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Eventual linear convergence rate of an exchange algorithm for superresolution.

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Abstract—We study an iterative discretization algorithm for solving optimization problems regularized by the total variation norm. Its design relies on ideas from the Frank-Wolfe algorithm, as well as from semi-infinite programming. For smooth regularity terms, we prove an eventual linear convergence rate guarantee.

I. EXCHANGE ALGORITHMS

For a bounded subset $\Omega \subseteq \mathbb{R}^d$, let $\mathcal{M}(\Omega)$ denote the space of Radon measures of bounded total variation $\|\cdot\|_{\mathcal{M}}$. For an operator $A : \mathcal{M}(\Omega) \rightarrow \mathbb{R}^m$ and a function $f : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\infty\}$, we consider the following optimization problem

$$\min_{\mu \in \mathcal{M}(\Omega)} \|\mu\|_{\mathcal{M}} + f(A\mu). \quad (\mathcal{P})$$

This formulation encompasses for instance the infinite-dimensional Basis Pursuit and the BLASSO, which in particular can be applied in superresolution [1], [2], [11], [4]. The dual program of (\mathcal{P}) is given by

$$\sup_{q \in \mathbb{R}^m, \|A^*q\|_{\infty} \leq 1} -f^*(q). \quad (\mathcal{D})$$

In this work, we present theoretical properties of numerically finding a solution of (\mathcal{P}) provided one follows:

Algorithm A

- 1) Given a grid $\Omega_k \subseteq \Omega$, find a solution pair (μ_k, q_k) of the (finite-dimensional) restrictions of (\mathcal{P}) and (\mathcal{D}) ;

$$\min_{\mu \in \mathcal{M}(\Omega_k)} \|\mu\|_{\mathcal{M}} + f(A\mu) \quad (\mathcal{P}_k)$$

$$\sup_{q \in \mathbb{R}^m, \sup_{x \in \Omega_k} |(A^*q)(x)| \leq 1} -f^*(q). \quad (\mathcal{D}_k)$$

- 2) Calculate the set X_k of local maximizers x of $|A^*q_k|$ with $|(A^*q_k)(x)| > 1$. Define new grid $\Omega_{k+1} := \Omega_k \cup X_k$.
- 3) Repeat until convergence.

This strategy is an example of a so-called *exchange algorithm*, applied to solve the semi-infinite program (\mathcal{D}) [9], [7], [10]. Notably, it also has interesting connections to the so-called Frank-Wolfe or Conditional gradient method [6], [5], [3].

II. EVENTUAL LINEAR CONERGENCE

It is not hard to prove that our exchange algorithm in great generality will converge towards a solution pair of (\mathcal{P}) – (\mathcal{D}) . Our main finding is that this will occur at an eventually linear rate under the following reasonable regularity assumptions:

- 1) The function f is differentiable, with an L -Lipschitz gradient

- 2) The measurement functions a_j are all C_0^2 .
- 3) (\mathcal{P}) has a unique solution $\mu^* = \sum_{i=1}^s \alpha_i^* \delta_{\xi_i}$.
- 4) The associated dual solution q^* obeys the following :
 - a) $|A^*q^*(x)| = 1$ only if $x \in \xi$
 - b) $x \mapsto |A^*q^*(x)|$ is strongly concave around each ξ_i .

Statements like 4) have been proven to be true in the important special case of $f(v) = \frac{\lambda}{2} \|v - b\|_2^2$ for a $b = A\mu_0$ for an atomic μ_0 with well-separated support, at least for large values of λ , for many classes of measurement operators A (see [4], [8]).

We may now formulate our main result. We say that the algorithm has entered a τ -*regime* if in all future iterations, only points within a distance τ from ξ will be added to the grid.

Theorem II.1. *Under the above assumptions, Algorithm A obeys:*

- 1) For each $\tau > 0$, the algorithm will enter a τ -regime after finitely many iterations.
- 2) After a finite number of iterations N , the algorithm will add exactly one point x_k^i in the proximity of each ξ_i .
- 3) For small $\tau > 0$, the algorithm will enter a τ -regime after at most $N + C \log(\tau^{-1})$ iterations. For such k , we will furthermore have $\inf(\mathcal{P}_k) \leq \inf(\mathcal{P}) + \kappa\tau^2$. That is, the algorithm will eventually converge linearly.

III. STOPPING CRITERION AND MERGING

Since the grid Ω_k will get increasingly fine around the points ξ_i , the discretized problems will inevitably become increasingly ill-conditioned, and we must hence typically stop the algorithm quite early. A computable stopping criterion is given by the following proposition.

Proposition III.1. *Assume that each local maximizer x_k^i is at a distance at most δ from the grid Ω_k . We then have*

$$\sup(\mathcal{D}_k) \leq \sup(\mathcal{D}) + C\delta^2, \quad \|q_k - q^*\|_2 \leq C_2\delta,$$

If f is strongly convex, δ is small and A obeys an additional regularity condition, μ_k will furthermore be supported on clusters K_i with $\sup_{x \in K_i} \|x - \xi_i\|_2 \leq C_3\delta$. If we merge such clusters to their centers of mass, we obtain a measure $\sum_{i=1}^s \tilde{\alpha}_i^0 \delta_{\tilde{x}_i^0}$ with

$$\sup_i \|\tilde{x}_i^0 - \xi_i\|_2 \leq C_3\delta, \quad \|\tilde{\alpha} - \alpha^*\|_2 \leq C_4\delta.$$

The last assertion of Proposition III.1 allows efficient merging operations that pave the way to postprocessing the measure.

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