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# Eventual linear convergence rate of an exchange algorithm for superresolution.

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**Abstract**—We study an iterative discretization algorithm for solving optimization problems regularized by the total variation norm. Its design relies on ideas from the Frank-Wolfe algorithm, as well as from semi-infinite programming. For smooth regularity terms, we prove an eventual linear convergence rate guarantee.

## I. EXCHANGE ALGORITHMS

For a bounded subset  $\Omega \subseteq \mathbb{R}^d$ , let  $\mathcal{M}(\Omega)$  denote the space of Radon measures of bounded total variation  $\|\cdot\|_{\mathcal{M}}$ . For an operator  $A : \mathcal{M}(\Omega) \rightarrow \mathbb{R}^m$  and a function  $f : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\infty\}$ , we consider the following optimization problem

$$\min_{\mu \in \mathcal{M}(\Omega)} \|\mu\|_{\mathcal{M}} + f(A\mu). \quad (\mathcal{P})$$

This formulation encompasses for instance the infinite-dimensional Basis Pursuit and the BLASSO, which in particular can be applied in superresolution [1], [2], [11], [4]. The dual program of  $(\mathcal{P})$  is given by

$$\sup_{q \in \mathbb{R}^m, \|A^*q\|_{\infty} \leq 1} -f^*(q). \quad (\mathcal{D})$$

In this work, we present theoretical properties of numerically finding a solution of  $(\mathcal{P})$  provided one follows:

### Algorithm A

- 1) Given a grid  $\Omega_k \subseteq \Omega$ , find a solution pair  $(\mu_k, q_k)$  of the (finite-dimensional) restrictions of  $(\mathcal{P})$  and  $(\mathcal{D})$ ;

$$\min_{\mu \in \mathcal{M}(\Omega_k)} \|\mu\|_{\mathcal{M}} + f(A\mu) \quad (\mathcal{P}_k)$$

$$\sup_{q \in \mathbb{R}^m, \sup_{x \in \Omega_k} |(A^*q)(x)| \leq 1} -f^*(q). \quad (\mathcal{D}_k)$$

- 2) Calculate the set  $X_k$  of local maximizers  $x$  of  $|A^*q_k|$  with  $|(A^*q_k)(x)| > 1$ . Define new grid  $\Omega_{k+1} := \Omega_k \cup X_k$ .
- 3) Repeat until convergence.

This strategy is an example of a so-called *exchange algorithm*, applied to solve the semi-infinite program  $(\mathcal{D})$  [9], [7], [10]. Notably, it also has interesting connections to the so-called Frank-Wolfe or Conditional gradient method [6], [5], [3].

## II. EVENTUAL LINEAR CONVERGENCE

It is not hard to prove that our exchange algorithm in great generality will converge towards a solution pair of  $(\mathcal{P})$ – $(\mathcal{D})$ . Our main finding is that this will occur at an eventually linear rate under the following reasonable regularity assumptions:

- 1) The function  $f$  is differentiable, with an  $L$ -Lipschitz gradient

- 2) The measurement functions  $a_j$  are all  $C_0^2$ .
- 3)  $(\mathcal{P})$  has a unique solution  $\mu^* = \sum_{i=1}^s \alpha_i^* \delta_{\xi_i}$ .
- 4) The associated dual solution  $q^*$  obeys the following :
  - a)  $|A^*q^*(x)| = 1$  only if  $x \in \xi$
  - b)  $x \mapsto |A^*q^*(x)|$  is strongly concave around each  $\xi_i$ .

Statements like 4) have been proven to be true in the important special case of  $f(v) = \frac{\lambda}{2} \|v - b\|_2^2$  for a  $b = A\mu_0$  for an atomic  $\mu_0$  with well-separated support, at least for large values of  $\lambda$ , for many classes of measurement operators  $A$  (see [4], [8]).

We may now formulate our main result. We say that the algorithm has entered a  $\tau$ -regime if in all future iterations, only points within a distance  $\tau$  from  $\xi$  will be added to the grid.

**Theorem II.1.** *Under the above assumptions, Algorithm A obeys:*

- 1) For each  $\tau > 0$ , the algorithm will enter a  $\tau$ -regime after finitely many iterations.
- 2) After a finite number of iterations  $N$ , the algorithm will add exactly one point  $x_k^i$  in the proximity of each  $\xi_i$ .
- 3) For small  $\tau > 0$ , the algorithm will enter a  $\tau$ -regime after at most  $N + C \log(\tau^{-1})$  iterations. For such  $k$ , we will furthermore have  $\inf(\mathcal{P}_k) \leq \inf(\mathcal{P}) + \kappa\tau^2$ . That is, the algorithm will eventually converge linearly.

## III. STOPPING CRITERION AND MERGING

Since the grid  $\Omega_k$  will get increasingly fine around the points  $\xi_i$ , the discretized problems will inevitably become increasingly ill-conditioned, and we must hence typically stop the algorithm quite early. A computable stopping criterion is given by the following proposition.

**Proposition III.1.** *Assume that each local maximizer  $x_k^i$  is at a distance at most  $\delta$  from the grid  $\Omega_k$ . We then have*

$$\sup(\mathcal{D}_k) \leq \sup(\mathcal{D}) + C\delta^2, \quad \|q_k - q^*\|_2 \leq C_2\delta,$$

*If  $f$  is strongly convex,  $\delta$  is small and  $A$  obeys an additional regularity condition,  $\mu_k$  will furthermore be supported on clusters  $K_i$  with  $\sup_{x \in K_i} \|x - \xi_i\|_2 \leq C_3\delta$ . If we merge such clusters to their centers of mass, we obtain a measure  $\sum_{i=1}^s \tilde{\alpha}_i^0 \delta_{\tilde{x}_i^0}$  with*

$$\sup_i \|\tilde{x}_i^0 - \xi_i\|_2 \leq C_3\delta, \quad \|\tilde{\alpha} - \alpha^*\|_2 \leq C_4\delta.$$

The last assertion of Proposition III.1 allows efficient merging operations that pave the way to postprocessing the measure.

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