

Speed of convergence towards attracting sets for endomorphisms of P k

Johan Taflin

► To cite this version:

Johan Taflin. Speed of convergence towards attracting sets for endomorphisms of P k. Indiana University Mathematics Journal, 2013, 62 (1), pp.33-44. 10.1512/iumj.2013.62.4895 . hal-02375556

HAL Id: hal-02375556 https://hal.science/hal-02375556v1

Submitted on 22 Nov 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Speed of convergence towards attracting sets for endomorphisms of \mathbb{P}^k

Johan Taflin

April 22, 2014

Abstract

Let f be a non-invertible holomorphic endomorphism of \mathbb{P}^k having an attracting set A. We show that, under some natural assumptions, A supports a unique invariant positive closed current τ , of the right bidegree and of mass 1. Moreover, if R is a current supported in a small neighborhood of A then its push-forwards by f^n converge to τ exponentially fast. We also prove that the equilibrium measure on Ais hyperbolic.

1 Introduction

Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ on the complex projective space \mathbb{P}^k . A compact subset A of \mathbb{P}^k is called an *attracting* set if it has a trapping neighborhood U i.e. $f(U) \Subset U$ and $A = \bigcap_{n\geq 0} f^n(U)$ where $f^n := f \circ \cdots \circ f$, n times. It follows that A is invariant, f(A) = A. Furthermore, if A contains a dense orbit then A is a trapped attractor. Typical examples of such objects are fractal and their underlying dynamics are hard to study. We refer to [Mil85], [Rue89] for general discussions on attractors and to [FW99], [JW00], [FS01], [BDM07], [Taf10] and references therein for examples of different types of attractors in \mathbb{P}^2 .

Attracting sets are stable under small perturbations. Indeed, if f has an attracting set $A = \bigcap_{n\geq 0} f^n(U)$ then any small perturbation f_{ϵ} of f has an attracting set defined by $A_{\epsilon} = \bigcap_{n\geq 0} f_{\epsilon}^n(U)$. For example, when f restricted to \mathbb{C}^k defines a polynomial self-map then the hyperplane at infinity $\mathbb{P}^k \setminus \mathbb{C}^k$ is an attracting set. In the same way, it is easy to create examples where the attracting set is a projective subspace of arbitrary dimension. In this paper, we consider a family of endomorphisms, stable under small perturbations,

which contains these examples. It was introduced by Dinh in [Din07] and we briefly recall the context.

In the sequel, we always assume that f possesses an attracting set A which has a trapped neighborhood U satisfying the following properties. There exist an integer $1 \leq p \leq k-1$ and two projective subspaces I and L of dimension p-1 and k-p respectively such that $I \cap U = \emptyset$ and $L \subset U$. We do not assume that L and I are invariant. Since $I \cap L = \emptyset$, for each $x \in L$ there exists a unique projective subspace I(x) of dimension p which contains I and such that $L \cap I(x) = \{x\}$. Furthermore, for each $x \in L$ we ask that $U \cap I(x)$ is strictly convex as a subset of $I(x) \setminus I \simeq \mathbb{C}^p$. All these assumptions are stable under small perturbations of f. The geometric assumption on U is slightly stronger than the one of Dinh, who only requires $U \cap I(x)$ to be star-shaped at x. We need convexity in order the solve the $\overline{\partial}$ -equation on U. Indeed, under our assumption U is a (p-1)-convex domain in the sense of [HL88].

If E is a subset of \mathbb{P}^k , let $\mathscr{C}_q(E)$ denote the set of all positive closed currents of bidegree (q,q), supported in E and of mass 1. It is well known that for any integer $1 \leq q \leq k$ and any smooth form S in $\mathscr{C}_q(\mathbb{P}^k)$, the sequence $d^{-qn}(f^n)^*(S)$ converges to a positive closed current T^q of bidegree (q,q) called the *Green current of order* q of f. We refer to [DS10] for a detailed exposition on these currents and their effectiveness in holomorphic dynamics.

When q = k, it gives the equilibrium measure of f, $\mu := T^k$. It is exponentially mixing and it is the unique measure of maximal entropy $k \log d$ on \mathbb{P}^k . Moreover, it is hyperbolic and all its Lyapunov exponents are larger or equal to $(\log d)/2$. The dynamics outside the support of μ is not very well understood. The aim of this paper is to continue the investigation started in [Din07] on the attracting sets described above, which do not intersect $\operatorname{supp}(\mu)$. Indeed, since $I \cap U = \emptyset$, by regularization there exists a smooth form $S \in \mathscr{C}_{k-p+1}(\Omega)$, where $\Omega := \mathbb{P}^k \setminus \overline{U}$. As $f^{-1}(\Omega) \subset \Omega$, it follows that $\operatorname{supp}(T^{k-p+1}) \cap U = \emptyset$, and hence $\operatorname{supp}(T^q) \cap U = \emptyset$ if $q \geq k - p + 1$.

The set $\mathscr{C}_p(U)$ is non-empty since it contains the current [L] of integration on L and its regularizations in U. In the situation described above, Dinh proved that if R is a continuous element of $\mathscr{C}_p(U)$ then its normalized push-forwards by f^n , $d^{-(k-p)n}(f^n)_*(R)$, converge to a current τ which is independent of the choice of R. Moreover, the current τ gives us information on the geometry of A and on the dynamics of $f_{|A}$: it is woven, supported in A and invariant i.e. $f_*(\tau) = d^{k-p}\tau$. Our main result is that, with a natural additional assumption on $f_{|U}$, stable under small perturbations, we obtain an explicit exponential speed of the above convergence for any R in $\mathscr{C}_p(U)$.

Theorem 1.1. Let f and τ be as above and assume that $\|\wedge^{k-p+1} Df(z)\| < 1$ on \overline{U} . There is a constant $0 < \lambda < 1$ such that for each $0 < \alpha \leq 2$ the following property holds. There exists C > 0 such that for any element R of $\mathscr{C}_p(U)$ and any (k - p, k - p)-form φ of class \mathcal{C}^{α} on \mathbb{P}^k we have

$$\left| \left\langle d^{-(k-p)n}(f^n)_*(R) - \tau, \varphi \right\rangle \right| \le C \lambda^{n\alpha/2} \|\varphi\|_{\mathcal{C}^{\alpha}}.$$
 (1.1)

In particular, τ is the unique invariant current in $\mathscr{C}_p(U)$ and $d^{-(k-p)n}(f^n)_*(R)$ converge to τ uniformly on $R \in \mathscr{C}_p(U)$.

Recall that f induces a self-map Df on the tangent bundle $T\mathbb{P}^k$ which also gives a self-map $\wedge^q Df$ on the exterior power $\wedge^q T\mathbb{P}^k$, $1 \leq q \leq k$. In the sequel, all the norms on \mathcal{C}^{α} , L^r , etc. are with respect to the Fubini-Study metric on \mathbb{P}^k . It also gives a uniform norm which induces an operator norm for $\wedge^q Df$.

In the same spirit as Theorem 1.1, we proved in [Taf11] that for a generic current S in $\mathscr{C}_1(\mathbb{P}^k)$, the sequence $d^{-n}(f^n)^*(S)$ converges to T exponentially fast. However, the contexts are quite different. Here, we consider currents of arbitrary bidegree which are in general much harder to handle. Moreover, in [Taf11] we deeply use that T has Hölder continuous local potentials. In the present situation, we can expect that the attracting current τ is always more singular. The idea to prove Theorem 1.1 is to use Henkin-Leiterer's solution with estimates of the dd^c -equation on U in order to study separately the harmonic and non-harmonic parts of the left hand side term of (1.1). When $dd^c \varphi = 0$ on U, we use the "geometry" of $\mathscr{C}_p(U)$, introduced in [Din07] and [DS06], and Harnack's inequality to obtain exponential estimates. In order to deal with the non-harmonic part, we use the assumption on $\| \wedge^{k-p+1} Df \|$. This assumption comes naturally in several basic examples and their perturbations.

In [Din07], Dinh also showed that the equilibrium measure associated to A, defined by $\nu := \tau \wedge T^{k-p}$, is invariant, mixing and of maximal entropy $(k-p) \log d$ on A. Theorem 1.1 is a first step in order to obtain other ergodic and stochastic properties on ν as exponential mixing or central limit theorem. We postpone this question in a future work.

Under the same assumptions, we deduce from the work of de Thélin [dT08], see also [Dup09], the following result on ν .

Theorem 1.2. If f is as in Theorem 1.1, then the measure ν is hyperbolic. More precisely, counting with multiplicity it has k - p Lyapunov exponents larger than or equal to $(\log d)/2$ and p Lyapunov exponents strictly smaller than $-(k-p)(\log d)/2$.

2 Structural discs of currents

In this section we recall the notion of structural varieties of currents. It was introduced by Dinh and Sibony in order to put a geometry on the space $\mathscr{C}_p(U)$ which is of infinite dimension, see [DS06] and [Din07]. The definition of structural varieties is based on slicing theory and they can be seen as complex subvarieties inside $\mathscr{C}_p(U)$. In [DS09], the authors developed the notion of super-potential which involves more deeply this geometry.

Slicing theory can be seen as a generalization to currents of restriction of smooth forms to submanifolds. We will briefly explain it in our context and refer to [Fed69] for a more complete account.

Let U be an open subset of \mathbb{P}^k satisfying the geometric hypothesis as above. Let V be a complex manifold of dimension l. We denote by π_U and π_V the canonical projections of $U \times V$ to U and V respectively. If \mathscr{R} is a positive closed current of bidegree (p, p) on $U \times V$ with $\pi_U(\operatorname{supp}(\mathscr{R})) \subseteq U$ then, for all θ in V, the slice $\langle \mathscr{R}, \pi_V, \theta \rangle$ is well defined. For any (k - p, k - p)form ϕ on $U \times V$ we have

$$\langle \mathscr{R}, \pi_V, \theta \rangle(\phi) = \lim_{\epsilon \to 0} \langle \mathscr{R} \wedge \pi_V^*(\psi_{\theta,\epsilon}), \phi \rangle$$

where $\psi_{\theta,\epsilon}$ is an appropriate approximation in V of the Dirac mass at θ . It is a (p + l, p + l)-current on $U \times V$ supported on $U \times \{\theta\}$ which can be identified to a (p, p)-current on U. A family of currents $(R_{\theta})_{\theta \in V}$ in $\mathscr{C}_p(U)$ is a *structural variety* if there exists a positive closed current \mathscr{R} in $U \times V$ such that $R_{\theta} = \langle \mathscr{R}, \pi_V, \theta \rangle$. When V is isomorphic to the unit disc of \mathbb{C} , we call $(R_{\theta})_{\theta \in V}$ a *structural disc*.

Recall that in our situation $f(U) \in U$. Under the geometrical assumption on U, Dinh constructed in [Din07, p.233] a family of structural discs in $\mathscr{C}_p(U)$. He uses that for each $x \in L$ the set $I(x) \cap U$ is star-shaped at x to obtain a property similar to star-shappens for $\mathscr{C}_p(U)$.

More precisely, up to an automorphism, we can assume that

$$I = \{ x \in \mathbb{P}^k \mid x_i = 0, \ 0 \le i \le k - p \}, \ L = \{ x \in \mathbb{P}^k \mid x_i = 0, \ k - p + 1 \le i \le k \},$$

where $x = [x_0 : \cdots : x_k]$ are the homogeneous coordinates of \mathbb{P}^k . For $\theta \in \mathbb{C}$, $A_{\theta}(x) := [x_0 : \cdots : x_{k-p} : \theta x_{k-p+1} : \cdots : \theta x_k]$ is an automorphism of \mathbb{P}^k except for $\theta = 0$ where it is the projection of $\mathbb{P}^k \setminus I$ on L. Let set U' := f(U). As $I(x) \cap U$ is star-shaped at $x \in L$, there exists a simply connected open neighborhood $V \subset \mathbb{C}$ of [0, 1] such that $A_{\theta}(U') \Subset U$ for all θ in \overline{V} . If S is in $\mathscr{C}_p(U')$ then the family $(S_{\theta})_{\theta \in V}$ with $S_{\theta} := (A_{\theta})_*(S)$ defined a structural disc. Indeed, if $\Lambda : \mathbb{P}^k \times V \to \mathbb{P}^k$ is the meromorphic map defined by $\Lambda(x, \theta) = (A_{\theta})^{-1}(x)$ and $\mathscr{S} := \Lambda^*S$ then $S_{\theta} = \langle \mathscr{S}, \pi_V, \theta \rangle$, see [Din07] for more details. For any S in $\mathscr{C}_p(U')$, we have that $S_1 = S$ and $S_0 = [L]$ which is independent of S. In other words, any current in $\mathscr{C}_p(U')$ is linked to [L] by a structural disc in $\mathscr{C}_p(U)$. Moreover, S_{θ} depends continuously on θ and we have the following important property.

Lemma 2.1 ([Din07]). Let S be in $\mathscr{C}_p(U')$ and $(S_{\theta})_{\theta \in V}$ be the structural disc described above. If ϕ is a real continuous (k - p, k - p)-form with $dd^c \phi = 0$ on U then the function $\theta \mapsto \langle S_{\theta}, \phi \rangle$ is harmonic on V.

3 q-Convex set and d-bar equation

The concept of q-convexity generalizes both Stein and compact manifolds. Andreotti and Grauert [AG62] obtained vanishing or finiteness theorems for q-convex manifolds and, in [HL88], Henkin and Leiterer developed a similar theory using integral representations. In particular, they obtained solutions of the $\overline{\partial}$ -equation with explicit estimates, which play a key role in our proof. For this reason, we use the conventions of [HL88].

If $1 \leq q \leq k$ is an integer then a real \mathcal{C}^2 function ρ on an open subset $V \subset \mathbb{P}^k$ is called *q*-convex if, in any holomorphic local coordinates, the Hermitian form

$$L_{\rho}(x)t = \sum_{i,j=1}^{k} \frac{\partial^2 \rho(x)}{\partial z_i \partial \overline{z}_j} t_i \overline{t}_j$$

has at least q strictly positive eigenvalues at any point $x \in V$.

Let $0 \le q \le k-1$. We say that an open subset D of \mathbb{P}^k is *strictly q-convex* if there exists a (q+1)-convex function ρ in a neighborhood V of ∂D such that

$$D \cap V = \{ x \in V \, | \, \rho(x) < 0 \}.$$

Moreover, if the same condition is satisfied with V a neighborhood of D then D is called *completely strictly q-convex*.

The strict *q*-convexity has the following important consequence, see [HL88, Theorem 11.2].

Theorem 3.1. Let D be a strictly q-convex open subset of \mathbb{P}^k with \mathcal{C}^2 boundary. If ϕ is a continuous $\overline{\partial}$ -exact form of bidegree (r, s) in a neighborhood of \overline{D} with $0 \leq s \leq k, k-q \leq r \leq k$, then there exists a continuous (s, r-1)-form ψ on D such that $\overline{\partial}\psi = \phi$ and

$$\|\psi\|_{\infty,D} \le C \|\phi\|_{\infty,D}$$

for some C > 0 independent of ϕ .

Furthermore, Andreotti and Grauert proved the following vanishing theorem, see [AG62] and [HL88, Theorem 12.7].

Theorem 3.2. If D is a completely strictly q-convex open subset of \mathbb{P}^k with \mathcal{C}^2 boundary then $H^{s,r}(D,\mathbb{C}) = 0$ for any $0 \leq s \leq k$ and $k - q \leq r \leq k$.

Henkin and Leiterer [HL88, Theorem 5.13] give the following criteria of q-convexity, which is closely related to our geometric assumption on U with q = p - 1.

Theorem 3.3. Let D be an open subset of \mathbb{P}^k with \mathcal{C}^2 boundary. If for each $x \in \partial D$ there exists a complex submanifold $Y \subset \mathbb{P}^k$ of dimension q + 1, transverse to ∂D and such that $Y \cap D$ is a strictly pseudoconvex domain in Y, then D is strictly q-convex.

This result applies to our trapping neighborhood U with q = p-1. Indeed, observe that, possibly by exchanging U by a slightly smaller open set which contains f(U), we can assume that ∂U is smooth and the intersection of ∂U with I(x) is transverse for all $x \in L$. The projective space I(x) has dimension p = q + 1 and $U \cap I(x)$ is strictly convex in $I(x) \setminus I \simeq \mathbb{C}^p$, so in particular strictly pseudoconvex in I(x). Therefore, by Theorem 3.3, U is strictly (p-1)-convex. In the sequel, we always choose such an attracting neighborhood U.

Up to an automorphism of \mathbb{P}^k , I is defined in homogeneous coordinates by $I = \{x \in \mathbb{P}^k \mid x_i = 0, 0 \le i \le k - p\}$. The function

$$\eta(x) = \frac{|x_{k-p+1}|^2 + \dots + |x_k|^2}{|x_0|^2 + \dots + |x_{k-p}|^2},$$

is a (q + 1)-convex exhausting function of $\mathbb{P}^k \setminus I$, i.e. $\mathbb{P}^k \setminus I$ is completely *q*-convex. In general, strictly *q*-convex subsets of a completely *q*-convex domain are not completely strictly *q*-convex. However, in our case it is easy to construct from a *q*-convex function ρ such that

$$U \cap V = \{ x \in V \, | \, \rho(x) < 0 \}$$

for some neighborhood V of ∂U , a q-convex defining function defined in a neighborhood of \overline{U} . Indeed, it is enough to compose (η, ρ) with a good approximation of the maximum function (see [HL88, Definition 4.12]). It will give a (q + 1)-convex function since the positive eigenvalues of the complex Hessians of ρ and η are in the same directions. Therefore, U is completely strictly (p-1)-convex and we have the following solution for the dd^c -equation in symmetric bidegrees. **Theorem 3.4.** Let U be as above. If φ is a $C^2(r,r)$ -form in a neighborhood of \overline{U} with $k - p \leq r \leq k$, then there exists a continuous (r,r)-form ψ on U such that $dd^c\psi = dd^c\varphi$ and

$$\|\psi\|_{\infty,U} \le C \|dd^c\varphi\|_{\infty,U}$$

for some C > 0 independent of φ .

Proof. The proof follows closely the proof of Theorem 2.7 in [DNS08].

Without loss of generality, we can assume that φ is real and therefore $dd^c \varphi$ is also real. First, we solve the equation $d\xi = dd^c \varphi$ with estimates. Let W be a small neighborhood of \overline{U} , with the same geometric property and such that φ is defined on W. The maps A_{θ} defined in Section 2 give a homotopy $A : [0,1] \times W \to W$, $A(\theta, x) = A_{\theta}(x)$, between $A_1 = \text{Id}$ and the projection A_0 of W on L. Since L has dimension k-p, A_0^* vanish identically on (r+1, r+1)-forms if $r \geq k - p$. Therefore, by homotopy formula (see e.g. [BT82, p38]), there exists a form ξ on W such that $d\xi = dd^c \varphi$ and $\|\xi\|_{\infty,U} \lesssim \|dd^c \varphi\|_{\infty,U}$. Moreover, possibly by exchanging ξ by $(\xi + \overline{\xi})/2$, we can assume that $\xi = \Xi + \overline{\Xi}$ where Ξ is a (r, r+1)-form. As $d\xi$ is a (r+1, r+1)-form, it follows that $\overline{\partial \Xi} = 0$ and $d\xi = \partial \Xi + \overline{\partial \Xi}$. Therefore, by Theorem 3.2, Ξ is $\overline{\partial}$ -exact and by Theorem 3.1, there exists a continuous (r, r)-form Ψ such that $\overline{\partial \Psi} = \Xi$ and $\|\Psi\|_{\infty,U} \lesssim \|\Xi\|_{\infty,U}$.

Finally, if $\psi = -i\pi(\Psi - \overline{\Psi})$ we have

$$dd^{c}\psi = \partial\overline{\partial}(\Psi - \overline{\Psi}) = \partial\Xi + \overline{\partial\Xi} = dd^{c}\varphi,$$

and

$$\|\psi\|_{\infty,U} \lesssim \|\Xi\|_{\infty,U} \lesssim \|dd^c\varphi\|_{\infty,U}.$$

4 Attracting speed

For R in $\mathscr{C}_p(U)$, we denote by R_n its normalized push-forward by f^n , i.e. $R_n := d^{-(k-p)n}(f^n)_*(R)$. To obtain (1.1), the first observation is that the norm of $R_n - \tau$, seen as a linear form on the space of continuous test (k-p, k-p)-forms, is bounded independently of n and R. Therefore, it is sufficient to establish (1.1) for $\alpha = 2$ and then apply interpolation theory between Banach spaces, see e.g. [Tri95], in order to obtain the general case.

Let denote by X the set of all real continuous (k - p, k - p)-forms ϕ on U such that $dd^c\phi = 0$ and $|\langle R - \tau, \phi \rangle| \leq 1$ for all $R \in \mathscr{C}_p(U)$. Observe that, since $f(U) \Subset U$, if ϕ is in X then $f^*(\phi)$ is defined on U where it is still a

real continuous form with $dd^c(f^*(\phi)) = 0$. The set X is a truncated convex cone and the first part of the proof of Theorem 1.1 is to show that $d^{-(k-p)}f^*$ acts by contraction on it. This result is available without any assumption on $\| \wedge^{k-p+1} Df \|$. It is based on Lemma 2.1 and Harnack's inequality for harmonic functions.

Lemma 4.1. There exists a constant $0 < \lambda_1 < 1$ such that for any R in $\mathscr{C}_p(U)$, ϕ in X and n in \mathbb{N} we have

$$|\langle R_n - \tau, \phi \rangle| \le \lambda_1^n.$$

Proof. If R is in $\mathscr{C}_p(U)$ and ϕ in X, $R_1 := d^{-(k-p)} f_*(R)$ is in $\mathscr{C}_p(U')$ and we define the function $h_{R,\phi}$ on V by $h_{R,\phi}(\theta) := \langle R_{1,\theta} - \tau, \phi \rangle$, where $\theta \mapsto R_{1,\theta}$ is the structural disc described in Section 2. The definition of X implies that $|h_{R,\phi}| \leq 1$ on V, for all $R \in \mathscr{C}_p(U)$ and $\phi \in X$. Moreover, since R_1 is in $\mathscr{C}_p(U')$, it follows from Lemma 2.1 that all these functions are harmonic on V.

Now, observe that if we take $R = \tau$ then $h_{\tau,\phi}(1) = 0$ for all $\phi \in X$, since $d^{p-k}f_*\tau = \tau$. Hence, as $|h_{\tau,\phi}| \leq 1$ on V, Harnack's inequality says that there exists $0 \leq a < 1$ such that $|h_{\tau,\phi}(0)| \leq a$ for all ϕ in X. On the other hand, $R_{1,0}$ is a current independent of R. So, for all $R \in \mathscr{C}_p(U)$ and $\phi \in X$ we have $h_{R,\phi}(0) = h_{\tau,\phi}(0)$ and therefore $|h_{R,\phi}(0)| \leq a$. Once again, we deduce from Harnack's inequality there exists $0 < \lambda_1 < 1$, independent of R and ϕ , such that $|h_{R,\phi}(1)| \leq \lambda_1$ or equivalently

$$\left|\left\langle R_1 - \tau, \frac{\phi}{\lambda_1}\right\rangle\right| = \left|\left\langle R - \tau, \phi_1\right\rangle\right| \le 1,$$

where $\phi_1 = d^{-(k-p)} f^*(\phi/\lambda_1)$. Moreover, ϕ_1 is defined on U and $dd^c \phi_1 = 0$. It follows that ϕ_1 is in X. Using the same arguments with ϕ_1 instead of ϕ gives that $|\langle R_1 - \tau, \phi_1 \rangle| \leq \lambda_1$ which can be rewrite $|\langle R_2 - \tau, \phi \rangle| \leq \lambda_1^2$. Inductively, we obtain that $|\langle R_n - \tau, \phi \rangle| \leq \lambda_1^n$.

Remark 4.2. The constant λ_1 is not directly related to f. Indeed, it only depends on V i.e. on the size of U and the distance between ∂U and $\partial f(U)$. If h is the unique biholomorphism between V and the unit disc in \mathbb{C} such that h(0) = 0 and $h(1) = \alpha \in]0, 1[$ then Harnack's inequality gives explicitly that we can take $a = 2\alpha/(1+\alpha)$ and $\lambda_1 = 4\alpha/(1+\alpha)^2$.

In order to prove Theorem 1.1, we use Theorem 3.4 together with the assumption on $\| \wedge^{k-p+1} Df \|$ and Lemma 4.1.

If $\|\wedge^{k-p+1} Df(z)\| < 1$ on \overline{U} then by continuity, there exists a constant $0 < \lambda_2 < 1$ such that $\|\wedge^{k-p+1} Df(z)\| < \lambda_2$ on U. Hence, if φ is a (k-p, k-p)-form of class \mathcal{C}^2 , we have for $\varphi_i := d^{-i(k-p)}(f^i)^*(\varphi)$ with $i \in \mathbb{N}$

$$\|dd^{c}\varphi_{i}\|_{\infty,U} \lesssim \frac{\lambda_{2}^{2i}}{d^{i(k-p)}}\|\varphi\|_{\mathcal{C}^{2}}$$

Here, the symbol \leq means inequality up to a constant which only depends on our conventions and on U. By Theorem 3.4 with r = k - p, there exists a continuous (k - p, k - p)-form ψ_i on U such that

$$dd^c\psi_i = dd^c\varphi_i$$

and

$$\|\psi_i\|_{\infty,U} \lesssim \|dd^c \varphi_i\|_{\infty,U} \lesssim \frac{\lambda_2^{2i}}{d^{i(k-p)}} \|\varphi\|_{\mathcal{C}^2}.$$

We can now complete the proof of our main result.

End of the proof of Theorem 1.1. Let R be in $\mathscr{C}_p(U)$ and φ be a (k-p, k-p)form of class \mathcal{C}^2 . Without loss of generality, we can assume that φ is real. Let $0 \leq i \leq n$ be two arbitrary integers. We set l := n - i. If R_n , φ_i and ψ_i are defined as above then we have

$$\langle R_n - \tau, \varphi \rangle = \langle R_l - \tau, \varphi_i \rangle = \langle R_l - \tau, \varphi_i - \psi_i \rangle + \langle R_l - \tau, \psi_i \rangle,$$

since τ is invariant. On the one hand,

$$\langle R_l - \tau, \psi_i \rangle \lesssim \|\psi_i\|_{\infty, U} \lesssim \frac{\lambda_2^{2i}}{d^{i(k-p)}} \|\varphi\|_{\mathcal{C}^2},$$

$$(4.1)$$

since R_l and τ are supported on U. On the other hand, observe that there exists a constant $M \geq 1$ independent of φ such that $\|d^{-(k-p)}f^*(\varphi)\|_{\infty} \leq M \|\varphi\|_{\infty}$. Therefore,

$$\begin{aligned} \|\varphi_i - \psi_i\|_{\infty, U} &\leq M^i \|\varphi\|_{\infty} + \|\psi_i\|_{\infty, U} \leq M^i \|\varphi\|_{\infty} + C \frac{\lambda_2^{2i}}{d^{i(k-p)}} \|\varphi\|_{\mathcal{C}^2} \\ &\lesssim M^i \|\varphi\|_{\mathcal{C}^2}, \end{aligned}$$

and in particular

$$|\langle S - \tau, \varphi_i - \psi_i \rangle| \lesssim M^i \|\varphi\|_{\mathcal{C}^2},$$

for any S in $\mathscr{C}_p(U)$.

Moreover, $\varphi_i - \psi_i$ is a real continuous (k - p, k - p)-form on U and $dd^c(\varphi_i - \psi_i) = 0$. Hence, $(\varphi_i - \psi_i)/(CM^i \|\varphi\|_{C^2})$ belongs to X where C > 0

is a constant depending only on U and on our conventions. It follows by Lemma 4.1 that

$$|\langle R_l - \tau, \varphi_i - \psi_i \rangle| \le CM^i ||\varphi||_{\mathcal{C}^2} \lambda_1^l.$$
(4.2)

To summarize, equations (4.1) and (4.2) imply that there are constants $0 < \lambda_1, \lambda_2 < 1$, and $M \ge 1$ such that

$$|\langle R_n - \tau, \varphi \rangle| \lesssim \|\varphi\|_{\mathcal{C}^2} \left(M^i \lambda_1^l + \frac{\lambda_2^{2i}}{d^{i(k-p)}} \right).$$

If $q \in \mathbb{N}$ is large enough then $M\lambda_1^q < 1$. Therefore, if we choose $n \simeq (q+1)i$, we obtain $l \simeq iq$ and

$$|\langle R_n - \tau, \varphi \rangle| \lesssim \|\varphi\|_{\mathcal{C}^2} \lambda^n$$

where $\lambda := \max(\lambda_2^2 d^{-(k-p)}, M \lambda_1^q)^{1/(q+1)} < 1$. This estimate holds for arbitrary n in \mathbb{N} and is uniform on φ and R.

Remark 4.3. In Theorem 1.1, it is enough to assume that $\|\wedge^{k-p+1} Df(z)\| < d^{(k-p)/2}$ on \overline{U} . Moreover, it is easy using small perturbations of a suitable polynomial map to construct examples with $\|\wedge^{k-p+1} Df(z)\|$ as small as we want on \overline{U} .

5 Hyperbolicity of the equilibrium measure

In this section, we prove Theorem 1.2. Recall that the equilibrium measure associated to A is given by $\nu := \tau \wedge T^{k-p}$. It has maximal entropy on A equal to $(k-p) \log d$, [Din07]. On the other hand, we have the following powerful result, see [dT08] and [Dup09].

Theorem 5.1. If the Lyapunov exponents of ν are ordered so that

$$\chi_1 \geq \cdots \geq \chi_{a-1} > \chi_a \geq \cdots \geq \chi_k,$$

then

$$h(\nu) \le (a-1)\log d + 2\sum_{i=a}^{k} \chi_i^+,$$
 (5.1)

where $h(\nu)$ denotes the entropy of ν and $\chi_i^+ := \max(\chi_i, 0)$.

Now, let $1 \le c \le k$ be such that

$$\chi_1 \geq \cdots \geq \chi_c > 0 \geq \chi_{c+1} \geq \cdots \geq \chi_k.$$

If we take a = c + 1 in Theorem 5.1, we obtain $h(\nu) \le c \log d$. Since $h(\nu) = (k-p) \log d$, it follows that $c \ge (k-p)$. It means there are at least k-p strictly

positive Lyapunov exponents. Moreover, if we have equality, c = k - p, Theorem 5.1 applied to the smallest *a* such that $\chi_a = \chi_c$ gives

$$(k-p)\log d = h(\nu) \le (a-1)\log d + 2(k-p-a+1)\chi_c.$$

Hence, $\chi_c \geq (\log d)/2$. Note that in this part we do not need the assumption on $\| \wedge^{k-p+1} Df \|$.

It remains to prove that the assumptions of Theorem 1.1 imply that $c \leq k - p$ and $\chi_{c+1} < -(k - p)(\log d)/2$. It is not hard to deduce form Oseledec theorem [Ose68] that the sum of the q largest Lyapunov exponents verifies

$$\chi_1 + \dots + \chi_q = \lim_{n \to \infty} \frac{1}{n} \log \| \wedge^q Df^n(z) \|,$$

for ν -almost all z. Moreover, we have

$$\|\wedge^q Df^{n+m}(z)\| \le \|\wedge^q Df^n(z)\| \|\wedge^q Df^m(f^n(z))\|$$

Therefore, it follows that

$$\|\wedge^q Df^n(z)\| \le (\max_{z \in U} \|\wedge^q Df(z)\|)^n$$

and

$$\chi_1 + \dots + \chi_q \le \log \max_{z \in U} \| \wedge^q Df(z) \| =: \gamma.$$

Hence, if $\| \wedge^{k-p+1} Df(z) \| < 1$ on \overline{U} then

$$\chi_1 + \dots + \chi_{k-p+1} \le \gamma < 0.$$

Therefore, $c \leq k - p$ and we have seen above that in this case c = k - p and $\chi_c \geq (\log d)/2$. Finally, we have

$$\gamma \ge \chi_1 + \dots + \chi_{k-p} + \chi_{k-p+1} \ge \frac{k-p}{2} \log d + \chi_{k-p+1},$$

which implies

$$\chi_{k-p+1} \le \gamma - \frac{k-p}{2} \log d.$$

Remark 5.2. Theorem 5.1 with a = 1 implies the Ruelle inequality, i.e.

$$\chi_1 + \dots + \chi_c \ge \frac{k-p}{2} \log d.$$

Therefore, it is enough to assume that $\|\wedge^{k-p+1} Df(z)\| < d^{(\frac{k-p}{2})(\frac{k-p+1}{k})}$ on \overline{U} since

$$\chi_1 + \dots + \chi_{k-p+1} \ge \frac{k-p+1}{c}(\chi_1 + \dots + \chi_c),$$

if $c \ge k - p + 1$.

References

- [AG62] A. Andreotti and H. Grauert. Théorème de finitude pour la cohomologie des espaces complexes. Bull. Soc. Math. France, 90:193– 259, 1962.
- [BDM07] A. Bonifant, M. Dabija, and J. Milnor. Elliptic curves as attractors in P². I. Dynamics. *Experiment. Math.*, 16(4):385–420, 2007.
- [BT82] R. Bott and L. Tu. Differential forms in algebraic topology, volume 82 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1982.
- [Din07] T.-C. Dinh. Attracting current and equilibrium measure for attractors on \mathbb{P}^k . J. Geom. Anal., 17(2):227–244, 2007.
- [DNS08] T.-C. Dinh, V.-A. Nguyên, and N. Sibony. Dynamics of horizontallike maps in higher dimension. Adv. Math., 219(5):1689–1721, 2008.
- [DS06] T.-C. Dinh and N. Sibony. Geometry of currents, intersection theory and dynamics of horizontal-like maps. Ann. Inst. Fourier (Grenoble), 56(2):423–457, 2006.
- [DS09] T.-C. Dinh and N. Sibony. Super-potentials of positive closed currents, intersection theory and dynamics. *Acta Math.*, 203(1):1–82, 2009.
- [DS10] T.-C. Dinh and N. Sibony. Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. In *Holomorphic dynamical systems*, volume 1998 of *Lecture Notes* in Math., pages 165–294. Springer, Berlin, 2010.
- [dT08] H. de Thélin. Sur les exposants de Lyapounov des applications méromorphes. *Invent. Math.*, 172(1):89–116, 2008.
- [Dup09] C. Dupont. Large entropy measures for endomorphisms of CP(k). arXiv/0911.4675, 2009.
- [Fed69] H. Federer. Geometric measure theory. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

- [FS01] J.-E. Fornæss and N. Sibony. Dynamics of P² (examples). In Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998), volume 269 of Contemp. Math., pages 47–85. Amer. Math. Soc., Providence, RI, 2001.
- [FW99] J.-E. Fornæss and B. Weickert. Attractors in P². In Several complex variables (Berkeley, CA, 1995–1996), volume 37 of Math. Sci. Res. Inst. Publ., pages 297–307. Cambridge Univ. Press, Cambridge, 1999.
- [HL88] G. Henkin and J. Leiterer. Andreotti-Grauert theory by integral formulas, volume 74 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, MA, 1988.
- [JW00] M. Jonsson and B. Weickert. A nonalgebraic attractor in \mathbb{P}^2 . Proc. Amer. Math. Soc., 128(10):2999–3002, 2000.
- [Mil85] J. Milnor. On the concept of attractor. Comm. Math. Phys., 99(2):177–195, 1985.
- [Ose68] V. I. Oseledec. A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems. Trudy Moskov. Mat. Obšč., 19:179–210, 1968.
- [Rue89] D. Ruelle. Elements of differentiable dynamics and bifurcation theory. Academic Press Inc., Boston, MA, 1989.
- [Taf10] J. Taflin. Invariant elliptic curves as attractors in the projective plane. J. Geom. Anal., 20(1):219–225, 2010.
- [Taf11] J. Taflin. Equidistribution speed towards the Green current for endomorphisms of \mathbb{P}^k . Adv. Math., 227:2059–2081, 2011.
- [Tri95] H. Triebel. Interpolation theory, function spaces, differential operators. Johann Ambrosius Barth, Heidelberg, second edition, 1995.

J. Taflin, UPMC Univ Paris 06, UMR 7586, Institut de Mathématiques de Jussieu, F-75005 Paris, France. taflin@math.jussieu.fr Universitetet i Oslo, Mathematisk Institutt, Postok 1053 Blindern, 0316 Oslo, Norway. johantaf@math.uio.no