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ON THE COARSE GEOMETRY OF JAMES SPACES

G. LANCIEN, C. PETITJEAN, AND A. PROCHÁZKA

ABSTRACT. In this note we prove that the Kalton interlaced graphs do not equi-coarsely embed into the James space \mathcal{J} nor into its dual \mathcal{J}^* . It is a particular case of a more general result on the non equi-coarse embeddability of the Kalton graphs into quasi-reflexive spaces with a special asymptotic structure. This allows us to exhibit a coarse invariant for Banach spaces, namely the non equi-coarse embeddability of this family of graphs, which is very close to but different from the celebrated property \mathcal{Q} of Kalton. We conclude with a remark on the coarse geometry of the James tree space \mathcal{JT} and of its predual.

1. INTRODUCTION

In a fundamental paper on the coarse geometry of Banach spaces ([14]), N. Kalton introduced a property of metric spaces that he named property \mathcal{Q} . In particular, its absence served as an obstruction to coarse embeddability into reflexive Banach spaces. This property is related to the behavior of Lipschitz maps defined on a particular family of metric graphs that we shall denote $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$. We will recall the precise definitions of these graphs and of property \mathcal{Q} in section 2.2. Let us just say, vaguely speaking for the moment, that a Banach space X has property \mathcal{Q} if for every Lipschitz map f from $([\mathbb{N}]^k, d_{\mathbb{K}}^k)$ to X , there exists a full subgraph $[\mathbb{M}]^k$ of $[\mathbb{N}]^k$, with \mathbb{M} infinite subset of \mathbb{N} , on which f satisfies a strong concentration phenomenon. It is then easy to see that if a Banach space X has property \mathcal{Q} , then the family of graphs $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$ does not equi-coarsely embed into X (see the definition in section 2.1). One of the main results in [14] is that any reflexive Banach space has property \mathcal{Q} . It then readily follows that a reflexive Banach space cannot contain a coarse copy of all separable metric spaces, or equivalently does not contain a coarse copy of the Banach space c_0 . In fact, with a sophistication of this argument, Kalton proved an even stronger result in [14]: if a separable Banach space contains a coarse copy of c_0 , then there is an integer k such that the dual of order k of X is non separable. In particular, a quasi-reflexive Banach space does not contain a coarse copy of c_0 . However, Kalton proved that the most famous example of a quasi-reflexive space, namely the James space \mathcal{J} , as well as its dual \mathcal{J}^* , fail property \mathcal{Q} .

The main purpose of this paper is to show that, although they do not obey the concentration phenomenon described by property \mathcal{Q} , neither \mathcal{J} nor \mathcal{J}^* equi-coarsely contains the family of graphs $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$ (Corollary

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5.3). This provides a coarse invariant, namely “not containing equi-coarsely the Kalton graphs”, that is very close to but different from property \mathcal{Q} . This could allow to find obstructions to coarse embeddability between seemingly close Banach spaces. Our result is actually more general. We prove in Theorem 4.1 that a quasi-reflexive Banach space X such that both X and X^* admit an equivalent p -asymptotically uniformly smooth norm (see the definition in section 3), for some p in $(1, \infty)$, does not equi-coarsely contain the Kalton graphs.

We conclude this note by showing that if the James tree space \mathcal{JT} or its predual coarsely embeds into a separable Banach space X , then there exists $k \in \mathbb{N}$ so that the dual of order k of X is non separable. This extends slightly Theorem 3.5 in [14].

2. METRIC NOTIONS

2.1. Coarse embeddings. Let M, N be two metric spaces and $f: M \rightarrow N$ be a map. We define the compression modulus ρ_f and the expansion modulus ω_f as follows. For $t \in [0, \infty)$, we set

$$\begin{aligned}\rho_f(t) &= \inf\{d_N(f(x), f(y)) : d_M(x, y) \geq t\}, \\ \omega_f(t) &= \sup\{d_N(f(x), f(y)) : d_M(x, y) \leq t\}.\end{aligned}$$

We adopt the convention $\sup(\emptyset) = 0$ and $\inf(\emptyset) = \infty$. Note that for every $x, y \in M$,

$$\rho_f(d_M(x, y)) \leq d_N(f(x), f(y)) \leq \omega_f(d_M(x, y)).$$

We say that f is a *coarse embedding* if $\omega_f(t) < \infty$ for every $t \in [0, +\infty)$ and $\lim_{t \rightarrow \infty} \rho_f(t) = \infty$.

Next, let $(M_i)_{i \in I}$ be a family of metric spaces. We say that the family $(M_i)_{i \in I}$ *equi-coarsely embeds* into a metric space N if there exist two maps $\rho, \omega: [0, +\infty) \rightarrow [0, +\infty)$ and maps $f_i: M_i \rightarrow N$ for $i \in I$ such that:

- (i) $\lim_{t \rightarrow \infty} \rho(t) = \infty$,
- (ii) $\omega(t) < \infty$ for every $t \in [0, +\infty)$,
- (iii) $\rho(t) \leq \rho_{f_i}(t)$ and $\omega_{f_i}(t) \leq \omega(t)$ for every $i \in I$ and $t \in [0, \infty)$.

2.2. The Kalton interlaced graphs and property \mathbf{Q} . For $k \in \mathbb{N}$ and \mathbb{M} an infinite subset of \mathbb{N} , we put $[\mathbb{M}]^{\leq k} = \{S \subset \mathbb{M} : |S| \leq k\}$, $[\mathbb{M}]^k = \{S \subset \mathbb{M} : |S| = k\}$, $[\mathbb{M}]^\omega = \{S \subset \mathbb{M} : S \text{ is infinite}\}$, and $[\mathbb{M}]^{<\omega} = \{S \subset \mathbb{M} : S \text{ is finite}\}$. We always list the elements of some \overline{m} in $[\mathbb{N}]^{<\omega}$ or in $[\mathbb{N}]^\omega$ in increasing order, meaning that if we write $\overline{m} = (m_1, m_2, \dots, m_l)$ or $\overline{m} = (m_1, m_2, m_3, \dots)$, we tacitly assume that $m_1 < m_2 < \dots$.

For $\overline{m} = (m_1, m_2, \dots, m_r) \in [\mathbb{N}]^{<\omega}$ and $\overline{n} = (n_1, n_2, \dots, n_s) \in [\mathbb{N}]^{<\omega}$, we write $\overline{m} \prec \overline{n}$, if $r < s \leq k$ and $m_i = n_i$, for $i = 1, 2, \dots, r$, and we write $\overline{m} \preceq \overline{n}$ if $\overline{m} \prec \overline{n}$ or $\overline{m} = \overline{n}$. Thus $\overline{m} \preceq \overline{n}$ if \overline{m} is an initial segment of \overline{n} .

Following Kalton [14], for $\mathbb{M} \in [\mathbb{N}]^\omega$, we equip $[\mathbb{M}]^k$ with a graph structure by declaring $\overline{m} \neq \overline{n} \in [\mathbb{M}]^k$ adjacent if and only if

$$n_1 \leq m_1 \leq n_2 \leq \dots \leq n_k \leq m_k \quad \text{or} \quad m_1 \leq n_1 \leq m_2 \leq \dots \leq m_k \leq n_k.$$

For any $\overline{m}, \overline{n} \in [\mathbb{M}]^k$, the distance $d_{\mathbb{K}}^k(\overline{m}, \overline{n})$ is then defined as the shortest path distance in the graph $[\mathbb{M}]^k$.

Remark 2.1. The distance $d_{\mathbb{K}}^k$ is independent of the set \mathbb{M} and therefore $[\mathbb{M}_1]^k$ is a metric subspace of $[\mathbb{M}_2]^k$ whenever $\mathbb{M}_1 \in [\mathbb{M}_2]^\omega$.

This last claim is an immediate consequence of the following explicit formula for the distance.

Proposition 2.2. *Let $k \in \mathbb{N}$ and $\mathbb{M} \in [\mathbb{N}]^\omega$. Then $d_{\mathbb{K}}^k(\bar{n}, \bar{m}) = d(\bar{n}, \bar{m})$ for all $\bar{n}, \bar{m} \in [\mathbb{M}]^k$ where $d(\bar{n}, \bar{m}) = \sup\{|\bar{n} \cap S| - |\bar{m} \cap S| : S \text{ segment of } \mathbb{N}\}$.*

Proof. It is easily seen that d is a metric on $[\mathbb{M}]^k$. Since $d_{\mathbb{K}}^k$ is a graph metric on $[\mathbb{M}]^k$, in order to show $d_{\mathbb{K}}^k = d$ it is enough to verify that $d_{\mathbb{K}}^k(\bar{n}, \bar{m}) = 1$ if and only if $d(\bar{n}, \bar{m}) = 1$ and that d is a graph metric.

For $A \subset \mathbb{N}$ let us denote $\mathbf{1}_A : \mathbb{N} \rightarrow \{0, 1\}$ the indicator function of A and let us first observe the following fact.

Fact: *For every $\bar{n}, \bar{m} \in [\mathbb{M}]^k$,*

$$d(\bar{n}, \bar{m}) = \max_i F(i) - \min_i F(i)$$

where $F(i) = F_{\bar{n}, \bar{m}}(i) = \sum_{j=1}^i \mathbf{1}_{\bar{n}}(j) - \mathbf{1}_{\bar{m}}(j)$ (and $F(0) = 0$).

Indeed, we have for any segment $S = [a, b]$ that

$$|S \cap \bar{n}| - |S \cap \bar{m}| = \sum_{j \in S} (\mathbf{1}_{\bar{n}}(j) - \mathbf{1}_{\bar{m}}(j)) = F(b) - F(a-1).$$

In particular $\max_S |S \cap \bar{n}| - |S \cap \bar{m}| \leq \max F - \min F$. On the other hand if $S = [a, b]$ is such that $\{F(a-1), F(b)\} = \{\max F, \min F\}$ then $||S \cap \bar{n}| - |S \cap \bar{m}|| \geq \max F - \min F$ which finishes the proof of the fact.

It is clear that $d_{\mathbb{K}}^k(\bar{n}, \bar{m}) = 1$ if and only if $\max F - \min F = 1$. Thus it only remains to prove that d is a graph metric. Now given \bar{n}, \bar{m} such that $d(\bar{n}, \bar{m}) \geq 2$ we are looking for $\bar{\ell} \in [\mathbb{M}]^k \setminus \{\bar{m}, \bar{n}\}$ such that $d(\bar{m}, \bar{n}) = d(\bar{n}, \bar{\ell}) + d(\bar{\ell}, \bar{m})$. Without loss of generality we will assume that $\max F_{\bar{n}, \bar{m}} > 0$. Notice that the sets $\arg \max(F)$ and $\arg \min(F)$ are disjoint. We select inductively $\{a_1 < \dots < a_p\} \subset \arg \max(F)$ and $\{b_1 < \dots < b_q\} \subset \arg \min(F)$ (with $p \geq 1$ and $q \geq 0$) with the property that

- $a_1 = \min \arg \max(F)$,
- For $i \geq 1$, $b_i = \min(\{n > a_i\} \cap \arg \min(F))$, if this is not empty.
- $a_{i+1} = \min(\{n > b_i\} \cap \arg \max(F))$, if this set is not empty.

Notice that $\{a_1, \dots, a_p\} \subset \bar{n} \setminus \bar{m}$ and $\{b_1, \dots, b_q\} \subset \bar{m} \setminus \bar{n}$. Notice also that either $p = q$ or $p = q + 1$. In the latter case we define $b_p := r$ for some r such that $r > a_p$ and $F(r-1) > F(r)$. Such r must exist since $F(\max\{n_k, m_k\}) = 0$. Also we have $r \in \bar{m} \setminus \bar{n}$. We will set

$$\bar{\ell} = \bar{n} \cup \{b_1, \dots, b_p\} \setminus \{a_1, \dots, a_p\}.$$

It is clear that $\bar{\ell} \in [\mathbb{M}]^k$. We also have $\max F_{\bar{\ell}, \bar{m}} = \max F_{\bar{n}, \bar{m}} - 1$ and $\min F_{\bar{\ell}, \bar{m}} = \min F_{\bar{n}, \bar{m}}$. Indeed, the point $\bar{\ell}$ is constructed in such a way that when $F_{\bar{n}, \bar{m}}$ attains its maximum for the first time (going from the left), $F_{\bar{\ell}, \bar{m}}$ is reduced by one and stays reduced by 1 until the next time the minimum of $F_{\bar{n}, \bar{m}}$ is attained (or until the point r) where this reduction is corrected back; and so on. Thus $d(\bar{\ell}, \bar{m}) = d(\bar{n}, \bar{m}) - 1$. Also, since the sets $\{a_1, \dots, a_p\}$ and $\{b_1, \dots, b_p\}$ are interlaced we have $F_{\bar{n}, \bar{m}} - 1 \leq F_{\bar{\ell}, \bar{m}} \leq F_{\bar{n}, \bar{m}}$. Therefore, since

$F_{\bar{n}, \bar{m}} = F_{\bar{n}, \bar{\ell}} + F_{\bar{\ell}, \bar{m}}$, we have that $0 \leq F_{\bar{n}, \bar{\ell}} \leq 1$ and so finally $d(\bar{n}, \bar{\ell}) = 1$, since it is clear that $\bar{n} \neq \bar{\ell}$. \square

Note that if X is a Banach space and $f: ([\mathbb{M}]^k, d_{\mathbb{K}}^k) \rightarrow X$ is a map with finite expansion modulus ω_f , then $\omega_f(1)$ is actually the Lipschitz constant of f as $d_{\mathbb{K}}^k$ is a graph distance on $[\mathbb{M}]^k$.

In [14] the property \mathcal{Q} is defined in the setting of metric spaces. For homogeneity reasons, its definition can be simplified for Banach spaces. Let us recall it here.

Definition 2.3. Let X be a Banach space. We say that X has *property \mathcal{Q}* if there exists $C \geq 1$ such that for every $k \in \mathbb{N}$ and every Lipschitz map $f: ([\mathbb{N}]^k, d_{\mathbb{K}}^k) \rightarrow X$, there exists an infinite subset \mathbb{M} of \mathbb{N} such that:

$$\forall \bar{n}, \bar{m} \in [\mathbb{M}]^k, \|f(\bar{n}) - f(\bar{m})\| \leq C\omega_f(1).$$

The following proposition should be clear from the definitions. We shall however include its short proof.

Proposition 2.4. *Let X be a Banach space. If X has property \mathcal{Q} , then the family of graphs $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$ does not equi-coarsely embed into X .*

Proof. Let $C \geq 1$ be given by the definition of property \mathcal{Q} . Aiming for a contradiction, assume that the family $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$ equi-coarsely embeds into X . That is, there are maps $f_k: ([\mathbb{N}]^k, d_{\mathbb{K}}^k) \rightarrow X$ and two functions $\rho, \omega: [0, +\infty) \rightarrow [0, +\infty)$ such that $\lim_{t \rightarrow \infty} \rho(t) = \infty$ and

$$\forall k \in \mathbb{N} \quad \forall t > 0 \quad \rho(t) \leq \rho_{f_k}(t) \quad \text{and} \quad \omega_{f_k}(t) \leq \omega(t) < \infty.$$

Thus, for every $k \in \mathbb{N}$, there exists an infinite subset \mathbb{M}_k of \mathbb{N} such that $\text{diam}(f([\mathbb{M}_k]^k)) \leq C\omega(1)$. Since $\text{diam}([\mathbb{M}_k]^k) = k$, this implies that for all $k \in \mathbb{N}$, $\rho(k) \leq C\omega(1)$. This contradicts the fact that $\lim_{t \rightarrow \infty} \rho(t) = \infty$. \square

A concrete bi-Lipschitz copy of the metric spaces $([\mathbb{N}]^k, d_{\mathbb{K}}^k)$ in c_0 is given by the following proposition.

Proposition 2.5. *Let $(s_n)_{n=1}^\infty$ be the summing basis of c_0 , that is $s_n = \sum_{i=1}^n e_i$, where $(e_i)_{i=1}^\infty$ is the canonical basis of c_0 .*

For $k \in \mathbb{N}$, define $f_k: ([\mathbb{N}]^k, d_{\mathbb{K}}^k) \rightarrow c_0$ by $f_k(\bar{n}) = \sum_{i=1}^k s_{n_i}$. Then

$$\frac{1}{2}d_{\mathbb{K}}^k(\bar{n}, \bar{m}) \leq \|f_k(\bar{n}) - f_k(\bar{m})\|_\infty \leq d_{\mathbb{K}}^k(\bar{n}, \bar{m})$$

for all $\bar{n}, \bar{m} \in [\mathbb{N}]^k$.

Proof. Since $d_{\mathbb{K}}^k = d$, one can show (as in the Fact in the proof of Proposition 2.2) that $d_{\mathbb{K}}^k(\bar{n}, \bar{m}) = \max(f_k(\bar{n}) - f_k(\bar{m})) - \min(f_k(\bar{n}) - f_k(\bar{m}))$. The result then follows easily since $\min(f_k(\bar{n}) - f_k(\bar{m})) \leq 0 \leq \max(f_k(\bar{n}) - f_k(\bar{m}))$ for all $\bar{n}, \bar{m} \in [\mathbb{N}]^k$. \square

Remark 2.6. We already explained that c_0 cannot coarsely embed into any Banach space with property \mathcal{Q} (in particular into any reflexive Banach space) and that Kalton even showed with additional arguments that if c_0 coarsely embeds into a separable Banach space X , then one of the iterated

duals of X has to be non separable. An inspection of his proof shows that the uniformly discrete metric spaces

$$M_k = \left\{ \sum_{i=1}^k s_{n_i} \times \mathbf{1}_A : (n_1, \dots, n_k) \in [\mathbb{N}]^k, A \in [\mathbb{N}]^\omega \right\} \subset c_0$$

do not equi-coarsely embed into any Banach space X such that $X^{(r)}$ is separable for all r . See Theorem 6.1 below for more on this subject.

Studying further the property \mathcal{Q} in [14], Kalton exhibited non reflexive quasi-reflexive spaces with the property \mathcal{Q} but showed that \mathcal{J} and \mathcal{J}^* fail property \mathcal{Q} . It is worth noticing that a theorem of Schoenberg [21] implies that ℓ_1 coarsely embeds into ℓ_2 , and therefore ℓ_1 provides a simple example of a non-reflexive Banach space with property \mathcal{Q} .

We conclude this section with two propositions that we state here for future reference. We start with a classical version of Ramsey's theorem.

Proposition 2.7 (Corollary 1.2 in [10]). *Let (K, d) be a compact metric space, $k \in \mathbb{N}$ and $f: [\mathbb{N}]^k \rightarrow K$. Then for every $\varepsilon > 0$, there exists an infinite subset \mathbb{M} of \mathbb{N} such that $d(f(\bar{n}), f(\bar{m})) < \varepsilon$ for every $\bar{n}, \bar{m} \in [\mathbb{M}]^k$.*

For a Banach space X , we call *tree of height k in X* any family $(x(\bar{n}))_{\bar{n} \in [\mathbb{N}]^{\leq k}}$, with $x(\bar{n}) \in X$. Then, if $\mathbb{M} \in [\mathbb{N}]^\omega$, $(x(\bar{n}))_{\bar{n} \in [\mathbb{M}]^{\leq k}}$ will be called a *full subtree* of $(x(\bar{n}))_{\bar{n} \in [\mathbb{N}]^{\leq k}}$. A tree $(x^*(\bar{n}))_{\bar{n} \in [\mathbb{M}]^{\leq k}}$ in X^* is called *weak*-null* if for any $\bar{n} \in [\mathbb{M}]^{\leq k-1}$, the sequence $(x^*(n_1, \dots, n_{k-1}, t))_{t > n_{k-1}, t \in \mathbb{M}}$ is weak*-null.

The next proposition is based on a weak*-compactness argument and will be crucial for our proofs. Although the distance considered on $[\mathbb{N}]^k$ is different, the proof follows the same lines as Lemma 4.1 in [3]. We therefore state it now without further detail.

Proposition 2.8. *Let X be a separable Banach space, $k \in \mathbb{N}$, and $f: ([\mathbb{N}]^k, d_{\mathbb{K}}^k) \rightarrow X^*$ a Lipschitz map. Then there exist $\mathbb{M} \in [\mathbb{N}]^\omega$ and a weak*-null tree $(x^*(\bar{m}))_{\bar{m} \in [\mathbb{M}]^{\leq k}}$ in X^* with $\|x_{\bar{m}}^*\| \leq \omega_f(1)$ for all $\bar{m} \in [\mathbb{M}]^{\leq k} \setminus \{\emptyset\}$ and so that*

$$\forall \bar{n} \in [\mathbb{M}]^k, f(\bar{n}) = \sum_{i=0}^k x^*(n_1, \dots, n_i) = \sum_{\bar{m} \leq \bar{n}} x^*(\bar{m}).$$

3. UNIFORM ASYMPTOTIC PROPERTIES OF NORMS AND RELATED ESTIMATES

We recall the definitions that will be considered in this paper. For a Banach space $(X, \|\cdot\|)$ we denote by B_X the closed unit ball of X and by S_X its unit sphere. The following definitions are due to V. Milman [19] and we adopt the notation from [13]. For $t \in [0, \infty)$ we define

$$\bar{\rho}_X(t) = \sup_{x \in S_X} \inf_Y \sup_{y \in S_Y} (\|x + ty\| - 1),$$

where Y runs through all closed subspaces of X of finite codimension. Then, the norm $\|\cdot\|$ is said to be *asymptotically uniformly smooth* (in short AUS) if

$$\lim_{t \rightarrow 0} \frac{\bar{\rho}_X(t)}{t} = 0.$$

For $p \in (1, \infty)$ it is said to be *p-asymptotically uniformly smooth* (in short *p-AUS*) if there exists $c > 0$ such that for all $t \in [0, \infty)$, $\bar{\rho}_X(t) \leq ct^p$.

We will also need the dual modulus defined by

$$\bar{\delta}_X^*(t) = \inf_{x^* \in S_{X^*}} \sup_E \inf_{y^* \in S_E} (\|x^* + ty^*\| - 1),$$

where E runs through all finite-codimensional weak*-closed subspaces of X^* . The norm of X^* is said to be *weak* asymptotically uniformly convex* (in short *AUC**) if $\bar{\delta}_X^*(t) > 0$ for all t in $(0, \infty)$. If there exists $c > 0$ and $q \in [1, \infty)$ such that for all $t \in [0, 1]$ $\bar{\delta}_X^*(t) \geq ct^q$, we say that the norm of X^* is *q-AUC**. The following proposition is elementary.

Proposition 3.1. *Let X be a Banach space. For any $t \in (0, 1)$, any weakly null sequence $(x_n)_{n=1}^\infty$ in B_X and any $x \in S_X$ we have:*

$$\limsup_{n \rightarrow \infty} \|x + tx_n\| \leq 1 + \bar{\rho}_X(t).$$

For any weak-null sequence $(x_n^*)_{n=1}^\infty \subset X^*$ and for any $x^* \in X^* \setminus \{0\}$ we have*

$$\limsup_{n \rightarrow \infty} \|x^* + x_n^*\| \geq \|x^*\| \left(1 + \bar{\delta}_X^* \left(\frac{\limsup_{n \rightarrow \infty} \|x_n^*\|}{\|x^*\|} \right) \right).$$

We will also need the following refinement (see Proposition 2.1 in [18]).

Proposition 3.2. *Let X be a Banach space. Then the bidual norm on X^{**} has the following property. For any $t \in (0, 1)$, any weak*-null sequence $(x_n^{**})_{n=1}^\infty$ in $B_{X^{**}}$ and any $x \in S_X$ we have:*

$$\limsup_{n \rightarrow \infty} \|x + tx_n^{**}\| \leq 1 + \bar{\rho}_X(t).$$

Let us now recall the following classical duality result concerning these moduli (see for instance [8] Corollary 2.3 for a precise statement).

Proposition 3.3. *Let X be a Banach space. Then $\|\cdot\|_X$ is AUS if and only if $\|\cdot\|_{X^*}$ is AUC*.*

If $p, q \in (1, \infty)$ are conjugate exponents, then $\|\cdot\|_X$ is p-AUS if and only if $\|\cdot\|_{X^}$ is q-AUC*.*

We conclude this section with a list of a few classical properties of Orlicz functions and norms that are related to these moduli. A map $\varphi : [0, \infty) \rightarrow [0, \infty)$ is called an *Orlicz function* if it is continuous, non decreasing, convex and so that $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. The *Orlicz norm* $\|\cdot\|_{\ell_\varphi}$, associated with φ is defined on c_{00} , the space of finitely supported sequences, as follows:

$$\forall x = (x_n)_{n=1}^\infty \in c_{00}, \quad \|x\|_{\ell_\varphi} = \inf \left\{ r > 0, \sum_{n=1}^\infty \varphi(x_n/r) \leq 1 \right\}.$$

The following is immediate from the definition.

Lemma 3.4. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be an Orlicz function and $p \in [1, \infty)$.*

- (i) *If there exists $C > 0$ such that $\varphi(t) \leq Ct^p$, for all $t \in [0, 1]$, then there exists $A > 0$ such that $\|x\|_{\ell_\varphi} \leq A\|x\|_{\ell_p}$, for all $x \in c_{00}$.*
- (ii) *If there exists $c > 0$ such that $\varphi(t) \geq ct^p$, for all $t \in [0, 1]$, then there exists $a > 0$ such that $\|x\|_{\ell_\varphi} \geq a\|x\|_{\ell_p}$, for all $x \in c_{00}$.*

Assume now that $\varphi : [0, \infty) \rightarrow [0, \infty)$ is an Orlicz function which is 1-Lipschitz and such that $\lim_{t \rightarrow \infty} \varphi(t)/t = 1$. Consider for $(s, t) \in \mathbb{R}^2$,

$$N_2^\varphi(s, t) = \begin{cases} |s| + |s|\varphi(|t|/|s|) & \text{if } s \neq 0, \\ |t| & \text{if } s = 0. \end{cases}$$

Then define by induction for all $n \geq 3$:

$$\forall (s_1, \dots, s_n) \in \mathbb{R}^n, \quad N_n^\varphi(s_1, \dots, s_n) = N_2^\varphi(N_{n-1}^\varphi(s_1, \dots, s_{n-1}), s_n).$$

The following is proved in [16] (see Lemma 4.3 and its preparation).

Lemma 3.5.

- (i) For any $n \geq 2$, the function N_n^φ is an absolute (or lattice) norm on \mathbb{R}^n , meaning that $N_n(s_1, \dots, s_n) \leq N_n(t_1, \dots, t_n)$, whenever $|s_i| \leq |t_i|$ for all $i \leq n$.
- (ii) For any $n \in \mathbb{N}$ and any $x \in \mathbb{R}^n$:

$$\frac{1}{2} \|s\|_{\ell_\varphi} \leq N_n^\varphi(s) \leq e \|s\|_{\ell_\varphi}.$$

When X is a Banach space, it is easy to see that $\bar{\rho}_X$ is a 1-Lipschitz Orlicz function such that $\lim_{t \rightarrow \infty} \rho(t)/t = 1$. But due to its lack of convexity, $\bar{\delta}_X^*$ is not an Orlicz function and we need to modify it. Following [16], we define

$$\delta(t) = \int_0^t \frac{\bar{\delta}_X^*(s)}{s} ds.$$

It is easy to see that $\bar{\delta}_X^*(t)/t$ is increasing and tends to 1 as t tends to ∞ . Therefore, δ is an Orlicz function which is 1-Lipschitz, such that $\lim_{t \rightarrow \infty} \delta(t)/t = 1$ and satisfying:

$$\forall t \in [0, \infty), \quad \bar{\delta}_X^*(t/2) \leq \delta(t) \leq \bar{\delta}_X^*(t).$$

The following statement is now a direct consequence of Lemmas 3.4 and 3.5.

Lemma 3.6. *Let X be a Banach space and $p \in [1, \infty)$.*

- (i) *If there exists $C > 0$ such that $\bar{\rho}_X(x) \leq Ct^p$, for all $t \in [0, 1]$, then there exists $A > 0$ such that*

$$\forall n \in \mathbb{N} \quad \forall x \in \mathbb{R}^n, \quad N_n^{\bar{\rho}_X}(x) \leq A \|x\|_{\ell_p^n}.$$

- (ii) *If there exists $c > 0$ such that $\bar{\delta}_X^*(t) \geq ct^p$, for all $t \in [0, 1]$, then there exists $a > 0$ such that*

$$\forall n \in \mathbb{N} \quad \forall x \in \mathbb{R}^n, \quad N_n^\delta(x) \geq a \|x\|_{\ell_p^n}.$$

We will also use the following reformulation of Propositions 3.1 and 3.2 in terms of the norms N_2^δ and $N_2^{\bar{\rho}_X}$.

Lemma 3.7. *Let X be a Banach space.*

- (i) *Let $(x_n^*) \subset X^*$ be weak*-null. Then for any $x^* \in X^*$ we have*

$$\limsup_{n \rightarrow \infty} \|x^* + x_n^*\| \geq N_2^\delta(\|x^*\|, \limsup_{n \rightarrow \infty} \|x_n^*\|).$$

- (ii) *Similarly, if $(x_n^{**}) \subset X^{**}$ is weak*-null and $x \in X$, then*

$$\liminf_{n \rightarrow \infty} \|x + x_n^{**}\| \leq N_2^{\bar{\rho}_X}(\|x\|, \liminf_{n \rightarrow \infty} \|x_n^{**}\|).$$

Proof. If $x^* = 0$ there is nothing to do, so we may assume that $x^* \neq 0$. By application of Proposition 3.1 we see that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|x^* + x_n^*\| &\geq \|x^*\| \left(1 + \bar{\delta}_X \left(\frac{\limsup \|x_n^*\|}{\|x^*\|} \right) \right) \\ &\geq \|x^*\| \left(1 + \delta \left(\frac{\limsup \|x_n^*\|}{\|x^*\|} \right) \right) = N_2^\delta(\|x^*\|, \limsup \|x_n^*\|) \end{aligned}$$

The proof of the second claim is even simpler so we leave it to the reader. \square

4. THE GENERAL RESULT

Let us first recall that a Banach space is said to be *quasi-reflexive* if the image of its canonical embedding into its bidual is of finite codimension in its bidual. We can now state our main result.

Theorem 4.1. *Let X be a quasi-reflexive Banach space, let $p \in (1, \infty)$ and denote q its conjugate exponent. Assume that X admits an equivalent p -AUS norm and that X^* admits an equivalent q -AUS norm. Then the family $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$ does not equi-coarsely embed into X^{**} .*

We immediately deduce the following.

Corollary 4.2. *Let X be a quasi-reflexive Banach space, let $p \in (1, \infty)$ and denote q its conjugate exponent. Assume that X admits an equivalent p -AUS norm and that X^* admits an equivalent q -AUS norm. Then the family $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$ does not equi-coarsely embed into X , nor does it equi-coarsely embed into any iterated dual $X^{(r)}$ ($r \geq 0$) of X .*

Proof. Since X is quasi reflexive we infer that $X^{(r)}$ admits an equivalent p -AUS norm when r is even and it admits an equivalent q -AUS norm when r is odd. Indeed, note that when r is even $X^{(r)}$ is isomorphic to $X \oplus_p F$ where F is finite-dimensional (resp. $X^{(r)} \simeq X^* \oplus_q F$ when r is odd). Now it is obvious from Theorem 4.1 that $([\mathbb{N}]^k)_{k \in \mathbb{N}}$ do not equi-coarsely embed into $X^{(r)}$ when r is even. When r is odd, we just exchange the roles of p and q . \square

Before going into the detailed proof of Theorem 4.1 let us briefly indicate the main idea. We assume that there is an equi-coarse family of embeddings (f_k) of $[\mathbb{N}]^k$ into X^{**} with moduli ρ and ω . We fix k sufficiently large and observe that, up to passing to a subgraph, f_k can be represented as the sum along the branches of a weak*-null countably branching tree of height k , say $(z_{\bar{n}})_{\bar{n} \in [\mathbb{N}]^{\leq k}}$. Moreover the norms of the elements of this tree stabilize on each level towards values $(K_i)_{i=1}^k \subset [0, \omega(1)]$. Applying the existence of a q -AUS norm on X^* one can show that $\sum_{i=1}^k K_i^p \leq c^p \omega(1)^p$ where c is a constant depending only on X . The benefit of this observation is twofold. On one hand we will be able to construct two elements $\bar{n}_0, \bar{m}_0 \in [\mathbb{N}]^l$ (with $l \leq k$) such that $\sum_{i=1}^l z_{(n_1, \dots, n_i)} - z_{(m_1, \dots, m_i)}$ is small in norm (say less than $2c\omega(1)$) while $d_{\mathbb{K}}^l(\bar{n}_0, \bar{m}_0)$ is large (say $\rho(d_{\mathbb{K}}^l(n_0, \bar{m}_0)) > 3c\omega(1)$). On the other hand the p -AUS renormability of X together with the quasi-reflexivity allows

to extend these elements to elements $\bar{n}, \bar{m} \in [\mathbb{N}]^k$ such that $d_{\mathbb{K}}^k(\bar{n}, \bar{m})$ is still large and

$$\begin{aligned} \left\| \sum_{i=l+1}^k z_{(n_1, \dots, n_i)} - z_{(m_1, \dots, m_i)} \right\| &\sim \left(\sum_{i=l+1}^k \|z_{(n_1, \dots, n_i)} - z_{(m_1, \dots, m_i)}\|^p \right)^{1/p} \\ &\sim \left(\sum_{i=l+1}^k K_i^p \right)^{1/p} \leq c\omega(1). \end{aligned}$$

Eventually, summing the tree from 1 to k over the branches ending by \bar{n} and \bar{m} we get the desired contradiction

$$3c\omega(1) < \rho(d_{\mathbb{K}}^k(\bar{n}, \bar{m})) \leq \|f_k(\bar{n}) - f_k(\bar{m})\| \leq 3c\omega(1).$$

Proof of Theorem 4.1. Let us assume that there are two maps $\rho, \omega: [0, +\infty) \rightarrow [0, +\infty)$ and maps $f_k([\mathbb{N}]^k, d_{\mathbb{K}}^k): \rightarrow (X^{**}, \|\cdot\|)$ for $k \in \mathbb{N}$ such that:

- (i) $\lim_{t \rightarrow \infty} \rho(t) = \infty$,
- (ii) $\omega(t) < \infty$ for every $t \in (0, +\infty)$,
- (iii) $\rho(t) \leq \rho_{f_k}(t)$ and $\omega_{f_k}(t) \leq \omega(t)$ for every $k \in \mathbb{N}$ and $t \in (0, \infty)$.

Note that all f_k 's are $\omega(1)$ -Lipschitz for $\|\cdot\|$ and so $\omega(1) > 0$. Since all the sets $[\mathbb{N}]^k$ are countable, we may and will assume that X and therefore, by the quasi-reflexivity of X , that all its iterated duals are separable.

Let us fix $N \in \mathbb{N}$. Pick $\alpha \in \mathbb{N}$ such that $\alpha \geq \frac{p}{q}$ and set $k = N^{1+\alpha} \in \mathbb{N}$. We also fix $\eta > 0$. We shall provide at the end of our proof a contradiction if N is chosen large enough and η small enough. We denote $\|\cdot\|$ the original norm on X , as well as its dual and bidual norms. Let us assume, as we may, that $\|\cdot\|$ is p -AUS on X . We denote its modulus of asymptotic uniform smoothness $\bar{\rho}_{\|\cdot\|}$ or simply $\bar{\rho}_X$.

For the first step of the proof we shall exploit the existence of an equivalent q -AUS norm $|\cdot|$ on X^* (we also denote $|\cdot|$ its dual norm on X^{**}). It is worth mentioning that if X is not reflexive, $|\cdot|$ cannot be the dual norm of an equivalent norm on X (see for instance Proposition 2.6 in [7]). Assume also that there exists $b > 0$ such that

$$(4.1) \quad \forall z \in X^{**} \quad b\|z\| \leq |z| \leq \|z\|.$$

Then we have that all f_k 's are also $\omega(1)$ -Lipschitz for $|\cdot|$.

By Proposition 3.3, we have that there exists $c > 0$ such that for all $t \in [0, 1]$, $\bar{\delta}_{|\cdot|}^*(t) \geq ct^p$. We denote again

$$\delta(t) = \int_0^t \frac{\bar{\delta}_{|\cdot|}^*(s)}{s} ds.$$

Recall that Lemma 3.6 ensures the existence of $a > 0$ such that for all $n \in \mathbb{N}$, $N_n^\delta \geq 2a\|\cdot\|_{\ell_p^n}$.

First, using the separability of X^* and Proposition 2.8, we may assume by passing to a full subtree, that there exist a weak*-null tree $(z(\bar{m}))_{\bar{m} \in [\mathbb{N}]^{\leq k}}$ in X^{**} with $|z(\bar{m})| \leq \omega(1)$ for all $\bar{m} \in [\mathbb{N}]^{\leq k} \setminus \{\emptyset\}$ and so that

$$\forall \bar{n} \in [\mathbb{N}]^k, \quad f_k(\bar{n}) = \sum_{i=0}^k z(n_1, \dots, n_i) = \sum_{\bar{m} \leq \bar{n}} z(\bar{m}).$$

For $r \in \mathbb{N}$ we denote $E_r = \{\bar{m} = (m_1, \dots, m_j) \in [\mathbb{N}]^{\leq k} \setminus \{\emptyset\}, m_j = r\}$ and $F_r = \bigcup_{u=1}^r E_u$. Fix a sequence $(\lambda_r)_{r=1}^\infty$ in $(0, 1)$ such that $\prod_{r=1}^\infty \lambda_r > \frac{1}{2}$. We now use Lemma 3.7 (i) and the fact that $(z(\bar{m}))_{\bar{m} \in [\mathbb{N}]^{\leq k}}$ is a weak*-null tree to build inductively $n_1 < \dots < n_r$ so that for all $\bar{n}^1, \dots, \bar{n}^L \in F_{n_r-1}$, for all $\varepsilon_1, \dots, \varepsilon_L \in \{-1, 1\}$ and all $\bar{n} \in E_{n_r}$, we have

$$|z(\bar{n}) + \sum_{l=1}^L \varepsilon_l z(\bar{n}^l)| \geq \lambda_r N_2^\delta \left(\left| \sum_{l=1}^L \varepsilon_l z(\bar{n}^l) \right|, |z(\bar{n})| \right).$$

Therefore, using the fact that N_2^δ is an absolute norm and after passing to a full subtree, we may assume that for all $r_1 < \dots < r_L$ in \mathbb{N} , all $\varepsilon_1, \dots, \varepsilon_L \in \{-1, 1\}$ and all $\bar{n}^1, \dots, \bar{n}^L$ so that $\bar{n}^l \in E_{r_l}$ for $1 \leq l \leq L$, we have

$$(4.2) \quad \left| \sum_{l=1}^L \varepsilon_l z(\bar{n}^l) \right| \geq \frac{1}{2} N_L^\delta (|z(\bar{n}^1)|, \dots, |z(\bar{n}^L)|) \geq a \left(\sum_{i=1}^L |z(\bar{n}^i)|^p \right)^{1/p}.$$

Assume now that $\bar{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ is such that $n_1 < \dots < n_k$ are even and choose $\bar{m} = (m_1, \dots, m_k)$ so that $n_1 < m_1 < \dots < n_k < m_k$. It follows from (4.2) that

$$\begin{aligned} |f(\bar{n}) - f(\bar{m})| &= \left| \sum_{i=1}^k z(n_1, \dots, n_i) - z(m_1, \dots, m_i) \right| \\ &\geq a \left(\sum_{i=1}^k |z(n_1, \dots, n_i)|^p + |z(m_1, \dots, m_i)|^p \right)^{1/p}. \end{aligned}$$

We now use the fact that $d_{\mathbb{K}}^k(\bar{n}, \bar{m}) = 1$ and f is $\omega(1)$ -Lipschitz, to deduce

$$\left(\sum_{i=1}^k |z(n_1, \dots, n_i)|^p \right)^{1/p} \leq \frac{1}{a} \omega(1).$$

So replacing \mathbb{N} with $2\mathbb{N}$ and setting $A = 1/a$, we may assume that

$$(4.3) \quad \forall \bar{n} \in [\mathbb{N}]^k, \quad \left(\sum_{i=1}^k |z(n_1, \dots, n_i)|^p \right)^{1/p} \leq A \omega(1).$$

By Ramsey's theorem (Proposition 2.7), we may also assume, after passing again to a full subtree, that for all $i \in \{1, \dots, k\}$ there exists $K_i \in [0, \omega(1)]$ such that

$$\forall (n_1, \dots, n_i) \in [\mathbb{N}]^i, \quad K_i \leq |z(n_1, \dots, n_i)| \leq K_i + \eta.$$

The estimate (4.3) yields

$$(4.4) \quad \sum_{i=1}^k K_i^p \leq A^p \omega(1)^p.$$

Therefore, since $k = N^{1+\alpha}$, there exists $j \in \{0, N, \dots, N(N^\alpha - 1)\}$ such that

$$\sum_{i=j+1}^{j+N} K_i^p \leq \frac{A^p \omega(1)^p}{N^\alpha}.$$

Then we deduce from Hölder's inequality that

$$(4.5) \quad \sum_{i=j+1}^{j+N} K_i \leq N^{1/q} \frac{A\omega(1)}{N^{\alpha/p}} \leq A\omega(1).$$

We now use the assumption that X is quasi-reflexive, so that $X^{**} = X \oplus F$, where F is of finite dimension. Thus, for each $(n_1, \dots, n_i) \in [\mathbb{N}]^{\leq k}$, we can decompose $z(n_1, \dots, n_i) = x(n_1, \dots, n_i) + e(n_1, \dots, n_i)$, with $x(n_1, \dots, n_i) \in X$ and $e(n_1, \dots, n_i) \in F$. Then, the compactness of bounded sets in F and another application of Proposition 2.7 allows us to assume, after passing to a full subtree, that

$$\forall i \in \{1, \dots, k\} \quad \forall \bar{n}, \bar{v} \in [\mathbb{N}]^i, \quad \|e(\bar{n}) - e(\bar{v})\| < \eta,$$

Which implies that for all $i \in \{1, \dots, k\}$ and all $\bar{n}, \bar{v} \in [\mathbb{N}]^i$ we have

$$(4.6) \quad \left| \|z(\bar{n}) - z(\bar{v})\| - \|x(\bar{n}) - x(\bar{v})\| \right| < \eta.$$

We are now ready for the last step of the proof, where we shall build \bar{m} and \bar{u} in $[\mathbb{N}]^k$ so that $d_{\mathbb{K}}^k(\bar{m}, \bar{u}) = N$, but $|f(\bar{m}) - f(\bar{u})|$ is bounded by a constant depending only on $\omega(1)$ and on X . This will yield a contradiction with the fact $\lim_{N \rightarrow \infty} \rho(N) = \infty$.

First, we set $m_i = u_i = i$, for all $1 \leq i \leq j$. Then, for $j+1 \leq i \leq j+N$, we set $m_i = i$ and $u_i = i+N$. Finally, we shall build $m_i = u_i$ inductively, for $j+N < i \leq k$. Note, that when this will be done, we will indeed have $d_{\mathbb{K}}^k(\bar{m}, \bar{u}) = N$.

First, we obviously have

$$(4.7) \quad \sum_{i=1}^j z(m_1, \dots, m_i) - z(u_1, \dots, u_i) = 0.$$

The next estimate follows from (4.5).

$$(4.8) \quad \left| \sum_{i=j+1}^{j+N} z(m_1, \dots, m_i) - z(u_1, \dots, u_i) \right| \leq \sum_{i=j+1}^{j+N} 2(K_i + \eta) \leq 3A\omega(1),$$

if η was initially chosen small enough.

We now select the remaining coordinates of \bar{m} and \bar{u} inductively using the fact that $\|\cdot\|$ is p -AUS. To shorten the notation for the end of the proof, we shall now denote $x_i = x(m_1, \dots, m_i)$, $z_i = z(m_1, \dots, m_i)$, $x'_i = x(u_1, \dots, u_i)$ and $z'_i = z(u_1, \dots, u_i)$. First, we simply set $m_{j+N+1} = u_{j+N+1} = j+2N+1$. We now use the fact that the tree $(z(\bar{m}))_{\bar{m} \in [\mathbb{N}]^{\leq k}}$ is weak*-null and Lemma 3.7 (ii) to find $m_{j+N+2} = u_{j+N+2} > j+2N+1$ such that

$$\begin{aligned} & \|x_{j+N+1} - x'_{j+N+1} + z_{j+N+2} - z'_{j+N+2}\| \\ & \leq N_2^{\bar{p}X} (\|x_{j+N+1} - x'_{j+N+1}\|, \|z_{j+N+2} - z'_{j+N+2}\|) + \eta \end{aligned}$$

It follows from (4.6) that

$$\begin{aligned} & \|z_{j+N+1} - z'_{j+N+1} + z_{j+N+2} - z'_{j+N+2}\| \\ & \leq N_2^{\bar{p}X} (\|z_{j+N+1} - z'_{j+N+1}\| + \eta, \|z_{j+N+2} - z'_{j+N+2}\|) + 2\eta \\ & \leq N_2^{\bar{p}X} \left(\frac{2}{b} (K_{j+N+1} + \eta) + \eta, \frac{2}{b} (K_{j+N+2} + \eta) \right) + 2\eta. \end{aligned}$$

Similarly, we can inductively find $m_{j+N+2} = u_{j+N+2} < \dots < m_k = u_k$ such that,

$$\left\| \sum_{i=j+N+1}^k (z_i - z'_i) \right\| \leq \frac{2}{b} N_{k-j-N}^{\bar{p}_X} (K_{j+N+1}, \dots, K_k) + \omega(1)$$

provided η is chosen small enough. Since Lemma 3.6 ensures the existence of $C > 0$ such that $N_n^{\bar{p}_X} \leq C \| \cdot \|_{\ell_p^n}$ for all $n \in \mathbb{N}$ the above inequality yields

$$\left\| \sum_{i=j+N+1}^k (z_i - z'_i) \right\| \leq \frac{2C}{b} \left(\sum_{i=j+N+1}^k K_i^p \right)^{1/p} + \omega(1) \leq \left(\frac{2CA}{b} + 1 \right) \omega(1).$$

Finally, combining the above estimate with (4.7) and (4.8), we get that

$$\|f(\bar{m}) - f(\bar{u})\| \leq \frac{3A + 2CA + b}{b} \omega(1).$$

As announced at the beginning of the proof, this yields a contradiction if N was initially chosen, as it was possible, so that $\rho(N) > \frac{3A+2CA+b}{b} \omega(1)$. \square

Unlike reflexivity, quasi-reflexivity itself is not enough to prevent the Kalton graphs from embedding into a Banach space. We thank P. Motakis for showing us the next example.

Proposition 4.3 (Motakis). *There exists a quasi-reflexive Banach space X such that the family of graphs $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$ equi-Lipschitz embeds into X .*

Proof. The proof relies on the existence of a quasi-reflexive Banach space X of order one which admits a spreading model, generated by a basis of X that is equivalent to the summing basis $(s_n)_{n=1}^\infty$ of c_0 . This is shown in [9] (Proposition 3.2) and based on a construction given in [6]. We refer the reader to [5] for the necessary definitions. Consequently, there exists a sequence $(x_n)_{n=1}^\infty$ in S_X and constants $A, B > 0$ such that for all $k \leq n_1 < \dots < n_k$ and all $\varepsilon_1, \dots, \varepsilon_k$ in $\{-1, 0, 1\}$ one has

$$(4.9) \quad A \left\| \sum_{i=1}^k \varepsilon_i s_i \right\|_{c_0} \leq \left\| \sum_{i=1}^k \varepsilon_i x_{n_i} \right\|_X \leq B \left\| \sum_{i=1}^k \varepsilon_i s_i \right\|_{c_0}.$$

For $k \in \mathbb{N}$ and $\bar{n} = (n_1, \dots, n_k) \in [\mathbb{N}]^k$ we define

$$g_k(\bar{n}) = \sum_{i=1}^k x_{2k+n_i}.$$

It follows easily from Proposition 2.5, the inequality (4.9) and the fact that $(s_n)_{n=1}^\infty$ is a spreading sequence that

$$\frac{A}{2} d_{\mathbb{K}}^k(\bar{n}, \bar{m}) \leq \|g_k(\bar{n}) - g_k(\bar{m})\|_X \leq B d_{\mathbb{K}}^k(\bar{n}, \bar{m})$$

for all $\bar{n}, \bar{m} \in [\mathbb{N}]^k$. \square

Remark 4.4. Let us mention that, more generally, it is proved in [2] that for any conditional normalized spreading sequence $(e_n)_{n=1}^\infty$, there exists a quasi-reflexive Banach space X of order 1 with a normalized basis $(x_i)_{i=1}^\infty$ which generates $(e_n)_{n=1}^\infty$ as a spreading model.

5. THE JAMES SEQUENCE SPACES

Let $p \in (1, \infty)$. We now recall the definition and some basic properties of the James space \mathcal{J}_p . We refer the reader to [1](Section 3.4) and references therein for more details on the classical case $p = 2$. The James space \mathcal{J}_p is the real Banach space of all sequences $x = (x(n))_{n \in \mathbb{N}}$ of real numbers with finite p -variation and verifying $\lim_{n \rightarrow \infty} x(n) = 0$. The space \mathcal{J}_p is endowed with the following norm

$$\|x\|_{\mathcal{J}_p} = \sup \left\{ \left(\sum_{i=1}^{k-1} |x(p_{i+1}) - x(p_i)|^p \right)^{1/p} : 1 \leq p_1 < p_2 < \dots < p_k \right\}.$$

This is the historical example, constructed for $p = 2$ by R.C. James in [11], of a quasi-reflexive Banach space which is isomorphic to its bidual. In fact \mathcal{J}_p^{**} can be seen as the space of all sequences $x = (x(n))_{n \in \mathbb{N}}$ of real numbers with finite p -variation, which is $\mathcal{J}_p \oplus \mathbb{R}\mathbf{1}$, where $\mathbf{1}$ denotes the constant sequence equal to 1.

The standard unit vector basis $(e_n)_{n=1}^\infty$ ($e_n(i) = 1$ if $i = n$ and $e_n(i) = 0$ otherwise) is a monotone shrinking basis for \mathcal{J}_p . Hence, the sequence $(e_n^*)_{n=1}^\infty$ of the associated coordinate functionals is a basis of its dual \mathcal{J}_p^* . Then the weak* topology $\sigma(\mathcal{J}_p^*, \mathcal{J}_p)$ is easy to describe. A sequence $(x_n^*)_{n=1}^\infty$ in \mathcal{J}_p^* converges to 0 in the $\sigma(\mathcal{J}_p^*, \mathcal{J}_p)$ topology if and only if it is bounded and $\lim_{n \rightarrow \infty} x_n^*(i) = 0$ for every $i \in \mathbb{N}$.

For $x \in \mathcal{J}_p$, we define $\text{supp } x = \{i \in \mathbb{N} : x(i) \neq 0\}$. For $x, y \in \mathcal{J}_p$, we denote: $x \prec y$ whenever $\max \text{supp } x < \min \text{supp } y$.

Similarly, an element x^* of \mathcal{J}_p^* will be written $x^* = \sum_{n=1}^\infty x^*(n)e_n^*$ and $\text{supp } x^* = \{i \in \mathbb{N} : x^*(i) \neq 0\}$ and we shall denote $x^* \prec y^*$ whenever $\max \text{supp } x^* < \min \text{supp } y^*$.

The detailed proof of the following proposition can be found in [20] (Proposition 2.3). This is a consequence of the following fact: there exists $C \geq 1$ such that $\|\sum_{i=1}^n x_i\|_{\mathcal{J}_p}^p \leq C \sum_{i=1}^n \|x_i\|_{\mathcal{J}_p}^p$, for all $x_1 \prec \dots \prec x_n$ in \mathcal{J}_p .

Proposition 5.1. *There exists an equivalent norm $|\cdot|$ on \mathcal{J}_p such that its dual norm $|\cdot|_*$ has the following property. For any $x^*, y^* \in \mathcal{J}_p^*$ such that $x^* \prec y^*$, we have that*

$$|x^* + y^*|_*^q \geq |x^*|_*^q + |y^*|_*^q.$$

In particular, $|\cdot|_$ is q -AUC* for the weak* topology induced by \mathcal{J}_p and therefore $|\cdot|$ is p -AUS on \mathcal{J}_p .*

There is also a natural weak* topology on \mathcal{J}_p . Indeed, the summing basis $(s_n)_{n=1}^\infty$ ($s_n(i) = 1$ if $i \leq n$ and $s_n(i) = 0$ otherwise) is a monotone and boundedly complete basis for \mathcal{J}_p . Thus, \mathcal{J}_p is naturally isometric to a dual Banach space: $\mathcal{J}_p = X^*$ with X being the closed linear span of the biorthogonal functionals $(e_n^* - e_{n+1}^*)_{n=1}^\infty$ in \mathcal{J}_p^* associated with $(s_n)_{n=1}^\infty$. Note that $X = \{x^* \in \mathcal{J}_p^*, \sum_{n=1}^\infty x^*(n) = 0\}$. Thus, a sequence $(x_n)_{n=1}^\infty$ in \mathcal{J}_p converges to 0 in the $\sigma(\mathcal{J}_p, X)$ topology if and only if it is bounded and $\lim_{n \rightarrow \infty} (x_n(i) - x_n(j)) = 0$ for every $i \neq j \in \mathbb{N}$. The next proposition is easy (see Proposition 2.3 in [17] for the case $p = 2$).

Proposition 5.2. *The usual norm on \mathcal{J}_p is p -AUC* for the weak* topology induced by X . In other words, the restriction to X of the usual norm on \mathcal{J}_p^* is q -AUS.*

Then, since X is one codimensional in \mathcal{J}_p^* , we have that \mathcal{J}_p^* is isomorphic to $X \oplus \mathbb{R}$ and therefore also admits an equivalent q -AUS norm.

The above remarks combined with Corollary 4.2 immediately yield the following.

Corollary 5.3. *Let $p \in (1, \infty)$. Then, the family $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_{k \in \mathbb{N}}$ does not equi-coarsely embed into \mathcal{J}_p , nor does it equi-coarsely embed into \mathcal{J}_p^* .*

6. A REMARK ON THE JAMES TREE SPACE

Let us recall the construction of the James tree space \mathcal{JT} . We denote $T = 2^{<\omega}$ the tree of all finite sequences with coefficients in $\{0, 1\}$ equipped with its natural order: for $s, t \in T$, we say that $s \leq t$ if the sequence t extends s . The set of all infinite sequences with coefficients in $\{0, 1\}$ will be denoted 2^ω . For $s \in T$, the length of s is denoted $|s|$. We call *segment* of T any set of the form $\{s \in T, t \leq s \leq t'\}$ with $t \leq t'$ in T . For a map $x : T \rightarrow \mathbb{R}$, we define

$$\|x\|_{\mathcal{JT}} = \sup \left\{ \left(\sum_{i=1}^n \left(\sum_{s \in S_i} x(s) \right)^2 \right)^{1/2} \right\},$$

where the supremum is taken over all pairwise disjoint segments S_1, \dots, S_n of T . Then the *James tree space* is the space $\mathcal{JT} = \{x : T \rightarrow \mathbb{R}, \|x\|_{\mathcal{JT}} < \infty\}$ equipped with the norm $\|\cdot\|_{\mathcal{JT}}$. For $s \in T$, we denote $e_s : T \rightarrow \mathbb{R}$ defined by $e_s(t) = \delta_{s,t}$, $t \in T$. If $\psi : \mathbb{N} \rightarrow T$ is a bijection such that $|\psi(n)| \leq |\psi(m)|$ whenever $n \leq m$, then $(e_{\psi(n)})_{n=1}^\infty$ is a normalized, monotone and boundedly complete basis of \mathcal{JT} . For $s \in T$, the coordinate functional e_s^* is defined by $e_s^*(x) = x(s)$, $x \in \mathcal{JT}$. Then the closed linear span of $\{e_s^*, s \in T\}$ in \mathcal{JT}^* is denoted \mathcal{B} and \mathcal{B}^* is isometric to \mathcal{JT} . The space \mathcal{JT} was built by R.C. James in [12] to serve as the first example of a separable Banach space with non separable dual, which does not contain an isomorphic copy of ℓ_1 .

In [14] it is shown that if a Banach space X coarsely contains c_0 then there exists $k \in \mathbb{N}$ such that $X^{(k)}$, the dual of order k of X , is non separable. A close look at the proof of Theorem 3.5 in [14] allows to state the following.

Theorem 6.1 (Kalton). *Let X and Y be two Banach spaces such that X coarsely embeds into Y . Assume moreover that there exist an uncountable set I and for every $i \in I$ and $k \in \mathbb{N}$, a 1-Lipschitz map $f_i^k : ([\mathbb{N}]^k, d_{\mathbb{K}}^k) \rightarrow X$ such that*

$$\lim_{k \rightarrow \infty} \inf_{i \neq j \in I} \inf_{\mathbb{M} \in [\mathbb{N}]^\omega} \sup_{\overline{n} \in [\mathbb{M}]^k} \|f_i^k(\overline{n}) - f_j^k(\overline{n})\| = \infty.$$

Then there exists $r \in \mathbb{N}$ such that $Y^{(r)}$ is not separable.

As an application, we can show the following.

Theorem 6.2. *Let Y be a Banach space such that \mathcal{B} or \mathcal{JT} coarsely embeds into Y . Then there exists $r \in \mathbb{N}$ such that $Y^{(r)}$ is not separable.*

Proof. For $\sigma \in 2^\omega$, we denote $\sigma|_n = (\sigma_1, \dots, \sigma_n)$. Then, for $k \in \mathbb{N}$, we define $f_\sigma^k : [\mathbb{N}]^k \rightarrow \mathcal{B}$ as follows. For $\bar{n} = (n_1, \dots, n_k) \in [\mathbb{N}]^k$ let

$$f_\sigma^k(\bar{n}) = \frac{1}{\sqrt{k}} \sum_{i=1}^k \sum_{s \leq \sigma|_{n_i}} e_s^*.$$

Assume for instance that $n_1 \leq m_1 \leq \dots \leq n_k \leq m_k$. Then we can write

$$f_\sigma^k(\bar{m}) - f_\sigma^k(\bar{n}) = \frac{1}{\sqrt{k}} \sum_{i=1}^k \sum_{s \in S_i} e_s^*,$$

where S_1, \dots, S_k are pairwise disjoint segments in T . Note that for any segment S_i the sum $\sum_{s \in S_i} e_s^*$ belongs to the unit ball of \mathcal{JT}^* . It then follows from the Cauchy-Schwarz inequality that f_σ^k is 1-Lipschitz on $([\mathbb{N}]^k, d_{\mathbb{K}}^k)$. Assume now that $\sigma \neq \tau \in 2^\omega$. Pick $r \in \mathbb{N}$ such that $\sigma_r \neq \tau_r$. Then for any $\bar{m} \in [\mathbb{N}]^\omega$ and any $\bar{n} = (n_1, \dots, n_k) \in [\mathbb{M}]^k$ with $n_1 \geq r$, we have

$$\|f_\sigma^k(\bar{n}) - f_\tau^k(\bar{n})\|_{\mathcal{B}} \geq |\langle f_\sigma^k(\bar{n}) - f_\tau^k(\bar{n}), e_{\sigma|_{n_1}} \rangle| \geq \sqrt{k}.$$

By Theorem 6.1 and the uncountability of 2^ω , this finishes our proof for \mathcal{B} .

For $\sigma \in 2^\omega$ and $k \in \mathbb{N}$ define now $g_\sigma^k : [\mathbb{N}]^k \rightarrow \mathcal{JT}$ by

$$\forall \bar{n} = (n_1, \dots, n_k) \in [\mathbb{N}]^k, \quad g_\sigma^k(\bar{n}) = \frac{1}{\sqrt{2k}} \sum_{i=1}^k e_{\sigma|_{n_i}}.$$

It is easily checked that g_σ^k is 1-Lipschitz on $([\mathbb{N}]^k, d_{\mathbb{K}}^k)$. Assume now that $\sigma \neq \tau \in 2^\omega$. Pick $r \in \mathbb{N}$ such that $\sigma_r \neq \tau_r$. Then for any $\bar{m} \in [\mathbb{N}]^\omega$ and any $\bar{n} = (n_1, \dots, n_k) \in [\mathbb{M}]^k$ with $n_1 \geq r$, denote $S = \{s \in T, \sigma|_{n_1} \leq s \leq \sigma|_{n_k}\}$. The set S is a segment in T and $x^* = \sum_{s \in S} e_s^*$ is in the unit ball of \mathcal{JT}^* . Therefore

$$\|g_\sigma^k(\bar{n}) - g_\tau^k(\bar{n})\|_{\mathcal{JT}} \geq \langle g_\sigma^k(\bar{n}) - g_\tau^k(\bar{n}), x^* \rangle \geq \frac{\sqrt{k}}{\sqrt{2}}.$$

This concludes our proof for \mathcal{JT} . \square

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