ZERO-CYCLES ON HOMOGENEOUS SPACES AND THE INVERSE GALOIS PROBLEM

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(report on joint work with Yonatan Harpaz)

Let Γ be a finite group and k be a number field.

It has been understood since Hilbert and Noether that the problem of realising Γ as a Galois group over k is really a problem about the arithmetic of the unirational variety $\mathbf{A}^m_k/\Gamma$, where Γ acts by permuting the coordinates once an embedding $\Gamma \hookrightarrow \mathfrak{S}_m$ has been chosen. Equivalently, it is really a problem about the arithmetic of the homogeneous space $\mathrm{SL}_{n,k}/\Gamma$, where Γ acts by multiplication on the right, an embedding $\Gamma \hookrightarrow \mathrm{SL}_n(k)$ having been chosen. (Indeed, by Speiser's lemma, these two varieties are stably birational.) Letting X denote a smooth compactification of $\mathrm{SL}_{n,k}/\Gamma$, a positive answer to the inverse Galois problem for Γ would follow, in particular, from the density of $X(k)$ in the Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.

Theorem 1. *Let X be a smooth compactification of a homogeneous space V of a connected linear algebraic group G over k. Let $\bar{x} \in V(\bar{k})$ be a geometric point and let $H_{\bar{x}} \subseteq G(\bar{k})$ denote its stabiliser.*

(1) *The image of the natural map $\mathrm{CH}_0(X) \to \prod_{v\in\Omega} \mathrm{CH}_0(X \otimes_k k_v)$, where Ω denotes the set of places of k, consists of those families $(z_v)_{v\in\Omega}$ in the kernel of the Brauer–Manin pairing such that $\deg(z_v)$ is independent of v. In particular, if $X(\mathbf{A}_k)^{\mathrm{Br}(X)} \neq \varnothing$, then X admits a zero-cycle of degree 1.*

*Assume now that G is semi-simple and simply connected (*e.g.*, that $G = \mathrm{SL}_{n,k}$) and that $H_{\bar{x}}$ is finite.*

(2) *Assume that* [4, Conjecture 9.1] *is true. If $H_{\bar{x}}$ is solvable, then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.*

(3) *If $H_{\bar{x}}$ admits a filtration, with cyclic graded quotients, by normal subgroups of $H_{\bar{x}}$ which are stable under the natural outer action of $\mathrm{Gal}(\bar{k}/k)$, then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.*

The density of $X(k)$ in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ may well hold for all rationally connected varieties (a conjecture put forward by Colliot-Thélène) and was previously known for smooth compactifications of homogeneous spaces of linear algebraic groups with connected geometric stabiliser and for smooth compactifications of homogeneous spaces of semi-simple simply connected linear algebraic groups with finite abelian geometric stabiliser (Sansuc, Borovoi). Assertion (1) was previously known under these same assumptions (Liang). Furthermore, Neukirch had proved the density of $X(k)$ in $X(\mathbf{A}_k)$ when $V = \mathrm{SL}_{n,k}/\Gamma$ and the order of Γ and the number of roots of unity contained in k are coprime (a condition which excludes, *e.g.*, 2-groups).

Assertion (3) applies to $V = \mathrm{SL}_{n,k}/\Gamma$ when Γ is supersolvable (*e.g.*, when it is nilpotent). Both (2) and (3) rely on an induction on the order of $H_{\bar{x}}$. Even if one

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focuses on the inverse Galois problem, and therefore on homogeneous spaces of the form $\mathrm{SL}_{n,k}/\Gamma$, it is an essential point for this induction that the theorem can be applied to homogeneous spaces of $\mathrm{SL}_{n,k}$ which need not have a rational point and whose geometric stabiliser need not be constant.

Assertions (2) and (3) thus recover Shafarevich's theorem on the inverse Galois problem for finite solvable groups, in the supersolvable case (and, conditionally, in general). They also go further, as they yield answers to Grunwald's problem, which asks for Galois extensions of k with group Γ whose completions at a finite set S of places of k are prescribed. In particular, combining (2) and (3) with recent work of Lucchini Arteche on the bad places for the Brauer–Manin obstruction on such X shows that Grunwald's problem admits a positive answer when Γ is supersolvable (conditionally, when Γ is solvable) and S avoids the places dividing the order of Γ. We recall that some condition on S must appear even in the simplest situations: Wang gave a counterexample for $k = \mathbf{Q}$, $\Gamma = \mathbf{Z}/8\mathbf{Z}$, $S = \{2\}$.

The strategy of the proof of Theorem 1 relies on a geometric dévissage of these homogeneous spaces, which we now explain. For the sake of simplicity, we focus on (2) and assume that $G = \mathrm{SL}_{n,k}$ and that [4, Conjecture 9.1] holds. In this situation, we establish the following inductive step, which does not require any assumption on the finite group $H_{\bar{x}}$ and which clearly implies (2): *if $X'(k)$ is dense in $X'(\mathbf{A}_k)^{\mathrm{Br}(X')}$ for any smooth compactification X' of any homogeneous space of G with geometric stabiliser isomorphic to the derived subgroup of $H_{\bar{x}}$, then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.*

The starting point is a version, for rationally connected varieties, of the theory of descent that Colliot-Thélène and Sansuc developed in the 80's for geometrically rational varieties. Just as in the theory of descent for elliptic curves, the idea is to reduce questions about the rational points of a given variety X to the same questions for auxiliary varieties which lie above it. The auxiliary varieties we use are the *universal torsors* of Colliot-Thélène and Sansuc; for a smooth and proper variety X over k such that $\mathrm{Pic}(X \otimes_k \bar{k})$ is torsion-free, they are torsors $Y \to X$ under the torus over k whose character group is $\mathrm{Pic}(X \otimes_k \bar{k})$.

Proposition 2. *If X is a smooth and proper variety over k such that $\mathrm{Pic}(X \otimes_k \bar{k})$ is torsion-free, then $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ is contained in the union, over all universal torsors $f : Y \to X$, of $f'\big(Y'(\mathbf{A}_k)^{\mathrm{Br}(Y')}\big)$, where Y' denotes a smooth compactification of Y such that f extends to a morphism $f' : Y' \to X$.*

This statement, which can also be deduced from recent work of Cao, was first proved by Colliot-Thélène and Sansuc when X is geometrically rational, a generality which is insufficient for our purposes since there exist finite nilpotent groups Γ such that $\mathrm{SL}_{n,k}/\Gamma$ fails to be geometrically rational (Saltman, Bogomolov). Here, the point is that the full Brauer group of X is taken into account. Dealing with the *algebraic* subgroup $\mathrm{Br}_1(X) = \mathrm{Ker}\big(\mathrm{Br}(X) \to \mathrm{Br}(X \otimes_k \bar{k})\big)$ is not enough, as recent examples of transcendental Brauer–Manin obstructions to weak approximation on $\mathrm{SL}_{n,k}/\Gamma$ for certain p-groups Γ show (Demarche, Lucchini Arteche, Neftin [2]).

The next result elucidates the geometry of the universal torsors in our situation.

Proposition 3. *Let V and X be as in Theorem 1, with $G = \mathrm{SL}_{n,k}$ and $H_{\bar{x}}$ finite. Let $f : Y \to X$ be a universal torsor of X. There exist a dense open subset $W \subseteq Y$, a quasi-trivial torus Q and a morphism $\pi : W \to Q$ whose fibres are homogeneous spaces of G with geometric stabiliser isomorphic to the derived subgroup of $H_{\bar{x}}$.*

Putting together the above two propositions, we see that in order to prove assertion (2) of Theorem 1, the only missing ingredient is a positive answer to the following question, where Z, B are smooth compactifications of W, Q such that π extends to a morphism $p : Z \to B$.

Question 4. *Let $p : Z \to B$ be a dominant morphism between smooth and proper varieties over k. Assume that B and the generic fibre of p are rationally connected. If $B(k)$ is dense in $B(\mathbf{A}_k)^{\mathrm{Br}(B)}$ and $Z_b(k)$ is dense in $Z_b(\mathbf{A}_k)^{\mathrm{Br}(Z_b)}$ for all rational points b of a dense open subset of B, is $Z(k)$ dense in $Z(\mathbf{A}_k)^{\mathrm{Br}(Z)}$?*

An answer in a very special case (the case in which $B = \mathbf{P}^1_k$ and in which for any $b \in \mathbf{P}^1(k) \setminus \{0, \infty\}$, the variety Z_b is smooth and the map $\mathrm{Br}(k) \to \mathrm{Br}(Z_b)$ is onto) was a key step in Hasse's proof of the Hasse–Minkowski theorem on the rational points of quadrics—namely, the step which relies on Dirichlet's theorem on primes in arithmetic progressions (see [5, Proposition 3.17]). More general situations have been considered ever since. The general case, however, seems entirely out of reach. In [4], we provide a positive answer when B is rational over k, assuming the validity of [4, Conjecture 9.1], and we provide a positive answer to the analogous question for zero-cycles, under the same assumption on B, unconditionally.

As the torus Q is quasi-trivial, it is rational over k, and so is B. This completes the strategy for establishing the first two assertions of the theorem in the case of a homogeneous space of $\mathrm{SL}_{n,k}$ with finite solvable geometric stabiliser. To deduce (1) in full generality, a first reduction, using Sylow subgroups, allows us to deal with homogeneous spaces of $\mathrm{SL}_{n,k}$ with arbitrary finite geometric stabiliser; the general case then follows thanks to the work of Demarche and Lucchini Arteche [1]. The proof of (3) rests on a variant of this strategy. Universal torsors are replaced with torsors of a certain intermediate type built out of the cyclic quotients appearing in (3); they lead to fibrations $\pi : W \to Q$ admitting a rational section over $\bar{k}$. On the other hand, for fibrations into rationally connected varieties which are smooth over a quasi-trivial torus and which admit a rational section over $\bar{k}$, we provide an unconditional positive answer to Question 4, by performing a descent (in the form of a slight generalisation of Proposition 2) and applying the work of Harari [3].

References

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