HAL
open science

Diagrams for nonabelian Hodge spaces on the affine line

Philip Boalch, Daisuke Yamakawa

To cite this version:

Philip Boalch, Daisuke Yamakawa. Diagrams for nonabelian Hodge spaces on the affine line. Comptes Rendus. Mathématique, 2020, 358 (1), pp.59-65. hal-02377190

HAL Id: hal-02377190

https://hal.science/hal-02377190v1

Submitted on 23 Nov 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DIAGRAMS FOR NONABELIAN HODGE SPACES ON THE AFFINE LINE

PHILIP BOALCH AND DAISUKE YAMAKAWA

Abstract. In this announcement a diagram will be defined for each nonabelian Hodge space on the affine line.

1. Introduction

In a previous paper ([3] Apx. C), a diagram was defined for each algebraic connection on a vector bundle on the affine line, under the condition that the connection was untwisted at infinity (i.e. had unramified irregular type). In that case the diagram was a graph (or a "doubled quiver"). It is known that such a connection determines a triple of complex algebraic moduli spaces $\mathcal{M}_{\mathrm{DR}}, \mathcal{M}_{\mathrm{Dol}}, \mathcal{M}_{\mathrm{B}}$, which are algebraically distinct yet naturally diffeomorphic. In other words the connection determines a single *nonabelian Hodge space* $\mathfrak{M}$, with a triple of distinct algebraic structures. Hence this gives a way to attach a diagram to a class of nonabelian Hodge spaces. The simplest examples of such graphs match up with the affine Dynkin diagrams corresponding to the Okamoto Weyl group symmetries of the Painlevé equations (corresponding to some of the H3 surfaces, i.e. the spaces $\mathfrak{M}$ of real dimension four). More background and motivation related to integrable systems and isomonodromy is recounted in [6].

The purpose of this note is to extend this story by defining a diagram for any algebraic connection on a vector bundle on the affine line, i.e. for any nonabelian Hodge space attached to the affine line. One can show using the Fourier–Laplace transform that any moduli space $\mathcal{M}_{\mathrm{DR}}$ of meromorphic connections on a smooth affine curve of genus zero is isomorphic to one on the affine line (i.e. with just one puncture), and this is expected to hold for the full nonabelian Hodge triple. It is thus hoped that these diagrams serve a useful purpose in the classification of nonabelian Hodge spaces (and this is certainly the case in the examples considered in [3, 4, 5]).

2. The construction

Let $\Sigma = \mathbb{P}^1(\mathbb{C})$ and fix a point $p = \infty \in \Sigma$ so that $\Sigma^\circ = \Sigma \setminus p$ is the affine line. The diagram of an algebraic connection $(E, \nabla) \to \Sigma^\circ$ is determined by its formal isomorphism class at ∞. This formal class is equivalent to the irregular class plus the formal monodromy conjugacy classes, as follows.

Let ∂ be the circle of real oriented directions at p. Recall that the exponential local system $\mathcal{I}$ is a covering space $\pi : \mathcal{I} \to \partial$, consisting of a disjoint union of circles $\langle q \rangle$ each of which is a finite cover of ∂. (This notation is from [8, 7] where further discussion

and references may be found). Deligne's way of stating the Hukuhara–Turritin–Levelt formal classification of connections is as follows:

Theorem 1 ([15] Thm 2.3)**.** *The category of connections on vector bundles on the formal punctured disk at p is equivalent to the category of $\mathcal{I}$-graded local systems $V^0 \to \partial$ of finite dimensional complex vector spaces.*

Such a graded local system V^0 is the same thing as a local system (of finite dimensional complex vector spaces) on the topological space $\mathcal{I}$, having *compact support* (in the sense that it has rank zero on all but a finite number of component circles of $\mathcal{I}$).

By definition the *irregular class* of a connection is the map $\Theta : \pi_0(\mathcal{I}) \to \mathbb{N}$ taking the rank of V^0 on each circle ([8] §3.5). In down to earth terms the circles $\langle q \rangle$ correspond to the exponential factors e^q of the corresponding connection, so fixing the irregular class amounts to fixing the exponential factors plus their integer multiplicities. Thus an irregular class can be written as a formal sum

$$\Theta = n_1\langle q_1 \rangle + \cdots + n_m\langle q_m \rangle$$

of a finite number of distinct circles $\langle q_i \rangle$, with integer multiplicities $n_i = \Theta(q_i) \geq 1$.

Now a local system of rank n on a circle $\langle q \rangle$ is a very simple object, and is classified by the conjugacy class in $\mathrm{GL}_n(\mathbb{C})$ of its monodromy (in a positive sense once around the circle). Thus the graded/formal local system V^0 determines conjugacy classes

$$\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_m), \qquad \mathcal{C}_i \subset \mathrm{GL}_{n_i}(\mathbb{C})$$

where $\mathcal{C}_i$ is the class of the monodromy of the local system on the circle $\langle q_i \rangle$.

Now there is a well-known method due to Kraft–Procesi and others of attaching a graph L (a type A Dynkin graph) to a *marked* conjugacy class in $\mathrm{GL}_n(\mathbb{C})$. It is reviewed in [5] Defn 9.2. Moreover the graph is independent of the marking if the marking is chosen to be *minimal*, in the sense of [5] Defn 9.2. The number of nodes of L is the degree of a minimal polynomial of any element of the class. Thus V^0 determines *legs* $L_1, \ldots, L_m$, where L_i is the type A Dynkin graph determined by a minimal marking of the class $\mathcal{C}_i$.

The next step is to define the *core diagram* Γ. For this first recall (e.g. from [7]) that:

1) For any circle $\langle q \rangle \subset \mathcal{I}$ the ramification degree $\mathrm{ram}(q)$ is the degree of the covering map $\pi : \langle q \rangle \to \partial$,

2) The set of *points of maximal decay* is a discrete subset $\mathfrak{d} \subset \mathcal{I}$. It consists of a finite subset $\mathfrak{d}(q) \subset \langle q \rangle$ in each circle, where the function e^q has maximal decay,

3) The size of the set $\mathfrak{d}(q)$ is called the irregularity $\mathrm{Irr}(q)$ of the irregular class $\langle q \rangle$ (and is zero if and only if $q = 0$). For an arbitrary class $\Theta = \sum n_i\langle q_i \rangle$ the irregularity is $\mathrm{Irr}(\Theta) = \sum n_i \mathrm{Irr}(q_i)$,

4) For any pair of circles $\langle q_1\rangle, \langle q_2\rangle$ the irregular class $\mathrm{Hom}(\langle q_1\rangle, \langle q_2\rangle)$ is well-defined (the definition is straightforward if one thinks in terms of corresponding graded local systems).

Now the *core diagram* Γ is defined as follows:

- Γ has m nodes, labelled by the circles $\langle q_1\rangle, \ldots, \langle q_m\rangle$,
- If $i \neq j$ then the number of arrows from $\langle q_i\rangle$ to $\langle q_j\rangle$ is given by

$$B_{ij} = A_{ij} - \beta_i\beta_j \tag{1}$$

where $\beta_i = \mathrm{ram}(q_i), A_{ij} = \mathrm{Irr}(\mathrm{Hom}(\langle q_i\rangle, \langle q_j\rangle))$,

- If $i = j$ the number of arrows from $\langle q_i\rangle$ to $\langle q_i\rangle$ (oriented loops) is

$$B_{ii} = A_{ii} - \beta_i^2 + 1. \tag{2}$$

Definition 2. *The diagram $\widehat{\Gamma}$ of $(\Theta, \mathcal{C})$ is obtained by gluing the end node of the leg L_i to the node $\langle q_i\rangle$ of the core diagram Γ, for $i = 1, \ldots, m$.*

As usual (when drawing diagrams) a pair of oppositely oriented arrows is identified with a single unoriented edge. In particular a pair of oriented loops is the same thing as a single (unoriented) edge loop. One can show (e.g. using the symplectic results of [8]) that the integer $A_{ii}-\beta_i^2+1$ is always even, and so all the diagrams here only involve unoriented edges (it is clear that $B_{ij} = B_{ji}$). It should be noted that we call this a diagram, and not a graph, since some of the edges may have *negative* multiplicity. (The negative edges will be indicated by dashed lines in figures.) The meaning of a negative edge is that one has more relations than linear maps—the diagram arises since there are explicit matrix presentations of the wild character varieties, many of which date back to Birkhoff (cf. the history discussed in [7]). It is something of a surprise that there are (lots of) perfectly good moduli spaces whose diagrams have negative edges.

The untwisted case considered in [3] is the case where each $\beta_i = 1$, so that $\langle q_i\rangle \to \partial$ is a trivial (degree one) cover. In this case each q_i can be identified with a polynomial in x with zero constant term (where x is a coordinate on $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$). In this case $\mathrm{End}(\langle q_i\rangle) = \langle 0\rangle$ so that $A_{ii} = 0$ and so there are no edge loops. Further $\mathrm{Hom}(\langle q_i\rangle, \langle q_j\rangle) = \langle q_j - q_i\rangle$ is again unramified, and its irregularity is just the degree of the polynomial $q_j - q_i$, so that

$$A_{ij} - \beta_i\beta_j = \deg(q_i - q_j) - 1. \tag{3}$$

It follows that the core diagram coincides with the graph Γ defined in [3] Apx C. The irregular type $\sum A_i dz/z^{k-i}$ considered there is dQ where Q is the diagonal matrix with entries given by the q_i, written in the coordinate $z = 1/x$. The simple expression $\deg(q_i - q_j) - 1$ for the edge multiplicities of this graph appears in [12] §3.3.

2.1. **Adding some tame singularities.** Here is how to extend this construction to the case where a finite number of tame singularities on $\mathbb{A}^1$ are included as well.

(This is similar to the procedure for adding tame singularities at finite distance in [3, 4, 5]). As mentioned in the introduction, this case (and any other case on $\mathbb{P}^1$) can be reduced to the case already considered above via Fourier–Laplace.

Let $n = \mathrm{rk}(E) = \mathrm{rk}(V^0) = \sum_1^m n_i\beta_i$ be the rank of the irregular class Θ considered above. Choose points $a_1, \ldots, a_k \in \mathbb{A}^1(\mathbb{C})$, and fix a tame formal class at each point. This is the same as fixing conjugacy classes $\widehat{\mathcal{C}}_1, \ldots, \widehat{\mathcal{C}}_k \subset \mathrm{GL}_n(\mathbb{C})$, i.e. the local monodromy conjugacy classes. (This is the same as fixing the isomorphism class of a graded local system at each point, but graded entirely by the corresponding tame circle $\langle 0\rangle$ with multiplicity n).

Let $\widehat{L}_i$ be the leg determined by a minimal marking of $\widehat{\mathcal{C}}_i$ (in the sense of [5] Defn 9.2). Assuming each conjugacy class is non-central, each $\widehat{L}_i$ has at least two nodes.

Let $\beta = \sum \beta_i = \mathrm{ram}(\Theta)$ so that $\beta \leq n$. Now splay the end node of $\widehat{L}_i$ into β nodes, thus replacing the end node by β nodes (cf. [3] Figure 6 §A.5). Glue the first β_1 such nodes to the core node $\langle q_1\rangle$. Then glue the next β_2 such nodes to the core node $\langle q_2\rangle$, etc, thus gluing each of the β nodes to one of the core nodes. In effect the second node of $\widehat{L}_i$ is now linked to $\langle q_j\rangle$ by β_j (unoriented) edges for $j = 1, \ldots, m$. Repeat this process for each $i = 1, \ldots, k$.

This defines directly the diagram $\widehat{\Gamma}$ of any meromorphic connection (E, ∇) on $\mathbb{P}^1$ that is tame at all but one point (i.e. associated to the choice of the formal class at ∞ plus tame classes at each a_i).

3. Cartan matrix and dimensions

Given a diagram $\widehat{\Gamma}$ with nodes N and "adjacency matrix" B (so $B_{ij} \in \mathbb{Z}$ is the possibly negative number of arrows from node i to j), define the Cartan matrix of $\widehat{\Gamma}$ to be $C = 2.\mathrm{Id} - B$. Let $(\cdot,\cdot) : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{Z}$ be the resulting symmetric bilinear form defined by $(\varepsilon_i, \varepsilon_j) = C_{ij}$, where the ε_i are the basis vectors of $\mathbb{Z}^N$. A *dimension vector* for $\widehat{\Gamma}$ is a vector $\mathbf{d} \in \mathbb{N}^N$, i.e. a map $N \to \mathbb{N}$ assigning a nonnegative integer to each node.

Now suppose $\widehat{\Gamma}$ is the diagram determined by a wild Riemann surface $\mathbf{\Sigma} = (\mathbb{P}^1, \infty, \Theta)$ with marked formal monodromy classes $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_m)$ as above. Then $\widehat{\Gamma}$ comes equipped with a dimension vector $\mathbf{d}$ (the dimension of a core node $\langle q_i\rangle$ is just $n_i = \Theta(q_i)$ and the procedure described in [5] Defn 9.2 gives the dimensions down the legs). On the other hand one can consider the (symplectic) wild character variety

$$\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, \mathcal{C}) \subset \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/H$$

determined by this data, as in [8] (except here we only consider the stable points). It is the symplectic leaf of the Poisson wild character variety $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)/H$ determined by the classes $\mathcal{C}$ (see also [7] which focuses on the general linear case). Here $\Pi = \pi_1(\widetilde{\Sigma}, b)$ is the wild surface group (the fundamental group of the auxiliary surface $\widetilde{\Sigma}$), $G = \mathrm{GL}_n(\mathbb{C})$ and $H = \prod_1^m \mathrm{GL}_{n_i}(\mathbb{C})^{\beta_i} \subset G$. The classes $\mathcal{C}$ determine a twisted

conjugacy class of H (by saying the monodromy around the circle $\langle q_i \rangle$ is in $\mathcal{C}_i$), and thus a symplectic leaf of $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)/H$. (Fixing a symplectic leaf is the same as fixing the isomorphism class of the corresponding formal/$\mathcal{I}$-graded local system V^0.) The explicit presentations of the wild character varieties then leads to the statement:

Proposition 3. *If $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, \mathcal{C})$ is nonempty then* $\dim_{\mathbb{C}}(\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, \mathcal{C})) = 2 - (\mathbf{d}, \mathbf{d})$.

The idea is that $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, \mathcal{C})$ is thus a type of multiplicative quiver variety for the "doubled quiver $\widehat{\Gamma}$". This proposition can be proved directly, as sketched in the appendix.

4. Examples

It is easy to compute many examples. Here are some of the simplest. Note that if the multiplicities $n_i = 1$ (so the formal monodromies are scalars) and there are no tame singularities, then we just need compute the core diagram. Note also that everything is invariant under admissible deformations so we won't worry about constant factors—for example the (modern) Airy equation has $\Theta = \langle (2/3)x^{3/2} \rangle$, and the version $y'' = 9xy$ used by Stokes in 1857 has $\Theta = \langle 2x^{3/2} \rangle$, both of which are admissible deformations of $\langle x^{3/2} \rangle$. In the Painlevé cases below, unless explicitly stated otherwise, we use the linear equation in the standard Lax pair, due to Garnier/Jimbo–Miwa [10, 13]. (Recall the Painlevé equations arise as the isomonodromy equations of linear connections—the diagram of the Painlevé equation is that of the corresponding linear connection.)

• Airy: $\Theta = \langle x^{3/2} \rangle$. $\mathrm{End}(\Theta) = \langle 2x^{3/2} \rangle + 2\langle 0 \rangle$ so $A_{11} = 3$ and in turn $B_{11} = 3 - 4 + 1 = 0$ so the Cartan matrix $C = (2)$ is the same as that for $\mathfrak{sl}_2$, i.e. A_1. The diagram has one node with zero loops (the A_1 Dynkin diagram). $\dim(\mathcal{M}_{\mathrm{B}}) = 0$.

• Painlevé one: $\Theta = \langle x^{5/2} \rangle$. $\mathrm{End}(\Theta) = \langle 2x^{5/2} \rangle + 2\langle 0 \rangle$ so $A_{11} = 5$ and in turn $B_{11} = 5-4+1 = 2$ so the Cartan matrix $C = (0)$ is the zero 1×1 matrix. The diagram has one node with one loop (the affine A_0 Dynkin diagram $\widehat{A}_0$). $\dim(\mathcal{M}_{\mathrm{B}}) = 2$. This diagram appears also on the additive side—the corresponding additive moduli space $\mathcal{M}^*$ is isomorphic to the affine plane $\mathbb{A}^2$ (the $\widehat{A}_0$ ALE space, familiar from the ADHM construction).

• Weber: $\Theta = \langle x^2 \rangle + \langle -x^2 \rangle$. This is untwisted so fits in to the set-up of [3]. $A = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ so that $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. This is the Cartan matrix of $\mathfrak{sl}_3$, i.e. A_2. The diagram has two nodes connected by one edge (the A_2 Dynkin diagram). $\dim(\mathcal{M}_{\mathrm{B}}) = 0$.

• Painlevé two: $\Theta = \langle x^3 \rangle + \langle -x^3 \rangle$. This is untwisted so fits in to the set-up of [3]. $A = \left(\begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix}\right)$ so that $B = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. The diagram has two nodes connected by two edges (the $\widehat{A}_1$ Dynkin diagram). This diagram appears also on the additive side ([2] Ex. 3)—the corresponding additive moduli space $\mathcal{M}^*$ is diffeomorphic to the Eguchi–Hanson space (the $\widehat{A}_1$ ALE space). $\dim(\mathcal{M}_{\mathrm{B}}) = 2$.

• Painlevé two revisited: $\Theta = \langle x^{3/2} \rangle$ plus a tame pole at $x = 0$ (this is the Flaschka–Newell Lax pair, from the modified KdV equation). The procedure of §2.1 again gives the $\widehat{A}_1$ diagram: as in the Airy equation we get one node with no loops at ∞ (but with ramification $\beta = 2$). At the simple pole we get a leg of length 2. We splay its end node into two nodes, and glue both of them to the node from ∞ yielding $\widehat{A}_1$.

• Bessel–Clifford equation (${}_0F_1$-equation/confluent hypergeometric limit equation/Kummer's second equation, $xy'' + ay' = y$): $\Theta = \langle x^{1/2} \rangle$ plus a tame pole at $x = 0$. The procedure of §2.1 gives a diagram with two nodes attached by two edges. One node has no loops and the other has a single *negative* loop. $C = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 4 \end{smallmatrix}\right)$. $\dim(\mathcal{M}_{\mathrm{B}}) = 0$.

• Painlevé three: $\Theta = \langle x^{1/2} \rangle$ plus two tame poles at $x = 0, 1$. (This is the Lax pair for P_3 known as "degenerate Painlevé five", [14] (6.17)). The procedure of §2.1 gives a diagram with three nodes: two nodes each attached with two edges to a central node, and the central node has a single *negative* loop:

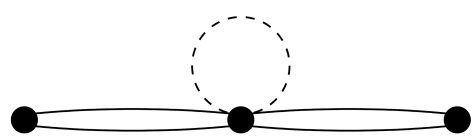

FIGURE 1. The diagram of the Painlevé three equation.

The dashed line indicates that the loop has negative multiplicity. The corresponding Cartan matrix is

$$C = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}. \tag{4}$$

The corresponding additive moduli space $\mathcal{M}^*$ for Painlevé three (from the standard Lax pair) is known[1] to be the affine D_2 ALF space, and so it is natural to view this graph as the Dynkin diagram of type $\widehat{D}_2$. As a further consistency check one can consider the intersection form from the corresponding De Rham moduli space (the Okamato space of initial conditions). It is known ([17] p.182) that the intersection form is the negative of $\widehat{A}_1 \oplus A_1$, i.e. $-\left(\begin{smallmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right)$. This fits since one can choose another $\mathbb{Z}$-basis in which the intersection form is given by the negative of our $\widehat{D}_2$ Cartan matrix (4):

Lemma 4. *The bilinear forms with matrices $\widehat{D}_2$ and $\widehat{A}_1 \oplus A_1$ are equivalent over $\mathbb{Z}$.*

[1]This is stated in [3]—it follows immediately since [1] implies $\mathcal{M}^* \cong \mathbb{C}^* \setminus\!\!\setminus T^*\mathrm{SL}_2(\mathbb{C}) /\!\!/ \mathbb{C}^*$, and this is the description of the $\widehat{D}_2$ ALF space in [9] p.88.

Proof. $g^T \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} g = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}$ where $g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$. □

This diagram should also be compared/contrasted with the "shape" for Painlevé 3 suggested in [11] Example 6.17 (and last shape in figure on p.928), and with that in the approach of [18].

The diagrams for Painlevé 4,5,6 have already been discussed in detail in [2, 3, 4, 5]. The diagrams for the six Painlevé equations are thus as follows:

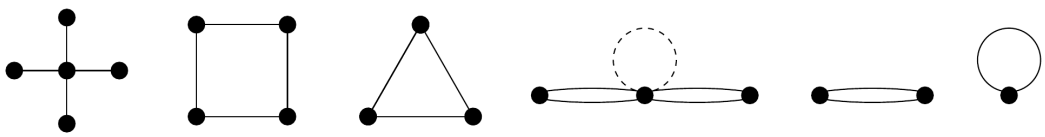

FIGURE 2. The diagrams of the six Painlevé equations.

Note that the number of nodes minus one is always the number of parameters in the corresponding Painlevé equation. Each node has dimension one, except for the central node in the Painlevé VI case, which has dimension 2.

Remark 5. Each of the Painlevé equations 2-6 has a special class of solutions coming from solutions of a linear differential equation ([16] p.310). We now understand the key statement: *In each case the diagram of the linear equation is obtained by removing one node of the diagram for the corresponding Painlevé equation:*

Painlevé equation	P_6	P_5	P_4	P_3	P_2
Special solutions	Gauss ${}_2F_1$	Kummer ${}_1F_1$	Weber	Bessel–Clifford ${}_0F_1$	Airy

Note that the discussion in [3, 4] implies the Kummer equation has diagram A_3 and the Gauss equation has diagram D_4. Also [16] just writes "Bessel" for the special solutions of P_3, but the Bessel equation $x^2y'' + xy' = (\nu^2 - x^2)y$ is really just a special one parameter subfamily of the Kummer (${}_1F_1$) equations (up to a twist)—it is the subfamily that are pullbacks of a ${}_0F_1$ equation (studied by Clifford): If f satisfies $xf'' + af' = f$ then $x^{a-1} \cdot f(-x^2/4)$ satisfies the Bessel equation with parameter $\nu = a - 1$, for example:

$$J_\nu(x) = \frac{(\frac{1}{2}x)^\nu}{\Gamma(\nu+1)} \cdot {}_0F_1(\nu+1; -x^2/4).$$

APPENDIX A. SKETCH OF PROOF OF PROP. 3

Recall from [8] that the space $\mathcal{B} := \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ is isomorphic to $\mathcal{A}/\!\!/G$, where $\mathcal{A} \cong G \times H(\partial) \times \mathbb{S}\mathrm{to}$ is the fission space, $H(\partial)$ is a twist of H and $\mathbb{S}\mathrm{to}$ is the product of the Stokes groups (which has dimension $\mathrm{Irr}\,\mathrm{End}(\Theta)$). [8] shows that $\mathcal{A}$ is a twisted quasi-Hamiltonian $G \times H$ space, with a moment map $\mu = (\mu_G, \mu_H)$

to $G \times H(\partial)$. The G action is free which implies $\dim \mathcal{B} = \dim \mathcal{A} - 2\dim G$. In turn $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, \mathcal{C}) = \mathcal{B}/\!\!/_{\widetilde{\mathcal{C}}} H = \mu_H^{-1}(\widetilde{\mathcal{C}})/H$ is the tq-Hamiltonian reduction (of the stable points) by H at the twisted conjugacy class $\widetilde{\mathcal{C}} \subset H(\partial)$ determined by $\mathcal{C}$. This has dimension $\dim(\mathcal{B} \times \widetilde{\mathcal{C}}) - 2\dim(H) + 2$, since $H/Z(G)$ acts effectively on stable points, and the result follows.

A better approach is to frame the corresponding Stokes local systems slightly differently, as follows (this won't work for arbitrary reductive groups): Just choose one basepoint on each circle $\langle q_i \rangle$, and frame there. The resulting space of framed Stokes local systems has the form $\mathcal{E} := \mathcal{B}/H^{\perp}$ where $H^{\perp} \cong \prod_1^m \mathrm{GL}_{n_i}(\mathbb{C})^{\beta_i - 1} \subset H$ (forgetting most of the old framings). This has a residual action of $\check{H} := \prod_1^m \mathrm{GL}_{n_i}(\mathbb{C})$ (changing the remaining framings), and one can deduce from [8] that $\mathcal{E}$ is a quasi-Hamiltonian $\check{H}$-space, with moment map given by the formal monodromy, all the way around each circle $\langle q_i \rangle$ (cf. [7] p.1—there is only one outer boundary circle). Then $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, \mathcal{C}) = \mathcal{E}/\!\!/_{\mathcal{C}} \check{H}$ is just the q-Hamiltonian reduction at the class $\mathcal{C} \subset \check{H}$. The space $\mathcal{E}$ behaves as if it were the space of invertible representations of the "doubled quiver" given by the core Γ—one can identify directly the positive terms in (1),(2) with generators (*Stokes arrows* [7], plus formal monodromies) and the negative terms with relations (from $\mu_G = 1$). The dimension count is then standard, as in [5] §9.1.

Acknowledgments. The second named author was supported by JSPS KAKENHI Grant Number 18K03256.

References

1. P. P. Boalch, *Symplectic manifolds and isomonodromic deformations*, Adv. in Math. **163** (2001), 137–205.
2. ———, *Quivers and difference Painlevé equations*, Groups and symmetries: From the Neolithic Scots to John McKay, CRM Proc. Lect. Notes, 47, 2009, arXiv:0706.2634, 2007.
3. ———, *Irregular connections and Kac–Moody root systems,* arXiv:0806.1050, 2008.
4. ———, *Simply-laced isomonodromy systems*, Publ. Math. I.H.E.S. **116** (2012), no. 1, 1–68.
5. ———, *Global Weyl groups and a new theory of multiplicative quiver varieties*, Geometry and Topology **19** (2015), 3467–3536.
6. ———, *Wild character varieties, meromorphic Hitchin systems and Dynkin diagrams*, (2018), Geometry and Physics II: A Festschrift in honour of Nigel Hitchin, pp.425-446, arXiv:1703.10376.
7. ———, *Topology of the Stokes phenomenon* arXiv:1903.12612, 2019.
8. P. P. Boalch and D. Yamakawa, *Twisted wild character varieties*, arXiv:1512.08091.
9. A. S. Dancer, *Dihedral singularities and gravitational instantons*, J. Geom. Phys. **12** (1993), no. 2, 77–91.
10. R. Garnier, *Sur des équations ... et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes*, Ann. Sci. ÉNS (3) **29** (1912), 1–126.
11. K. Hiroe, *Linear differential equations on the Riemann sphere and representations of quivers*, Duke Math. J. **166** (2017), no. 5, 855–935.
12. K. Hiroe and D. Yamakawa, *Moduli spaces of meromorphic connections and quiver varieties*, Adv. Math. **266** (2014), 120–151, arXiv:1305.4092.
13. M. Jimbo and T. Miwa, *Monodromy preserving deformations of linear differential equations with rational coefficients II*, Physica 2D (1981), 407–448.

14. N. Joshi, A. V. Kitaev, and P. A. Treharne, *On the linearization of the Painlevé III-VI equations and reductions of the three-wave resonant system*, J. of Math. Phys. **48** (2007), 42.
15. B. Malgrange, *Équations différentielles à coefficients polynomiaux*, Progr. in Math., vol. 96, 1991.
16. K. Okamoto, *The Painlevé equations and the Dynkin diagrams*, Painlevé transcendents (Sainte-Adèle, PQ, 1990), NATO ASI Ser. B, vol. 278, Plenum, New York, 1992, pp. 299–313.
17. H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001), no. 1, 165–229.
18. D. Yamakawa, *Quiver varieties with multiplicities, Weyl groups of non-symmetric Kac-Moody algebras, and Painlevé equations*, SIGMA **6** (2010), Paper 087, 43.

Laboratoire de Mathématiques d'Orsay,
Univ. Paris-Sud, CNRS,
Université Paris-Saclay,
91405 Orsay, France

philip.boalch@math.u-psud.fr

Department of Mathematics,
Faculty of Science Division I,
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku,
Tokyo 162-8601, Japan

yamakawa@rs.tus.ac.jp