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## ON THE COARSE GEOMETRY OF JAMES SPACES

G. LANCIEN, C. PETITJEAN, AND A. PROCHÁZKA

ABSTRACT. In this note we prove that the Kalton interlaced graphs do not equi-coarsely embed into the James space  $\mathcal{J}$  nor into its dual  $\mathcal{J}^*$ . It is a particular case of a more general result on the non equi-coarse embeddability of the Kalton graphs into quasi-reflexive spaces with a special asymptotic structure. This allows us to exhibit a coarse invariant for Banach spaces, namely the non equi-coarse embeddability of this family of graphs, which is very close to but different from the celebrated property  $\mathcal{Q}$  of Kalton. We conclude with a remark on the coarse geometry of the James tree space  $\mathcal{JT}$  and of its predual.

## 1. INTRODUCTION

In a fundamental paper on the coarse geometry of Banach spaces ([14]), N. Kalton introduced a property of metric spaces that he named property  $\mathcal{Q}$ . In particular, its absence served as an obstruction to coarse embeddability into reflexive Banach spaces. This property is related to the behavior of Lipschitz maps defined on a particular family of metric graphs that we shall denote  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$ . We will recall the precise definitions of these graphs and of property  $\mathcal{Q}$  in section 2.2. Let us just say, vaguely speaking for the moment, that a Banach space  $X$  has property  $\mathcal{Q}$  if for every Lipschitz map  $f$  from  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)$  to  $X$ , there exists a full subgraph  $[\mathbb{M}]^k$  of  $[\mathbb{N}]^k$ , with  $\mathbb{M}$  infinite subset of  $\mathbb{N}$ , on which  $f$  satisfies a strong concentration phenomenon. It is then easy to see that if a Banach space  $X$  has property  $\mathcal{Q}$ , then the family of graphs  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$  does not equi-coarsely embed into  $X$  (see the definition in section 2.1). One of the main results in [14] is that any reflexive Banach space has property  $\mathcal{Q}$ . It then readily follows that a reflexive Banach space cannot contain a coarse copy of all separable metric spaces, or equivalently does not contain a coarse copy of the Banach space  $c_0$ . In fact, with a sophistication of this argument, Kalton proved an even stronger result in [14]: if a separable Banach space contains a coarse copy of  $c_0$ , then there is an integer  $k$  such that the dual of order  $k$  of  $X$  is non separable. In particular, a quasi-reflexive Banach space does not contain a coarse copy of  $c_0$ . However, Kalton proved that the most famous example of a quasi-reflexive space, namely the James space  $\mathcal{J}$ , as well as its dual  $\mathcal{J}^*$ , fail property  $\mathcal{Q}$ .

The main purpose of this paper is to show that, although they do not obey the concentration phenomenon described by property  $\mathcal{Q}$ , neither  $\mathcal{J}$  nor  $\mathcal{J}^*$  equi-coarsely contains the family of graphs  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$  (Corollary

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5.3). This provides a coarse invariant, namely “not containing equi-coarsely the Kalton graphs”, that is very close to but different from property  $\mathcal{Q}$ . This could allow to find obstructions to coarse embeddability between seemingly close Banach spaces. Our result is actually more general. We prove in Theorem 4.1 that a quasi-reflexive Banach space  $X$  such that both  $X$  and  $X^*$  admit an equivalent  $p$ -asymptotically uniformly smooth norm (see the definition in section 3), for some  $p$  in  $(1, \infty)$ , does not equi-coarsely contain the Kalton graphs.

We conclude this note by showing that if the James tree space  $\mathcal{JT}$  or its predual coarsely embeds into a separable Banach space  $X$ , then there exists  $k \in \mathbb{N}$  so that the dual of order  $k$  of  $X$  is non separable. This extends slightly Theorem 3.5 in [14].

## 2. METRIC NOTIONS

**2.1. Coarse embeddings.** Let  $M, N$  be two metric spaces and  $f: M \rightarrow N$  be a map. We define the compression modulus  $\rho_f$  and the expansion modulus  $\omega_f$  as follows. For  $t \in [0, \infty)$ , we set

$$\begin{aligned}\rho_f(t) &= \inf\{d_N(f(x), f(y)) : d_M(x, y) \geq t\}, \\ \omega_f(t) &= \sup\{d_N(f(x), f(y)) : d_M(x, y) \leq t\}.\end{aligned}$$

We adopt the convention  $\sup(\emptyset) = 0$  and  $\inf(\emptyset) = \infty$ . Note that for every  $x, y \in M$ ,

$$\rho_f(d_M(x, y)) \leq d_N(f(x), f(y)) \leq \omega_f(d_M(x, y)).$$

We say that  $f$  is a *coarse embedding* if  $\omega_f(t) < \infty$  for every  $t \in [0, +\infty)$  and  $\lim_{t \rightarrow \infty} \rho_f(t) = \infty$ .

Next, let  $(M_i)_{i \in I}$  be a family of metric spaces. We say that the family  $(M_i)_{i \in I}$  *equi-coarsely embeds* into a metric space  $N$  if there exist two maps  $\rho, \omega: [0, +\infty) \rightarrow [0, +\infty)$  and maps  $f_i: M_i \rightarrow N$  for  $i \in I$  such that:

- (i)  $\lim_{t \rightarrow \infty} \rho(t) = \infty$ ,
- (ii)  $\omega(t) < \infty$  for every  $t \in [0, +\infty)$ ,
- (iii)  $\rho(t) \leq \rho_{f_i}(t)$  and  $\omega_{f_i}(t) \leq \omega(t)$  for every  $i \in I$  and  $t \in [0, \infty)$ .

**2.2. The Kalton interlaced graphs and property  $\mathbf{Q}$ .** For  $k \in \mathbb{N}$  and  $\mathbb{M}$  an infinite subset of  $\mathbb{N}$ , we put  $[\mathbb{M}]^{\leq k} = \{S \subset \mathbb{M} : |S| \leq k\}$ ,  $[\mathbb{M}]^k = \{S \subset \mathbb{M} : |S| = k\}$ ,  $[\mathbb{M}]^\omega = \{S \subset \mathbb{M} : S \text{ is infinite}\}$ , and  $[\mathbb{M}]^{<\omega} = \{S \subset \mathbb{M} : S \text{ is finite}\}$ . We always list the elements of some  $\bar{m}$  in  $[\mathbb{N}]^{<\omega}$  or in  $[\mathbb{N}]^\omega$  in increasing order, meaning that if we write  $\bar{m} = (m_1, m_2, \dots, m_l)$  or  $\bar{m} = (m_1, m_2, m_3, \dots)$ , we tacitly assume that  $m_1 < m_2 < \dots$ .

For  $\bar{m} = (m_1, m_2, \dots, m_r) \in [\mathbb{N}]^{<\omega}$  and  $\bar{n} = (n_1, n_2, \dots, n_s) \in [\mathbb{N}]^{<\omega}$ , we write  $\bar{m} \prec \bar{n}$ , if  $r < s \leq k$  and  $m_i = n_i$ , for  $i = 1, 2, \dots, r$ , and we write  $\bar{m} \preceq \bar{n}$  if  $\bar{m} \prec \bar{n}$  or  $\bar{m} = \bar{n}$ . Thus  $\bar{m} \preceq \bar{n}$  if  $\bar{m}$  is an initial segment of  $\bar{n}$ .

Following Kalton [14], for  $\mathbb{M} \in [\mathbb{N}]^\omega$ , we equip  $[\mathbb{M}]^k$  with a graph structure by declaring  $\bar{m} \neq \bar{n} \in [\mathbb{M}]^k$  adjacent if and only if

$$n_1 \leq m_1 \leq n_2 \leq m_2 \leq n_3 \leq m_3 \leq n_k \leq m_k \quad \text{or} \quad m_1 \leq n_1 \leq m_2 \leq n_2 \leq m_k \leq n_k.$$

For any  $\bar{m}, \bar{n} \in [\mathbb{M}]^k$ , the distance  $d_{\mathbb{K}}^k(\bar{m}, \bar{n})$  is then defined as the shortest path distance in the graph  $[\mathbb{M}]^k$ .

**Remark 2.1.** The distance  $d_{\mathbb{K}}^k$  is independent of the set  $\mathbb{M}$  and therefore  $[\mathbb{M}_1]^k$  is a metric subspace of  $[\mathbb{M}_2]^k$  whenever  $\mathbb{M}_1 \in [\mathbb{M}_2]^\omega$ .

This last claim is an immediate consequence of the following explicit formula for the distance.

**Proposition 2.2.** *Let  $k \in \mathbb{N}$  and  $\mathbb{M} \in [\mathbb{N}]^\omega$ . Then  $d_{\mathbb{K}}^k(\bar{n}, \bar{m}) = d(\bar{n}, \bar{m})$  for all  $\bar{n}, \bar{m} \in [\mathbb{M}]^k$  where  $d(\bar{n}, \bar{m}) = \sup\{|\bar{n} \cap S| - |\bar{m} \cap S| : S \text{ segment of } \mathbb{N}\}$ .*

*Proof.* It is easily seen that  $d$  is a metric on  $[\mathbb{M}]^k$ . Since  $d_{\mathbb{K}}^k$  is a graph metric on  $[\mathbb{M}]^k$ , in order to show  $d_{\mathbb{K}}^k = d$  it is enough to verify that  $d_{\mathbb{K}}^k(\bar{n}, \bar{m}) = 1$  if and only if  $d(\bar{n}, \bar{m}) = 1$  and that  $d$  is a graph metric.

For  $A \subset \mathbb{N}$  let us denote  $\mathbf{1}_A : \mathbb{N} \rightarrow \{0, 1\}$  the indicator function of  $A$  and let us first observe the following fact.

**Fact:** *For every  $\bar{n}, \bar{m} \in [\mathbb{M}]^k$ ,*

$$d(\bar{n}, \bar{m}) = \max_i F(i) - \min_i F(i)$$

where  $F(i) = F_{\bar{n}, \bar{m}}(i) = \sum_{j=1}^i \mathbf{1}_{\bar{n}}(j) - \mathbf{1}_{\bar{m}}(j)$  (and  $F(0) = 0$ ).

Indeed, we have for any segment  $S = [a, b]$  that

$$|S \cap \bar{n}| - |S \cap \bar{m}| = \sum_{j \in S} (\mathbf{1}_{\bar{n}}(j) - \mathbf{1}_{\bar{m}}(j)) = F(b) - F(a-1).$$

In particular  $\max_S |S \cap \bar{n}| - |S \cap \bar{m}| \leq \max F - \min F$ . On the other hand if  $S = [a, b]$  is such that  $\{F(a-1), F(b)\} = \{\max F, \min F\}$  then  $||S \cap \bar{n}| - |S \cap \bar{m}|| \geq \max F - \min F$  which finishes the proof of the fact.

It is clear that  $d_{\mathbb{K}}^k(\bar{n}, \bar{m}) = 1$  if and only if  $\max F - \min F = 1$ . Thus it only remains to prove that  $d$  is a graph metric. Now given  $\bar{n}, \bar{m}$  such that  $d(\bar{n}, \bar{m}) \geq 2$  we are looking for  $\bar{\ell} \in [\mathbb{M}]^k \setminus \{\bar{m}, \bar{n}\}$  such that  $d(\bar{m}, \bar{n}) = d(\bar{n}, \bar{\ell}) + d(\bar{\ell}, \bar{m})$ . Without loss of generality we will assume that  $\max F_{\bar{n}, \bar{m}} > 0$ . Notice that the sets  $\arg \max(F)$  and  $\arg \min(F)$  are disjoint. We select inductively  $\{a_1 < \dots < a_p\} \subset \arg \max(F)$  and  $\{b_1 < \dots < b_q\} \subset \arg \min(F)$  (with  $p \geq 1$  and  $q \geq 0$ ) with the property that

- $a_1 = \min \arg \max(F)$ ,
- For  $i \geq 1$ ,  $b_i = \min(\{n > a_i\} \cap \arg \min(F))$ , if this is not empty.
- $a_{i+1} = \min(\{n > b_i\} \cap \arg \max(F))$ , if this set is not empty.

Notice that  $\{a_1, \dots, a_p\} \subset \bar{n} \setminus \bar{m}$  and  $\{b_1, \dots, b_q\} \subset \bar{m} \setminus \bar{n}$ . Notice also that either  $p = q$  or  $p = q + 1$ . In the latter case we define  $b_p := r$  for some  $r$  such that  $r > a_p$  and  $F(r-1) > F(r)$ . Such  $r$  must exist since  $F(\max\{n_k, m_k\}) = 0$ . Also we have  $r \in \bar{m} \setminus \bar{n}$ . We will set

$$\bar{\ell} = \bar{n} \cup \{b_1, \dots, b_p\} \setminus \{a_1, \dots, a_p\}.$$

It is clear that  $\bar{\ell} \in [\mathbb{M}]^k$ . We also have  $\max F_{\bar{\ell}, \bar{m}} = \max F_{\bar{n}, \bar{m}} - 1$  and  $\min F_{\bar{\ell}, \bar{m}} = \min F_{\bar{n}, \bar{m}}$ . Indeed, the point  $\bar{\ell}$  is constructed in such a way that when  $F_{\bar{n}, \bar{m}}$  attains its maximum for the first time (going from the left),  $F_{\bar{\ell}, \bar{m}}$  is reduced by one and stays reduced by 1 until the next time the minimum of  $F_{\bar{n}, \bar{m}}$  is attained (or until the point  $r$ ) where this reduction is corrected back; and so on. Thus  $d(\bar{\ell}, \bar{m}) = d(\bar{n}, \bar{m}) - 1$ . Also, since the sets  $\{a_1, \dots, a_p\}$  and  $\{b_1, \dots, b_p\}$  are interlaced we have  $F_{\bar{n}, \bar{m}} - 1 \leq F_{\bar{\ell}, \bar{m}} \leq F_{\bar{n}, \bar{m}}$ . Therefore, since

$F_{\bar{n}, \bar{m}} = F_{\bar{n}, \bar{\ell}} + F_{\bar{\ell}, \bar{m}}$ , we have that  $0 \leq F_{\bar{n}, \bar{\ell}} \leq 1$  and so finally  $d(\bar{n}, \bar{\ell}) = 1$ , since it is clear that  $\bar{n} \neq \bar{\ell}$ .  $\square$

Note that if  $X$  is a Banach space and  $f: ([\mathbb{M}]^k, d_{\mathbb{K}}^k) \rightarrow X$  is a map with finite expansion modulus  $\omega_f$ , then  $\omega_f(1)$  is actually the Lipschitz constant of  $f$  as  $d_{\mathbb{K}}^k$  is a graph distance on  $[\mathbb{M}]^k$ .

In [14] the property  $\mathcal{Q}$  is defined in the setting of metric spaces. For homogeneity reasons, its definition can be simplified for Banach spaces. Let us recall it here.

**Definition 2.3.** Let  $X$  be a Banach space. We say that  $X$  has *property  $\mathcal{Q}$*  if there exists  $C \geq 1$  such that for every  $k \in \mathbb{N}$  and every Lipschitz map  $f: ([\mathbb{N}]^k, d_{\mathbb{K}}^k) \rightarrow X$ , there exists an infinite subset  $\mathbb{M}$  of  $\mathbb{N}$  such that:

$$\forall \bar{n}, \bar{m} \in [\mathbb{M}]^k, \|f(\bar{n}) - f(\bar{m})\| \leq C\omega_f(1).$$

The following proposition should be clear from the definitions. We shall however include its short proof.

**Proposition 2.4.** *Let  $X$  be a Banach space. If  $X$  has property  $\mathcal{Q}$ , then the family of graphs  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$  does not equi-coarsely embed into  $X$ .*

*Proof.* Let  $C \geq 1$  be given by the definition of property  $\mathcal{Q}$ . Aiming for a contradiction, assume that the family  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$  equi-coarsely embeds into  $X$ . That is, there are maps  $f_k: ([\mathbb{N}]^k, d_{\mathbb{K}}^k) \rightarrow X$  and two functions  $\rho, \omega: [0, +\infty) \rightarrow [0, +\infty)$  such that  $\lim_{t \rightarrow \infty} \rho(t) = \infty$  and

$$\forall k \in \mathbb{N} \quad \forall t > 0 \quad \rho(t) \leq \rho_{f_k}(t) \quad \text{and} \quad \omega_{f_k}(t) \leq \omega(t) < \infty.$$

Thus, for every  $k \in \mathbb{N}$ , there exists an infinite subset  $\mathbb{M}_k$  of  $\mathbb{N}$  such that  $\text{diam}(f([\mathbb{M}_k]^k)) \leq C\omega(1)$ . Since  $\text{diam}([\mathbb{M}_k]^k) = k$ , this implies that for all  $k \in \mathbb{N}$ ,  $\rho(k) \leq C\omega(1)$ . This contradicts the fact that  $\lim_{t \rightarrow \infty} \rho(t) = \infty$ .  $\square$

A concrete bi-Lipschitz copy of the metric spaces  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)$  in  $c_0$  is given by the following proposition.

**Proposition 2.5.** *Let  $(s_n)_{n=1}^{\infty}$  be the summing basis of  $c_0$ , that is  $s_n = \sum_{i=1}^n e_i$ , where  $(e_i)_{i=1}^{\infty}$  is the canonical basis of  $c_0$ .*

*For  $k \in \mathbb{N}$ , define  $f_k: ([\mathbb{N}]^k, d_{\mathbb{K}}^k) \rightarrow c_0$  by  $f_k(\bar{n}) = \sum_{i=1}^k s_{n_i}$ . Then*

$$\frac{1}{2}d_{\mathbb{K}}^k(\bar{n}, \bar{m}) \leq \|f_k(\bar{n}) - f_k(\bar{m})\|_{\infty} \leq d_{\mathbb{K}}^k(\bar{n}, \bar{m})$$

*for all  $\bar{n}, \bar{m} \in [\mathbb{N}]^k$ .*

*Proof.* Since  $d_{\mathbb{K}}^k = d$ , one can show (as in the Fact in the proof of Proposition 2.2) that  $d_{\mathbb{K}}^k(\bar{n}, \bar{m}) = \max(f_k(\bar{n}) - f_k(\bar{m})) - \min(f_k(\bar{n}) - f_k(\bar{m}))$ . The result then follows easily since  $\min(f_k(\bar{n}) - f_k(\bar{m})) \leq 0 \leq \max(f_k(\bar{n}) - f_k(\bar{m}))$  for all  $\bar{n}, \bar{m} \in [\mathbb{N}]^k$ .  $\square$

**Remark 2.6.** We already explained that  $c_0$  cannot coarsely embed into any Banach space with property  $\mathcal{Q}$  (in particular into any reflexive Banach space) and that Kalton even showed with additional arguments that if  $c_0$  coarsely embeds into a separable Banach space  $X$ , then one of the iterated

duals of  $X$  has to be non separable. An inspection of his proof shows that the uniformly discrete metric spaces

$$M_k = \left\{ \sum_{i=1}^k s_{n_i} \times \mathbf{1}_A : (n_1, \dots, n_k) \in [\mathbb{N}]^k, A \in [\mathbb{N}]^\omega \right\} \subset c_0$$

do not equi-coarsely embed into any Banach space  $X$  such that  $X^{(r)}$  is separable for all  $r$ . See Theorem 6.1 below for more on this subject.

Studying further the property  $\mathcal{Q}$  in [14], Kalton exhibited non reflexive quasi-reflexive spaces with the property  $\mathcal{Q}$  but showed that  $\mathcal{J}$  and  $\mathcal{J}^*$  fail property  $\mathcal{Q}$ . It is worth noticing that a theorem of Schoenberg [21] implies that  $\ell_1$  coarsely embeds into  $\ell_2$ , and therefore  $\ell_1$  provides a simple example of a non-reflexive Banach space with property  $\mathcal{Q}$ .

We conclude this section with two propositions that we state here for future reference. We start with a classical version of Ramsey's theorem.

**Proposition 2.7** (Corollary 1.2 in [10]). *Let  $(K, d)$  be a compact metric space,  $k \in \mathbb{N}$  and  $f: [\mathbb{N}]^k \rightarrow K$ . Then for every  $\varepsilon > 0$ , there exists an infinite subset  $\mathbb{M}$  of  $\mathbb{N}$  such that  $d(f(\bar{n}), f(\bar{m})) < \varepsilon$  for every  $\bar{n}, \bar{m} \in [\mathbb{M}]^k$ .*

For a Banach space  $X$ , we call *tree of height  $k$  in  $X$*  any family  $(x(\bar{n}))_{\bar{n} \in [\mathbb{N}]^{\leq k}}$ , with  $x(\bar{n}) \in X$ . Then, if  $\mathbb{M} \in [\mathbb{N}]^\omega$ ,  $(x(\bar{n}))_{\bar{n} \in [\mathbb{M}]^{\leq k}}$  will be called a *full subtree* of  $(x(\bar{n}))_{\bar{n} \in [\mathbb{N}]^{\leq k}}$ . A tree  $(x^*(\bar{n}))_{\bar{n} \in [\mathbb{M}]^{\leq k}}$  in  $X^*$  is called *weak\*-null* if for any  $\bar{n} \in [\mathbb{M}]^{\leq k-1}$ , the sequence  $(x^*(n_1, \dots, n_{k-1}, t))_{t > n_{k-1}, t \in \mathbb{M}}$  is weak\*-null.

The next proposition is based on a weak\*-compactness argument and will be crucial for our proofs. Although the distance considered on  $[\mathbb{N}]^k$  is different, the proof follows the same lines as Lemma 4.1 in [3]. We therefore state it now without further detail.

**Proposition 2.8.** *Let  $X$  be a separable Banach space,  $k \in \mathbb{N}$ , and  $f: ([\mathbb{N}]^k, d_{\mathbb{R}}^k) \rightarrow X^*$  a Lipschitz map. Then there exist  $\mathbb{M} \in [\mathbb{N}]^\omega$  and a weak\*-null tree  $(x^*(\bar{m}))_{\bar{m} \in [\mathbb{M}]^{\leq k}}$  in  $X^*$  with  $\|x_{\bar{m}}^*\| \leq \omega_f(1)$  for all  $\bar{m} \in [\mathbb{M}]^{\leq k} \setminus \{\emptyset\}$  and so that*

$$\forall \bar{n} \in [\mathbb{M}]^k, f(\bar{n}) = \sum_{i=0}^k x^*(n_1, \dots, n_i) = \sum_{\bar{m} \leq \bar{n}} x^*(\bar{m}).$$

### 3. UNIFORM ASYMPTOTIC PROPERTIES OF NORMS AND RELATED ESTIMATES

We recall the definitions that will be considered in this paper. For a Banach space  $(X, \|\cdot\|)$  we denote by  $B_X$  the closed unit ball of  $X$  and by  $S_X$  its unit sphere. The following definitions are due to V. Milman [19] and we adopt the notation from [13]. For  $t \in [0, \infty)$  we define

$$\bar{\rho}_X(t) = \sup_{x \in S_X} \inf_Y \sup_{y \in S_Y} (\|x + ty\| - 1),$$

where  $Y$  runs through all closed subspaces of  $X$  of finite codimension. Then, the norm  $\|\cdot\|$  is said to be *asymptotically uniformly smooth* (in short AUS) if

$$\lim_{t \rightarrow 0} \frac{\bar{\rho}_X(t)}{t} = 0.$$

For  $p \in (1, \infty)$  it is said to be *p-asymptotically uniformly smooth* (in short *p-AUS*) if there exists  $c > 0$  such that for all  $t \in [0, \infty)$ ,  $\bar{\rho}_X(t) \leq ct^p$ .

We will also need the dual modulus defined by

$$\bar{\delta}_X^*(t) = \inf_{x^* \in S_{X^*}} \sup_E \inf_{y^* \in S_E} (\|x^* + ty^*\| - 1),$$

where  $E$  runs through all finite-codimensional weak\*-closed subspaces of  $X^*$ . The norm of  $X^*$  is said to be *weak\* asymptotically uniformly convex* (in short *AUC\**) if  $\bar{\delta}_X^*(t) > 0$  for all  $t$  in  $(0, \infty)$ . If there exists  $c > 0$  and  $q \in [1, \infty)$  such that for all  $t \in [0, 1]$   $\bar{\delta}_X^*(t) \geq ct^q$ , we say that the norm of  $X^*$  is *q-AUC\**. The following proposition is elementary.

**Proposition 3.1.** *Let  $X$  be a Banach space. For any  $t \in (0, 1)$ , any weakly null sequence  $(x_n)_{n=1}^\infty$  in  $B_X$  and any  $x \in S_X$  we have:*

$$\limsup_{n \rightarrow \infty} \|x + tx_n\| \leq 1 + \bar{\rho}_X(t).$$

*For any weak\*-null sequence  $(x_n^*)_{n=1}^\infty \subset X^*$  and for any  $x^* \in X^* \setminus \{0\}$  we have*

$$\limsup_{n \rightarrow \infty} \|x^* + x_n^*\| \geq \|x^*\| \left( 1 + \bar{\delta}_X^* \left( \frac{\limsup_{n \rightarrow \infty} \|x_n^*\|}{\|x^*\|} \right) \right).$$

We will also need the following refinement (see Proposition 2.1 in [18]).

**Proposition 3.2.** *Let  $X$  be a Banach space. Then the bidual norm on  $X^{**}$  has the following property. For any  $t \in (0, 1)$ , any weak\*-null sequence  $(x_n^{**})_{n=1}^\infty$  in  $B_{X^{**}}$  and any  $x \in S_X$  we have:*

$$\limsup_{n \rightarrow \infty} \|x + tx_n^{**}\| \leq 1 + \bar{\rho}_X(t).$$

Let us now recall the following classical duality result concerning these moduli (see for instance [8] Corollary 2.3 for a precise statement).

**Proposition 3.3.** *Let  $X$  be a Banach space. Then  $\|\cdot\|_X$  is AUS if and only if  $\|\cdot\|_{X^*}$  is AUC\*.*

*If  $p, q \in (1, \infty)$  are conjugate exponents, then  $\|\cdot\|_X$  is p-AUS if and only if  $\|\cdot\|_{X^*}$  is q-AUC\*.*

We conclude this section with a list of a few classical properties of Orlicz functions and norms that are related to these moduli. A map  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is called an *Orlicz function* if it is continuous, non decreasing, convex and so that  $\varphi(0) = 0$  and  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ . The *Orlicz norm*  $\|\cdot\|_{\ell_\varphi}$ , associated with  $\varphi$  is defined on  $c_{00}$ , the space of finitely supported sequences, as follows:

$$\forall x = (x_n)_{n=1}^\infty \in c_{00}, \quad \|x\|_{\ell_\varphi} = \inf \left\{ r > 0, \sum_{n=1}^\infty \varphi(x_n/r) \leq 1 \right\}.$$

The following is immediate from the definition.

**Lemma 3.4.** *Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be an Orlicz function and  $p \in [1, \infty)$ .*

- (i) *If there exists  $C > 0$  such that  $\varphi(t) \leq Ct^p$ , for all  $t \in [0, 1]$ , then there exists  $A > 0$  such that  $\|x\|_{\ell_\varphi} \leq A\|x\|_{\ell_p}$ , for all  $x \in c_{00}$ .*
- (ii) *If there exists  $c > 0$  such that  $\varphi(t) \geq ct^p$ , for all  $t \in [0, 1]$ , then there exists  $a > 0$  such that  $\|x\|_{\ell_\varphi} \geq a\|x\|_{\ell_p}$ , for all  $x \in c_{00}$ .*

Assume now that  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is an Orlicz function which is 1-Lipschitz and such that  $\lim_{t \rightarrow \infty} \varphi(t)/t = 1$ . Consider for  $(s, t) \in \mathbb{R}^2$ ,

$$N_2^\varphi(s, t) = \begin{cases} |s| + |s|\varphi(|t|/|s|) & \text{if } s \neq 0, \\ |t| & \text{if } s = 0. \end{cases}$$

Then define by induction for all  $n \geq 3$ :

$$\forall (s_1, \dots, s_n) \in \mathbb{R}^n, N_n^\varphi(s_1, \dots, s_n) = N_2^\varphi(N_{n-1}^\varphi(s_1, \dots, s_{n-1}), s_n).$$

The following is proved in [16] (see Lemma 4.3 and its preparation).

**Lemma 3.5.**

- (i) For any  $n \geq 2$ , the function  $N_n^\varphi$  is an absolute (or lattice) norm on  $\mathbb{R}^n$ , meaning that  $N_n(s_1, \dots, s_n) \leq N_n(t_1, \dots, t_n)$ , whenever  $|s_i| \leq |t_i|$  for all  $i \leq n$ .
- (ii) For any  $n \in \mathbb{N}$  and any  $x \in \mathbb{R}^n$ :

$$\frac{1}{2} \|s\|_{\ell_\varphi} \leq N_n^\varphi(s) \leq e \|s\|_{\ell_\varphi}.$$

When  $X$  is a Banach space, it is easy to see that  $\bar{\rho}_X$  is a 1-Lipschitz Orlicz function such that  $\lim_{t \rightarrow \infty} \rho(t)/t = 1$ . But due to its lack of convexity,  $\bar{\delta}_X^*$  is not an Orlicz function and we need to modify it. Following [16], we define

$$\delta(t) = \int_0^t \frac{\bar{\delta}_X^*(s)}{s} ds.$$

It is easy to see that  $\bar{\delta}_X^*(t)/t$  is increasing and tends to 1 as  $t$  tends to  $\infty$ . Therefore,  $\delta$  is an Orlicz function which is 1-Lipschitz, such that  $\lim_{t \rightarrow \infty} \delta(t)/t = 1$  and satisfying:

$$\forall t \in [0, \infty), \quad \bar{\delta}_X^*(t/2) \leq \delta(t) \leq \bar{\delta}_X^*(t).$$

The following statement is now a direct consequence of Lemmas 3.4 and 3.5.

**Lemma 3.6.** *Let  $X$  be a Banach space and  $p \in [1, \infty)$ .*

- (i) *If there exists  $C > 0$  such that  $\bar{\rho}_X(x) \leq Ct^p$ , for all  $t \in [0, 1]$ , then there exists  $A > 0$  such that*

$$\forall n \in \mathbb{N} \quad \forall x \in \mathbb{R}^n, \quad N_n^{\bar{\rho}_X}(x) \leq A \|x\|_{\ell_p^n}.$$

- (ii) *If there exists  $c > 0$  such that  $\bar{\delta}_X^*(t) \geq ct^p$ , for all  $t \in [0, 1]$ , then there exists  $a > 0$  such that*

$$\forall n \in \mathbb{N} \quad \forall x \in \mathbb{R}^n, \quad N_n^\delta(x) \geq a \|x\|_{\ell_p^n}.$$

We will also use the following reformulation of Propositions 3.1 and 3.2 in terms of the norms  $N_2^\delta$  and  $N_2^{\bar{\rho}_X}$ .

**Lemma 3.7.** *Let  $X$  be a Banach space.*

- (i) *Let  $(x_n^*) \subset X^*$  be weak\*-null. Then for any  $x^* \in X^*$  we have*

$$\limsup_{n \rightarrow \infty} \|x^* + x_n^*\| \geq N_2^\delta(\|x^*\|, \limsup \|x_n^*\|).$$

- (ii) *Similarly, if  $(x_n^{**}) \subset X^{**}$  is weak\*-null and  $x \in X$ , then*

$$\liminf_{n \rightarrow \infty} \|x + x_n^{**}\| \leq N_2^{\bar{\rho}_X}(\|x\|, \liminf \|x_n^{**}\|).$$



*Proof.* If  $x^* = 0$  there is nothing to do, so we may assume that  $x^* \neq 0$ . By application of Proposition 3.1 we see that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|x^* + x_n^*\| &\geq \|x^*\| \left( 1 + \bar{\delta}_X \left( \frac{\limsup \|x_n^*\|}{\|x^*\|} \right) \right) \\ &\geq \|x^*\| \left( 1 + \delta \left( \frac{\limsup \|x_n^*\|}{\|x^*\|} \right) \right) = N_2^\delta(\|x^*\|, \limsup \|x_n^*\|) \end{aligned}$$

The proof of the second claim is even simpler so we leave it to the reader.  $\square$

#### 4. THE GENERAL RESULT

Let us first recall that a Banach space is said to be *quasi-reflexive* if the image of its canonical embedding into its bidual is of finite codimension in its bidual. We can now state our main result.

**Theorem 4.1.** *Let  $X$  be a quasi-reflexive Banach space, let  $p \in (1, \infty)$  and denote  $q$  its conjugate exponent. Assume that  $X$  admits an equivalent  $p$ -AUS norm and that  $X^*$  admits an equivalent  $q$ -AUS norm. Then the family  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$  does not equi-coarsely embed into  $X^{**}$ .*

We immediately deduce the following.

**Corollary 4.2.** *Let  $X$  be a quasi-reflexive Banach space, let  $p \in (1, \infty)$  and denote  $q$  its conjugate exponent. Assume that  $X$  admits an equivalent  $p$ -AUS norm and that  $X^*$  admits an equivalent  $q$ -AUS norm. Then the family  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$  does not equi-coarsely embed into  $X$ , nor does it equi-coarsely embed into any iterated dual  $X^{(r)}$  ( $r \geq 0$ ) of  $X$ .*

*Proof.* Since  $X$  is quasi reflexive we infer that  $X^{(r)}$  admits an equivalent  $p$ -AUS norm when  $r$  is even and it admits an equivalent  $q$ -AUS norm when  $r$  is odd. Indeed, note that when  $r$  is even  $X^{(r)}$  is isomorphic to  $X \oplus_p F$  where  $F$  is finite-dimensional (resp.  $X^{(r)} \simeq X^* \oplus_q F$  when  $r$  is odd). Now it is obvious from Theorem 4.1 that  $([\mathbb{N}]^k)_{k \in \mathbb{N}}$  do not equi-coarsely embed into  $X^{(r)}$  when  $r$  is even. When  $r$  is odd, we just exchange the roles of  $p$  and  $q$ .  $\square$

Before going into the detailed proof of Theorem 4.1 let us briefly indicate the main idea. We assume that there is an equi-coarse family of embeddings  $(f_k)$  of  $[\mathbb{N}]^k$  into  $X^{**}$  with moduli  $\rho$  and  $\omega$ . We fix  $k$  sufficiently large and observe that, up to passing to a subgraph,  $f_k$  can be represented as the sum along the branches of a weak\*-null countably branching tree of height  $k$ , say  $(z_{\bar{n}})_{\bar{n} \in [\mathbb{N}]^{\leq k}}$ . Moreover the norms of the elements of this tree stabilize on each level towards values  $(K_i)_{i=1}^k \subset [0, \omega(1)]$ . Applying the existence of a  $q$ -AUS norm on  $X^*$  one can show that  $\sum_{i=1}^k K_i^p \leq c^p \omega(1)^p$  where  $c$  is a constant depending only on  $X$ . The benefit of this observation is twofold. On one hand we will be able to construct two elements  $\bar{n}_0, \bar{m}_0 \in [\mathbb{N}]^l$  (with  $l \leq k$ ) such that  $\sum_{i=1}^l z_{(n_1, \dots, n_i)} - z_{(m_1, \dots, m_i)}$  is small in norm (say less than  $2c\omega(1)$ ) while  $d_{\mathbb{K}}^l(\bar{n}_0, \bar{m}_0)$  is large (say  $\rho(d_{\mathbb{K}}^l(n_0, \bar{m}_0)) > 3c\omega(1)$ ). On the other hand the  $p$ -AUS renormability of  $X$  together with the quasi-reflexivity allows

to extend these elements to elements  $\bar{n}, \bar{m} \in [\mathbb{N}]^k$  such that  $d_{\mathbb{K}}^k(\bar{n}, \bar{m})$  is still large and

$$\begin{aligned} \left\| \sum_{i=l+1}^k z_{(n_1, \dots, n_i)} - z_{(m_1, \dots, m_i)} \right\| &\sim \left( \sum_{i=l+1}^k \|z_{(n_1, \dots, n_i)} - z_{(m_1, \dots, m_i)}\|^p \right)^{1/p} \\ &\sim \left( \sum_{i=l+1}^k K_i^p \right)^{1/p} \leq c\omega(1). \end{aligned}$$

Eventually, summing the tree from 1 to  $k$  over the branches ending by  $\bar{n}$  and  $\bar{m}$  we get the desired contradiction

$$3c\omega(1) < \rho(d_{\mathbb{K}}^k(\bar{n}, \bar{m})) \leq \|f_k(\bar{n}) - f_k(\bar{m})\| \leq 3c\omega(1).$$

*Proof of Theorem 4.1.* Let us assume that there are two maps  $\rho, \omega: [0, +\infty) \rightarrow [0, +\infty)$  and maps  $f_k([\mathbb{N}]^k, d_{\mathbb{K}}^k): \rightarrow (X^{**}, \|\cdot\|)$  for  $k \in \mathbb{N}$  such that:

- (i)  $\lim_{t \rightarrow \infty} \rho(t) = \infty$ ,
- (ii)  $\omega(t) < \infty$  for every  $t \in (0, +\infty)$ ,
- (iii)  $\rho(t) \leq \rho_{f_k}(t)$  and  $\omega_{f_k}(t) \leq \omega(t)$  for every  $k \in \mathbb{N}$  and  $t \in (0, \infty)$ .

Note that all  $f_k$ 's are  $\omega(1)$ -Lipschitz for  $\|\cdot\|$  and so  $\omega(1) > 0$ . Since all the sets  $[\mathbb{N}]^k$  are countable, we may and will assume that  $X$  and therefore, by the quasi-reflexivity of  $X$ , that all its iterated duals are separable.

Let us fix  $N \in \mathbb{N}$ . Pick  $\alpha \in \mathbb{N}$  such that  $\alpha \geq \frac{p}{q}$  and set  $k = N^{1+\alpha} \in \mathbb{N}$ . We also fix  $\eta > 0$ . We shall provide at the end of our proof a contradiction if  $N$  is chosen large enough and  $\eta$  small enough. We denote  $\|\cdot\|$  the original norm on  $X$ , as well as its dual and bidual norms. Let us assume, as we may, that  $\|\cdot\|$  is  $p$ -AUS on  $X$ . We denote its modulus of asymptotic uniform smoothness  $\bar{\rho}_{\|\cdot\|}$  or simply  $\bar{\rho}_X$ .

For the first step of the proof we shall exploit the existence of an equivalent  $q$ -AUS norm  $|\cdot|$  on  $X^*$  (we also denote  $|\cdot|$  its dual norm on  $X^{**}$ ). It is worth mentioning that if  $X$  is not reflexive,  $|\cdot|$  cannot be the dual norm of an equivalent norm on  $X$  (see for instance Proposition 2.6 in [7]). Assume also that there exists  $b > 0$  such that

$$(4.1) \quad \forall z \in X^{**} \quad b\|z\| \leq |z| \leq \|z\|.$$

Then we have that all  $f_k$ 's are also  $\omega(1)$ -Lipschitz for  $|\cdot|$ .

By Proposition 3.3, we have that there exists  $c > 0$  such that for all  $t \in [0, 1]$ ,  $\bar{\delta}_{|\cdot|}^*(t) \geq ct^p$ . We denote again

$$\delta(t) = \int_0^t \frac{\bar{\delta}_{|\cdot|}^*(s)}{s} ds.$$

Recall that Lemma 3.6 ensures the existence of  $a > 0$  such that for all  $n \in \mathbb{N}$ ,  $N_n^\delta \geq 2a\|\cdot\|_{\ell_p^n}$ .

First, using the separability of  $X^*$  and Proposition 2.8, we may assume by passing to a full subtree, that there exist a weak\*-null tree  $(z(\bar{m}))_{\bar{m} \in [\mathbb{N}]^{\leq k}}$  in  $X^{**}$  with  $|z_{\bar{m}}| \leq \omega(1)$  for all  $\bar{m} \in [\mathbb{N}]^{\leq k} \setminus \{\emptyset\}$  and so that

$$\forall \bar{n} \in [\mathbb{N}]^k, \quad f_k(\bar{n}) = \sum_{i=0}^k z(n_1, \dots, n_i) = \sum_{\bar{m} \leq \bar{n}} z(\bar{m}).$$

For  $r \in \mathbb{N}$  we denote  $E_r = \{\bar{m} = (m_1, \dots, m_j) \in [\mathbb{N}]^{\leq k} \setminus \{\emptyset\}, m_j = r\}$  and  $F_r = \bigcup_{u=1}^r E_u$ . Fix a sequence  $(\lambda_r)_{r=1}^\infty$  in  $(0, 1)$  such that  $\prod_{r=1}^\infty \lambda_r > \frac{1}{2}$ . We now use Lemma 3.7 (i) and the fact that  $(z(\bar{m}))_{\bar{m} \in [\mathbb{N}]^{\leq k}}$  is a weak\*-null tree to build inductively  $n_1 < \dots < n_r$  so that for all  $\bar{n}^1, \dots, \bar{n}^L \in F_{n_r-1}$ , for all  $\varepsilon_1, \dots, \varepsilon_L \in \{-1, 1\}$  and all  $\bar{n} \in E_{n_r}$ , we have

$$\left| z(\bar{n}) + \sum_{l=1}^L \varepsilon_l z(\bar{n}^l) \right| \geq \lambda_r N_2^\delta \left( \left| \sum_{l=1}^L \varepsilon_l z(\bar{n}^l) \right|, |z(\bar{n})| \right).$$

Therefore, using the fact that  $N_2^\delta$  is an absolute norm and after passing to a full subtree, we may assume that for all  $r_1 < \dots < r_L$  in  $\mathbb{N}$ , all  $\varepsilon_1, \dots, \varepsilon_L \in \{-1, 1\}$  and all  $\bar{n}^1, \dots, \bar{n}^L$  so that  $\bar{n}^l \in E_{r_l}$  for  $1 \leq l \leq L$ , we have

$$(4.2) \quad \left| \sum_{l=1}^L \varepsilon_l z(\bar{n}^l) \right| \geq \frac{1}{2} N_L^\delta (|z(\bar{n}^1)|, \dots, |z(\bar{n}^L)|) \geq a \left( \sum_{i=1}^L |z(\bar{n}^i)|^p \right)^{1/p}.$$

Assume now that  $\bar{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$  is such that  $n_1 < \dots < n_k$  are even and choose  $\bar{m} = (m_1, \dots, m_k)$  so that  $n_1 < m_1 < \dots < n_k < m_k$ . It follows from (4.2) that

$$\begin{aligned} |f(\bar{n}) - f(\bar{m})| &= \left| \sum_{i=1}^k z(n_1, \dots, n_i) - z(m_1, \dots, m_i) \right| \\ &\geq a \left( \sum_{i=1}^k |z(n_1, \dots, n_i)|^p + |z(m_1, \dots, m_i)|^p \right)^{1/p}. \end{aligned}$$

We now use the fact that  $d_{\mathbb{K}}^k(\bar{n}, \bar{m}) = 1$  and  $f$  is  $\omega(1)$ -Lipschitz, to deduce

$$\left( \sum_{i=1}^k |z(n_1, \dots, n_i)|^p \right)^{1/p} \leq \frac{1}{a} \omega(1).$$

So replacing  $\mathbb{N}$  with  $2\mathbb{N}$  and setting  $A = 1/a$ , we may assume that

$$(4.3) \quad \forall \bar{n} \in [\mathbb{N}]^k, \quad \left( \sum_{i=1}^k |z(n_1, \dots, n_i)|^p \right)^{1/p} \leq A \omega(1).$$

By Ramsey's theorem (Proposition 2.7), we may also assume, after passing again to a full subtree, that for all  $i \in \{1, \dots, k\}$  there exists  $K_i \in [0, \omega(1)]$  such that

$$\forall (n_1, \dots, n_i) \in [\mathbb{N}]^i, \quad K_i \leq |z(n_1, \dots, n_i)| \leq K_i + \eta.$$

The estimate (4.3) yields

$$(4.4) \quad \sum_{i=1}^k K_i^p \leq A^p \omega(1)^p.$$

Therefore, since  $k = N^{1+\alpha}$ , there exists  $j \in \{0, N, \dots, N(N^\alpha - 1)\}$  such that

$$\sum_{i=j+1}^{j+N} K_i^p \leq \frac{A^p \omega(1)^p}{N^\alpha}.$$

Then we deduce from Hölder's inequality that

$$(4.5) \quad \sum_{i=j+1}^{j+N} K_i \leq N^{1/q} \frac{A\omega(1)}{N^{\alpha/p}} \leq A\omega(1).$$

We now use the assumption that  $X$  is quasi-reflexive, so that  $X^{**} = X \oplus F$ , where  $F$  is of finite dimension. Thus, for each  $(n_1, \dots, n_i) \in [\mathbb{N}]^{\leq k}$ , we can decompose  $z(n_1, \dots, n_i) = x(n_1, \dots, n_i) + e(n_1, \dots, n_i)$ , with  $x(n_1, \dots, n_i) \in X$  and  $e(n_1, \dots, n_i) \in F$ . Then, the compactness of bounded sets in  $F$  and another application of Proposition 2.7 allows us to assume, after passing to a full subtree, that

$$\forall i \in \{1, \dots, k\} \quad \forall \bar{n}, \bar{v} \in [\mathbb{N}]^i, \quad \|e(\bar{n}) - e(\bar{v})\| < \eta,$$

Which implies that for all  $i \in \{1, \dots, k\}$  and all  $\bar{n}, \bar{v} \in [\mathbb{N}]^i$  we have

$$(4.6) \quad \left| \|z(\bar{n}) - z(\bar{v})\| - \|x(\bar{n}) - x(\bar{v})\| \right| < \eta.$$

We are now ready for the last step of the proof, where we shall build  $\bar{m}$  and  $\bar{u}$  in  $[\mathbb{N}]^k$  so that  $d_{\mathbb{K}}^k(\bar{m}, \bar{u}) = N$ , but  $|f(\bar{m}) - f(\bar{u})|$  is bounded by a constant depending only on  $\omega(1)$  and on  $X$ . This will yield a contradiction with the fact  $\lim_{N \rightarrow \infty} \rho(N) = \infty$ .

First, we set  $m_i = u_i = i$ , for all  $1 \leq i \leq j$ . Then, for  $j+1 \leq i \leq j+N$ , we set  $m_i = i$  and  $u_i = i+N$ . Finally, we shall build  $m_i = u_i$  inductively, for  $j+N < i \leq k$ . Note, that when this will be done, we will indeed have  $d_{\mathbb{K}}^k(\bar{m}, \bar{u}) = N$ .

First, we obviously have

$$(4.7) \quad \sum_{i=1}^j z(m_1, \dots, m_i) - z(u_1, \dots, u_i) = 0.$$

The next estimate follows from (4.5).

$$(4.8) \quad \left| \sum_{i=j+1}^{j+N} z(m_1, \dots, m_i) - z(u_1, \dots, u_i) \right| \leq \sum_{i=j+1}^{j+N} 2(K_i + \eta) \leq 3A\omega(1),$$

if  $\eta$  was initially chosen small enough.

We now select the remaining coordinates of  $\bar{m}$  and  $\bar{u}$  inductively using the fact that  $\| \cdot \|$  is  $p$ -AUS. To shorten the notation for the end of the proof, we shall now denote  $x_i = x(m_1, \dots, m_i)$ ,  $z_i = z(m_1, \dots, m_i)$ ,  $x'_i = x(u_1, \dots, u_i)$  and  $z'_i = z(u_1, \dots, u_i)$ . First, we simply set  $m_{j+N+1} = u_{j+N+1} = j+2N+1$ . We now use the fact that the tree  $(z(\bar{m}))_{\bar{m} \in [\mathbb{N}]^{\leq k}}$  is weak\*-null and Lemma 3.7 (ii) to find  $m_{j+N+2} = u_{j+N+2} > j+2N+1$  such that

$$\begin{aligned} & \|x_{j+N+1} - x'_{j+N+1} + z_{j+N+2} - z'_{j+N+2}\| \\ & \leq N_2^{\bar{p}X} (\|x_{j+N+1} - x'_{j+N+1}\|, \|z_{j+N+2} - z'_{j+N+2}\|) + \eta \end{aligned}$$

It follows from (4.6) that

$$\begin{aligned} & \|z_{j+N+1} - z'_{j+N+1} + z_{j+N+2} - z'_{j+N+2}\| \\ & \leq N_2^{\bar{p}X} (\|z_{j+N+1} - z'_{j+N+1}\| + \eta, \|z_{j+N+2} - z'_{j+N+2}\|) + 2\eta \\ & \leq N_2^{\bar{p}X} \left( \frac{2}{b} (K_{j+N+1} + \eta) + \eta, \frac{2}{b} (K_{j+N+2} + \eta) \right) + 2\eta. \end{aligned}$$

Similarly, we can inductively find  $m_{j+N+2} = u_{j+N+2} < \dots < m_k = u_k$  such that,

$$\left\| \sum_{i=j+N+1}^k (z_i - z'_i) \right\| \leq \frac{2}{b} N_{k-j-N}^{\bar{\rho}_X} (K_{j+N+1}, \dots, K_k) + \omega(1)$$

provided  $\eta$  is chosen small enough. Since Lemma 3.6 ensures the existence of  $C > 0$  such that  $N_n^{\bar{\rho}_X} \leq C \| \ell_n^p \|$  for all  $n \in \mathbb{N}$  the above inequality yields

$$\left\| \sum_{i=j+N+1}^k (z_i - z'_i) \right\| \leq \frac{2C}{b} \left( \sum_{i=j+N+1}^k K_i^p \right)^{1/p} + \omega(1) \leq \left( \frac{2CA}{b} + 1 \right) \omega(1).$$

Finally, combining the above estimate with (4.7) and (4.8), we get that

$$\|f(\bar{m}) - f(\bar{u})\| \leq \frac{3A + 2CA + b}{b} \omega(1).$$

As announced at the beginning of the proof, this yields a contradiction if  $N$  was initially chosen, as it was possible, so that  $\rho(N) > \frac{3A+2CA+b}{b} \omega(1)$ .  $\square$

Unlike reflexivity, quasi-reflexivity itself is not enough to prevent the Kalton graphs from embedding into a Banach space. We thank P. Motakis for showing us the next example.

**Proposition 4.3** (Motakis). *There exists a quasi-reflexive Banach space  $X$  such that the family of graphs  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$  equi-Lipschitz embeds into  $X$ .*

*Proof.* The proof relies on the existence of a quasi-reflexive Banach space  $X$  of order one which admits a spreading model, generated by a basis of  $X$  that is equivalent to the summing basis  $(s_n)_{n=1}^{\infty}$  of  $c_0$ . This is shown in [9] (Proposition 3.2) and based on a construction given in [6]. We refer the reader to [5] for the necessary definitions. Consequently, there exists a sequence  $(x_n)_{n=1}^{\infty}$  in  $S_X$  and constants  $A, B > 0$  such that for all  $k \leq n_1 < \dots < n_k$  and all  $\varepsilon_1, \dots, \varepsilon_k$  in  $\{-1, 0, 1\}$  one has

$$(4.9) \quad A \left\| \sum_{i=1}^k \varepsilon_i s_i \right\|_{c_0} \leq \left\| \sum_{i=1}^k \varepsilon_i x_{n_i} \right\|_X \leq B \left\| \sum_{i=1}^k \varepsilon_i s_i \right\|_{c_0}.$$

For  $k \in \mathbb{N}$  and  $\bar{n} = (n_1, \dots, n_k) \in [\mathbb{N}]^k$  we define

$$g_k(\bar{n}) = \sum_{i=1}^k x_{2k+n_i}.$$

It follows easily from Proposition 2.5, the inequality (4.9) and the fact that  $(s_n)_{n=1}^{\infty}$  is a spreading sequence that

$$\frac{A}{2} d_{\mathbb{K}}^k(\bar{n}, \bar{m}) \leq \|g_k(\bar{n}) - g_k(\bar{m})\|_X \leq B d_{\mathbb{K}}^k(\bar{n}, \bar{m})$$

for all  $\bar{n}, \bar{m} \in [\mathbb{N}]^k$ .  $\square$

**Remark 4.4.** Let us mention that, more generally, it is proved in [2] that for any conditional normalized spreading sequence  $(e_n)_{n=1}^{\infty}$ , there exists a quasi-reflexive Banach space  $X$  of order 1 with a normalized basis  $(x_i)_{i=1}^{\infty}$  which generates  $(e_n)_{n=1}^{\infty}$  as a spreading model.

## 5. THE JAMES SEQUENCE SPACES

Let  $p \in (1, \infty)$ . We now recall the definition and some basic properties of the James space  $\mathcal{J}_p$ . We refer the reader to [1](Section 3.4) and references therein for more details on the classical case  $p = 2$ . The James space  $\mathcal{J}_p$  is the real Banach space of all sequences  $x = (x(n))_{n \in \mathbb{N}}$  of real numbers with finite  $p$ -variation and verifying  $\lim_{n \rightarrow \infty} x(n) = 0$ . The space  $\mathcal{J}_p$  is endowed with the following norm

$$\|x\|_{\mathcal{J}_p} = \sup \left\{ \left( \sum_{i=1}^{k-1} |x(p_{i+1}) - x(p_i)|^p \right)^{1/p} : 1 \leq p_1 < p_2 < \dots < p_k \right\}.$$

This is the historical example, constructed for  $p = 2$  by R.C. James in [11], of a quasi-reflexive Banach space which is isomorphic to its bidual. In fact  $\mathcal{J}_p^{**}$  can be seen as the space of all sequences  $x = (x(n))_{n \in \mathbb{N}}$  of real numbers with finite  $p$ -variation, which is  $\mathcal{J}_p \oplus \mathbb{R}\mathbf{1}$ , where  $\mathbf{1}$  denotes the constant sequence equal to 1.

The standard unit vector basis  $(e_n)_{n=1}^{\infty}$  ( $e_n(i) = 1$  if  $i = n$  and  $e_n(i) = 0$  otherwise) is a monotone shrinking basis for  $\mathcal{J}_p$ . Hence, the sequence  $(e_n^*)_{n=1}^{\infty}$  of the associated coordinate functionals is a basis of its dual  $\mathcal{J}_p^*$ . Then the weak\* topology  $\sigma(\mathcal{J}_p^*, \mathcal{J}_p)$  is easy to describe. A sequence  $(x_n^*)_{n=1}^{\infty}$  in  $\mathcal{J}_p^*$  converges to 0 in the  $\sigma(\mathcal{J}_p^*, \mathcal{J}_p)$  topology if and only if it is bounded and  $\lim_{n \rightarrow \infty} x_n^*(i) = 0$  for every  $i \in \mathbb{N}$ .

For  $x \in \mathcal{J}_p$ , we define  $\text{supp } x = \{i \in \mathbb{N} : x(i) \neq 0\}$ . For  $x, y \in \mathcal{J}_p$ , we denote:  $x \prec y$  whenever  $\max \text{supp } x < \min \text{supp } y$ .

Similarly, an element  $x^*$  of  $\mathcal{J}_p^*$  will be written  $x^* = \sum_{n=1}^{\infty} x^*(n)e_n^*$  and  $\text{supp } x^* = \{i \in \mathbb{N} : x^*(i) \neq 0\}$  and we shall denote  $x^* \prec y^*$  whenever  $\max \text{supp } x^* < \min \text{supp } y^*$ .

The detailed proof of the following proposition can be found in [20] (Proposition 2.3). This a consequence of the following fact: there exists  $C \geq 1$  such that  $\|\sum_{i=1}^n x_i\|_{\mathcal{J}_p}^p \leq C \sum_{i=1}^n \|x_i\|_{\mathcal{J}_p}^p$ , for all  $x_1 \prec \dots \prec x_n$  in  $\mathcal{J}_p$ .

**Proposition 5.1.** *There exists an equivalent norm  $|\cdot|$  on  $\mathcal{J}_p$  such that its dual norm  $|\cdot|_*$  has the following property. For any  $x^*, y^* \in \mathcal{J}_p^*$  such that  $x^* \prec y^*$ , we have that*

$$|x^* + y^*|_*^q \geq |x^*|_*^q + |y^*|_*^q.$$

*In particular,  $|\cdot|_*$  is  $q$ -AUC\* for the weak\* topology induced by  $\mathcal{J}_p$  and therefore  $|\cdot|$  is  $p$ -AUS on  $\mathcal{J}_p$ .*

There is also a natural weak\* topology on  $\mathcal{J}_p$ . Indeed, the summing basis  $(s_n)_{n=1}^{\infty}$  ( $s_n(i) = 1$  if  $i \leq n$  and  $s_n(i) = 0$  otherwise) is a monotone and boundedly complete basis for  $\mathcal{J}_p$ . Thus,  $\mathcal{J}_p$  is naturally isometric to a dual Banach space:  $\mathcal{J}_p = X^*$  with  $X$  being the closed linear span of the biorthogonal functionals  $(c_n^* - c_{n+1}^*)_{n=1}^{\infty}$  in  $\mathcal{J}_p^*$  associated with  $(s_n)_{n=1}^{\infty}$ . Note that  $X = \{x^* \in \mathcal{J}_p^*, \sum_{n=1}^{\infty} x^*(n) = 0\}$ . Thus, a sequence  $(x_n)_{n=1}^{\infty}$  in  $\mathcal{J}_p$  converges to 0 in the  $\sigma(\mathcal{J}_p, X)$  topology if and only if it is bounded and  $\lim_{n \rightarrow \infty} (x_n(i) - x_n(j)) = 0$  for every  $i \neq j \in \mathbb{N}$ . The next proposition is easy (see Proposition 2.3 in [17] for the case  $p = 2$ ).

**Proposition 5.2.** *The usual norm on  $\mathcal{J}_p$  is  $p$ -AUC\* for the weak\* topology induced by  $X$ . In other words, the restriction to  $X$  of the usual norm on  $\mathcal{J}_p^*$  is  $q$ -AUS.*

Then, since  $X$  is one codimensional in  $\mathcal{J}_p^*$ , we have that  $\mathcal{J}_p^*$  is isomorphic to  $X \oplus \mathbb{R}$  and therefore also admits an equivalent  $q$ -AUS norm.

The above remarks combined with Corollary 4.2 immediately yield the following.

**Corollary 5.3.** *Let  $p \in (1, \infty)$ . Then, the family  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$  does not equi-coarsely embed into  $\mathcal{J}_p$ , nor does it equi-coarsely embed into  $\mathcal{J}_p^*$ .*

## 6. A REMARK ON THE JAMES TREE SPACE

Let us recall the construction of the James tree space  $\mathcal{JT}$ . We denote  $T = 2^{<\omega}$  the tree of all finite sequences with coefficients in  $\{0, 1\}$  equipped with its natural order: for  $s, t \in T$ , we say that  $s \leq t$  if the sequence  $t$  extends  $s$ . The set of all infinite sequences with coefficients in  $\{0, 1\}$  will be denoted  $2^\omega$ . For  $s \in T$ , the length of  $s$  is denoted  $|s|$ . We call *segment* of  $T$  any set of the form  $\{s \in T, t \leq s \leq t'\}$  with  $t \leq t'$  in  $T$ . For a map  $x : T \rightarrow \mathbb{R}$ , we define

$$\|x\|_{\mathcal{JT}} = \sup \left\{ \left( \sum_{i=1}^n \left( \sum_{s \in S_i} x(s) \right)^2 \right)^{1/2} \right\},$$

where the supremum is taken over all pairwise disjoint segments  $S_1, \dots, S_n$  of  $T$ . Then the *James tree space* is the space  $\mathcal{JT} = \{x : T \rightarrow \mathbb{R}, \|x\|_{\mathcal{JT}} < \infty\}$  equipped with the norm  $\|\cdot\|_{\mathcal{JT}}$ . For  $s \in T$ , we denote  $e_s : T \rightarrow \mathbb{R}$  defined by  $e_s(t) = \delta_{s,t}$ ,  $t \in T$ . If  $\psi : \mathbb{N} \rightarrow T$  is a bijection such that  $|\psi(n)| \leq |\psi(m)|$  whenever  $n \leq m$ , then  $(e_{\psi(n)})_{n=1}^\infty$  is a normalized, monotone and boundedly complete basis of  $\mathcal{JT}$ . For  $s \in T$ , the coordinate functional  $e_s^*$  is defined by  $e_s^*(x) = x(s)$ ,  $x \in \mathcal{JT}$ . Then the closed linear span of  $\{e_s^*, s \in T\}$  in  $\mathcal{JT}^*$  is denoted  $\mathcal{B}$  and  $\mathcal{B}^*$  is isometric to  $\mathcal{JT}$ . The space  $\mathcal{JT}$  was built by R.C. James in [12] to serve as the first example of a separable Banach space with non separable dual, which does not contain an isomorphic copy of  $\ell_1$ .

In [14] it is shown that if a Banach space  $X$  coarsely contains  $c_0$  then there exists  $k \in \mathbb{N}$  such that  $X^{(k)}$ , the dual of order  $k$  of  $X$ , is non separable. A close look at the proof of Theorem 3.5 in [14] allows to state the following.

**Theorem 6.1** (Kalton). *Let  $X$  and  $Y$  be two Banach spaces such that  $X$  coarsely embeds into  $Y$ . Assume moreover that there exist an uncountable set  $I$  and for every  $i \in I$  and  $k \in \mathbb{N}$ , a 1-Lipschitz map  $f_i^k : ([\mathbb{N}]^k, d_{\mathbb{K}}^k) \rightarrow X$  such that*

$$\lim_{k \rightarrow \infty} \inf_{i \neq j \in I} \inf_{M \in [\mathbb{N}]^\omega} \sup_{\bar{n} \in [M]^k} \|f_i^k(\bar{n}) - f_j^k(\bar{n})\| = \infty.$$

*Then there exists  $r \in \mathbb{N}$  such that  $Y^{(r)}$  is not separable.*

As an application, we can show the following.

**Theorem 6.2.** *Let  $Y$  be a Banach space such that  $\mathcal{B}$  or  $\mathcal{JT}$  coarsely embeds into  $Y$ . Then there exists  $r \in \mathbb{N}$  such that  $Y^{(r)}$  is not separable.*

*Proof.* For  $\sigma \in 2^\omega$ , we denote  $\sigma|_n = (\sigma_1, \dots, \sigma_n)$ . Then, for  $k \in \mathbb{N}$ , we define  $f_\sigma^k : [\mathbb{N}]^k \rightarrow \mathcal{B}$  as follows. For  $\bar{n} = (n_1, \dots, n_k) \in [\mathbb{N}]^k$  let

$$f_\sigma^k(\bar{n}) = \frac{1}{\sqrt{k}} \sum_{i=1}^k \sum_{s \leq \sigma|_{n_i}} e_s^*.$$

Assume for instance that  $n_1 \leq m_1 \leq \dots \leq n_k \leq m_k$ . Then we can write

$$f_\sigma^k(\bar{m}) - f_\sigma^k(\bar{n}) = \frac{1}{\sqrt{k}} \sum_{i=1}^k \sum_{s \in S_i} e_s^*,$$

where  $S_1, \dots, S_k$  are pairwise disjoint segments in  $T$ . Note that for any segment  $S_i$  the sum  $\sum_{s \in S_i} e_s^*$  belongs to the unit ball of  $\mathcal{JT}^*$ . It then follows from the Cauchy-Schwarz inequality that  $f_\sigma^k$  is 1-Lipschitz on  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)$ . Assume now that  $\sigma \neq \tau \in 2^\omega$ . Pick  $r \in \mathbb{N}$  such that  $\sigma_r \neq \tau_r$ . Then for any  $\bar{m} \in [\mathbb{N}]^\omega$  and any  $\bar{n} = (n_1, \dots, n_k) \in [\mathbb{M}]^k$  with  $n_1 \geq r$ , we have

$$\|f_\sigma^k(\bar{n}) - f_\tau^k(\bar{n})\|_{\mathcal{B}} \geq \left| \langle f_\sigma^k(\bar{n}) - f_\tau^k(\bar{n}), e_{\sigma|_{n_1}} \rangle \right| \geq \sqrt{k}.$$

By Theorem 6.1 and the uncountability of  $2^\omega$ , this finishes our proof for  $\mathcal{B}$ .

For  $\sigma \in 2^\omega$  and  $k \in \mathbb{N}$  define now  $g_\sigma^k : [\mathbb{N}]^k \rightarrow \mathcal{JT}$  by

$$\forall \bar{n} = (n_1, \dots, n_k) \in [\mathbb{N}]^k, \quad g_\sigma^k(\bar{n}) = \frac{1}{\sqrt{2k}} \sum_{i=1}^k e_{\sigma|_{n_i}}.$$

It is easily checked that  $g_\sigma^k$  is 1-Lipschitz on  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)$ . Assume now that  $\sigma \neq \tau \in 2^\omega$ . Pick  $r \in \mathbb{N}$  such that  $\sigma_r \neq \tau_r$ . Then for any  $\bar{m} \in [\mathbb{N}]^\omega$  and any  $\bar{n} = (n_1, \dots, n_k) \in [\mathbb{M}]^k$  with  $n_1 \geq r$ , denote  $S = \{s \in T, \sigma|_{n_1} \leq s \leq \sigma|_{n_k}\}$ . The set  $S$  is a segment in  $T$  and  $x^* = \sum_{s \in S} e_s^*$  is in the unit ball of  $\mathcal{JT}^*$ . Therefore

$$\|g_\sigma^k(\bar{n}) - g_\tau^k(\bar{n})\|_{\mathcal{JT}} \geq \langle g_\sigma^k(\bar{n}) - g_\tau^k(\bar{n}), x^* \rangle \geq \frac{\sqrt{k}}{\sqrt{2}}.$$

This concludes our proof for  $\mathcal{JT}$ .  $\square$

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