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IDENTIFICATION OF TIME-VARYING SOURCE TERM IN TIME-FRACTIONAL DIFFUSION EQUATIONS

YAVAR KIAN[1], ÉRIC SOCCORSI[1], QI XUE[2] AND MASAHIRO YAMAMOTO[3,4,5]

Abstract. This paper is concerned with the inverse problem of determining the time and space dependent source term of diffusion equations with constant-order time-fractional derivative in $(0, 2)$. We examine two different cases. In the first one, the source is the product of two spatial and temporal terms, and we prove that both of them can be retrieved by knowledge of one arbitrary internal measurement of the solution for all times. In the second case, we assume that the first term of the product varies with one fixed space variable, while the second one is a function of all the remaining space variables and the time variable, and we show that both terms are uniquely determined by two arbitrary lateral measurements of the solution over the entire time span. These two source identification results boil down to a weak unique continuation principle in the first case and a unique continuation principle for Cauchy data in the second one, that are preliminarily established. Finally, numerical reconstruction of spatial term of source terms in the form of the product of two spatial and temporal terms, is carried out through an iterative algorithm based on the Tikhonov regularization method.

Keywords: Inverse source problems, diffusion equation, fractional diffusion equation, uniqueness result, numerical reconstruction, Tikhonov regularization method.

Mathematics subject classification 2010 : 35R30, 35R11.

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1. Introduction

1.1. **Settings.** Let Ω be a bounded and connected open subset of $\mathbb{R}^d$, $d \geqslant 2$, with C^2 boundary $\partial\Omega$. Given $a := (a_{i,j})_{1\leqslant i,j\leqslant d} \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^{d^2})$, symmetric, i.e.,

$$a_{i,j}(x) = a_{j,i}(x),\ x \in \Omega,\ i,j = 1,\dots,d,$$

and fulfilling the ellipticity condition

$$\exists c > 0,\ \sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \geqslant c|\xi|^2,\ x \in \Omega,\ \xi = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d, \tag{1.1}$$

we introduce the formal differential operator

$$\mathcal{A}_0 u(x) := -\sum_{i,j=1}^{d} \partial_{x_i}\left(a_{i,j}(x)\partial_{x_j}u(x)\right),\ x := (x_1,\dots,x_d) \in \Omega,$$

where ∂_{x_i} denotes the partial derivative with respect to x_i, $i = 1,\dots,d$. We perturb $\mathcal{A}_0$ by a potential function $q \in L^\kappa(\Omega)$, $\kappa \in (d,+\infty]$, that is lower bounded by some positive constant,

$$\exists r \in (0,+\infty),\ q(x) \geqslant r,\ x \in \Omega, \tag{1.2}$$

and define the operator $\mathcal{A}_q := \mathcal{A}_0 + q$, where the notation q is understood as the multiplication operator by the corresponding function.

Next, for $T \in (0,+\infty)$, $\alpha \in (0,2)$ and $\rho \in L^\infty(\Omega)$ obeying

$$0 < \rho_0 \leqslant \rho(x) \leqslant \rho_M < +\infty,\ x \in \Omega, \tag{1.3}$$

we consider the following initial boundary value problem (IBVP) with source term $f \in L^1(0,T;L^2(\Omega))$,

$$\begin{cases} (\rho(x)\partial_t^\alpha + \mathcal{A}_q)u(t,x) &= f(t,x), & (t,x) \in Q := (0,T)\times\Omega, \\ B_\star u(t,x) &= 0, & (t,x) \in \Sigma := (0,T)\times\partial\Omega, \\ \partial_t^k u(0,x) &= 0, & x \in \Omega,\ k = 0,\dots,N_\alpha, \end{cases} \tag{1.4}$$

where

$$N_\alpha := \begin{cases} 0 & \text{if } \alpha \in (0,1], \\ 1 & \text{if } \alpha \in (1,2), \end{cases}$$

and ∂_t^α denotes the fractional Caputo derivative of order α with respect to t, defined by

$$\partial_t^\alpha u(t,x) := \frac{1}{\Gamma(N_\alpha + 1 - \alpha)}\int_0^t (t-s)^{N_\alpha-\alpha}\partial_s^{N_\alpha+1}u(s,x)\mathrm{d}s,\ (t,x) \in Q, \tag{1.5}$$

when $\alpha \in (0,1)\cup(1,2)$, while ∂_t^α is the usual first order derivative ∂_t when $\alpha = 1$. In the second line of (1.4), $B_\star$ is either of the two following boundary operators:

(a) $B_\star u := u$,

(b) $B_\star u := \partial_{\nu_a} u$, where ∂_{ν_a} stands for the normal derivative with respect to $a = (a_{i,j})_{1\leqslant i,j\leqslant d}$, expressed by

$$\partial_{\nu_a} h(x) := \sum_{i,j=1}^{d} a_{i,j}(x)\partial_{x_j} h(x)\nu_i(x),\ x \in \partial\Omega,$$

and $\nu = (\nu_1, \ldots, \nu_d)$ is the outward unit normal vector to $\partial\Omega$.

Otherwise stated, the IBVP (1.4) is endowed with homogeneous Dirichlet (resp., Neumann) boundary condition when $B_\star$ is given by (a) (resp., (b)).

1.2. **Weak solution.** With reference to [21, 34], we say that u is a weak solution to (1.4) if u is the restriction to Q of a function $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\Omega))$, i.e. $u = v_{|Q}$, fulfilling the two following conditions:

(a) $\inf\{\varepsilon > 0 :\ e^{-\varepsilon t} v \in L^1(\mathbb{R}_+; L^2(\Omega))\} = 0$;

(b) For all $p \in (0, +\infty)$, the Laplace transform of v with respect to t, computed at p,

$$V(p) = \mathcal{L}[v](p) := \int_0^{+\infty} e^{-pt} v(t, .)dt,$$

solves the following boundary value problem (BVP)

$$\begin{cases} (\mathcal{A}_q + \rho p^\alpha)V(p) = F(p), & \text{in } \Omega, \\ B_\star V(p) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where F denotes the Laplace transform with respect to t of f multiplied by the characteristic function $t \mapsto \mathbb{1}_{(0,T)}(t)$ of the interval $(0, T)$, i.e.

$$F(p) := \mathcal{L}[f\mathbb{1}_{(0,T)}](p) = \int_0^T e^{-pt} f(t, .)dt. \tag{1.7}$$

Here and in the remaining part of this text, we use the notation $v(t, \cdot)$ as a shorthand for the function $x \mapsto v(t, x)$.

The weak solution to (1.4) exists and is unique within the class $\mathcal{C}((0, T], L^2(\Omega))$, and it enjoys a Duhamel representation formula, given in Section 2. We refer the reader to [21, 20, 34] for the existence and the uniqueness issue of such a solution to (1.4), as well as for its classical properties. We point out that for $\alpha = 1$, the weak solution to (1.4) coincides with the classical variational $\mathcal{C}^1([0, T]; H^{-1}(\Omega)) \cap \mathcal{C}([0, T]; H^1(\Omega))$-solution to the corresponding parabolic equation.

1.3. **Inverse problem.** In this paper we examine the inverse problem of determining the source term f appearing in the first line of (1.4), from either internal or lateral measurement of the weak solution u to (1.4). But it turns out that this problem is ill-posed in the sense that the above data do not uniquely determine f.

1.3.1. *A natural obstruction to identifiability.* This can be seen upon taking a proper subset Ω' of Ω, picking a function $u_0 \in \mathcal{C}_0^\infty((0,T) \times (\Omega \setminus \overline{\Omega'}))$ that is not identically zero, and considering the IBVP (1.4) associated with the source term $f_0 := \rho \partial_t^\alpha \tilde{u}_0 - \mathcal{A}_q \tilde{u}_0$, where $\tilde{u}_0$ is the extension by zero of u_0 on $(0,T) \times \overline{\Omega'}$, i.e.

$$\tilde{u}_0(t,x) := \begin{cases} u_0(t,x) & \text{if } (t,x) \in (0,T) \times (\Omega \setminus \overline{\Omega'}), \\ 0 & \text{if } (t,x) \in (0,T) \times \overline{\Omega'}. \end{cases}$$

Evidently, $\tilde{u}_0$ is a weak solution to (1.4), so we have $u = \tilde{u}_0$ from the uniqueness of the solution to (1.4) with $f = f_0$. Moreover, since $\tilde{u}_0$ is not identically zero in Q, then the same is true for f_0 (otherwise $\tilde{u}_0$ would be zero everywhere by uniqueness of the solution to (1.4), in contradiction to the definition of u_0). Thus, we have $u_{|(0,T)\times\Omega'} = 0$, despite of the fact that f_0 is not identically zero in Q. This establishes that the recovery of the unknown source term f by partial knowledge of u, is completely hopeless, or, otherwise stated, that full knowledge of the solution u to (1.4) (i.e. measurement of u performed on the entire time-space cylinder Q) is needed in order to uniquely determine general unknown source terms f in $L^1(0,T;L^2(\Omega))$.

As a consequence, the inverse source problem under investigation has to be reformulated. Different lines of research can be pursued. One possible direction is the one of assuming that the unknown function $f : Q \to \mathbb{R}$ depends on a restricted number of parameters of $(t,x) \in Q$. Another direction is the one of considering source terms with separated variables. In this paper, we follow the second direction.

1.3.2. *Motivations.* Depending on whether $\alpha = 1$ or $\alpha \in (0,1) \cup (1,2)$, the system (1.4) models typical or anomalous diffusion phenomena appearing in several areas of applied sciences, such as geophysics, environmental science and biology, see e.g. [17, 31]. In this context, sub-diffusive (resp., super-diffusive) processes are described by (1.4) with $\alpha \in (0,1)$ (resp., $\alpha \in (1,2)$), and kinetic equation (1.4) may be seen as a corresponding macroscopic model to microscopic diffusion phenomena driven by continuous time random walk, see e.g., [30]. The inverse problem under examination here, is to know whether time and space varying source terms can be retrieved by either internal or lateral data, in presence of typical or anomalous

diffusion. We point out that such a framework can be adapted to the recovery of moving sources as in [23].

1.4. **A short review of inverse source problems.** Inverse problems are generally nonlinear in the sense that the unknown parameter of the problem depends in a nonlinear way on the data. For instance, this is the case for inverse coefficients problems or inverse spectral problems, see e.g. [13, 27]. However, this is no longer true for inverse source problems, as the dependence of the unknown source term is linear with respect to the (internal or lateral) data. When this remarkable feature of inverse source problems does not guarantee that they are easy to solve, it certainly does explain why they have become increasingly popular among the mathematical community.

This is particularly true when typical diffusion is considered, i.e. when $\alpha = 1$ in (1.4), where the inverse problem of determining a time independent source term has been extensively studied by several authors in [5, 6, 19, 38, 39] and in [12], which is based on the celebrated Bukhgeim-Klibanov approach introduced in [3], the list being non exhaustive. As for inverse time independent source problem with $\alpha \in (0,1) \cup (1,2)$, we refer the reader to [15], and to [16, 17, 18, 20, 26, 33] for inverse coefficient problems in the context of anomalous diffusion equations.

In all the above mentioned inverse source results, the source term was stationary. The stability issue in determining the temporal source term of time-fractional diffusion equations was examined in [7, 34], and in the same context, the time and space dependent factor of suitable source terms is reconstructed in [22]. As for the determination of time dependent sources in parabolic equations, we refer the reader to [1, 9, 14, 24], and to [2, 10, 11] for the same problem with hyperbolic equations.

Let us now collect the main achievements of this article in the coming section.

1.5. **Main results.** We start by stating a weak uniqueness principle (UP) for the IBVP (1.4) whose source term f is the product of two functions, each of them depending only on either the time-variable or the space-variable.

Theorem 1.1. *Let $\sigma \in L^1(0,T)$ be supported in $[0,T)$, let $g \in L^2(\Omega)$ and assume that*

$$f(t,x) := \sigma(t)g(x), \ (t,x) \in Q. \tag{1.8}$$

Denote by u the weak solution to (1.4)*. Then, for all $\alpha \in (0,2)$ and an arbitrary non-empty open subset $\Omega' \subset \Omega$, we have:*

$$(u_{|(0,T)\times\Omega'} = 0) \Longrightarrow (f = 0 \text{ in } Q).$$

As a corollary, we have the following unique identification result for the corresponding inverse source problem.

Corollary 1.2. *For $j = 1,2$, let $\sigma_j \in L^1(0,T)$ be such that $\operatorname{supp}\sigma_j \subset [0,T)$, let $g_j \in L^2(\Omega)$, and assume that either of the two following conditions is fulfilled:*

(i) $\sigma_1 = \sigma_2$ is not identically zero in $(0,T)$,

(ii) $g_1 = g_2$ is not identically zero in Ω.

Let u_j denote the solution to (1.4) *with $f = f_j$, where*

$$f_j(t,x) := \sigma_j(t) g_j(x),\ (t,x) \in Q.$$

Then, the following implication holds for any non-empty open subset $\Omega' \subset \Omega$:

$$\left(u_1 = u_2 \text{ in } (0,T)\times\Omega'\right) \Longrightarrow \left(\sigma_1 = \sigma_2 \text{ in } (0,T) \text{ and } g_1 = g_2 \text{ in } \Omega\right). \tag{1.9}$$

Actually, the result of Corollary 1.2 remains valid upon removing the hypothesis on the support of σ_j, $j = 1,2$, which was inherited from Theorem 1.1, but this is at the expense of a greater regularity assumption on these two functions.

Theorem 1.3. *For $j = 1,2$, let $g_j \in L^2(\Omega)$ and let $\sigma_j \in L^1(0,T)$ fulfill*

$$\sigma_1(t) = \sigma_2(t) = \sigma(t),\ t \in (0,t_0),$$

where $t_0 \in (0,T)$ and σ is a non-zero holomorphic function in the complex half-strip $\mathcal{S}_\delta := \{x+iy:\ x \in (-\delta,+\infty),\ y \in (-\delta,\delta)\}$ of fixed width $\delta \in (0,+\infty)$, which grows no faster than polynomials, i.e. such that

$$|\sigma(t)| \leqslant C(1+t)^N,\ t \in (0,+\infty), \tag{1.10}$$

for some positive constant C and some natural number N, that are both independent of t. We assume also that g_1 is not identically zero in Ω. Then (1.9) *holds true.*

We also investigate the case where the function g, appearing in (1.8), depends on one real parameter only, assumed to be, without restricting the generality of the foregoing, the last component x_n of the space variable $x = (x_1,\ldots,x_{n-1},x_n) \in \Omega$, while α is a function of the

time variable t and $x' := (x_1, \ldots, x_{n-1})$. More precisely, we assume existence of $L \in (0, +\infty)$ and of $\omega \subset \mathbb{R}^{d-1}$, open and bounded, such that $\Omega_0 := \omega \times (-L, L)$ verifies

$$\overline{\Omega_0} \subset \Omega, \tag{1.11}$$

and we consider source terms f supported in $[0, T] \times \overline{\Omega_0}$, of the following form,

$$f(t, x', x_n) := \begin{cases} \sigma(t, x') g(x_n) & \text{if } (t, x', x_n) \in Q_0 := (0, T) \times \Omega_0, \\ 0 & \text{if } (t, x', x_n) \in Q \setminus Q_0, \end{cases} \tag{1.12}$$

where σ and g will be made precise below. In this case, we have the following UP for local Cauchy data:

Theorem 1.4. *Assume* (1.11) *and assume that* $\Omega \setminus \overline{\Omega_0}$ *is connected. For* $\zeta \in \left(\frac{3}{4}, 1\right)$ *and* $r \in \left(\frac{1}{\alpha(1-\zeta)}, +\infty\right)$, *let* $\sigma \in L^r(0, T; L^2(\omega))$ *be supported in* $[0, T) \times \overline{\omega}$, *and let* $g \in L^2(-L, L)$. *Then for each* $\alpha \in (0, 2)$, *there exists a unique solution* $u \in \mathcal{C}([0, T]; H^{2\zeta}(\Omega))$ *to the IBVP* (1.4) *associated with density* $\rho = 1$ *a.e. in* Ω, *elliptic operator* $\mathcal{A}_q = -\Delta$, *and source term* f *defined by* (1.12). *Moreover, the following implication*

$$\left(u_{|(0,T)\times\gamma} = \partial_\nu u_{|(0,T)\times\gamma} = 0\right) \Longrightarrow (f = 0 \text{ in } Q) \tag{1.13}$$

holds for any non-empty subset γ *of* $\partial\Omega$.

We point out that the statement of Theorem 1.1 (resp., Theorem 1.4) can be adapted to the framework of distributed order fractional diffusion equations, and we refer to Theorem 6.2 (resp., Theorem 6.3) in Section 6 for the corresponding result.

The coming result is a byproduct of Theorem 1.4, likewise Corollary 1.2 follows from Theorem 1.1.

Corollary 1.5. *Let* Ω *be the same as in Theorem 1.4. For* $j = 1, 2$, *let* $\sigma_j \in L^r(0, T; L^2(\omega))$ *be such that* $\operatorname{supp} \sigma_j \subset [0, T) \times \overline{\omega}$, *where* r *is as in Theorem 1.4, and let* $g_j \in L^2(-L, L)$, *fulfill either of the two following conditions:*

(i) $\sigma_1 = \sigma_2$,

(ii) $g_1 = g_2$.

Denote by u_j, $j = 1, 2$, *the solution to* (1.4) *associated with* $\rho = 1$ *a.e. in* Ω, $\mathcal{A}_q = -\Delta$ *and* $f = f_j$, *where* f_j *is obtained by substituting* (σ_j, g_j) *for* (σ, g) *in the right hand side of* (1.12).

Then, we have $\sigma_1 = \sigma_2$ in $(0,T)\times\omega$ and $g_1 = g_2$ in $(-L,L)$, whenever the two following identities

$$\partial_\nu^k u_1(t,x) = \partial_\nu^k u_2(t,x),\ (t,x)\in(0,T)\times\gamma,\ k=0,1, \tag{1.14}$$

hold for an arbitrary non-empty subset γ of $\partial\Omega$.

In a similar fashion as the identification result of Corollary 1.2 was adapted to non-compactly supported time-dependent source terms of the form (1.8) in Theorem 1.3, we translate Corollary 1.5 into the following statement for non-compactly supported sources of the form (1.12).

Theorem 1.6. *For $j=1,2$, let $g_j\in L^2(-L,L)$ and let $\sigma_j\in L^1(0,T;L^2(\omega))$ fulfill*

$$\sigma_1(t,x') = \sigma_2(t,x') = \sigma(t),\ t\in(0,t_0)\times\omega, \tag{1.15}$$

where $t_0\in(0,T)$ and $t\mapsto\sigma(t,\cdot)$ is a non-zero holomorphic $L^2(\omega)$-valued function in the complex half-strip $\mathcal{S}_\delta$ introduced in Theorem 1.3, which grows no faster than polynomials:

$$\|\sigma(t,\cdot)\|_{L^2(\omega)} \leqslant C(1+t)^N,\ t\in(0,+\infty),$$

for some positive constant C and some natural number N, that are both independent of t. We assume also that g_1 is not identically zero in Ω. Then (1.14) implies that $\sigma_1=\sigma_2$ and $g_1=g_2$.

In the particular case where $\alpha=1$, the PDE in the first line of (1.4) is of parabolic type, and the obstruction to unique determination of f by partial measurement of the solution, manifested in Section 1.1, can be further described for source terms of the form

$$f(t,x) := \sigma(t)g(x) + \beta(t)h(x). \tag{1.16}$$

Namely, given a suitable internal boundary observation of the solution to (1.4), we aim to characterize all source terms of the form (1.16), generating the exact same data. To this purpose, we define A_q as the self-adjoint operator in $L^2(\Omega)$, generated by the closed sesquilinear form

$$(u,v)\mapsto \sum_{i,j=1}^d \left(a_{i,j}(x)\partial_{x_i}u(x)\overline{\partial_{x_j}v(x)} + q(x)u(x)\overline{v(x)}\right)dx,\ u,v\in V,$$

where $V := H_0^1(\Omega)$ if $\mathcal{A}_q$ is endowed with a homogeneous Dirichlet boundary condition, while $V := H^1(\Omega)$ if the boundary condition attached to $\mathcal{A}_q$ is of Neumann type. Otherwise stated, A_q is the (positive) self-adjoint operator in $L^2(\Omega)$, acting as $\mathcal{A}_q$ on its domain $D(A_q)$, dense in

$L^2(\Omega)$. We denote by A_0 the operator A_q when $q = 0$ a.e. in Ω. In light of (1.1)-(1.2), $D(A_q)$ is independent of q (see e.g. [20, Section 2.1]) and it is embedded in $H^2(\Omega)$:

$$D(A_q) = D(A_0) \subset H^2(\Omega). \tag{1.17}$$

Next we introduce the operator $A_{q,\rho} := \rho^{-1} A_q$, with domain

$$D(A_{q,\rho}) = D(A_q), \tag{1.18}$$

positive and self-adjoint in the weighted-space $L^2_\rho(\Omega) := L^2(\Omega; \rho dx)$. Evidently, $A_{q,\rho}$ is self-adjoint in $L^2_\rho(\Omega)$. If we suppose that the function β does not change sign and that it is not-identically zero in $(0,T)$, then the operator $\int_0^T \beta(t) e^{A_{q,\rho} t} dt$ is boundedly invertible in $L^2_\rho(\Omega)$. Let us denote its inverse by $\left(\int_0^T \beta(t) e^{A_{q,\rho} t} dt\right)^{-1}$. Then, by the operatorial calculus, the following operator

$$H_{q,\rho} := -\left(\int_0^T \beta(t) e^{A_{q,\rho} t} dt\right)^{-1} \left(\int_0^T \sigma(t) e^{A_{q,\rho} t} dt\right), \tag{1.19}$$

is self-adjoint in $L^2_\rho(\Omega)$.

Theorem 1.7. *Let $\sigma \in L^2(0,T)$ and $\beta \in L^1(0,T)$ be supported in $[0,T)$. Assume further that β is not-identically zero and does not change sign in $(0,T)$. Given g and h in $L^2(\Omega)$, denote by u the solution to (1.4) associated with $\alpha = 1$ and source term f expressed by (1.16). Then, for any non-empty open subset $\Omega' \subset \Omega$, we have the implication:*

$$\left(u_{|(0,T)\times\Omega'} = 0\right) \Longrightarrow \left(h = \rho H_{q,\rho} \rho^{-1} g \text{ in } \Omega\right), \tag{1.20}$$

where $H_{q,\rho}$ is the operator defined in (1.19).

Although Theorem 1.7 is interesting in its own right, the main benefice of the above statement is the following characterization of the set of source terms expressed by (1.8), which generate the same specified data.

Corollary 1.8. *For $\ell \in \mathbb{N}$ fixed, let β be in $H^\ell_0(0,T)$, the closure of $\mathcal{C}^\infty_0(0,T)$ in the $H^\ell(0,T)$-norm topology. Assume that β is supported in $[0,T)$, that it is not-identically zero and that it does not change sign in $(0,T)$. Suppose moreover that $\alpha = 1$ and that $\rho(x) = 1$ for a.e. $x \in \Omega$. For g and h in $L^2(\Omega)$, let u_g denote the weak-solution to (1.4) associated with $f(t,x) = \frac{d^\ell \beta}{dt^\ell}(t) g(x)$ and u_h be the weak-solution to (1.4) with source term $f(t,x) = -\beta(t) h(x)$. Then, for any non-empty open subset $\Omega' \subset \Omega$, we have the implication:*

$$\left(u_g = u_h \text{ in } (0,T)\times\Omega'\right) \Longrightarrow \left(h = (-1)^{\ell+1} A_q^\ell\, g \text{ in } \Omega\right). \tag{1.21}$$

Moreover, in the particular case where $\ell = 1$, we have in addition:

$$g = h = 0 \text{ in } \Omega'. \tag{1.22}$$

Having stated the main results of this article, we briefly comment on them in the coming section.

1.6. **Comments.** The derivation of a source identification result such as Corollary 1.2 from a UP such as the one stated in Theorem 1.1, is rather standard in the analysis of inverse source problems for diffusion equations, see e.g. [5, 6, 10, 15]. The strategy used in these four articles to determine the spatial part of the source (1.8) under the assumption that its temporal part σ is known, is to turn the non-homogeneous diffusion equation under study into a homogeneous one, by moving the source information into the initial data. This firstly requires that $\sigma(0) \neq 0$, and secondly that $\sigma \in \mathcal{C}^1([0,T])$. The condition $\sigma(0) \neq 0$ suggests that the source should be switched on before the data are collected. This is surprising considering that only the (stationary) spatial part of the source is retrieved here. Indeed, from a practical point of view, one may wonder why starting the observation early might be a problem for determining g. In any case, it turns out that this rather unnatural condition $\sigma(0) \neq 0$ was removed for an evolutionary equation in the infinite time range $(0, +\infty)$ in [10, Theorem 2]. As far as we know, Theorem 1.1 is the first mathematical result doing the same when the time evolution is restricted to a bounded interval $(0, T)$. Furthermore, we stress out that the second condition $\sigma \in \mathcal{C}^1([0,T])$ requested by [5, 6, 10, 15] is weakened to $\sigma \in L^1(0,T)$ in Theorem 1.1 and Corollary 1.2. In this paper, we assume only that $\sigma \in L^1(0,T)$. This regularity is weak and for example, the method in [15] does not work directly and so we need extra arguments.

In Theorem 1.1, either of the two terms σ or g appearing in the right hand side of (1.8), is retrieved when the other one is known, which, in this connection, is very similar to the results of [5, 6, 10, 15]. On the other hand, in Theorem 1.3, we are able to identify simultaneously σ and g, but this is at the expense of greater regularity on α and upon assuming partial knowledge of σ, as it is requested that up to some fixed time $t_0 \in (0,T)$, the function $t \mapsto \sigma(t)$ be known and depend analytically on t. In this respect, Theorem 1.4 and Corollary 1.5 may be seen as an alternative approach to Theorem 1.3 for recovering a source term depending on the time variable and all the space variables excepting one.

A key ingredient in the derivation of Theorems 1.1 and 1.4 is the time analyticity property of the solution to (1.4), exhibited in Proposition 2.1. While this is classical for $\alpha = 1$, the proof

requires a more careful treatment for $\alpha \in (0,1) \cup (1,2)$, which is based on the representation formula [20, Theorem 1.1 and Remark 1] of the solution to (1.4).

Notice that the obstruction to identifiability manifested in Section 1.3.1 is made explicit in Theorem 1.7 and Corollary 1.8 for source terms expressing as the superposition of two functions of the form (1.8), as we are able in this case to characterize the set of all source terms of this type, which are associated with the same data.

1.7. **Outline.** The proofs of Theorems 1.1, 1.4 and 1.7 rely on suitable analytic properties of the solution to (1.4), that are established in Section 2. The proofs of Theorem 1.1, Corollary 1.2 and Theorem 1.3 can be found in Section 3, while the ones of Theorem 1.4, Corollary 1.5 and Theorem 1.6 are displayed in Section 4. Section 5 contains the proofs of Theorem 1.7 and Corollary 1.8. In Section 6, the UP stated in Theorem 1.1 and the UP for local Cauchy data of Theorem 1.4 are adapted to the framework of distributed order diffusion equations. Finally, numerical reconstruction of the unknown spatial term of sources identified by Corollary 1.2 is carried out in Section 7, by means of an iterative method.

2. Direct problem: representation and time-analyticity of the solution

In this section we establish time-analytic properties of the weak solution u to (1.4). Their derivation is based on an appropriate representation formula of u, that is borrowed from [20].

2.1. **A representation formula.** Assume (1.1), let $\alpha \in (0,2)$, let $\rho \in L^\infty(\Omega)$ fulfill (1.3), and let $q \in L^\kappa(\Omega)$, with $\kappa \in (d, +\infty]$, satisfy (1.2). Then, for all $p \in \mathbb{C} \setminus \mathbb{R}_-$, the operator $A_q + \rho p^\alpha$, where A_q is defined in Section 1.5, is boundedly invertible in $L^2(\Omega)$, by virtue of [20, Proposition 2.1]. Moreover, in view of [20, Eq. (2.4)-(2.5)], the following resolvent estimate

$$\left\|(A_q + \rho p^\alpha)^{-1}\right\|_{\mathcal{B}(L^2(\Omega))} \leqslant C\,|p|^{-\alpha}\,,\ p \in \mathbb{C} \setminus \mathbb{R}_-, \tag{2.1}$$

holds with $C = \rho_0^{-1} \max\left(2, \sin(\alpha \arctan((3\rho_M)^{-1}\rho_0))^{-1}\right)$. Here, $(A_q + \rho p^\alpha)^{-1}$ denotes the resolvent operator of $A_q + \rho p^\alpha$ and $\mathcal{B}(L^2(\Omega))$ is the space of linear bounded operators in $L^2(\Omega)$. These two results were established for $\alpha \in (0,1)$ and for the form domain $V = H_0^1(\Omega)$ in [20], but they extend to $\alpha \in [1,2)$ and $V = H^1(\Omega)$ in a straightforward way.

Next, for all $f \in L^1(0,T; L^2(\Omega))$, the weak solution u to (1.4) reads

$$u(t,\cdot) = \int_0^t S(t-s) f(s,\cdot) ds,\ t \in (0,T), \tag{2.2}$$

where we have set

$$S(t)h := \int_{\gamma(\varepsilon,\theta)} e^{tp}(A_q + \rho p^\alpha)^{-1} h dp,\ t \in (0,+\infty),\ h \in L^2(\Omega). \tag{2.3}$$

Here, ε is arbitrary in $(0,+\infty)$, θ can be any angle in $\left(\frac{\pi}{2}, \min\left(\pi, \frac{\pi}{\alpha}\right)\right)$ and $\gamma(\varepsilon,\theta)$ is the following modified Haenkel contour in $\mathbb{C}$,

$$\gamma(\varepsilon,\theta) := \gamma_-(\varepsilon,\theta) \cup \gamma_0(\varepsilon,\theta) \cup \gamma_+(\varepsilon,\theta), \tag{2.4}$$

where

$$\gamma_\pm(\varepsilon,\theta) := \{se^{\pm i\theta} :\ s \in [\varepsilon,+\infty)\} \text{ and } \gamma_0(\varepsilon,\theta) := \{\varepsilon e^{i\beta} :\ \beta \in [-\theta,\theta]\} \tag{2.5}$$

are traversed in the positive sense. The Duhamel representation formula (2.2)-(2.3) is a direct consequence of [20, Theorem 1.1 and Remark 1] and the density of $L^\infty(0,T;L^2(\Omega)) \cap \mathcal{C}((0,T],L^2(\Omega))$ in $L^1(0,T;L^2(\Omega))$. It is our main tool in the derivation of the time-analytic properties of the weak solution u to the IBVP (1.4).

2.2. **Time-analyticity.** The main result of this section is as follows.

Proposition 2.1. *Assume that $f \in L^1(0,T;L^2(\Omega))$ is supported in $[0,T-3\varepsilon_\star] \times \overline{\Omega}$, for some fixed $\varepsilon_\star \in \left(0, \frac{T}{4}\right)$. Then, there exists $\theta_\star \in \left(0, \min\left(\frac{\pi}{4}, \frac{\pi}{2\alpha} - \frac{\pi}{4}\right)\right)$, such that the weak solution u to (1.4), given by (2.2)-(2.3), extends to a $L^2(\Omega)$-valued map still denoted by u, which is analytic in $\mathcal{C}_{\theta_\star}$, where $\mathcal{C}_{\theta_\star} := \{\tau e^{i\psi} :\ \tau \in (T-\varepsilon_\star,+\infty),\ \psi \in (-\theta_\star,\theta_\star)\}$.*

Moreover, $t \mapsto u(t,\cdot)$ is holomorphic in $\mathcal{C}_{\theta_\star}$, its Laplace transform $p \mapsto U(p) = \mathcal{L}[u](p) : x \mapsto \int_0^{+\infty} e^{-pt}u(t,x)dt$ is well defined for all $p \in (0,+\infty)$, and each $U(p)$ is a solution to the following BVP

$$\begin{cases} (\mathcal{A}_q + \rho p^\alpha)U(p) &= F(p), \quad \text{in } \Omega, \\ B_\star U(p) &= 0, \qquad \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

where $F(p)$ is the Laplace transform defined in (1.7).

Proof. Bearing in mind that the weak solution u to (1.4) is expressed by (2.2)-(2.3) for some fixed $(\varepsilon,\theta) \in (0,1) \times \left(\frac{\pi}{2}, \min\left(\pi, \frac{\pi}{\alpha}\right)\right)$, we pick $\theta_\star \in \left(0, \frac{\theta-\pi/2}{2}\right)$ in such a way that

$$(T-\varepsilon_\star)\cos(\theta-\theta_\star) \leqslant (T-2\varepsilon_\star)\cos\theta. \tag{2.7}$$

Then, for all $p = re^{\pm i\theta} \in \gamma_\pm(\varepsilon,\theta)$, all $z = \tau e^{i\psi} \in \mathcal{C}_{\theta_\star}$ and all $s \in (0,T-3\varepsilon_\star)$, we have

$$\Re((z-s)p) = r\left(\tau\cos(\psi \pm \theta) - s\cos\theta\right) \leqslant r\left((T-\varepsilon_\star)\cos(\theta-\theta_\star) - (T-3\varepsilon_\star)\cos\theta\right),$$

where the symbol $\mathfrak{R}$ denotes the real part. This and (2.7) yield

$$\mathfrak{R}((z-s)p) \leqslant \varepsilon_\star \, |p| \cos\theta, \; p \in \gamma_\pm(\varepsilon,\theta), \; z \in \mathcal{C}_{\theta_\star}, \; s \in (0, T-3\varepsilon_\star). \tag{2.8}$$

Hence, in light of (2.1) and (2.4)-(2.5), we see for every $z \in \mathcal{C}_{\theta_\star}$ that the function

$$v(z,\cdot) := \int_0^{T-3\varepsilon_\star} \int_{\gamma(\varepsilon,\theta)} e^{(z-s)p} (A_q + \rho p^\alpha)^{-1} f(s,\cdot) dp \, ds, \tag{2.9}$$

is well-defined in Ω. Further, since $f(t,\cdot) = 0$ for all $t \in (T-3\varepsilon_\star, T)$, we infer from this and (2.2)-(2.3) that

$$v(t,\cdot) = \int_0^{T-3\varepsilon_\star} S(t-s) f(s,\cdot) ds = \int_0^t S(t-s) f(s,\cdot) ds = u(t,\cdot), \; t \in (T-3\varepsilon_\star, T).$$

Therefore, putting $v := u$ on $(0, T-\varepsilon_\star) \times \Omega$, we obtain that $u = v_{|Q}$. Moreover, by [20, Theorem 1.1 and Remark 1], the Laplace transform $V(p) = \int_0^{+\infty} e^{-pt} v(t,\cdot) dt$ of v, is solution to the BVP (1.6).

It remains to show that $z \mapsto v(z,\cdot)$ is a holomorphic $L^2(\Omega)$-valued function in $\mathcal{C}_{\theta_\star}$. To do that, we refer to (2.4) and (2.9), and we decompose v into the sum $v_0 + v_+ + v_-$, where

$$v_j(z,\cdot) := \int_0^{T-3\varepsilon_\star} \int_{\gamma_j(\varepsilon,\theta)} e^{(z-s)p} (A_q + \rho p^\alpha)^{-1} f(s,\cdot) dp \, ds, \; j = 0, +, -.$$

Since $z \mapsto v_0(z,\cdot)$ is obviously holomorphic in $\mathcal{C}_{\theta_\star}$, we are thus left with the task of proving that this is also the case for $z \mapsto v_\pm(z,\cdot)$. This can be done upon noticing that the $L^2(\Omega)$-valued function

$$z \mapsto e^{(z-s)p} (A_q + \rho p^\alpha)^{-1} f(s,\cdot), \; p \in \gamma_\pm(\varepsilon,\theta), \; s \in (0, T-3\varepsilon_\star),$$

is holomorphic in $\mathcal{C}_{\theta_\star}$, that the two following estimates,

$$\left\| \partial_z^k e^{(z-s)p} (A_q + \rho p^\alpha)^{-1} f(s,\cdot) \right\|_{L^2(\Omega)} \leqslant C(1+|p|) \, |p|^{-\alpha} \, e^{\varepsilon_\star |p| \cos\theta} \, \|f(s,\cdot)\|_{L^2(\Omega)}, \; z \in \mathcal{C}_{\theta_\star}$$

hold for $k = 0, 1$ and some constant C that is independent of p and s, by virtue of (2.1) and (2.8), and that the function $(r,s) \mapsto (1+r) r^{-\alpha} e^{\varepsilon_\star r \cos\theta} \|f(s,\cdot)\|_{L^2(\Omega)}$ belongs to $L^1((\varepsilon, +\infty) \times (0, T-3\varepsilon_\star))$. □

Remark 1. *Since $F(p) \in L^2(\Omega)$ for all $p \in (0, +\infty)$, then, in accordance with Section 2.1, we may reformulate the claim of Proposition 2.1 that $U(p)$ solves* (2.6)*, as:*

$$U(p) = (A_q + \rho p^\alpha)^{-1} F(p). \tag{2.10}$$

Since the multiplication operator by ρ is invertible in $\mathcal{B}(L^2(\Omega))$, according to (1.3)*, then $A_{q,\rho}+p^\alpha$ is boundedly invertible in $L^2_\rho(\Omega)$ for each $p\in(0,+\infty)$, and* (2.10) *may thus be equivalently rewritten as*

$$U(p)=(A_{q,\rho}+p^\alpha)^{-1}\rho^{-1}F(p).$$

Armed with Proposition 2.1, we turn now to proving the main results of this article.

3. Proof of Theorems 1.1 and 1.3, and proof of Corollary 1.2

We start by establishing Theorem 1.1.

3.1. **Proof of Theorem 1.1.** We split the proof into 4 steps. In the first one, we establish a family of resolvent identities for the Laplace transform of the solution to (1.4), indexed by the Laplace variable $p\in(0,+\infty)$. The second step is to express these identities in terms of the spectral decomposition of the operator $A_{q,\rho}$, introduced in Section 1.5. The third step, based on a weak unique continuation principle for second order elliptic equations, provides the desired result, while Step 4 contains the proof of a technical claim, used in Step 3.

3.1.1. *Step 1: A p-indexed family of resolvent identities.* As $\operatorname{supp}\sigma\subset[0,T)$ by assumption, we pick $\varepsilon_*\in(0,T/4)$ such that $\operatorname{supp}\sigma\subset[0,T-3\varepsilon_*]$. Then, with reference to Proposition 2.1, we extend the weak solution to (1.4) into a $L^2(\Omega)$-valued function $z\mapsto u(z,\cdot)$, defined in $(0,T-\varepsilon_*]\cup\mathcal{C}_{\theta_\star}$ for some $\theta_\star\in\big(0,\min\big(\frac{\pi}{4},\frac{\pi}{2}-\frac{\pi}{2\alpha}\big)\big)$, which is holomorphic in $\mathcal{C}_{\theta_\star}$. Evidently, the $L^2(\Omega')$-valued function $z\mapsto u(z,\cdot)_{|\Omega'}$ is holomorphic in $\mathcal{C}_{\theta_\star}$ as well. Bearing in mind that $u_{|Q'}=0$, where we have set $Q':=(0,T)\times\Omega'$, and that $(0,T)\cap\mathcal{C}_{\theta_\star}=[T-\varepsilon_*,T)$, we get that

$$u(z,x)=0,\ (z,x)\in((0,T)\cup\mathcal{C}_{\theta_\star})\times\Omega',$$

from the unique continuation principle for holomorphic functions. In particular, this entails that $u(t,x)=0$ for all $(t,x)\in(0+\infty)\times\Omega'$ and consequently that the Laplace transform $U(p)$ of u with respect to t, vanishes a.e. in Ω' for every $p\in(0,+\infty)$. Putting this together with the second statement of Proposition 2.1, we obtain that each $U(p)$, $p\in(0,+\infty)$, is solution to

$$\left\{\begin{array}{rcll}(\mathcal{A}_q+\rho p^\alpha)U(p) &=& \widehat{\sigma}(p)f, & \text{in }\Omega,\\ B_\star U(p) &=& 0, & \text{on }\partial\Omega,\\ U(p) &=& 0, & \text{in }\Omega',\end{array}\right. \tag{3.1}$$

where we have set $\widehat{\sigma}(p) := \int_0^T e^{-pt}\sigma(t)dt$. Since $f \in L^2(\Omega)$, then, in accordance with Remark 1, (3.1) may be equivalently reformulated , as

$$\begin{cases} U(p) &= \widehat{\sigma}(p)(A_{q,\rho} + p^\alpha)^{-1}\rho^{-1}f \quad \text{in } L^2_\rho(\Omega), \\ U(p) &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{in } L^2_\rho(\Omega'). \end{cases} \tag{3.2}$$

3.1.2. *Step 2: Spectral representation.* Since the injection $V \hookrightarrow L^2(\Omega)$ is compact, the resolvent of the operator $A_{q,\rho}$, defined in Section 1.5, is compact in $L^2_\rho(\Omega)$. Let $\{\lambda_n : n \in \mathbb{N}\}$ be the increasing sequence of the eigenvalues of $A_{q,\rho}$. For each $n \in \mathbb{N}$, we denote by $m_n \in \mathbb{N}$ the algebraic multiplicity of the eigenvalue λ_n and we introduce a family $\{\varphi_{n,k} : \ k = 1, \ldots, m_n\}$ of eigenfunctions of $A_{q,\rho}$, which satisfy

$$A_{q,\rho}\varphi_{n,k} = \lambda_n \varphi_{n,k},$$

and form an orthonormal basis in $L^2_\rho(\Omega)$ of the eigenspace of $A_{q,\rho}$ associated with λ_n (i.e. the kernel of $A_{q,\rho} - \lambda_n I$, where the notation I stands for the identity operator of $L^2_\rho(\Omega)$). The first line in (3.2) then yields for all $p \in (0, +\infty)$, that the following equality

$$U(p) = \widehat{\sigma}(p)\left(\sum_{n=1}^{+\infty} \frac{\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}}{\lambda_n + p^\alpha}\right),$$

holds in $L^2_\rho(\Omega)$ with $g_{n,k} := \langle \rho^{-1}f, \varphi_{n,k}\rangle_{L^2_\rho(\Omega)}$. From this, the second line of (3.2) and the continuity of the projection from $L^2_\rho(\Omega)$ into $L^2_\rho(\Omega')$, it then follows that

$$\widehat{\sigma}(p)\left(\sum_{n=1}^{+\infty} \frac{\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}(x)}{\lambda_n + p^\alpha}\right) = 0, \ x \in \Omega', \ p \in (0, +\infty). \tag{3.3}$$

3.1.3. *Step 3: End of the proof.* Since $p \mapsto \widehat{\sigma}(p)$ is holomorphic in $\mathbb{C}_+ := \{z \in \mathbb{C} : \ \Re z > 0\}$, then either of the two following conditions is true:

(a) For all $p \in \mathbb{C}_+$ we have $\widehat{\sigma}(p) = 0$;

(b) There exists an open interval $I \subset (0, +\infty)$, such that $\widehat{\sigma}(p) \neq 0$ for each $p \in I$.

The first case is easily treated as we get that $\sigma = 0$ a.e. in $(0, T)$ from (a) and the injectivity of the Laplace transform, which entails the desired result. In the second case, we combine (b) with (3.3) and obtain that

$$\sum_{n=1}^{+\infty} \frac{\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}(x)}{\lambda_n + p^\alpha} = 0, \ x \in \Omega', \ p \in I. \tag{3.4}$$

Let us introduce the following $L^2_\rho(\Omega')$-valued function,

$$R(z) := \sum_{n=1}^{+\infty} \frac{\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}}{\lambda_n + z}, \ z \in \mathbb{C} \setminus \{-\lambda_n : \ n \in \mathbb{N}\}, \tag{3.5}$$

meromorphic in $\mathbb{C} \setminus \{-\lambda_n : \ n \in \mathbb{N}\}$ with simple poles $\{-\lambda_n : \ n \in \mathbb{N}\}$. Evidently, (3.4) can be equivalently rewritten as

$$R(p^\alpha) = \sum_{n=1}^{+\infty} \frac{\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}}{\lambda_n + p^\alpha} = 0, \ p \in I,$$

the above identity being understood in $L^2_\rho(\Omega')$. Therefore, we necessarily have $R(z) = 0$ for all $z \in \mathbb{C} \setminus \{-\lambda_n : \ n \in \mathbb{N}\}$, and consequently it holds true for all $n \in \mathbb{N}$ that

$$\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}(x) = 0, \ x \in \Omega'. \tag{3.6}$$

Assume for a while that for each $n \in \mathbb{N}$, the eigenfunctions $\varphi_{n,k}$, $k = 1, \ldots, m_n$, are linearly independent in $L^2_\rho(\Omega')$, the proof of this claim being postponed to Section 3.1.4, below. Then, we infer from (3.6) that $g_{n,k} = 0$ for all $n \in \mathbb{N}$ and all $k = 1, \ldots, m_n$. Therefore, we find that

$$g = \sum_{n=1}^{+\infty} \sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k} = 0$$

in $L^2_\rho(\Omega)$, which proves the desired result.

3.1.4. *Step 4: The $\varphi_{n,k}$, $k = 1, \ldots, m_n$, are linearly independent in $L^2_\rho(\Omega')$.* For $n \in \mathbb{N}$ fixed, we consider m_n complex numbers α_k, for $k = 1, \ldots, m_n$, such that

$$\sum_{k=1}^{m_n} \alpha_k\varphi_{n,k}(x) = 0, \ x \in \Omega', \tag{3.7}$$

and we put $\varphi := \sum_{k=1}^{m_n} \alpha_k\varphi_{n,k}$. Since each $\varphi_{n,k}$ lies in $D(A_{q,\rho})$, the domain of the operator $A_{q,\rho}$, then the same is true for φ, i.e.

$$\varphi \in D(A_{q,\rho}) = D(A_q), \tag{3.8}$$

according to (1.17), and we have $A_{q,\rho}\varphi = \lambda_n\varphi$ in $L^2_\rho(\Omega)$. This and (3.7) translate into the fact that

$$\begin{cases} (\mathcal{A}_q - \lambda_n\rho)\varphi &= 0, \quad \text{in } \Omega, \\ \varphi &= 0, \quad \text{in } \Omega'. \end{cases}$$

Moreover, as we have $\varphi \in H^2(\Omega)$ from (1.17)-(1.18) and (3.8), the weak unique continuation principle for second order elliptic partial differential equations (see e.g. [35, Theorem 1]) then yields that $\varphi = 0$ a.e. in Ω, i.e.

$$\varphi(x) = \sum_{k=1}^{m_n} \alpha_k \varphi_{n,k}(x) = 0, \ x \in \Omega.$$

Bearing in mind that $\{\varphi_{n,k} : \ k = 1, \ldots, m_n\}$ is orthonormal in $L^2_\rho(\Omega)$, we deduce from the above line that $\alpha_k = 0$ for all $k = 1, \ldots, m_n$, which establishes that the $\varphi_{n,k}$, $k = 1, \ldots, m_n$, are linearly independent in $L^2_\rho(\Omega')$.

Having completed the proof of Theorem 1.1, we turn now to showing Corollary 1.2.

3.2. **Proof of Corollary 1.2 and Theorem 1.3.** In light of (1.4), $u := u_1 - u_2$ is a weak solution is solution to

$$\begin{cases} (\rho(x)\partial_t^\alpha + \mathcal{A}_q)u(t,x) &= f(t,x), \quad (t,x) \in Q, \\ B_\star u(t,x) &= 0, \qquad (t,x) \in \Sigma, \\ \partial_t^k u(0,\cdot) &= 0, \qquad \text{in } \Omega, \ k = 0, \ldots, N_\alpha, \end{cases} \tag{3.9}$$

with $f(t,x) = \sigma_1(t)g_1(x) - \sigma_2(t)g_2(x)$ for a.e. $(t,x) \in Q$.

3.2.1. *Proof of Corollary 1.2.* In the first (resp., second) case (i) (resp., (ii)), we have $f(t,x) = \sigma_1(t)(g_1 - g_2)(x)$ where $\sigma_1 \in L^1(0,T)$ is supported in $[0,T)$ and $g_1 - g_2 \in L^2(\Omega)$ (resp., $f(t,x) = (\sigma_1 - \sigma_2)(t)g_1(x)$ where $\sigma_1 - \sigma_2 \in L^1(0,T)$ is supported in $[0,T)$ and $g_1 \in L^2(\Omega)$). Since $u = 0$ in Q', then, under Condition (i), an application of Theorem 1.1 yields $\sigma_1(t)(g_1 - g_2)(x) = 0$ for a.e. $(t,x) \in Q$ and hence $g_1 = g_2$ in Ω. Similarly, under Condition (ii), we obtain that $(\sigma_1 - \sigma_2)(t)g_1(x) = 0$ for a.e. $(t,x) \in Q$ and consequently that $\sigma_1 = \sigma_2$ in $(0,T)$. The proof of Corollary 1.5 is thus complete and we turn now to proving Theorem 1.3.

3.2.2. *Proof of Theorem 1.3.* With reference to (3.9), we consider the following IBVP

$$\begin{cases} (\rho(x)\partial_t^\alpha + \mathcal{A}_q)w(t,x) &= \sigma(t)(g_1(x) - g_2(x)), \quad (t,x) \in (0,+\infty) \times \Omega, \\ B_\star w(t,x) &= 0, \qquad (t,x) \in (0,+\infty) \times \partial\Omega, \\ \partial_t^k w(0,x) &= 0, \qquad x \in \Omega, \ k = 0, \ldots, N_\alpha, \end{cases} \tag{3.10}$$

for $\alpha \in (0,2)$. With reference to Section 2.1, (3.10) admits a unique solution $w \in \mathcal{C}([0,+\infty), L^2(\Omega))$, which is expressed by (2.2)-(2.3). Moreover, due to (1.10), we get upon arguing as in the derivation of [26, Theorem 1.4] that the $L^2(\Omega)$-valued function $t \mapsto w(t,\cdot)$ is analytic in $(0,+\infty)$.

On the other hand, from the uniqueness of the solution to (3.9) with $T = t_0$ and $f(t,x) = \sigma(t)(g_1(x) - g_2(x))$, we get that $w(t,x) = u(t,x)$ for a.e. $(t,x) \in (0,t_0) \times \Omega$. Since $u = 0$ in Q', by assumption, the analyticity of $t \mapsto w(t,\cdot)$ in $(0,+\infty)$ then yields

$$w(t,x) = 0, \ (t,x) \in (0,+\infty) \times \Omega'. \tag{3.11}$$

Thus, taking the Laplace transform with respect to $t \in (0,+\infty)$ in (3.10) and in (3.11), we obtain in the same way as in the derivation of (3.2) in Section 3.1.1, that for every $p \in (0,+\infty)$,

$$\begin{cases} W(p) &= \widehat{\sigma}(p)(A_{q,\rho} + p^\alpha)^{-1}\rho^{-1}(g_1 - g_2) \quad \text{in } L^2_\rho(\Omega), \\ W(p) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{in } L^2_\rho(\Omega'), \end{cases}$$

where $W(p) := \int_0^{+\infty} e^{-tp} w(t,\cdot)dt$ and $\widehat{\sigma}(p) := \int_0^{+\infty} e^{-tp}\sigma(t,\cdot)dt$ are the Laplace transforms of w and σ, respectively. Notice from (1.10) that $\widehat{\sigma}(p)$ is well-defined for each $p \in (0,+\infty)$.

Now, arguing in the exact same way as in the three last steps of the derivation of Theorem 1.1, we obtain that $g_1 = g_2$ in Ω. Therefore, condition (ii) of Corollary 1.2 is fulfilled and, by invoking Corollary 1.2, we deduce that (1.9) holds true.

4. Proof of Theorems 1.4 and 1.6, and proof of Corollary 1.5

4.1. **Proof of Theorem 1.4.** We split the proof into two main parts. The first one is to prove existence of a $\mathcal{C}([0,T]; H^{2\zeta}(\Omega))$-solution to the IBVP (1.4) with $\rho = 1$ a.e. in Ω and $\mathcal{A}_q = -\Delta$.

4.1.1. *Improved space-regularity result.* We start by establishing that the weak-solution to (1.4) associated with $\rho = 1$, $q = 0$ and source term $f \in L^r(0,T; L^2(\Omega))$, lies in $\mathcal{C}([0,T]; H^{2\zeta}(\Omega))$.

As a preamble, we set $A := A_0$, where we recall that A_0 is the self-adjoint realization of the (opposite of) the Laplace operator in $L^2(\Omega)$, endowed with either Dirichlet or Neumann boundary condition. Otherwise stated, A is the self-adjoint operator in $L^2(\Omega)$, acting as $-\Delta$ on its domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ when the boundary operator $B_\star$ appearing in (1.4) reads $B_\star u = u$, while it is $D(A) = H^2(\Omega)$ when $B_\star u = \partial_{\nu_a} u$. We denote by $(\lambda_n)_{n\in\mathbb{N}}$ the sequence of eigenvalues of A, arranged in non-decreasing order and repeated with the multiplicity, and we introduce an orthonormal basis $(\varphi_n)_{n\ \in\mathbb{N}}$ in $L^2(\Omega)$ of eigenfunctions of A, obeying $A\varphi_n = \lambda_n \varphi_n$ for all $n \in \mathbb{N}$.

Since the operator A is nonnegative, we recall from the functional calculus, that

$$(A+1)^s h = \sum_{n=1}^{+\infty} (1+\lambda_n)^s \langle h, \varphi_n \rangle_{L^2(\Omega)} \varphi_n, \ h \in D((1+A)^s), \ s \in [0,+\infty),$$

where $D((1+A)^s) = \left\{ h \in L^2(\Omega) : \ \sum_{n=1}^{+\infty} \left| \langle h, \varphi_n \rangle_{L^2(\Omega)} \right|^2 (1+\lambda_n)^{2s} < \infty \right\}$. For further reference, we set

$$\|h\|_{D((A+1)^s)} := \left(\sum_{n=1}^{+\infty} (1+\lambda_n)^{2s} \left| \langle h, \varphi_n \rangle_{L^2(\Omega)} \right|^2 \right)^{\frac{1}{2}}, \ h \in D((A+1)^s).$$

As $f \in L^r(0,T;L^2(\Omega))$ with $r > \frac{1}{\alpha}$, then the weak solution u to (1.4) reads

$$u(t,\cdot) = \sum_{n=1}^{\infty} u_n(t)\varphi_n, \ t \in (0,T), \tag{4.1}$$

where $u_n(t) := \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) \langle f(s,\cdot), \varphi_n \rangle_{L^2(\Omega)} ds$ and $E_{\alpha,\beta}$ is the Mittag-Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ z \in \mathbb{C}, \ \alpha \in (0,+\infty), \beta \in \mathbb{R}.$$

We refer the reader to [34, Theorem 2.4], [21, Theorem 1.1] or [7, Lemma 3.3] for the derivation of the representation formula (4.1) of u. Next, we recall from [32, Theorem 1.6] that

$$|E_{\alpha,\alpha}(-\lambda_n t^\alpha)| \leqslant C \frac{t^{-\alpha\zeta}+1}{(1+\lambda_n)^\zeta}, \ t \in (0,T), \ n \in \mathbb{N},$$

for some positive constant C which is independent of n and t. Thus, for all $n \in \mathbb{N}$ we have

$$\left| t^{\alpha-1} (1+\lambda_n)^\zeta E_{\alpha,\alpha}(-\lambda_n t^\alpha) \right| \leqslant C t^{\alpha(1-\zeta)-1}, \ t \in (0,T),$$

and consequently $t \mapsto t^{\alpha-1}(1+\lambda_n)^\zeta E_{\alpha,\alpha}(-\lambda_n t^\alpha) \in L^{r'}(0,T)$, where r' is the real number conjugated to r, i.e. r' is such that $\frac{1}{r'} = 1 - \frac{1}{r} > 1 - \alpha(1-\zeta)$. Therefore, using that $s \mapsto \langle f(s,\cdot), \varphi_n \rangle_{L^2(\Omega)} \in L^r(0,T)$, we obtain that $t \mapsto (1+\lambda_n)^\zeta u_n(t) \in \mathcal{C}([0,T])$, and the following estimate

$$\begin{aligned}
\left\| \sum_{k=n}^{m} u_k(t)\varphi_k \right\|_{D((1+A)^\zeta)} &\leqslant \int_0^t (t-s)^{\alpha-1} \left(\sum_{k=n}^{m} (1+\lambda_n)^{2\zeta} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha)^2 \left| \langle f(s,\cdot), \varphi_n \rangle_{L^2(\Omega)} \right|^2 \right)^{\frac{1}{2}} ds \\
&\leqslant C \int_0^t (t-s)^{\alpha(1-\zeta)-1} \left(\sum_{k=n}^{m} \left| \langle f(s,\cdot), \varphi_n \rangle_{L^2(\Omega)} \right|^2 \right)^{\frac{1}{2}} ds \\
&\leqslant C \left(\int_0^T s^{(\alpha(1-\zeta)-1)r'} ds \right)^{\frac{1}{r'}} \left(\int_0^T \left(\sum_{k=n}^{m} \left| \langle f(s,\cdot), \varphi_n \rangle_{L^2(\Omega)} \right|^2 \right)^{\frac{r}{2}} ds \right)^{\frac{1}{r}}, \quad (4.2)
\end{aligned}$$

which is true for all $t \in [0,T]$ and for all natural numbers m and n with $n \leqslant m$.

On the other hand, since $\lim_{n,m\to+\infty} \left(\int_0^T \left(\sum_{k=n}^{m} \left| \langle f(s,\cdot), \varphi_n \rangle_{L^2(\Omega)} \right|^2 \right)^{\frac{r}{2}} ds \right)^{\frac{1}{r}} = 0$ as we have $f \in L^r(0,T;L^2(\Omega))$ by assumption, we derive from (4.2) that $\left(\sum_{k=1}^{n} u_k \varphi_k \right)_{n\in\mathbb{N}}$ is a Cauchy

sequence in $\mathcal{C}([0,T];D((A+1)^\zeta))$. Therefore, we have $u \in \mathcal{C}([0,T],D((A+1)^\zeta)))$ by (4.1) and consequently $u \in \mathcal{C}([0,T],H^{2\zeta}(\Omega))$ from the embedding $D((A+1)^\zeta)) \subset H^{2\zeta}(\Omega)$.

Having established the first claim of Theorem 1.4, we turn now to proving (1.13).

4.1.2. *Proof of the UP.* Put $\Omega_0 := \omega \times (-L,L)$ and pick an open subset $\Omega_\star \subset \mathbb{R}^d$ with $\mathcal{C}^2$ boundary, fulfilling all the following conditions simultaneously:

$$\text{(a) } \Omega \subset \Omega_\star, \ \text{(b) } \partial\Omega \setminus \partial\Omega_\star \subset \gamma, \ \text{(c) } \Omega' := \Omega_\star \setminus \overline{\Omega} \text{ is not empty}, \ \text{(d) } \Omega_\star \setminus \overline{\Omega_0} \text{ is connected.} \tag{4.3}$$

Notice that such a subset $\Omega_\star$ exists in $\mathbb{R}^d$ as $\Omega \setminus \overline{\Omega_0}$ is connected and $\partial\Omega$ is $\mathcal{C}^2$. We split the proof into two steps.

Step 1: Elliptic BVPs indexed by p. Setting $f(t,x) := 0$ and $u(t,x) := 0$ for a.e. $(t,x) \in Q'$, we infer from (4.3)(b) and the assumption $u_{|(0,T)\times\gamma} = \partial_\nu u_{|(0,T)\times\gamma} = 0$, that

$$\left\{ \begin{array}{rcll} (\partial_t^\alpha - \Delta)u(t,x) & = & f(t,x), & (t,x) \in (0,T) \times \Omega_\star, \\ u & = & 0, & (t,x) \in (0,T) \times \partial\Omega_\star, \\ \partial_t^k u(0,\cdot) & = & 0, & \text{in } \Omega_\star, \ k = 0,\ldots,N_\alpha. \end{array} \right. \tag{4.4}$$

We have $r > 2$ as $\alpha(1-\zeta) < \frac{1}{2}$, whence $f \in L^1(0,T;L^2(\Omega_\star))$. Moreover, f being supported in $[0,T) \times \overline{\Omega_\star}$, hence in $[0,T-3\varepsilon_\star] \times \overline{\Omega_\star}$ for some fixed $\varepsilon_\star \in \left(0,\frac{T}{3}\right)$, we extend $t \mapsto u(t,\cdot)$ to a $L^2(\Omega_\star)$-valued function in $(0,+\infty)$ which is analytic in $(T-\varepsilon_\star,+\infty)$, by invoking Proposition 2.1 where Ω is replaced by $\Omega_\star$. Bearing in mind that u vanishes in Q', by assumption, we find that

$$u(t,x) = 0, \ (t,x) \in (0,+\infty) \times \Omega'. \tag{4.5}$$

Moreover, in light of Proposition 2.1, we get for all $p \in (0,+\infty)$ that the Laplace transform $U(p) = \int_0^{+\infty} e^{-pt}u(t)dt$ of u, is solution to the following BVP

$$\left\{ \begin{array}{rcll} (-\Delta + p^\alpha)U(p) & = & F(p), & \text{in } \Omega_\star, \\ U(p) & = & 0, & \text{on } \partial\Omega_\star, \end{array} \right. \tag{4.6}$$

where $\nu_\star$ is the outward unit normal vector to $\partial\Omega_\star$ and F is defined by (1.7). Since $F(p) \in L^2(\Omega_\star)$ for each $p \in (0,+\infty)$ and since $\partial\Omega_\star$ is $\mathcal{C}^2$, then $U(p) \in H^2(\Omega_\star)$ by elliptic regularity.

Next, as f is supported in $[0,T] \times \overline{\Omega_0}$, we have $F(p) = 0$ in $\Omega_\star \setminus \overline{\Omega_0}$ for all $p \in (0,+\infty)$, and consequently

$$\left\{ \begin{array}{rcll} (-\Delta + p^\alpha)U(p) & = & 0, & \text{in } \Omega_\star \setminus \overline{\Omega_0}, \\ U(p) & = & 0, & \text{on } \Omega', \end{array} \right. \tag{4.7}$$

by (4.5)-(4.6). Since $\Omega_\star \setminus \overline{\Omega_0}$ is connected and $\Omega' \subset \Omega_\star \setminus \overline{\Omega_0}$, and since $U(p) \in H^2(\Omega_\star \setminus \overline{\Omega_0})$, then the weak unique continuation principle for elliptic equations to (4.7) yields that $U(p) = 0$ in $\Omega_\star \setminus \overline{\Omega_0}$. Thus, taking into account that $\overline{\Omega_0} \subset \Omega_\star$ and that $U(p) \in H^2(\Omega_\star)$, we have

$$U(p) = \partial_{\nu_0} U(p) = 0 \text{ in } \partial\Omega_0,$$

where ν_0 denotes the outward unit normal vector to Ω_0. From this and the first line of (4.6), it then follows that

$$\begin{cases} (-\Delta + p^\alpha)U(p,x) &= G(p,x')h(x_n), & x = (x', x_n) \in \Omega_0, \\ U(p) = \partial_{\nu_0} U(p) &= 0, & \text{on } \partial\Omega_0, \end{cases} \tag{4.8}$$

where $G(p) := \int_0^{+\infty} e^{-pt} g(t)dt$.

Step 2: Fourier transform. For all $(k,\theta) \in \mathbb{R} \times \mathbb{S}^{d-2}$, where $\mathbb{S}^{d-2}$ is the unit sphere of $\mathbb{R}^{d-1}$, we notice that

$$(-\Delta + p^\alpha)e^{-ik\theta\cdot x'} e^{\omega(p,k)x_n} = (k^2 - \omega(p,k)^2 + p^\alpha)e^{-ik\theta\cdot x'} e^{\omega(p,k)x_n} = 0,\ p \in (0,+\infty),$$

where $\omega(p,k) := \left(p^\alpha + k^2\right)^{\frac{1}{2}}$. This and (4.8) yield

$$\int_{\Omega_0} G(p,x')h(x_n)e^{-ik\theta\cdot x'} e^{\omega(p,k)x_n} dx'dx_n = \int_{\Omega_0} (-\Delta + p^\alpha)U(p,x)e^{-ik\theta\cdot x'} e^{\omega(p,k)x_n} dx'dx_n = 0,$$

upon integrating by parts, and hence we get that $\left(\int_\omega G(p,x')e^{-ik\theta\cdot x'}dx'\right)\left(\int_{-L}^{L} h(x_n)e^{\omega(p,k)x_n}dx_n\right) = 0$ from Fubini's theorem. Putting $G(p,\cdot) = 0$ in $\mathbb{R}^{d-1} \setminus \omega$ and $h = 0$ in $\mathbb{R} \setminus (-L,L)$, we thus find that

$$\left(\int_{\mathbb{R}^{n-1}} G(p,x')e^{-ik\theta\cdot x'}dx'\right)\left(\int_{\mathbb{R}} h(x_n)e^{\omega(p,k)x_n}dx_n\right) = 0,\ \theta \in \mathbb{S}^{d-2},\ k \in \mathbb{R}. \tag{4.9}$$

Next, $h \in L^1(\mathbb{R})$ being compactly supported and not identically zero in $\mathbb{R}$, its Fourier transform $z \mapsto \int_{\mathbb{R}} h(x_n)e^{zx_n}dx_n$ is holomorphic and not identically zero in $\mathbb{C}$. Therefore, there exists a non empty interval $(a,b) \subset (0,+\infty)$, with $a < b$, such that we have

$$\int_{\mathbb{R}} h(x_n)e^{\omega(p,k)x_n}dx_n \neq 0,\ k \in (a,b).$$

This and (4.9) yield $\int_{\mathbb{R}^{d-1}} G(p,x')e^{-ik\theta\cdot x'}dx' = 0$ for all $\theta \in \mathbb{S}^{d-2}$ and $k \in (a,b)$. Otherwise stated, the partial Fourier transform of $x' \mapsto G(p,x')$ vanishes in the concentric ring $C_{a,b} := \{y \in \mathbb{R}^{d-1} :\ a < |y| < b\}$, where $|y|$ denotes the Euclidian norm of $y \in \mathbb{R}^{d-1}$, i.e.

$$\int_{\mathbb{R}^{d-1}} G(p,x')e^{-i\xi\cdot x'}dx' = 0,\ \xi \in C_{a,b}. \tag{4.10}$$

Next, since $x' \mapsto G(p,x')$ is supported in the compact subset $\overline{\omega}$, then the function $\xi \mapsto \int_{\mathbb{R}^{d-1}} G(p,x')e^{-i\xi\cdot x'}dx'$ is real-analytic in $\mathbb{R}^{d-1}$, so we infer from (4.10) that $\int_{\mathbb{R}^{d-1}} G(p,x')e^{-i\xi\cdot x'}dx' =$

0 for all $\xi \in \mathbb{R}^{d-1}$. Therefore, we have $G(p,\cdot) = 0$ in $\mathbb{R}^{d-1}$, by the injectivity of the partial Fourier transform with respect to x', and since this equality holds for all $p \in (0,+\infty)$, we obtain that $g = 0$ in $(0,T)\times\omega$, from the injectivity of the Laplace transform with respect to t. This completes the proof of Theorem 1.4.

4.1.3. *Proof of Corollary 1.4 and Theorem 1.6.* The derivation of Corollary 1.5 from Theorem 1.4 follows the same path as the one of Corollary 1.2 from Theorem 1.1. Analogously, the proof of Theorem 1.6 is easily adapted from the one of Theorem 1.3.

5. Proof of Theorem 1.7 and Corollary 1.8

In this section we display the proof of the UP of Theorem 1.7 and its byproduct stated in Corollary 1.8. We start with Theorem 1.7.

5.1. **Proof of Theorem 1.7.** We argue as in the derivation of (3.2) and get for every $p \in (0,+\infty)$ that the Laplace transform $U(p)$ of the solution u to to (1.4) with $\alpha = 1$ and source term f, given by (1.16), solves

$$\begin{cases} U(p) &= \widehat{\sigma}(p)(A_{q,\rho}+p^\alpha)^{-1}\rho^{-1}f + \widehat{\beta}(p)(A_{q,\rho}+p^\alpha)^{-1}\rho^{-1}g \quad \text{in } L^2_\rho(\Omega), \\ U(p) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } L^2_\rho(\Omega'). \end{cases} \tag{5.1}$$

Here we stick with the notations of Section 3.1.1 and denote by $\widehat{\sigma}$ the Laplace transform of σ, i.e. $\widehat{\sigma}(p) := \int_0^T e^{-pt}\sigma(t)dt$, and we set $\widehat{\beta}(p) := \int_0^T e^{-pt}\beta(t)dt$. From the spectral representation of the operator $A_{q,\rho}$, introduced in Section 3.1.2, we infer from (5.1) that the identity

$$\widehat{\sigma}(p)\sum_{n=1}^{+\infty}\frac{\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}}{\lambda_n+p} + \widehat{\beta}(p)\sum_{n=1}^{+\infty}\frac{\sum_{k=1}^{m_n} h_{n,k}\varphi_{n,k}}{\lambda_n+p} = 0, \tag{5.2}$$

holds in $L^2_\rho(\Omega')$ for every $p \in (0,+\infty)$, with $g_{n,k} := \langle \rho^{-1}g, \varphi_{n,k}\rangle_{L^2_\rho(\Omega)}$ and $h_{n,k} := \langle \rho^{-1}h, \varphi_{n,k}\rangle_{L^2_\rho(\Omega)}$. Moreover, as $p \mapsto \widehat{\sigma}(p)\sum_{n=1}^{+\infty}\frac{\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}}{\lambda_n+p}$ and $p \mapsto \widehat{\beta}(p)\sum_{n=1}^{+\infty}\frac{\sum_{k=1}^{m_n} h_{n,k}\varphi_{n,k}}{\lambda_n+p}$ can be meromorphically continued to $\mathbb{C}\setminus\{-\lambda_n :\ n\in\mathbb{N}\}$, then the same is true for the left-hand-side of (5.2). Therefore, for each $N \in \mathbb{N}$ fixed, we get upon multiplying (5.2) by $\lambda_N + p$ and sending p to $-\lambda_N$, that

$$\sum_{k=1}^{m_N}\left(\widehat{\sigma}(-\lambda_N)g_{N,k} + \widehat{\beta}(-\lambda_N)h_{N,k}\right)\varphi_{N,k} = 0,$$

in $L^2_\rho(\Omega')$. Since the function β is not identically zero and does not change sign in $(0,T)$, we have $\widehat{\beta}(-\lambda_N) \neq 0$, so the above line can be reformulated as

$$\sum_{k=1}^{m_N}\left(h_{N,k} + \frac{\widehat{\sigma}(-\lambda_N)}{\widehat{\beta}(-\lambda_N)}g_{N,k}\right)\varphi_{N,k} = 0,$$

the equality being understood in the $L^2_\rho(\Omega')$-sense. Next, since the family $\{\varphi_{N,k},\ k=1,\dots,m_N\}$ is linearly independent in $L^2_\rho(\Omega')$, by virtue of Section 3.1.4, we necessarily

$$\frac{\widehat{\sigma}(-\lambda_N)}{\widehat{\beta}(-\lambda_N)} g_{N,k} = -h_{N,k},\ k=1,\dots,m_N. \tag{5.3}$$

Now, as (5.3) is valid for all $N\in\mathbb{N}$, it follows from the Parseval identity $\sum_{n=1}^{+\infty}\sum_{k=1}^{m_n}|h_{n,k}|^2 = \left\|\rho^{-1}h\right\|^2_{L^2_\rho(\Omega)}$ for $h\in L^2(\Omega)$, that

$$\left|\frac{\widehat{\sigma}(-\lambda_n)}{\widehat{\beta}(-\lambda_n)}\right|^2 |g_{n,k}|^2 < \infty.$$

Therefore, g lies in the domain of the operator of $H_{\rho,q}$ and fulfills (1.20), by the operatorial calculus.

5.2. **Proof of Corollary 1.8.** Since $u := u_g - u_h$ is a solution to the IBVP (1.4) associated with $\alpha=1$ and source term f of the form (1.16), where $\sigma = \frac{d^\ell\beta}{dt^\ell}$, then we have $h = H_{q,1}g$ by Theorem 1.7. This entails that

$$-\widehat{\beta}(-\lambda_n)^{-1}\widehat{\frac{d^\ell\beta}{dt^\ell}}(-\lambda_n)\langle g,\varphi_{n,k}\rangle_{L^2(\Omega)} = \langle h,\varphi_{n,k}\rangle_{L^2(\Omega)},\ n\in\mathbb{N},\ k=1,\dots,m_n. \tag{5.4}$$

Moreover, we have $\widehat{\frac{d^\ell\beta}{dt^\ell}}(p) = p^\ell\widehat{\beta}(p)$ for each $p\in\mathbb{R}$, as $\beta\in H^\ell_0(0,T)$, so (5.4) becomes

$$(-1)^{\ell+1}\lambda_n^\ell\langle g,\varphi_{n,k}\rangle_{L^2(\Omega)} = \langle h,\varphi_{n,k}\rangle_{L^2(\Omega)},\ n\in\mathbb{N},\ k=1,\dots,m_n.$$

This entails that $g\in D(A_q^\ell)$ verifies (1.21).

In the particular case where $\ell=1$, we have $h=-A_q g$ whence u is a solution to the IBVP (1.4) with $\alpha=1$ and $f(t,x) = (\partial_t - \mathcal{A}_q)\beta(t)g(x)$ for a.e. $(t,x)\in Q$. As $(t,x)\mapsto\beta(t)g(x)$ is a weak-solution to the exact same problem, we have $u(t,x)=\beta(t)g(x)$ in Q, by uniqueness of the solution to (1.4), and (1.22) follows directly from this.

6. UP for distributed order diffusion equations

In this section we consider the IBVP

$$\begin{cases} (\rho(x)\mathbb{D}_t^{(\mu)} + \mathcal{A}_q)u(t,x) &= f(t,x), \quad (t,x)\in Q,\\ B_\star u(t,x) &= 0, \qquad (t,x)\in\Sigma,\\ v(0,x) &= 0, \qquad x\in\Omega. \end{cases} \tag{6.1}$$

where $\mathbb{D}_t^{(\mu)}$ denotes the distributed order fractional derivative

$$\mathbb{D}_t^{(\mu)}h(t) := \int_0^1 \mu(\alpha)\partial_t^\alpha h(t)d\alpha,$$

induced by a non-negative weight function $\mu \in L^\infty(0,1)$, obeying the following condition:

$$\exists \alpha_0 \in (0,1),\ \exists \delta \in (0,\alpha_0),\ \forall \alpha \in (\alpha_0-\delta,\alpha_0),\ \mu(\alpha) \geq \frac{\mu(\alpha_0)}{2} > 0. \tag{6.2}$$

Here and in the remaining part of this section, ρ and $\mathcal{A}_q$ are the same as in Section 1.1 and ∂_t^α is the Caputo derivative of order α defined by (1.5).

6.1. **The direct problem.** Let $f \in L^1(0,T;L^2(\Omega))$. We stick with the definition [26, Definition 1.1] of a weak solution to (6.1), that is to say that u is a weak solution to (6.1) if we have $u = v_{|Q}$ for some $v \in \mathcal{S}'(\mathbb{R}_+, L^2(\Omega))$ whose Laplace transform V verifies the following BVP for all $p \in (0,+\infty)$,

$$\left\{ \begin{array}{rcl} (\mathcal{A}_q + \rho p \vartheta(p))V(p) & = & F(p) \text{ in } \Omega, \\ B_\star V(p) & = & 0 \text{ on } \partial\Omega, \end{array} \right. \tag{6.3}$$

where $\vartheta(p) := \int_0^1 p^{\alpha-1}\mu(\alpha)d\alpha$ and F is given by (1.7).

We recall from [26, Theorems 1.1 and 1.2] that under the more restrictive assumption $f \in L^\infty(0,T;L^2(\Omega))$, the IBVP (6.1) admits a unique solution $u \in \mathcal{C}([0,T],L^2(\Omega)) \cap L^1(0,T;H^{2\zeta}(\Omega))$ for every $\zeta \in (0,1)$. Moreover, by [26, Proposition 2.1], u enjoys the following representation formula

$$u(t,\cdot) = \int_0^t S_\mu(t-s)f(s,\cdot)ds,\ t \in (0,T), \tag{6.4}$$

where

$$S_\mu(t)\psi := \frac{1}{2i\pi}\sum_{n=1}^{+\infty}\sum_{k=1}^{m_n}\left(\int_{\gamma(\varepsilon,\theta)} \frac{e^{pt}}{\vartheta(p)+\lambda_n}dp\right)\langle \rho^{-1}\psi, \varphi_{n,k}\rangle_{L^2_\rho(\Omega)}\varphi_{n,k},\ \psi \in L^2(\Omega). \tag{6.5}$$

In (6.5), the pair (ε,θ) is arbitrary in $(0,+\infty)\times\left(\frac{\pi}{2},\pi\right)$, the contour $\gamma(\varepsilon,\theta)$ is given by (2.4)-(2.5), and the λ_n, m_n and $\varphi_{n,k}$ are the same as in Section 3.1.2.

Let us now extend (6.4)-(6.5) to the case of source terms $f \in L^1(0,T;L^2(\Omega))$.

Proposition 6.1. *Assume* (6.2) *and let* $f \in L^1(0,T;L^2(\Omega))$. *Then, for every* $\zeta \in (0,1)$, *there exists a unique weak solution* $u \in \mathcal{C}([0,T],L^2(\Omega)) \cap L^1(0,T;H^{2\zeta}(\Omega))$ *to* (6.1), *which is expressed by* (6.4)-(6.5).

Proof. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{C}_0^\infty(0,T;L^2(\Omega))^{\mathbb{N}}$ be an approximating sequence of f in $L^1(0,T;L^2(\Omega))$, i.e. such that

$$\lim_{n\to\infty}\|f_n - f\|_{L^1(0,T;L^2(\Omega))} = 0. \tag{6.6}$$

Next, with reference to (6.4), we introduce for all $n \in \mathbb{N}$

$$v_n(t,\cdot) := \int_0^t S_\mu(t-s)\mathbb{1}_{(0,T)}(s)f_n(s,\cdot)ds,\ t \in [0,+\infty),$$

in $\mathcal{S}'(\mathbb{R}_+, L^2(\Omega))$, where S_μ is given by (6.5) and $\mathbb{1}_{(0,T)}$ denotes the characteristic function of the interval $(0,T)$.

As $f_n \in L^\infty(0,T;L^2(\Omega))$ for all $n \in \mathbb{N}$, the Laplace transform V_n of v_n, verifies

$$(A_{q,\rho} + p\vartheta(p))V_n(p) = F_n(p),\ p \in (0,+\infty), \tag{6.7}$$

according to [26, Proposition 2.1], where F_n is the expression obtained by substituting f_n for f in the right hand side of (1.7). Moreover, we have

$$\limsup_{n\to\infty} \|F_n(p) - F(p)\|_{L^2(\Omega)} \leqslant \limsup_{n\to\infty} \|f_n - f\|_{L^1(0,T;L^2(\Omega))} = 0,\ p \in (0,+\infty), \tag{6.8}$$

from (6.6).

The next step of the proof is to establish for all $p \in (0,+\infty)$ that the Laplace transform $V(p)$ of the $L^2(\Omega)$-valued tempered distribution in $[0,+\infty)$,

$$t \mapsto v(t,\cdot) := \int_0^t S_\mu(t-s)\mathbb{1}_{(0,T)}(s)f(s,\cdot)ds \tag{6.9}$$

is well-defined in $L^2(\Omega)$ and verifies

$$\limsup_{n\to\infty} \|V_n(p) - V(p)\|_{L^2(\Omega)} = 0. \tag{6.10}$$

To this purpose, we recall the following estimate from [26, Lemma 2.2],

$$\frac{1}{|\vartheta(p)+\lambda_n|} \leqslant C\max(|p|^{-\alpha_0+\delta}, |p|^{-\alpha_0}),\ p \in \mathbb{C}\setminus(-\infty,0],\ n \in \mathbb{N}, \tag{6.11}$$

where the positive constant C is independent of n and p. Indeed, for all $t \in (0,+\infty)$ and all $\psi \in L^2(\Omega)$, we infer from (6.11) upon taking $\varepsilon = t^{-1}$ in (2.5), that

$$\begin{aligned} &\left\| \sum_{n=1}^{+\infty}\sum_{k=1}^{m_n} \left(\int_{\gamma_0(\varepsilon,\theta)} \frac{e^{pt}}{\vartheta(p)+\lambda_n} dp \right) \langle \rho^{-1}\psi, \varphi_{n,k}\rangle_{L^2_\rho(\Omega)} \varphi_{n,k} \right\|_{L^2(\Omega)} \\ \leqslant\ & C\max(t^{\alpha_0-\delta-1}, t^{\alpha_0-1}) \left| \int_{-\theta}^{\theta} e^{\cos\beta} d\beta \right|^2 \|\psi\|_{L^2(\Omega)} \end{aligned}$$

and

$$\left\| \sum_{n=1}^{+\infty}\sum_{k=1}^{m_n} \left(\int_{\gamma_\pm(\varepsilon,\theta)} \frac{e^{pt}}{\vartheta(p)+\lambda_n} dp \right) \langle \rho^{-1}\psi, \varphi_{n,k}\rangle_{L^2_\rho(\Omega)} \varphi_{n,k} \right\|_{L^2(\Omega)}$$

$$
\begin{aligned}
&\leqslant C\left|\int_{t^{-1}}^{+\infty}\max(r^{-\alpha_0+\delta},r^{-\alpha_0})e^{tr\cos\theta}dr\right|\|\psi\|_{L^2(\Omega)}\\
&\leqslant Ct^{-1}\left|\int_{1}^{+\infty}\max((t^{-1}r)^{-\alpha_0+\delta},(t^{-1}r)^{-\alpha_0})e^{r\cos\theta}dr\right|\|\psi\|_{L^2(\Omega)}\\
&\leqslant C\max(t^{\alpha_0-\delta-1},t^{\alpha_0-1})\left|\int_{1}^{+\infty}e^{r\cos\theta}dr\right|\|\psi\|_{L^2(\Omega)}.
\end{aligned}
$$

Putting these two estimates together with (2.4) and (6.5), we obtain that

$$
\|S_\mu(t)\|_{\mathcal{B}(L^2(\Omega))}\leqslant C\max(t^{\alpha_0-\delta-1},t^{\alpha_0-1}),\ t\in(0,+\infty), \tag{6.12}
$$

for some constant C that is independent of t. Thus, it holds true for all $p\in(0,+\infty)$, that

$$
t\mapsto y_p(t):=e^{-pt}\|S_\mu(t)\|_{\mathcal{B}(L^2(\Omega))}\in L^1(0,+\infty).
$$

Moreover, setting $\tilde{f}_p(t):=\mathbb{1}_{(0,T)}(t)e^{-pt}\|f(t,\cdot)\|_{L^2(\Omega)}$ for a.e. $t\in(0,+\infty)$, we get for each $p\in(0,+\infty)$ that

$$
\begin{aligned}
&e^{-pt}\left\|\int_0^t S_\mu(t-s)\mathbb{1}_{(0,T)}(s)f(s,\cdot)ds\right\|_{L^2(\Omega)}\\
\leqslant\ &\int_0^t e^{-p(t-s)}\|S_\mu(t-s)\|_{\mathcal{B}(L^2(\Omega))}\mathbb{1}_{(0,T)}(s)e^{-ps}\|f(s,\cdot)\|_{L^2(\Omega)}ds\\
\leqslant\ &(y_p*\tilde{f}_p)(t),
\end{aligned}
$$

where the symbol $*$ denotes the convolution in $(0,+\infty)$. Therefore, we find for every fixed $p\in(0,+\infty)$ that $\int_0^{+\infty}e^{-pt}\left\|\int_0^t S_\mu(t-s)\mathbb{1}_{(0,T)}(s)f(s,\cdot)ds\right\|_{L^2(\Omega)}dt$ is upper bounded by $\left\|y_p*\tilde{f}_p\right\|_{L^1(0,+\infty)}$, and hence by $\|y_p\|_{L^1(0,+\infty)}\left\|\tilde{f}_p\right\|_{L^1(0,+\infty)}$, which combined with (6.12), yields

$$
\int_0^{+\infty}e^{-pt}\left\|\int_0^t S_\mu(t-s)\mathbb{1}_{(0,T)}(s)f(s,\cdot)ds\right\|_{L^2(\Omega)}dt\leqslant C\max(p^{\delta-\alpha_0},p^{-\alpha_0})\|f\|_{L^1(0,T;L^2(\Omega))}, \tag{6.13}
$$

for some positive constant C that is independent of p. As a consequence $V(p)$ is well-defined in $L^2(\Omega)$ and satisfies

$$
\|V(p)\|_{L^2(\Omega)}\leqslant C\max(p^{\delta-\alpha_0},p^{-\alpha_0})\|f\|_{L^1(0,T;L^2(\Omega))},\ p\in(0,+\infty).
$$

Arguing as before with $f-f_n$ instead of f, we have

$$
\|V(p)-V_n(p)\|_{L^2(\Omega)}\leqslant C\max(p^{\delta-\alpha_0},p^{-\alpha_0})\|f-f_n\|_{L^1(0,T;L^2(\Omega))},\ p\in(0,+\infty),\ n\in\mathbb{N},
$$

which together with (6.6), yields (6.10).

With reference to (6.3) we are left with the task of proving that $V(p)$ lies in $D(A_{q,\rho})$, the domain of the operator $A_{q,\rho}$, and verifies

$$(A_{q,\rho} + p\vartheta(p))V(p) = F(p),\ p \in (0, +\infty), \tag{6.14}$$

To do that, we recall from the very definition of the function ϑ that $p\vartheta(p) > 0$ for all $p \in (0, +\infty)$, and hence that the operator $A_{q,\rho} + p\vartheta(p)$ is lower bounded by $\lambda_1 > 0$ in $L^2_\rho(\Omega)$, according to (1.1), (1.2) and (1.3). Thus, $A_{q,\rho} + p\vartheta(p)$ is boundedly invertible in $L^2_\rho(\Omega)$ and we have $\left\|(A_{q,\rho} + p\vartheta(p))^{-1}\right\|_{\mathcal{B}(L^2_\rho(\Omega))} \leqslant \lambda_1^{-1}$ for all $p \in (0, +\infty)$. Therefore, since $V_n(p) = (A_{q,\rho} + p\vartheta(p))^{-1}F_n(p)$ for all $n \in \mathbb{N}$ and all $p \in (0 + \infty)$, from (6.7), we get that

$$\begin{aligned}\left\|(A_{q,\rho} + p\vartheta(p))^{-1}F(p) - V_n(p)\right\|_{L^2_\rho(\Omega)} &\leqslant \left\|(A_{q,\rho} + p\vartheta(p))^{-1}\right\|_{\mathcal{B}(L^2(\Omega)} \left\|F(p) - F_n(p)\right\|_{L^2_\rho(\Omega)} \\ &\leqslant \rho_M^{\frac{1}{2}}\lambda_1 \left\|F(p) - F_n(p)\right\|_{L^2(\Omega)}.\end{aligned}$$

In light of (6.8), this entails that

$$\lim_{n\to+\infty} \left\|(A_{q,\rho} + p\vartheta(p))^{-1}F(p) - V_n(p)\right\|_{L^2(\Omega)} = 0,\ p \in (0, +\infty).$$

From this, (6.10) and the uniqueness of the limit in $L^2(\Omega)$, it then follows that $V(p) = (A_{q,\rho} + p\vartheta(p))^{-1}F(p)$, which is the claim of (6.14).

Therefore, $u = v_{|Q}$, where v is defined by (6.9), is a weak solution to (6.1). Now, in accordance with [26, Section 1.4], the end of the proof follows the same lines as the one in [26, Theorem 1.2]. □

Remark 2. *The representation formula* (6.4)-(6.5) *of the solution to* (6.1) *was obtained by replacing S by S_μ in* (2.2)-(2.3)*. Therefore, if $f \in L^1(0, T; L^2(\Omega))$ is supported in $[0, T-3\varepsilon_\star]\times\overline{\Omega}$ for some $\varepsilon_\star \in \left(0, \frac{T}{4}\right)$, then by substituting S_μ for S in the derivation of Proposition 2.1, we see that the weak solution u to* (6.1) *extends to a $L^2(\Omega)$-valued map which is analytic in $\mathcal{C}_{\theta_\star}$. Here, $\theta_\star$ can be any angle in $\left(0, \min\left(\frac{\pi}{4}, \frac{\pi}{2\alpha} - \frac{\pi}{4}\right)\right)$ and $\mathcal{C}_{\theta_\star}$ is defined in Proposition 2.1. Moreover, the extended function $t \mapsto u(t, \cdot)$ is holomorphic in $\mathcal{C}_{\theta_\star}$ and its Laplace transform $U(p)$ is a solution to* (6.3) *for all $p \in (0, +\infty)$.*

6.2. **Unique continuation.** The first result is reminiscent of Theorem 1.1.

Theorem 6.2. *Assume* (6.2)*, let σ, g and f be as in Theorem 1.1, and denote by u the weak-solution to* (6.1)*. Then, for any non-empty open subset $\Omega' \subset \Omega$, we have the implication:*

$$\left(u = 0 \text{ in } (0, T) \times \Omega'\right) \Longrightarrow (f = 0 \text{ in } \Omega).$$

Proof. With reference to Remark 2 we get by following the same lines as in the derivation of (3.3) that

$$\widehat{\sigma}(p)\left(\sum_{n=1}^{+\infty}\frac{\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}(x')}{\lambda_n+\vartheta(p)}\right)=0,\ x\in\Omega',\ p\in(0,+\infty), \tag{6.15}$$

where we used the notations introduced in Section 3.1.2. Since the function $\widehat{\sigma}$ is holomorphic in $\mathbb{C}_+$, then, leaving aside the trivial case where $\widehat{\sigma}$ is identically zero, which yields $\sigma=0$ in $(0,T)$ by injectivity of the Laplace transform, we may assume that $\widehat{\sigma}(p)\neq 0$ for all $p\in I$, where I is a non-empty subinterval of $(0,+\infty)$. In light of (6.15), this entails that

$$\sum_{n=1}^{+\infty}\frac{\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}(x')}{\lambda_n+\vartheta(p)}=0,\ x\in\Omega',\ p\in I,$$

and consequently that

$$R(\vartheta(p))=0,\ p\in I, \tag{6.16}$$

in the $L^2_\rho(\Omega')$-sense, where the function R is defined by (3.5). Next, bearing in mind that $\vartheta'(p)=\int_0^1\alpha p^{\alpha-1}\mu(\alpha)d\alpha$, we infer from (6.2) that

$$\vartheta'(p)\geqslant\int_{\alpha_0-\delta}^{\alpha_0}\alpha p^{\alpha-1}\mu(\alpha)d\alpha\geqslant\frac{\delta(\alpha_0-\delta)\mu(\alpha_0)}{2}\min_{p\in I}(p^{\alpha_0-1},p^{\alpha_0-\delta-1})>0,\ p\in I.$$

From this, (6.16) and the fact that R is a meromorphic function in $\mathbb{C}\setminus\{\lambda_n,\ n\in\mathbb{N}\}$, it then follows that

$$\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}(x')=0,\ x'\in\Omega',\ n\in\mathbb{N}.$$

Therefore, we have $g_{n,k}=0$ for all $k=0,\dots,m_n$ and all $n\in\mathbb{N}$, from Section 3.1.4, and consequently $g=\sum_{n=1}^{+\infty}\sum_{k=1}^{m_n} g_{n,k}\varphi_{n,k}=0$ in $L^2_\rho(\Omega)$. □

The second statement is inspired by Theorem 1.4.

Theorem 6.3. *Let T, Ω, ω, L, ρ and $\mathcal{A}_q$ be the same as in Theorem 1.4. Denote by u the solution to* (6.1) *where f is defined by* (1.12) *for $\sigma\in L^1(0,T;L^2(\omega))$ and $g\in L^2(-L,L)$, given by Proposition 6.1. Assume moreover that f is supported in $[0,T)\times\omega$. Then, for any non-empty open subset $\gamma\subset\partial\Omega$, we have the implication:*

$$(u=\partial_\nu u=0 \text{ on } (0,T)\times\gamma)\Longrightarrow(f=0 \text{ in } (0,T)\times\Omega).$$

Proof. Let the function u be extended as in Remark 2. Since its Laplace transform $U(p)$, $p\in(0,\infty)$, is solution to the BVP

$$\begin{cases}(-\Delta+\vartheta(p))U(p,\cdot) &= F(p,\cdot), \quad \text{in } \Omega,\\ B_*U(p) &= 0, \quad \text{on } \partial\Omega,\end{cases}$$

where $F(p) \in L^2(\Omega)$, then we have $U(p) \in H^2(\Omega)$ by the elliptic regularity theorem. Therefore, taking into account that

$$(-\Delta + \vartheta(p))e^{-ik\theta\cdot x'}e^{(k^2+\vartheta(p))^{1/2}x_n} = \left(k^2 - (k^2+\vartheta(p)) + \vartheta(p)\right)e^{-ik\theta\cdot x'}e^{(k^2+\vartheta(p))^{1/2}x_n} = 0,$$

for any $k \in \mathbb{R}$ and any $\theta \in \mathbb{S}^{d-2}$, and that

$$\begin{cases} (-\Delta + \vartheta(p))U(p,x) &= \hat{\sigma}(p,x')g(x_n), & x = (x', x_n) \in \Omega_0, \\ U(p,x) = \partial_{\nu_0}U(p,x) &= 0, & x \in \partial\Omega_0, \end{cases} \tag{6.17}$$

for all $p \in (0, +\infty)$, we find upon multiplying the first line in (6.17) by $e^{(k^2+\vartheta(p))^{1/2}x_n}$ and integrating by parts in Ω_0, that:

$$\int_{\Omega_0} \hat{\sigma}(p,x')h(x_n)g(x_n)e^{-ik\theta\cdot x'}e^{(k^2+\vartheta(p))^{1/2}x_n}dx'dx_n = 0.$$

By the Fubini theorem, the above equality immediately leads to

$$\left(\int_\omega \hat{\sigma}(p,x')e^{-ik\theta\cdot x'}dx'\right)\left(\int_{-L}^{L} g(x_n)e^{(k^2+\vartheta(p))^{1/2}x_n}dx_n\right) = 0,$$

for all $p \in (0, +\infty)$, all $k \in \mathbb{R}$ and all $\theta \in \mathbb{S}^{d-2}$, so the result follows from this upon arguing in the same as in the proof of Theorem 1.4. □

7. Numerical reconstruction method

In this section we reconstruct numerically the spatial part of the source term that was identified through internal data by Corollary 1.2. This is by means of an iterative scheme based on the Tikhonov regularization method, which is defined in the coming section.

7.1. **Iterative method.** We aim for building an efficient iterative scheme for numerical reconstruction of the spatial term $g(x)$ of the source, from knowledge of the temporal term $\sigma(t)$ and internal measurements for all time in a subregion Ω'. A reconstruction algorithm in the case $\alpha \in (0,1)$ has been studied in [15]. We extend the algorithm to the case $\alpha \in (1,2)$. According to Theorem 1.1, we choose a compactly supported time component instead of the non-supported one in [15]. Let us recall that, for $\alpha \in (1,2)$, the fractional Caputo derivative of order α is defined by

$$\partial_t^\alpha u(t,x) := \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\partial_s^2 u(s,x)ds, \quad (s,x) \in Q,$$

and the backward fractional Caputo derivative is defined by

$$\partial_{t_-}^\alpha u(t,x) := \frac{1}{\Gamma(2-\alpha)}\int_t^T (s-t)^{1-\alpha}\partial_s^2 u(s,x)ds, \quad (s,x) \in Q.$$

The backward Riemann-Liouville fractional derivative of order α is defined by

$$\partial_t^{\alpha *} u(t,x) := \frac{1}{\Gamma(2-\alpha)} \partial_t^2 \int_t^T (s-t)^{1-\alpha} u(s,x) ds, \quad (s,x) \in Q.$$

We introduce also maps J_0 and J_1 given by

$$\begin{aligned} J_0 u(t,x) &= \lim_{t \to T} \frac{1}{\Gamma(2-\alpha)} \int_t^T (s-t)^{1-\alpha} u(s,x) ds, \\ J_1 u(t,x) &= \lim_{t \to T} \frac{1}{\Gamma(2-\alpha)} \partial_t \int_t^T (s-t)^{1-\alpha} u(s,x) ds. \end{aligned}$$

Considering the homogeneous Neumann boundary condition, the forward problem is given by

$$\begin{cases} (\partial_t^\alpha + \mathcal{A}_q) u(t,x) = g(x)\sigma(t), & \text{in } Q, \\ \partial_{\nu_a} u(t,x) = 0, & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = \partial_t u(0,x) = 0, & \text{in } \Omega, \end{cases} \tag{7.1}$$

The weak formula of the system (7.1) is given by

$$\int_Q \left(\sum_{i,j} a_{i,j} \partial_{x_i} u \partial_{x_j} w + quw + u \partial_t^{\alpha *} w \right) dxdt = \int_Q g\sigma w dxdt.$$

for any test function $w \in H^\alpha(0,T; L^2(\Omega)) \bigcap L^2(0,T; H^1(\Omega))$ with $J_0 w = J_1 w = 0$ in Ω.

Assume that we have noise contaminated measurement u^δ in a subregion $\Omega' \subset \Omega$ with characteristic function $\chi_{\Omega'}$ from $t=0$ to $t=T$ satisfying $\|u^\delta - u(f_{\text{true}})\|_{L^2(\Omega)} \leqslant \delta$, where δ is the noise level. The reconstruction of the source term can be formulated as a least squares problem with Tikhonov regularization

$$\min_{g \in L^2(\Omega)} \Phi(g), \quad \Phi(g) := \|u(g) - u^\delta\|^2_{L^2((0,T)\times\Omega')} + \rho \|g\|^2_{L^2(\Omega)}. \tag{7.2}$$

Traditional iterative methods to solve the least squares problem (7.2) require the computation of the Fréchet derivative $\Phi'(g)$ of the object function $\Phi(g)$. For an arbitrary direction $h \in L^2(\Omega)$, $\Phi'(g)h$ is given by

$$\Phi'(g)h = 2\int_0^T \int_{\Omega'} (u(g) - u^\delta)(u'(g)h) dxdt + 2\rho \int_\Omega ghdx \tag{7.3}$$

$$= 2\int_0^T \int_{\Omega'} (u(g) - u^\delta) u(h) dxdt + 2\rho \int_\Omega ghdx \tag{7.4}$$

because of the linear dependence of u on g. By defining the adjoint system

$$\begin{cases} (\partial_t^{\alpha *} + \mathcal{A}_q) z(t,x) = \chi_{\Omega'}(u(g) - u^\delta), & \text{in } Q, \\ \partial_{\nu_a} z(t,x) = 0, & \text{on } (0,T) \times \partial\Omega, \\ J_0 z = J_1 z = 0, & \text{in } \Omega, \end{cases} \tag{7.5}$$

the first term of the equation (7.4) is equal to

$$\begin{aligned}\int_0^T \int_{\Omega'} (u(g) - u^\delta) u(h) dx dt &= \int_Q \chi_{\Omega'} (u(g) - u^\delta) u(h) dx dt \\ &= \int_Q (\partial_t^{\alpha *} + \mathcal{A}_q) z(t,x) u(h) dx dt \\ &= \int_Q g \sigma z(g) dx dt\end{aligned}$$

implying

$$\Phi'(g)h = 2 \int_\Omega \left(\int_0^T \sigma z(g) dt + \rho g \right) h dx.$$

Therefore at the optimal point we have

$$\rho g = - \int_0^T \sigma z(g) dt. \tag{7.6}$$

By adding Mg on both sides of (7.6), an iterative algorithm from [8, 15] is used for the reconstruction

$$g_{k+1} = \frac{M}{M+\rho} g_k - \frac{1}{M+\rho} \int_0^T \sigma z(g_k) dt, \quad k = 0, 1, 2, \ldots,$$

where $M > 0$ is a parameter controlling the convergence. The iterative stops after

$$\|g_{k+1} - g_k\|_{L^2(\Omega)} / \|g_k\|_{L^2(\Omega)} < \varepsilon$$

with ε the precision parameter.

By careful choice of final conditions and integration by parts, we know that the adjoint system (7.5) coincides with the following problem with a backward Caputo fractional derivative

$$\begin{cases} (\partial_{t_-}^\alpha + \mathcal{A}_q) z(t,x) = \chi_{\Omega'} (u(g) - u^\delta), & \text{in } Q, \\ \partial_{\nu_a} z(t,x) = 0, & \text{on } (0,T) \times \partial\Omega, \\ z(T,x) = z(T,x) = 0, & \text{in } \Omega. \end{cases} \tag{7.7}$$

Therefore, during each iteration we solve the fractional derivative equation twice.

7.2. **Numerical computations.** This section provides several results of numerical computation of g by the iterative scheme introduced in Section 7.1, in the particular case where:

$$\Omega = (0,1) \times (0,1), \quad T = 1, \quad \mathcal{A}_q u = -0.1 \Delta u + u.$$

The temporal term $\sigma(t)$ of the source and the initial guess $g_0(x)$ are chosen as follows

$$\sigma(t) = \frac{1}{\sqrt{2\pi} s} e^{-\frac{(t-0.4)^2}{2s^2}}, \ s = 0.12, \quad g_0(x) = 2.$$

Notice that $\sigma(t) \approx 0$ near $t = 0$ and $t = T$. From numerical point of view, σ is compactly supported in $(0, T)$. To obtain the noisy observation u^δ, we solve the forward problem numerically and add uniformly distributed random noise to the solution, i.e.,

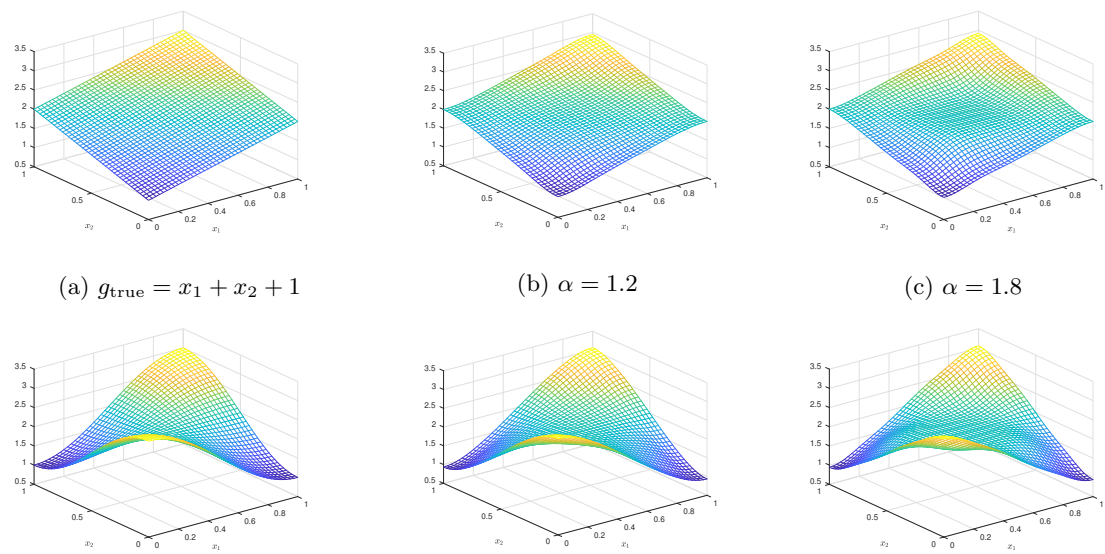
$$u^\delta(t, x) = \big(1 + \delta \mathrm{rand}(-1, 1)\big) u(g_{\mathrm{true}})(t, x).$$

Here $\mathrm{rand}(-1, 1)$ is a uniformly distributed number in $[-1, 1]$ and δ is the noise level. For parameters in the iterative method, we fix $\rho = 10^{-5}$ and $M = 4$. To evaluate the performance of the reconstruction, we compute the relative error $Res := \|g_k - g_{\mathrm{true}}\|_{L^2(\Omega)} / \|g_{\mathrm{true}}\|_{L^2(\Omega)}$.

Example 7.1. *In this example we compare reconstructed results for different g with different α. We choose the noise level $\delta = 2\%$, the stopping criterion $\varepsilon = \delta/50$ and the observation subregion $\Omega' = \overline{\Omega} \setminus (0.1, 0.9)^2$. We choose two pairs of fractional orders $\alpha = 1.2$ and $\alpha = 1.8$, and two true source terms*

$$g_{true}(x_1, x_2) = x_1 + x_2 + 1 \quad \textit{and} \quad g_{true}(x_1, x_2) = \cos(\pi x_1)\cos(\pi x_2) + 2.$$

Figure 1 demonstrates the result with iteration steps K and relative errors Res. Reconstruction with $\alpha = 1.2$ takes fewer steps and the result is more accurate than $\alpha = 1.8$.



(a) $g_{\mathrm{true}} = x_1 + x_2 + 1$ (b) $\alpha = 1.2$ (c) $\alpha = 1.8$

(d) $g_{\mathrm{true}} = \cos(\pi x_1)\cos(\pi x_2) + 2$ (e) $\alpha = 1.2$ (f) $\alpha = 1.8$

FIGURE 1. True solutions (left), reconstructions for $\alpha = 1.2$ (middle) and reconstructions for $\alpha = 1.8$ (right). Figure (1b) corresponds to $K = 91, Res = 2.71\%$; Figure (1c) corresponds to $K = 139, Res = 5.77\%$; Figure (1e) corresponds to $K = 113, Res = 3.65\%$; Figure (1f) corresponds to $K = 166, Res = 7.07\%$.

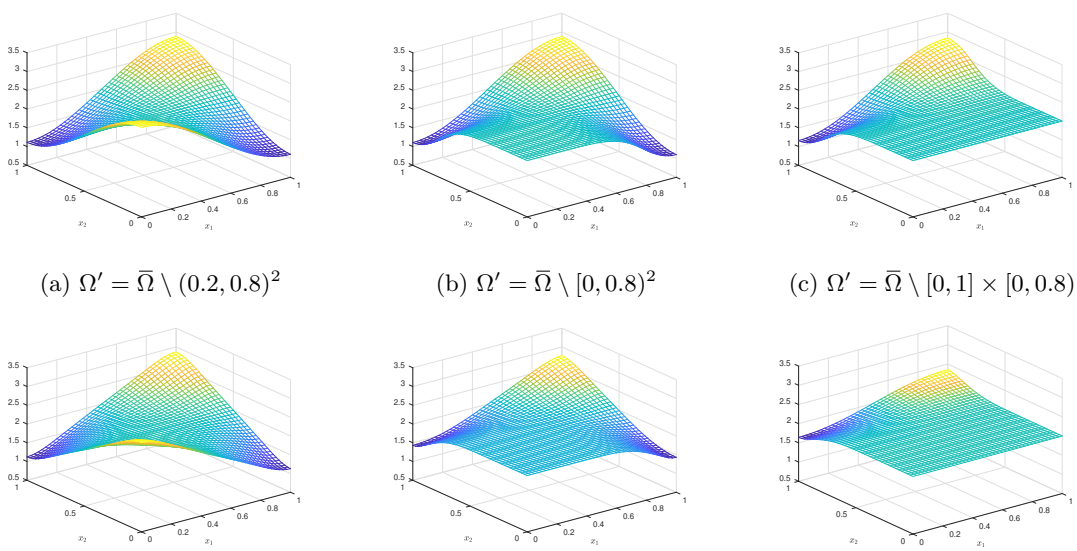
Example 7.2. *In this example we fix* $\alpha = 1.5$, $\delta = 10\%$, $\varepsilon = 10^{-3}$ *and*

$$g_{true} = \cos(\pi x_1)\cos(\pi x_2) + 2.$$

We study the effect of observation regions to reconstructed results by choosing six different regions:

$$\Omega' = \overline{\Omega} \setminus (0.2, 0.8)^2, \qquad \Omega' = \overline{\Omega} \setminus (0.05, 0.95)^2,$$
$$\Omega' = \overline{\Omega} \setminus [0, 0.8)^2, \qquad \Omega' = \overline{\Omega} \setminus [0, 0.95)^2,$$
$$\Omega' = \overline{\Omega} \setminus [0, 1] \times [0, 0.8), \qquad \Omega' = \overline{\Omega} \setminus [0, 1] \times [0, 0.95).$$

Figure (2) shows reconstructed results with different observation regions and Table (1) lists the number of steps and relative errors. With the increasing of the observation region, the reconstructed result becomes more accurate. If we are lack of observation near some boundaries, it is hard to obtain a good reconstruction near those boundaries.



(a) $\Omega' = \overline{\Omega} \setminus (0.2, 0.8)^2$ (b) $\Omega' = \overline{\Omega} \setminus [0, 0.8)^2$ (c) $\Omega' = \overline{\Omega} \setminus [0, 1] \times [0, 0.8)$

(d) $\Omega' = \overline{\Omega} \setminus (0.05, 0.95)^2$ (e) $\Omega' = \overline{\Omega} \setminus [0, 0.95)^2$ (f) $\Omega' = \overline{\Omega} \setminus [0, 1] \times [0, 0.95)$

FIGURE 2. Effect of observation regions to reconstructed results.

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Ω'	K	Res
$\overline{\Omega} \setminus (0.2, 0.8)^2$	73	3.95%
$\overline{\Omega} \setminus (0.05, 0.95)^2$	92	9.09%
$\overline{\Omega} \setminus [0, 0.8)^2$	72	13.54%
$\overline{\Omega} \setminus [0, 0.95)^2$	73	17.49%
$\overline{\Omega} \setminus [0, 1] \times [0, 0.8)$	63	18.42%
$\overline{\Omega} \setminus [0, 1] \times [0, 0.95)$	40	22.09%

TABLE 1. Number of steps and relative errors for different regions of observation.

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