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# Equidistribution speed towards the Green current for endomorphisms of $\mathbb{P}^k$

Johan Taflin

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## Abstract

Let  $f$  be a non-invertible holomorphic endomorphism of  $\mathbb{P}^k$ . For a hypersurface  $H$  of  $\mathbb{P}^k$ , generic in the Zariski sense, we give an explicit speed of convergence of  $f^{-n}(H)$  towards the dynamical Green  $(1, 1)$ -current of  $f$ .

**Key words:** Green current, equidistribution speed.

**AMS 2000 subject classification:** 37F, 32H.

## 1 Introduction

Let  $f$  be a holomorphic endomorphism of algebraic degree  $d \geq 2$  on the complex projective space  $\mathbb{P}^k$ . The iterates  $f^n = f \circ \dots \circ f$  define a dynamical system on  $\mathbb{P}^k$ . It is well known that, if  $\omega$  denotes the normalized Fubini-Study form on  $\mathbb{P}^k$  then the sequence  $d^{-n}(f^n)^*(\omega)$  converges to a positive closed current  $T$  of bidegree  $(1, 1)$  called the Green current of  $f$  (see e.g. [9]). It is a totally invariant current, whose support is the Julia set of  $f$  and that exhibits interesting dynamical properties. In particular, for a generic hypersurface  $H$  of degree  $s$ , the sequence  $d^{-n}(f^n)^*[H]$  converges to  $sT$  [7]. Here,  $[H]$  denotes the current of integration on  $H$  and the convergence is in the sense of currents. In fact, if we denote by  $T^p$  the self-intersection  $T \wedge \dots \wedge T$ , Dinh and Sibony proposed the following conjecture on equidistribution.

**Conjecture 1.1.** *Let  $f$  be a holomorphic endomorphism of  $\mathbb{P}^k$  of algebraic degree  $d \geq 2$  and  $T$  its Green current. If  $H$  is an analytic set of pure codimension  $p$  and of degree  $s$  which is generic in the Zariski sense, then the sequence  $d^{-pn}(f^n)^*[H]$  converges to  $sT^p$  exponentially fast.*

The aim of the paper is to prove the conjecture for  $p = 1$ . It is a direct consequence of the following more precise result on currents. Indeed, we only have to apply the theorem to  $S := s^{-1}[H]$  for hypersurfaces  $H$  which does not contain any element of  $\mathcal{A}_\lambda$ .

**Theorem 1.2.** *Let  $f, T$  be as above and let  $1 < \lambda < d$ . There exists a finite family  $\mathcal{A}_\lambda$  of periodic irreducible analytic sets such that if  $S$  is a positive closed  $(1, 1)$ -current of mass 1, whose dynamical potential  $u$  verifies  $\|u\|_{L^1(X)} \leq C$  for all  $X$  in  $\mathcal{A}_\lambda$ , then the sequence  $S_n := d^{-n}(f^n)^*(S)$  converges exponentially fast to  $T$ . More precisely, for every  $0 < \beta \leq 2$  and  $\phi \in \mathcal{C}^\beta(\mathbb{P}^k)$  we get*

$$|\langle S_n - T, \phi \rangle| \leq A \|\phi\|_{\mathcal{C}^\beta} \left( \frac{\lambda}{d} \right)^{n\beta/2}, \quad (1.1)$$

where  $A > 0$  depends on the constants  $C$  and  $\beta$  but is independent of  $S$ ,  $\phi$  and  $n$ .

Here, the space  $L^1(X)$  is with respect to the volume form  $\omega^{\dim(X)}$  on  $X$  and  $\mathcal{C}^\beta(\mathbb{P}^k)$  denotes the space of  $(k-1, k-1)$ -forms whose coefficients are of class  $\mathcal{C}^\beta$ , equipped with the norm induced by a fixed atlas. The dynamical potential of  $S$  is the unique quasi-plurisubharmonic function  $u$  such that  $S = dd^c u + T$  and  $\max_{\mathbb{P}^k} u = 0$ . Note that  $\mathcal{A}_\lambda$  will be explicitly constructed. Theorem 1.2 still holds if we replace  $\mathcal{A}_\lambda$  by an analytic subset, e.g. a finite set, which intersects all components of  $\mathcal{A}_\lambda$ .

**Remark 1.3.** *We can prove a similar result for a regular polynomial automorphism  $f : \mathbb{C}^k \rightarrow \mathbb{C}^k$  outside its indeterminacy set, see [20].*

Equidistribution problem without speed was considered in dimension 1 by Brolin [3] for polynomials and by Lyubich [19] and Freire, Lopes and Mañé [16] for rational maps. They proved that for every point  $a$  in  $\mathbb{P}^1$ , with maybe two exceptions, the preimages of  $a$  by  $f^n$  converge towards the equilibrium measure, which is the counterpart of the Green current in dimension 1. In this context, a result in the direction of Theorem 1.2 was obtained in [11] (see also references therein).

In higher dimension, for  $p = k$ , simple convergence in Conjecture 1.1 was established by Fornæss and Sibony [14], Briend and Duval [2], Dinh and Sibony [6]. Recently in [10], Dinh and Sibony give exponential speed of convergence, which completes Conjecture 1.1 for  $p = k$ . The equidistribution of hypersurfaces was proved by Fornæss and Sibony for generic maps [15] and by Favre and Jonsson in dimension 2 [13]. The convergence for general endomorphisms and Zariski generic hypersurfaces was obtained by Dinh and Sibony in [7]. These papers state convergence but without speed. In other

codimensions, the problem is much more delicate. However, the conjecture was solved for generic maps in [8], using the theory of super-potentials.

We partially follow the strategy developed in [15], [13] and [7], which is based on pluripotential theory together with volume estimates, i.e. a lower bound to the contraction of volume by  $f$ . These estimates are available outside some exceptional sets which are treated using hypothesis on the map  $f$  or on the current  $S$ .

The exceptional set  $\mathcal{A}_\lambda$  will be defined in Section 5. It is in general a union of periodic analytic sets possibly singular. In our proof of Theorem 1.2, it is necessary to obtain the convergence of the trace of  $S_n$  to these analytic sets. So, we have to prove an analog of Theorem 1.2 where  $\mathbb{P}^k$  is replaced with an invariant analytic set. The geometry of the analytic set near singularities is the source of important technical difficulties. We will collect in Section 2 and Section 3 several versions of Lojasiewicz's inequality which will allow us to work with singular analytic sets and also to obtain good estimates on the size of a ball under the action of  $f^n$ . Such estimates are crucial in order to obtain the convergence outside exceptional sets.

Theorem 1.2 can be reformulated as an  $L^1$  estimate of the dynamical potential  $u_n$  of  $S_n$  (see Theorem 6.1). The problem is equivalent to a size control of the sublevel set  $K_n = \{u_n \leq -(\lambda/d)^n\}$ . Since  $T$  is totally invariant, we get that  $u_n = d^{-n}u \circ f^n$  and  $f^n(K_n) = \{u \leq -\lambda^n\}$ . The above estimate on the size of ball can be applied provided that  $K_n$  be not concentrated near the exceptional sets. The last property will be obtained using several generalizations of exponential Hörmander's estimate for plurisubharmonic functions that will be stated in Section 4. A key point in our approach is that, by reducing the domain of integration, we obtain uniform exponential estimates for non-compact families of quasi-plurisubharmonic functions.

We close this introduction by setting some notations and conventions. The symbols  $\lesssim$  and  $\gtrsim$  mean inequalities up to constants which only depend on  $f$  or on the ambient space. To desingularize an analytic subset of  $\mathbb{P}^k$ , we always use a finite sequence of blow-ups of  $\mathbb{P}^k$ . Unless otherwise specified, the distances that we consider are naturally induced by embedding or smooth metrics for compact manifolds. For  $K > 0$  and  $0 < \alpha \leq 1$ , we say that a function  $u : X \rightarrow \mathbb{C}$  is  $(K, \alpha)$ -Hölder continuous if for all  $x$  and  $y$  in  $X$ , we have  $|u(x) - u(y)| \leq K \text{dist}(x, y)^\alpha$ . We denote by  $\mathbb{B}$  the unit ball of  $\mathbb{C}^k$  and for  $r > 0$ , by  $\mathbb{B}_r$  the ball centered at the origin with radius  $r$ . In  $\mathbb{P}^k$ , we denote by  $B(x, r)$  the ball of center  $x$  and of radius  $r$ . And, for  $X \subset \mathbb{P}^k$  an analytic subset, we denote by  $B_X(x, r)$  the connected component of  $B(x, r) \cap X$  which contains  $x$ . We call it the ball of center  $x$  and of radius  $r$  in  $X$ . It may have more than one irreducible component. Finally, for a subset  $Z \subset X$ , we denote by  $Z_{X,t}$  or simply  $Z_t$ , the tubular  $t$ -neighborhood of  $Z$  in  $X$ , i.e. the union of

$B_X(z, t)$  for all  $z$  in  $Z$ . A function on  $X$  is called (strongly) holomorphic if it has locally a holomorphic extension to a neighborhood of the ambient space.

## 2 Lojasiewicz's inequality and consequences

One of the main technical difficulties of our approach is related to singularities of analytic sets that we will handle using blow-ups along smooth varieties. In this section, we study the behavior of metric properties under blow-ups. It will allow us to establish volume and exponential estimates onto singular analytic sets.

We will frequently use the following Lojasiewicz inequalities. We refer to [1] for further details.

**Theorem 2.1.** *Let  $U$  be an open subset of  $\mathbb{C}^k$  and let  $h, g$  be subanalytic functions in  $U$ . If  $h^{-1}(0) \subset g^{-1}(0)$ , then for any compact subset  $K$  of  $U$ , there exists a constant  $N \geq 1$  such that, for all  $z$  in  $K$ , we have*

$$|h(z)| \gtrsim |g(z)|^N.$$

In this paper, we only use the notion of subanalytic function in the following case. Let  $U$  be an open subset of  $\mathbb{C}^k$  and  $A \subset U$  be an analytic set. Every compact of  $U$  has a neighborhood  $V \subset U$  such that the function  $x \mapsto \text{dist}(x, A)$  and analytic functions on  $U$  are subanalytic on  $V$ . Moreover, the composition or the sum of two such functions is still subanalytic on  $V$ . In particular, we have the following property.

**Corollary 2.2.** *Let  $U$  be an open subset of  $\mathbb{C}^k$  and let  $A, B$  be analytic subsets of  $U$ . Then for any compact subset  $K$  of  $U$ , there exists a constant  $N \geq 1$  such that, for all  $z$  in  $K$ , we have*

$$\text{dist}(z, A) + \text{dist}(z, B) \gtrsim \text{dist}(z, A \cap B)^N.$$

We briefly recall the construction of blow-up that we will use later. If  $U$  is an open subset of  $\mathbb{C}^k$  which contains 0, the blow-up  $\widehat{U}$  of  $U$  at 0 is the submanifold of  $U \times \mathbb{P}^{k-1}$  defined by the equations  $z_i w_j = z_j w_i$  for  $1 \leq i, j \leq k$  where  $(z_1, \dots, z_k)$  are the coordinates of  $\mathbb{C}^k$  and  $[w_1 : \dots : w_k]$  are the homogeneous coordinates of  $\mathbb{P}^{k-1}$ . The sets  $w_i \neq 0$  define local charts on  $\widehat{U}$  where the canonical projection  $\pi : \widehat{U} \rightarrow U$ , if we set for simplicity  $i = 1$  and  $w_1 = 1$ , is given by

$$\pi(z_1, w_2, \dots, w_k) = (z_1, z_1 w_2, \dots, z_1 w_k).$$

If  $V \subset \mathbb{C}^p$  is an open subset, the blow-up of  $U \times V$  along  $\{0\} \times V$  is defined by  $\widehat{U} \times V$ . This is the local model of a blow-up.

Finally, if  $X$  is a complex manifold, the blow-up  $\widehat{X}$  of  $X$  along a submanifold  $Y$  is obtained by sticking copies of the above model and by using suitable atlas of  $X$ . The natural projection  $\pi : \widehat{X} \rightarrow X$  defines a biholomorphism between  $\widehat{X} \setminus \widehat{Y}$  and  $X \setminus Y$  where the set  $\widehat{Y} := \pi^{-1}(Y)$  is called the exceptional hypersurface. If  $A$  is an analytic subset of  $X$  not contained in  $Y$ , the strict transform of  $A$  is defined as the closure of  $\pi^{-1}(A \setminus Y)$ .

We have the following elementary lemma.

**Lemma 2.3.** *Let  $\pi : \widehat{U} \times V \rightarrow U \times V$  be as above and  $\widehat{Y}$  denote the exceptional hypersurface in  $\widehat{U} \times V$ . Assume that  $U \times V$  is bounded in  $\mathbb{C}^k \times \mathbb{C}^p$ . Then for all  $\widehat{z}, \widehat{z}' \in \widehat{U} \times V$ , we have*

$$\text{dist}(z, z') \gtrsim \text{dist}(\widehat{z}, \widehat{z}')(\text{dist}(\widehat{z}, \widehat{Y}) + \text{dist}(\widehat{z}', \widehat{Y})),$$

where  $z = \pi(\widehat{z})$  and  $z' = \pi(\widehat{z}')$ .

*Proof.* Since  $\pi$  leaves invariant the second coordinate, considering the maximum norm on  $U \times V$ , the general setting is reduced to the case of blow-up of a point. Hence we can take  $V = \{0\}$ . The lemma is obvious if  $z$  or  $z'$  is equal to 0. Since  $\pi$  is a biholomorphism outside  $\widehat{Y}$ , we can assume that  $0 < \|z\|, \|z'\| < 1$ . Moreover, up to an isometry of  $\mathbb{C}^k$ , we can assume that  $\max |z_i| = |z_1|$  and  $\max |z'_i| = |z'_1|$ . Indeed, we can send  $z$  and  $z'$  into the plane generated by the first two coordinates and then use the rotation group of this plane. Therefore, in the chart  $w_1 = 1$ , we have  $\widehat{z} = (z_1, w_2, \dots, w_k)$ ,  $\widehat{z}' = (z'_1, w'_2, \dots, w'_k)$  with  $|w_i|, |w'_i| \leq 1$ .

By triangle inequality, we have

$$|z_1 w_i - z'_1 w'_i| \geq |z_1 w_i - z_1 w'_i| - |z_1 w'_i - z'_1 w'_i| \geq |z_1| |w_i - w'_i| - |z_1 - z'_1|.$$

Hence, by symmetry in  $z_1$  and  $z'_1$  we get

$$2|z_1 w_i - z'_1 w'_i| \geq (|z_1| + |z'_1|) |w_i - w'_i| - 2|z_1 - z'_1|.$$

Therefore, there is a constant  $a > 0$  independent of  $z$  and  $z'$  such that

$$|z_1 - z'_1| + \sum_{i=2}^k |z_1 w_i - z'_1 w'_i| \geq a(|z_1| + |z'_1|)(|z_1 - z'_1| + \sum_{i=2}^k |w_i - w'_i|).$$

The left-hand side corresponds to  $\text{dist}(z, z')$ . We also have that  $\text{dist}(\widehat{z}, \widehat{z}') \simeq |z_1 - z'_1| + \sum_{i=2}^k |w_i - w'_i|$  and  $\text{dist}(\widehat{z}, \widehat{Y}) + \text{dist}(\widehat{z}', \widehat{Y}) \simeq |z_1| + |z'_1|$ . The result follows.  $\square$

A similar result holds for analytic sets. More precisely, consider an irreducible analytic subset  $X$  of  $\mathbb{P}^k$  of dimension  $l$  and a smooth variety  $Y$  contained in  $X$ . Let  $\bar{\pi} : \widehat{\mathbb{P}^k} \rightarrow \mathbb{P}^k$  be the blow-up along  $Y$  and  $\pi$  the restriction of  $\bar{\pi}$  to the strict transform  $\widehat{X}$  of  $X$ . Denote by  $\bar{Y}$  and  $\widehat{Y}$  the exceptional hypersurfaces in  $\widehat{\mathbb{P}^k}$  and in  $\widehat{X}$  respectively.

**Lemma 2.4.** *There exists  $N \geq 1$  such that for all  $\widehat{z}$  and  $\widehat{z}'$  in  $\widehat{X}$*

$$\text{dist}(z, z') \gtrsim \text{dist}(\widehat{z}, \widehat{z}')(\text{dist}(\widehat{z}, \widehat{Y}) + \text{dist}(\widehat{z}', \widehat{Y}))^N,$$

*Proof.* The previous lemma gives the inequality with  $N = 1$  if we substitute  $\widehat{Y}$  by  $\bar{Y}$ . By Corollary 2.2 applied to  $A = \widehat{X}$  and  $B = \bar{Y}$ , there exists  $N \geq 1$  such that  $\text{dist}(\widehat{x}, \bar{Y}) \gtrsim \text{dist}(\widehat{x}, \widehat{Y})^N$  for all  $\widehat{x}$  in  $\widehat{X}$ . The result follows.  $\square$

Here is the first estimate on contraction for blow-ups.

**Lemma 2.5.** *There exists a constant  $N \geq 1$  such that for all  $0 < t \leq 1/2$ , if  $\text{dist}(\widehat{x}, \widehat{Y}) > t$  and  $r < t/2$ , then  $\pi(B_{\widehat{X}}(\widehat{x}, r))$  contains  $B_X(\pi(\widehat{x}), t^N r)$ . Moreover, if  $N$  is large enough then the image by  $\pi$  of a ball of radius  $0 < r \leq 1/2$  contains a ball of radius  $r^N$  in  $X$ .*

*Proof.* Let  $\widehat{y}$  be a point in  $\widehat{X}$  such that  $\text{dist}(\widehat{x}, \widehat{y}) = r$  and set  $x = \pi(\widehat{x})$ ,  $y = \pi(\widehat{y})$ . The assumption on  $r$  gives that  $\text{dist}(\widehat{y}, \widehat{Y}) > t/2$ . Therefore, we deduce from Lemma 2.4 that

$$\text{dist}(x, y) \gtrsim rt^N.$$

The first assertion follows since  $t \leq 1/2$  and  $\pi$  is a biholomorphism outside  $\widehat{Y}$ .

For a general ball  $B$  of radius  $r$  in  $\widehat{X}$ , we can reduce the ball in order to avoid  $\widehat{Y}$  and then apply the first statement. More precisely, as  $\dim(\widehat{Y}) \leq l-1$  there is a constant  $c > 0$  such that for all  $\rho > 0$ ,  $\widehat{Y}$  is covered by  $c\rho^{-2(l-1)}$  balls of radius  $\rho$ . On the other hand, by a theorem of Lelong [18, 4], the volume of a ball of radius  $\rho$  in  $\widehat{X}$  varies between  $c'^{-1}\rho^{2l}$  and  $c'\rho^{2l}$  for some  $c' > 0$ . Hence, the volume of  $\widehat{Y}_\rho$  is of order  $\rho^{2l}$ . Take  $\rho = c''r^l$  with  $c'' > 0$  small enough. By counting the volume, we see that  $B$  is not contained in  $\widehat{Y}_\rho$ . Therefore,  $B \setminus \widehat{Y}$  contains a ball of radius  $\rho/3$ . We obtain the result using the first assertion.  $\square$

In the same spirit, we have the following lemma.

**Lemma 2.6.** *Let  $\widehat{Z}$  be a compact manifold,  $Z$  be an irreducible analytic subset of  $\mathbb{P}^k$  and  $\pi : \widehat{Z} \rightarrow Z$  be a surjective holomorphic map. Let  $A$  be an irreducible analytic subset of  $Z$  and define  $\widehat{A} := \pi^{-1}(A)$ . There exists  $N \geq 1$  such that  $A_{tN}$  is included in  $\pi(\widehat{A}_t)$  for all  $t > 0$  small enough. Moreover, if  $\widehat{A}$  is the union on two analytic sets  $\widehat{A}_1, \widehat{A}_2$  such that  $A_2 := \pi(\widehat{A}_2)$  is strictly contained in  $A$  then  $\pi(\widehat{A}_{1,t})$  contains  $A_{tN} \setminus A_{2,t^{1/2}}$ .*

*Proof.* Since  $\text{dist}(\widehat{z}, \widehat{A}) = 0$  if and only if  $\text{dist}(\pi(\widehat{z}), A) = 0$ , we can apply Theorem 2.1 to these functions which implies the existence of  $N \geq 1$  such that

$$\text{dist}(\widehat{z}, \widehat{A})^N \lesssim \text{dist}(\pi(\widehat{z}), A).$$

Therefore, since  $\pi$  is surjective,  $\pi(\widehat{A}_t)$  contains  $A_{tN}$  for  $t > 0$  small enough.

Now, let  $\widehat{A} = \widehat{A}_1 \cup \widehat{A}_2$  be as above. Since  $\pi$  is holomorphic, there exists  $c > 0$  such that  $\pi(\widehat{A}_{2,t}) \subset A_{2,ct}$ . Therefore,  $\pi(\widehat{A}_{1,t})$  contains  $A_{tN} \setminus A_{2,t^{1/2}}$  for  $t > 0$  sufficiently small.  $\square$

In the sequel, we will constantly use desingularization of analytic sets. The following lemma allows us to conserve integral estimates.

**Lemma 2.7.** *Let  $Z$  and  $\widehat{Z}$  be irreducible analytic subsets of Kähler manifolds. Let  $\pi : \widehat{Z} \rightarrow Z$  be a surjective proper holomorphic map. Then, for every compact  $\widehat{L}$  of  $\widehat{Z}$  there exists  $q \geq 1$  such that if  $v$  is in  $L^q_{loc}(Z)$  then  $\widehat{v} := v \circ \pi$  is in  $L^1(\widehat{L})$ . Moreover, there exists  $c > 0$ , depending on  $\widehat{L}$ , such that*

$$\|\widehat{v}\|_{L^1(\widehat{L})} \leq c \|v\|_{L^q(\pi(\widehat{L}))}.$$

*Proof.* Using a desingularization, we can assume that  $\widehat{Z}$  is a smooth Kähler manifold with a Kähler form  $\widehat{\omega}$ . Denote by  $\omega, n, m$  a Kähler form on  $Z$  and the dimensions of  $Z$  and  $\widehat{Z}$  respectively. Generic fibers of  $\pi$  are compact of dimension  $m - n$  and form a continuous family. It follows that the integral of  $\widehat{\omega}^{m-n}$  on that fibers is a constant.

Consider  $\widehat{\lambda} = \pi^*(\omega^n) \wedge \widehat{\omega}^{m-n}$  on  $\widehat{Z}$ . The last observation implies that  $\pi_*(\widehat{\lambda}) = \omega^n$  up to a constant. Therefore, if  $v$  is in  $L^q_{loc}(Z)$  then  $\widehat{v}$  is in  $L^q_{loc}(\widehat{Z}, \widehat{\lambda})$ . Moreover, we can write  $\widehat{\lambda} = h\widehat{\omega}^m$  where  $h$  is a positive function. If there exists  $\tau > 0$  such that  $h^{-\tau}$  is integrable on  $\widehat{L}$  with respect to  $\widehat{\omega}^m$ , we obtain for  $p = 1 + \tau$  and  $q$  its conjugate that

$$\begin{aligned} \int_{\widehat{L}} |\widehat{v}| \widehat{\omega}^m &= \int_{\widehat{L}} |\widehat{v}| h^{-1} \widehat{\lambda} \leq \left( \int_{\widehat{L}} |\widehat{v}|^q \widehat{\lambda} \right)^{1/q} \left( \int_{\widehat{L}} h^{-p} \widehat{\lambda} \right)^{1/p} \\ &\lesssim \left( \int_{\pi(\widehat{L})} |v|^q \omega^n \right)^{1/q} \left( \int_{\widehat{L}} h^{-\tau} \widehat{\omega}^m \right)^{1/p} \\ &\lesssim \|v\|_{L^q(\pi(\widehat{L}))}. \end{aligned}$$

It remains to show the existence of  $\tau$ . The set  $\{h = 0\}$  is contained in the complex analytic set  $A$  where the rank of  $\pi$  is not maximal. More precisely, if  $\pi$  has maximal rank at  $z$ , then we can linearize  $\pi$  in a neighborhood of  $z$ . Therefore,  $\widehat{\lambda}$  and  $\widehat{\omega}^m$  are comparable in that neighborhood.

Since  $\widehat{L}$  is compact, the problem is local. Let  $z_0$  be in  $\widehat{L}$ . We can find a small chart  $U$  at  $z_0$  and a holomorphic function  $\phi$  on  $U$  such that  $A \cap U$  is contained in  $\{\phi = 0\}$ . We can also replace  $\widehat{\omega}$  and  $\omega$  by standard Euclidean forms on  $U$  and on a neighborhood of  $\pi(U)$ . So, we can assume that  $h$  is analytic. By Lojasiewicz inequality, for every compact  $K$  of  $U$ , there exists  $N \geq 1$  such that  $h(z) \gtrsim |\phi(z)|^N$  for all  $z$  in  $K$ . On the other hand, exponential estimate (cf. [17] and Section 4) applied to the plurisubharmonic function  $\log |\phi|$  says that  $\phi^{-\alpha}$  is in  $L^1(K, \widehat{\omega}^m)$  for some  $\alpha > 0$ . Therefore,  $h^{-\alpha/N}$  belongs to  $L^1(K, \widehat{\omega}^m)$ . We obtain the desired property near  $z_0$  by taking  $\tau \leq \alpha/N$ .  $\square$

Finally, the following results establish a relation between the regularity of functions on an analytic set and that of their lifts to a desingularization.

**Proposition 2.8.** *Let  $Z, \widehat{Z}, \pi$  be as in Lemma 2.7 and  $v$  be a function on  $Z$ . Assume that the lift of  $v$  to  $\widehat{Z}$  is  $(K, \alpha)$ -Hölder continuous. Then, for every compact  $L$  of  $Z$ , there exist constants  $0 < \alpha' \leq 1$  and  $a > 0$ , independent of  $v$ , such that  $v$  is  $(aK, \alpha')$ -Hölder continuous on  $L$ .*

*Proof.* Let  $\Delta$  be the diagonal of  $Z \times Z$ . We still denote by  $\pi$  the map induced on the product  $\widehat{Z} \times \widehat{Z}$  and we set  $\widehat{\Delta} = \pi^{-1}(\Delta)$ . As in Lemma 2.6, by Lojasiewicz inequality, we have

$$\text{dist}(\widehat{a}, \widehat{\Delta})^M \lesssim \text{dist}(\pi(\widehat{a}), \Delta),$$

if  $\pi(\widehat{a}) \in L \times L$ . Therefore, if we set  $\widehat{a} = (\widehat{x}, \widehat{x}')$  and  $\pi(\widehat{a}) = (x, x')$ , then we can rewrite this inequality as

$$(\text{dist}(\widehat{x}, \widehat{z}) + \text{dist}(\widehat{x}', \widehat{z}'))^M \lesssim \text{dist}(x, x'), \quad (2.1)$$

for some  $\widehat{z}, \widehat{z}' \in \widehat{Z}$  such that  $\pi(\widehat{z}) = \pi(\widehat{z}')$ .

Now, let  $v$  be as in the proposition and  $\widehat{v} = v \circ \pi$  denote its lift. Taking the same notation as above we have

$$|v(x) - v(x')| = |\widehat{v}(\widehat{x}) - \widehat{v}(\widehat{x}')| \leq |\widehat{v}(\widehat{x}) - \widehat{v}(\widehat{z})| + |\widehat{v}(\widehat{z}') - \widehat{v}(\widehat{x}')|,$$

since  $\pi(\widehat{z}) = \pi(\widehat{z}')$  which implies that  $\widehat{v}(\widehat{z}) = \widehat{v}(\widehat{z}')$ . Therefore, the assumption on  $\widehat{v}$  implies that

$$|v(x) - v(x')| \leq K(\text{dist}(\widehat{x}, \widehat{z})^\alpha + \text{dist}(\widehat{z}', \widehat{x}')^\alpha),$$

and finally (2.1) gives

$$|v(x) - v(x')| \leq aK \operatorname{dist}(x, x')^{\alpha/M},$$

where  $a > 0$  depends only on  $\alpha$ ,  $L$  and  $\pi$ .  $\square$

**Corollary 2.9.** *For every compact  $L$  of  $Z$ , there exists  $0 < \alpha \leq 1$  such that every continuous weakly holomorphic function on  $Z$  is  $\alpha$ -Hölder continuous on  $L$ . Moreover, every uniformly bounded family of such functions is uniformly  $\alpha$ -Hölder continuous on  $L$ .*

*Proof.* Recall that a continuous function on  $Z$  is weakly holomorphic if it is holomorphic on the regular part of  $Z$ . The result is known if  $Z$  is smooth with  $\alpha = 1$ . Therefore, in general, it is enough to apply Proposition 2.8 to a desingularization of  $Z$ .  $\square$

In Proposition 3.4, we will need a similar result in a local setting but with a uniform control of the constants. It is the aim of the two following results.

**Proposition 2.10.** *Let  $Z$ ,  $\widehat{Z}$ ,  $\pi$  and  $L$  be as in Proposition 2.8. Let  $v$  be a function defined on a ball  $B_Z(y, r) \subset L$  with  $0 < r \leq 1/2$ . Assume that  $\widehat{v} = v \circ \pi$  is  $(K, \alpha)$ -Hölder continuous. Then, there exist constants  $0 < \alpha' \leq 1$ ,  $a > 0$  and  $N \geq 1$  independent of  $v$ ,  $y$  and  $r$  such that  $v$  is  $(aK, \alpha')$ -Hölder continuous on  $B_Z(y, r^N)$ .*

*Proof.* The proof is the same as that of Proposition 2.8 except that we have to check that if  $x$  and  $x'$  are in  $B_Z(y, r^N)$  with  $N \geq 1$  large enough, then  $\widehat{v}$  is well defined at the points  $\widehat{x}$  and  $\widehat{x}'$  defined in (2.1).

Since  $\pi$  is holomorphic, we have  $\operatorname{dist}(x, \pi(\widehat{x})) \lesssim \operatorname{dist}(\widehat{x}, \widehat{x})$ . Therefore, by (2.1) we have

$$\operatorname{dist}(x, \pi(\widehat{x})) \lesssim \operatorname{dist}(\widehat{x}, \widehat{x}) + \operatorname{dist}(\widehat{x}', \widehat{x}') \lesssim \operatorname{dist}(x, x')^{1/M}.$$

Hence, if  $N$  is large enough, then  $\widehat{x}$  and  $\widehat{x}'$  belong to  $\pi^{-1}(B_Z(y, r))$ . Then, the result follows as in Proposition 2.8.  $\square$

**Corollary 2.11.** *For every compact  $L$  of  $Z$ , there are constants  $0 < \alpha \leq 1$ ,  $K > 0$  and  $N \geq 1$  such that if  $v$  is a continuous weakly holomorphic function on  $B_Z(y, r) \subset L$  with  $|v| \leq 1$  then  $v$  is  $(K/r, \alpha)$ -Hölder continuous on  $B_Z(y, r^N)$ .*

*Proof.* Let  $\pi : \widehat{Z} \rightarrow Z$  be a desingularization. Let  $\widehat{x}$  be in  $\pi^{-1}(B_Z(y, r/2))$ . Since  $\pi$  is holomorphic, there is  $a > 0$  such that  $B_{\widehat{Z}}(\widehat{x}, r/a)$  is contained in  $\pi^{-1}(B_Z(y, r))$ . Therefore, by Cauchy's inequality,  $\widehat{v}$  is  $a/r$ -Lipschitz on  $\pi^{-1}(B_Z(y, r/2))$ . Hence, the result follows from Proposition 2.10.  $\square$

### 3 Volume estimate for endomorphisms

The multiplicities of an endomorphism  $f$  are strongly related to volume estimates which were used successfully to solve equidistribution problems. In what follows, we generalize Lojasiewicz type inequalities obtained in [15], [7] and [10] to analytic sets, possibly singular. The aim is to control the size of a ball under iterations of  $f$  in an invariant analytic set. Singularities, in particular the points where the analytic sets are not locally irreducible, lead to technical difficulties.

In this section,  $X$  always denotes an irreducible analytic set of a smooth manifold. In order to avoid some problems related to the local connectedness of analytic sets, instead of the distance induced by an embedding of  $X$ , we consider the distance  $\rho$  defined by paths in  $X$ . Namely, if  $x, y \in X$  then  $\rho(x, y)$  is the length of the shortest path in  $X$  between  $x$  and  $y$ . These two distances on  $X$  are related by the following result (see e.g. [1]).

**Theorem 3.1.** *Let  $K$  be a compact subset of  $X$ . There exists a constant  $r > 0$  such that for all  $x, y \in K$  we have*

$$\text{dist}(x, y) \leq \rho(x, y) \lesssim \text{dist}(x, y)^r.$$

The first step to state volume estimate is the following result.

**Proposition 3.2.** *Let  $\Gamma \subset \mathbb{B} \times \mathbb{B}$  and  $X \subset \mathbb{B}$  be two analytic subsets with  $X$  locally irreducible and such that the first projection  $\pi : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$  defines a ramified covering of degree  $m$  from  $\Gamma$  to  $X$ . There exist constants  $a > 0$  and  $b \geq 1$  such that if  $x, y \in X \cap \mathbb{B}_{1/2}$  then we can write*

$$\pi^{-1}(x) \cap \Gamma = \{x^1, \dots, x^m\} \text{ and } \pi^{-1}(y) \cap \Gamma = \{y^1, \dots, y^m\},$$

with  $\text{dist}(x, y) \geq a \text{dist}(x^i, y^i)^{bm}$ . Moreover,  $a$  increases with  $m$  but is independent of  $\Gamma$  and  $b$  depends only on  $X$ .

*Proof.* By Theorem 3.1, to establish the proposition, we can replace  $\text{dist}(x, y)$  by  $\rho(x, y)$  on  $X$ . For  $w \in X_{\text{Reg}}$ , we can define the  $j$ -th Weierstrass polynomial,  $k + 1 \leq j \leq 2k$ , on  $t \in \mathbb{C}$

$$P_j(t, w) = \prod_{z \in \pi^{-1}(w) \cap \Gamma} (t - z_j) = \sum_{l=0}^m a_{j,l}(w) t^l,$$

where  $z = ((z_1, \dots, z_k), (z_{k+1}, \dots, z_{2k})) \in \mathbb{B} \times \mathbb{B}$ . The coefficients  $a_{j,l}$  are holomorphic on  $X_{\text{Reg}}$  and uniformly bounded by  $m!$  since  $\Gamma \subset \mathbb{B} \times \mathbb{B}$ . As  $X$  is locally irreducible, they can be extended continuously to  $X$  (see e.g. [4]). It

gives a continuous extension of the polynomials  $P_j$  to  $X$  with  $P_j(z_j, \pi(z)) = 0$  if  $z$  is in  $\Gamma$ . Moreover, by Corollary 2.9 there exists  $\alpha > 0$  such that the coefficients  $a_{j,l}$  are uniformly  $\alpha$ -Hölder continuous on  $X \cap \mathbb{B}_{3/4}$  with respect to  $\rho$ .

We claim that there is a constant  $a > 0$  such that if  $x, y \in X \cap \mathbb{B}_{1/2}$  and  $x' \in \mathbb{B}$  with  $\tilde{x} := (x, x') \in \Gamma$  then there is  $\tilde{y} \in \pi^{-1}(y) \cap \Gamma$  with

$$\rho(x, y) \geq a \text{dist}(\tilde{x}, \tilde{y})^{bm},$$

where  $b = 1/\alpha$ . From this, the result follows exactly as in the end of the proof of [7, Lemma 4.3].

It remains to prove the claim. Fix  $c > 0$  large enough and let  $r = \rho(x, y)$ . We can assume that  $r$  is non-zero and sufficiently small, otherwise the result is obvious. Since the covering has degree  $m$ , we can find an integer  $2 \leq l \leq 4km$  such that for all  $k+1 \leq j \leq 2k$ , no root of the polynomial  $P_j(t, x)$  satisfies

$$c(l-1)r^{1/bm} \leq |\tilde{x}_j - t| \leq c(l+1)r^{1/bm}.$$

It gives a *security ring* over  $x$  which does not intersect  $\Gamma$ . Using the regularity of  $P_j$ , it can be extended to a neighborhood of  $x$ . More precisely, for  $\theta \in \mathbb{R}$  we define  $\xi_j = \tilde{x}_j + cle^{i\theta}r^{1/bm}$  and

$$G_{j,c,\theta}(w) = c^{-m+1}P_j(\xi_j, w).$$

The choice of  $l$  implies that

$$|G_{j,c,\theta}(x)| = c^{-m+1}|P_j(\xi_j, x)| \geq c^{-m+1} \prod_{z \in \pi^{-1}(x)} |\xi_j - z_j| \geq cr^\alpha.$$

Moreover, we deduce from the properties on the coefficients  $a_{j,l}$  that the functions  $G_{j,c,\theta}$  are uniformly  $\alpha$ -Hölder continuous on  $X \cap \mathbb{B}_{3/4}$  with respect to  $(j, c, \theta)$ . Hence, if  $c$  is large enough, they do not vanish on  $B := \{z \in X \mid \rho(x, z) < 2r\}$  which contains  $y$ .

It implies that  $P_j(t, w) \neq 0$  if  $w$  is in  $B$  and  $|t - \tilde{x}_j| = lcr^{1/bm}$ . Therefore, if we denote by  $\Sigma$  the boundary of the polydisc of center  $x'$  and of radius  $lcr^{1/bm}$ , we have  $\Gamma \cap (B \times \Sigma) = \emptyset$ . Hence, since  $B$  is connected and contains  $y$ , by continuity there is a point  $\tilde{y}$  in  $\pi^{-1}(y) \cap \Gamma$  with  $|\tilde{x}_j - \tilde{y}_j| \leq lcr^{1/bm}$ . This completes the proof of the claim.  $\square$

**Remark 3.3.** *Our proof shows that if we assume that  $m \leq m_0$  for a fixed  $m_0$ , then  $a$  depends only on the Hölder constants  $(K, \alpha)$  associated with  $a_{j,l}$ . Furthermore, if  $\alpha$  is fixed then  $a$  is proportional to  $K$ . Note that without assumption on the constants  $a$  and  $b$ , Proposition 3.2 can be deduced from Theorem 2.1.*

The control of multiplicities of the covering gives the following more precise result.

**Proposition 3.4.** *Let  $X$ ,  $\Gamma$  and  $\pi$  be as above. Let  $Z \subset X$  be a proper analytic subset. Assume that the multiplicity at each point of  $\pi^{-1}(x)$  is at most equal to  $s$  if  $x \in X \setminus Z$ . Then, there exist constants  $a > 0$ ,  $b \geq 1$  and  $N \geq 1$  such that for all  $0 < t \leq 1/2$  and  $x, y \in X \cap \mathbb{B}_{1/2}$  with  $\text{dist}(x, Z) > t$  and  $\text{dist}(y, Z) > t$ , we can write*

$$\pi^{-1}(x) \cap \Gamma = \{x^1, \dots, x^m\} \text{ and } \pi^{-1}(y) \cap \Gamma = \{y^1, \dots, y^m\},$$

with  $\text{dist}(x, y) \geq at^N \text{dist}(x^i, y^i)^{bs}$ .

*Proof.* Let  $t > 0$  and  $x \in X \cap \mathbb{B}_{1/2}$  with  $\text{dist}(x, Z) > t$ . We want to find a neighborhood  $B \subset X$  of  $x$  such that each component of  $\Gamma \cap \pi^{-1}(B)$  defines a ramified covering of degree at most equal to  $s$  over  $B$ .

We construct an analytic set  $Y$  associated to the multiplicities of  $\Gamma$ . Namely, first define  $Y' \subset \Gamma^{s+1}$  by  $\{z \in \Gamma^{s+1} \mid \pi \circ \tau_i(z) = \pi \circ \tau_j(z), 1 \leq i, j \leq s+1\}$ , where  $\tau_i, 1 \leq i \leq s+1$ , are the canonical projections of  $\Gamma^{s+1}$  onto  $\Gamma$ . For  $i \neq j$  we set  $A_{i,j} = \{z \in Y' \mid \tau_i(z) = \tau_j(z)\}$ . Then,  $Y$  is defined by  $\overline{Y' \setminus \cup_{i \neq j} A_{i,j}}$ . The map  $\pi_1 = \pi \circ \tau_1 : Y \rightarrow X$  defines a ramified covering. If  $x$  is generic in  $X$  then a point  $z \in Y$  over  $x \in X$  represents a family of  $(s+1)$  distinct points in  $\pi^{-1}(x) \cap \Gamma$ .

If  $\pi'$  denotes the second projection of  $\mathbb{B} \times \mathbb{B}$  onto  $\mathbb{B}$ , consider the map  $h : Y \rightarrow \mathbb{C}^{(s+1)^2}$  define by

$$h(z) = (\pi' \circ \tau_i(z) - \pi' \circ \tau_j(z))_{1 \leq i, j \leq s+1}.$$

By construction,  $h(z) = 0$  means precisely that there is a point in  $\Gamma$  with multiplicity greater than  $s$  over  $\pi_1(z)$ . It implies that  $\pi_1(h^{-1}(0)) \subset Z$ . Hence, Theorem 2.1 implies there is a constant  $M > 0$  such that if  $\pi(z) \in X \cap \mathbb{B}_{1/2}$  then

$$\|h(z)\| \gtrsim \text{dist}(z, h^{-1}(0))^M \gtrsim \text{dist}(\pi_1(z), Z)^M. \quad (3.1)$$

Let  $a, b > 0$  be the constants as in Proposition 3.2. As in the proof of that proposition, we use the distance function  $\rho$  on  $X$ . Fix  $\gamma > 0$  small enough and set  $B = \{z \in X \mid \rho(x, z) < \gamma t^{Mbm}\}$ . For  $\tilde{x} = (x, x')$  in  $\Gamma$ , we can choose  $2 \leq l \leq 8m$  such that  $\pi^{-1}(x) \cap \Gamma$  do not intersect the ring

$$r(l-2) \leq \|\tilde{x} - w\| \leq r(l+2),$$

where  $r = (a^{-1}\gamma)^{1/bm}t^M$ . Hence, if  $H$  denotes the ball of center  $x'$  and of radius  $rl$ , we have  $\text{dist}(\pi^{-1}(x) \cap \Gamma, B \times \partial H) > r$ . Therefore, by Proposition

3.2  $\Gamma \cap B \times \partial H = \emptyset$ . It assures that  $\pi$  is proper on  $\Gamma \cap B \times H$  and then defines a ramified covering.

Moreover, this covering has degree at most equal to  $s$ . Otherwise, according to the radius of  $H$ , we have

$$\min_{z \in \pi_1^{-1}(x)} \|h(z)\| \leq (s+1)^2 2rl,$$

which is in contradiction with (3.1) if  $\gamma$  is small enough, since  $\text{dist}(x, Z) > t$  and  $r = (a^{-1}\gamma)^{1/bm} t^M$ .

Now, we want to apply Proposition 3.2 to this covering. But, in order to control the constants, we have to reduce  $B$ . Indeed, according to Remark 3.3, the constants of that proposition depend only on the Hölder continuity of the coefficients  $a_{j,l}$  (and on the degree of the covering). These coefficients are bounded continuous weakly holomorphic functions defined on  $B$  then by Corollary 2.11 with  $L = \overline{X} \cap \mathbb{B}_{3/4}$ , they are  $(Kt^{-M'}, \alpha)$ -Hölder continuous on  $B(x, t^{M'})$  for some  $M' \geq 1$  large enough,  $0 < \alpha \leq 1$  and  $K > 0$  independent of  $x$  and  $t$ . Therefore, after a coordinates dilation by  $t^{-M'}$  at  $x$ , we can apply Proposition 3.2 with the same exponent  $b$  and the second constant proportional to some power of  $t$ . Finally, let  $y \in X$ . We can assume that  $\rho(x, y) \leq t^{M'}/2$ , otherwise the proposition is obvious. Hence, the previous observation implies that

$$\pi^{-1}(x) \cap \Gamma \cap B \times H = \{x^1, \dots, x^s\} \text{ and } \pi^{-1}(y) \cap \Gamma \cap B \times H = \{y^1, \dots, y^s\},$$

with  $\rho(x, y) \geq a't^N \text{dist}(x^i, y^i)^{bs}$  where  $a' > 0, b \geq 1$  and  $N \geq 1$  are independent of  $x$  and  $t$ . More precisely, we can write  $N = M' + N'$ , where the contribution in  $t^{M'}$  comes from the dilation and that in  $t^{N'}$  comes from the estimate on Hölder continuity. The construction can be applied to each component of  $\Gamma \cap \pi^{-1}(B)$ . This gives the result.  $\square$

We now consider the dynamical context. Assume that  $X \subset \mathbb{P}^k$  is an irreducible analytic set of dimensions  $l$  which is invariant by  $f$ . Denote by  $g$  the restriction of  $f$  to  $X$ . For  $x$  in  $X$ , we define *the local multiplicity* of  $g$  at  $x$  as the maximal number of points in  $g^{-1}(z)$  which are near  $x$  for  $z \in X$  close enough to  $g(x)$ . The local multiplicity is smaller than *the topological degree* i.e. the number of points in  $g^{-1}(x)$  for  $x$  generic in  $X$ . In our case, the topological degree of  $g$  is equal to  $d^l$ , see [7].

There exists a finite covering  $(U_i)_{i \in I}$  of  $X$  by open subsets of  $\mathbb{P}^k$  such that  $X \cap U_i$  can be decomposed into locally irreducible components. Hence, we can apply Proposition 3.4 to the graph of  $g$  over each component of  $X \cap U_i$ . It gives the following corollary.

**Corollary 3.5.** *Let  $\eta > 1$  and  $Z \subset X$  be a proper analytic subset. Assume that the local multiplicity of  $g$  is less than  $\eta$  outside  $g^{-1}(Z)$ . Then there are constants  $a > 0$ ,  $b \geq 1$  and  $N \geq 1$  such that if  $0 < t < 1$  and  $x, y$  are two points outside  $Z_t$  where  $X$  is locally irreducible, then we can write*

$$g^{-1}(x) = \{x^1, \dots, x^{d^l}\} \text{ and } g^{-1}(y) = \{y^1, \dots, y^{d^l}\}$$

with  $\text{dist}(x, y) \geq at^N \text{dist}(x^j, y^j)^{b\eta}$ .

From this, we obtain the following size estimate for image of balls which is crucial in the proof of our main result.

**Corollary 3.6.** *Let  $\delta > 1$  and  $E \subset X$  be a proper analytic subset. Denote by  $\tilde{E}$  the preimage of  $E$  by  $g$ . Assume that the local multiplicity is less than  $\delta$  outside  $\tilde{E}$ . There exist constants  $0 \leq A \leq 1$ ,  $b \geq 1$  and  $N \geq 1$  such that if  $0 < t \leq 1/2$ ,  $r < t/2$  and  $x \in X \setminus \tilde{E}_t$ , then  $g(B_X(x, r))$  contains a ball of radius  $At^N r^{b\delta}$ . Moreover,  $b$  depends only on  $X$ .*

*Proof.* Fix  $t > 0$  and  $r < t/2$ . As in the proof of Lemma 2.5 with  $\hat{Y} = X_{\text{Sing}} \cup g^{-1}(X_{\text{Sing}})$ , possibly after replacing  $r$  by  $cr^l$  for some  $c > 0$ , we can assume that  $B_X(x, r)$  and  $g(B_X(x, r))$  are contained in  $X_{\text{Reg}}$ . The local multiplicity of  $f$  on  $X$  is bounded on  $d^l$ . So, there exists  $2 \leq i \leq 4d^l$ , such that the ring  $\{\frac{r(i-1)}{4d^l+1} \leq \text{dist}(x, x') \leq \frac{r(i+1)}{4d^l+1}\}$  contains no preimage of  $g(x)$ . Thus, if  $x' \in \partial B_X(x, r \frac{i}{4d^l+1})$ , then

$$\text{dist}(x', g^{-1}(g(x))) \geq \frac{r}{4d^l + 1}.$$

Moreover, we can apply Corollary 3.5 with  $\eta = d^l$  and  $Z = \emptyset$ . Hence, there exist  $a, b > 0$  such that  $g(\overline{B_X(x, r \frac{i}{4d^l+1})}) \subset X \setminus E_{a(t/2)^{bd^l}}$ . Therefore, we can apply once again Corollary 3.5 with  $\eta = \delta$ ,  $Z = E$  and  $a(t/2)^{bd^l}$  instead of  $t$ . We get, for some constants  $a' > 0$  and  $N_0 \geq 1$

$$\text{dist}(g(x'), g(x)) \geq a' \left( a \left( \frac{t}{2} \right)^{bd^l} \right)^{N_0} \left( \frac{r}{4d^l + 1} \right)^{b\delta},$$

and, since  $g$  is an open mapping near  $x$

$$B_X(g(x), At^N r^{b\delta}) \subset g(B_X(x, r))$$

with  $A = \frac{a^{N_0} a'}{2^{N_0 b d^l} (4d^l + 1)^{b\delta}}$  and  $N = N_0 b d^l$ . □

**Remark 3.7.** *When  $X$  is smooth, the ball in  $g(B_X(x, r))$  can be chosen centered at  $g(x)$ .*

## 4 Psh functions and exponential estimates

We refer to [4] for basics on currents and plurisubharmonic (psh for short) functions. Let  $T$  be a positive closed  $(1,1)$ -current of mass 1 on  $\mathbb{P}^k$  with continuous local potentials. Let us recall briefly the associated notions of psh and weakly psh (wpsh for short) modulo  $T$  functions introduced in [7].

Let  $Y$  be an analytic space. A function  $v : Y \rightarrow \mathbb{R} \cup \{-\infty\}$  is *wpsh* if it is psh on  $Y_{Reg}$  and for  $y$  in  $Y$ , we have  $v(y) = \limsup v(z)$  with  $z \in Y_{Reg}$  and  $z \rightarrow y$ . These functions coincide with psh functions if  $Y$  is smooth. On compact spaces, the notion is very restrictive. However, if  $X$  is an analytic subset of  $\mathbb{P}^k$ , we have the more flexible notion of *wpsh modulo  $T$  function* on  $X$ . Locally, it is the difference of a wpsh function on  $X$  and a potential of  $T$ . If  $X$  is smooth, we say that the function is psh modulo  $T$ .

Note that the restriction of a psh modulo  $T$  function to an analytic subset is either wpsh modulo  $T$  or equal to  $-\infty$  on an irreducible component. If  $u$  is wpsh modulo  $T$  on  $X$  then  $dd^c(u[X]) + T \wedge [X]$  is a positive closed current supported on  $X$ . On the other hand, if  $S$  is a positive closed  $(1,1)$ -current on  $\mathbb{P}^k$  with mass 1, there is a psh modulo  $T$  function  $u$  on  $\mathbb{P}^k$ , unique up to a constant, such that  $S = dd^c u + T$ .

These notions bring good compactness properties which permit to obtain uniform estimates. We have the following statements established in [7].

**Proposition 4.1.** *Let  $(u_n)$  be a sequence of wpsh modulo  $T$  functions on  $X$ , uniformly bounded from above. Then there is a subsequence  $(u_{n_i})$  satisfying one of the following properties:*

- *There is an irreducible component  $Y$  of  $X$  such that  $(u_{n_i})$  converges uniformly to  $-\infty$  on  $Y \setminus X_{Sing}$ .*
- *$(u_{n_i})$  converges in  $L^p(X)$  to a wpsh modulo  $T$  function  $u$  for every  $1 \leq p < +\infty$ .*

*In the last case,  $\limsup u_{n_i} \leq u$  on  $X$  with equality almost everywhere.*

It implies the following lemma.

**Lemma 4.2.** *Let  $\mathcal{G}$  be a family of psh modulo  $T$  functions on  $\mathbb{P}^k$  uniformly bounded from above. Assume that each irreducible component of  $X$  contains an analytic subset  $Y$  such that the restriction of  $\mathcal{G}$  to  $Y$  is bounded in  $L^1(Y)$ . Then, the restriction of  $\mathcal{G}$  to  $X$  is bounded in  $L^1(X)$ .*

A classical result of Hörmander [17] gives a uniform bound to  $\exp(-u)$  in  $L^1(\mathbb{B}_{1/2})$  for  $u$  in a class of psh functions in the unit ball of  $\mathbb{C}^k$ . Similar

estimates can be obtained for compact families of quasi-psh functions. From now, we assume that  $T$  has  $(K, \alpha)$ -Hölder continuous local potentials, with  $0 < \alpha \leq 1$  and  $K > 0$ . In the rest of this section, we establish exponential estimates for psh modulo  $T$  functions in different situations. A key observation in our approach is that Hölder continuity allows us to work with non-compact families. In the sequel, we will apply these estimates to  $T$ , the Green current associated to  $f$ . They allow us to control the volume of some sublevel sets of potentials of currents near exceptional sets.

As a consequence of classical Hörmander's estimate, we have the following lemma which will be of constant use. Here,  $\nu$  denotes the standard volume form on  $\mathbb{C}^k$  and  $T$  is seen as a fixed current on the unit ball  $\mathbb{B}$  of  $\mathbb{C}^k$ . We assume that it admits a potential  $g$  which is  $(K, \alpha)$ -Hölder continuous on  $\mathbb{B}$ .

**Lemma 4.3.** *Let  $v$  be a psh modulo  $T$  function in  $\mathbb{B}_t$  with  $v \leq 0$  and  $v(0) > -\infty$ . Let  $0 < s < -v(0)^{-1}$  and  $t > 0$  be such that  $Kt^\alpha \leq s^{-1}$ . There is a constant  $c > 0$  independent of  $v$ ,  $s$  and  $t$  such that*

$$\int_{\mathbb{B}_{t/2}} \exp\left(-\frac{sv}{2}\right) \nu \leq ct^{2k}. \quad (4.1)$$

*Proof.* As  $v$  is psh modulo  $T$ , we have  $v = v' - g$  with  $v'$  psh. We set  $\tilde{v}(z) = v'(z) - g(0) - Kt^\alpha$ . Then  $\tilde{v}$  is psh in  $\mathbb{B}_t$ ,  $\tilde{v}(0) = v(0) - Kt^\alpha \geq -2s^{-1}$  and  $\tilde{v} \leq v \leq 0$  because  $g(z) - g(0) \leq Kt^\alpha$  on  $\mathbb{B}_t$ . By [17, Theorem 4.4.5] there exists  $c > 0$  such that

$$\int_{\mathbb{B}_{1/2}} \exp\left(-\frac{s\tilde{v}(tz)}{2}\right) \nu \leq c,$$

thus, by a change of variables  $z \mapsto tz$ , we get

$$\int_{\mathbb{B}_{t/2}} \exp\left(-\frac{sv}{2}\right) \nu \leq \int_{\mathbb{B}_{t/2}} \exp\left(-\frac{s\tilde{v}}{2}\right) \nu \leq ct^{2k}.$$

□

For the rest of the section,  $X$  always denotes an irreducible analytic subset of  $\mathbb{P}^k$  of dimension  $l$  and  $v$  is a psh modulo  $T$  function in  $\mathbb{P}^k$  with  $v \leq 0$ . In Section 6 we will extend the previous result to the neighborhood of  $X$ , where the condition at 0 is replaced by an integrability condition on  $X$ . For this purpose, we have to control the size of sublevel sets of  $v$  in  $X$ . This is the aim of the following global result.

**Lemma 4.4.** *For  $X$  as above, there exists  $q_0 \geq 1$  with the following property. For  $q > q_0$  we set  $\epsilon = 2lq_0/q\alpha$  and take  $M > 0$  and  $s \geq 1$  such that  $s^{1+\epsilon}\|v\|_{L^q(X)} \leq M$ . Then, there exist constants  $a, c > 0$  independent of  $v, q$  and  $s$  such that*

$$\int_X \exp(-asv)\omega^l \leq c. \quad (4.2)$$

If  $X$  is smooth, we can choose  $q_0 = 1$ .

*Proof.* First, assume that  $X$  is a compact smooth manifold with a volume form  $\eta$ . Since  $X$  has dimension  $l$ , for  $t > 0$  we can cover it by balls  $(B_i)_{i \in I}$  with  $B_i := B_X(x_i, t)$  and such that  $|I| \leq c't^{-2l}$  for some  $c' > 0$ . Let  $t = s^{-1/\alpha}$ . As above, in each ball  $B_X(x_i, 2t)$  we can write  $v = v'_i - g_i$ , where  $g_i$  is a local potential of  $T$ . Using local charts at  $x_i$ , we can identify  $B_X(x_i, 2t)$  with  $\mathbb{B}_{2t}$  in  $\mathbb{C}^l$ . We consider  $\tilde{v}_i(z) = s(v'_i(tz) - g_i(0))$ . These functions are psh in  $\mathbb{B}_2$ . We show that they belong to a compact family, independent of  $v$  and  $s$ . Using a change of variables  $z \mapsto tz$  and Hölder's inequality, we get

$$\begin{aligned} \|\tilde{v}_i\|_{L^1(\mathbb{B}_2)} &\leq \int_{\mathbb{B}_2} s|v(tz)|\nu + \int_{\mathbb{B}_2} s|g_i(tz) - g_i(0)|\nu \\ &\leq st^{-2l}\|v\|_{L^q(X)}|\mathbb{B}_2|^{1/p}t^{2l/p} + 2^\alpha K|\mathbb{B}_2| \\ &= s^{1+\epsilon}\|v\|_{L^q(X)}|\mathbb{B}_2|^{1/p} + 2^\alpha K|\mathbb{B}_2| \\ &\leq M|\mathbb{B}_2|^{1/p} + 2^\alpha K|\mathbb{B}_2| \leq M', \end{aligned}$$

where  $p$  is the conjugate of  $q$ ,  $|\mathbb{B}_2|$  is the volume of  $\mathbb{B}_2$  and  $M'$  is a positive constant. The family  $\mathcal{U} = \{u \in PSH(\mathbb{B}_2) \mid \|u\|_{L^1(\mathbb{B}_2)} \leq M'\}$  is compact so there exists a constant  $a > 0$  such that  $\|\exp(-au)\|_{L^1(\mathbb{B}_2)}$  is uniformly bounded for all  $u \in \mathcal{U}$ . Therefore, for  $i \in I$

$$\int_{\mathbb{B}_t} \exp(-as(v'_i(z) - g_i(0)))\nu \lesssim t^{2l}.$$

Moreover, the Hölder continuity implies that  $-sv(z) \leq K - s(v'_i(z) - g_i(x_i))$  in  $B_i$ . Hence, since  $(B_i)_{i \in I}$  is a covering of  $X$  we obtain

$$\begin{aligned} \int_X \exp(-asv)\eta &\leq \sum_{i \in I} \int_{B_i} \exp(-asv)\eta \\ &\leq \sum_{i \in I} \int_{B_i} \exp(a(K - s(v'_i(z) - g_i(x_i))))\eta \\ &\lesssim \sum_{i \in I} t^{2l} \leq c'. \end{aligned}$$

This implies the lemma if  $X$  is smooth with  $q_0 = 1$ .

In the general case, we consider a desingularization  $\pi : \widehat{X} \rightarrow X$  with a volume form  $\eta$  on  $\widehat{X}$ . The map  $\pi$  is surjective, then by Lemma 2.7, there exists  $q_0 \geq 1$  such that

$$\|\widehat{v}\|_{L^{q_0/q_0}(\widehat{X}, \eta)} \lesssim \|v\|_{L^q(X, \omega^l)}.$$

Moreover,  $\pi^*(T)$  possesses  $\alpha$ -Hölder local potentials and  $\widehat{v} \leq 0$  is psh modulo  $\pi^*(T)$ . Therefore, this choice of  $q_0$  allows us to apply the lemma on  $\widehat{X}$  and get

$$\int_{\widehat{X}} \exp(-as\widehat{v})\eta \leq c.$$

The result follows since

$$\int_X \exp(-asv)\omega^l = \int_{\widehat{X}} \exp(-as\widehat{v})\pi^*(\omega^l) \leq \|h\|_\infty \int_{\widehat{X}} \exp(-as\widehat{v})\eta,$$

where we write  $\pi^*(\omega^l) = h\eta$ .  $\square$

The following estimate is a consequence of Lemma 4.3 and is related to the geometry of sublevel sets of psh modulo  $T$  functions. In Section 6, it will establish the existence of balls where we can apply our volume estimates.

**Lemma 4.5.** *For  $s \geq 2$  set  $F_s = \{x \in X \mid v(x) \leq -s^{-1}\}$ . There are constants  $\beta, c > 0$  independent of  $v$  and  $s$  such that if  $F_s$  contains no ball of radius  $s^{-\beta}$  then*

$$\int_X \exp\left(-\frac{sv}{2}\right)\omega^l \leq c.$$

*Proof.* We first consider the case where  $X$  is smooth. Let  $t = 4^{-1}(Ks)^{-1/\alpha}$ . As in the proof of the previous lemma, we cover  $X$  by balls  $(B_i)_{i \in I}$  of radius  $t$  with  $|I| \leq c't^{-2l}$ ,  $c' > 0$ . Assume there is no ball of radius  $t$  in  $F_s$ . Hence, for each  $i \in I$  there exists  $x_i$  in  $B_i$  such that  $v(x_i) > -s^{-1}$ . The balls  $B'_i$  of center  $x_i$  and of radius  $2t$  cover  $X$ . Thus

$$\int_X \exp\left(-\frac{sv}{2}\right)\omega^l \leq \sum_{i \in I} \int_{B'_i} \exp\left(-\frac{sv}{2}\right)\omega^l.$$

But,  $s < -v(x_i)^{-1}$  and  $K(4t)^\alpha \leq s^{-1}$  therefore we can apply Lemma 4.3 on each ball

$$\int_{B'_i} \exp\left(-\frac{sv}{2}\right)\omega^l \lesssim t^{2l}.$$

Hence, we get

$$\int_X \exp\left(-\frac{sv}{2}\right)\omega^l \lesssim \sum_{i \in I} t^{2l} \leq c',$$

which gives the result when  $X$  is smooth with  $\beta > 1/\alpha$  such that  $s^{-\beta} < t$ .

If  $X$  is singular, we consider a desingularization  $\pi : \widehat{X} \rightarrow X$ . By Lemma 2.5, there exists  $N \geq 1$  such that the image of a ball of radius  $r$  under  $\pi$  contains a ball of radius  $r^N$ . Hence, if  $\beta$  is large enough, the hypothesis on  $F_s$  assures there is no ball of radius  $t$  in  $\widehat{F}_s = \pi^{-1}(F_s)$ . Then, we can apply the lemma to  $\widehat{v} = v \circ \pi$  which is psh modulo  $\pi^*(T)$ . We get

$$\int_X \exp\left(-\frac{sv}{2}\right)\omega^l = \int_{\widehat{X}} \exp\left(-\frac{s\widehat{v}}{2}\right)\pi^*(\omega^l) \leq c,$$

for some  $c > 0$ , since  $\pi^*(\omega^l)$  is smooth.  $\square$

## 5 Exceptional sets

Let  $f$  be an endomorphism of  $\mathbb{P}^k$  of algebraic degree  $d \geq 2$ . The aim of this section is to construct two families  $\mathcal{A}_\lambda$  and  $\mathcal{B}_\lambda$  of analytic sets where the iterate sequence  $f^n$  has important local multiplicities. Let  $X \subset \mathbb{P}^k$  be an irreducible invariant analytic set. Define  $\kappa_{X,n}(x)$ , or simply  $\kappa_n(x)$  if no confusion is possible, as the local multiplicity of  $f^n|_X$  at  $x$ . It is a *sub-multiplicative cocycles*, namely it is upper semi-continuous for the Zariski topology on  $X$ ,  $\min_X \kappa_n = 1$  and for any  $m, n \geq 0$  and  $x \in X$  we have the following sub-multiplicative relation

$$\kappa_{n+m}(x) \leq \kappa_m(f^n(x))\kappa_n(x).$$

The inequality may be strict when  $X$  is singular. Define

$$\kappa_{-n}(x) := \max_{y \in (f|_X)^{-n}(x)} \kappa_n(y).$$

We recall the following theorem of Dinh [5], see also [12].

**Theorem 5.1.** *The sequence of functions  $\kappa_{-n}^{1/n}$  converges pointwise to a function  $\kappa_-$ . Moreover, for every  $\lambda > 1$ , the level set  $E_\lambda(X) = \{\kappa_- \geq \lambda\}$  is a proper analytic subset of  $X$  which is invariant under  $f|_X$ . In particular,  $\kappa_-$  is upper semi-continuous in the Zariski sense.*

For a generic endomorphism of  $\mathbb{P}^k$ ,  $E_\lambda(\mathbb{P}^k)$  is empty. In this case, Theorem 1.2 is already known in all bidegrees [8]. In our proof, we will proceed by

induction, proving the exponentially fast convergence on  $X$  if it is already established on each irreducible component of  $E_\lambda(X)$ . But, even if  $E_\lambda(X)$  is invariant, its irreducible components are periodic and not invariant in general. Therefore, if  $X$  is only periodic, we define  $E_\lambda(X)$  in the same way, replacing  $f$  by  $f^p$  and  $\lambda$  by  $\lambda^p$ , where  $p$  is a period of  $X$ . By Theorem 5.1, this definition is independent of the choice of  $p$ .

Fix  $1 < \lambda < d$ . Define the family  $\mathcal{B}_\lambda$  of exceptional sets as follows. First, we set  $\mathbb{P}^k \in \mathcal{B}_\lambda$ . If  $X$  is in  $\mathcal{B}_\lambda$ , we add to  $\mathcal{B}_\lambda$  all irreducible components of  $E_\lambda(X)$ . This family is finite and since the functions  $\kappa_-$  are upper semi-continuous in the Zariski sense, there exists  $1 < \delta < \lambda$  such that  $\mathcal{B}_\lambda = \mathcal{B}_\delta$ , or equivalently  $E_\lambda(X) = E_\delta(X)$  if  $X \in \mathcal{B}_\lambda$ . This will give us some flexibility in order to obtain estimates using an induction process.

As all elements of  $\mathcal{B}_\lambda$  are periodic, they are invariant under some iterate  $f^{n_0}$ . Let us remark that it is sufficient to prove Theorem 1.2 for an iterate of  $f$ . Hence, we can assume that  $n_0 = 1$ , replacing  $f$  and  $\lambda$  by  $f^{n_0}$  and  $\lambda^{n_0}$ . Dinh also proved that  $\kappa_{n_1} < \delta^{n_1}$  outside  $(f|_X)^{-n_1}(E_\lambda(X))$  for some  $n_1 \geq 1$ . Once again, we can assume that  $n_1 = 1$ .

The second family  $\mathcal{A}_\lambda$ , that takes place in Theorem 1.2, is defined as the set of minimal elements for the inclusion in  $\mathcal{B}_\lambda$ . This family is not empty and each element of  $\mathcal{B}_\lambda$  contains at least one element of  $\mathcal{A}_\lambda$ . Note that no element of  $\mathcal{A}_\lambda$  is contained in another one. These analytic sets play a special role in the next section, to start induction and to obtain compactness properties. When  $\mathbb{P}^k$  is an element of  $\mathcal{A}_\lambda$ , it is the only element in  $\mathcal{A}_\lambda$  and the exceptional set is empty. Otherwise, define the exceptional set as the union of all the elements of  $\mathcal{A}_\lambda$ .

## 6 Equidistribution speed

This section is devoted to the proof of Theorem 1.2. Fix an endomorphism  $f$  of algebraic degree  $d \geq 2$  of  $\mathbb{P}^k$ , and denote by  $T$  its Green current. Recall that  $T$  is totally invariant i.e.  $d^{-1}f^*(T) = T$ , and has  $(K, \alpha)$ -Hölder continuous local potentials for some  $0 < \alpha \leq 1$ ,  $K > 0$ .

Fix  $C > 0$  and  $1 < \lambda < d$ , and let  $\mathcal{A}_\lambda, \mathcal{B}_\lambda$  be as in Section 5. Define  $\mathcal{F}_\lambda(C)$  as the family of psh modulo  $T$  functions  $v$  on  $\mathbb{P}^k$  such that  $\max_{\mathbb{P}^k} v = 0$  and  $\|v\|_{L^1(X)} \leq C$  for all  $X \in \mathcal{A}_\lambda$ . By construction of  $\mathcal{A}_\lambda$ , Lemma 4.2 implies that  $\mathcal{F}_\lambda(C)$  is compact for each  $C > 0$ . Moreover, if  $X$  is an element of  $\mathcal{B}_\lambda$ , the restriction of  $\mathcal{F}_\lambda(C)$  to  $X$  forms a family of wpsH modulo  $T$  functions on  $X$  which is relatively compact in  $L^p(X)$  for every  $1 \leq p < +\infty$ .

If  $S$  is a positive closed  $(1,1)$ -current of mass 1, it is cohomologous to  $T$ . Hence, there exists a unique psh modulo  $T$  function  $u$  on  $\mathbb{P}^k$  such that

$S = dd^c u + T$  and  $\max_{\mathbb{P}^k} u = 0$ . We call  $u$  the dynamical potential of  $S$ . As  $T$  is totally invariant, the dynamical potential of  $S_n = d^{-n}(f^n)^*(S)$  is  $u_n = d^{-n}u \circ f^n$ .

Since  $S_n - T$  is a continuous linear operator on  $\mathcal{C}^0(\mathbb{P}^k)$  whose norm is bounded, by interpolation theory between Banach spaces we have

$$\|S_n - T\|_{\mathcal{C}^\beta} \lesssim \|S_n - T\|_{\mathcal{C}^2}^{\beta/2},$$

uniformly in  $S$  and  $n$ , see [21]. Consequently, in order to prove Theorem 1.2 we can assume that  $\beta = 2$ .

Moreover, it is easy to see that  $\|dd^c \phi\|_\infty \lesssim \|\phi\|_{\mathcal{C}^2}$  for  $\phi$  in  $\mathcal{C}^2(\mathbb{P}^k)$ . Therefore,

$$\begin{aligned} |\langle S_n - T, \phi \rangle| &= |\langle dd^c u_n, \phi \rangle| = |\langle u_n, dd^c \phi \rangle| \\ &\lesssim \|\phi\|_{\mathcal{C}^2} \|u_n\|_{L^1(\mathbb{P}^k)}. \end{aligned}$$

Hence, Theorem 1.2 is a direct consequence of the following theorem applied to  $p = 1$  and  $X = \mathbb{P}^k$ .

**Theorem 6.1.** *For each  $1 \leq p < +\infty$  and  $X \in \mathcal{B}_\lambda$  there exists a constant  $A_{X,p}$  such that for all  $u \in \mathcal{F}_\lambda(C)$  and  $n \geq 0$  we have*

$$\|u_n\|_{L^p(X)} \leq A_{X,p} \left( \frac{\lambda}{d} \right)^n,$$

where  $u_n = d^{-n}u \circ f^n$ .

As in Section 5, we can assume that each element of  $\mathcal{B}_\lambda$  is invariant by  $f$ , and there is  $1 < \delta < \lambda$  satisfying the following properties for all  $X$  in  $\mathcal{B}_\lambda$ :

- $E_\lambda(X) = E_\delta(X)$ ,
- $\kappa_{X,1} < \delta$  outside  $\tilde{E}_\lambda(X) = (f|_X)^{-1}(E_\lambda(X))$ .

Let  $X$  be an element  $\mathcal{B}_\lambda$  of dimension  $l$  and  $\lambda_1 > 0$  with  $\delta < \lambda_1 < \lambda$ . Assume that Theorem 6.1 is true on each irreducible component of  $E_\lambda = E_\lambda(X)$  for  $\lambda_1$  and all  $p \geq 1$ . To prove it on  $X$ , we consider the sublevel set  $K_n = \{x \in X \mid u_n(x) \leq -s_n\}$  for a suitable constant  $s_n$ . Exponential estimates on  $\tilde{E}_\lambda$  will prove that its image by  $f^i$ ,  $0 \leq i \leq n$ , cannot be concentrated near  $\tilde{E}_\lambda$ . Therefore, volume estimates will imply that  $f^n(K_n) = \{x \in X \mid u(x) \leq -d^n s_n\}$  is large if Theorem 6.1 is false on  $X$ . Hence, a good choice of  $s_n$ , allowed by the gap between  $\lambda_1$  and  $\lambda$ , will give a contradiction.

We first fix some constants. In Corollary 3.6 the constant  $b$  depends only on  $X$ . Then, by replacing  $f$  by  $f^n$  and  $\delta$  by  $\delta^n$  with  $b\delta^n < \delta^n < \lambda_1^n$ , we can assume that  $b = 1$ . Let  $0 < A \leq 1$ ,  $N \geq 1$  be the other constants of Corollary 3.6. Fix  $\lambda_2, \lambda_3 > 0$  such that

- $\delta < \lambda_1 < \lambda_2 < \lambda_3 < \lambda$ , and
- $q > q_0$  large enough such that  $\lambda_1/d < (\lambda_2/d)^{1+\epsilon}$  where  $\epsilon$  and  $q_0$  are defined in Lemma 4.4.

Multiplicities of  $f|_X$  are controlled outside  $\tilde{E}_\lambda$ . By induction hypothesis, we have a control of  $u_n$  on  $E_\lambda$ . We want to extend it to  $\tilde{E}_\lambda$ . Let  $E$  be an irreducible component of  $E_\lambda$ . The restriction of  $f$  to each component of  $(f|_X)^{-1}(E)$  is surjective onto  $E$ . Therefore, we deduce from Lemma 2.7 that there exists  $q' \geq 1$  such that

$$\|v \circ f\|_{L^q((f|_X)^{-1}(E))} \lesssim \|v\|_{L^{qq'}(E)},$$

for all psh modulo  $T$  function  $v$  on  $\mathbb{P}^k$ . Hence, by induction hypothesis, there is a constant  $M > 0$  such that  $\|u_n\|_{L^q(\tilde{E}_\lambda)} \leq M(\lambda_1/d)^n$  for  $n \geq 1$ . The next step is to obtain exponential estimates in a neighborhood of  $\tilde{E}_\lambda$ .

**Lemma 6.2.** *There exist constants  $c, \eta \geq 1$  and  $n_0 \geq 1$  such that if  $n \geq n_0$  then for all  $u \in \mathcal{F}_\lambda(C)$  we have*

$$\int_{\tilde{E}_{\lambda, t_n}} \exp(-(d/\lambda_2)^n u_n) \omega^l \leq c,$$

where  $t_n = (\lambda_2/d)^{n\eta}$ .

*Proof.* Let  $E$  be an irreducible component of  $\tilde{E}_\lambda$  of dimension  $i$ . According to the choice of  $q$ , we can find  $\lambda'_2 < \lambda_2$  such that  $\lambda_1/d < (\lambda'_2/d)^{1+\epsilon}$ . Hence

$$\|u_n\|_{L^q(E)} (d/\lambda'_2)^{(1+\epsilon)n} \leq M(\lambda_1/d)^n (d/\lambda'_2)^{(1+\epsilon)n} \leq M.$$

and by Lemma 4.4 with  $s = (d/\lambda'_2)^n$  we have

$$\int_E \exp(-a'(d/\lambda'_2)^n u_n) \omega^i \leq c',$$

for some constants  $a', c' > 0$ . Therefore, if we set  $\rho_n = (\lambda_2/d)^n$ , the volume in  $E$  of  $F_n = \{x \in E \mid u_n(x) \leq -\rho_n\}$  is smaller than  $c' \exp(-a'(\lambda_2/\lambda'_2)^n)$ . In particular,  $F_n$  contains no ball of radius  $\rho_n^{2/\alpha}$  for  $n$  large enough.

If  $X$  is smooth then set  $t_n = \rho_n^{1/\alpha}$ . As in Lemma 4.5, for  $n$  large enough, we can find a covering of  $E_{t_n}$  by balls with center in  $E$  and of radius  $2t_n$  on which Lemma 4.3 holds. Hence, we get

$$\int_{E_{t_n}} \exp(-au_n/\rho_n) \omega^l \leq c,$$

for some  $a, c > 0$ . The same argument with  $\lambda_2$  slightly smaller shows that we can choose  $a = 1$ . We conclude the proof by summing on all irreducible components of  $\widetilde{E}_\lambda$ .

When  $X$  is singular, we consider a desingularization  $\pi : \widehat{X} \rightarrow X$ . In order to establish the estimate near  $E$ , we proceed inductively as follows. Assume that there exists a triplet  $(A, a, \theta)$  with  $a > 0$ ,  $\theta \geq 1$  and an analytic set  $A \subset E$  such that

$$\int_{E_{t\theta} \setminus A_{t^{1/\theta}}} \exp(-au_n/\rho_n) \omega^l$$

is uniformly bounded in  $n \geq 0$  for  $t \leq \rho_n$ . Then, using the properties of the elements of  $\mathcal{B}_\lambda$  and dynamical arguments, we claim that a similar estimate holds if we substitute  $(A, a, \theta)$  by some  $(A', a', \theta')$  with  $\dim(A') < \dim(A)$ . It will give the result for  $\eta$  large enough after less than  $l$  steps since  $\dim(E) < l$ .

More precisely, let  $V$  be an irreducible component of  $A$  with maximal dimension. We distinguish two cases, according to whether  $V$  is in  $\mathcal{B}_\lambda$  or not. In the first case, we know that for all  $p \geq 1$ ,  $\|u_n\|_{L^p(V)} \lesssim (\lambda_1/d)^n$ . We set  $\widehat{V} := \pi^{-1}(V)$ . We denote by  $\widehat{V}_1$  the union of all components of  $\widehat{V}$  which are mapped onto  $V$  and by  $\widehat{V}_2$  the union of the other components of  $\widehat{V}$ . Therefore, Lemma 2.7 implies that

$$\|\widehat{u}_n\|_{L^p(\widehat{V}_1)} \lesssim (\lambda_1/d)^n,$$

for all  $p \geq 1$ , where  $\widehat{u}_n = u_n \circ \pi$ . Hence, the smooth version of the lemma implies that

$$\int_{\widehat{V}_{1, \rho_n^{1/\alpha}}} \exp(-a'\widehat{u}_n/\rho_n) \pi^*(\omega^l)$$

is uniformly bounded for  $a' > 0$  small enough. Moreover, by Lemma 2.6, there exists a constant  $\theta' \geq 1$  such that  $\pi(\widehat{V}_{1,t})$  contains  $V_{t^{\theta'}} \setminus V_{2,t^{1/2}}$ , where  $V_2 = \pi(\widehat{V}_2)$ . It gives the desired result near  $V$ , since  $\dim(V_2) < \dim(V)$ .

From now, we can assume that no irreducible component of  $A$  with maximal dimension belongs to  $\mathcal{B}_\lambda$  (in particular  $A \neq E$ ). Let  $V$  denote the union of all irreducible components of  $A$  with maximal dimension. In particular, these components are not totally invariant for  $f|_E$ , therefore there exist an analytic set  $Z \subset E$  containing no component of  $V$  and an integer  $m \geq 1$  such that  $f^m(Z) = V$ . We set  $Z' = Z \cap A$ . The assumption on  $A$  and  $\theta$  implies that if  $t \leq \rho_n$  then

$$\int_{Z_{t\theta} \setminus A_{t^{1/\theta}}} \exp(-au_n/\rho_n) \omega^l$$

is bounded uniformly on  $n$ . By Corollary 2.2,  $Z_t \cap A_{t^{1/\theta}}$  is contained in  $Z'_{t^{1/\theta'}}$  for some  $\theta' > \theta$ . So,

$$\int_{Z_{t^{\theta''}} \setminus Z'_{t^{1/\theta''}}} \exp(-au_n/\rho_n) \omega^l$$

is bounded uniformly on  $n$ . Fix a constant  $B > 1$  large enough. We deduce from Corollary 3.6 applied to  $\mathbb{P}^k$  that for all  $t > 0$ ,  $f^m(Z_t)$  contains  $V_{B^{-1}t^{dmk}}$ . Moreover, since  $f^m$  is Lipschitz,  $f^m(Z'_t)$  is contained in  $V'_{Bt}$ , where  $V' = f^m(Z')$ . So, we have

$$f^m(Z_{t^{\theta''}} \setminus Z'_{t^{1/\theta''}}) \supset V_{t^{\theta'}} \setminus V'_{t^{1/\theta'}},$$

for  $t > 0$  small enough and  $\theta' > \theta''$  large enough. It follows that

$$\begin{aligned} \int_{V_{t^{\theta'}} \setminus V'_{t^{1/\theta'}}} \exp(-a'u_n/\rho_n) \omega^l &\leq \int_{Z_{t^{\theta''}} \setminus Z'_{t^{1/\theta''}}} \exp(-a' \frac{u_{n+m} \lambda_2^m}{\rho_{n+m}}) (f^m_{|X})^*(\omega^l) \\ &\lesssim \int_{Z_{t^{\theta''}} \setminus Z'_{t^{1/\theta''}}} \exp(-a' \frac{u_{n+m} \lambda_2^m}{\rho_{n+m}}) \omega^l, \end{aligned} \quad (6.1)$$

since  $(f^m_{|X})^*(\omega^l) \lesssim \omega^l$ . Moreover, for  $a'$  small enough the right-hand side in (6.1) is bounded uniformly on  $n$  and  $\dim(V') = \dim(Z') < \dim(V)$  since  $Z$  contains no component of  $V$ . This together with the estimate outside  $A$  prove the claim with  $A' = V'$ .  $\square$

From now, we fix  $p \geq 1$  and for  $u$  in  $\mathcal{F}_\lambda(C)$  denote  $\mathcal{N}(u) = \{n \geq 1 \mid \|u_n\|_{L^p(X)} \geq (\lambda/d)^n\}$  and denote by  $\mathcal{N}$  the union of  $\mathcal{N}(u)$  for all  $u$ . Our goal is to prove that  $\mathcal{N}$  is finite, which will imply Theorem 6.1. For this purpose, we have the following result.

**Lemma 6.3.** *There are constants  $n_1 \geq 1$  and  $\beta \geq 1$  such that if  $n$  is in  $\mathcal{N}(u)$  with  $n \geq n_1$  then  $K_n = \{x \in X \mid u_n(x) \leq -(\lambda_3/d)^n\}$  contains a ball of radius  $(\lambda_3/d)^{\beta n}$ .*

*Proof.* Since  $x^p \lesssim \exp(x)$  if  $x \geq 0$ , we deduce from the assumption on  $\|u_n\|_{L^p(X)}$  that

$$\begin{aligned} (\lambda/\lambda_3)^n &\lesssim \left( \int_X (-(d/\lambda_3)^n u_n)^p \omega^l \right)^{1/p} \\ &\lesssim \left( \int_X \exp(-(d/\lambda_3)^n u_n/2) \omega^l \right)^{1/p}. \end{aligned} \quad (6.2)$$

On the other hand, let  $\beta$  be the constant in Lemma 4.5. For  $n$  sufficiently large we have  $(d/\lambda_3)^n \geq 2$ . Hence, Lemma 4.5 with  $s = (d/\lambda_3)^n$  imply that  $K_n$  has to contain a ball of radius  $(\lambda_3/d)^{n\beta}$ , otherwise the right-hand side of (6.2) would be bounded uniformly on  $n$ , which is impossible since  $\lambda_3 < \lambda$ .  $\square$

We can now complete the proof of the main theorem.

*End of the proof of Theorem 6.1.* If  $B \subset X$  is a Borel set then  $|B|$  denotes its volume with respect to the measure  $\omega^l$ . As we have already seen, the volume of a ball of radius  $r$  in  $X$  is larger than  $c'r^{2l}$ ,  $0 < c' \leq 1$ . Therefore, observe that if  $x$  is in  $\tilde{E}_{\lambda, t_n/2}$  then  $|\tilde{E}_{\lambda, t_n} \cap B_X(x, r)| = |B_X(x, r)| \geq c'(r/2)^{2l}$  for  $r < t_n/2$ .

From now, assume in order to obtain a contradiction that  $\mathcal{N}$  is infinite. Consider  $u \in \mathcal{F}_\lambda(C)$  and  $n \in \mathcal{N}(u)$  large enough. Fix also  $\beta$  large enough. So, we have  $(\lambda_3/d)^{\beta n} < t_n/4$  and

$$c \exp(-(\lambda_3/\lambda_2)^n) \leq c'(A^\delta(t_n/2)^{N\delta}(\lambda_3/d)^{\beta n}/2)^{2l}, \quad (6.3)$$

where  $c$  is defined in Lemma 6.2. Let  $r_0 = (\lambda_3/d)^{\beta n}$ , and for  $1 \leq i \leq n$  let  $r_i = A(t_n/2)^N r_{i-1}^\delta$ . We will prove by induction that for  $0 \leq i \leq n$ ,  $f^i(K_n) = \{x \in X \mid u_{n-i}(x) \leq -\lambda_3^n/d^{n-i}\}$  contains a ball  $B_i$  of radius  $r_i$ .

Since  $\beta$  is large, Lemma 6.3 implies that the assertion is true for  $i = 0$ . Let  $0 \leq i \leq n-1$  and assume the property is true for  $i$ . We deduce from Lemma 6.2 that

$$\int_{\tilde{E}_{\lambda, t_{n-i}}} \exp(-(d/\lambda_2)^{n-i} u_{n-i}) \omega^l \leq c,$$

and in particular

$$|\tilde{E}_{\lambda, t_n} \cap B_i| \leq |\tilde{E}_{\lambda, t_{n-i}} \cap f^i(K_n)| < c \exp(-(\lambda_3/\lambda_2)^n \lambda_2^i),$$

since  $t_n \leq t_{n-i}$ . This and (6.3) imply that

$$|B_i| \geq c'r_i^{2l} > 2^{2l} |\tilde{E}_{\lambda, t_n} \cap B_i|,$$

since  $r_i \geq (A^\delta(t_n/2)^{N\delta} r_0)^{\delta^i}$  and  $\delta < \lambda_2$ . Consequently, the center of  $B_i$  is not in  $\tilde{E}_{\lambda, t_n/2}$  and by Corollary 3.6,  $f(B_i) \subset f^{i+1}(K_n)$  contains a ball  $B_{i+1}$  of radius  $r_{i+1} = A(t_n/2)^N r_i^\delta$ . Note that we already reduced the problem to the case where the constant  $b$  in Corollary 3.6 is equal to 1.

Therefore, for all  $n$  in  $\mathcal{N}(u)$  sufficiently large, the volume of  $f^n(K_n) = \{x \in X \mid u(x) \leq -\lambda_3^n\}$  is greater than  $D^{n\delta^n}$ , with  $0 < D < 1$  independent

of  $u$  and  $n$ . This contradicts the inequality  $\delta < \lambda_3$ . Indeed, since  $\mathcal{F}_\lambda(C)$  is bounded in  $L^q(X)$ , by Lemma 4.4 there exists  $a' > 0$  such that

$$\int_X \exp(-a'u)\omega^l$$

is uniformly bounded for  $u$  in  $\mathcal{F}_\lambda(C)$ .

Hence,  $\mathcal{N}$  is finite and in particular bounded by some  $n_2 \geq 1$ . We conclude using the fact that the restriction of  $\cup_{n=0}^{n_2} d^{-n}(f^n)^*(\mathcal{F}_\lambda(C))$  to  $X$  is a relatively compact family of wpsH modulo  $T$  functions and then bounded in  $L^p(X)$ . Therefore, we have

$$\|u_n\|_{L^p(X)} \lesssim \left(\frac{\lambda}{d}\right)^n,$$

if  $n \leq n_2$  and thus for every  $n \geq 0$  by the definition of  $\mathcal{N}$ . □

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