example when

$$\begin{aligned} b(v', v) &= \langle v' \rangle^\beta \langle v \rangle^\beta && \text{with } |\beta| \leq \gamma, \\ b(v', v) &= |v' - v|^\beta && \text{with } \beta \in (-d/2, 0]. \end{aligned}$$

**1.2. Main results: decay rates.** — Let  $\gamma > 0$ . When  $\gamma + \beta < 2$ , let

$$\alpha = \frac{\gamma - \beta}{1 - \beta} \in (0, 2) \quad \text{and} \quad \begin{cases} \alpha' := \alpha & \text{when } \gamma + \beta < 2 \\ \alpha' := 2 & \text{when } \gamma + \beta \geq 2. \end{cases}$$

By defining  $\|f\|_{X \cap Y}^2 := \|f\|_X^2 + \|f\|_Y^2$ , our main theorem is as follows.

**Theorem 1.** — Let  $d \geq 2$ ,  $\gamma > 0$  and assume that  $\beta$  and  $\gamma$  are such that

$$\beta < \gamma, \quad \gamma + \beta \neq 2.$$

Let  $f$  be a solution of (1) with initial condition  $f^{\text{in}} \in L^1(dx dv) \cap L^2(dx d\mu)$ . Then the following alternative holds true

$$\|f\|_{L^2(dx d\mu)}^2 \leq C \langle t \rangle^{-\frac{d}{\alpha}} \|f^{\text{in}}\|_{L^1(dx dv) \cap L^2(dx d\mu)}^2, \quad \text{if } \beta \geq 0,$$

$$\|f\|_{L^2(dx d\mu)}^2 \leq C \langle t \rangle^{-\min\left(\frac{d}{\alpha'}, \frac{k}{|\beta|}\right)} \|f^{\text{in}}\|_{L^1(dx dv) \cap L^2(\langle v \rangle^k dx d\mu)}^2 \quad \text{if } \beta < 0,$$

for any  $k \in (0, \gamma)$ .

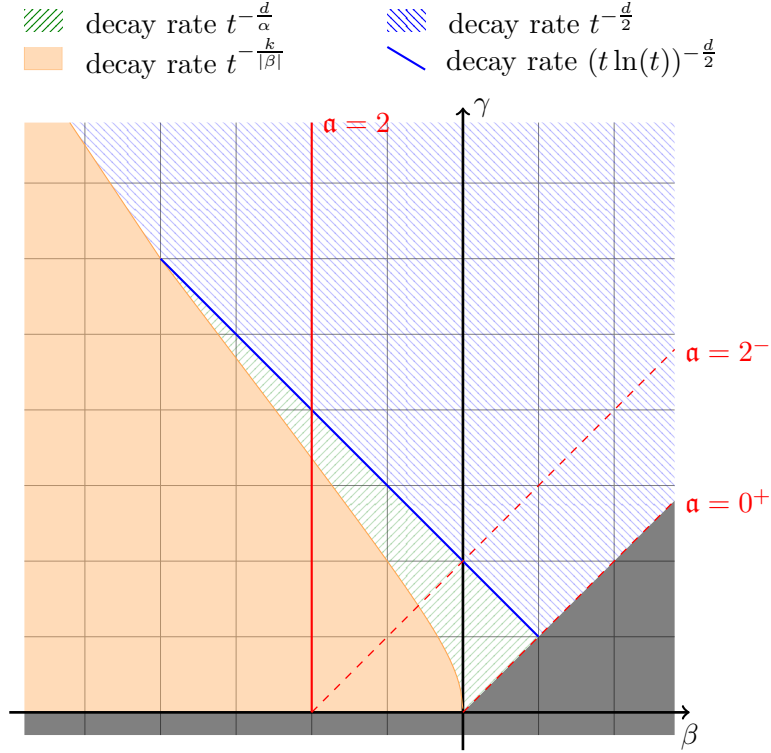


FIGURE 1. Decay rates of Theorem 1 and 2 depending on  $\beta$  and  $\gamma$  in the case of dimension  $d = 3$ . When  $\beta \leq 0$ ,  $k$  is chosen close to  $\gamma$ .

In the limiting case  $\gamma + \beta = 2$ , there is a logarithmic correction for the decay rate.

**Theorem 2.** — Let  $d \geq 2$ ,  $\gamma > 0$  and  $\beta = 2 - \gamma < 1$  and let  $f$  be a solution of (1) with initial condition  $f^{\text{in}} \in L^1(dx dv) \cap L^2(dx d\mu)$ . Then if  $\beta \geq 0$  it holds

$$\|f\|_{L^2(dx d\mu)}^2 \leq C \langle t \ln(t) \rangle^{-\frac{d}{2}} \|f^{\text{in}}\|_{L^1(dx dv) \cap L^2(dx d\mu)}^2$$

If  $\beta \leq 0$ , then for any  $k \in (0, \gamma)$

$$\|f\|_{L^2(dx d\mu)}^2 \leq C \langle t \ln(t) \rangle^{-\frac{d}{2}} \|f^{\text{in}}\|_{L^1(dx dv) \cap L^2(\langle v \rangle^k dx d\mu)}^2 \quad \text{if } k > \frac{d|\beta|}{2}$$

$$\|f\|_{L^2(dx d\mu)}^2 \leq C \langle t \rangle^{-\frac{k}{|\beta|}} \|f^{\text{in}}\|_{L^1(dx dv) \cap L^2(\langle v \rangle^k dx d\mu)}^2 \quad \text{if } k \leq \frac{d|\beta|}{2}.$$

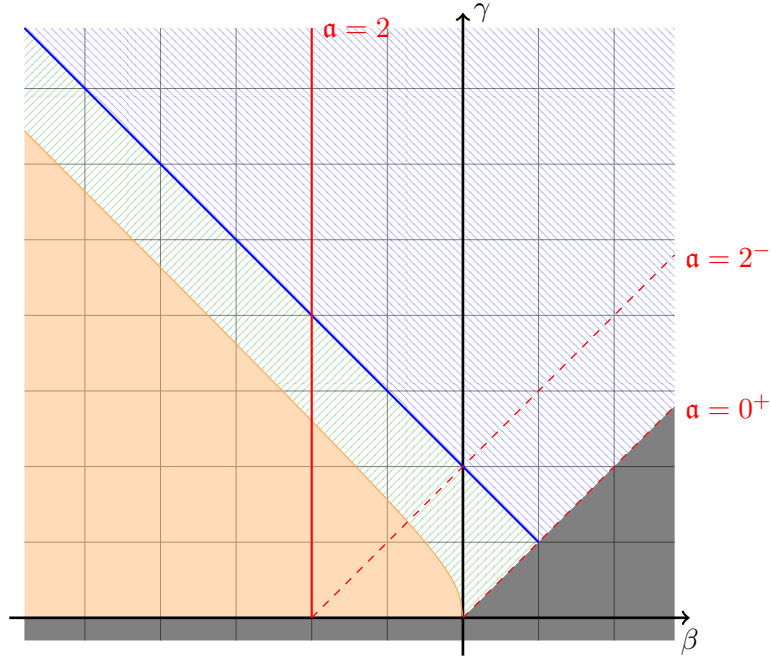


FIGURE 2. Decay rates of Theorem 1 and 2 depending on  $\beta$  and  $\gamma$  in the case of dimension  $d = 2$ . When  $\beta \leq 0$ ,  $k$  is chosen close to  $\gamma$ . The legend is the same as for Figure 1

The result in dimension  $d = 1$  is slightly different.

**Theorem 3.** — Let  $d = 1$ ,  $\gamma > 0$  and assume that  $\beta$  and  $\gamma$  are such that

$$\beta < \gamma, \quad \gamma + \beta \neq 2.$$

Let  $f$  be a solution of (1) with initial condition  $f^{\text{in}} \in L^1(dx dv) \cap L^2(dx d\mu)$ . If  $\beta \geq 0$ , then

$$\|f\|_{L^2(dx d\mu)}^2 \leq C \langle t \rangle^{-\frac{d}{\alpha'}} \|f^{\text{in}}\|_{L^1(dx dv) \cap L^2(dx d\mu)}^2.$$

Now if  $\beta < 0$ ,

$$\|f\|_{L^2(dx d\mu)}^2 \leq C \langle t \rangle^{-\min\left(\frac{d}{\alpha'}, \frac{k}{|\beta|}\right)} \|f^{\text{in}}\|_{L^1(dx dv) \cap L^2(\langle v \rangle^k dx d\mu)}^2$$

for any  $k \in (0, \gamma)$ , except when  $\gamma \in (1, -\beta)$  and  $\beta < -1$  for which  $k \in (0, \frac{\gamma}{\alpha})$ . Lastly, if  $\beta < -1$ ,  $\gamma \in (1, -\beta)$  and  $k \in [\frac{\gamma}{\alpha}, \gamma)$ , then for any  $\tau < \frac{k+\gamma}{k\alpha-\gamma+|\beta|(\alpha+1)}$ ,

there exists a constant  $C > 0$  such that

$$\|f\|_{L^2(dx d\mu)}^2 \leq C \langle t \rangle^{-\tau} \|f^{\text{in}}\|_{L^1(dx dv) \cap L^2(\langle v \rangle^k dx d\mu)}^2.$$

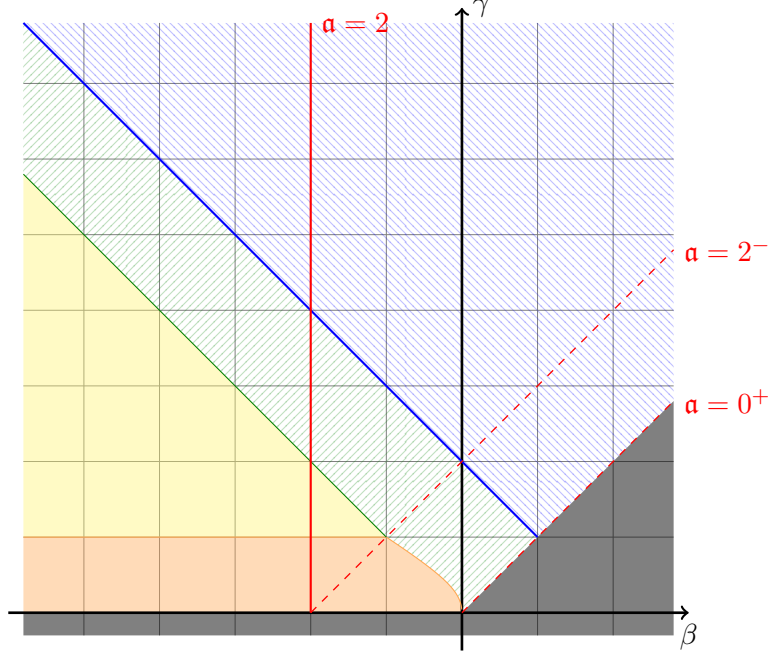


FIGURE 3. Decay rates of Theorem 3 depending on  $\beta$  and  $\gamma$  in the case of dimension  $d = 1$ . When  $\beta \leq 0$ ,  $k$  is chosen close to  $\gamma$ . The legend is the same as for Figure 1 except for the yellow part which corresponds to the decay of the form  $t^{-\tau}$  with  $\tau < \frac{k+\gamma}{k\alpha-\gamma+|\beta|(\alpha+1)}$ .

As in [8], the case of the flat  $d$ -dimensional torus  $\mathbb{T}^d \ni x$  follows from our method without further efforts. Then (1) admits a global equilibrium given by  $f_\infty = \rho_\infty F(v)$  with  $\rho_\infty = \frac{1}{|\mathbb{T}^d|} \iint_{\mathbb{T}^d \times \mathbb{R}^d} f^{\text{in}} dx dv$ , and the rate of convergence to the equilibrium is just given by the microscopic dynamics

$$\|f - f_\infty\|_{L^2(dx d\mu)} \lesssim e^{-\lambda t} \|f^{\text{in}} - f_\infty\|_{L^2(dx d\mu)} \quad \text{when } 0 \leq \beta < \gamma$$

$$\|f - f_\infty\|_{L^2(dx d\mu)} \lesssim \langle t \rangle^{-k/|\beta|} \|f^{\text{in}} - f_\infty\|_{L^2(\langle v \rangle^k dx d\mu)} \quad \text{when } \beta < 0,$$

with  $k \in (0, \gamma)$ . In particular, if  $f = f(t, v)$  does not depend on  $x$ , then we obtain the homogeneous equation

$$\partial_t f = \mathbb{L}f,$$

which gives the same rate of convergence of  $f$  to  $F$  in the norm  $L^2(d\mu)$ . This is coherent with the results in [2, 25, 24, 13, 18]. Moreover, we point out that our result is a bit stronger since it uses a finite  $\|f^{\text{in}}\|_{L^2(\langle v \rangle^k dx d\mu)}$  norm for the initial condition, which is smaller than the usual  $\|f^{\text{in}} F^{-1}\|_{L^\infty(dx dv)}$  norm.

**1.3. The hypocoercivity strategy.** — Let us consider the measure  $d\mu = F^{-1}(v) dv$  and the Fourier transform of  $f$  in  $x$  defined by

$$\widehat{f}(t, \xi, v) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(t, x, v) dx.$$

If  $f$  solves (1), then in the equation satisfied by  $\widehat{f}$ ,

$$\partial_t \widehat{f} + \mathbb{T} \widehat{f} = \mathbb{L} \widehat{f}, \quad \widehat{f}(0, \xi, v) = \widehat{f}^{\text{in}}(\xi, v), \quad \mathbb{T} \widehat{f} = i(v \cdot \xi) \widehat{f},$$

$\xi \in \mathbb{R}^d$  can be seen as a parameter and for each Fourier mode  $\xi$ , one can study the decay of  $f$ . This is what is called a *mode-by-mode analysis*, as in [8]. Therefore, for any given  $\xi \in \mathbb{R}^d$ , we consider  $g : (t, v) \mapsto \widehat{f}(t, \xi, v)$  on the complex valued Hilbert space  $\mathcal{H} = L^2(d\mu)$  with scalar product and norm given by

$$\langle f, g \rangle = \int_{\mathbb{R}^d} \bar{f} g d\mu \quad \text{and} \quad \|f\| = \int_{\mathbb{R}^d} |f|^2 d\mu.$$

Notice that this is a simple Hilbert extension of the scalar product and norm considered previously. As above,  $\Pi$  denotes the orthogonal projection on the subspace generated by  $F$ . Let us observe that the property  $\Pi \mathbb{T} \Pi = 0$  holds as a consequence of the radial symmetry of  $F$ . Always in Fourier variable, we can define the entropy as follows. First we define

$$\varphi(\xi, v) := \frac{\langle v \rangle^{-\beta}}{1 + \langle v \rangle^{2|1-\beta|} |\xi|^2} \quad \text{and} \quad \psi(v) := \langle v \rangle^{-2}.$$

The entropy  $\mathbb{H}_\xi$  is then defined by

$$(4) \quad \begin{aligned} \mathbb{H}_\xi(f) &:= \|\widehat{f}\|^2 + \delta \operatorname{Re} \langle \mathbb{A}_\xi \widehat{f}, \widehat{f} \rangle \\ \mathbb{A}_\xi \widehat{f} &:= \psi \Pi \mathbb{T}^* \varphi \widehat{f}, \end{aligned}$$

for a given constant  $\delta \in (0, 1)$ .

With all these definitions we obtain  $|\psi| \leq 1$  and  $|(v \cdot \xi) \varphi(\xi, v)| \leq 1$ , so that the Cauchy-Schwarz inequality yields

$$|\langle \mathbb{A}_\xi g, g \rangle| \leq \left| \int_{\mathbb{R}^d} \psi(\xi, v) g(\xi, v) dv \right| \left| \int_{\mathbb{R}^d} (v \cdot \xi) \varphi(\xi, v) g(\xi, v) dv \right| \leq \|g\|^2.$$

Thus,  $\mathbb{H}_\xi$  is equivalent to the  $L^2(d\mu)$  norm and more precisely we get

$$(1 - \delta) \|f\|^2 \leq \mathbb{H}_\xi(f) \leq (1 + \delta) \|f\|^2.$$

One can compare this new entropy to the one introduced in [14] and used in previous works where the diffusion limit is not fractional [14, 8, 9]. In these works, the operator  $\mathbb{A}$  was defined by

$$\mathbb{A} = \left(1 + |\mathbb{T} \Pi|^2\right)^{-1} (\mathbb{T} \Pi)^*,$$

which can be written in Fourier variable as

$$\mathbb{A} = \Pi \frac{-iv \cdot \xi}{1 + \Theta |\xi|^2},$$



where  $\Theta = \int_{\mathbb{R}^d} |v \cdot \mathbf{e}|^2 F(v) dv$  for an arbitrary unitary vector  $\mathbf{e}$ . One directly remarks that this operator is not defined if  $F$  does not have at least finite moments of order 2. The new operator  $A_\xi$  can be written

$$A_\xi = \frac{1}{\langle v \rangle^2} \Pi \frac{(-iv \cdot \xi) \langle v \rangle^{-\beta}}{1 + \langle v \rangle^{2|1-\beta|} |\xi|^2}.$$

The function on the left of  $\Pi$  was added to control the cases when  $F$  does not have 2 moments, and the apparition of the  $\beta$  in the operator provides the good scaling corresponding to the fractional diffusion limit in [20]. In this paper, the symbol of the limiting diffusion operator is indeed obtained as

$$\int_{\mathbb{R}^d} a(\xi, v) F(v) dv,$$

where

$$a(\xi, v) = \frac{\nu(v)}{\nu(v) - iv \cdot \xi} = \frac{\nu(v) (\nu(v) + iv \cdot \xi)}{\nu(v)^2 + |v \cdot \xi|^2},$$

and by assuming that  $\nu(v) = \langle v \rangle^\beta$ , the leading order term is given by

$$\frac{(iv \cdot \xi) \langle v \rangle^\beta}{\langle v \rangle^{2\beta} + |v \cdot \xi|^2} = \frac{(iv \cdot \xi) \langle v \rangle^{-\beta}}{1 + \langle v \rangle^{-2\beta} |v \cdot \xi|^2}.$$

The new operator  $A_\xi$  can be seen as a mix between these two approaches.

**1.4. A brief review of the literature.** — *Fractional diffusion limits* of kinetic equations attracted a considerable interest in the recent years. The microscopic jump processes are indeed easy to encode in kinetic equations and the diffusion limit provides a simple procedure to justify the use of fractional operators at macroscopic level. We refer to [3, 20] for an introduction to the topic in the case of operators of scattering type and a discussion of earlier results on standard, *i.e.*, non-fractional, diffusion limits. In a recent paper, [19], the case of a Fokker-Planck equation with heavy tails local equilibria has been considered (only when  $d = 1$ ) and, in from a probabilistic point of view, [15] collects various related results. The diffusion limit of the fractional Fokker-Planck equation has been studied in [1].

In the homogeneous case, that is, when there is no  $x$ -dependence, it is classical to introduce a potential function  $\Phi(v) = -\log F(v)$  and classify the behavior of the solution  $h$  to (1) according to the growth rate of  $\Phi$ . Assume that the collision operator is either the Fokker-Planck operator of case (a) or the scattering operator of Case (b). Schematically, if

$$\Phi(v) = \left(1 + |v|^2\right)^{\gamma/2},$$

we obtain that  $\|h(t, \cdot) - F\|_{L^2(\mathbb{R}^d, d\mu)}$  decays exponentially if  $\gamma \geq 1$ . Here we assume that  $\|h(t, \cdot)\|_{L^1(\mathbb{R}^d)} = \|F\|_{L^1(\mathbb{R}^d)}$ . In the range  $\gamma \in (0, 1)$ , the Poincaré inequality of case (a) has to be replaced by a *weak Poincaré* or a *weighted Poincaré inequality*: see [22, 17] and rates of convergence are typically algebraic in  $t$ . Summarizing, the lowest is the rate of growth as

$|x| \rightarrow +\infty$ , the slowest is the rate of convergence of  $h$  to  $F$ . The turning point occurs for the minimal growth which guarantees that  $F$  is integrable, at least for solutions of the homogeneous equation with initial data in  $L^1(\mathbb{R}^d)$ . If we consider for instance

$$\Phi(v) = \frac{\gamma}{2} \log(1 + |v|^2),$$

with  $\gamma < d$ , then diffusive effects win over confinement and the unique stationary solution with finite mass is 0. To measure the sharp rate of decay of  $h$  towards 0, one can replace the Poincaré inequality and the weak Poincaré inequality by *Nash's inequality*. See [9].

Standard diffusion limits provide an interesting insight into the *micro/macro decomposition* which has been the key of the  $L^2$  *hypocoercive approach* of [14]. Another parameter has now to be taken into account: the confinement in the spatial variable  $x$ . In presence of a confining potential  $V = V(x)$  with sufficient growth and when  $F$  has a fast decay rate, typically a quadratic growth, the rate of convergence is found to be exponential. A milder growth of  $V$  gives a slower convergence rate as analyzed in [11]. With  $e^{-V}$  not integrable, diffusion again wins in the hypocoercive picture, and the rate of convergence of a finite mass solution of (1) towards 0 can be captured by Nash and related Caffarelli-Kohn-Nirenberg inequalities: see [8, 9].

A typical regime for fractional diffusion limits is given by local equilibria with fat tails which behave according to (2) with  $\gamma \in (0, 2)$ :  $F$  is integrable but has no second moment. Whenever fractional diffusion limits can be obtained, we claim that rates of convergence can also be produced in an adapted  $L^2$  hypocoercive approach. To simplify the exposition, we shall consider only the case  $V = 0$  and measure the rate of convergence to 0. It is natural to expect that a fractional Nash type approach has to play the central role, and this is indeed what happens. The mode-by-mode hypocoercivity estimate shows that rates are of the order of  $|\xi|^\alpha$  as  $\xi \rightarrow 0$  which results in the expected time decay. In this direction, let us mention that the spectral information associated with  $|\xi|^\alpha$  is very natural in connection with the fractional heat equation as was recently shown in [4].

## 2. Mode by mode hypocoercivity

In this section, we present the first step towards the proof of our main theorems, which is an energy estimate coming from the hypocoercivity functional. To help the readability, we recall that

$$A_\xi \hat{f} = \psi(\xi, \cdot) F \int_{\mathbb{R}^d} (-iv \cdot \xi) \varphi(\xi, v) \hat{f}(\xi, v) dv.$$

We denote by  $L^*$  the dual of  $L$  in  $L^2(d\mu)$  and we define  $\alpha' := 2 + \frac{\min(\gamma+\beta-2, 0)}{|1-\beta|}$  so that  $\alpha' = \alpha$  when  $\gamma + \beta \leq 2$  and  $\alpha' = 2$  when  $\gamma + \beta \geq 2$ . We also define the weighted norms

$$\|g\|_k^2 := \int_{\mathbb{R}^d} |g|^2 \langle v \rangle^k d\mu,$$

so that in particular  $\|g\| = \|g\|_0$ .

**Proposition 4.** — *Let  $\gamma > 0$ ,  $\gamma > \beta$  and  $\eta \in (-\gamma, \gamma)$  verifying  $\eta \geq \beta$ , and assume that*

$$\begin{aligned}\mu_L(\xi) &:= \|\mathbf{L}^*((v \cdot \xi)\varphi(\xi, v)F)\|_{-\eta} \lesssim \left(\frac{|\xi|}{\langle \xi \rangle}\right)^{\frac{\alpha'}{2}}, \\ \lambda_L(\xi) &:= \|\mathbf{L}^*(\psi F)\|_{-\eta} \lesssim 1.\end{aligned}$$

Then if  $\gamma + \beta \neq 2$

$$\frac{d}{dt} \langle \mathbf{A}_\xi \widehat{f}, \widehat{f} \rangle \lesssim - \left(\frac{|\xi|}{\langle \xi \rangle}\right)^{\alpha'} \|\Pi \widehat{f}\|_\eta^2 + C \|(1 - \Pi)\widehat{f}\|_\eta^2,$$

and if  $\gamma + \beta = 2$ , then

$$\frac{d}{dt} \langle \mathbf{A}_\xi \widehat{f}, \widehat{f} \rangle \lesssim - \frac{|\xi|^2 \ln(|\xi|)}{1 + |\xi|^2 \ln(|\xi|)} \|\Pi \widehat{f}\|_\eta^2 + C \|(1 - \Pi)\widehat{f}\|_\eta^2.$$

*Proof.* — Note that since  $\varphi$  and  $\psi$  commute with  $\mathbb{T}$  and  $\Pi\mathbb{T}\Pi = 0$ , we get

$$\mathbf{A}_\xi \Pi = -\psi \Pi \mathbb{T} \Pi \varphi = 0, \quad \mathbf{A}_\xi^* \mathbb{T} \Pi = \varphi \mathbb{T} \Pi \mathbb{T} \Pi \psi = 0.$$

Moreover,  $\mathbf{L}\Pi = 0$ . With these identities, using the micro-macro decomposition  $\widehat{f} = \Pi \widehat{f} + (1 - \Pi)\widehat{f}$ , we obtain

$$\begin{aligned}\frac{d}{dt} \langle \mathbf{A}_\xi \widehat{f}, \widehat{f} \rangle &= - \langle \mathbf{A}_\xi \mathbb{T} \Pi \widehat{f}, \Pi \widehat{f} \rangle - \langle \mathbf{A}_\xi \mathbb{T} \Pi \widehat{f}, (1 - \Pi)\widehat{f} \rangle - \langle \mathbf{A}_\xi \mathbb{T} (1 - \Pi)\widehat{f}, \Pi \widehat{f} \rangle \\ &\quad - \langle \mathbf{A}_\xi \mathbb{T} (1 - \Pi)\widehat{f}, (1 - \Pi)\widehat{f} \rangle - \langle \mathbf{A}_\xi (1 - \Pi)\widehat{f}, \mathbb{T} (1 - \Pi)\widehat{f} \rangle \\ &\quad + \langle \mathbf{A}_\xi \mathbf{L} (1 - \Pi)\widehat{f}, \widehat{f} \rangle + \langle \mathbf{A}_\xi (1 - \Pi)\widehat{f}, \mathbf{L} (1 - \Pi)\widehat{f} \rangle \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.\end{aligned}$$

• **Step 1:  $I_1$  gives the macroscopic coercivity.**

First note that since  $\int_{\mathbb{R}^d} F = 1$ , it holds

$$|\rho_{\widehat{f}}(\xi)|^2 = \int_{\mathbb{R}^d} |\rho_{\widehat{f}}(\xi) F|^2 d\mu = \|\Pi \widehat{f}\|_\eta^2.$$

Then, since  $\Pi \widehat{f}(\xi, v) = \rho_{\widehat{f}}(\xi) F(v)$  and  $\mathbf{A}_\xi = \psi \Pi \mathbb{T}^* \varphi$ , we get

$$I_1 = - \langle \mathbf{A}_\xi \mathbb{T} F, F \rangle |\rho_{\widehat{f}}|^2 = - \langle \psi \Pi |\mathbb{T}|^2 \varphi F, F \rangle \|\Pi \widehat{f}\|_\eta^2,$$

where we used the fact that  $\int_{\mathbb{R}^d} F = 1$ . Since  $\Pi = |F\rangle\langle F|$ , it yields

$$I_1 = - \langle F, \psi F \rangle \langle F, |\mathbb{T}|^2 \varphi F \rangle \|\Pi \widehat{f}\|_\eta^2 = -\lambda_0 \mu_2 \|\Pi \widehat{f}\|_\eta^2,$$

where we define

$$\begin{aligned}\lambda_k &:= \left\| |v \cdot \xi|^k \psi F \right\|_{L^1(dv)} = \langle F, |\mathbb{T}|^k \psi F \rangle \\ \mu_k &:= \left\| |v \cdot \xi|^k \varphi F \right\|_{L^1(dv)} = \langle F, |\mathbb{T}|^k \varphi F \rangle.\end{aligned}$$

• **Step 2: a bound on micro-macro terms.**

Using again the definition of  $\Pi$ , the second term can be written

$$I_2 = - \left\langle \psi \Pi \mathbb{T}^* \varphi \mathbb{T} \Pi \widehat{f}, (1 - \Pi) \widehat{f} \right\rangle = - \left\langle F, |\mathbb{T}|^2 \varphi F \right\rangle \rho_{\widehat{f}} \left\langle \psi F, (1 - \Pi) \widehat{f} \right\rangle,$$

and using the Cauchy-Schwarz inequality to control the last scalar product, we get

$$|I_2| \leq \mu_2 \|\psi F\|_{-\eta} |\rho_{\widehat{f}}| \left\| (1 - \Pi) \widehat{f} \right\|_{\eta} \leq \tilde{\lambda}_0 \mu_2 \left\| \Pi \widehat{f} \right\| \left\| (1 - \Pi) \widehat{f} \right\|_{\eta},$$

where

$$\tilde{\lambda}_k := \left\| |v \cdot \xi|^k \psi F \right\|_{-\eta} \quad \text{and} \quad \tilde{\mu}_k := \left\| |v \cdot \xi|^k \varphi F \right\|_{-\eta}.$$

In the same way, we obtain for the next terms

$$|I_3| \leq \lambda_0 \tilde{\mu}_2 \left\| \Pi \widehat{f} \right\| \left\| (1 - \Pi) \widehat{f} \right\|_{\eta}$$

$$|I_4| \leq \tilde{\lambda}_0 \tilde{\mu}_2 \left\| (1 - \Pi) \widehat{f} \right\|_{\eta}^2$$

$$|I_5| \leq \tilde{\lambda}_1 \tilde{\mu}_1 \left\| (1 - \Pi) \widehat{f} \right\|_{\eta}^2$$

To get a bound on  $I_6$ , we use the fact that  $\mathbb{T}^* = -\mathbb{T}$  to obtain

$$\begin{aligned} I_6 &= - \left\langle \psi \Pi \mathbb{T} \varphi \mathbb{L} (1 - \Pi) \widehat{f}, \widehat{f} \right\rangle = - \left\langle F, \mathbb{T} \varphi \mathbb{L} (1 - \Pi) \widehat{f} \right\rangle \left\langle F \psi, \widehat{f} \right\rangle \\ &= \left\langle \mathbb{L}^* \mathbb{T} \varphi F, (1 - \Pi) \widehat{f} \right\rangle \left\langle F \psi, \widehat{f} \right\rangle. \end{aligned}$$

With a micro-macro decomposition, one may write

$$\left\langle F \psi, \widehat{f} \right\rangle = \lambda_0 \rho_{\widehat{f}} + \left\langle F \psi, (1 - \Pi) \widehat{f} \right\rangle.$$

Moreover, the Cauchy-Schwarz inequality gives

$$\left\langle \mathbb{L}^* \mathbb{T} \varphi F, (1 - \Pi) \widehat{f} \right\rangle \leq \mu_{\mathbb{L}} \left\| (1 - \Pi) \widehat{f} \right\|_{\eta}.$$

It yields

$$\begin{aligned} |I_6| &\leq \left( \lambda_0 \left\| \Pi \widehat{f} \right\| + \tilde{\lambda}_0 \left\| (1 - \Pi) \widehat{f} \right\|_{\eta} \right) \mu_{\mathbb{L}} \left\| (1 - \Pi) \widehat{f} \right\|_{\eta} \\ &\leq \lambda_0 \mu_{\mathbb{L}} \left\| \Pi \widehat{f} \right\| \left\| (1 - \Pi) \widehat{f} \right\|_{\eta} + \tilde{\lambda}_0 \mu_{\mathbb{L}} \left\| (1 - \Pi) \widehat{f} \right\|_{\eta}^2. \end{aligned}$$

In the same way, we get

$$|I_7| \leq \lambda_{\mathbb{L}} \tilde{\mu}_1 \left\| (1 - \Pi) \widehat{f} \right\|_{\eta}^2.$$

• **Step 3: the differential inequality.**

Gathering all the estimates and defining  $X := \left\| \Pi \widehat{f} \right\|$  and  $Y := \left\| (1 - \Pi) \widehat{f} \right\|_{\eta}$ , we deduce

$$\begin{aligned} \frac{d}{dt} \left\langle \mathbb{A}_{\xi} \widehat{f}, \widehat{f} \right\rangle &\leq -\lambda_0 \mu_2 X^2 + \left( \tilde{\lambda}_0 \mu_2 + \lambda_0 \tilde{\mu}_2 + \lambda_0 \mu_{\mathbb{L}} \right) XY \\ &\quad + \left( \tilde{\lambda}_0 \tilde{\mu}_2 + \tilde{\lambda}_1 \tilde{\mu}_1 + \tilde{\lambda}_0 \mu_{\mathbb{L}} + \lambda_{\mathbb{L}} \tilde{\mu}_1 \right) Y^2, \end{aligned}$$

which by Young's inequality leads to

$$\begin{aligned} \frac{d}{dt} \langle A_\xi \widehat{f}, \widehat{f} \rangle &\leq \left( \frac{\sigma}{2} (\tilde{\lambda}_0 \mu_2 + \lambda_0 \tilde{\mu}_2 + \lambda_0 \mu_L) - \lambda_0 \mu_2 \right) X^2 \\ &\quad + \left( \tilde{\lambda}_0 \tilde{\mu}_2 + \tilde{\lambda}_1 \tilde{\mu}_1 + \tilde{\lambda}_0 \mu_L + \lambda_L \tilde{\mu}_1 + \frac{1}{2\sigma} (\tilde{\lambda}_0 \mu_2 + \lambda_0 \tilde{\mu}_2 + \lambda_0 \mu_L) \right) Y^2. \end{aligned}$$

One may choose  $\sigma = \frac{\lambda_0 \mu_2}{\tilde{\lambda}_0 \mu_2 + \lambda_0 \tilde{\mu}_2 + \lambda_0 \mu_L}$  to get

$$\frac{d}{dt} \langle A_\xi \widehat{f}, \widehat{f} \rangle \leq -\frac{1}{2} \lambda_0 \mu_2 \left\| \Pi \widehat{f} \right\|_\eta^2 + \mathcal{C}(\xi) \left\| (1 - \Pi) \widehat{f} \right\|_\eta^2,$$

with

$$\mathcal{C}(\xi) = \tilde{\lambda}_0 \tilde{\mu}_2 + \tilde{\lambda}_1 \tilde{\mu}_1 + \tilde{\lambda}_0 \mu_L + \lambda_L \tilde{\mu}_1 + \frac{\left( \tilde{\lambda}_0 \mu_2 + \lambda_0 \tilde{\mu}_2 + \lambda_0 \mu_L \right)^2}{2\lambda_0 \mu_2}.$$

To infer that  $\mathcal{C}(\xi)$  is in fact bounded in  $\xi$ , we need to estimate carefully various coefficients. This needs the following lemmas.

**Lemma 5.** — For  $\gamma > \beta$ , the coefficient  $\mu_2$ , defined by

$$\mu_2(\xi) = \int_{\mathbb{R}^d} \frac{|v \cdot \xi|^2 \langle v \rangle^{-\beta}}{1 + \langle v \rangle^{2|1-\beta|} |\xi|^2} \frac{c_\gamma dv}{\langle v \rangle^{d+\gamma}},$$

is bounded by above and below when  $|\xi|$  is large and satisfies

$$\begin{aligned} \mu_2(\xi) &\underset{\xi \rightarrow 0}{\sim} C |\xi|^{\min(2, 2 + \frac{\gamma+\beta-2}{|1-\beta|})} && \text{if } \gamma + \beta \neq 2, \\ &\underset{\xi \rightarrow 0}{\sim} \frac{-1}{d|1-\beta|} |\xi|^2 \ln(|\xi|) && \text{if } \gamma + \beta = 2. \end{aligned}$$

The coefficients  $\tilde{\mu}_1$  and  $\tilde{\mu}_2$  have been respectively defined by taking  $k = 1$  and  $k = 2$  in

$$\tilde{\mu}_k(\xi) = \left( \int_{\mathbb{R}^d} \left( \frac{|v \cdot \xi|^k \langle v \rangle^{-\beta}}{1 + \langle v \rangle^{2|1-\beta|} |\xi|^2} \right)^2 \frac{c_\gamma dv}{\langle v \rangle^{d+\gamma+\eta}} \right)^{\frac{1}{2}}.$$

Whenever  $\eta + \gamma + 2\beta + 4|1-\beta| > 2k$ , they satisfy

$$\begin{aligned} \tilde{\mu}_k(\xi) &\underset{\xi \rightarrow 0}{\sim} C |\xi|^{\min(k, k + \frac{\gamma+\eta+2\beta-2k}{2|1-\beta|})} && \text{when } \gamma + 2\beta + \eta \neq 2k \\ \tilde{\mu}_k(\xi) &\underset{\xi \rightarrow 0}{\sim} -C |\xi|^k \ln(|\xi|) && \text{when } \gamma + 2\beta + \eta = 2k \\ \tilde{\mu}_k(\xi) &\underset{\xi \rightarrow \infty}{\sim} C |\xi|^{k-2}, \end{aligned}$$

where  $a \simeq b$  means that there exists a constant  $C > 0$  such that  $C^{-1}a \leq b \leq Ca$ .

**Remark 1.** — The coefficients  $\tilde{\mu}_1$  and  $\tilde{\mu}_2$  are well defined when  $\beta \leq \eta < \gamma$  since  $(\beta - 1) + |\beta - 1| = 2(\beta - 1)_+$  so that for  $k \leq 2$

$$\eta + \gamma + 2\beta + 4|\beta - 1| = (\eta - \beta) + (\gamma - \beta) + 8(\beta - 1)_+ + 4 > 2k.$$

**Lemma 6.** — *The coefficients  $\lambda_0$  and  $\lambda_1$  have been respectively defined by taking  $k = 0$  and  $k = 1$  in*

$$\lambda_k(\xi) := \int_{\mathbb{R}^d} \frac{|v \cdot \xi|^k}{\langle v \rangle^2} \frac{c_\gamma dv}{\langle v \rangle^{d+\gamma}} = |\xi|^k \int_{\mathbb{R}^d} \frac{c_\gamma |v_1|^k dv}{\langle v \rangle^{d+\gamma+2}}.$$

*The coefficients  $\tilde{\lambda}_0$  and  $\tilde{\lambda}_1$  have been respectively defined by taking  $k = 0$  and  $k = 1$  in*

$$\tilde{\lambda}_k(\xi) := \left( \int_{\mathbb{R}^d} \left( \frac{|v \cdot \xi|^k}{\langle v \rangle^2} \right)^2 \frac{c_\gamma dv}{\langle v \rangle^{d+\gamma+\eta}} \right)^{\frac{1}{2}} = |\xi|^k \left( \int_{\mathbb{R}^d} \frac{|v_1|^{2k}}{\langle v \rangle^4} \frac{c_\gamma dv}{\langle v \rangle^{d+\gamma+\eta}} \right)^{\frac{1}{2}},$$

*which is a well defined integral since  $\eta > -\gamma$ .*

*Proof of Lemma 5.* — Let  $c := |1 - \beta| \geq 0$  and start with the first claim. We first obtain the equivalence when  $\xi \rightarrow 0$ . Note that if  $\gamma + \beta > 2$ , then

$$\mu_2(\xi) = \int_{\mathbb{R}^d} \frac{|v \cdot \xi|^2 \langle v \rangle^{-\beta}}{1 + \langle v \rangle^{2c} |\xi|^2} \frac{c_\gamma dv}{\langle v \rangle^{d+\gamma}} \underset{\xi \rightarrow 0}{\sim} |\xi|^2 \int_{\mathbb{R}^d} \frac{c_\gamma |v_1|^2}{\langle v \rangle^{d+\gamma+\beta}} dv,$$

Now, if  $\gamma + \beta < 2$ , then  $c > 0$  and we can use the change of variable  $v = |\xi|^{\frac{-1}{c}} u$  in the integral defining  $\mu_2$  to get

$$\mu_2(\xi) = \int_{\mathbb{R}^d} \frac{||\xi|^{\frac{c-1}{c}} u_1|^2 \langle |\xi|^{\frac{-1}{c}} u \rangle^{-\beta}}{1 + \langle |\xi|^{\frac{-1}{c}} u \rangle^{2c} |\xi|^2} \frac{|\xi|^{\frac{-d}{c}} c_\gamma du}{\langle |\xi|^{\frac{-1}{c}} u \rangle^{d+\gamma}}.$$

Note that for any  $u \in \mathbb{R}^d \setminus \{0\}$ , we have  $\langle |\xi|^{\frac{-1}{c}} u \rangle \underset{\xi \rightarrow 0}{\sim} |u| |\xi|^{\frac{-1}{c}}$ . As a consequence,

$$\mu_2(\xi) \underset{\xi \rightarrow 0}{\sim} |\xi|^{2+\frac{\gamma+\beta-2}{c}} \int_{\mathbb{R}^d} \frac{|u_1|^2}{1 + |u|^{2c}} \frac{c_\gamma du}{|u|^{d+\gamma+\beta}}.$$

Note that the later integral is finite, since  $\gamma + \beta - 2 < 0$  and  $\gamma > \beta$  (which implies  $\beta < 1$  so that  $\gamma + \beta + 2 + 2c = \gamma - \beta > 0$ ).

Finally, assume that  $\gamma + \beta = 2$  and  $\gamma > \beta$ . Then it holds  $\beta < 1$  so that  $c = 1 - \beta > 0$  and

$$\mu_2 = |\xi|^2 \int_{\mathbb{R}^d} \frac{c_\gamma |v_1|^2}{1 + \langle v \rangle^{2c} |\xi|^2} \frac{dv}{\langle v \rangle^{d+2}}.$$

Now looking at the part where  $|v| \geq |\xi|^{\frac{-1}{c}}$  and by the same change of variable as above, we get

$$I_0 := \int_{|v| \geq |\xi|^{\frac{-1}{c}}} \frac{|v_1|^2}{1 + \langle v \rangle^{2c} |\xi|^2} \frac{dv}{\langle v \rangle^{d+2}} \underset{\xi \rightarrow 0}{\sim} \int_{|u| \geq 1} \frac{|u_1|^2}{1 + |u|^{2c}} \frac{du}{|u|^{d+2}},$$

which is finite since  $c > 0$ . For the other part of the integral, since it is invariant by rotation of  $\xi$ , for any  $j \in \{1, \dots, d\}$ , it holds

$$\begin{aligned} \int_{|v| < |\xi|^{-\frac{1}{c}}} \frac{|v_1|^2}{1 + \langle v \rangle^{2c} |\xi|^2} \frac{dv}{\langle v \rangle^{d+2}} &= \int_{|v| < |\xi|^{-\frac{1}{c}}} \frac{|v_j|^2}{1 + \langle v \rangle^{2c} |\xi|^2} \frac{dv}{\langle v \rangle^{d+2}} \\ &= \frac{1}{d} \int_{|v| < |\xi|^{-\frac{1}{c}}} \frac{|v|^2}{1 + \langle v \rangle^{2c} |\xi|^2} \frac{dv}{\langle v \rangle^{d+2}} \\ &= \frac{1}{d} (I_1 - I_2), \end{aligned}$$

with

$$\begin{aligned} I_2 &= \int_{|v| < |\xi|^{-\frac{1}{c}}} \frac{1}{1 + \langle v \rangle^{2c} |\xi|^2} \frac{dv}{\langle v \rangle^{d+2}} \underset{\xi \rightarrow 0}{\sim} \int_{\mathbb{R}^d} \frac{dv}{\langle v \rangle^{d+2}} \\ I_1 &= \int_{|v| < |\xi|^{-\frac{1}{c}}} \frac{1}{1 + \langle v \rangle^{2c} |\xi|^2} \frac{dv}{\langle v \rangle^d}, \end{aligned}$$

since  $|v|^2 = \langle v \rangle^2 - 1$ . Then by using the fact  $\frac{1}{1+X} = 1 - \frac{1}{1+X}$  we can also write  $I_1 = I_{11} + I_{12}$  with

$$I_{11} = \int_{|v| < |\xi|^{-\frac{1}{c}}} \frac{dv}{\langle v \rangle^d} \underset{\xi \rightarrow 0}{\sim} -\frac{1}{c} \ln(|\xi|),$$

and

$$I_{12} = \int_{|v| < |\xi|^{-\frac{1}{c}}} \frac{1}{1 + \langle v \rangle^{-2c} |\xi|^{-2}} \frac{dv}{\langle v \rangle^d}.$$

But this last term is bounded when  $\xi \rightarrow 0$  since by the change of variables  $v = |\xi|^{-\frac{1}{c}} u$ , we obtain

$$I_{12} = \int_{|u| < 1} \frac{|\xi|^{-\frac{d}{c}}}{1 + \langle |\xi|^{-\frac{1}{c}} u \rangle^{-2c} |\xi|^{-2} \langle |\xi|^{-\frac{1}{c}} u \rangle} \frac{du}{|\xi|^{-2} \langle |\xi|^{-\frac{1}{c}} u \rangle^d} \underset{\xi \rightarrow 0}{\sim} \int_{|u| < 1} \frac{|u|^{2c}}{1 + |u|^{2c} |u|^d} du.$$

Altogether, since  $I_{11}$  is the only unbounded term when  $\xi \rightarrow 0$ , this yields

$$\mu_2 = |\xi|^2 \left( I_0 + \frac{1}{d} (I_{11} + I_{12} - I_2) \right) \underset{\xi \rightarrow 0}{\sim} -\frac{1}{cd} |\xi|^2 \ln(|\xi|).$$

Now turn to the equivalence when  $\xi \rightarrow \infty$ . Note that since  $\gamma - \beta > 0$ , then

$$\mu_2(\xi) = \int_{\mathbb{R}^d} \frac{|v \cdot \xi|^2 \langle v \rangle^\beta}{\langle v \rangle^{2\beta} + \langle v \rangle^2 |\xi|^2} \frac{c_\gamma dv}{\langle v \rangle^{d+\gamma}} \underset{\xi \rightarrow \infty}{\sim} \int_{\mathbb{R}^d} \frac{c_\gamma |v_1|^2}{\langle v \rangle^{d+\gamma+2-\beta}} dv.$$

The first claim on  $\mu_2$  is now completed. The end of the lemma follows the same techniques, we omit the proof.  $\square$

• **Step 4: uniform bound on  $\mathcal{C}(\xi)$ .**

From these lemmas we deduce that  $\lambda_0$  and  $\tilde{\lambda}_0$  are bounded by above and by below by a positive constant so that

$$\mathcal{C}(\xi) \lesssim \tilde{\mu}_2 + \tilde{\lambda}_1 \tilde{\mu}_1 + \mu_L + \lambda_L \tilde{\mu}_1 + 1 + \frac{\tilde{\mu}_2^2}{\mu_2} + \frac{\mu_L^2}{\mu_2}.$$

We also deduce that for  $|\xi|$  large, it holds

$$\begin{aligned}\mu_2 &\underset{\xi \rightarrow \infty}{\sim} \tilde{\mu}_2 \underset{\xi \rightarrow \infty}{\sim} C \\ \tilde{\mu}_1 &\underset{\xi \rightarrow \infty}{\sim} C |\xi|^{-1} \\ \tilde{\lambda}_1 &\underset{\xi \rightarrow \infty}{\sim} C |\xi|,\end{aligned}$$

so that for  $|\xi| \geq 1$  and since by hypothesis  $\mu_L$  and  $\lambda_L$  are bounded by above,

$$C(\xi) \lesssim 1 + \mu_L + \lambda_L |\xi|^{-1} + \mu_L^2 \lesssim 1.$$

Again from the previous lemmas, we also deduce that for  $\beta < 1$ ,  $\gamma + \beta \neq 2$ ,  $\gamma + \eta + 2\beta \neq 4$  and  $\gamma + \eta + 2\beta \neq 2$

$$\begin{aligned}\mu_2 &\underset{\xi \rightarrow 0}{\sim} C |\xi|^{2+\min(0, \frac{\gamma+\beta-2}{|1-\beta|})} = C |\xi|^{\min(2, \frac{\gamma-\beta}{1-\beta})} \\ \tilde{\mu}_2 &\underset{\xi \rightarrow 0}{\sim} C |\xi|^{2+\min(0, \frac{\gamma+2\beta+\eta-4}{2|1-\beta|})} = C |\xi|^{\min(2, \frac{\gamma-2\beta+\eta}{2(1-\beta)})} \\ \tilde{\mu}_1 &\underset{\xi \rightarrow 0}{\sim} C |\xi|^{1+\min(0, \frac{\gamma+2\beta+\eta-2}{2|1-\beta|})} = C |\xi|^{\min(1, \frac{\gamma+\eta}{2(1-\beta)})} \\ \tilde{\lambda}_1 &\underset{\xi \rightarrow 0}{\sim} C |\xi|.\end{aligned}$$

Since  $\eta \geq \beta \geq 2\beta - \gamma$  and  $\eta \geq -\gamma$ , it implies that for  $|\xi| \leq 1$

$$C(\xi) \lesssim 1 + \mu_L + \lambda_L + \frac{\tilde{\mu}_2^2}{\mu_2} + \frac{\mu_L^2}{\mu_2} \lesssim 1 + \frac{\tilde{\mu}_2^2}{\mu_2},$$

where the second inequality follows from the hypotheses on  $\lambda_L$  and  $\mu_L$ . Moreover, for  $|\xi| \leq 1$ , and  $\gamma + \beta < 2$ , it holds

$$\frac{\tilde{\mu}_2^2}{\mu_2} \underset{\xi \rightarrow 0}{\sim} C |\xi|^{\frac{\min(\eta-\beta, 4-3\beta-\gamma)}{1-\beta}},$$

which is bounded near 0 since  $\eta \geq \beta$  and  $4-3\beta-\gamma = 2(1-\beta)+2-(\gamma+\beta) \geq 0$ . If  $\gamma + \beta > 2$ , then

$$\frac{\tilde{\mu}_2^2}{\mu_2} \underset{\xi \rightarrow 0}{\sim} C |\xi|^{\min(2, \frac{\gamma+\eta-2}{1-\beta})},$$

which is bounded near 0 since  $\eta \geq \beta > 2 - \gamma$ . When  $\beta > 1$ ,  $\gamma + 2\beta + \eta - 4 \neq 0$  and  $\gamma + 2\beta + \eta - 2 \neq 0$ , then  $\gamma \geq \beta > 1$  and

$$\begin{aligned}\mu_2 &\underset{\xi \rightarrow 0}{\sim} C |\xi|^2 \\ \tilde{\mu}_2 &\underset{\xi \rightarrow 0}{\sim} C |\xi|^{\min(2, \frac{\gamma+\eta+6\beta-8}{2(\beta-1)})} \\ \tilde{\mu}_1 &\underset{\xi \rightarrow 0}{\sim} C |\xi|^{\min(1, \frac{\gamma+\eta+4\beta-4}{2(\beta-1)})} \\ \tilde{\lambda}_1 &\underset{\xi \rightarrow 0}{\sim} C |\xi|.\end{aligned}$$



Since  $\eta \geq \beta > 1$ , we get  $\gamma + \eta + 6\beta - 8 > 0$  and  $\gamma + \eta + 4\beta - 4 > 0$  so that we obtain once again  $\mathcal{C}(\xi) \lesssim 1 + \frac{\tilde{\mu}_2^2}{\mu_2}$ . Moreover

$$\frac{\tilde{\mu}_2^2}{\mu_2} \underset{\xi \rightarrow 0}{\sim} C |\xi|^{\min(2, \frac{\gamma + \eta + 4\beta - 6}{\beta - 1})},$$

which is bounded near 0 since  $\gamma + \eta + 4\beta - 6 > 0$ . In the critical cases when a  $\ln(|\xi|)$  appears in the expression of  $\mu_2$ ,  $\tilde{\mu}_1$  or  $\tilde{\mu}_2$ , we obtain expressions of the form  $|\xi|^\varepsilon \ln(|\xi|)$  for some  $\varepsilon > 0$ , so that all the terms also remain bounded. We conclude that in all the cases,  $\mathcal{C}(\xi)$  is bounded from above uniformly with respect to  $\xi$ , which ends the proof of Proposition 4.  $\square$

### 3. Estimates in weighted $L^2$ spaces

In this section, we assume that  $\beta \leq 0$ . We show the propagation of weighted norms  $L^2(\langle v \rangle^k dx d\mu)$  with power law of order  $k \in (0, \gamma)$ . This will be crucially used to prove our main Theorems 1–3. The result is the following.

**Proposition 7.** — *Let  $k \in (0, \gamma)$  and  $f$  be solution of equation (1) with initial condition  $f^{\text{in}} \in L^2(\langle v \rangle^k dx d\mu)$ . Then, there exists a constant  $\mathcal{C}_k = C_{d, \gamma, \beta, k} > 0$  such that for any  $t \in \mathbb{R}_+$  it holds*

$$\|f(t, \cdot, \cdot)\|_{L^2(\langle v \rangle^k dx d\mu)} \leq \mathcal{C}_k \|f^{\text{in}}\|_{L^2(\langle v \rangle^k dx d\mu)}.$$

This strategy of using the conservation of weighted norms has also been used in a companion paper [7] when  $F$  has a sub-exponential form. There, any value of  $k$  was authorized, and this was implicitly a consequence of the fact that such a  $F$  had finite weighted norms  $L^2(\langle v \rangle^k dx d\mu)$  for any  $k \in \mathbb{R}_+$ . In the present case of this paper, one has to be really careful with the order of the weighted norms at stage. Note that we may not expect global propagation of higher moments than those that  $F$  has, and this justifies the range of values of  $k$ .

We thus choose to adopt the following interpolation strategy. Note that for any function  $h \in L^2(\langle v \rangle^k dx d\mu)$ , one has immediately

$$\|h\|_{L^2(\langle v \rangle^k dx d\mu)} = \|F^{-1}h\|_{L^2(F \langle v \rangle^k dx dv)}.$$

As such, it is equivalent to control the semi-group  $e^{(L-\mathbb{T})t}$  in  $L^2(\langle v \rangle^k dx d\mu)$  and  $F^{-1}e^{(L-\mathbb{T})t}$  in  $L^2(F \langle v \rangle^k dx dv)$ . Since  $L^2(\langle v \rangle^k F dx dv)$  is an interpolation space between  $L^1(F \langle v \rangle^k dx dv)$  and  $L^\infty(dx dv)$  (see [23]), if we prove that  $F^{-1}e^{(L-\mathbb{T})t}$  is bounded onto  $L^\infty(dx dv)$  and onto  $L^1(F \langle v \rangle^k dx dv)$ , it will be automatically bounded onto  $L^2(F \langle v \rangle^k dx dv)$ , which is exactly the result of Proposition 7.

The rest of this section thus goes as follows. We will separately show boundedness of  $F^{-1}e^{(L-\mathbb{T})t}$  onto  $L^\infty(dx dv)$  and onto  $L^1(F \langle v \rangle^k dx dv)$ . The first point is relatively immediate whereas the second point requires a Lyapunov and splitting strategy, as previously used in [17, 18, 7].

**3.1. The boundedness in  $L^\infty(dx dv)$ . —**

**Lemma 8.** — *Let  $\mathsf{L}$  be as in case (a), (b) or (c). Then, in all these cases,*

$$\left\| F^{-1} e^{t(\mathsf{L}-\mathsf{T})} \right\|_{L^\infty(dx dv) \rightarrow L^\infty(dx dv)} \leq 1.$$

*Proof.* — This is a consequence of the maximum principle.  $\square$

**3.2. The boundedness in  $L^1(F \langle v \rangle^k dx dv)$ .** — Remark that bounding the operator  $F^{-1} e^{t(\mathsf{L}-\mathsf{T})}$  in  $L^1(F \langle v \rangle^k dx dv)$  is equivalent to bounding  $e^{t(\mathsf{L}-\mathsf{T})}$  in  $L^1(\langle v \rangle^k dx dv)$ . To obtain such a bound, we first write a Lyapunov type estimate.

**Lemma 9.** — *Let  $\mathsf{L}$  be as in case (a), (b) or (c). Then, in all these cases, for any  $k \in (0, \gamma - \beta)$ , there exists  $(a, b, R) \in \mathbb{R} \times \mathbb{R}_+^2$  such that for any  $f \in L^1(\langle v \rangle^k)$  the following inequality holds*

$$\iint_{\mathbb{R}^{2d}} \frac{f}{|f|} \mathsf{L} f \langle v \rangle^k dx dv \leq \iint_{\mathbb{R}^{2d}} (a \mathbf{1}_{B_R} - b \langle v \rangle^\beta) |f| \langle v \rangle^k dx dv.$$

*Proof.* — First assume that  $f \geq 0$ . Then one may write,

$$\iint_{\mathbb{R}^{2d}} \mathsf{L} f \langle v \rangle^k dx dv = \iint_{\mathbb{R}^{2d}} \mathsf{L} f F \langle v \rangle^k dx d\mu = \iint_{\mathbb{R}^{2d}} F^{-1} \mathsf{L}^*(F \langle v \rangle^k) f dx dv.$$

In case (a),

$$\begin{aligned} F^{-1} \mathsf{L}^*(F \langle \cdot \rangle^k)(v) &= F^{-1} \mathsf{L}(F \langle \cdot \rangle^k)(v) \\ &= \langle v \rangle^{d+\gamma} \nabla_v \cdot \left( \langle v \rangle^{-d-\gamma} \nabla_v \left( \langle v \rangle^k \right) \right) \\ &= k \langle v \rangle^{d+\gamma} \nabla_v \cdot \left( \langle v \rangle^{-d-\gamma+k-2} v \right) \\ &= k(d + \gamma - k + 2) \langle v \rangle^{k-4} - k(\gamma + 2 - k) \langle v \rangle^{k-2} \end{aligned}$$

which indeed implies the result for  $k < \gamma - \beta$  since in this case  $\beta = -2$ .

In case (b), using assumption (H1), one obtains

$$\begin{aligned} F^{-1} \mathsf{L}^*(F \langle \cdot \rangle^k)(v) &= \int_{\mathbb{R}^d} b(v', v) \left( \langle v' \rangle^k F(v') - \langle v \rangle^k F(v') \right) dv' \\ &= \left( \int_{\mathbb{R}^d} b(v', v) \frac{\langle v' \rangle^k}{\langle v \rangle^k} F(v') dv' - \nu(v) \right) \langle v \rangle^k. \end{aligned}$$

By hypothesis (H4), it yields

$$F^{-1} \mathsf{L}^*(F \langle \cdot \rangle^k)(v) \leq \left( \frac{\mathcal{C}_b}{\langle v \rangle^k} - 1 \right) \langle v \rangle^\beta.$$

We conclude that inequality (9) holds for any  $k < \gamma - \beta$ .

In case (c),

$$\begin{aligned} F^{-1}\mathbf{L}^*(F\langle\cdot\rangle^k)(v) &= \Delta_v^{\frac{\alpha}{2}}(\langle v\rangle^k) - E(v) \cdot \nabla_v(\langle v\rangle^k) \\ &= \left( \langle v\rangle^{-k} \Delta_v^{\frac{\alpha}{2}}(\langle v\rangle^k) - k(v \cdot E(v)) \langle v\rangle^{-2} \right) \langle v\rangle^k, \\ &\leq \left( \langle v\rangle^{-k} \Delta_v^{\frac{\alpha}{2}}(\langle v\rangle^k) - C_{F,\alpha} k \langle v\rangle^\beta \right) \langle v\rangle^k, \end{aligned}$$

where we used Proposition 20 that gives  $v \cdot E \geq C_{F,\alpha} \langle v\rangle^{\beta+2}$ . Since  $k < \alpha = \gamma - \beta$ , we control  $\langle v\rangle^{-k} \Delta_v^{\frac{\alpha}{2}}(\langle v\rangle^k)$  using a classical estimate for the fractional Laplacian of weights (see e.g. [6, 5, 18]),

$$\langle v\rangle^{-k} \Delta_v^{\frac{\alpha}{2}}(\langle v\rangle^k) \lesssim \langle v\rangle^{-\alpha}.$$

If now we remove the assumption that  $f \geq 0$ , the result still follows by using Kato's inequality (see e.g. [10] for the Laplacian and [12] for the fractional Laplacian) for the operator  $\mathbf{L}$  since it implies

$$\iint_{\mathbb{R}^{2d}} \frac{f}{|f|} \mathbf{L}f \langle v\rangle^k dx dv \leq \iint_{\mathbb{R}^{2d}} \mathbf{L}(|f|) \langle v\rangle^k dx dv,$$

and we can use the estimate by replacing  $f$  by  $|f|$ .  $\square$

**3.3. A factorization method.** — We employ a "shrinking" strategy, as in [16, 17, 21]. We write  $\mathbf{L} - \mathbf{T}$  as the sum of a dissipative part  $\mathbf{C}$  and a bounded part  $\mathbf{B}$  such that  $\mathbf{L} - \mathbf{T} = \mathbf{B} + \mathbf{C}$ .

**Lemma 10.** — *With the notation of Lemma 9, let  $(k, k_2) \in (0, \gamma) \times (0, \gamma - \beta)$  such that  $k_2 > k - \beta$ ,  $a = \max(a_k, a_{k_2})$ ,  $R = \max\{R_k, R_{k_2}\}$ ,  $\mathbf{C} = a \mathbf{1}_{B_R}$  and  $\mathbf{B} = \mathbf{L} - \mathbf{T} - \mathbf{C}$ . Then for any  $t \in \mathbb{R}_+$ , we have:*

- (i)  $\|\mathbf{C}\|_{L^1(dx d\mu) \rightarrow L^1(\langle v\rangle^{k_2} dx d\mu)} \leq a \langle R \rangle^{k_2}$ ,
- (ii)  $\|e^{t\mathbf{B}}\|_{L^1(\langle v\rangle^k dx d\mu) \rightarrow L^1(\langle v\rangle^k dx d\mu)} \leq 1$ ,
- (iii)  $\|e^{t\mathbf{B}}\|_{L^1(\langle v\rangle^{k_2} dx d\mu) \rightarrow L^1(\langle v\rangle^k dx d\mu)} \leq C(1+t)^{\frac{k_2-k}{\beta}}$  for some  $C > 0$ .

*Proof.* — Property (i) is a consequence of the definition of  $\mathbf{C}$ . Property (ii) follows from Lemma 9. Indeed, for any  $g \in L^1(\langle v\rangle^k)$ ,

$$\begin{aligned} \iint_{\mathbb{R}^{2d}} \mathbf{B}g \langle v\rangle^k dx dv &\leq \iint_{\mathbb{R}^{2d}} \left( a_k \mathbf{1}_{B_{R_k}} - a \mathbf{1}_{B_R} - b_k \langle v\rangle^\beta \right) |g| \langle v\rangle^k dx dv \\ &\leq -b_k \|g\|_{L^1(\langle v\rangle^{k-\beta} dx dv)}. \end{aligned}$$

To prove (iii), define  $g = e^{t\mathbf{B}}g^{\text{in}}$ . By Hölder's inequality and the above contraction property, we get

$$\|g\|_{L^1(\langle v\rangle^k dx dv)} \leq \|g\|_{L^1(\langle v\rangle^{k-\beta} dx dv)}^{\frac{k_2-k}{k_2-k-\beta}} \|g^{\text{in}}\|_{L^1(\langle v\rangle^{k_2} dx dv)}^{\frac{|\beta|}{k_2-k-\beta}}.$$

As a consequence,

$$\iint_{\mathbb{R}^{2d}} \mathbf{B}g \langle v \rangle^k dx dv \leq -b_k \left( \|g\|_{L^1(\langle v \rangle^k dx dv)} \right)^{1 + \frac{|\beta|}{k_2 - k}} \|g^{\text{in}}\|_{L^1(\langle v \rangle^{k_2} dx dv)}^{-\frac{|\beta|}{k_2 - k}},$$

so that by Grönwall's Lemma, we obtain

$$\begin{aligned} \|f\|_{L^1(\langle v \rangle^k dx dv)} &\leq \left( \|f^{\text{in}}\|_{L^1(\langle v \rangle^k dx dv)}^{-\frac{|\beta|}{k_2 - k}} + \frac{k_2 - k}{b_k |\beta|} t \|f^{\text{in}}\|_{L^1(\langle v \rangle^{k_2} dx dv)}^{-\frac{|\beta|}{k_2 - k}} \right)^{-\frac{k_2 - k}{|\beta|}} \\ &\leq \frac{1}{\left(1 + \frac{b_k |\beta|}{k_2 - k} t\right)^{\frac{k_2 - k}{|\beta|}}} \|f^{\text{in}}\|_{L^1(\langle v \rangle^{k_2} dx dv)}, \end{aligned}$$

which implies (iii).  $\square$

**Lemma 11.** — *Let  $k \in (0, \gamma)$  and  $f$  be a solution of (1) with initial condition  $f^{\text{in}} \in L^1(\langle v \rangle^k dx dv)$ . Then, there exists a constant  $\mathcal{C}_k = C_{d, \gamma, \beta, k} > 0$  such that*

$$\|f\|_{L^1(\langle v \rangle^k dx dv)} \leq \mathcal{C}_k \|f^{\text{in}}\|_{L^1(\langle v \rangle^k dx dv)}.$$

*Proof.* — Defining the convolution of two operators by  $\mathbf{U} \star \mathbf{V} = \int_0^t \mathbf{U}(t-s) \mathbf{V}(s) ds$ , we can write the following Duhamel's formula

$$e^{t(\mathbf{L}-\mathbf{T})} = e^{t\mathbf{B}} + e^{t\mathbf{B}} \star \mathbf{C} e^{t(\mathbf{L}-\mathbf{T})}.$$

Combining the formulas from Lemma 10 and the fact that

$$\|e^{t(\mathbf{L}-\mathbf{T})}\|_{L^1(dx dv) \rightarrow L^1(dx dv)} \leq 1,$$

we get

$$\|e^{t\mathbf{L}}\|_{L^1(\langle v \rangle^k dx dv) \rightarrow L^1(\langle v \rangle^k dx dv)} \leq 1 + a \langle R \rangle^{k_2} \int_0^t \frac{ds}{(1 + cs)^{\frac{k_2 - k}{|\beta|}}},$$

which is bounded uniformly in time since  $k_2 - k > -\beta = |\beta|$ .  $\square$

#### 4. Interpolation inequalities of Poincaré type

Let us introduce the quadratic forms associated to the three types of microscopic operators. In the Fokker-Planck case  $\mathbf{L}_1$ ,

$$\Gamma_1(h) := |\nabla_v h|^2,$$

in the scattering case  $\mathbf{L}_2$ ,

$$\Gamma_2(h)(\cdot) := \int_{\mathbb{R}^d} b(\cdot, v') |h(v) - h(v')|^2 F(v') dv',$$

and in the fractional Fokker-Planck case  $\mathbf{L}_3$ ,

$$\Gamma_3(h)(\cdot) := \int_{\mathbb{R}^d} \frac{|h(v') - h(\cdot)|^2}{|v' - \cdot|^{d+\alpha}} dv'.$$

**Theorem 12.** — Let  $\gamma > 0$ ,  $\eta \in (-\gamma, \gamma)$  and  $k \in [\eta, \gamma)$ . Then there exists  $\mathcal{C} = \mathcal{C}_{d,\gamma,k} > 0$  such that any  $h$  verifying  $\langle h \rangle_F = 0$ ,

$$\int_{\mathbb{R}^d} |h|^2 \langle v \rangle^\eta F \, dv \leq \mathcal{C} \left( \int_{\mathbb{R}^d} \Gamma_i(h) F \, dv \right)^{\frac{k-\eta}{k-\min(\eta,\beta)}} \left( \int_{\mathbb{R}^d} |h|^2 \langle v \rangle^k F \, dv \right)^{\frac{(\eta-\beta)_+}{k-\min(\eta,\beta)}},$$

for  $i = 1, 2, 3$ .

*Proof.* — For legibility, define  $w(v) = \langle v \rangle^\eta$ . Note that the fact that  $\langle h \rangle_F = 0$  and the Cauchy-Schwarz inequality yield

$$\begin{aligned} \int_{\mathbb{R}^d} |h|^2 w F \, dv &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (h - h') F' \, dv' \right|^2 w F \, dv \\ &\leq \Theta_{-\eta} \iint_{\mathbb{R}^{2d}} |h - h'|^2 \langle v' \rangle^\eta F' w F \, dv \, dv' \\ &\leq \Theta_{-\eta} \iint_{\mathbb{R}^{2d}} |h - h'|^2 w w' F F' \, dv \, dv', \end{aligned}$$

where  $\Theta_{-\eta} = \int_{\mathbb{R}^d} \langle v \rangle^{-\eta} F \, dv$ . Take any  $R > 1$ ,

$$\begin{aligned} \iint_{|v|>R} |h - h'|^2 w w' F F' \, dv \, dv' &\leq \frac{2}{\langle R \rangle^{k-\eta}} \iint_{|v|>R} (|h|^2 + |h'|^2) \langle v \rangle^k F F' \, dv \, dv' \\ &\lesssim \frac{1}{R^{k-\eta}} \int_{\mathbb{R}^d} |h|^2 (\Theta_k + \langle v \rangle^k) F \, dv, \end{aligned}$$

and the same bound holds for  $|v'| > R$ , so that

$$(5) \quad \iint_{|v|\geq R \text{ or } |v'|\geq R} |h' - h|^2 w w' F F' \, dv \, dv' \lesssim \frac{\Theta_k}{R^{k-\eta}} \int_{\mathbb{R}^d} |h|^2 \langle v \rangle^k F \, dv.$$

• **Case**  $\Gamma = \Gamma_1$

Now define  $h_R := h\chi_R$  and  $\chi_R(v) := \chi(|v|/R)$  where  $\chi$  is a truncation function on  $\mathbb{R}_+$  with the following properties:  $0 \leq \chi \leq 1$ ,  $\chi \equiv 1$  on  $[0, 1]$ ,  $\chi \equiv 0$  on  $[2, +\infty)$ . By the fact that  $\chi_R = 1$  on  $B_R$ , we obtain

$$\iint_{B_R^2} |h' - h|^2 w w' F F' \, dv \, dv' \leq \iint_{\mathbb{R}^{2d}} |h'_R - h_R|^2 w w' F F' \, dv \, dv'.$$

However, by expanding the square, we get

$$\begin{aligned} \iint_{\mathbb{R}^{2d}} |h'_R - h_R|^2 w w' F F' \, dv \, dv' &= 2 \left( \int_{\mathbb{R}^d} |h_R|^2 w F \, dv - \langle h \rangle_w^2 \right) \\ &= 2 \int_{\mathbb{R}^d} |h_R - \langle h_R \rangle_{wF}|^2 w F \, dv, \end{aligned}$$

so that we can use the Hardy-Poincaré inequality, which leads to

$$\begin{aligned} \int_{\mathbb{R}^d} |h_R - \langle h_R \rangle_{wF}|^2 w F \, dv &\leq \mathcal{C}_{\text{HP}} \int_{\mathbb{R}^d} |\nabla_v h_R|^2 \langle v \rangle^2 w F \, dv \\ &\leq 2 \mathcal{C}_{\text{HP}} \int_{B_{2R}} (|\nabla_v h|^2 + |\nabla_v \chi_R|^2 |h|^2) \langle v \rangle^2 w F \, dv \end{aligned}$$

where  $\mathcal{C}_{\text{HP}} = C_{d,\gamma-\eta}$  is the constant of the Hardy-Poincaré inequality. For the first term in the last integral, we use the fact that  $|v| < 2R$  to get

$$\int_{B_{2R}} |\nabla_v h|^2 \langle v \rangle^2 w F \, dv \lesssim R^{(2+\eta)_+} \int_{\mathbb{R}^d} |\nabla_v h|^2 F \, dv$$

For the second term, the fact that  $\nabla_v \chi_R = R^{-1} \chi'$  is supported in  $B_{2R} \setminus B_R$  yields

$$\int_{\mathbb{R}^d} |\nabla_v \chi_R|^2 |h|^2 \langle v \rangle^2 w F \, dv \leq \frac{C_\chi}{R^{k-\eta}} \int_{\mathbb{R}^d} |h|^2 \langle v \rangle^k F \, dv$$

with  $C_\chi = C_{k-\eta} \|\chi'\|_{L^\infty([1,2])}$ . Gathering all the inequalities, we get

$$\begin{aligned} \int_{\mathbb{R}^d} |h|^2 w F \, dv &\leq \iint_{\mathbb{R}^{2d}} |h - h'|^2 w' F' w F \, dv \, dv' \\ &\lesssim \Theta_{-\eta} \left( \mathcal{C}_{\text{HP}} R^{(2+\eta)_+} \int_{\mathbb{R}^d} |\nabla_v h|^2 F \, dv + \frac{\mathcal{C}_{\text{HP}} + \Theta_k}{R^{k-\eta}} \int_{\mathbb{R}^d} |h|^2 \langle v \rangle^k F \, dv \right). \end{aligned}$$

Optimizing with respect to  $R$  yields the result. More precisely, this last inequality can be written under the form  $a \leq R^{(2+\eta)_+} b + R^{-(k-\eta)} c$  with  $a \leq c$ . If  $c \leq b$ , then we can simply write  $a \leq c = c^\theta c^{1-\theta} \leq b^\theta c^{1-\theta}$ . If it is not the case, then we can take  $R := \left(\frac{c}{b}\right)^{\frac{1}{k+\max(\eta,2)}} \geq 1$ , which leads to  $a \leq 2b^\theta c^{1-\theta}$ .

- **Case**  $\Gamma = \Gamma_3$

In this case, following the idea of the proof in [25], it holds

$$\begin{aligned} &\iint_{B_R^2} |h' - h|^2 w w' F F' \, dv \, dv' \\ &\leq \iint_{B_R^2} \frac{|h' - h|^2}{|v' - v|^{d+\mathbf{a}}} |v' - v|^{d+\mathbf{a}} w w' F F' \, dv \, dv' \\ &\leq C_F(R) \iint_{B_R^2} \frac{|h' - h|^2}{|v' - v|^{d+\mathbf{a}}} (F + F') \, dv \, dv' \\ &\leq 2C_F(R) \int_{\mathbb{R}^d} \Gamma(h) F \, dv, \end{aligned}$$

where  $C_F(R) = \sup_{(v,v') \in B_R^2} \left( \frac{|v'-v|^{d+\mathbf{a}} w w' F F'}{F+F'} \right)$ . Since  $F = c_\gamma \langle v \rangle^{-(d+\gamma)}$  and  $|v - v'| \leq \langle v \rangle + \langle v' \rangle$ , we get

$$\begin{aligned} \frac{|v' - v|^{d+\mathbf{a}} w w' F F'}{F + F'} &\leq \frac{2^{d+\mathbf{a}} c_\gamma \left( \langle v' \rangle^{d+\mathbf{a}} + \langle v \rangle^{d+\mathbf{a}} \right) \langle v \rangle^\eta \langle v' \rangle^\eta}{\langle v' \rangle^{d+\gamma} + \langle v \rangle^{d+\gamma}} \\ &\lesssim c_\gamma R^{(\eta-\beta)_+}, \end{aligned}$$

since  $\beta = \gamma - \mathbf{a}$ . The result then follows by the same steps as in the Fokker-Planck case.

- **Case**  $\Gamma = \Gamma_2$

In this case, when  $v$  and  $v'$  are in  $B_R$ , we use hypothesis (H2) to get

$$\iint_{B_R^2} |h' - h|^2 w \tilde{w}' F F' \, dv \, dv' \lesssim R^{(\tilde{\eta}-\beta)_+} \iint_{\mathbb{R}^{2d}} \mathfrak{b}(v, v') |h' - h|^2 F F' \, dv \, dv'.$$

The result follows as in the other cases.  $\square$

**Remark 2.** — In particular, if  $\gamma + \beta > 0$  and  $\eta \leq \beta$ , we obtain a weighted Poincaré inequality

$$\int_{\mathbb{R}^d} |h|^2 \langle v \rangle^\eta F \, dv \leq \mathcal{C} \int_{\mathbb{R}^d} \Gamma(h) F \, dv.$$

If  $\gamma + \beta \leq 0$ , then  $\beta \leq -\gamma < \eta$ , which yields

$$\int_{\mathbb{R}^d} |h|^2 \langle v \rangle^\eta F \, dv \leq \mathcal{C} \left( \int_{\mathbb{R}^d} \Gamma(h) F \, dv \right)^{\frac{k-\eta}{k-\beta}} \left( \int_{\mathbb{R}^d} |h|^2 \langle v \rangle^k F \, dv \right)^{\frac{\eta-\beta}{k-\beta}}.$$

## 5. Proof of the theorems

### 5.1. Proof of Theorem 1 in the case $\gamma + \beta > 0$ . —

**Proposition 13.** — *There exists  $\delta > 0$  such that if  $\mathbf{H}_\xi$  is defined by formula (4) and  $\mathbf{H}(f) := \int_{\mathbb{R}^d} \mathbf{H}_\xi(f) \, d\xi$ , it holds*

$$\frac{d}{dt} \mathbf{H}(f) \lesssim -\delta \left( \left\| \left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \Pi \hat{f} \right\|_{L^2(d\xi \, d\mu)}^2 + \|(1 - \Pi)f\|_{L^2(\langle v \rangle^\beta \, dx \, d\mu)}^2 \right).$$

*Proof of Proposition 13.* — By the microscopic weighted coercivity one has

$$\frac{1}{2} \partial_t \|\hat{f}\|^2 \leq -\mathcal{C}_P \|(1 - \Pi)\hat{f}\|_{L^2(\langle v \rangle^\beta \, d\mu)}^2.$$

Using the result of Proposition 4 with  $\eta = -\beta$ , this leads to

$$\partial_t \mathbf{H}_\xi(f) \lesssim -\delta \left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \|\Pi \hat{f}\|^2 + (\mathcal{C} \delta - \mathcal{C}_P) \|(1 - \Pi)\hat{f}\|_\beta^2.$$

Taking  $\delta$  sufficiently small yields

$$\partial_t \mathbf{H}_\xi(f) \leq -\mathcal{C} \delta \left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \|\Pi \hat{f}\|^2 - \frac{1}{2} \mathcal{C}_P \|(1 - \Pi)\hat{f}\|_\beta^2.$$

The result is then obtained by integrating with respect to  $\xi$  and using Plancherel's formula.  $\square$

To control the macroscopic part, we shall control  $\left\| \left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \Pi \hat{f} \right\|_{L^2(d\xi \, d\mu)}$  in terms of  $\|\Pi f\|$ , which can be done in the spirit of the fractional Nash inequality

$$(6) \quad \|u\|_{L^2} \leq \mathcal{C}_{\text{Nash}} \|u\|_{L^1}^{\frac{\alpha'}{d+\alpha'}} \left\| |\xi|^{\frac{\alpha'}{2}} u \right\|_{L^2}^{\frac{d}{d+\alpha'}}.$$

This is done in the following lemma.

**Lemma 14.** — *For any  $f \in L^2(dx \, d\mu) \cap L^1(dx \, dv)$ ,*

$$\|\Pi f\|_{L^2(dx \, d\mu)} \leq \mathcal{C} \|f\|_{L^2(dx \, d\mu) \cap L^1(dx \, dv)}^{\frac{\alpha'}{d+\alpha'}} \left\| \left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \Pi \hat{f} \right\|_{L^2(d\xi \, d\mu)}^{\frac{d}{d+\alpha'}} ,$$

where  $\mathcal{C} = \max(\sqrt{2}, \mathcal{C}_{\text{Nash}})$  and we recall that  $\|f\|_{X \cap Y}^2 = \|f\|_X^2 + \|f\|_Y^2$ .

*Proof.* — For  $\xi \in \mathbb{R}^d$ , define  $u_f$  through

$$(1 + |\xi|^{\alpha'}) \widehat{u}_f = \widehat{\rho}_f.$$

From this definition, one sees that

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|\xi|^{\alpha'}}{1 + |\xi|^{\alpha'}} \|\Pi \widehat{f}\|^2 d\xi &= \int_{\mathbb{R}^d} |\xi|^{\alpha'} (1 + |\xi|^{\alpha'}) \widehat{u}_f \overline{\widehat{u}_f} d\xi \\ (7) \quad &= \int_{\mathbb{R}^d} |\xi|^{\frac{\alpha'}{2}} |\widehat{u}_f|^2 d\xi + \int_{\mathbb{R}^d} |\xi|^{\alpha'} |\widehat{u}_f|^2 d\xi. \end{aligned}$$

$$(8) \quad \geq \int_{\mathbb{R}^d} |\xi|^{\frac{\alpha'}{2}} |\widehat{u}_f|^2 d\xi$$

Moreover, after taking the modulus, squaring, and using identity (7), it also gives

$$\begin{aligned} \|\Pi f\|^2 &= \|\widehat{u}_f\|^2 + 2 \left\| |\xi|^{\frac{\alpha'}{2}} \widehat{u}_f \right\|^2 + \left\| |\xi|^{\alpha'} \widehat{u}_f \right\|^2 \\ &\leq \|\widehat{u}_f\|^2 + 2 \int_{\mathbb{R}^d} \frac{|\xi|^{\alpha'}}{1 + |\xi|^{\alpha'}} \|\Pi \widehat{f}\|^2 d\xi. \end{aligned}$$

For the first term, using the fractional Nash inequality (6) on  $u_f$  and inequality (8), we get the following bound

$$\begin{aligned} \|\widehat{u}_f\|_{L^2} = \|u_f\|_{L^2} &\leq \mathcal{C}_{\text{Nash}} \|u_f\|_{L^1}^{\frac{\alpha'}{d+\alpha'}} \left\| |\xi|^{\frac{\alpha'}{2}} \widehat{u}_f \right\|_{L^2}^{\frac{d}{d+\alpha'}} \\ &\leq \mathcal{C}_{\text{Nash}} \|f\|_{L^1(dx dv)}^{\frac{\alpha'}{d+\alpha'}} \left( \int_{\mathbb{R}^d} \frac{|\xi|^{\alpha'}}{1 + |\xi|^{\alpha'}} \|\Pi \widehat{f}\|^2 d\xi \right)^{\frac{d}{d+\alpha'}}. \end{aligned}$$

where we used the fact that  $\|u_f\|_{L^1} = \|\rho_f\|_{L^1} = \|f\|_{L^1(dx dv)}$ . Since  $\langle \xi \rangle^{\alpha'} \leq 1 + |\xi|^{\alpha'}$ , this leads to

$$(9) \quad \begin{aligned} \|\Pi f\|_{L^2(dx d\mu)}^2 &\leq 2 \left\| \left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \Pi \widehat{f} \right\|_{L^2(dx d\mu)}^2 \\ &\quad + \mathcal{C}_{\text{Nash}}^2 \|f\|_{L^1(dx dv)}^{\frac{2\alpha'}{d+\alpha'}} \left\| \left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \Pi \widehat{f} \right\|_{L^2(dx d\mu)}^{\frac{2d}{d+\alpha'}}. \end{aligned}$$

To conclude, we just have to remark that since  $\left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \leq 1$ , it holds

$$\left\| \left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \Pi \widehat{f} \right\|_{L^2(dx d\mu)} \leq \left\| \Pi \widehat{f} \right\|_{L^2(dx d\mu)}^{\frac{\alpha'}{d+\alpha'}} \left\| \left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \Pi \widehat{f} \right\|_{L^2(dx d\mu)}^{\frac{d}{d+\alpha'}},$$

so that inequality (9) becomes

$$\|\Pi f\|_{L^2(dx d\mu)}^2 \leq \left( 2 \|f\|_{L^2(dx d\mu)}^{\frac{2\alpha'}{d+\alpha'}} + \mathcal{C}_{\text{Nash}}^2 \|f\|_{L^1(dx dv)}^{\frac{2\alpha'}{d+\alpha'}} \right) \left\| \left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \Pi \widehat{f} \right\|_{L^2(dx d\mu)}^{\frac{2d}{d+\alpha'}},$$



which yields the result.  $\square$

*Proof of Theorem 1 when  $\gamma + \beta > 0$ .* — The conclusion of the proof of the theorem in this case is now as in [8, 9, 7]. For legibility, we shall split cases according to the sign of  $\beta$ .

**If  $\beta$  is positive**, then  $\langle v \rangle^\beta \geq 1$  and using the estimate of Lemma 14 together with the fact that the  $L^2(dx d\mu) \cap L^1(dx dv)$  norm is bounded by a constant, we obtain that

$$\begin{aligned} \left\| \left( \frac{|\xi|}{\langle \xi \rangle} \right)^{\alpha'} \Pi \widehat{f} \right\|_{L^2(d\xi d\mu)}^2 + \|(1 - \Pi)f\|_{L^2(\langle v \rangle^\beta dx d\mu)}^2 \\ \geq C_1 \| \Pi f \|_{L^2(dx d\mu)}^{2\frac{d+\alpha'}{d}} + \|(1 - \Pi)f\|_{L^2(dx d\mu)}^2 \cdot \\ \geq \min\{C_1, \|f^{\text{in}}\|_{L^2(dx d\mu)}^{-2\frac{\alpha'}{d}}\} \|f\|_{L^2(dx d\mu)}^{2(1+\frac{\alpha'}{d})} \end{aligned}$$

Collecting terms, we have

$$\frac{d}{dt} \mathbf{H}(f) \leq -C \mathbf{H}(f)^{1+\alpha'/d},$$

using the norm equivalence between  $\|f\|$  and  $\mathbf{H}(f)$ .

**If  $\beta$  is negative**, the proof is actually very similar, except that one has to use the conservation of weighted norms given by Proposition 7 for  $k \leq \gamma$  to control  $\|f\|_{L^2(\langle v \rangle^k dx d\mu)}$ . Since we can note that

$$\begin{aligned} \|(1 - \Pi)f\|_{L^2(\langle v \rangle^k dx d\mu)} &\leq \|f\|_{L^2(\langle v \rangle^k dx d\mu)} + \Theta_k \|\rho\|_{L^2} \\ &\leq (1 + \Theta_k) \|f^{\text{in}}\|_{L^2(\langle v \rangle^k dx d\mu)} =: C_2, \end{aligned}$$

then by Hölder's inequality, indeed,

$$\|(1 - \Pi)f\|_{L^2(\langle v \rangle^\beta dx d\mu)}^2 \geq C_2^{\frac{2\beta}{k}} \left( \|(1 - \Pi)f\|_{L^2(\langle v \rangle^k dx d\mu)}^2 \right)^{1-\frac{\beta}{k}}.$$

We obtain in this case

$$\frac{d}{dt} \mathbf{H}(f) \leq -C \mathbf{H}(f)^{1+1/\tau},$$

with  $\tau := \max\left(\frac{\alpha'}{d}, \frac{|\beta|}{k}\right)$ . In both cases, the proof follows from a Grönwall's estimate.  $\square$

## 5.2. Proof of Theorems 1–3 in the case $\gamma + \beta \leq 0$ . —

*Proof of Theorem 1 in the case  $\gamma + \beta \leq 0$ .* — In this case,  $\alpha' = \alpha$  and  $\beta \leq -\gamma < 0$ . By Young's inequality, Theorem 12 implies that for any  $r > 0$

$$\langle \mathbf{L}\widehat{f}, \widehat{f} \rangle \leq -r \left\| (1 - \Pi)\widehat{f} \right\|_\eta^2 + C_\tau r^{1+\tau} \left\| (1 - \Pi)\widehat{f} \right\|_k,$$

with  $\tau = \frac{k-\eta}{\eta-\beta}$ . Then by Proposition 7, we get that  $\iint_{\mathbb{R}^{2d}} |f|^2 \langle v \rangle^k F d\xi d\mu$  is uniformly bounded in time if it is initially bounded, so that integrating with

respect to  $\xi$  yields

$$(10) \quad \int_{\mathbb{R}^d} \langle \mathbf{L}\widehat{f}, \widehat{f} \rangle d\xi \leq -r \left\| (1 - \Pi)\widehat{f} \right\|_{L^2(\langle v \rangle^\eta d\xi d\mu)}^2 + \mathcal{C}^{\text{in}} r^{1+\tau}.$$

where  $\mathcal{C}^{\text{in}} = C_\tau \left\| (1 - \Pi)\widehat{f} \right\|_{L^\infty(\mathbb{R}_+, L^2(\langle v \rangle^k d\xi d\mu))}$ . From Proposition 4 we obtain

$$\begin{aligned} \partial_t \mathbf{H}_\xi(f) &= \langle \mathbf{L}\widehat{f}, \widehat{f} \rangle + \delta \operatorname{Re} \partial_t \langle \mathbf{A}_\xi \widehat{f}, \widehat{f} \rangle \\ &\leq \langle \mathbf{L}\widehat{f}, \widehat{f} \rangle - C \delta \left( \frac{|\xi|}{\langle \xi \rangle} \right)^\alpha \left\| \Pi \widehat{f} \right\|^2 + C \delta \left\| (1 - \Pi)\widehat{f} \right\|_\eta^2, \end{aligned}$$

Now by taking  $\delta = \frac{r}{2C}$  in  $\mathbf{H}(f) = \mathbf{H}(f)(r, t)$ , and integrating in  $\xi$ , we obtain

$$\begin{aligned} \partial_t \mathbf{H}(f) &\leq \int_{\mathbb{R}^d} \langle \mathbf{L}\widehat{f}, \widehat{f} \rangle d\xi - \frac{C r}{2C} \int_{\mathbb{R}^d} \left( \frac{|\xi|}{\langle \xi \rangle} \right)^\alpha \left\| \Pi \widehat{f} \right\|^2 d\xi \\ &\quad + \frac{r}{2} \left\| (1 - \Pi)\widehat{f} \right\|_{L^2(\langle v \rangle^\eta d\xi d\mu)}^2. \end{aligned}$$

Combined with inequality (10), this yields

$$\partial_t \mathbf{H}(f) \lesssim -r \left( \int_{\mathbb{R}^d} \left( \frac{|\xi|}{\langle \xi \rangle} \right)^\alpha \left\| \Pi \widehat{f} \right\|^2 d\xi + \left\| (1 - \Pi)f \right\|_{L^2(\langle v \rangle^\eta dx d\mu)}^2 \right) + \mathcal{C}^{\text{in}} r^{1+\tau},$$

and by Lemma 14 and Hölder's inequality, we obtain

$$\partial_t \mathbf{H}(f) \lesssim -r \left( \mathcal{C}_1^{\text{in}} \left\| \Pi f \right\|_{L^2(dx d\mu)}^{2(1+\frac{\alpha}{d})} + \mathcal{C}_2^{\text{in}} \left\| (1 - \Pi)f \right\|_{L^2(dx d\mu)}^{2(1+\frac{\eta_-}{k})} \right) + \mathcal{C}^{\text{in}} r^{1+\tau},$$

where  $\mathcal{C}_1^{\text{in}} = C \left( \left\| f^{\text{in}} \right\|_{L^2(dx d\mu)}^2 + \left\| f^{\text{in}} \right\|_{L^1(dx dv)}^2 \right)^{-\frac{\alpha}{d}}$  and  $\mathcal{C}_2^{\text{in}} = \left\| f^{\text{in}} \right\|_{L^2(\langle v \rangle^k dx d\mu)}^{-\frac{2\eta_-}{k}}$ .

This yields

$$\partial_t \mathbf{H}(f) \lesssim -\mathcal{C}_0^{\text{in}} r \mathbf{H}(f)^{1+1/\tau'} + \mathcal{C}^{\text{in}} r^{1+\tau}.$$

with  $\tau' = \min\left(\frac{d}{\alpha}, \frac{k}{\eta_-}\right)$ . Define

$$u(t) := C_0 r^{-\frac{\tau\tau'}{1+\tau'}} \mathbf{H}(f) \left( r, C_1^{-1} r^{\frac{-1-\tau-\tau'}{1+\tau'}} t \right)$$

$$C_0 = \left( \frac{\mathcal{C}_0^{\text{in}}}{\mathcal{C}^{\text{in}}} \right)^{\frac{\tau'}{1+\tau'}}$$

$$C_1 = \left( \mathcal{C}_0^{\text{in}} \right)^{\frac{\tau'}{1+\tau'}} \left( \mathcal{C}^{\text{in}} \right)^{\frac{1}{1+\tau'}}.$$

We are led to study the following Cauchy problem

$$\begin{cases} u'(t) = -u(t)^{1+1/\tau'} + 1, & t \in \mathbb{R}^+, \\ u(0) = u_0. \end{cases}$$

Here,  $u(0) = C_0 r^{-\frac{\tau\tau'}{1+\tau'}} \mathbf{H}(f)(r, 0)$  is greater than 1 as long as  $r \leq (C_0 \mathbf{H}(f)(r, 0))^{\frac{1}{\tau}(1+\frac{1}{\tau'})}$ . It is true for  $r$  sufficiently small since  $\mathbf{H}(f)(r, 0) \geq (1 - \delta) \left\| f^{\text{in}} \right\|_{L^2(dx d\mu)}^2 = (1 - \frac{r}{2C}) \left\| f^{\text{in}} \right\|_{L^2(dx d\mu)}^2$ . Therefore, there exists  $r_0 \in (0, 1)$  such that for any

$r \in (0, r_0]$  and  $t \in \mathbb{R}^+$ ,  $u(0) > \bar{u} > 1$  and

$$u(t) = S^{-1}(t + S(u(0))), \quad S(u) := \int_u^\infty \frac{ds}{s^{1+1/\tau'} - 1},$$

which implies

$$\mathbf{H}(f)(r, t) \leq C_0 r^{\frac{\tau\tau'}{1+\tau'}} S^{-1}\left(C_1 r^{\frac{1+\tau+\tau'}{1+\tau'}} t + S(u(0))\right).$$

Now choose  $r(t) := r_0 \langle t \rangle^{-\frac{1+\tau'}{1+\tau+\tau'}}$ . Then since

$$u(0) \geq C_0 r_0^{-\frac{\tau\tau'}{1+\tau'}} \langle t \rangle^{\frac{\tau\tau'}{1+\tau+\tau'}} \left(1 - \frac{r_0}{2\mathcal{C}}\right) \|f^{\text{in}}\|_{L^2(dx d\mu)^2}^2 \xrightarrow[t \rightarrow \infty]{} \infty,$$

we deduce that  $S(u(0))$  remains bounded so that

$$\mathbf{H}(f)(r(t), t) \leq C_0 \langle t \rangle^{-\frac{\tau\tau'}{1+\tau+\tau'}} S^{-1}\left(C_1 r_0^{\frac{1+\tau+\tau'}{1+\tau'}} \frac{t}{\langle t \rangle} + S(u(0))\right) \lesssim \langle t \rangle^{-\tau''},$$

with  $\tau'' = \frac{\tau\tau'}{1+\tau+\tau'}$  or equivalently

$$1 + \frac{1}{\tau''} = \left(1 + \frac{1}{\tau}\right) \left(1 + \frac{1}{\tau'}\right) = \left(\frac{k-\beta}{k-\eta}\right) \left(1 + \frac{1}{\min\left(\frac{d}{\alpha}, \frac{k}{\eta_-}\right)}\right),$$

where we recall that  $\beta \leq -\gamma < \eta \leq k < \gamma$ . If  $\eta < 0$  and  $\min\left(\frac{d}{\alpha}, \frac{k}{\eta_-}\right) = \frac{k}{\eta_-}$ , then

$$\frac{1}{\tau''} = \left(\frac{k-\beta}{k-\eta}\right) \left(1 + \frac{\eta_-}{k}\right) - 1 = \left(\frac{k-\beta}{k-\eta}\right) \left(\frac{k-\eta}{k}\right) - 1 = \frac{|\beta|}{k}.$$

This is in particular the case as soon as  $d \geq 2$  since in this case,  $\alpha \leq d$ , so that one can take  $\eta = -k \in (-\gamma, 0)$ , which implies that  $\frac{k}{\eta_-} = 1 \leq \frac{d}{\alpha}$ . When  $\alpha > d$ , then  $d = 1$  and one can still get the same result if  $k < \frac{\gamma}{\alpha}$ . In this case, one can indeed take  $\eta \in (-\gamma, -k\alpha)$  so that  $\frac{d}{\alpha} = \frac{1}{\alpha} \geq \frac{k}{\eta_-}$ . If this is not the case, so that  $\alpha > d = 1$  and  $k \in \left[\frac{\gamma}{\alpha}, \gamma\right)$ , then

$$\frac{1}{\tau''} = \left(\frac{k-\beta}{k-\eta}\right) (1 + \alpha) - 1 = \frac{\eta + \alpha k - \beta(1 + \alpha)}{k - \eta}.$$

Since one can take  $\eta$  as close as wanted to  $-\gamma$ , one can get any  $\tau'' < \frac{k+\gamma}{k\alpha-\gamma+|\beta|(\alpha+1)}$ . This finishes the proof of the theorem in this case.  $\square$

### 5.3. Proof of Theorem 2: the critical case $\gamma + \beta = 2$ .—

*Proof of Theorem 2.* — In the case when  $\gamma + \beta = 2$ , then recall that by Proposition 4 it holds

$$\frac{d}{dt} \langle \mathbf{A}_\xi \widehat{f}, \widehat{f} \rangle \lesssim -\mu_2(\xi) \|\Pi \widehat{f}\|^2 + \mathcal{C} \|(1 - \Pi) \widehat{f}\|_\beta^2,$$

where  $\mu_2$  is a positive increasing function of  $|\xi|$  bounded by 1 and such that

$$\mu_2(\xi) \underset{\xi \rightarrow 0}{\sim} \mathcal{C}_{d,\beta} |\xi|^2 |\ln(|\xi|)|.$$

Therefore, as in the case when  $\gamma + \beta \in \mathbb{R}_+ \setminus \{2\}$  (Proposition 13), by the weak microscopic coercivity

$$(11) \quad - \langle \mathbf{L}g, g \rangle \geq \mathcal{C}_m \| (1 - \Pi)g \|_{L^2(\langle v \rangle^\eta d\mu)}^{\frac{2}{\theta}} \| (1 - \Pi)g \|_{L^2(\langle v \rangle^k d\mu)}^{2 - \frac{2}{\theta}}.$$

and taking  $\delta \leq \frac{\mathcal{C}_P}{2\mathcal{C}}$ , we deduce

$$\frac{d}{dt} \mathbf{H}(f) \leq -\delta \left( \left\| \mu_2(\xi)^{\frac{1}{2}} \Pi \widehat{f} \right\|_{L^2(d\xi d\mu)}^2 + \mathcal{C} \| (1 - \Pi)f \|_{L^2(\langle v \rangle^\beta dx d\mu)}^2 \right).$$

Now, as in the classical proof of Nash's inequality, since  $\mu_2$  is increasing, we can write for any  $R > 0$

$$\begin{aligned} \|\Pi f\|_{L^2(d\xi d\mu)}^2 &= \|\rho_f\|_{L^2}^2 = \int_{|\xi| < R} |\widehat{\rho}_f|^2 d\xi + \int_{|\xi| \geq R} |\widehat{\rho}_f|^2 d\xi \\ &\leq \frac{\omega_d}{d} R^d \|\widehat{\rho}_f\|_{L^\infty(d\xi)}^2 + \frac{1}{\mu_2(R)} \int_{|\xi| \geq R} \mu_2(\xi) |\widehat{\rho}_f|^2 d\xi \\ &\leq \frac{\omega_d}{d} R^d \|f\|_{L^1(dx d\mu)}^2 + \frac{1}{\mu_2(R)} \left\| \mu_2(\xi)^{\frac{1}{2}} \Pi \widehat{f} \right\|_{L^2(dx d\mu)}^2, \end{aligned}$$

which can be written under the form  $X \leq R^d a + \frac{b}{\mu_2(R)}$ . Remarking that  $R \mapsto R^d \mu_2(R)$  is a bijection of  $\mathbb{R}_+^*$ , one can take  $R$  such that  $R^d \mu_2(R) = \frac{b}{a}$  which yields  $X \leq 2R^d a$  and since  $R^d \mu_2(R)$  is increasing, we get

$$\frac{X}{2a} \mu_2 \left( \left( \frac{X}{2a} \right)^{\frac{1}{d}} \right) \leq R^d \mu_2(R) = \frac{b}{a},$$

so that

$$b \geq \frac{1}{2} X \mu_2 \left( \left( \frac{X}{2a} \right)^{\frac{1}{d}} \right),$$

which translates in our case to

$$\begin{aligned} \left\| \mu_2(\xi)^{\frac{1}{2}} \Pi \widehat{f} \right\|_{L^2(dx d\mu)}^2 &\geq \frac{1}{2} \|\Pi f\|_{L^2(d\xi d\mu)}^2 \mu_2 \left( \left( \frac{\|\Pi f\|_{L^2(d\xi d\mu)}}{\mathcal{C}_d \|f\|_{L^1(dx d\mu)}} \right)^{\frac{2}{d}} \right) \\ &=: \Phi \left( \|\Pi f\|_{L^2(d\xi d\mu)}^2 \right). \end{aligned}$$

Moreover, denoting by  $a := \|f\|_{L^1(dx d\mu)}$ , it holds

$$\begin{aligned} \Phi(X) &\underset{X \rightarrow 0}{\sim} \frac{\mathcal{C}_{d,\beta}}{2} X \left( \frac{X}{2a} \right)^{\frac{2}{d}} \left| \ln \left( \left( \frac{X}{2a} \right)^{\frac{1}{d}} \right) \right| \\ &\underset{X \rightarrow 0}{\sim} \mathcal{C}_{d,\beta} a^{-\frac{2}{d}} X^{1 + \frac{2}{d}} |\ln(X)|, \end{aligned}$$

and  $\Phi(X) \underset{X \rightarrow \infty}{\sim} X/2$ . With the same method of proof as the non-critical cases, this yields

$$\begin{aligned} \frac{d}{dt} \mathbf{H}(f) &\lesssim -H(f)^{1 + \frac{2}{d}} \langle \ln(H(f)) \rangle && \text{if } \beta \geq 0 \text{ or if } \beta \leq 0 \text{ and } \frac{k}{|\beta|} \geq \frac{d}{2} \\ \frac{d}{dt} \mathbf{H}(f) &\lesssim -H(f)^{1 + \frac{|\beta|}{k}} && \text{if } \beta \leq 0 \text{ and } \frac{k}{|\beta|} < \frac{d}{2}. \end{aligned}$$

The result is then deduced by Gronwall's Lemma. Let us detail the first case. Since  $H(f)$  is easily seen to converge to 0, we are looking for the long time behavior of the solution of the following differential equation  $\frac{d}{dt}y(t) = -y(t)^{1+\frac{2}{d}} |\ln(y(t))|$  with initial condition  $y(0)$  that we can assume smaller than 1. Its solution is the solution of the following integral equation

$$\int_{y(0)}^{y(t)} \frac{ds}{s^{1+\frac{2}{d}} \ln(s)} = t.$$

Since  $\frac{d}{ds} \frac{s^{-\frac{2}{d}}}{\ln(s)} = \frac{-1}{s^{1+\frac{2}{d}} \ln(s)} \left( \frac{2}{d} + \frac{1}{\ln(s)} \right) \underset{s \rightarrow 0}{\sim} \frac{-2/d}{s^{1+\frac{2}{d}} \ln(s)}$  and  $y(t) \xrightarrow{t \rightarrow \infty} 0$ , we deduce that

$$(12) \quad t = \int_{y(0)}^{y(t)} \frac{ds}{s^{1+\frac{2}{d}} \ln(s)} \underset{t \rightarrow \infty}{\sim} -\frac{d}{2} \frac{y(t)^{-\frac{2}{d}}}{\ln(y(t))}.$$

In particular, by taking the logarithm of the above equation, we obtain that  $\ln(t) \underset{t \rightarrow \infty}{\sim} \ln\left(y(t)^{-\frac{2}{d}}\right) \underset{t \rightarrow \infty}{\sim} -\frac{2}{d} \ln(y(t))$ , which, injected back in equation (12) yields

$$t \underset{t \rightarrow \infty}{\sim} \frac{y(t)^{-\frac{2}{d}}}{\ln(t)},$$

or equivalently  $y(t) \underset{t \rightarrow \infty}{\sim} (t \ln(t))^{-\frac{d}{2}}$ .  $\square$

## 6. Quantitative estimates of $\mu_{\mathbf{L}}$ and $\lambda_{\mathbf{L}}$ .

In this section, we show that the three types of operators we consider in the paper satisfy the hypothesis on  $\mu_{\mathbf{L}}$  and  $\lambda_{\mathbf{L}}$  needed in Proposition 4, that is

$$\mu_{\mathbf{L}}(\xi) \lesssim |\xi|^{\min(1, 1+\frac{\gamma+\beta-2}{2|1-\beta|})} \mathbf{1}_{|\xi| \leq 1} + \mathbf{1}_{|\xi| \geq 1}, \quad \lambda_{\mathbf{L}}(\xi) \lesssim 1.$$

For readability, let us recall that

$$\mu_{\mathbf{L}}(\xi) := \|\mathbf{L}^*((v \cdot \xi)\varphi(\xi, \cdot)F)\|_{L^2(\langle v \rangle^{-\eta} d\mu)}, \quad \lambda_{\mathbf{L}}(\xi) := \|\mathbf{L}^*(\psi(\xi, \cdot)F)\|_{L^2(\langle v \rangle^{-\eta} d\mu)}.$$

### 6.1. Fokker-Planck operators. —

**Lemma 15.** — *Consider  $\mathbf{L} = \mathbf{L}_1$ , a Fokker-Planck operator. Then*

$$\mu_{\mathbf{L}} \lesssim |\xi|^{\min(1, \frac{\gamma+4-\eta}{6})} \mathbf{1}_{|\xi| \leq 1} + |\xi|^{-1} \mathbf{1}_{|\xi| \geq 1}, \quad \lambda_{\mathbf{L}} \lesssim 1.$$

*Proof.* — Note that in this case,  $\mathbf{L}$  is self-adjoint. Start with estimating  $\mu_{\mathbf{L}}$ .

$$\begin{aligned} F^{-1}\mathbf{L}((v \cdot \xi)\varphi F) &= \nabla_v \cdot (F \nabla_v ((v \cdot \xi)\varphi)) \\ &= \Delta_v((v \cdot \xi)\varphi) - (d + \gamma) \frac{v}{\langle v \rangle^2} \cdot \nabla_v((v \cdot \xi)\varphi). \end{aligned}$$

We can then compute

$$\begin{aligned} \nabla_v((v \cdot \xi)\varphi) &= \varphi \xi + (v \cdot \xi) \nabla_v \varphi \\ \Delta_v((v \cdot \xi)\varphi) &= 2\xi \cdot \nabla_v \varphi + (v \cdot \xi) \Delta_v \varphi. \end{aligned}$$

We end up with

$$F^{-1}\mathbf{L}((v \cdot \xi)\varphi) = 2\xi \cdot \nabla_v \varphi + (v \cdot \xi) \left( \Delta_v \varphi - \frac{(d+\gamma)}{\langle v \rangle^2} (\varphi + v \cdot \nabla_v \varphi) \right).$$

Recalling that  $\varphi = \frac{\langle v \rangle^2}{1 + \langle v \rangle^6 |\xi|^2}$ , we may for legibility write  $A = 1 + \langle v \rangle^6 |\xi|^2$  so that  $\varphi = \langle v \rangle^2 A^{-1}$ . Yields

$$\nabla_v \varphi = \left( 2A^{-1} - 6 \langle v \rangle^6 |\xi|^2 A^{-2} \right) v = 2 \left( 1 - 2 \langle v \rangle^6 |\xi|^2 \right) \frac{v}{A^2}.$$

As a consequence,

$$\begin{aligned} \xi \cdot \nabla_v \varphi &= 2 \left( 1 - 2 \langle v \rangle^6 |\xi|^2 \right) \frac{v \cdot \xi}{A^2} \\ v \cdot \nabla_v \varphi &= 2 \left( 1 - 2 \langle v \rangle^6 |\xi|^2 \right) \frac{|v|^2}{A^2}. \end{aligned}$$

From this we can readily estimate, since  $\langle v \rangle^6 |\xi|^2 \leq A$ ,

$$|\xi \cdot \nabla_v \varphi| \lesssim |v \cdot \xi| A^{-1}, \quad |v \cdot \nabla_v \varphi| \lesssim \langle v \rangle^2 A^{-1}.$$

The last part to estimate is

$$\begin{aligned} \Delta_v \varphi &= 2 \left( 1 - 2 \langle v \rangle^6 |\xi|^2 \right) \nabla_v \cdot \left( \frac{v}{A^2} \right) + 2 \nabla_v \cdot \left( 1 - 2 \langle v \rangle^6 |\xi|^2 \right) \frac{v}{A^2}, \\ &= \frac{2}{A^2} \left( 1 - 2 \langle v \rangle^6 |\xi|^2 \right) \left( d + 12|v|^2 \langle v \rangle^4 |\xi|^2 A^{-1} \right) - 24|v|^2 \langle v \rangle^4 |\xi|^2 A^{-2} \end{aligned}$$

that gives

$$|\Delta \varphi| \lesssim A^{-1}.$$

Combining previous estimates, we thus end up with

$$\left| F^{-1}\mathbf{L}((v \cdot \xi)\varphi) \right| \lesssim |v \cdot \xi| A^{-1}.$$

This allows to estimate  $\mu_{\mathbf{L}}(\xi)$  as follows,

$$\|\mathbf{L}((v \cdot \xi)\varphi(\xi, \cdot)F)\|_{L^2(\langle v \rangle^{-\eta} d\mu)} \lesssim \left( \int_{\mathbb{R}^d} \frac{|v \cdot \xi|^2}{\left( 1 + \langle v \rangle^6 |\xi|^2 \right)^2 \langle v \rangle^{d+\gamma+\eta}} dv \right)^{\frac{1}{2}}.$$

The conclusion then comes from following exactly the same steps as in the proof of Lemma 5, that give

$$\mu_{\mathbf{L}} \lesssim |\xi|^{\min(1, \frac{\gamma+\eta+4}{6})} \mathbf{1}_{|\xi| \leq 1} + |\xi|^{-1} \mathbf{1}_{|\xi| \geq 1}.$$

We now estimate  $\lambda_{\mathbf{L}}$ . Recalling that  $\psi = \langle v \rangle^{-2}$ , one has

$$\left| F^{-1}\mathbf{L}(\psi F) \right| = \left| \Delta_v \psi - (d+\gamma) \frac{v}{\langle v \rangle^2} \cdot \nabla_v \psi \right|,$$

which is a bounded quantity. Since  $\langle v \rangle^{-\eta} F \in L^1$ , we conclude that  $\lambda_{\mathbf{L}}$  is bounded.  $\square$

## 6.2. Scattering collision operators. —

**Lemma 16.** — *Let  $\mathbf{L} = \mathbf{L}_2$  and  $\gamma > \beta$ . Then*

$$\mu_{\mathbf{L}} \lesssim |\xi|^{\min(1, 1 + \frac{\gamma + \eta - 2}{2|1 - \beta|})} \mathbf{1}_{|\xi| < 1} + |\xi|^{-1} \mathbf{1}_{|\xi| \geq 1}, \quad \lambda_{\mathbf{L}} \lesssim 1.$$

*Proof.* — To estimate  $\mu_{\mathbf{L}}$ , we write

$$\begin{aligned} F^{-1} \mathbf{L}^* ((v \cdot \xi) \varphi F) &= \int_{\mathbb{R}^d} \mathfrak{b}(v', v) ((v' \cdot \xi) \varphi(v') - (v \cdot \xi) \varphi(v)) F(v') \, dv' \\ &= \int_{\mathbb{R}^d} \mathfrak{b}(v', v) (v' \cdot \xi) \varphi(v') F(v') \, dv' - (v \cdot \xi) \varphi(v) \nu(v) \end{aligned}$$

Then we first remark that the inequality of Cauchy-Schwarz yields

$$\begin{aligned} &\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \mathfrak{b}(v', v) (v' \cdot \xi) \varphi(v') F' \, dv' \right|^2 \langle v \rangle^{-\eta} F \, dv \\ &\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |(v' \cdot \xi) \varphi(v')|^2 \langle v \rangle^{\beta} F' \, dv' \right) \left( \int_{\mathbb{R}^d} \frac{\mathfrak{b}(v', v)^2}{\nu(v')} F' \, dv' \right) \langle v \rangle^{-\eta} F \, dv \\ &\leq \mathcal{C}_b \int_{\mathbb{R}^d} |\nu(v) (v \cdot \xi) \varphi(v)|^2 \langle v \rangle^{-\eta} F \, dv, \end{aligned}$$

where, by assumption (H3),

$$\mathcal{C}_b := \iint_{\mathbb{R}^{2d}} \frac{\mathfrak{b}(v', v)^2}{\nu(v') \nu(v)} F F' \, dv \, dv' < \infty.$$

Then, since

$$\int_{\mathbb{R}^d} |\nu(v) (v \cdot \xi) \varphi|^2 \langle v \rangle^{-\eta} F \, dv \leq C \int_{\mathbb{R}^d} \frac{|v \cdot \xi|^2}{(1 + \langle v \rangle^{2|1 - \beta|} |\xi|^2)^2} \frac{dv}{\langle v \rangle^{d + \gamma + \eta}},$$

the result follows again by adapting the proof of Lemma 5. The estimate for  $\lambda_{\mathbf{L}}(\xi)$  comes fairly easily from

$$\begin{aligned} F^{-1} \mathbf{L}^* (\psi F) &= \int_{\mathbb{R}^d} \mathfrak{b}(v', v) (\psi(v') - \psi(v)) F(v') \, dv' \\ &= \int_{\mathbb{R}^d} \mathfrak{b}(v', v) \psi(v') F(v') \, dv' - \nu(v) \psi(v) \end{aligned}$$

Again, the inequality of Cauchy-Schwarz yields

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \mathfrak{b}(v', v) \psi(v') F' \, dv' \right|^2 &\leq \left( \int_{\mathbb{R}^d} |\psi(v')|^2 \langle v' \rangle^{\beta} F' \, dv' \right) \left( \int_{\mathbb{R}^d} \frac{\mathfrak{b}(v', v)^2}{\nu(v')} F' \, dv' \right) \\ &\leq \mathcal{C}_b |\nu(v) \psi(v)|^2, \end{aligned}$$

so that  $|F^{-1} \mathbf{L}^* (\psi F)| \leq (\mathcal{C}_b^{1/2} + 1) |\nu(v) \psi(v)|$ . Then, since

$$\int_{\mathbb{R}^d} |\nu(v) \psi(v)|^2 \langle v \rangle^{-\eta} F \, dv \leq C \int_{\mathbb{R}^d} \frac{dv}{\langle v \rangle^{d + \gamma + \eta - 2\beta + 4}},$$

we deduce that  $\lambda_{\mathbf{L}} \lesssim 1$  since  $\gamma + \eta - 2\beta + 4 > \gamma - \beta + \eta - \beta > 0$ .  $\square$

**6.3. Fractional Fokker-Planck operator.** — In this section, we set  $\mathbf{L} = \mathbf{L}_3$ . Since computations are more involved, we split the estimates of  $\mu_{\mathbf{L}}$  and  $\lambda_{\mathbf{L}}$ .

**Proposition 17.** — For any  $\gamma > |\beta|$ , we have,

$$\mu_{\mathbf{L}} \lesssim |\xi|^{\frac{\alpha'}{2}} \mathbf{1}_{|\xi| \leq 1} + \mathbf{1}_{|\xi| \geq 1}.$$

Recall  $F^{-1}\mathbf{L}^*(F\cdot) = \Delta_v^{\frac{\alpha}{2}} - E \cdot \nabla_v$ , we have

$$\begin{aligned} \mu_{\mathbf{L}}^2 &= \int_{\mathbb{R}^d} \left| \Delta_v^{\frac{\alpha}{2}} ((v \cdot \xi)\varphi(v)) - E(v) \cdot \nabla_v ((v \cdot \xi)\varphi(v)) \right|^2 \frac{dv}{\langle v \rangle^{d+\gamma+\eta}} \\ &\leq 2 \int_{\mathbb{R}^d} \left| \Delta_v^{\frac{\alpha}{2}} ((v \cdot \xi)\varphi) \right|^2 \frac{dv}{\langle v \rangle^{d+\gamma+\eta}} + 2 \int_{\mathbb{R}^d} |E(v) \cdot \nabla_v ((v \cdot \xi)\varphi)|^2 \frac{dv}{\langle v \rangle^{d+\gamma+\eta}}. \end{aligned}$$

We shall estimate the two integrals of the latter r.h.s. separately. Start with the easiest, the second integral, which is estimated as follows. Recalling Proposition 20 which helps estimating  $E$ ,

$$\begin{aligned} |E(v) \cdot \nabla_v ((v \cdot \xi)\varphi)| &\lesssim |E(v) \cdot \xi \varphi| + |(v \cdot \xi)E(v) \cdot \nabla_v \varphi| \\ &\lesssim |v \cdot \xi| \langle v \rangle^\beta \varphi + |v \cdot \xi| \varphi \langle v \rangle^{-2} \langle v \rangle^\beta |v|^2 \\ &\lesssim |v \cdot \xi| \langle v \rangle^\beta \varphi, \end{aligned}$$

so that

$$\|E(v) \cdot \nabla_v ((v \cdot \xi)\varphi)\|_{L^2(\langle v \rangle^{-\eta} F dv)}^2 \lesssim \int_{\mathbb{R}^d} \frac{|v \cdot \xi|^2}{(1 + \langle v \rangle^{2|1-\beta|} |\xi|^2)^2} \frac{dv}{\langle v \rangle^{d+\gamma+\eta}},$$

and

$$\left( \int_{\mathbb{R}^d} \frac{|v \cdot \xi|^2}{(1 + \langle v \rangle^{2|1-\beta|} |\xi|^2)^2} \frac{dv}{\langle v \rangle^{d+\gamma+\eta}} \right)^{\frac{1}{2}} \lesssim |\xi|^{\frac{\alpha' - \frac{\beta-\eta}{1-\beta}}{2}} \mathbf{1}_{|\xi| \leq 1} + \mathbf{1}_{|\xi| \geq 1},$$

follows.

The estimation of the first integral is more involved and requires estimating  $\Delta_v^{\frac{\alpha}{2}}((v \cdot \xi)\varphi)$ , that we do now in the following lemma.

**Lemma 18.** — For  $\alpha \in (0, 2)$  it holds

$$(13) \quad \left| \Delta_v^{\frac{\alpha}{2}}((v \cdot \xi)\varphi) \right| \lesssim |\xi|^{\frac{\alpha'}{2}} \mathbf{1}_{|\xi| \leq 1} + \mathbf{1}_{|\xi| \geq 1}.$$

Since the proof of this lemma is technical, we postpone it for a while and show now how to conclude the proof of the proposition with this lemma. From it we deduce since  $\gamma + \eta > 0$

$$\int_{\mathbb{R}^d} \left| \Delta_v^{\frac{\alpha}{2}} ((v \cdot \xi)\varphi(v)) \right|^2 \frac{dv}{\langle v \rangle^{d+\gamma+\eta}} \leq |\xi|^{\frac{\alpha'}{2}} \mathbf{1}_{|\xi| \leq 1} + \mathbf{1}_{|\xi| \geq 1}.$$

Gathering all previous estimates finishes the proof of Proposition 17. The only remaining thing is to present the proof of technical Lemma 18.

*Proof of Lemma 18.* — Only locally in that proof, we use the notation

$$\forall v \in \mathbb{R}^d, \quad m(v) = (v \cdot \xi) \varphi(v)$$



for legibility. For any  $v \in \mathbb{R}^d$ , we can write the fractional Laplacian as

$$\begin{aligned} \Delta_v^{\frac{\alpha}{2}}(m)(v) &= \int_{|v-v'| < \frac{\langle v \rangle}{2}} \frac{m(v') - m(v) - (v' - v) \cdot \nabla m(v)}{|v - v'|^{d+\alpha}} dv' \\ &\quad + \int_{|v-v'| \geq \frac{\langle v \rangle}{2}} \frac{m(v') - m(v)}{|v - v'|^{d+\alpha}} dv' \\ &= I_1 + I_2. \end{aligned}$$

The rest of the proof of the lemma consists in bounding both latter integrals.

• **Step 1 : a bound of  $I_1$ .**

The first integral is controlled by a second order Taylor approximation. We thus estimate the Hessian of  $m$ . Computing the gradient of  $\varphi$  yields

$$\nabla_v \varphi = - \left( \beta + (\beta + 2|1 - \beta|) \langle v \rangle^{2|1-\beta|} |\xi|^2 \right) \langle v \rangle^{\beta-2} \varphi^2 v,$$

from which we deduce that  $|\nabla_v \varphi| \lesssim \langle v \rangle^{-1} \varphi$ . Then to estimate the Hessian of  $\varphi$ , we write

$$\begin{aligned} \left| \nabla_v^2 \varphi(v) \right| &= \left| \nabla_v \left( \left( \beta + (\beta + 2|1 - \beta|) \langle v \rangle^{2|1-\beta|} |\xi|^2 \right) \langle v \rangle^{\beta-2} \varphi^2 v \right) \right|, \\ &\lesssim \langle v \rangle^{-2} \varphi^2 + \langle v \rangle^{-1} |\nabla_v \varphi|, \end{aligned}$$

from which we deduce that  $|\nabla_v^2 \varphi| \lesssim \langle v \rangle^{-2} \varphi$ . Then, since  $\nabla_v((v \cdot \xi)\varphi(v)) = \varphi(v)\xi + (v \cdot \xi)\nabla_v \varphi(v)$ , it turns out that

$$\left| \nabla_v^2((v \cdot \xi)\varphi(v)) \right| \lesssim |\nabla_v \varphi(v)| |\xi| + |v \cdot \xi| \left| \nabla_v^2(\varphi(v)) \right| \lesssim |\xi| \langle v \rangle^{-1} \varphi.$$

Therefore,

$$\begin{aligned} |I_1| &\leq \int_{|z| \leq \frac{\langle v \rangle}{2}} \frac{\|\nabla_v^2 m\|_{L^\infty(B(v, \frac{\langle v \rangle}{2}))}}{|z|^{d+\alpha-2}} dz \leq \frac{|\xi| 2^{\alpha-2} \omega_d}{(2-\alpha) \langle v \rangle^{\alpha-2}} \left\| \langle \cdot \rangle^{-1} \varphi \right\|_{L^\infty(B(v, \frac{\langle v \rangle}{2}))} \\ &\lesssim \frac{|\xi| \varphi(v)}{(2-\alpha) \langle v \rangle^{\alpha-1}}, \end{aligned}$$

since any  $\langle v' \rangle$  is uniformly comparable to  $\langle v \rangle$  on  $B(v, \frac{\langle v \rangle}{2})$ .

• **Step 2: a bound of  $I_2$ .**

The way of estimating the second integral will actually depend on the value of  $|\xi|$ . We thus split this step into two,  $|\xi| \leq 1$  and  $|\xi| \geq 1$ .

Start with  $|\xi| \geq 1$ . Then, one starts splitting  $I_2$  into three pieces,

$$\begin{aligned} |I_2| &\leq \int_{\substack{|v-v'| \geq \frac{\langle v \rangle}{2} \\ |v'| < \langle v \rangle}} \frac{|m(v')| dv'}{|v - v'|^{d+\alpha}} + \int_{\substack{|v-v'| \geq \frac{\langle v \rangle}{2} \\ |v'| \geq \langle v \rangle}} \frac{|m(v')| dv'}{|v - v'|^{d+\alpha}} + \int_{|v-v'| \geq \frac{\langle v \rangle}{2}} \frac{|m(v)| dv'}{|v - v'|^{d+\alpha}} \\ &\leq \frac{2^{d+\alpha}}{\langle v \rangle^{d+\alpha}} \|m\|_{L^1(B_0(\langle v \rangle))} + \frac{2^\alpha \omega_d \|m\|_{L^\infty(B_0^c(\langle v \rangle))}}{\alpha \langle v \rangle^\alpha} + \frac{2^\alpha \omega_d |m(v)|}{\alpha \langle v \rangle^\alpha} \\ &\lesssim \langle v \rangle^{-\alpha} \left( \langle v \rangle^{-d} \|m\|_{L^1(B_0(\langle v \rangle))} + \|m\|_{L^\infty(B_0^c(\langle v \rangle))} + |m(v)| \right). \end{aligned}$$

To proceed further, it is mandatory to estimate  $\langle v \rangle^{-d} \|m\|_{L^1(B_0(\langle v \rangle))}$  and  $\|m\|_{L^\infty(B_0^c(\langle v \rangle))}$  for any  $v \in \mathbb{R}^d$  and this where  $|\xi| \geq 1$  will help.

Indeed,

$$\begin{aligned} \langle v \rangle^{-d} \|m\|_{L^1(B_0(\langle v \rangle))} &\leq \langle v \rangle^{-d} |\xi|^{-1} && \text{if } \beta + 2|1 - \beta| - 1 > d, \\ &\leq \langle v \rangle^{-d} \langle v \rangle^{d+1-\beta-2|1-\beta|} |\xi|^{-1} && \text{if } \beta + 2|1 - \beta| - 1 < d, \\ &\leq 2|\xi| \langle v \rangle \varphi(v) && \text{if } \beta + 2|1 - \beta| - 1 < d. \end{aligned}$$

For all  $v' \in B_0^c(\langle v \rangle)$ ,

$$\begin{aligned} |(v' \cdot \xi)\varphi(v')| &\leq \langle v' \rangle |\xi| \frac{\langle v' \rangle^{-\beta}}{1 + \langle v' \rangle^{2|1-\beta|} |\xi|^2} \leq \frac{\langle v' \rangle^{1-\beta} |\xi|}{1 + \langle v' \rangle^{2|1-\beta|} |\xi|^2} \\ &\leq \frac{\langle v \rangle^{1-\beta} |\xi|}{1 + \langle v \rangle^{2|1-\beta|} |\xi|^2} = \langle v \rangle |\xi| \varphi(v), \end{aligned}$$

where we have used that  $\langle v' \rangle \mapsto \frac{\langle v' \rangle^{1-\beta} |\xi|}{1 + \langle v' \rangle^{2|1-\beta|} |\xi|^2}$  is decreasing for  $\langle v' \rangle \geq \langle v \rangle$ . Indeed, when  $1 - \beta \leq 0$  this latter fact is immediate and when  $1 - \beta \geq 0$  it results from the fact that  $\langle v' \rangle^{1-\beta} |\xi| \geq 1$  since  $|\xi| \geq 1$ .

We deduce,

$$\begin{aligned} |I_2| &\lesssim \langle v \rangle^{-\alpha} \left( \langle v \rangle^{-d} |\xi|^{-1} + \langle v \rangle |\xi| \varphi(v) \right) && \text{if } \beta + 2|1 - \beta| - 1 > d \\ &\lesssim \langle v \rangle^{-\alpha} (\langle v \rangle |\xi| \varphi(v)) && \text{if } \beta + 2|1 - \beta| - 1 < d. \end{aligned}$$

Assume now that  $|\xi| < 1$ . Write

$$\begin{aligned} |I_2| &\leq \int_{|z| \geq \frac{\langle v \rangle}{2}} \sup_{|v-v'| > \frac{\langle v \rangle}{2}} \left( \frac{|m(v) - m(v')|}{|v - v'|^\ell} \right) \frac{dz}{|z|^{d+\alpha-\ell}} \\ &\leq \frac{2^{\alpha-\ell} \omega_d}{(\alpha - \ell) \langle v \rangle^{\alpha-\ell}} \sup_{|v-v'| > \frac{\langle v \rangle}{2}} \left( \frac{|m(v) - m(v')|}{|v - v'|^\ell} \right). \end{aligned}$$

Here,  $\ell$  will be chosen later. The next step is thus to estimate the  $\ell$ -Hölder seminorm of  $m$ . For  $\beta < 1$  and any  $w \in \mathbb{R}^d$ , write

$$\begin{aligned} |m(w)| &\leq |\xi| \langle w \rangle \varphi = \frac{\langle w \rangle^{1-\beta} |\xi|}{1 + \langle w \rangle^{2|1-\beta|} |\xi|^2} \\ &\leq |\xi|^{\frac{\alpha'}{2}} \langle w \rangle^{\frac{\alpha'(1-\beta)}{2}} \frac{\langle w \rangle^{\frac{(1-\beta)(2-\alpha')}{2}} |\xi|^{\frac{2-\alpha'}{2}}}{1 + \langle w \rangle^{2|1-\beta|} |\xi|^2} \lesssim |\xi|^{\frac{\alpha'}{2}} \langle w \rangle^\ell, \end{aligned}$$

with  $\ell = \frac{\alpha'(1-\beta)}{2} \in (0, 1)$ . We deduce that for any  $(v, v')$  such that  $|v - v'| > \frac{\langle v \rangle}{2}$ ,

$$|m(v) - m(v')| \lesssim |\xi|^{\frac{\alpha'}{2}} (2 \langle v \rangle^\ell + |v' - v|^\ell) \lesssim |\xi|^{\frac{\alpha'}{2}} |v - v'|^\ell,$$

and thus  $|I_2| \leq |\xi|^{\frac{\alpha'}{2}} \frac{2^{\alpha-\ell} \omega_d}{(\alpha-\ell) \langle v \rangle^{\alpha-\ell}}$ .

When  $\beta \geq 1$ , the estimate can be performed exactly as in the  $|\xi| \geq 1$  situation, we do not repeat the argument. The proof of Lemma 18 is now complete.  $\square$

**Proposition 19.** — *Let  $\gamma \geq \beta_+$ . Then there exists a constant  $C > 0$  independent from  $\xi$  such that*

$$\lambda_{\mathbf{L}} \leq C$$

*Proof.* — We follow the same steps to estimate  $\lambda_{\mathbf{L}}$ . We have

$$\begin{aligned} \lambda_{\mathbf{L}}(\xi)^2 &= \int_{\mathbb{R}^d} \left( \Delta_v^{\frac{\mathfrak{a}}{2}}(\psi(v)) - E(v) \cdot \nabla_v \psi(v) \right)^2 \frac{dv}{\langle v \rangle^{d+\gamma+\eta}} \\ &\leq 2 \int_{\mathbb{R}^d} \left| \Delta_v^{\frac{\mathfrak{a}}{2}}(\psi(v)) \right|^2 \frac{dv}{\langle v \rangle^{d+\gamma+\eta}} + 2 \int_{\mathbb{R}^d} |E(v) \cdot \nabla_v \psi(v)|^2 \frac{dv}{\langle v \rangle^{d+\gamma+\eta}}. \end{aligned}$$

We now estimate both integrals of the last right hand side separately. By Lemma 18,  $\Delta_v^{\frac{\mathfrak{a}}{2}}(\langle v \rangle^{-2})$  is a bounded function, from which we deduce that the first integral is bounded since  $\gamma + \eta > 0$ . For the second integral we get

$$\int_{\mathbb{R}^d} |E(v) \cdot \nabla_v \psi(v)|^2 \frac{dv}{\langle v \rangle^{d+\gamma+\eta}} \leq \int_{\mathbb{R}^d} \frac{|v|^2 dv}{\langle v \rangle^{d+\gamma+\eta-2\beta+8}}$$

which is bounded since  $\gamma + \eta - 2\beta + 6 > 0$ .  $\square$

## Appendix A

### Steady states and force field for the fractional Laplacian with drift

By definition of the force field  $E$ , we get

$$\nabla_v \cdot (EF) = (-\Delta_v)^{\frac{\mathfrak{a}}{2}} F = - \left( \nabla_v \cdot \left( \nabla_v \left( (-\Delta_v)^{\frac{\mathfrak{a}-2}{2}} F \right) \right) \right),$$

and this implies

$$E(v)F(v) = -\nabla_v \left( (-\Delta_v)^{\frac{\mathfrak{a}-2}{2}} F \right) = -C_{d,\gamma,\mathfrak{a}} \nabla_v \left( \frac{1}{|v|^{d+\mathfrak{a}-2}} * \frac{1}{\langle v \rangle^{d+\gamma}} \right),$$

when  $E$  is radial. From this expression we deduce the following result.

**Proposition 20.** — *Assume  $\mathfrak{a} \in (0, 2)$ . Then*

$$E(v) = G(v) \langle v \rangle^\beta v,$$

where  $G \in L^\infty(\mathbb{R}^d)$  is a positive function such that  $1/G \in L^\infty(B_0^c(1))$ .

*Proof.* — Let  $u(v) = -\nabla_v \left( \frac{1}{|v|^{d+\mathfrak{a}-2}} * \frac{1}{\langle v \rangle^{d+\gamma}} \right) (v)$  so that  $E$  is proportional to  $\langle \cdot \rangle^{d+\gamma} u$ . We shall estimate  $u(v) \cdot v$  from above and below to prove the proposition.

We have

$$u(v) = (d + \gamma) \left( \frac{1}{|v|^{d+\mathfrak{a}-2}} * \frac{v}{\langle v \rangle^{d+\gamma+2}} \right).$$

Note that from this expression, and since  $\frac{v}{\langle v \rangle^{d+\gamma+2}} \in C^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  and  $\mathfrak{a} < 2$ , one has  $u \in L_{\text{loc}}^\infty(\mathbb{R}^d)$  so that the remaining work is to focus on  $B_0^c(1)$ .

Assume from now on that  $v$  is such that  $|v| > 1$ . We now split the cases depending on the integrability at infinity of  $\frac{v}{|v|^{d+\mathfrak{a}}}$ , that is  $\mathfrak{a} \in (0, 1)$  or not.

**Start with  $\mathfrak{a} \in (0, 1)$ .** Write

$$(14) \quad u(v) = (d + \mathfrak{a} - 2) \left( \frac{v}{|v|^{d+\mathfrak{a}}} * \frac{1}{\langle v \rangle^{d+\gamma}} \right).$$

Then since

$$\int_{|v'| \geq \frac{\langle v \rangle}{2}} \frac{v'}{|v'|^{d+\mathfrak{a}}} \frac{dv'}{\langle v' - v \rangle^{d+\gamma}} \leq \left( \int_{\mathbb{R}^d} \frac{dv'}{\langle v' \rangle^{d+\gamma}} \right) \frac{2^{d+\mathfrak{a}-1}}{|v|^{d+\mathfrak{a}-1}},$$

and

$$\int_{|v'| < \frac{\langle v \rangle}{2}} \frac{v'}{|v'|^{d+\mathfrak{a}}} \frac{dv'}{\langle v' - v \rangle^{d+\gamma}} \leq \left( \int_{B_0(\frac{\langle v \rangle}{2})} \frac{dv'}{|v'|^{d+\mathfrak{a}-1}} \right) \frac{2^{d+\mathfrak{a}-1}}{|v|^{d+\gamma}} \leq \frac{2^{d+\mathfrak{a}-1} \omega_d}{1 - \mathfrak{a}} \frac{1}{|v|^{d+\gamma+\mathfrak{a}-1}},$$

we deduce that

$$|u(v) \cdot v| \leq |u(v)| |v| \lesssim |v|^{-(d+\mathfrak{a}-2)}.$$

To get a bound by below on  $u(v) \cdot v$ , we cut the integral in two parts and we use the fact that  $|v| > 1$  and  $|v' - v| < 1/2$  implies  $v' \cdot v > 0$ . First

$$\begin{aligned} \int_{\substack{|v'-v| > 1/2 \\ |v'+v| > 1/2}} \frac{v' \cdot v}{|v'|^{d+\mathfrak{a}}} \frac{dv'}{\langle v' - v \rangle^{d+\gamma}} &= \left( \int_{\substack{|v'-v| > 1/2 \\ v' \cdot v > 0}} + \int_{\substack{|v'+v| > 1/2 \\ v' \cdot v < 0}} \right) \frac{v' \cdot v}{|v'|^{d+\mathfrak{a}}} \frac{dv'}{\langle v' - v \rangle^{d+\gamma}} \\ &= \int_{\substack{|v'-v| > 1/2 \\ v' \cdot v > 0}} \frac{v' \cdot v}{|v'|^{d+\mathfrak{a}}} \left( \frac{1}{\langle v' - v \rangle^{d+\gamma}} - \frac{1}{\langle v' + v \rangle^{d+\gamma}} \right) dv', \end{aligned}$$

which is positive since  $\langle v' + v \rangle^2 - \langle v' - v \rangle^2 = 2v' \cdot v \geq 0$ . The remaining terms are treated as follows

$$\begin{aligned} \int_{\substack{|v'-v| \leq 1/2 \\ \text{or} \\ |v'+v| \leq 1/2}} \frac{v' \cdot v}{|v'|^{d+\mathfrak{a}}} \frac{dv'}{\langle v' - v \rangle^{d+\gamma}} &= \int_{|v'-v| < \frac{1}{2}} \frac{v' \cdot v}{|v'|^{d+\mathfrak{a}}} \left( \frac{1}{\langle v' - v \rangle^{d+\gamma}} - \frac{1}{\langle v' + v \rangle^{d+\gamma}} \right) dv' \\ &\geq \left( \frac{1}{\langle 1/2 \rangle^{d+\gamma}} - \frac{1}{\langle 3/2 \rangle^{d+\gamma}} \right) \int_{|v'-v| < \frac{1}{2}} \frac{v' \cdot v}{|v'|^{d+\mathfrak{a}}} dv', \end{aligned}$$

since  $|v' + v| \geq 2|v| - |v' - v| \geq \frac{3}{2}$ . Finally, since  $|v| > 1$ , when  $|v' - v| < \frac{1}{2}$  we get

$$\begin{aligned} 2v' \cdot v &= |v|^2 + |v'|^2 - |v' - v|^2 \geq |v|^2 - \frac{1}{2} \geq \frac{|v|^2}{2} \\ |v'| &\leq |v| + |v' - v| \leq 2|v|, \end{aligned}$$

so that

$$\int_{|v'-v| < \frac{1}{2}} \frac{v' \cdot v}{|v'|^{d+\mathfrak{a}}} dv' \geq \frac{|B_0(1/2)|}{2^{d+\mathfrak{a}+2}} \frac{1}{|v|^{d+\mathfrak{a}-2}}.$$

This implies  $u(v) \cdot v \geq C|v|^{-(d+\mathbf{a}-2)}$ . Therefore, we can define  $G(v) := (u(v) \cdot v) |v|^{d+\mathbf{a}-2}$  and we just proved that  $G \in L^\infty(\mathbb{R}^d)$  and  $G^{-1} \in L^\infty(B_0^c(1))$ . Since  $u$  is radial, we can write

$$u(v) = G(v) \frac{v}{|v|^{d+\mathbf{a}}},$$

and we deduce the result by writing  $E(v) \propto F(v)^{-1}u(v)$  and using the fact that  $\beta = \gamma - \mathbf{a}$ .

**Now consider the case  $\mathbf{a} \in [1, 2)$ .** The gradient of  $\frac{1}{|v|^{d+\mathbf{a}-2}}$  is then a distribution of order 1 that can be defined as a principal value. Indeed, in the sense of distributions, it holds for any  $\varphi \in \mathcal{D}(\mathbb{R}^d)$

$$\begin{aligned} \left\langle \nabla_v \left( \frac{1}{|v|^{d+\mathbf{a}-2}} \right), \varphi \right\rangle_{\mathcal{D}', \mathcal{D}} &= - \int_{\mathbb{R}^d} \frac{\nabla_v \varphi(v)}{|v|^{d+\mathbf{a}-2}} dv \\ &= - \int_{\mathbb{R}^d} \frac{\nabla_v (\varphi(v) - \varphi(0))}{|v|^{d+\mathbf{a}-2}} dv \\ &= (d + \mathbf{a} - 2) \int_{\mathbb{R}^d} \frac{v}{|v|^{d+\mathbf{a}}} (\varphi(v) - \varphi(0)) dv \\ &=: (d + \mathbf{a} - 2) \left\langle \text{pv} \left( \frac{v}{|v|^{d+\mathbf{a}}} \right), \varphi \right\rangle_{\mathcal{D}', \mathcal{D}}. \end{aligned}$$

We deduce that formula (14) becomes

$$\begin{aligned} u(v) &= (d + \mathbf{a} - 2) \text{pv} \left( \frac{v}{|v|^{d+\mathbf{a}}} \right) * \frac{1}{\langle v \rangle^{d+\gamma}} \\ &= (d + \mathbf{a} - 2) \int_{\mathbb{R}^d} \frac{v'}{|v'|^{d+\mathbf{a}}} \left( \frac{1}{\langle v - v' \rangle^{d+\gamma}} - \frac{1}{\langle v \rangle^{d+\gamma}} \right) dv', \end{aligned}$$

so that

$$|u(v)| \leq \int_{\mathbb{R}^d} \frac{1}{|v' - v|^{d+\mathbf{a}-1}} \left| \frac{1}{\langle v' \rangle^{d+\gamma}} - \frac{1}{\langle v \rangle^{d+\gamma}} \right| dv',$$

which in the same way as in Lemma 18 may be estimated

$$|u(v)| \leq \frac{C}{\langle v \rangle^{\min(d+\gamma, d)+\mathbf{a}-1}} = \frac{C}{\langle v \rangle^{d+\mathbf{a}-1}}.$$

Now estimate  $u(v) \cdot v$ . We have

$$\begin{aligned} \int_{|v'| \geq \frac{1}{2}} \frac{v' \cdot v}{|v'|^{d+\mathbf{a}}} \left( \frac{1}{\langle v - v' \rangle^{d+\gamma}} - \frac{1}{\langle v \rangle^{d+\gamma}} \right) dv' &= \int_{|v'| \geq \frac{1}{2}} \frac{v' \cdot v}{|v'|^{d+\mathbf{a}}} \frac{1}{\langle v - v' \rangle^{d+\gamma}} dv' \\ &\geq \frac{C}{\langle v \rangle^{d+\mathbf{a}-2}}. \end{aligned}$$

The integral over the complement is small (since  $\mathbf{a}-2 < 0 < \gamma$ ,  $d+\gamma > d+\mathbf{a}-2$ ),

$$\begin{aligned} & \left| \int_{|v'| < \frac{1}{2}} \frac{v' \cdot v}{|v'|^{d+\mathbf{a}}} \left( \frac{1}{\langle v-v' \rangle^{d+\gamma}} - \frac{1}{\langle v \rangle^{d+\gamma}} \right) dv' \right| \\ & \leq (d+\gamma) \int_{|v'| < \frac{1}{2}} \frac{|v|}{|v'|^{d+\mathbf{a}-2}} \sup_{w \in B_v(1/2)} \langle w \rangle^{-(d+\gamma+1)} dv' \leq \frac{C}{\langle v \rangle^{d+\gamma}}. \end{aligned}$$

and we obtain  $u(v) \cdot v \geq C|v|^{-(d+\mathbf{a}-2)}$ . The result follows.  $\square$

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