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# Speed of convergence towards attracting sets for endomorphisms of $\mathbb{P}^k$

Johan Taflin

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## Abstract

Let  $f$  be a non-invertible holomorphic endomorphism of  $\mathbb{P}^k$  having an attracting set  $A$ . We show that, under some natural assumptions,  $A$  supports a unique invariant positive closed current  $\tau$ , of the right bidegree and of mass 1. Moreover, if  $R$  is a current supported in a small neighborhood of  $A$  then its push-forwards by  $f^n$  converge to  $\tau$  exponentially fast. We also prove that the equilibrium measure on  $A$  is hyperbolic.

## 1 Introduction

Let  $f$  be a holomorphic endomorphism of algebraic degree  $d \geq 2$  on the complex projective space  $\mathbb{P}^k$ . A compact subset  $A$  of  $\mathbb{P}^k$  is called an *attracting set* if it has a *trapping neighborhood*  $U$  i.e.  $f(U) \Subset U$  and  $A = \bigcap_{n \geq 0} f^n(U)$  where  $f^n := f \circ \dots \circ f$ ,  $n$  times. It follows that  $A$  is invariant,  $f(A) = A$ . Furthermore, if  $A$  contains a dense orbit then  $A$  is a *trapped attractor*. Typical examples of such objects are fractal and their underlying dynamics are hard to study. We refer to [Mil85], [Rue89] for general discussions on attractors and to [FW99], [JW00], [FS01], [BDM07], [Taf10] and references therein for examples of different types of attractors in  $\mathbb{P}^2$ .

Attracting sets are stable under small perturbations. Indeed, if  $f$  has an attracting set  $A = \bigcap_{n \geq 0} f^n(U)$  then any small perturbation  $f_\epsilon$  of  $f$  has an attracting set defined by  $A_\epsilon = \bigcap_{n \geq 0} f_\epsilon^n(U)$ . For example, when  $f$  restricted to  $\mathbb{C}^k$  defines a polynomial self-map then the hyperplane at infinity  $\mathbb{P}^k \setminus \mathbb{C}^k$  is an attracting set. In the same way, it is easy to create examples where the attracting set is a projective subspace of arbitrary dimension. In this paper, we consider a family of endomorphisms, stable under small perturbations,

which contains these examples. It was introduced by Dinh in [Din07] and we briefly recall the context.

In the sequel, we always assume that  $f$  possesses an attracting set  $A$  which has a trapped neighborhood  $U$  satisfying the following properties. There exist an integer  $1 \leq p \leq k-1$  and two projective subspaces  $I$  and  $L$  of dimension  $p-1$  and  $k-p$  respectively such that  $I \cap U = \emptyset$  and  $L \subset U$ . We do not assume that  $L$  and  $I$  are invariant. Since  $I \cap L = \emptyset$ , for each  $x \in L$  there exists a unique projective subspace  $I(x)$  of dimension  $p$  which contains  $I$  and such that  $L \cap I(x) = \{x\}$ . Furthermore, for each  $x \in L$  we ask that  $U \cap I(x)$  is strictly convex as a subset of  $I(x) \setminus I \simeq \mathbb{C}^p$ . All these assumptions are stable under small perturbations of  $f$ . The geometric assumption on  $U$  is slightly stronger than the one of Dinh, who only requires  $U \cap I(x)$  to be star-shaped at  $x$ . We need convexity in order to solve the  $\bar{\partial}$ -equation on  $U$ . Indeed, under our assumption  $U$  is a  $(p-1)$ -convex domain in the sense of [HL88].

If  $E$  is a subset of  $\mathbb{P}^k$ , let  $\mathcal{C}_q(E)$  denote the set of all positive closed currents of bidegree  $(q, q)$ , supported in  $E$  and of mass 1. It is well known that for any integer  $1 \leq q \leq k$  and any smooth form  $S$  in  $\mathcal{C}_q(\mathbb{P}^k)$ , the sequence  $d^{-qn}(f^n)^*(S)$  converges to a positive closed current  $T^q$  of bidegree  $(q, q)$  called the *Green current of order  $q$*  of  $f$ . We refer to [DS10] for a detailed exposition on these currents and their effectiveness in holomorphic dynamics.

When  $q = k$ , it gives the *equilibrium measure* of  $f$ ,  $\mu := T^k$ . It is exponentially mixing and it is the unique measure of maximal entropy  $k \log d$  on  $\mathbb{P}^k$ . Moreover, it is hyperbolic and all its Lyapunov exponents are larger or equal to  $(\log d)/2$ . The dynamics outside the support of  $\mu$  is not very well understood. The aim of this paper is to continue the investigation started in [Din07] on the attracting sets described above, which do not intersect  $\text{supp}(\mu)$ . Indeed, since  $I \cap U = \emptyset$ , by regularization there exists a smooth form  $S \in \mathcal{C}_{k-p+1}(\Omega)$ , where  $\Omega := \mathbb{P}^k \setminus \bar{U}$ . As  $f^{-1}(\Omega) \subset \Omega$ , it follows that  $\text{supp}(T^{k-p+1}) \cap U = \emptyset$ , and hence  $\text{supp}(T^q) \cap U = \emptyset$  if  $q \geq k-p+1$ .

The set  $\mathcal{C}_p(U)$  is non-empty since it contains the current  $[L]$  of integration on  $L$  and its regularizations in  $U$ . In the situation described above, Dinh proved that if  $R$  is a continuous element of  $\mathcal{C}_p(U)$  then its normalized push-forwards by  $f^n$ ,  $d^{-(k-p)n}(f^n)_*(R)$ , converge to a current  $\tau$  which is independent of the choice of  $R$ . Moreover, the current  $\tau$  gives us information on the geometry of  $A$  and on the dynamics of  $f|_A$ : it is woven, supported in  $A$  and invariant i.e.  $f_*(\tau) = d^{k-p}\tau$ . Our main result is that, with a natural additional assumption on  $f|_U$ , stable under small perturbations, we obtain an explicit exponential speed of the above convergence for any  $R$  in  $\mathcal{C}_p(U)$ .

**Theorem 1.1.** *Let  $f$  and  $\tau$  be as above and assume that  $\|\wedge^{k-p+1} Df(z)\| < 1$  on  $\bar{U}$ . There is a constant  $0 < \lambda < 1$  such that for each  $0 < \alpha \leq 2$  the*

following property holds. There exists  $C > 0$  such that for any element  $R$  of  $\mathcal{C}_p(U)$  and any  $(k-p, k-p)$ -form  $\varphi$  of class  $\mathcal{C}^\alpha$  on  $\mathbb{P}^k$  we have

$$|\langle d^{-(k-p)n}(f^n)_*(R) - \tau, \varphi \rangle| \leq C\lambda^{n\alpha/2}\|\varphi\|_{\mathcal{C}^\alpha}. \quad (1.1)$$

In particular,  $\tau$  is the unique invariant current in  $\mathcal{C}_p(U)$  and  $d^{-(k-p)n}(f^n)_*(R)$  converge to  $\tau$  uniformly on  $R \in \mathcal{C}_p(U)$ .

Recall that  $f$  induces a self-map  $Df$  on the tangent bundle  $T\mathbb{P}^k$  which also gives a self-map  $\wedge^q Df$  on the exterior power  $\wedge^q T\mathbb{P}^k$ ,  $1 \leq q \leq k$ . In the sequel, all the norms on  $\mathcal{C}^\alpha$ ,  $L^r$ , etc. are with respect to the Fubini-Study metric on  $\mathbb{P}^k$ . It also gives a uniform norm which induces an operator norm for  $\wedge^q Df$ .

In the same spirit as Theorem 1.1, we proved in [Taf11] that for a generic current  $S$  in  $\mathcal{C}_1(\mathbb{P}^k)$ , the sequence  $d^{-n}(f^n)^*(S)$  converges to  $T$  exponentially fast. However, the contexts are quite different. Here, we consider currents of arbitrary bidegree which are in general much harder to handle. Moreover, in [Taf11] we deeply use that  $T$  has Hölder continuous local potentials. In the present situation, we can expect that the attracting current  $\tau$  is always more singular. The idea to prove Theorem 1.1 is to use Henkin-Leiterer's solution with estimates of the  $dd^c$ -equation on  $U$  in order to study separately the harmonic and non-harmonic parts of the left hand side term of (1.1). When  $dd^c\varphi = 0$  on  $U$ , we use the "geometry" of  $\mathcal{C}_p(U)$ , introduced in [Din07] and [DS06], and Harnack's inequality to obtain exponential estimates. In order to deal with the non-harmonic part, we use the assumption on  $\|\wedge^{k-p+1} Df\|$ . This assumption comes naturally in several basic examples and their perturbations.

In [Din07], Dinh also showed that the *equilibrium measure associated to*  $A$ , defined by  $\nu := \tau \wedge T^{k-p}$ , is invariant, mixing and of maximal entropy  $(k-p)\log d$  on  $A$ . Theorem 1.1 is a first step in order to obtain other ergodic and stochastic properties on  $\nu$  as exponential mixing or central limit theorem. We postpone this question in a future work.

Under the same assumptions, we deduce from the work of de Thélin [dT08], see also [Dup09], the following result on  $\nu$ .

**Theorem 1.2.** *If  $f$  is as in Theorem 1.1, then the measure  $\nu$  is hyperbolic. More precisely, counting with multiplicity it has  $k-p$  Lyapunov exponents larger than or equal to  $(\log d)/2$  and  $p$  Lyapunov exponents strictly smaller than  $-(k-p)(\log d)/2$ .*

## 2 Structural discs of currents

In this section we recall the notion of structural varieties of currents. It was introduced by Dinh and Sibony in order to put a geometry on the space  $\mathcal{C}_p(U)$  which is of infinite dimension, see [DS06] and [Din07]. The definition of structural varieties is based on slicing theory and they can be seen as complex subvarieties inside  $\mathcal{C}_p(U)$ . In [DS09], the authors developed the notion of super-potential which involves more deeply this geometry.

Slicing theory can be seen as a generalization to currents of restriction of smooth forms to submanifolds. We will briefly explain it in our context and refer to [Fed69] for a more complete account.

Let  $U$  be an open subset of  $\mathbb{P}^k$  satisfying the geometric hypothesis as above. Let  $V$  be a complex manifold of dimension  $l$ . We denote by  $\pi_U$  and  $\pi_V$  the canonical projections of  $U \times V$  to  $U$  and  $V$  respectively. If  $\mathcal{R}$  is a positive closed current of bidegree  $(p, p)$  on  $U \times V$  with  $\pi_U(\text{supp}(\mathcal{R})) \Subset U$  then, for all  $\theta$  in  $V$ , the slice  $\langle \mathcal{R}, \pi_V, \theta \rangle$  is well defined. For any  $(k-p, k-p)$ -form  $\phi$  on  $U \times V$  we have

$$\langle \mathcal{R}, \pi_V, \theta \rangle(\phi) = \lim_{\epsilon \rightarrow 0} \langle \mathcal{R} \wedge \pi_V^*(\psi_{\theta, \epsilon}), \phi \rangle,$$

where  $\psi_{\theta, \epsilon}$  is an appropriate approximation in  $V$  of the Dirac mass at  $\theta$ . It is a  $(p+l, p+l)$ -current on  $U \times V$  supported on  $U \times \{\theta\}$  which can be identified to a  $(p, p)$ -current on  $U$ . A family of currents  $(R_\theta)_{\theta \in V}$  in  $\mathcal{C}_p(U)$  is a *structural variety* if there exists a positive closed current  $\mathcal{R}$  in  $U \times V$  such that  $R_\theta = \langle \mathcal{R}, \pi_V, \theta \rangle$ . When  $V$  is isomorphic to the unit disc of  $\mathbb{C}$ , we call  $(R_\theta)_{\theta \in V}$  a *structural disc*.

Recall that in our situation  $f(U) \Subset U$ . Under the geometrical assumption on  $U$ , Dinh constructed in [Din07, p.233] a family of structural discs in  $\mathcal{C}_p(U)$ . He uses that for each  $x \in L$  the set  $I(x) \cap U$  is star-shaped at  $x$  to obtain a property similar to star-sharpness for  $\mathcal{C}_p(U)$ .

More precisely, up to an automorphism, we can assume that

$$I = \{x \in \mathbb{P}^k \mid x_i = 0, 0 \leq i \leq k-p\}, \quad L = \{x \in \mathbb{P}^k \mid x_i = 0, k-p+1 \leq i \leq k\},$$

where  $x = [x_0 : \cdots : x_k]$  are the homogeneous coordinates of  $\mathbb{P}^k$ . For  $\theta \in \mathbb{C}$ ,  $A_\theta(x) := [x_0 : \cdots : x_{k-p} : \theta x_{k-p+1} : \cdots : \theta x_k]$  is an automorphism of  $\mathbb{P}^k$  except for  $\theta = 0$  where it is the projection of  $\mathbb{P}^k \setminus I$  on  $L$ . Let set  $U' := f(U)$ . As  $I(x) \cap U$  is star-shaped at  $x \in L$ , there exists a simply connected open neighborhood  $V \subset \mathbb{C}$  of  $[0, 1]$  such that  $A_\theta(U') \Subset U$  for all  $\theta$  in  $\bar{V}$ . If  $S$  is in  $\mathcal{C}_p(U')$  then the family  $(S_\theta)_{\theta \in V}$  with  $S_\theta := (A_\theta)_*(S)$  defined a structural disc. Indeed, if  $\Lambda : \mathbb{P}^k \times V \rightarrow \mathbb{P}^k$  is the meromorphic map defined by  $\Lambda(x, \theta) = (A_\theta)^{-1}(x)$  and  $\mathcal{S} := \Lambda^*S$  then  $S_\theta = \langle \mathcal{S}, \pi_V, \theta \rangle$ , see [Din07] for

more details. For any  $S$  in  $\mathcal{C}_p(U')$ , we have that  $S_1 = S$  and  $S_0 = [L]$  which is independent of  $S$ . In other words, any current in  $\mathcal{C}_p(U')$  is linked to  $[L]$  by a structural disc in  $\mathcal{C}_p(U)$ . Moreover,  $S_\theta$  depends continuously on  $\theta$  and we have the following important property.

**Lemma 2.1** ([Din07]). *Let  $S$  be in  $\mathcal{C}_p(U')$  and  $(S_\theta)_{\theta \in V}$  be the structural disc described above. If  $\phi$  is a real continuous  $(k-p, k-p)$ -form with  $dd^c\phi = 0$  on  $U$  then the function  $\theta \mapsto \langle S_\theta, \phi \rangle$  is harmonic on  $V$ .*

### 3 $q$ -Convex set and $\bar{\partial}$ -equation

The concept of  $q$ -convexity generalizes both Stein and compact manifolds. Andreotti and Grauert [AG62] obtained vanishing or finiteness theorems for  $q$ -convex manifolds and, in [HL88], Henkin and Leiterer developed a similar theory using integral representations. In particular, they obtained solutions of the  $\bar{\partial}$ -equation with explicit estimates, which play a key role in our proof. For this reason, we use the conventions of [HL88].

If  $1 \leq q \leq k$  is an integer then a real  $\mathcal{C}^2$  function  $\rho$  on an open subset  $V \subset \mathbb{P}^k$  is called  $q$ -convex if, in any holomorphic local coordinates, the Hermitian form

$$L_\rho(x)t = \sum_{i,j=1}^k \frac{\partial^2 \rho(x)}{\partial z_i \partial \bar{z}_j} t_i \bar{t}_j$$

has at least  $q$  strictly positive eigenvalues at any point  $x \in V$ .

Let  $0 \leq q \leq k-1$ . We say that an open subset  $D$  of  $\mathbb{P}^k$  is *strictly  $q$ -convex* if there exists a  $(q+1)$ -convex function  $\rho$  in a neighborhood  $V$  of  $\partial D$  such that

$$D \cap V = \{x \in V \mid \rho(x) < 0\}.$$

Moreover, if the same condition is satisfied with  $V$  a neighborhood of  $\bar{D}$  then  $D$  is called *completely strictly  $q$ -convex*.

The strict  $q$ -convexity has the following important consequence, see [HL88, Theorem 11.2].

**Theorem 3.1.** *Let  $D$  be a strictly  $q$ -convex open subset of  $\mathbb{P}^k$  with  $\mathcal{C}^2$  boundary. If  $\phi$  is a continuous  $\bar{\partial}$ -exact form of bidegree  $(r, s)$  in a neighborhood of  $\bar{D}$  with  $0 \leq s \leq k$ ,  $k-q \leq r \leq k$ , then there exists a continuous  $(s, r-1)$ -form  $\psi$  on  $D$  such that  $\bar{\partial}\psi = \phi$  and*

$$\|\psi\|_{\infty, D} \leq C \|\phi\|_{\infty, D}$$

for some  $C > 0$  independent of  $\phi$ .

Furthermore, Andreotti and Grauert proved the following vanishing theorem, see [AG62] and [HL88, Theorem 12.7].

**Theorem 3.2.** *If  $D$  is a completely strictly  $q$ -convex open subset of  $\mathbb{P}^k$  with  $\mathcal{C}^2$  boundary then  $H^{s,r}(D, \mathbb{C}) = 0$  for any  $0 \leq s \leq k$  and  $k - q \leq r \leq k$ .*

Henkin and Leiterer [HL88, Theorem 5.13] give the following criteria of  $q$ -convexity, which is closely related to our geometric assumption on  $U$  with  $q = p - 1$ .

**Theorem 3.3.** *Let  $D$  be an open subset of  $\mathbb{P}^k$  with  $\mathcal{C}^2$  boundary. If for each  $x \in \partial D$  there exists a complex submanifold  $Y \subset \mathbb{P}^k$  of dimension  $q + 1$ , transverse to  $\partial D$  and such that  $Y \cap D$  is a strictly pseudoconvex domain in  $Y$ , then  $D$  is strictly  $q$ -convex.*

This result applies to our trapping neighborhood  $U$  with  $q = p - 1$ . Indeed, observe that, possibly by exchanging  $U$  by a slightly smaller open set which contains  $f(U)$ , we can assume that  $\partial U$  is smooth and the intersection of  $\partial U$  with  $I(x)$  is transverse for all  $x \in L$ . The projective space  $I(x)$  has dimension  $p = q + 1$  and  $U \cap I(x)$  is strictly convex in  $I(x) \setminus I \simeq \mathbb{C}^p$ , so in particular strictly pseudoconvex in  $I(x)$ . Therefore, by Theorem 3.3,  $U$  is strictly  $(p - 1)$ -convex. In the sequel, we always choose such an attracting neighborhood  $U$ .

Up to an automorphism of  $\mathbb{P}^k$ ,  $I$  is defined in homogeneous coordinates by  $I = \{x \in \mathbb{P}^k \mid x_i = 0, 0 \leq i \leq k - p\}$ . The function

$$\eta(x) = \frac{|x_{k-p+1}|^2 + \cdots + |x_k|^2}{|x_0|^2 + \cdots + |x_{k-p}|^2},$$

is a  $(q + 1)$ -convex exhausting function of  $\mathbb{P}^k \setminus I$ , i.e.  $\mathbb{P}^k \setminus I$  is *completely  $q$ -convex*. In general, strictly  $q$ -convex subsets of a completely  $q$ -convex domain are not completely strictly  $q$ -convex. However, in our case it is easy to construct from a  $q$ -convex function  $\rho$  such that

$$U \cap V = \{x \in V \mid \rho(x) < 0\}$$

for some neighborhood  $V$  of  $\partial U$ , a  $q$ -convex defining function defined in a neighborhood of  $\bar{U}$ . Indeed, it is enough to compose  $(\eta, \rho)$  with a good approximation of the maximum function (see [HL88, Definition 4.12]). It will give a  $(q + 1)$ -convex function since the positive eigenvalues of the complex Hessians of  $\rho$  and  $\eta$  are in the same directions. Therefore,  $U$  is completely strictly  $(p - 1)$ -convex and we have the following solution for the  $dd^c$ -equation in symmetric bidegrees.

**Theorem 3.4.** *Let  $U$  be as above. If  $\varphi$  is a  $C^2$   $(r, r)$ -form in a neighborhood of  $\bar{U}$  with  $k - p \leq r \leq k$ , then there exists a continuous  $(r, r)$ -form  $\psi$  on  $U$  such that  $dd^c\psi = dd^c\varphi$  and*

$$\|\psi\|_{\infty, U} \leq C \|dd^c\varphi\|_{\infty, U}$$

for some  $C > 0$  independent of  $\varphi$ .

*Proof.* The proof follows closely the proof of Theorem 2.7 in [DNS08].

Without loss of generality, we can assume that  $\varphi$  is real and therefore  $dd^c\varphi$  is also real. First, we solve the equation  $d\xi = dd^c\varphi$  with estimates. Let  $W$  be a small neighborhood of  $\bar{U}$ , with the same geometric property and such that  $\varphi$  is defined on  $W$ . The maps  $A_\theta$  defined in Section 2 give a homotopy  $A : [0, 1] \times W \rightarrow W$ ,  $A(\theta, x) = A_\theta(x)$ , between  $A_1 = \text{Id}$  and the projection  $A_0$  of  $W$  on  $L$ . Since  $L$  has dimension  $k - p$ ,  $A_0^*$  vanish identically on  $(r + 1, r + 1)$ -forms if  $r \geq k - p$ . Therefore, by homotopy formula (see e.g [BT82, p38]), there exists a form  $\xi$  on  $W$  such that  $d\xi = dd^c\varphi$  and  $\|\xi\|_{\infty, U} \lesssim \|dd^c\varphi\|_{\infty, U}$ . Moreover, possibly by exchanging  $\xi$  by  $(\xi + \bar{\xi})/2$ , we can assume that  $\xi = \Xi + \bar{\Xi}$  where  $\Xi$  is a  $(r, r + 1)$ -form. As  $d\xi$  is a  $(r + 1, r + 1)$ -form, it follows that  $\bar{\partial}\Xi = 0$  and  $d\xi = \partial\Xi + \bar{\partial}\bar{\Xi}$ . Therefore, by Theorem 3.2,  $\Xi$  is  $\bar{\partial}$ -exact and by Theorem 3.1, there exists a continuous  $(r, r)$ -form  $\Psi$  such that  $\bar{\partial}\Psi = \Xi$  and  $\|\Psi\|_{\infty, U} \lesssim \|\Xi\|_{\infty, U}$ .

Finally, if  $\psi = -i\pi(\Psi - \bar{\Psi})$  we have

$$dd^c\psi = \partial\bar{\partial}(\Psi - \bar{\Psi}) = \partial\Xi + \bar{\partial}\bar{\Xi} = dd^c\varphi,$$

and

$$\|\psi\|_{\infty, U} \lesssim \|\Xi\|_{\infty, U} \lesssim \|dd^c\varphi\|_{\infty, U}.$$

□

## 4 Attracting speed

For  $R$  in  $\mathcal{C}_p(U)$ , we denote by  $R_n$  its normalized push-forward by  $f^n$ , i.e.  $R_n := d^{-(k-p)n}(f^n)_*(R)$ . To obtain (1.1), the first observation is that the norm of  $R_n - \tau$ , seen as a linear form on the space of continuous test  $(k - p, k - p)$ -forms, is bounded independently of  $n$  and  $R$ . Therefore, it is sufficient to establish (1.1) for  $\alpha = 2$  and then apply interpolation theory between Banach spaces, see e.g. [Tri95], in order to obtain the general case.

Let denote by  $X$  the set of all real continuous  $(k - p, k - p)$ -forms  $\phi$  on  $U$  such that  $dd^c\phi = 0$  and  $|\langle R - \tau, \phi \rangle| \leq 1$  for all  $R \in \mathcal{C}_p(U)$ . Observe that, since  $f(U) \Subset U$ , if  $\phi$  is in  $X$  then  $f^*(\phi)$  is defined on  $U$  where it is still a



real continuous form with  $dd^c(f^*(\phi)) = 0$ . The set  $X$  is a truncated convex cone and the first part of the proof of Theorem 1.1 is to show that  $d^{-(k-p)}f^*$  acts by contraction on it. This result is available without any assumption on  $\|\wedge^{k-p+1}Df\|$ . It is based on Lemma 2.1 and Harnack's inequality for harmonic functions.

**Lemma 4.1.** *There exists a constant  $0 < \lambda_1 < 1$  such that for any  $R$  in  $\mathcal{C}_p(U)$ ,  $\phi$  in  $X$  and  $n$  in  $\mathbb{N}$  we have*

$$|\langle R_n - \tau, \phi \rangle| \leq \lambda_1^n.$$

*Proof.* If  $R$  is in  $\mathcal{C}_p(U)$  and  $\phi$  in  $X$ ,  $R_1 := d^{-(k-p)}f_*(R)$  is in  $\mathcal{C}_p(U')$  and we define the function  $h_{R,\phi}$  on  $V$  by  $h_{R,\phi}(\theta) := \langle R_{1,\theta} - \tau, \phi \rangle$ , where  $\theta \mapsto R_{1,\theta}$  is the structural disc described in Section 2. The definition of  $X$  implies that  $|h_{R,\phi}| \leq 1$  on  $V$ , for all  $R \in \mathcal{C}_p(U)$  and  $\phi \in X$ . Moreover, since  $R_1$  is in  $\mathcal{C}_p(U')$ , it follows from Lemma 2.1 that all these functions are harmonic on  $V$ .

Now, observe that if we take  $R = \tau$  then  $h_{\tau,\phi}(1) = 0$  for all  $\phi \in X$ , since  $d^{p-k}f_*\tau = \tau$ . Hence, as  $|h_{\tau,\phi}| \leq 1$  on  $V$ , Harnack's inequality says that there exists  $0 \leq a < 1$  such that  $|h_{\tau,\phi}(0)| \leq a$  for all  $\phi$  in  $X$ . On the other hand,  $R_{1,0}$  is a current independent of  $R$ . So, for all  $R \in \mathcal{C}_p(U)$  and  $\phi \in X$  we have  $h_{R,\phi}(0) = h_{\tau,\phi}(0)$  and therefore  $|h_{R,\phi}(0)| \leq a$ . Once again, we deduce from Harnack's inequality there exists  $0 < \lambda_1 < 1$ , independent of  $R$  and  $\phi$ , such that  $|h_{R,\phi}(1)| \leq \lambda_1$  or equivalently

$$\left| \left\langle R_1 - \tau, \frac{\phi}{\lambda_1} \right\rangle \right| = |\langle R - \tau, \phi_1 \rangle| \leq 1,$$

where  $\phi_1 = d^{-(k-p)}f^*(\phi/\lambda_1)$ . Moreover,  $\phi_1$  is defined on  $U$  and  $dd^c\phi_1 = 0$ . It follows that  $\phi_1$  is in  $X$ . Using the same arguments with  $\phi_1$  instead of  $\phi$  gives that  $|\langle R_1 - \tau, \phi_1 \rangle| \leq \lambda_1$  which can be rewrite  $|\langle R_2 - \tau, \phi \rangle| \leq \lambda_1^2$ . Inductively, we obtain that  $|\langle R_n - \tau, \phi \rangle| \leq \lambda_1^n$ .  $\square$

**Remark 4.2.** *The constant  $\lambda_1$  is not directly related to  $f$ . Indeed, it only depends on  $V$  i.e. on the size of  $U$  and the distance between  $\partial U$  and  $\partial f(U)$ . If  $h$  is the unique biholomorphism between  $V$  and the unit disc in  $\mathbb{C}$  such that  $h(0) = 0$  and  $h(1) = \alpha \in ]0, 1[$  then Harnack's inequality gives explicitly that we can take  $a = 2\alpha/(1 + \alpha)$  and  $\lambda_1 = 4\alpha/(1 + \alpha)^2$ .*

In order to prove Theorem 1.1, we use Theorem 3.4 together with the assumption on  $\|\wedge^{k-p+1}Df\|$  and Lemma 4.1.

If  $\|\wedge^{k-p+1} Df(z)\| < 1$  on  $\bar{U}$  then by continuity, there exists a constant  $0 < \lambda_2 < 1$  such that  $\|\wedge^{k-p+1} Df(z)\| < \lambda_2$  on  $U$ . Hence, if  $\varphi$  is a  $(k-p, k-p)$ -form of class  $\mathcal{C}^2$ , we have for  $\varphi_i := d^{-i(k-p)}(f^i)^*(\varphi)$  with  $i \in \mathbb{N}$

$$\|dd^c \varphi_i\|_{\infty, U} \lesssim \frac{\lambda_2^{2i}}{d^{i(k-p)}} \|\varphi\|_{\mathcal{C}^2}.$$

Here, the symbol  $\lesssim$  means inequality up to a constant which only depends on our conventions and on  $U$ . By Theorem 3.4 with  $r = k - p$ , there exists a continuous  $(k - p, k - p)$ -form  $\psi_i$  on  $U$  such that

$$dd^c \psi_i = dd^c \varphi_i$$

and

$$\|\psi_i\|_{\infty, U} \lesssim \|dd^c \varphi_i\|_{\infty, U} \lesssim \frac{\lambda_2^{2i}}{d^{i(k-p)}} \|\varphi\|_{\mathcal{C}^2}.$$

We can now complete the proof of our main result.

*End of the proof of Theorem 1.1.* Let  $R$  be in  $\mathcal{C}_p(U)$  and  $\varphi$  be a  $(k-p, k-p)$ -form of class  $\mathcal{C}^2$ . Without loss of generality, we can assume that  $\varphi$  is real. Let  $0 \leq i \leq n$  be two arbitrary integers. We set  $l := n - i$ . If  $R_n$ ,  $\varphi_i$  and  $\psi_i$  are defined as above then we have

$$\langle R_n - \tau, \varphi \rangle = \langle R_l - \tau, \varphi_i \rangle = \langle R_l - \tau, \varphi_i - \psi_i \rangle + \langle R_l - \tau, \psi_i \rangle,$$

since  $\tau$  is invariant. On the one hand,

$$\langle R_l - \tau, \psi_i \rangle \lesssim \|\psi_i\|_{\infty, U} \lesssim \frac{\lambda_2^{2i}}{d^{i(k-p)}} \|\varphi\|_{\mathcal{C}^2}, \quad (4.1)$$

since  $R_l$  and  $\tau$  are supported on  $U$ . On the other hand, observe that there exists a constant  $M \geq 1$  independent of  $\varphi$  such that  $\|d^{-(k-p)} f^*(\varphi)\|_{\infty} \leq M \|\varphi\|_{\infty}$ . Therefore,

$$\begin{aligned} \|\varphi_i - \psi_i\|_{\infty, U} &\leq M^i \|\varphi\|_{\infty} + \|\psi_i\|_{\infty, U} \leq M^i \|\varphi\|_{\infty} + C \frac{\lambda_2^{2i}}{d^{i(k-p)}} \|\varphi\|_{\mathcal{C}^2} \\ &\lesssim M^i \|\varphi\|_{\mathcal{C}^2}, \end{aligned}$$

and in particular

$$|\langle S - \tau, \varphi_i - \psi_i \rangle| \lesssim M^i \|\varphi\|_{\mathcal{C}^2},$$

for any  $S$  in  $\mathcal{C}_p(U)$ .

Moreover,  $\varphi_i - \psi_i$  is a real continuous  $(k - p, k - p)$ -form on  $U$  and  $dd^c(\varphi_i - \psi_i) = 0$ . Hence,  $(\varphi_i - \psi_i)/(CM^i \|\varphi\|_{\mathcal{C}^2})$  belongs to  $X$  where  $C > 0$

is a constant depending only on  $U$  and on our conventions. It follows by Lemma 4.1 that

$$|\langle R_l - \tau, \varphi_i - \psi_i \rangle| \leq CM^i \|\varphi\|_{C^2} \lambda_1^l. \quad (4.2)$$

To summarize, equations (4.1) and (4.2) imply that there are constants  $0 < \lambda_1, \lambda_2 < 1$ , and  $M \geq 1$  such that

$$|\langle R_n - \tau, \varphi \rangle| \lesssim \|\varphi\|_{C^2} \left( M^i \lambda_1^l + \frac{\lambda_2^{2i}}{d^{i(k-p)}} \right).$$

If  $q \in \mathbb{N}$  is large enough then  $M\lambda_1^q < 1$ . Therefore, if we choose  $n \simeq (q+1)i$ , we obtain  $l \simeq iq$  and

$$|\langle R_n - \tau, \varphi \rangle| \lesssim \|\varphi\|_{C^2} \lambda^n,$$

where  $\lambda := \max(\lambda_2^2 d^{-(k-p)}, M\lambda_1^q)^{1/(q+1)} < 1$ . This estimate holds for arbitrary  $n$  in  $\mathbb{N}$  and is uniform on  $\varphi$  and  $R$ .  $\square$

**Remark 4.3.** *In Theorem 1.1, it is enough to assume that  $\|\wedge^{k-p+1} Df(z)\| < d^{(k-p)/2}$  on  $\bar{U}$ . Moreover, it is easy using small perturbations of a suitable polynomial map to construct examples with  $\|\wedge^{k-p+1} Df(z)\|$  as small as we want on  $\bar{U}$ .*

## 5 Hyperbolicity of the equilibrium measure

In this section, we prove Theorem 1.2. Recall that the equilibrium measure associated to  $A$  is given by  $\nu := \tau \wedge T^{k-p}$ . It has maximal entropy on  $A$  equal to  $(k-p) \log d$ , [Din07]. On the other hand, we have the following powerful result, see [dT08] and [Dup09].

**Theorem 5.1.** *If the Lyapunov exponents of  $\nu$  are ordered so that*

$$\chi_1 \geq \cdots \geq \chi_{a-1} > \chi_a \geq \cdots \geq \chi_k,$$

then

$$h(\nu) \leq (a-1) \log d + 2 \sum_{i=a}^k \chi_i^+, \quad (5.1)$$

where  $h(\nu)$  denotes the entropy of  $\nu$  and  $\chi_i^+ := \max(\chi_i, 0)$ .

Now, let  $1 \leq c \leq k$  be such that

$$\chi_1 \geq \cdots \geq \chi_c > 0 \geq \chi_{c+1} \geq \cdots \geq \chi_k.$$

If we take  $a = c+1$  in Theorem 5.1, we obtain  $h(\nu) \leq c \log d$ . Since  $h(\nu) = (k-p) \log d$ , it follows that  $c \geq (k-p)$ . It means there are at least  $k-p$  strictly

positive Lyapunov exponents. Moreover, if we have equality,  $c = k - p$ , Theorem 5.1 applied to the smallest  $a$  such that  $\chi_a = \chi_c$  gives

$$(k - p) \log d = h(\nu) \leq (a - 1) \log d + 2(k - p - a + 1) \chi_c.$$

Hence,  $\chi_c \geq (\log d)/2$ . Note that in this part we do not need the assumption on  $\|\wedge^{k-p+1} Df\|$ .

It remains to prove that the assumptions of Theorem 1.1 imply that  $c \leq k - p$  and  $\chi_{c+1} < -(k - p)(\log d)/2$ . It is not hard to deduce from Oseledec theorem [Ose68] that the sum of the  $q$  largest Lyapunov exponents verifies

$$\chi_1 + \cdots + \chi_q = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\wedge^q Df^n(z)\|,$$

for  $\nu$ -almost all  $z$ . Moreover, we have

$$\|\wedge^q Df^{n+m}(z)\| \leq \|\wedge^q Df^n(z)\| \|\wedge^q Df^m(f^n(z))\|.$$

Therefore, it follows that

$$\|\wedge^q Df^n(z)\| \leq \left( \max_{z \in U} \|\wedge^q Df(z)\| \right)^n$$

and

$$\chi_1 + \cdots + \chi_q \leq \log \max_{z \in U} \|\wedge^q Df(z)\| =: \gamma.$$

Hence, if  $\|\wedge^{k-p+1} Df(z)\| < 1$  on  $\bar{U}$  then

$$\chi_1 + \cdots + \chi_{k-p+1} \leq \gamma < 0.$$

Therefore,  $c \leq k - p$  and we have seen above that in this case  $c = k - p$  and  $\chi_c \geq (\log d)/2$ . Finally, we have

$$\gamma \geq \chi_1 + \cdots + \chi_{k-p} + \chi_{k-p+1} \geq \frac{k-p}{2} \log d + \chi_{k-p+1},$$

which implies

$$\chi_{k-p+1} \leq \gamma - \frac{k-p}{2} \log d.$$

**Remark 5.2.** *Theorem 5.1 with  $a = 1$  implies the Ruelle inequality, i.e.*

$$\chi_1 + \cdots + \chi_c \geq \frac{k-p}{2} \log d.$$

*Therefore, it is enough to assume that  $\|\wedge^{k-p+1} Df(z)\| < d^{(\frac{k-p}{2})(\frac{k-p+1}{k})}$  on  $\bar{U}$  since*

$$\chi_1 + \cdots + \chi_{k-p+1} \geq \frac{k-p+1}{c} (\chi_1 + \cdots + \chi_c),$$

*if  $c \geq k - p + 1$ .*

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