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HYPOCOERCIVITY AND SUB-EXPONENTIAL LOCAL EQUILIBRIA

E. BOUIN, J. DOLBEAULT, L. LAFLECHE, C. SCHMEISER

ABSTRACT. Hypocoercivity methods are applied to linear kinetic equations without any space confinement, when local equilibria have a sub-exponential decay. By Nash type estimates, global rates of decay are obtained, which reflect the behavior of the heat equation obtained in the diffusion limit. The method applies to Fokker-Planck and scattering collision operators. The main tools are a weighted Poincaré inequality (in the Fokker-Planck case) and norms with various weights. The advantage of weighted Poincaré inequalities compared to the more classical weak Poincaré inequalities is that the description of the convergence rates to the local equilibrium does not require extra regularity assumptions to cover the transition from super-exponential and exponential local equilibria to sub-exponential local equilibria.

1. INTRODUCTION

This paper is devoted a *hypocoercivity* method designed for obtaining decay rates in weighted L^2 norms of the solution to the Cauchy problem

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathbf{L} f, \\ f(0, x, v) = f^{\text{in}}(x, v), \end{cases} \quad (1)$$

for a distribution function $f(t, x, v)$, with *position* $x \in \mathbb{R}^d$, *velocity* $v \in \mathbb{R}^d$, and *time* $t \geq 0$. The collision operator \mathbf{L} acts only on the velocity variable and it is assumed to be such that its null space is spanned by a *local equilibrium* F . Let $e(v) := -\log F(v)$ and assume that the limit

$$\alpha = \lim_{|v| \rightarrow +\infty} \frac{\log e(v)}{\log |v|}$$

exists. Sub-exponential, exponential or super-exponential local equilibria correspond respectively to the cases $\alpha \in (0, 1)$, $\alpha = 1$ and $\alpha > 1$. In particular Gaussian local equilibria enter in the special case $\alpha = 2$, but notice that there are other equilibria with $\alpha = 2$, for instance, the ones obtained in Fermi-Dirac statistics. In this paper, we study specifically the *sub-exponential* case, with $F(v) = C_\alpha \exp(-(1 + |v|^2)^{\alpha/2})$ and $\alpha \in (0, 1)$.

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With no loss of generality, we can assume that F is nonnegative and of mass 1. We denote the measure associated with the inverse of the local equilibrium by

$$d\mu(v) := F(v)^{-1} dv.$$

We have in mind two linear collision operators: the *Fokker-Planck* operator

$$\mathbf{L}_1 f = \nabla_v \cdot \left(F \nabla_v (F^{-1} f) \right)$$

and a *scattering* collision operator

$$\mathbf{L}_2 f = \int_{\mathbb{R}^d} b(\cdot, v') \left(f(v') F(\cdot) - f(\cdot) F(v') \right) dv'.$$

Under assumptions on the *cross-section* b that will be given later and for a special choice of F in the Fokker-Planck case, these two operators \mathbf{L}_1 and \mathbf{L}_2 are responsible for the same asymptotic behavior. The operator \mathbf{L}_1 is a local operator with sharp estimates, while \mathbf{L}_2 is non-local but less restrictive in terms of local equilibria and of wider interest for applications in modeling in physics. Estimates for \mathbf{L}_2 are however not as tight as for \mathbf{L}_1 .

Our purpose is to consider a solution of (1) with a non-negative initial datum f^{in} and study its large time behavior. If f^{in} has finite mass, then mass is conserved for any $t \geq 0$. Since there is no stationary state with finite mass, it is expected that $f(t, \cdot, \cdot)$ locally vanishes as $t \rightarrow +\infty$. In order to state a result, we need some notations and a few additional assumptions. We shall denote by $\mathbb{1}_\Omega$ the characteristic function of a domain Ω , by Id the identity operator and define

$$\langle v \rangle := \sqrt{1 + |v|^2}.$$

Let us consider the norms

$$\|f\|_k = \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |f|^2 \langle v \rangle^k dx d\mu \right)^{1/2} \quad \text{and} \quad \|f\| = \|f\|_0.$$

We also define on $L^2(dx d\mu)$ the scalar product $\langle f_1, f_2 \rangle := \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_1 f_2 dx d\mu$ such that $\|f\|^2 = \langle f, f \rangle$. With these notations in hand, we can list our hypotheses. We first assume *local mass conservation*

$$\int_{\mathbb{R}^d} \mathbf{L} f dv = 0.$$

Such a property is always granted if $\mathbf{L} = \mathbf{L}_1$ and holds when $\mathbf{L} = \mathbf{L}_2$ if b satisfies

$$\int_{\mathbb{R}^d} \left(b(v, v') - b(v', v) \right) F' dv' = 0 \tag{H1}$$

where F' stands for $F(v')$. Notice that *micro-reversibility*, *i.e.*, the symmetry of b , is not required. Next, we assume that the *cross-section* b takes nonnegative values and

is such that the *collision frequency* ν verifies

$$\nu(v) := \int_{\mathbb{R}^d} \mathfrak{b}(v, v') F' dv' = B(v) |v|^{-\beta} \quad (\text{H2})$$

for some $\beta > 0$ and for some positive, continuous function B such that

$$\lim_{|v| \rightarrow \infty} B(v) = 1.$$

If $\mathbf{L} = \mathbf{L}_2$, we shall assume that there exists a finite positive constant \mathcal{C} such that

$$\forall h \in L^2(\mathbb{R}^d, d\mu), \quad \int_{\mathbb{R}^d} |h - \tilde{h}|^2 \nu F dv \leq \mathcal{C} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathfrak{b}(v, v') |h' - h|^2 F F' dv dv' \quad (2)$$

where $\tilde{h} = \int_{\mathbb{R}^d} h d\mu$. Sufficient conditions on \mathfrak{b} can be found for instance in [7, Proposition 2.2] or in [15, Lemma 1]. Notice that a Cauchy-Schwarz inequality yields

$$\begin{aligned} \int_{\mathbb{R}^d} |h - \tilde{h}|^2 \nu F dv &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (h - h') F' dv' \right|^2 \nu F dv \\ &\leq \int_{\mathbb{R}^d} \frac{F'}{\nu'} dv' \iint_{\mathbb{R}^d \times \mathbb{R}^d} |h - h'|^2 \nu \nu' F dv F' dv' \end{aligned}$$

so that (2) is obtained by assuming

$$\sup_{(v, v') \in \mathbb{R}^d \times \mathbb{R}^d} \frac{\nu(v) \nu(v')}{\mathfrak{b}(v, v')} \int_{\mathbb{R}^d} \frac{F'}{\nu'} dv' < \infty. \quad (\text{H3})$$

Notice that condition (H3) does not require the local integrability of \mathfrak{b} . Other sufficient conditions for (2) can be found in [15]. We shall additionally assume that \mathfrak{b} satisfies

$$\begin{aligned} \left\| \int_{\mathbb{R}^d} \mathfrak{b}(v, v') (v' - v) F F' dv' \right\|_{L^2((v)^\beta d\mu)} &< \infty \\ \text{and } \mathcal{C}_b := \sup_{v \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathfrak{b}(v', v)^2}{\nu(v)^2} F' dv' &< \infty. \quad (\text{H4}) \end{aligned}$$

So far we did not make any assumption on the *local equilibrium* F . If we assume that $\mathbf{L} = \mathbf{L}_1$ and consider the so-called *homogenous* case $f(t, x, v) = g(t, v)$ in which the distribution function is independent of x , then (1) is reduced to a simple Fokker-Planck equation and by a standard computation, we have that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |g - \bar{g}|^2 d\mu = -2 \int_{\mathbb{R}^d} |\nabla_v (F^{-1} g)|^2 F^2 d\mu \quad (3)$$

where $\bar{g} := (\int_{\mathbb{R}^d} g F d\mu) F = (\int_{\mathbb{R}^d} g dv) F$. To fix ideas, let us assume that

$$\forall v \in \mathbb{R}^d, \quad F(v) = C_\alpha e^{-(v)^\alpha} \quad (4)$$

where C_α is a normalisation constant such that F is a probability density. If $\alpha \geq 1$, it is easy to conclude using the Poincaré inequality

$$\forall g \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |\nabla_v (F^{-1} g)|^2 F^2 d\mu \leq \lambda \int_{\mathbb{R}^d} |g - \bar{g}|^2 d\mu,$$

for some positive constant λ and deduce that $\int_{\mathbb{R}^d} |g - \bar{g}|^2 d\mu$ is decaying at an exponential rate. In this paper, we are interested in the regime of *sub-exponential local equilibria* corresponding to

$$\alpha \in (0, 1). \quad (5)$$

In this range (5), we rely on the *weighted Poincaré inequality*

$$\forall g \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |\nabla_v (F^{-1} g)|^2 F^2 d\mu \geq \mathcal{C} \int_{\mathbb{R}^d} |g - \bar{g}|^2 \langle v \rangle^{-2(1-\alpha)} d\mu. \quad (6)$$

There is now a weight in the right-hand side, although we still define the average as $\bar{g} := (\int_{\mathbb{R}^d} g dv) F$ (i.e., the definition of \bar{g} does not involve the weight $\langle v \rangle^{-2(1-\alpha)}$): see Appendix A for details). Under appropriate conditions on the initial data, inequality (6) is enough to obtain that $\int_{\mathbb{R}^d} |g - \bar{g}|^2 d\mu$ has an algebraic decay rate. For such a problem, estimates based on *weak Poincaré inequalities* are also very popular in the scientific community of semi-group theory and Markov processes (see [20], [3, Proposition 7.5.10], [13] and Appendix B), but with the inconvenient that a uniform estimate of the solution is needed. Estimates based on weak Poincaré inequalities rely on a uniform bound for $\alpha < 1$ which is not present for $\alpha \geq 1$, while the approach developed in this paper only uses weighted L^2 norms with weights that vanish as $\alpha \rightarrow 1_-$.

In the Fokker-Planck case $\mathbf{L} = \mathbf{L}_1$, let us choose $\beta = 2(1 - \alpha)$ in order to provide unified statements. This choice arises from (6). In the case of the scattering operator $\mathbf{L} = \mathbf{L}_2$, notice that α can be chosen independently of β and that we can even consider more general local equilibria. However, as a simplifying assumption, we shall assume in both cases that F is a *sub-exponential equilibrium* given by (4) with α such that (5) holds. At least in the Fokker-Planck case, we can state a result for $\alpha \in (0, 1)$ which is consistent with the known results when $\alpha \geq 1$, with no extra regularity assumption. Our main result deals with the decay of a solution of (1) with finite mass in the non-homogeneous case and goes as follows.

Theorem 1. *Let $\alpha \in (0, 1)$, $\beta > 0$, $k > 0$ and let F be given by (4). Assume that either $\mathbf{L} = \mathbf{L}_1$ and $\beta = 2(1 - \alpha)$, or $\mathbf{L} = \mathbf{L}_2$ and (H1)–(H4) hold. There exists a numerical constant $\mathcal{C} > 0$ such that any solution f of (1) with initial datum $f^{\text{in}} \in L^2(\langle v \rangle^k dx d\mu) \cap L^1_+(dx dv)$ satisfies*

$$\forall t \geq 0, \quad \|f(t, \cdot, \cdot)\|^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(t, x, v)|^2 dx d\mu \leq \mathcal{C} \frac{\|f^{\text{in}}\|^2}{(1 + \kappa t)^\zeta}$$

with rate $\zeta = \min\{d/2, k/\beta\}$, for some positive κ which is an explicit function of the two quotients, $\|f^{\text{in}}\| / \|f^{\text{in}}\|_k$ and $\|f^{\text{in}}\|_{L^1(dx dv)} / \|f^{\text{in}}\|$.

If $k \geq d\beta/2$, we recover the rate of decay of the heat equation as in the case $\alpha \geq 1$. When $\mathbf{L} = \mathbf{L}_1$ and $\alpha \in (0, 1)$, this condition means $k \geq d(1 - \alpha)$ and it is automatically fulfilled for a given $k > 0$ when α approaches 1. With $\alpha = 1$, (6) is indeed the

usual Poincaré inequality associated with the measure $d\mu$. If $k \in (0, d\beta/2)$, a new limitation appears due to the lower rate of relaxation towards the local equilibrium: see Appendix B for details. The condition $k \geq d\beta/2$ makes sense because our L^2 hypocoercivity method is designed to capture the rate of the diffusion limit (which corresponds to the homogeneous equation) as in [4, 10]: explanations will be given in Appendix B.2. Our strategy of proof relies on the L^2 hypocoercivity method as in [9, 10] and the extension of [4, 5] to the case without confinement in x . Among related results, let us quote [13] with an approach based on the weak Poincaré inequality in the homogeneous case and sub-exponential equilibria, and also [6] for Gaussian local equilibria in presence of an external potential with sub-exponential growth in the variable x .

This paper is organized as follows. In Section 2, we prove an hypocoercive estimate which relates an entropy, which is equivalent to $\|f\|^2$, to an entropy production term involving a microscopic and a macroscopic component. Using weighted L^2 estimates established in Section 3, we obtain a new control by the microscopic component in Lemma 7 while the macroscopic component is estimated as in [4] using Nash's inequality, see Lemma 6. By collecting these estimates in Section 4, we complete the proof of Theorem 1. Two appendices are devoted to $L = L_1$: in Appendix A we provide a new proof of (6) and comment on the interplay with weak Poincaré inequalities, while the homogenous case is dealt with in Appendix B and rates of relaxation towards the local equilibrium are discussed using weighted L^2 norms, as an alternative approach to [13]. The main novelty of our approach is that we use new interpolations in order to exploit the entropy production term. As a consequence, with the appropriate weights, no other norm is needed than weighted L^2 norms. For simplicity, we assume that the distribution function is nonnegative but the extension to sign changing functions is straightforward.

2. AN ENTROPY–ENTROPY PRODUCTION ESTIMATE

We adapt the strategy of [10, 4]. Let Π be the orthogonal projection operator on $\text{Ker}(L)$ in $L^2(\mathbb{R}^d, d\mu)$, defined by

$$\Pi f := \rho_f F \quad \text{where} \quad \rho_f := \int_{\mathbb{R}^d} f \, dv .$$

To build a suitable Lyapunov functional, we introduce the operator

$$A := \left(\text{Id} + (\mathbb{T}\Pi)^*(\mathbb{T}\Pi) \right)^{-1} (\mathbb{T}\Pi)^*$$

and consider

$$\mathbb{H}[f] := \frac{1}{2} \|f\|^2 + \delta \langle Af, f \rangle .$$

It is known from [10, Lemma 1] that, for any $\delta \in (0, 1)$, $\mathbf{H}[f]$ and $\|f\|^2$ determine equivalent norms, in the sense that

$$\frac{1}{2}(1 - \delta) \|f\|^2 \leq \mathbf{H}[f] \leq \frac{1}{2}(1 + \delta) \|f\|^2. \quad (7)$$

A direct computation shows that

$$\frac{d}{dt} \mathbf{H}[f] = -\mathbf{D}[f] \quad (8)$$

with

$$\begin{aligned} \mathbf{D}[f] := & -\langle \mathbf{L}f, f \rangle + \delta \langle \mathbf{AT}\Pi f, \Pi f \rangle \\ & + \delta \langle \mathbf{AT}(\text{Id} - \Pi)f, \Pi f \rangle - \delta \langle \mathbf{TA}(\text{Id} - \Pi)f, (\text{Id} - \Pi)f \rangle - \delta \langle \mathbf{AL}(\text{Id} - \Pi)f, \Pi f \rangle \end{aligned}$$

where we have used that $\langle \mathbf{A}f, \mathbf{L}f \rangle = 0$. As a consequence of (6) and (2), there is a positive constant \mathcal{C} such that

$$\langle \mathbf{L}f, f \rangle \leq -\mathcal{C} \|(\text{Id} - \Pi)f\|_{-\beta}^2. \quad (9)$$

Proposition 2. *Under the assumptions of Theorem 1, there exists $\kappa > 0$ such that, for any $f \in \mathbf{L}^2(\langle v \rangle^{-\beta} dx d\mu) \cap \mathbf{L}^1(dx dv)$,*

$$\mathbf{D}[f] \geq \kappa \left(\|(\text{Id} - \Pi)f\|_{-\beta}^2 + \langle \mathbf{AT}\Pi f, \Pi f \rangle \right).$$

Notice that κ does not depend on $k > 0$ (the parameter k appears in the assumptions of Theorem 1). Expressing $\mathbf{D}[f]$ in terms of $\langle \mathbf{AT}\Pi f, \Pi f \rangle$ is standard, see for instance [4], but using the weighted norm $\|(\text{Id} - \Pi)f\|_{-\beta}$ is a new idea.

Proof. We have to prove that the three last terms in $\mathbf{D}[f]$ are controlled by the first two. The main difference with [10, 4] is the additional weight $\langle v \rangle^{-\beta}$ in the velocity variable.

• **Step 1: rewriting $\langle \mathbf{AT}\Pi f, \Pi f \rangle$.** Let $u = u_f$ be such that

$$u F = (\text{Id} + (\mathbf{T}\Pi)^*(\mathbf{T}\Pi))^{-1} \Pi f.$$

Then u solves $(u - \Theta \Delta u) F = \Pi f$, that is,

$$u - \Theta \Delta u = \rho_f \quad (10)$$

where $\Theta := \int_{\mathbb{R}^d} |v \cdot \mathbf{e}|^2 F(v) dv$ for an arbitrary unit vector \mathbf{e} . Since

$$\begin{aligned} \mathbf{AT}\Pi f &= (\text{Id} + (\mathbf{T}\Pi)^*(\mathbf{T}\Pi))^{-1} (\mathbf{T}\Pi)^*(\mathbf{T}\Pi) \Pi f \\ &= (\text{Id} + (\mathbf{T}\Pi)^*(\mathbf{T}\Pi))^{-1} (\text{Id} + (\mathbf{T}\Pi)^*(\mathbf{T}\Pi) - \text{Id}) \Pi f \\ &= \Pi f - (\text{Id} + (\mathbf{T}\Pi)^*(\mathbf{T}\Pi))^{-1} \Pi f = \Pi f - u F = (\rho_f - u) F, \end{aligned}$$

then by using equation (10), we obtain

$$\langle \mathbf{A}\mathbf{T}\Pi f, \Pi f \rangle = \langle \Pi f - u F, \Pi f \rangle = \langle -\Theta \Delta u F, (u - \Theta \Delta u) F \rangle ,$$

from which we deduce

$$\langle \mathbf{A}\mathbf{T}\Pi f, \Pi f \rangle = \Theta \|\nabla u\|_{L^2(dx)}^2 + \Theta^2 \|\Delta u\|_{L^2(dx)}^2 . \quad (11)$$

• **Step 2: a bound on $\langle \mathbf{A}\mathbf{T}(\text{Id} - \Pi)f, \Pi f \rangle$.** If u solves (10), we use the fact that

$$\mathbf{A}^*\Pi f = \mathbf{T}\Pi u F = \mathbf{T}u F \quad (12)$$

to compute

$$\langle \mathbf{A}\mathbf{T}(\text{Id} - \Pi)f, \Pi f \rangle = \langle (\text{Id} - \Pi)f, \mathbf{T}^*\mathbf{A}^*\Pi f \rangle = \langle (\text{Id} - \Pi)f, \mathbf{T}^*\mathbf{T}u F \rangle .$$

Therefore, since $\mathbf{T}^*\mathbf{T}u F = -v \cdot \nabla_x (v \cdot \nabla_x u) F$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} |\langle \mathbf{A}\mathbf{T}(\text{Id} - \Pi)f, \Pi f \rangle| &\leq \|(\text{Id} - \Pi)f\|_{-\beta} \left\| \sqrt{F} \langle v \rangle^{\frac{\beta}{2}} v \cdot \nabla_x (v \cdot \nabla_x u) \sqrt{F} \right\|_{L^2(dx dv)} \\ &\leq \Theta_{4+\beta} \|(\text{Id} - \Pi)f\|_{-\beta} \|\Delta u\|_{L^2(dx)} , \end{aligned}$$

hence

$$|\langle \mathbf{A}\mathbf{T}(\text{Id} - \Pi)f, \Pi f \rangle| \leq \mathcal{C}_4 \|(\text{Id} - \Pi)f\|_{-\beta} \langle \mathbf{A}\mathbf{T}\Pi f, \Pi f \rangle^{\frac{1}{2}} \quad (13)$$

where we have used identity (11), $\mathcal{C}_4 = \Theta_{4+\beta}/\Theta$ and

$$\Theta_k := \int_{\mathbb{R}^d} \langle v \rangle^k F(v) dv .$$

With this convention, notice that $\Theta_2 = d\Theta$.

• **Step 3: estimating $\langle \mathbf{T}\mathbf{A}(\text{Id} - \Pi)f, (\text{Id} - \Pi)f \rangle$.** As noticed in [10, Lemma 1], the equation $g = \Pi g = \mathbf{A}f$ is equivalent to

$$(\text{Id} + (\mathbf{T}\Pi)^*(\mathbf{T}\Pi))g = (\mathbf{T}\Pi)^*f$$

which, after multiplying by g and integrating, yields

$$\begin{aligned} \|g\|^2 + \|\mathbf{T}g\|^2 &= \langle g, g + (\mathbf{T}\Pi)^*(\mathbf{T}\Pi)g \rangle \\ &= \langle g, (\mathbf{T}\Pi)^*f \rangle = \langle \mathbf{T}\Pi g, f \rangle = \langle \mathbf{T}\mathbf{A}f, f \rangle \leq \|(\text{Id} - \Pi)f\|_{-\beta} \|\mathbf{T}\mathbf{A}f\|_{\beta} \end{aligned}$$

by the Cauchy-Schwarz inequality. We know that $(\mathbf{T}\Pi)^* = -\Pi\mathbf{T}$ so that $\mathbf{A}f = g = w F$ is determined by the equation

$$w - \Theta \Delta w = -\nabla_x \cdot \int_{\mathbb{R}^d} v f dv .$$

After multiplying by w and integrating in x , we obtain that

$$\begin{aligned} \Theta \int_{\mathbb{R}^d} |\nabla_x w|^2 dx &\leq \int_{\mathbb{R}^d} |w|^2 dx + \Theta \int_{\mathbb{R}^d} |\nabla_x w|^2 dx \\ &\leq \left(\int_{\mathbb{R}^d} |\nabla_x w|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} | \int_{\mathbb{R}^d} v f dv |^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

and notice that

$$\begin{aligned} \int_{\mathbb{R}^d} | \int_{\mathbb{R}^d} v f dv |^2 dx &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \langle v \rangle^{-\frac{\beta}{2}} \frac{(\text{Id} - \Pi) f}{\sqrt{F}} \cdot |v| \langle v \rangle^{\frac{\beta}{2}} \sqrt{F} dv \right|^2 dx \\ &\leq \Theta_{\beta+2} \|(\text{Id} - \Pi) f\|_{-\beta}^2 \end{aligned}$$

by the Cauchy-Schwarz inequality. Hence

$$\int_{\mathbb{R}^d} |\nabla_x w|^2 dx \leq \frac{\Theta_{\beta+2}}{\Theta^2} \|(\text{Id} - \Pi) f\|_{-\beta}^2$$

and

$$\|\text{TA}f\|_{\beta}^2 = \left\| \nabla_x w \cdot (v \langle v \rangle^{\beta/2} F) \right\|^2 = \Theta_{\beta+2} \int_{\mathbb{R}^d} |\nabla_x w|^2 dx \leq \mathcal{C}_2^2 \|(\text{Id} - \Pi) f\|_{-\beta}^2$$

with $\mathcal{C}_2 := \Theta_{\beta+2}/\Theta$. Since $g = \text{A}f$ so that $\|\text{A}f\|^2 + \|\text{TA}f\|^2 = \|g\|^2 + \|\text{T}g\|^2$, we obtain that

$$\begin{aligned} \langle \text{TA}f, f \rangle &= \langle \text{TA}(\text{Id} - \Pi)f, (\text{Id} - \Pi)f \rangle \\ &\leq \|(\text{Id} - \Pi)f\|_{-\beta} \|\text{TA}f\|_{\beta} \leq \mathcal{C}_2 \|(\text{Id} - \Pi)f\|_{-\beta}^2. \end{aligned} \quad (14)$$

We can also notice that

$$\begin{aligned} \langle \text{TA}f, f \rangle &= \langle (v \cdot \nabla_x w) F, f \rangle = \int_{\mathbb{R}^d} \nabla_x w \cdot \left(\int_{\mathbb{R}^d} v f dv \right) dx \\ &= \int_{\mathbb{R}^d} |w|^2 dx + \Theta \int_{\mathbb{R}^d} |\nabla_x w|^2 dx \geq 0. \end{aligned}$$

• **Step 4: bound for $\langle \text{AL}(\text{Id} - \Pi)f, \Pi f \rangle$.** We use again identity (12) to compute

$$\begin{aligned} |\langle \text{AL}(\text{Id} - \Pi)f, \Pi f \rangle| &= |\langle (\text{Id} - \Pi)f, \text{L}^* \text{A}^* \Pi f \rangle| = |\langle (\text{Id} - \Pi)f, \text{L}^* \text{T}u F \rangle| \\ &\leq \|(\text{Id} - \Pi)f\|_{-\beta} \|\text{L}^* \text{T}u F\|_{\beta}. \end{aligned}$$

In case $\text{L} = \text{L}_1$ we remark that

$$\begin{aligned} \|\text{L}^* \text{T}u F\|_{\beta}^2 &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| \nabla_v \cdot \left(F \nabla_v (v \cdot \nabla_x u) \right) \right|^2 \langle v \rangle^{\beta} dx d\mu \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla_v F \cdot \nabla_x u|^2 \langle v \rangle^{\beta} dx d\mu \leq \|\nabla_v F\|_{\text{L}^2(\langle v \rangle^{\beta} d\mu)}^2 \|\nabla_x u\|_{\text{L}^2(dx)}^2. \end{aligned}$$

In case $\mathbf{L} = \mathbf{L}_2$, notice first that

$$\mathbf{L}\mathbf{T}u F = \left(\int_{\mathbb{R}^d} \mathfrak{b}(\cdot, v') (v' - v) F' dv' \right) \nabla_x u F,$$

and thus, by the Cauchy-Schwarz inequality,

$$\|\mathbf{L}^*\mathbf{T}u F\|_{\beta} \leq \mathcal{B} \|\nabla_x u\|_{\mathbf{L}^2(dx)}, \quad \text{where } \mathcal{B} = \left\| \int_{\mathbb{R}^d} \mathfrak{b}(v, v') (v' - v) F' F dv' \right\|_{\mathbf{L}^2(\langle v \rangle^{\beta} d\mu)}$$

is bounded by assumption (H4). Combining these estimates with identity (11) we get

$$|\langle \mathbf{A}\mathbf{L}(\mathbf{Id} - \mathbf{\Pi})f, \mathbf{\Pi}f \rangle| \leq \mathcal{C}_F \|(\mathbf{Id} - \mathbf{\Pi})f\|_{-\beta} \langle \mathbf{A}\mathbf{T}\mathbf{\Pi}f, \mathbf{\Pi}f \rangle^{\frac{1}{2}} \quad (15)$$

where $\mathcal{C}_F = \mathcal{B}/\sqrt{\Theta}$.

• **Step 5: collecting all estimates.** Altogether, combining (9) and (13)–(15), we obtain

$$\begin{aligned} \frac{d}{dt} \mathbf{H}[f] &\leq -\mathcal{C} \|(\mathbf{Id} - \mathbf{\Pi})f\|_{-\beta}^2 - \delta \langle \mathbf{A}\mathbf{T}\mathbf{\Pi}f, \mathbf{\Pi}f \rangle \\ &\quad + \delta (\mathcal{C}_4 + \mathcal{C}_F) \|(\mathbf{Id} - \mathbf{\Pi})f\|_{-\beta} \langle \mathbf{A}\mathbf{T}\mathbf{\Pi}f, \mathbf{\Pi}f \rangle^{\frac{1}{2}} + \delta \mathcal{C}_2 \|(\mathbf{Id} - \mathbf{\Pi})f\|_{-\beta}^2 \end{aligned}$$

which by Young's inequality yields the existence of $\kappa > 0$ such that

$$\frac{d}{dt} \mathbf{H}[f] \leq -\kappa \left(\|(\mathbf{Id} - \mathbf{\Pi})f\|_{-\beta}^2 + \langle \mathbf{A}\mathbf{T}\mathbf{\Pi}f, \mathbf{\Pi}f \rangle \right)$$

for some $\delta \in (0, 1)$. Indeed, with $X := \|(\mathbf{Id} - \mathbf{\Pi})f\|_{-\beta}$ and $Y := \langle \mathbf{A}\mathbf{T}\mathbf{\Pi}f, \mathbf{\Pi}f \rangle^{\frac{1}{2}}$, it is enough to check that the quadratic form

$$\mathcal{Q}(X, Y) := (\mathcal{C} - \delta \mathcal{C}_2) X^2 - (\mathcal{C}_4 + \mathcal{C}_F) X Y + \delta Y^2$$

is negative, *i.e.*, $\mathcal{Q}(X, Y) \geq \kappa (X^2 + Y^2)$ for some $\kappa = \kappa(\delta)$ and $\delta \in (0, 1)$. \square

3. WEIGHTED \mathbf{L}^2 ESTIMATES

In this section, we show the propagation of weighted norms with weights $\langle v \rangle^k$ of arbitrary positive order $k \in \mathbb{R}^+$.

Proposition 3. *Let $k > 0$ and f be solution of (1) with $f^{\text{in}} \in \mathbf{L}^2(\langle v \rangle^k dx d\mu)$. Then there exists a constant $\mathcal{K}_k > 1$ such that*

$$\forall t \geq 0 \quad \|f(t, \cdot, \cdot)\|_{\mathbf{L}^2(\langle v \rangle^k dx d\mu)} \leq \mathcal{K}_k \|f^{\text{in}}\|_{\mathbf{L}^2(\langle v \rangle^k dx d\mu)}.$$

We shall state a technical lemma (Lemma 4 below) before proving a splitting result in Lemma 5, from which the proof of Proposition 3 easily follows (see section 3.3).

3.1. A technical lemma.

Lemma 4. *If either $\mathbf{L} = \mathbf{L}_1$ or $\mathbf{L} = \mathbf{L}_2$, then*

$$\forall t \geq 0, \quad \left\| e^{t(\mathbf{L}-\mathbf{T})} \right\|_{\mathbf{L}^2(dx d\mu) \rightarrow \mathbf{L}^2(dx d\mu)} \leq 1 \quad (16)$$

and there is some $\ell \in \mathbb{R}$ for which, for any $k \geq 0$, there exists $(a_k, b_k, R_k) \in \mathbb{R} \times \mathbb{R}_+^2$ such that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} f \mathbf{L} f \langle v \rangle^k dx d\mu \leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} (a_k \mathbb{1}_{B_{R_k}} - b_k \langle v \rangle^{-\ell}) |f|^2 \langle v \rangle^k dx d\mu \quad (17)$$

for any $f \in \mathbf{L}^2(\langle v \rangle^k dx d\mu) \cap \mathbf{L}^1(dx dv)$.

Proof. In the Fokker-Planck case $\mathbf{L} = \mathbf{L}_1$, the function $h := fF^{-1}$ solves

$$\partial_t h + v \cdot \nabla_x h = F^{-1} \nabla_v \cdot (F \nabla_v h)$$

so that

$$\begin{aligned} \iint_{\mathbb{R}^d \times \mathbb{R}^d} f \mathbf{L}_1 f \langle v \rangle^k dx d\mu &= \frac{d}{dt} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |h(t, v)|^2 \langle v \rangle^k F dx dv \\ &= -2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla_v h|^2 \langle v \rangle^k F dx dv - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v (h^2) \cdot (\nabla_v \langle v \rangle^k) F dx dv. \end{aligned}$$

This proves (16) if $k = 0$. Otherwise, (17) follows with $\ell = 2 - \alpha$ from

$$\nabla_v \log (F \nabla_v (\langle v \rangle^k)) = \frac{k}{\langle v \rangle^4} (d + (k + d - 2) |v|^2 - \alpha \langle v \rangle^\alpha |v|^2).$$

In the case of the scattering operator $\mathbf{L} = \mathbf{L}_2$, with $h := f/F$, we have

$$\begin{aligned} 2 \int_{\mathbb{R}^d} f \mathbf{L}_2 f \langle v \rangle^k d\mu &= 2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} b(v, v') (h' - h) h \langle v \rangle^k F F' dv dv' \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} b(v, v') ((h' - h) h + h' h) \langle v \rangle^k F F' dv dv' \\ &\quad - \iint_{\mathbb{R}^d \times \mathbb{R}^d} b(v, v') |h|^2 \langle v \rangle^k F F' dv dv', \\ 2 \int_{\mathbb{R}^d} f \mathbf{L}_2 f \langle v \rangle^k d\mu &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} b(v, v') ((h' - h) h + h' (h - h')) \langle v \rangle^k F F' dv dv' \\ &\quad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} b(v, v') (|h'|^2 - |h|^2) \langle v \rangle^k F F' dv dv' \\ &= - \iint_{\mathbb{R}^d \times \mathbb{R}^d} b(v, v') |h' - h|^2 \langle v \rangle^k F F' dv dv' \\ &\quad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} b(v, v') (|h'|^2 - |h|^2) \langle v \rangle^k F F' dv dv' \\ &\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} b(v, v') (|h'|^2 - |h|^2) \langle v \rangle^k F F' dv dv'. \end{aligned}$$

However, using assumption (H1), one can rearrange the last integral

$$\begin{aligned}
& \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathfrak{b}(v, v') (|h'|^2 - |h|^2) \langle v \rangle^k F F' \, dv \, dv' \\
&= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathfrak{b}(v, v') |h'|^2 \langle v \rangle^k F F' \, dv \, dv' - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathfrak{b}(v', v) |h|^2 \langle v \rangle^k F F' \, dv \, dv' \\
&= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathfrak{b}(v, v') |h'|^2 \langle v \rangle^k F F' \, dv \, dv' - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathfrak{b}(v, v') |h'|^2 \langle v' \rangle^k F F' \, dv \, dv' \\
&= \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \mathfrak{b}(v', v) (\langle v' \rangle^k - \langle v \rangle^k) F' \, dv' \right] |h|^2 F \, dv \\
&= \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \mathfrak{b}(v', v) \left(\frac{\langle v' \rangle^k}{\langle v \rangle^k} - 1 \right) F' \, dv' \right] |f|^2 \langle v \rangle^k \, d\mu.
\end{aligned}$$

This implies identity (16) by taking $k = 0$. To get inequality (17), we point out that

$$\begin{aligned}
2 \int_{\mathbb{R}^d} f \mathbf{L}_2 f \langle v \rangle^k \, d\mu &= \int_{\mathbb{R}^d} \mathfrak{b}(v', v) \frac{\langle v' \rangle^k}{\langle v \rangle^k} F' \, dv' - \nu(v) \\
&\leq \langle v \rangle^{-k} \sqrt{\Theta_{2k} \int_{\mathbb{R}^d} \mathfrak{b}(v', v)^2 F' \, dv'} - \nu(v) \leq \left(\langle v \rangle^{-k} \sqrt{\mathcal{C}_b \Theta_{2k} - 1} \right) \nu(v)
\end{aligned}$$

where we used assumption (H4). Finally, we conclude that inequality (17) holds for any $k > 0$ with $\ell = \beta$. \square

3.2. A splitting result. As in [11, 13, 17], we write $\mathbf{L} - \mathbf{T}$ as a dissipative part \mathbf{C} and a bounded part \mathbf{B} such that $\mathbf{L} - \mathbf{T} = \mathbf{B} + \mathbf{C}$.

Lemma 5. *With the notation of Lemma 4, let $k_1 > 0$, $k_2 > k_1 + 2\ell$, $a = \max\{a_{k_1}, a_{k_2}\}$, $R = \max\{R_{k_1}, R_{k_2}\}$, $\mathbf{C} = a \mathbf{1}_{B_R}$ and $\mathbf{B} = \mathbf{L} - \mathbf{T} - \mathbf{C}$. For any $t \in \mathbb{R}_+$, we have:*

- (i) $\|\mathbf{C}\|_{L^2(dx \, d\mu) \rightarrow L^2(\langle v \rangle^{k_2} \, dx \, d\mu)} \leq a \langle R \rangle^{k_2/2}$,
- (ii) $\|e^{t\mathbf{B}}\|_{L^2(\langle v \rangle^{k_1} \, dx \, d\mu) \rightarrow L^2(\langle v \rangle^{k_1} \, dx \, d\mu)} \leq 1$,
- (iii) $\|e^{t\mathbf{B}}\|_{L^2(\langle v \rangle^{k_2} \, dx \, d\mu) \rightarrow L^2(\langle v \rangle^{k_1} \, dx \, d\mu)} \leq C (1+t)^{-\frac{k_2-k_1}{2\ell}}$ for some $C > 0$.

Proof. Property (i) is a consequence of the definition of \mathbf{C} . Property (ii) follows from Lemma 4 according to

$$\begin{aligned}
\iint_{\mathbb{R}^d \times \mathbb{R}^d} f \mathbf{B} f \langle v \rangle^{k_1} \, dx \, d\mu &\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(a_{k_1} \mathbf{1}_{B_{R_{k_1}}} - a \mathbf{1}_{B_R} - b_{k_1} \langle v \rangle^{-\ell} \right) |f|^2 \langle v \rangle^{k_1} \, dx \, d\mu \\
&\leq -b_{k_1} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |f|^2 \langle v \rangle^{k_1 - \ell} \, dx \, d\mu.
\end{aligned}$$

Similarly, we know that $\|e^{t\mathbf{B}}\|_{L^2(\langle v \rangle^{k_2} \, dx \, d\mu) \rightarrow L^2(\langle v \rangle^{k_2} \, dx \, d\mu)} \leq 1$.

By combining Hölder's inequality

$$\|f\|_{L^2(\langle v \rangle^{k_1} \, dx \, d\mu)}^2 \leq \|f\|_{L^2(\langle v \rangle^{k_1 - \ell} \, dx \, d\mu)}^{\frac{2(k_2 - k_1)}{k_2 - k_1 + \ell}} \|f\|_{L^2(\langle v \rangle^{k_2} \, dx \, d\mu)}^{\frac{2\ell}{k_2 - k_1 + \ell}}$$

with Property (ii), we obtain

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} f \mathbf{B} f \langle v \rangle^{k_1} dx d\mu \leq -b_{k_1} \|f\|_{L^2(\langle v \rangle^{k_1} dx d\mu)}^2 \left(1 + \frac{\ell}{k_2 - k_1}\right) \|f^{\text{in}}\|_{L^2(\langle v \rangle^{k_2} dx d\mu)}^{-\frac{2\ell}{k_2 - k_1}}.$$

With $f = e^{t\mathbf{B}} f^{\text{in}}$, Property (iii) follows from Grönwall's lemma according to

$$\begin{aligned} \|f\|_{L^2(\langle v \rangle^{k_1} dx d\mu)}^2 &\leq \left(\|f^{\text{in}}\|_{L^2(\langle v \rangle^{k_1} dx d\mu)}^{-\frac{2\ell}{k_2 - k_1}} + \frac{2\ell b_{k_1} t}{k_2 - k_1} \|f^{\text{in}}\|_{L^2(\langle v \rangle^{k_2} dx d\mu)}^{-\frac{2\ell}{k_2 - k_1}} \right)^{-\frac{k_2 - k_1}{\ell}} \\ &\leq \left(\frac{k_2 - k_1}{k_2 - k_1 + 2\ell b_{k_1} t} \right)^{\frac{k_2 - k_1}{\ell}} \|f^{\text{in}}\|_{L^2(\langle v \rangle^{k_2} dx d\mu)}^2. \end{aligned}$$

□

3.3. Proof of Proposition 3. Using the convolution $U * V = \int_0^t U(t-s)V(s) ds$, Duhamel's formula asserts that

$$e^{t(\mathbf{L}-\mathbf{T})} = e^{t\mathbf{B}} + e^{t\mathbf{B}} * \mathbf{C} e^{t(\mathbf{L}-\mathbf{T})}.$$

By Lemma 5 and (16) with $k = k_1$, ℓ as in Lemma 4 and $k_2 > k + 2\ell$, we get that

$$\left\| e^{t(\mathbf{L}-\mathbf{T})} \right\|_{L^2(\langle v \rangle^{k_1} dx d\mu) \rightarrow L^2(\langle v \rangle^{k_1} dx d\mu)} \leq 1 + a \langle R \rangle^{\frac{k_2}{2}} \int_0^t \frac{C ds}{(1+s)^{\frac{k_2 - k_1}{2\ell}}}$$

is bounded uniformly in time. □

4. PROOF OF THEOREM 1

The control of the macroscopic part Πf by $\langle \mathbf{AT}\Pi f, \Pi f \rangle$ is achieved exactly as in [4]. We sketch a proof for the sake of completeness.

Lemma 6. *Under the assumptions of Theorem 1, for any $f \in L^1(dx d\mu) \cap L^2(dx dv)$,*

$$\langle \mathbf{AT}\Pi f, \Pi f \rangle \geq \Phi \left(\|\Pi f\|^2 \right)$$

with

$$\Phi^{-1}(y) := 2y + \left(\frac{y}{c} \right)^{\frac{d}{d+2}}, \quad c = \Theta \mathcal{C}_{\text{Nash}}^{-\frac{d+2}{d}} \|f\|_{L^1(dx dv)}^{-\frac{4}{d}}.$$

Proof. With u defined by (10), we control $\|\Pi f\|^2 = \|\rho_f\|_{L^2(dx)}^2$ by $\langle \mathbf{AT}\Pi f, \Pi f \rangle$ according to

$$\|\Pi f\|^2 = \|u\|_{L^2(dx)}^2 + 2\Theta \|\nabla u\|_{L^2(dx)}^2 + \Theta^2 \|\Delta u\|_{L^2(dx)}^2 \leq \|u\|_{L^2(dx)}^2 + 2\langle \mathbf{AT}\Pi f, f \rangle.$$

using (11). Then we observe that, for any $t \geq 0$,

$$\|u(t, \cdot)\|_{L^1} = \|\rho_f(t, \cdot)\|_{L^1} = \|f\|_{L^1(dx dv)}, \quad \|\nabla u(t, \cdot)\|_{L^2(dx)}^2 \leq \frac{1}{\Theta} \langle \mathbf{AT}\Pi f(t, \cdot, \cdot), f(t, \cdot, \cdot) \rangle$$

and use Nash's inequality

$$\|u\|_{L^2(dx)}^2 \leq \mathcal{C}_{\text{Nash}} \|u\|_{L^1(dx)}^{\frac{4}{d+2}} \|\nabla u\|_{L^2(dx)}^{\frac{2d}{d+2}}$$

to conclude the proof. \square

The control of $(\text{Id} - \Pi)f$ by the entropy production term relies on a new estimate.

Lemma 7. *Under the assumptions of Theorem 1, for any $f \in L^2(\langle v \rangle^k dx d\mu) \cap L^1(dx dv)$,*

$$\|(\text{Id} - \Pi)f\|_{-\beta}^2 \geq \Psi \left(\|(\text{Id} - \Pi)f\|^2 \right)$$

where \mathcal{K}_k is as in Proposition 3 and

$$\Psi(y) := C_0 y^{1+\beta/k}, \quad C_0 := \left(\mathcal{K}_k (1 + \Theta_k) \|f^{\text{in}}\|_k \right)^{-\frac{2\beta}{k}}.$$

Proof. Hölder's inequality

$$\|(\text{Id} - \Pi)f\| \leq \|(\text{Id} - \Pi)f\|_{-\beta}^{\frac{k}{k+\beta}} \|(\text{Id} - \Pi)f\|_k^{\frac{\beta}{k+\beta}}$$

and

$$\begin{aligned} \|(\text{Id} - \Pi)f\|_{L^2(\langle v \rangle^k dx d\mu)} &\leq \|f\|_k + \Theta_k \|\rho\|_{L^2(dx)} \\ &\leq (1 + \Theta_k) \|f\|_k \leq \mathcal{K}_k (1 + \Theta_k) \|f^{\text{in}}\|_k, \end{aligned}$$

where the last inequality holds by Proposition 3, provide us with the estimate. \square

Proof of Theorem 1. Using the estimates of Lemma 6 and Lemma 7, we obtain that

$$\|(\text{Id} - \Pi)f\|_{-\beta}^2 + \langle \text{AT}\Pi f, \Pi f \rangle \geq \Psi \left(\|(\text{Id} - \Pi)f\|^2 \right) + \Phi \left(\|\Pi f\|^2 \right).$$

Using (7), (8) and the fact that $\text{D}[f] \geq 0$ by Proposition 2, we know that

$$\|(\text{Id} - \Pi)f\|^2 \leq z_0 \quad \text{and} \quad \|\Pi f\|^2 \leq z_0 \quad \text{where} \quad z_0 := \frac{1 + \delta}{1 - \delta} \|f^{\text{in}}\|^2.$$

Thus, from

$$\Phi^{-1}(y) = y + \left(\frac{y}{c} \right)^{\frac{d}{d+2}} \leq \left(C_1^{-1} y \right)^{\frac{d}{d+2}} \quad \text{with} \quad C_1 := \left(\Phi(z_0)^{\frac{2}{d+2}} + c^{-\frac{d}{d+2}} \right)^{-\frac{d+2}{d}},$$

as long as $y \leq \Phi(z_0)$, we obtain

$$\Phi \left(\|\Pi f\|^2 \right) \geq C_1 \|\Pi f\|^{2\frac{d+2}{d}},$$

since $\|\Pi f\|^2 \leq z_0$. As a consequence,

$$\begin{aligned} \|(\text{Id} - \Pi)f\|_{-\beta}^2 + \langle \text{AT}\Pi f, \Pi f \rangle &\geq C_0 \|(\text{Id} - \Pi)f\|^{2\frac{k+\beta}{k}} + C_1 \|\Pi f\|^{2\frac{d+2}{d}} \\ &\geq 2^{\frac{1}{\zeta}} \min \left\{ C_0 z_0^{\frac{\beta}{k} - \frac{1}{\zeta}}, C_1 z_0^{\frac{2}{d} - \frac{1}{\zeta}} \right\} \|f\|^{2+\frac{2}{\zeta}} \end{aligned}$$

where $1/\zeta = \max\{2/d, \beta/k\}$, *i.e.*, $\zeta = \min\{d/2, k/\beta\}$. Collecting terms, we have

$$\frac{d}{dt} \mathbf{H}[f] \leq -C \zeta \mathbf{H}[f]^{1+\frac{1}{\zeta}}$$

using (7), (8) and Proposition 2, and

$$C := \frac{\kappa}{\zeta} 2^{\frac{1}{\zeta}} \min\left\{C_0 z_0^{\frac{\beta}{k}-\frac{1}{\zeta}}, C_1 z_0^{\frac{2}{d}-\frac{1}{\zeta}}\right\} \left(\frac{2}{1+\delta}\right)^{1+\frac{1}{\zeta}}.$$

Then the result of Theorem 1 follows from a Grönwall estimate.

$$\mathbf{H}[f(t, \cdot, \cdot)] \leq \mathbf{H}[f^{\text{in}}] \left(1 + C \mathbf{H}[f^{\text{in}}]^{\frac{1}{\zeta}} t\right)^{-\frac{1}{\zeta}}$$

The expression of κ can be explicitly computed in terms of $C_0 z_0^{\frac{\beta}{k}-\frac{1}{\zeta}} \mathbf{H}[f^{\text{in}}]^{\frac{1}{\zeta}}$, which is proportional to $(\|f^{\text{in}}\| / \|f^{\text{in}}\|_k)^{\frac{2\beta}{k}}$, and in terms of $C_1 z_0^{\frac{2}{d}-\frac{1}{\zeta}} \mathbf{H}[f^{\text{in}}]^{\frac{1}{\zeta}}$ which is a function of $(\|f^{\text{in}}\|_{L^1(dx dv)} / \|f^{\text{in}}\|)^{4/(d+2)}$, but it is of no practical interest. To see this, one has to take into account the expressions of C_0 , C_1 and \mathfrak{c} in terms of the initial datum f^{in} . \square

As a concluding remark, we emphasize that a control of the solution in the space $L^2(\langle v \rangle^k dx d\mu)$, based on Proposition 3, is enough to prove Theorem 1. In particular, there is no need of a uniform bound on f . This observation is new in L^2 hypocoercive methods, and consistent with the homogeneous case (see Appendix B).

APPENDIX A. WEIGHTED POINCARÉ INEQUALITIES

This appendix is devoted to a proof of (6) and considerations on related Poincaré inequalities. Inequality (6) is not a standard weighted Poincaré inequality because the average in the right-hand side of the inequality involves the measure of the left-hand side so that the right-hand side cannot be interpreted as a variance. Here we prove a generalization of (6) which relies on a purely spectral approach.

A.1. Continuous spectrum and weighted Poincaré inequalities. Let us consider two probability measures on \mathbb{R}^d

$$d\xi = e^{-\phi} dv \quad \text{and} \quad d\nu = \psi d\xi,$$

where ϕ and $\psi \geq 0$ are two measurable functions, and the *weighted Poincaré inequality*

$$\forall h \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |\nabla h|^2 d\xi \geq C_* \int_{\mathbb{R}^d} |h - \widehat{h}|^2 d\nu \quad (18)$$

where $\widehat{h} = \int_{\mathbb{R}^d} h d\nu$. The question we address here is: *on which conditions on ϕ and ψ do we know that (18) holds for some constant $C_* > 0$?* Our key example is

$$\phi(v) = \langle v \rangle^\alpha + \log Z_\alpha \quad \text{and} \quad \psi(v) = c_{\alpha,\beta}^{-1} \langle v \rangle^{-\beta} \quad (19)$$

with $\alpha > 0$, $\beta > 0$, $Z_\alpha = \int_{\mathbb{R}^d} e^{-\phi} dv$ and $c_{\alpha,\beta} = \int_{\mathbb{R}^d} \langle v \rangle^{-\beta} d\xi$.

Let us consider a potential Φ on \mathbb{R}^d and assume that it is a measurable function with

$$\sigma = \lim_{r \rightarrow +\infty} \inf_{w \in \mathcal{D}(B_r^c) \setminus \{0\}} \frac{\int_{\mathbb{R}^d} (|\nabla w|^2 + \Phi |w|^2) dv}{\int_{\mathbb{R}^d} |w|^2 dv} > 0,$$

where $B_r^c := \{v \in \mathbb{R}^d : |v| > r\}$ and $\mathcal{D}(B_r^c)$ denotes the space of smooth functions on \mathbb{R}^d with compact support in B_r^c . According to Persson's result [19, Theorem 2.1], the lower end σ of the continuous spectrum of the Schrödinger operator $-\Delta + \Phi$ is such that

$$\sigma \geq \lim_{r \rightarrow +\infty} \inf_{v \in B_r^c} \Phi(v) =: \sigma_0.$$

If we replace $\int_{\mathbb{R}^d} |w|^2 dv$ by the weighted integral $\int_{\mathbb{R}^d} |w|^2 \psi dv$ for some measurable function ψ , we have the modified result that the operator $\mathcal{L} = \psi^{-1}(-\Delta + \Phi)$ on $L^2(\mathbb{R}^d, \psi dv)$, associated with the quadratic form

$$w \mapsto \int_{\mathbb{R}^d} (|\nabla w|^2 + \Phi |w|^2) dv$$

has only discrete eigenvalues in the interval $(-\infty, \sigma)$ where

$$\sigma = \lim_{r \rightarrow +\infty} \inf_{w \in \mathcal{D}(B_r^c) \setminus \{0\}} \frac{\int_{\mathbb{R}^d} (|\nabla w|^2 + \Phi |w|^2) dv}{\int_{\mathbb{R}^d} |w|^2 \psi dv} > 0.$$

To prove it, it is enough to observe that 0 is the lower end of the continuous spectrum of $\mathcal{L} - \sigma$ by applying again [19, Theorem 2.1]. It is also straightforward to check that the lower end of the continuous spectrum of \mathcal{L} is such that

$$\sigma \geq \lim_{r \rightarrow +\infty} \mathfrak{q}(r) =: \sigma_0 \quad \text{where} \quad \mathfrak{q}(r) := \inf_{B_r^c} \frac{\Phi}{\psi}.$$

Notice that σ_0 is either finite or infinite. In the case of (19), we get that $\sigma_0 \in (0, +\infty]$ if and only if $\beta \geq 2(1 - \alpha)$. Relating the weighted Poincaré inequality (18) with the spectrum of \mathcal{L} is then classical. Let

$$h = w e^{\phi/2}, \quad \Phi = \frac{1}{4} |\nabla \phi|^2 - \frac{1}{2} \Delta \phi \tag{20}$$

and observe that

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla h|^2 d\xi &= Z_\alpha^{-1} \int_{\mathbb{R}^d} (|\nabla w|^2 + \Phi |w|^2) dv, \\ \int_{\mathbb{R}^d} |h - \hat{h}|^2 d\xi &= Z_\alpha^{-1} \int_{\mathbb{R}^d} |w - \tilde{w}|^2 \psi dv, \end{aligned}$$

where $\tilde{w} = \frac{\int_{\mathbb{R}^d} w \psi e^{-\phi/2} dv}{\int_{\mathbb{R}^d} \psi e^{-\phi} dv} e^{-\phi/2}$.

Proposition 8. *With the above notations, let Φ and ψ be two measurable functions such that $\sigma_0 > 0$. Then inequality (18) holds for some positive, finite, optimal constant $C_\star > 0$, which is at least equal to σ . Otherwise, if we have that $\lim_{r \rightarrow +\infty} \sup_{v \in B_r^c} \frac{\Phi(v)}{\psi(v)} = 0$, then inequality (18) does not hold.*

Proof. By construction, σ is nonnegative and the infimum of the Rayleigh quotient

$$w \mapsto \frac{\int_{\mathbb{R}^d} (|\nabla w|^2 + \Phi |w|^2) dv}{\int_{\mathbb{R}^d} |w|^2 \psi dv}$$

is achieved by $h \equiv \widehat{h} = 1$, that is, by $w = \widetilde{w} = e^{-\phi/2}$, which moreover generates the kernel of \mathcal{L} . Hence we can interpret \mathcal{C}_\star as the first positive eigenvalue, if there is any in the interval $(0, \sigma)$, or $\mathcal{C}_\star = \sigma$ if there is none. \square

In the case of (19), the condition $\beta \geq 2(1-\alpha)$ is a necessary and sufficient condition for the inequality (18) to hold. The threshold case $\beta = 2(1-\alpha)$ is remarkable: inequality (18) can be rewritten for any $\alpha \in (0, 1)$ as

$$\forall h \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |\nabla h|^2 e^{-\langle v \rangle^\alpha} dv \geq \mathcal{C}_\star \int_{\mathbb{R}^d} \frac{|h - \widehat{h}|^2 e^{-\langle v \rangle^\alpha}}{(1 + |v|^2)^{1-\alpha}} dv$$

for some constant $\mathcal{C}_\star \in (0, \alpha^2/4]$ and

$$\widehat{h} := \frac{1}{z_\alpha} \int_{\mathbb{R}^d} \frac{h e^{-\langle v \rangle^\alpha}}{(1 + |v|^2)^{1-\alpha}} dv, \quad z_\alpha = \int_{\mathbb{R}^d} \frac{e^{-\langle v \rangle^\alpha}}{(1 + |v|^2)^{1-\alpha}} dv.$$

A.2. A weighted Poincaré inequality with a non-classical average.

Corollary 9. *Let Φ and ψ be respectively a measurable function and a bounded positive function such that, with the notations of section A.1, $\sigma_0 > 0$ and $\psi^{-1} \in L^1(\mathbb{R}^d, d\xi)$. Then the inequality*

$$\forall h \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |\nabla h|^2 d\xi \geq \mathcal{C} \int_{\mathbb{R}^d} |h - \widetilde{h}|^2 dv \tag{21}$$

holds for some optimal constant $\mathcal{C} \in (0, \mathcal{C}_\star]$, where $\widetilde{h} := \int_{\mathbb{R}^d} h d\xi$. Here \mathcal{C}_\star denotes the optimal constant in (18).

As we shall see in the proof, our method provides us with an explicit lower bound \mathcal{C} in terms of \mathcal{C}_\star . We emphasize that in (21), the right-hand side is not the *variance* of h with respect of the measure dv because we subtract the average with respect to the measure $d\xi$ which appears in the left-hand side. In case $\phi(v) = \langle v \rangle^\alpha$, inequality (21) is equivalent to [13, inequality (1.12)], which can be deduced using the strategy of [1, 2]. Also see Appendix B.1 for further details.

Proof. Let us consider a function h . With no loss of generality, we can assume that $\widetilde{h} = \int_{\mathbb{R}^d} h d\xi = 0$ up to the replacement of h by $h - \widetilde{h}$. We use the IMS decomposition method (see [18, 21]), which goes as follows. Let χ be a truncation function on \mathbb{R}_+ with the following properties: $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $[0, 1]$, $\chi \equiv 0$ on $[2, +\infty)$ and $\chi'^2 / (1 - \chi^2) \leq \kappa$ for some $\kappa > 0$. Next, we define $\chi_R(v) = \chi(|v|/R)$, $h_{1,R} = h \chi_R$

and $h_{2,R} = h \sqrt{1 - \chi_R^2}$, so that $h_{1,R}$ is supported in the ball B_{2R} of radius $2R$ centered at $v = 0$ and $h_{2,R}$ is supported in $B_R^c = \mathbb{R}^d \setminus B_R$. Elementary computations show that $h^2 = h_{1,R}^2 + h_{2,R}^2$ and $|\nabla h|^2 = |\nabla h_{1,R}|^2 + |\nabla h_{2,R}|^2 - h^2 |\nabla \chi|^2 / (1 - \chi^2)$, so that $||\nabla h|^2 - |\nabla h_{1,R}|^2 - |\nabla h_{2,R}|^2| \leq \kappa h^2 / R^2$.

Since $h_{2,R}$ is supported in B_R^c , we know that

$$\int_{\mathbb{R}^d} |\nabla h_{2,R}|^2 d\xi \geq \mathfrak{q}(R) \int_{\mathbb{R}^d} |h_{2,R}|^2 d\nu$$

for any $R > 0$, where \mathfrak{q} is the quotient involved in the definition of σ_0 . We recall that $\lim_{r \rightarrow +\infty} \mathfrak{q}(r) = \sigma_0 > 0$. Using the method of the Holley-Stroock lemma (see [12] and [8] for a recent presentation), we deduce from inequality (18) that

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla h_{1,R}|^2 d\xi &\geq \mathcal{C}_\star \int_{\mathbb{R}^d} |h_{1,R} - \widehat{h}_{1,R}|^2 d\nu \\ &\geq \mathcal{C}_\star \int_{B_{2R}} |h_{1,R} - \widehat{h}_{1,R}|^2 \psi d\xi \\ &\geq \mathcal{C}_\star \inf_{B_{2R}} \psi \min_{c \in \mathbb{R}} \int_{B_{2R}} |h_{1,R} - c|^2 d\xi \\ &\geq \mathbf{Q}(R) \int_{\mathbb{R}^d} |h_{1,R}|^2 d\nu - \mathcal{C}_\star \frac{\inf_{B_{2R}} \psi}{\xi(B_{2R})} \left(\int_{\mathbb{R}^d} h_{1,R} d\xi \right)^2 \end{aligned}$$

where $\mathbf{Q}(R) := \mathcal{C}_\star \inf_{B_{2R}} \psi / \sup_{B_{2R}} \psi$. By the assumption $\widetilde{h} = 0$, we know that

$$\int_{B_R} h d\xi = - \int_{B_R^c} h d\xi,$$

from which we deduce that

$$\left(\int_{\mathbb{R}^d} h_{1,R} d\xi \right)^2 = \left(\int_{B_R} h d\xi + \int_{B_R^c} \chi h d\xi \right)^2 \leq \left(\int_{B_R^c} |h| d\xi \right)^2 \leq \int_{\mathbb{R}^d} h^2 d\nu \int_{B_R^c} \psi^{-1} d\xi$$

where the last inequality is simply a Cauchy-Schwarz inequality. Let

$$\varepsilon(R) := \mathcal{C}_\star \frac{\inf_{B_{2R}} \psi}{\xi(B_{2R})} \int_{B_R^c} \psi^{-1} d\xi.$$

By the assumption that $\psi^{-1} \in L^1(\mathbb{R}^d, d\xi)$, we know that

$$\lim_{R \rightarrow +\infty} \varepsilon(R) = 0 \quad \text{and} \quad \lim_{R \rightarrow +\infty} \frac{\varepsilon(R)}{\mathbf{Q}(R)} = 0.$$

Collecting all our assumptions, we have

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla h|^2 d\xi &\geq \int_{\mathbb{R}^d} \left(|\nabla h_{1,R}|^2 + |\nabla h_{2,R}|^2 - \frac{\kappa}{R^2} h^2 \right) d\xi \\ &\geq \left(\min \{ \mathbf{Q}(R), \mathfrak{q}(R) \} - \varepsilon(R) - \frac{\kappa}{R^2} \right) \int_{\mathbb{R}^d} |h|^2 d\nu \end{aligned}$$

where $\min \{ \mathbf{Q}(R), \mathfrak{q}(R) \} - \varepsilon(R) - \kappa/R^2$ is positive for $R > 0$, large enough.

Finally, let us notice that for any $c \in \mathbb{R}$ we have

$$\int_{\mathbb{R}^d} |h - c|^2 d\nu = \int_{\mathbb{R}^d} h^2 d\nu - 2c \int_{\mathbb{R}^d} h d\nu + c^2 \geq \int_{\mathbb{R}^d} |h - \widehat{h}|^2 d\nu$$

with equality if and only if $c = \widehat{h} = \int_{\mathbb{R}^d} h d\nu$. As a special case corresponding to $c = \widetilde{h} = \int_{\mathbb{R}^d} h d\xi$, we have

$$\int_{\mathbb{R}^d} |h - \widetilde{h}|^2 d\nu \geq \int_{\mathbb{R}^d} |h - \widehat{h}|^2 d\nu.$$

This proves that $\mathcal{C}_\star \geq \mathcal{C}$. □

In the special case of (19), it is possible to give a slightly shorter proof using the Poincaré inequality on B_R , for the measure $d\xi$: see [16, Chapter 6]. An independent proof of such an inequality is then needed for a general ϕ . The proof of Corollary 9 is more general and reduces everything to a comparison of the asymptotic behavior of ϕ and ψ . If these functions are given by (19), inequality (21) can be rewritten in the form of (6), we have an estimate of \mathcal{C} and we can characterize \mathcal{C}_\star as follows.

Proposition 10. *The optimal constant \mathcal{C}_\star is the ground state energy of the operator $\mathcal{L} = \psi^{-1}(-\Delta + \Phi)$ on $L^2(\mathbb{R}^d, \psi d\nu)$.*

The proof relies on (20). Details are left to the reader.

APPENDIX B. ALGEBRAIC DECAY RATES FOR THE FOKKER-PLANCK EQUATION

Here we consider simple estimates of the decay rates in the homogenous case given by $f(t, x, v) = g(t, v)$ of equation (1), that is, the Fokker-Planck equation

$$\partial_t g = \mathbf{L}_1 g. \tag{22}$$

After summarizing the standard approach based on the *weak Poincaré inequality* (see for instance [13]) in Section B.1, we introduce a new method which relies on *weighted L^2 estimates*. As already mentioned, the advantage of weighted Poincaré inequalities is that the description of the convergence rates to the local equilibrium does not require extra regularity assumptions to cover the transition from super-exponential ($\alpha > 1$) and exponential ($\alpha = 1$) local equilibria to sub-exponential local equilibria, with $\alpha \in (0, 1)$.

B.1. Weak Poincaré inequality. We assume that α satisfies (5) and $\eta \in (0, \beta)$ with $\beta = 2(1 - \alpha)$. By a simple Hölder inequality, with $(\tau + 1)/\tau = \beta/\eta$, we obtain

that

$$\begin{aligned} \int_{\mathbb{R}^d} |h - \tilde{h}|^2 d\xi &= \int_{\mathbb{R}^d} |h - \tilde{h}|^2 \langle v \rangle^{-\eta} \langle v \rangle^\eta d\xi \\ &\leq \left(\int_{\mathbb{R}^d} |h - \tilde{h}|^2 \langle v \rangle^{-\beta} d\xi \right)^{\frac{\tau}{\tau+1}} \left(\int_{\mathbb{R}^d} \|h - \tilde{h}\|_{L^\infty(\mathbb{R}^d)}^2 \langle v \rangle^{\beta\tau} d\xi \right)^{\frac{1}{1+\tau}}. \end{aligned}$$

Here we choose $\tilde{h} := \int_{\mathbb{R}^d} h d\xi$. Using (6), we end up with

$$\forall h \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |h - \tilde{h}|^2 d\xi \leq \mathcal{C}_{\alpha,\tau} \left(\int_{\mathbb{R}^d} |\nabla h|^2 d\xi \right)^{\frac{\tau}{1+\tau}} \|h - \tilde{h}\|_{L^\infty(\mathbb{R}^d)}^{\frac{2}{1+\tau}}, \quad (23)$$

for some explicit positive constant $\mathcal{C}_{\alpha,\tau}$. We learn from (3) that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |h(t, \cdot) - \tilde{h}|^2 d\xi = -2 \int_{\mathbb{R}^d} |\nabla_v h|^2 d\xi$$

if $g = hF$ is solves (22), and we also know that \tilde{h} does not depend on t . By a strategy that goes back at least to [14, Theorem 2.2] and is due, according to the author, to D. Stroock, we obtain that

$$\int_{\mathbb{R}^d} |h(t, \cdot) - \tilde{h}|^2 d\xi \leq \left(\left(\int_{\mathbb{R}^d} |h(0, \cdot) - \tilde{h}|^2 d\xi \right)^{-\frac{1}{\tau}} + \frac{2\tau^{-1}}{\mathcal{C}_{\alpha,\tau}^{1+1/\tau} \mathcal{M}} t \right)^{-\tau}$$

with $\mathcal{M} = \sup_{s \in (0,t)} \|h(s, \cdot) - \tilde{h}\|_{L^\infty(\mathbb{R}^d)}^{2/\tau}$. The limitation is of course that we need to restrict the initial conditions in order to have \mathcal{M} uniformly bounded with respect to t . Since η can be chosen arbitrarily close to β , the exponent τ can be taken arbitrarily large but to the price of a constant $\mathcal{C}_{\alpha,\tau}$ which explodes as $\eta \rightarrow \beta_-$.

Notice that (23) is equivalent to the *weak Poincaré inequality*

$$\forall h \in \mathcal{D}(\mathbb{R}^d), \quad \mathcal{C}_{\alpha,\tau}^{-1} \int_{\mathbb{R}^d} |h - \tilde{h}|^2 d\xi \leq \frac{\tau}{(1+\tau)^{1+\frac{1}{\tau}}} r^{-\frac{\tau}{1+\tau}} \int_{\mathbb{R}^d} |\nabla h|^2 d\xi + r \|h - \tilde{h}\|_{L^\infty(\mathbb{R}^d)}^2,$$

for all $r > 0$, as stated in [20, (1.6) and example 1.4 (c)]. The equivalence of this inequality and (23) is easily recovered by optimizing on $r > 0$. It is worth to remark that here we consider $\|h - \tilde{h}\|_{L^\infty(\mathbb{R}^d)}$ while various other quantities like, *e.g.*, the median can be used in weak Poincaré inequalities.

B.2. Weighted L^2 estimates. As an alternative approach to the *weak Poincaré inequality* method of Appendix B.1, we can consider for some arbitrary $k > 0$ the evolution according to equation (22) of $\int_{\mathbb{R}^d} |h(t, v)|^2 \langle v \rangle^k d\xi = \int_{\mathbb{R}^d} |h(t, v)|^2 \langle v \rangle^k F dv$ where $h := g/F$ solves

$$\partial_t h = F^{-1} \nabla_v \cdot (F \nabla_v h).$$

Let us compute

$$\frac{d}{dt} \int_{\mathbb{R}^d} |h(t, v)|^2 \langle v \rangle^k F dv + 2 \int_{\mathbb{R}^d} |\nabla_v h|^2 \langle v \rangle^k F dv = - \int_{\mathbb{R}^d} \nabla_v (h^2) \cdot (\nabla_v \langle v \rangle^k) F dv$$

and observe with $\ell = 2 - \alpha$ that

$$\nabla_v \cdot \left(F \nabla_v \langle v \rangle^k \right) = \frac{k}{\langle v \rangle^4} \left(d + (k + d - 2) |v|^2 - \alpha \langle v \rangle^\alpha |v|^2 \right) \leq a - b \langle v \rangle^{-\ell}$$

for some $a \in \mathbb{R}$, $b \in (0, +\infty)$. The same proof as in Proposition 3 shows that there exists a constant $\mathcal{K}_k > 0$ such that

$$\forall t \geq 0 \quad \|h(t, \cdot)\|_{L^2(\langle v \rangle^k d\xi)} \leq \mathcal{K}_k \|h^{\text{in}}\|_{L^2(\langle v \rangle^k d\xi)}.$$

Hence, if h solves (22) with initial value h^{in} , we can use (6) to write

$$\frac{d}{dt} \int_{\mathbb{R}^d} |h(t, \cdot) - \tilde{h}|^2 d\xi = -2 \int_{\mathbb{R}^d} |\nabla_v h|^2 d\xi \leq -2\mathcal{C} \int_{\mathbb{R}^d} |h - \tilde{h}|^2 \langle v \rangle^{-\beta} d\xi$$

with $\beta = 2(1 - \alpha)$ and $\tilde{h} = \int_{\mathbb{R}^d} h d\xi$. With $\theta = k/(k + \beta)$, Hölder's inequality

$$\int_{\mathbb{R}^d} |h - \tilde{h}|^2 d\xi \leq \left(\int_{\mathbb{R}^d} |h - \tilde{h}|^2 \langle v \rangle^{-\beta} d\xi \right)^\theta \left(\int_{\mathbb{R}^d} |h - \tilde{h}|^2 \langle v \rangle^k d\xi \right)^{1-\theta}$$

allows us to estimate the right hand side and obtain the following result.

Proposition 11. *Assume that α satisfies (5). Let $g^{\text{in}} \in L^1_+(\text{d}\mu) \cap L^2(\langle v \rangle^k \text{d}\mu)$ for some $k > 0$ and consider the solution g to (22) with initial datum g^{in} . With \mathcal{C} as in (6), if $\bar{g} = (\int_{\mathbb{R}^d} g dv) F$ where F is given by (4), then*

$$\int_{\mathbb{R}^d} |g(t, \cdot) - \bar{g}|^2 d\mu \leq \left(\left(\int_{\mathbb{R}^d} |g^{\text{in}} - \bar{g}|^2 d\mu \right)^{-\beta/k} + \frac{2\beta\mathcal{C}}{k\mathcal{K}^{\beta/k}} t \right)^{-k/\beta}$$

with $\beta = 2(1 - \alpha)$ and $\mathcal{K} := \mathcal{K}_k^2 \|g^{\text{in}}\|_{L^2(\langle v \rangle^k \text{d}\mu)}^2 + \Theta_k (\int_{\mathbb{R}^d} g^{\text{in}} dv)^2$.

We recall that $g = hF$, $\bar{g} = \tilde{h}F$ and $F d\mu = dv = F^{-1} d\xi$. We notice that arbitrarily large decay rates can be obtained under the condition that $k > 0$ is large enough. We recover that when $k < d\beta/2$, the rate of relaxation to the equilibrium is slower than $(1 + t)^{-d/2}$ and responsible for the limitation that appears in Theorem 1. However, the rate of the heat flow is recovered in Theorem 1 for a weight of order k with an arbitrarily small $k > 0$ if α is taken close enough to 1.

Proof. Using

$$\frac{1}{2} \int_{\mathbb{R}^d} |h - \tilde{h}|^2 \langle v \rangle^k d\xi \leq \int_{\mathbb{R}^d} |h|^2 \langle v \rangle^k d\xi + \Theta_k \tilde{h}^2 = \mathcal{K},$$

we obtain that $y(t) := \int_{\mathbb{R}^d} |g(t, \cdot) - \bar{g}|^2 d\mu$ obeys to $y' \leq -2\mathcal{C} \mathcal{K}^{1-1/\theta} y^{1/\theta}$ and conclude by a Grönwall estimate. \square

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