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# GLOBAL SOLUTIONS OF REACTION-DIFFUSION SYSTEMS ON 1D-NETWORKS

Frédéric Kuczma

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# GLOBAL SOLUTIONS OF REACTION-DIFFUSION SYSTEMS ON 1D-NETWORKS

FRÉDÉRIC KUCZMA

ABSTRACT. The purpose of this paper is to assess some results and the associated techniques for global existence of solutions of reaction-diffusion systems on networks. The motivation comes from the fact that phenomena can occur on ramified physical structures, of which one-dimensional networks are the simplest examples. We work in the setting of solutions provided by the classical semigroup theory. Local existence and uniqueness in this setting is ensured by the fixed-point argument, which is detailed. Construction of diffusion operators on networks via bilinear forms, generation of an analytic semigroup, ultracontractivity and maximal regularity properties, essential for the global existence analysis, are recalled or proved in detail, following in particular Mugnolo [25]. With these tools at hand, we exemplify the fact that fundamental results available in the literature on global existence and time asymptotics of reaction-diffusion systems extend from open domains of  $\mathbb{R}^n$  to networks. To be specific, here we deal with one dimensional networks with Kirchhoff conditions at nodes. In this setting, we first implement on a basic  $2 \times 2$  example the celebrated  $L^p$ -method of Martin-Pierre [22] which provided a large number of existence results in past 30 years. To give a second example, we revisit the system studied in Haraux-Youkana [17] where global existence results from construction of a Lyapunov function. Compactness properties of solutions, as  $t \rightarrow +\infty$ , are obtained adapting the arguments of Haraux-Kirane [16]. This permits to establish convergence of solutions to equilibria of the Haraux-Youkana systems on networks.

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## 0. INTRODUCTION

Reaction-diffusion has been mainly studied on an open subset  $\Omega \subset \mathbb{R}^n$  but reaction-diffusion phenomena occur in physical spaces having a ramified structure. One-dimensional network is the simplest case. Such problems appear, for example, in the study of bloodstream oxygenation, or blooms of algae on a metal mesh

plunged into the sea. A one-dimensional heat equation comprising a piecewise constant diffusion coefficient may also be seen as a heat equation on a one-dimensional network.

The main goal of this article is to demonstrate, via a systematic presentation of the underlying abstract theory and the detailed analysis of two concrete reaction-diffusion examples, that the key aspects of the global well-posedness theory naturally transfer to networks. To be precise, our analysis concerns the case where the coupling at the network vertices obeys the conditions of continuity of the state variables (population density) and of conservation of the fluxes. These conditions are often called Kirchhoff conditions. In the forthcoming work [3], we will explore the extension of these results to the setting of multidimensional networks, with possible coupling of one-dimensional and higher dimensional areas relevant in some ecological models of fragmented but connected patch-domains.

The analysis techniques we have in mind for reaction-diffusion problems require a good understanding of the semigroup generated by the linear diffusion operators. For the case of one-dimensional networks, properties of the semigroup generated by the edge-wise diffusion  $c\Delta$  coupled via the Kirchhoff conditions were established in a series of papers including [6], [20], [25]. The work of Mugnolo [25] is our main source. In section 2, we make a systematic presentation of this theory, combining precise reference to the underlying literature and accurate proofs of results that are classical in the community but for which we were not able to find a sharp reference. In particular, we pay attention to carefully define the notion of local  $L^\infty$ -solution and to justify its existence. The results contained in section 2 are also intended to provide the frame for the forthcoming work [3]. Indeed, with appropriate redefinition of the underlying measure-spaces, functional spaces and bilinear forms, the results we compiled from [25] and several other sources permit to deal with diffusion-generated semigroups in the setting of generalized network domains.

In the sections 4 and 5, we will take advantage of the theory developed in the section 2 to study the system

$$\begin{cases} \partial_t u_j - c_j^1 u_j'' = f_j(u_j, v_j), \\ \partial_t v_j - c_j^2 v_j'' = g_j(u_j, v_j). \end{cases}$$

Our goal is to study these reaction-diffusion equations with two very different approaches.

This system has already been treated in the standard case (i.e. on an open subset  $\Omega \subset \mathbb{R}^n$ ) by Martin-Pierre (see [22]), Hollis-Martin-Pierre (see [19]), Pierre (see [28]). It was the prototype of a wider class of systems studied via the  $L^p$ -method (see [28]). In particular, the system modelling the bloodstream oxygenation is one of this class (see section 4.5). Note also that our study contains in particular the case of the system studied in [28, Theorem 3.1, p. 425] for the case of diffusion  $-(c^1(x)u')', -(c^2(x)v')'$  with piecewise constant coefficients  $c^1, c^2$  in a one-dimensional domain.

Following the duality method (see [22]), we will prove, under a suitable mass-control assumption, that the previous system has global solutions. Although Kirchhoff conditions are analogous to homogeneous Neumann conditions in the standard case, using the duality method, we have to sum the two equations. The diffusion coefficients  $c_j^1$  and  $c_j^2$  being different, some cross vertex terms appear in the integrations by parts. These terms correspond rather to inhomogeneous Neumann

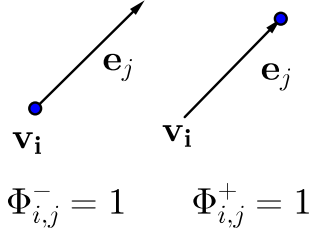


FIGURE 1.

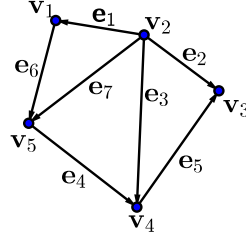


FIGURE 2.

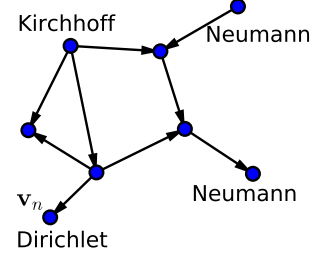


FIGURE 3.

conditions than to homogeneous Neumann conditions and we have to pay a special attention to them.

We develop a second example following Haraux and Youkana (see [17]), and we suppose that  $f_j$  and  $g_j$  are of the form  $f_j(r, s) = -r\varphi_j(r, s)$  and  $g_j(r, s) = r\varphi_j(r, s)$ . In the standard case, this system has first been studied by Alikakos for  $f(r, s) = -rs^\sigma$  (see [2]), followed by Masuda (see [23]) and then improved by Haraux and Youkana (see [17]). The method is based on use of a Lyapunov functional. Assuming that

$$\forall j \in \mathbb{N}_m, \frac{\ln(1 + \varphi_j(r))}{r} \xrightarrow{r \rightarrow +\infty} 0,$$

a global existence result will be proved. Secondly, we will deal with the asymptotic behavior of the solutions and prove a uniform convergence toward a constant solution. The key point is due to Haraux-Kirane (see [16]), which permits to estimate the  $C^1$ -norm of  $u(t, \cdot)$ . In Appendix C, we will carefully verify that the results of Haraux-Kirane transfer to one-dimensional networks.

Finally, let us note that, in the manner of Barabanova (see [8]), we can partially generalize the result of Haraux-Youkana, assuming an exponential growth of  $\varphi_j$  (whereas the exponential growth is not attained in the hypothesis of Haraux-Youkana). Note also that the asymptotic behaviour treated by Barabanova can easily be extended to one-dimensional networks.

## 1. NETWORKS AND GRAPHS

**1.1. Framework.** Now, we consider a finite connected network, to which we associate a graph  $G$ , comprising  $m$  edges  $\mathbf{e}_1, \dots, \mathbf{e}_m$  and  $n$  vertices  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . Edges will be systematically parameterized by the interval  $[0, 1]$  and are arbitrarily oriented. Following the standard terminology, let  $\mathbf{e}_j(0)$  denote the "tail" of the edge  $j$  (start end) and let  $\mathbf{e}_j(1)$  denote the "head" of the edge  $j$  (finish end). We define two matrices  $\Phi^+ = (\phi_{i,j}^+)_{1 \leq i \leq n, 1 \leq j \leq m} \in \mathcal{M}_{n,m}(\mathbb{R})$  and  $\Phi^- = (\phi_{i,j}^-)_{1 \leq i \leq n, 1 \leq j \leq m} \in \mathcal{M}_{n,m}(\mathbb{R})$  in the following way (see the figure 1):

$$\phi_{i,j}^+ := \begin{cases} 1 & \text{if } \mathbf{e}_j(1) = \mathbf{v}_i, \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad \phi_{i,j}^- := \begin{cases} 1 & \text{if } \mathbf{e}_j(0) = \mathbf{v}_i, \\ 0 & \text{else.} \end{cases}$$

The matrix  $\Phi = \Phi^+ - \Phi^-$  is the *incidence matrix* of the graph  $G$ . Note that each column of  $\Phi$  contains exactly one 1 and one -1. For every vertex  $\mathbf{v}_i$ , let  $\Gamma(\mathbf{v}_i)$

denote the set of indices  $j$  of the edges having an endpoint at  $\mathbf{v}_i$ :

$$\Gamma(\mathbf{v}_i) = \{j \in \llbracket 1, m \rrbracket : \mathbf{e}_j(0) = \mathbf{v}_i \text{ or } \mathbf{e}_j(1) = \mathbf{v}_i\}.$$

Lastly, we call *degree* of the vertex  $\mathbf{v}_i$  the integer  $|\Gamma(\mathbf{v}_i)|$ . For example, consider the graph illustrated by the figure 2. The incidence matrix is given by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}.$$

Moreover,  $\Gamma(\mathbf{v}_2) = \{1, 2, 3, 7\}$  and the degree of  $\mathbf{v}_2$  is equal to 4.

Generally, Dirichlet conditions are imposed on certain vertices. The topological structure of the network not interesting us, we will systematically identify all these vertices. From now on, we assume that the network has a unique vertex in which a Dirichlet condition is imposed. Without loss of generality, it is assumed that the Dirichlet condition holds on  $\mathbf{v}_n$ .

**1.2. Continuity at vertices.** Since every edge is parameterized by the interval  $[0, 1]$ , each such edge can be identified with  $[0, 1]$ . Thus, a function  $u$  defined on the network  $G$  will be described by a  $m$ -uplet  $u = (u_1, \dots, u_m)$ , each function  $u_j$  being defined on  $\mathbf{e}_j$  or, in an equivalent manner, on  $[0, 1]$ . When  $\mathbf{v}_i$  is an endpoint of  $\mathbf{e}_j$ , we denote  $u_j(\mathbf{v}_i) := u_j(1)$  if  $\phi_{i,j}^+ = 1$  and  $u_j(\mathbf{v}_i) := u_j(0)$  if  $\phi_{i,j}^- = -1$ . By abuse of notation, we set  $u_j(\mathbf{v}_i) := 0$  when  $j \notin \Gamma(\mathbf{v}_i)$ . We will be interested in functions  $u = (u_1, \dots, u_m)$  which are continuous at vertices, i.e. such that

$$\forall i \in \mathbb{N}_n, \forall (j, \ell) \in \Gamma(\mathbf{v}_i)^2, u_j(\mathbf{v}_i) = u_\ell(\mathbf{v}_i).$$

The following property gives a simple characterization of continuous functions on the graph (see [6, Lemma 3.1, p. 7]).

**Lemma 1.1.** *A function  $u \in (C([0, 1]))^m$  is continuous on the graph if and only if there exists  $d^u \in \mathbb{R}^n$  such that  ${}^t\Phi^- d^u = u(0)$  and  ${}^t\Phi^+ d^u = u(1)$ . So, the continuity of  $u$  at vertices and the Dirichlet condition at  $\mathbf{v}_n$  translate into*

$$\exists d^u \in \mathbb{R}^{n-1} \times \{0\} : {}^t\Phi^- d^u = u(0) \quad \text{and} \quad {}^t\Phi^+ d^u = u(1),$$

where  $d_i^u$  represents the value of  $u$  at vertex  $\mathbf{v}_i$ .

**Remark 1.2.** *Implicitly, we identify  $\mathbb{R}^n$  and  $\mathcal{M}_{n,1}(\mathbb{R})$ , i.e. row vectors and column vectors are identified.*

**1.3. Diffusion on network.** In order to introduce notations, let us start by describing a problem of diffusion on network. Consider a graph  $G$  satisfying the previous assumptions. Let  $u = (u_1, \dots, u_m)$  be a function depending on  $t$  (the "time variable") and on  $x$  (the "space variable"). Assume  $u$  is continuous on the graph with respect to  $x$ . Such a function  $u : (t, x) \mapsto u(t, x)$  being given,  $\partial_t u$ , or  $\frac{\partial u}{\partial t}$  denotes the time derivative whereas  $u'$  denotes the space derivative, considered edge-wise.

Then we are interested in the following diffusion problem:

$$(1) \quad \begin{cases} \partial_t u_j(t, x) = c_j u_j''(t, x) + f_j(t, x), & 1 \leq j \leq m, 0 < x < 1, \\ u_j(t, \mathbf{v}_i) = u_\ell(t, \mathbf{v}_i) := d_i^u(t), & j, \ell \in \Gamma(\mathbf{v}_i), 1 \leq i \leq n, \\ \sum_{j=1}^m \phi_{i,j} c_j u_j'(t, \mathbf{v}_i) = 0, & 1 \leq i \leq n-1, \\ d_n^u(t) = 0, \\ u_j(0, \cdot) = u_{0,j}, & 1 \leq j \leq m, 0 < x < 1. \end{cases}$$

The first equation is an inhomogeneous heat equation;  $c_j > 0$  is the diffusion coefficient relative to the edge  $j$ .

The second equation translates the continuity of  $u$  at each vertex, and the fourth translates the Dirichlet condition at  $\mathbf{v}_n$ .

Conditions imposed by the third equation are Kirchhoff conditions: they correspond to a conservation law. Note that the Kirchhoff condition is not imposed at the vertex  $\mathbf{v}_n$  but the homogeneous Dirichlet condition is assumed instead. Moreover, consider a vertex  $\mathbf{v}_i$  which degree is equal to 1. Then, the Kirchhoff condition is equivalent to the homogeneous Neumann condition at  $\mathbf{v}_i$ .

Now, let us introduce weighted incidence matrices  $\Phi_w^+ = (\omega_{i,j}^+) \in \mathcal{M}_{n,m}(\mathbb{R})$  and  $\Phi_w^- = (\omega_{i,j}^-) \in \mathcal{M}_{n,m}(\mathbb{R})$ :

$$\omega_{i,j}^+ = \begin{cases} c_j & \text{if } \phi_{i,j}^+ = 1 \text{ and } i \leq n-1, \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad \omega_{i,j}^- = \begin{cases} c_j & \text{if } \phi_{i,j}^- = 1 \text{ and } i \leq n-1, \\ 0 & \text{else.} \end{cases}$$

**Remark 1.3.** *In terms of weighted incidence matrices, continuity at vertices, Kirchhoff conditions and Dirichlet condition at  $\mathbf{v}_n$  translate to, in a more condensed way,*

$$\forall t, \exists d^u(t) \in \mathbb{R}^{n-1} \times \{0\} : \begin{cases} {}^t\Phi^+ d^u(t) = u(t, 1) & \text{and} & {}^t\Phi^- d^u(t) = u(t, 0), \\ \Phi_w^- u'(t, 0) = \Phi_w^+ u'(t, 1). \end{cases}$$

**1.4. Reaction-diffusion on network.** Now, let us consider a reaction-diffusion problem on the previously described network. To make it more concrete, it is supposed that  $N$  chemical components interact with each other on each edge  $\mathbf{e}_j$ .

The concentration of chemical component  $k, 1 \leq k \leq N$ , on edge  $\mathbf{e}_j$  is denoted by  $u_j^k$ . It is therefore a function with respect to the two variables  $t, x$ . Denote  $u^k = (u_1^k, \dots, u_m^k)$  and  $u_j = (u_j^1, \dots, u_j^N)$ . We propose to study the following reaction-diffusion system:

$$(2) \quad (\mathbf{S}): \begin{cases} \partial_t u_j^k = c_j^k (u_j^k)'' + f_j^k(u_j^1, \dots, u_j^N), & 1 \leq j \leq m, 1 \leq k \leq N, 0 < x < 1, \\ u_j^k(t, \mathbf{v}_i) = u_\ell^k(t, \mathbf{v}_i) := d_i^{u^k}(t), & j, \ell \in \Gamma(\mathbf{v}_i), 1 \leq i \leq n, 1 \leq k \leq N, \\ \sum_{j=1}^m \phi_{i,j} c_j^k (u_j^k)'(t, \mathbf{v}_i) = 0, & 1 \leq i \leq n-1, 1 \leq k \leq N, \\ d_n^{u^k}(t) = 0, & 1 \leq k \leq N, \\ u_j^k(0, \cdot) = u_{0,j}^k, & 1 \leq j \leq m, \end{cases}$$

where

- the functions  $f_j^k$  (these are the nonlinearities) have continuous derivative on  $\mathbb{R}^N$ . They represent interactions between various chemical components,
- $c_j^k > 0$  is the diffusion coefficient on the edge  $j$  with respect to the component  $k$ .

As above,

- The second equation translate an assumption of continuity of each function  $u_j^k$  at vertex  $\mathbf{v}_i$ .
- The third equation is a conservation law: these are the Kirchoff conditions.
- The fourth equation is a Dirichlet condition at vertex  $\mathbf{v}_n$ .

Moreover, let us assume that the initial condition is

- positive:  $\forall k \in \llbracket 1, N \rrbracket, \forall j \in \llbracket 1, m \rrbracket, u_{0,j}^k := u_j^k(0, \cdot) \geq 0$ ,
- uniformly bounded:  $\forall k \in \llbracket 1, N \rrbracket, \forall j \in \llbracket 1, m \rrbracket, u_{0,j}^k \in L^\infty(0; 1)$ .

We stress that the solutions of systems 1-2 must systematically be understood in the sense of Definitions 2.23-2.24.

**Remark 1.4.** *In terms of weighted incidence matrices, continuity at vertices, Kirchoff conditions and Dirichlet condition at  $\mathbf{v}_n$  translate to, in a more condensed way,*

$$\forall t, \forall k \in \llbracket 1, N \rrbracket, \exists d^{u^k}(t) \in \mathbb{R}^{n-1} \times \{0\} : \begin{cases} {}^t\Phi_w^{k,+} d^{u^k}(t) = u^k(t, 1), \\ {}^t\Phi_w^{k,-} d^{u^k}(t) = u^k(t, 0), \\ \Phi_w^{k,-}(u^k)'(t, 0) = \Phi_w^{k,+}(u^k)'(t, 1), \end{cases}$$

where  $\Phi_w^{k,+} = (\omega_{i,j}^{k,+}) \in \mathcal{M}_{n,m}(\mathbb{R})$  and  $\Phi_w^{k,-} = (\omega_{i,j}^{k,-}) \in \mathcal{M}_{n,m}(\mathbb{R})$  are given by

$$\omega_{i,j}^{k,+} = \begin{cases} c_j^k & \text{if } \phi_{i,j}^{k,+} = 1, i \leq n-1, \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad \omega_{i,j}^{k,-} = \begin{cases} c_j^k & \text{if } \phi_{i,j}^{k,-} = 1, i \leq n-1, \\ 0 & \text{else.} \end{cases}$$

## 2. GENERAL RESULTS ABOUT ONE-DIMENSIONAL NETWORKS

**2.1. Stating the problem.** Let us recall that we are interested in the system (2) where the initial conditions satisfy

$$(3) \quad \begin{cases} u_{0,j}^k \geq 0, & 1 \leq j \leq m, 1 \leq k \leq N, \\ u_{0,j}^k \in L^\infty(0; 1), & 1 \leq j \leq m, 1 \leq k \leq N, \end{cases}$$

and where the functions  $f_j^k$  have continuous derivative on  $\mathbb{R}^N$ . These are the nonlinearities, witch represent interactions between various chemical components on the edge  $\mathbf{e}_j$ . Let  $f := (f_j^k)_{1 \leq j \leq m, 1 \leq k \leq N}$  and, for  $u = (u_j^k)_{1 \leq j \leq m, 1 \leq k \leq N}$ , let  $f(u)$  denote the function  $t \mapsto f(u(t, \cdot))$ .

We say that the property of quasi-positivity **(P)** holds when

$$\mathbf{(P)} : \forall (k, j) \in \llbracket 1, N \rrbracket \times \llbracket 1, m \rrbracket, \forall (r^1, \dots, r^N) \in \mathbb{R}_+^N, f_j^k(r^1, \dots, r^{i-1}, 0, r^{i+1}, \dots, r^N) \geq 0.$$

**2.2. Functional framework: Hilbert spaces case.** Let

$$X_2 := L^2(0, 1)^m \quad \text{and} \quad \mathbb{X}_2 := \prod_{k=1}^N X_2 = L^2(0, 1)^{mN}.$$

Note that in the case  $N = 1$ ,  $\mathbb{X}_2 = X_2$ .  $X_2$  and  $\mathbb{X}_2$  are equipped with their usual inner product. For example

$$\forall u, v \in \mathbb{X}_2, (u|v)_{\mathbb{X}_2} = \sum_{k=1}^N \sum_{j=1}^m \int_0^1 u_j^k(x) v_j^k(x) dx$$

where  $u = (u_j^k)_{1 \leq j \leq m, 1 \leq k \leq N}$ . The vector function  $u_j := (u_j^1, \dots, u_j^N)$  is the concentration distribution of the chemical components on the edge  $j$ , whereas  $u^k := (u_1^k, \dots, u_m^k)$  is the concentration distribution of the chemical component



$k$  on all edges. The spaces  $X_2$  and  $\mathbb{X}_2$  are complete. Define the unbounded linear operator  $(A^k, D(A^k))$  on  $X_2$ :

$$A^k = \left( -\text{diag} \left( c_j^k \frac{d^2}{dx^2} \right), 1 \leq j \leq m \right),$$

$$D(A^k) = \left\{ u^k \in (H^2(0, 1))^m : \exists d^{u^k} \in \mathbb{R}^{n-1} \times \{0\}, \begin{cases} {}^t\phi^+ d^{u^k} = u^k(1), \\ {}^t\phi^- d^{u^k} = u^k(0), \\ \phi_w^{k,-}(u^k)'(0) = \phi_w^{k,+}(u^k)'(1). \end{cases} \right\}.$$

Likewise, we define on  $\mathbb{X}_2$  the unbounded linear operator  $(A, D(A))$  where  $D(A) = \prod_{k=1}^N D(A^k)$  and  $A = \prod_{k=1}^N A^k$ . This leads us naturally to study the homogeneous abstract Cauchy problem

$$(\mathbf{ACP}) \quad : \quad \begin{cases} \frac{du}{dt}(t) + Au(t) = 0, & (t > 0), \\ u(0) = u_0, \end{cases}$$

on  $\mathbb{X}_2$ , which is equivalent to the  $N$  abstract Cauchy problems on  $X_2$

$$(\mathbf{ACP}^k) \quad : \quad \forall k \in \llbracket 1, N \rrbracket, \begin{cases} \frac{du^k}{dt}(t) + A^k u^k(t) = 0, & (t > 0), \\ u^k(0) = u_0^k. \end{cases}$$

This problem enters the framework of Mugnolo: see [25].

Now, let us introduce the two new spaces

$$H := \left\{ u \in (H^1(0, 1))^m : \exists d^u \in \mathbb{R}^{n-1} \times \{0\}, \begin{cases} {}^t\phi^+ d^u = u(1), \\ {}^t\phi^- d^u = u(0), \end{cases} \right\},$$

and  $\mathbb{H} := H^N = H \times \cdots \times H$ .

**Lemma 2.1.**  *$H$  equipped with its usual inner product*

$$\forall (u, v) \in H \times H, (u|v) = \sum_{j=1}^m \int_0^1 [u_j v_j + u'_j v'_j]$$

*is densely and compactly embedded in  $X_2$ . Moreover, the bilinear form defined by*

$$(u|v)_H := \sum_{j=1}^m \int_0^1 u'_j(x) v'_j(x) dx$$

*is an inner product; the associated norm is equivalent to the standard  $H^1$  norm defined by*

$$\forall u \in H, \|u\| = \sum_{j=1}^m \int_0^1 [(u_j)^2(x) + (u'_j)^2(x)] dx.$$

*Equipped with one of this two equivalent norm,  $H$  is a Hilbert space. Evidently, such results naturally extends from one to  $N$  components.*

*Proof.* [25, Lemma 3.1, p. 4] The space  $H^1(0, 1)$  is a Hilbert space. Moreover, the embedding  $H^1(0, 1) \hookrightarrow C([0, 1])$  is continuous. So,  $H$  is a Hilbert space.

Inclusions  $(C_0^\infty(0, 1))^m \subset H \subset X_2$  show that  $H$  is dense in  $X_2$ . Furthermore, according to the Rellich-Kondrachov Theorem, the embedding  $(H^1(0, 1))^m \hookrightarrow X_2$  is compact and then  $H$  is compactly embedded in  $X_2$ . Finally, equivalence of the

two norms is a direct consequence of the Poincaré inequality for networks (see Property 2.35).  $\square$

**Definition 2.2.** For  $k \in [1, N]$ , let  $\mathbf{a}^k$  the bilinear form defined on  $H \times H$  by

$$\mathbf{a}^k(u^k, v^k) = \sum_{j=1}^m c_j^k \int_0^1 (u_j^k)'(x) (v_j^k)'(x) dx.$$

Likewise, let  $\mathbf{a}$  the bilinear form defined on  $\mathbb{H} \times \mathbb{H}$  by

$$\mathbf{a}(u, v) = \sum_{j=1}^m \sum_{k=1}^N c_j^k \int_0^1 (u_j^k)'(x) (v_j^k)'(x) dx.$$

**Property 2.3.** •  $\mathbf{a}$  is continuous on  $\mathbb{H} \times \mathbb{H}$ , i.e. :

$$\exists M \geq 0 : \forall (u, v) \in \mathbb{H} \times \mathbb{H}, |\mathbf{a}(u, v)| \leq M \|u\|_{\mathbb{H}} \cdot \|v\|_{\mathbb{H}},$$

- $\mathbf{a}$  is symmetric,
- $\mathbf{a}$  is coercive i.e. :  $\exists \alpha > 0 : \forall u \in \mathbb{H}, \mathbf{a}(u, u) \geq \alpha \|u\|_{\mathbb{H}}^2$ .

*Proof.* The bilinear forms  $\mathbf{a}^k$  enjoy the same properties (see [25, Property 3.2,p. 5]). The conclusion follows easily.  $\square$

Now, let us state an elementary but fundamental lemma. It claims that, in integrations by parts, the boundary terms cancel each other out. In other words, and roughly speaking, Kirchhoff conditions are equivalent to homogenous Neumann conditions for networks.

**Property 2.4.** Let  $u^k \in H$  such that Kirchhoff conditions hold. Then

$$(4) \quad \forall w^k \in H, \sum_{j=1}^m c_j^k \left[ (u_j^k)'(1) w_j^k(1) - (u_j^k)'(0) w_j^k(0) \right] = 0.$$

*Proof.* Thanks to continuity of  $w$ , we have<sup>1</sup>

$$c_j^k (u_j^k)'(1) w_j^k(1) = \sum_{\ell=1}^{n-1} c_j^k \phi_{\ell j}^+ (u_j^k)'(\mathbf{v}_\ell) w_j^k(\mathbf{v}_\ell) = \sum_{\ell=1}^{n-1} c_j^k \phi_{\ell j}^+ (u_j^k)'(\mathbf{v}_\ell) d_\ell^{w^k}$$

and

$$c_j^k (u_j^k)'(0) w_j^k(0) = \sum_{\ell=1}^{n-1} c_j^k \phi_{\ell j}^- (u_j^k)'(\mathbf{v}_\ell) w_j^k(\mathbf{v}_\ell) = \sum_{\ell=1}^{n-1} c_j^k \phi_{\ell j}^- (u_j^k)'(\mathbf{v}_\ell) d_\ell^{w^k}.$$

By difference, and summing for  $j$  from 1 to  $m$ , it follows

$$\begin{aligned} \sum_{j=1}^m c_j^k \left[ (u_j^k)'(1) w_j^k(1) - (u_j^k)'(0) w_j^k(0) \right] &= \sum_{j=1}^m \sum_{\ell=1}^{n-1} c_j^k \left[ \phi_{\ell j}^+ - \phi_{\ell j}^- \right] (u_j^k)'(\mathbf{v}_\ell) d_\ell^{w^k} \\ &= \sum_{\ell=1}^{n-1} d_\ell^k(w) \sum_{j=1}^m c_j^k \left[ \phi_{\ell j}^+ - \phi_{\ell j}^- \right] (u_j^k)'(\mathbf{v}_\ell). \end{aligned}$$

Yet, Kirchhoff condition for component  $k$  is exactly

$$\sum_{j=1}^m c_j^k \left[ \phi_{\ell j}^+ - \phi_{\ell j}^- \right] (u_j^k)'(\mathbf{v}_\ell) = 0$$

<sup>1</sup> $\ell$  runs from 1 to  $n-1$  because, by Dirichlet condition,  $d_n^w = 0$ .

and the proof is complete.  $\square$

From [25, Lemma 3.4, p. 7], we obtain the following property:

**Property 2.5.**  $\bullet$  *The unbounded linear operator associated with  $\mathfrak{a}^k$  is equal to  $(A^k, D(A^k))$ ,*  
 $\bullet$  *The unbounded linear operator associated with  $\mathfrak{a}$  is equal to  $(A, D(A))$ , where  $D(A) = \prod_{k=1}^N D(A^k)$  and  $A = \prod_{k=1}^N A^k$ .*

The Proposition A.25 leads directly to:

**Property 2.6.**  *$-A^k$  (respectively  $-A$ ) is the infinitesimal generator of an analytic semigroup of contractions on  $X_2$  (respectively on  $\mathbb{X}_2$ ). Interchangeably, we will denote by  $(T_2^k(t))_{t \geq 0}$  or  $(e^{-tA^k})_{t \geq 0}$  (respectively  $(T_2(t))_{t \geq 0}$  or  $(e^{-tA})_{t \geq 0}$ ) the semigroup generated by  $-A^k$  (respectively  $-A$ ).*

Remark that all the properties of  $A$  are inherited by  $A^k$  (take  $N = 1$ ).

**Property 2.7.**  *$A$  is a self-adjoint operator on  $\mathbb{X}_2$ . Consequently,  $(T_2(t))_{t \geq 0}$  is self-adjoint for all  $t \geq 0$ .*

*Proof.* Since  $\mathfrak{a}$  is symmetric, Property A.26 asserts that  $A$  is self-adjoint. Moreover, since  $\mathbb{X}_2$  is reflexive (as a finite product of reflexive spaces),  $(T_2(t))_{t \geq 0}$  is self-adjoint, according to [27, Corollary 10.6, p. 41].  $\square$

We will need a  $L^\infty$ -contractivity, in order to establish a local existence result.

**Definition 2.8.** *Take  $p \in [1, +\infty]$ . Consider the real vector space*

$$X_p := L^p(0, 1)^m = \{u = (u_1, \dots, u_m), u_j \in L^p(0, 1)\}.$$

$X_p$  is equipped with its usual norm defined by

$$\forall u \in X_p, \|u\|_{X_p} = \begin{cases} \left( \sum_{j=1}^m \int_0^1 |u_j(x)|^p dx \right)^{\frac{1}{p}}, & \text{if } p \neq +\infty, \\ \sup_{1 \leq j \leq m} \sup_{x \in \Omega} |u_j(x)|, & \text{if } p = +\infty. \end{cases}$$

*Provided with this norm,  $X$  is a Banach space. Consider also the product space  $\mathbb{X}_p = \prod_{k=1}^N X_p = (L^p(0, 1))^{mN}$  provided with its usual product norm.*

**Property 2.9.**

$$\forall p < +\infty, \forall u \in \mathbb{X}_\infty, \|u\|_{\mathbb{X}_p} \leq (mN)^{\frac{1}{p}} \|u\|_{\mathbb{X}_\infty} \leq mN \|u\|_{\mathbb{X}_\infty}.$$

*Proof.* Let  $u = (u_j^k)_{1 \leq j \leq m, 1 \leq k \leq N} \in \mathbb{X}_\infty$  and let  $p < +\infty$ . Then,

$$\|u\|_{\mathbb{X}_p}^p = \sum_{j=1}^m \sum_{k=1}^N \int_{\Omega} |u_j^k|^p \leq \sum_{j=1}^m \sum_{k=1}^N \int_{\Omega} \|u\|_{\mathbb{X}_\infty}^p = mN \|u\|_{\mathbb{X}_\infty}^p.$$

$\square$

**Property 2.10.** *The semigroup  $(e^{-tA})_{t \geq 0}$  associated with the bilinear form  $\mathfrak{a}$  is sub-markovian, i.e.*

- $\bullet$  *positive,*
- $\bullet$   *$\mathbb{X}_\infty$ -contractive i.e. the closed unit ball of  $\mathbb{X}_\infty$  is invariant under the action of  $e^{-tA}$ , for all  $t \geq 0$ .*

*Proof.* [25, Theorem 3.5, p. 7] Recall that the bilinear form  $\mathbf{a}$  is densely defined on  $\mathbb{H} \subset \mathbb{X}_2$ , continuous, accretive and closed. According to Theorem A.36, to prove that  $(e^{-tA})_{t \geq 0}$  is positive, it is sufficient to verify the implication

$$u \in D(\mathbf{a}) \Rightarrow u^+ \in D(\mathbf{a}) \quad \text{and} \quad \mathbf{a}(u^+, u^-) \leq 0.$$

Let  $u \in D(\mathbf{a})$ . The function  $u^+$  is clearly continuous on the graph and  $(u_j^k)^+ \in H^1(\Omega)$  (see Corollary B.3). Furthermore

$$\begin{aligned} \mathbf{a}(u^+, u^-) &= \sum_{j=1}^m \sum_{k=1}^N c_j^k \int_0^1 ((u_j^k)^+)'(x) ((u_j^k)^-)'(x) dx \\ &= \sum_{j=1}^m \sum_{k=1}^N c_j^k \int_0^1 (u_j^k)'(x) \mathbb{1}_{u_j^k > 0}(x) (-u_j^k)'(x) \mathbb{1}_{u_j^k < 0}(x) dx = 0 \end{aligned}$$

where, for every function  $u$ ,  $\mathbb{1}_{u > 0}$  denotes the characteristic function of the set  $\{u > 0\}$  etc.

Now,  $\mathbb{X}_\infty$ -contractivity follows from Theorem A.28. Indeed, since  $\mathbf{a}$  is symmetric, it suffices to verify that (recall that  $Tu = (1 \wedge |u|)\text{sign}(u)$ )

$$\forall u \in D(\mathbf{a}), Tu \in H \text{ and } \mathbf{a}(Tu, Tu) \leq \mathbf{a}(u, u).$$

Let  $u \in D(\mathbf{a})$ . Clearly,  $Tu \in (H^1(0, 1))^{mM}$  (see Corollary B.4). Moreover, by continuity of  $u$  at each vertex,  $Tu$  is also continuous at each vertex. Lastly,

$$Tu(x) = \begin{cases} u(x), & \text{if } |u(x)| \leq 1, \\ \frac{u(x)}{|u(x)|}, & \text{if } |u(x)| \geq 1, \end{cases} \quad \text{and} \quad (Tu)' = u' \mathbb{1}_{|u| < 1} \quad \text{a.e.}$$

As a consequence,

$$\begin{aligned} \mathbf{a}(Tu, Tu) &= \sum_{j=1}^m \sum_{k=1}^N c_j^k \int_0^1 \left( (Tu_j^k)' \right)^2(x) dx = \sum_{j=1}^m \sum_{k=1}^N c_j^k \int_0^1 \left( (u_j^k)' \mathbb{1}_{|u_j^k| < 1} \right)^2(x) dx \\ &\leq \sum_{j=1}^m \sum_{k=1}^N c_j^k \int_0^1 \left( (u_j^k)' \right)^2(x) dx = \mathbf{a}(u, u), \end{aligned}$$

which completes the proof.  $\square$

**2.3. Extrapolation; ultracontractivity.** Let us recall that

$$\forall (u, v) \in \mathbb{H} \times \mathbb{H}, \mathbf{a}(u, v) = \sum_{j=1}^m \sum_{k=1}^N c_j^k \int_0^1 (u_j^k)'(x) (v_j^k)'(x) dx,$$

and that  $T_2(t) = e^{-tA}$ , where  $A$  denotes the operator associated with  $\mathbf{a}$ . To prove that  $(e^{-tA})_{t \geq 0}$  is  $\mathbb{X}_1$ -contractive, we will recall the following property.

**Lemma 2.11.** *Let  $(\Omega, \mathcal{A}, \mu)$  be a measure space. Then*

$$\forall u \in L^1(\Omega), \|u\|_{L^1(\Omega)} = \sup_{\|v\|_{L^\infty(\Omega)} \leq 1} \left| \int_{\Omega} uv \right|.$$

**Property 2.12.**  $(e^{-tA})_{t \geq 0}$  is  $\mathbb{X}_1$ -contractive.

*Proof.* Following [25], we can identify  $(L^p(0, 1))^m$  and  $L^p(0, m)$  ( $1 \leq p \leq +\infty$ ). More precisely, let  $U : (L^p(0, 1))^m \rightarrow L^p(0, m)$  the function defined by

$$\forall u = (u_1, \dots, u_m) \in (L^p(0, 1))^m, \forall j \in \llbracket 1, m \rrbracket, \forall x \in (j-1, j), U(u)(x) = u_j(x-j+1).$$

Clearly,  $U$  is an isometry. As no confusion can arise, we will use the same letter to designate  $u$  and  $U(u)$ . For  $u \in \mathbb{X}_2$ , we have

$$\begin{aligned} \|e^{-tA}u\|_{\mathbb{X}_1} &= \sum_{k=1}^N \left\| e^{-tA^k} u^k \right\|_{X_1} = \sum_{k=1}^N \sup_{\|v^k\|_{X_\infty} \leq 1} \left| \int_{[0,m]} e^{-tA^k} u^k \cdot v^k \right| \\ &= \sum_{k=1}^N \sup_{\|v^k\|_{X_\infty} \leq 1} \left| \left( e^{-tA^k} u^k \mid v^k \right)_{X_2} \right| \\ &= \sum_{k=1}^N \sup_{\|v^k\|_{X_\infty} \leq 1} \left| \left( u^k \mid e^{-tA^k} v^k \right)_{X_2} \right| \quad \text{because } e^{-tA^k} = e^{-tA^k} \text{ is self-adjoint} \\ &= \sum_{k=1}^N \sup_{\|v^k\|_{X_\infty} \leq 1} \left| \int_{[0,m]} u^k \cdot e^{-tA^k} v^k \right| \leq \sum_{k=1}^N \sup_{\|v^k\|_{X_\infty} \leq 1} \int_{[0,m]} |u^k| \cdot |e^{-tA^k} v^k| \\ &\leq \sum_{k=1}^N \sup_{\|v^k\|_{X_\infty} \leq 1} \int_{[0,m]} |u^k| \cdot |v^k| \quad \text{because } e^{-tA^k} \text{ is } \mathbb{X}_\infty\text{-contractive} \\ &\leq \sum_{k=1}^N \int_{[0,m]} |u^k| = \|u\|_{\mathbb{X}_1}. \end{aligned}$$

□

**Theorem 2.13.** [25, Theorem 4.1, p. 12], [5, paragraph 7.2.2.]

- $(T_2(t))_{t \geq 0}$  extrapolates to a family of real contractive semigroups  $(T_p(t))_{t \geq 0}$  on  $\mathbb{X}_p$ ,  $1 \leq p \leq \infty$ .
- $(T_p(t))_{t \geq 0}$  is positive for  $p \in [1, +\infty]$ .
- $(T_p(t))_{t \geq 0}$  is strongly continuous for  $1 \leq p < \infty$  and analytic for  $1 < p < \infty$ .

**Lemma 2.14** (Nash inequality). *There exists a constant  $M > 0$  such that*

$$\forall u \in H^1(0, 1), \|u\|_{L^2(0,1)} \leq M \|u\|_{H^1(0,1)}^{\frac{1}{3}} \|u\|_{L^1(0,1)}^{\frac{2}{3}}.$$

See [7, Lemma 2.7] for a more general statement.

**Property 2.15.** • *We have*

$$\forall t > 0, \forall u \in \mathbb{X}_1, \|T_2(t)u\|_{\mathbb{X}_2} \leq Mt^{-\frac{1}{4}} \|u\|_{\mathbb{X}_1}.$$

- *Therefore, the semigroup  $(T_2(t))_{t \geq 0}$  associated with  $\mathbf{a}$  is ultracontractive.*
- *In particular, it satisfies*

$$\exists M > 0 : \forall t > 0, \forall u \in \mathbb{X}_2, \|T_2(t)u\|_{\mathbb{X}_\infty} \leq Mt^{-\frac{1}{4}} \|u\|_{\mathbb{X}_2}.$$

*Proof.* • Let us prove the first inequality. Since  $(T_2(t))_{t \geq 0}$  is  $\mathbb{X}_\infty$ -contractive, Theorem A.29 may be applied. Taking  $d = 1$ , it is enough to verify that

$$\exists M > 0 / \forall u \in \mathbb{H} \cap L^1(0, 1)^{mN} = \mathbb{H}, \|u\|_{\mathbb{X}_2}^6 \leq M \mathbf{a}(u, u) \|u\|_{\mathbb{X}_1}^4.$$

Let  $u \in \mathbb{H}$ . We have

$$\begin{aligned}
 \|u\|_{\mathbb{X}_2}^2 &= \sum_{j=1}^m \sum_{k=1}^N \|u_j^k\|_{L^2(0,1)}^2 \leq M^2 \sum_{j=1}^m \sum_{k=1}^N \|u_j^k\|_{H^1(0,1)}^{\frac{2}{3}} \|u_j^k\|_{L^1(0,1)}^{\frac{4}{3}} \\
 &\leq M^2 \left( \sum_{j=1}^m \sum_{k=1}^N \|u_j^k\|_{H^1(0,1)}^{\frac{2}{3}} \right) \left( \sum_{j=1}^m \sum_{k=1}^N \|u_j^k\|_{L^1(0,1)}^{\frac{4}{3}} \right) \\
 &\leq C \left( \sum_{j=1}^m \sum_{k=1}^N \|u_j^k\|_{H^1(0,1)} \right)^{\frac{2}{3}} \left( \sum_{j=1}^m \sum_{k=1}^N \|u_j^k\|_{L^1(0,1)} \right)^{\frac{4}{3}} \\
 &\leq C \|u\|_{\mathbb{H}}^{\frac{2}{3}} \|u\|_{\mathbb{X}_1}^{\frac{4}{3}} \leq C \mathbf{a}(u, u)^{\frac{1}{3}} \|u\|_{\mathbb{X}_1}^{\frac{4}{3}},
 \end{aligned}$$

$C$  denoting a constant depending on the data.

- So, we have proved that, for  $p = 1$ ,  $q = 2$  and  $d = 1$ ,

$$\exists c > 0 : \forall t \in ]0; 1], \|T(t)\|_{\mathcal{L}(L^p, L^q)} \leq ct^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}.$$

- According to Property A.18, again for  $d = 1$ ,

$$\exists c > 0 : \forall 1 \leq p < q \leq \infty / \forall t \in ]0; 1], \|T(t)\|_{\mathcal{L}(L^p, L^q)} \leq ct^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})},$$

which proves that the semigroup  $(T_2(t))_{t \geq 0}$  associated with  $\mathbf{a}$  is ultracontractive.

- The second inequality is obtained taking  $p = 2$  and  $q = +\infty$ . The proof is complete.  $\square$

The following property directly follows from [25, Property 4.6, p. 14].

**Property 2.16.** Let  $p \in [1; +\infty]$ .

- Let  $-A_p^k$  denote the infinitesimal generator of the semigroup  $(T_p^k(t))_{t \geq 0}$ . Then

$$D(A_p^k) = \left\{ u \in (W^{2,p}(\Omega))^m : \exists d^{u^k} \in \mathbb{R}^{n-1} \times \{0\} : \begin{cases} {}^t\phi^+ d^{u^k} = u^k(1), \\ {}^t\phi^- d^{u^k} = u^k(0), \\ \phi_w^{k,-}(u^k)'(0) = \phi_w^{k,-}(u^k)'(1), \end{cases} \right\}$$

and the action of  $A_p^k$  on  $D(A_p^k)$  is given by:  $A_p^k = -\text{diag} \left( c_j^k \frac{d^2}{dx^2}, 1 \leq j \leq m \right)$ .

- $D(A_p) = \prod_{k=1}^N D(A_p^k)$  and  $A_p = \prod_{k=1}^N A_p^k$ .

**Property 2.17.** Consider the following abstract Cauchy problem

$$(5) \quad (\mathbf{ACP}) : \frac{du}{dt} + Au = f(t) \quad , \quad u(0) = u^0 \in \mathbb{X}_\infty,$$

where  $A$  is the previously defined unbounded linear operator. Let  $u$  be a solution of  $(\mathbf{ACP})$  on the interval  $[0, T)$ ,  $0 < T < +\infty$ . Assume that

$$\forall p \in (1, +\infty), f \in L^p(0, T; \mathbb{X}_p).$$

Then,  $u \in L^\infty(0, T; \mathbb{X}_\infty)$ .

*Proof.* It is a straightforward application of Property A.31.  $\square$

## 2.4. Compactness.

**Property 2.18.**  $-A$  has compact resolvent.

*Proof.* Applying the Rellich-Kondrachov Theorem, we see that the embedding  $D(A) \hookrightarrow X_2$  is compact. Moreover, since  $-A$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions (in fact, we even have here an analytic semigroup),  $\rho(-A) \neq \emptyset$ . This follows, for example, from the Hille-Yosida Theorem: see Theorem A.6. It remains to apply Property A.20.  $\square$

**Property 2.19.** For all  $p \in (1, +\infty)$ ,  $-A_p$  has compact resolvent.

*Proof.* We have just seen that  $-A = -A_2$  has compact resolvent. According to [5, paragraph 7.2.2, p. 62], this property is inherited by  $-A_p$ ,  $1 < p < +\infty$ .  $\square$

**Property 2.20.** The semigroup  $(e^{-tA_p})_{t \geq 0}$  is compact for  $1 < p < +\infty$  and  $t > 0$ .

*Proof.* We already know that  $-A_p$  has compact resolvent. Moreover,  $(e^{-tA_p})_{t \geq 0}$  is an analytic semigroup (because  $1 < p < +\infty$ ), and so  $(e^{-tA_p})_{t \geq 0}$  is a differentiable semigroup. Conclusion follows from Properties A.21 and A.13.  $\square$

Ultracontractivity of  $(e^{-tA})_{t \geq 0}$  may also be used to improve the previous result. More precisely:

**Property 2.21.** The semigroup  $(e^{-tA_p})_{t \geq 0}$  is compact for  $1 \leq p \leq +\infty$  and  $t > 0$ . Moreover,  $\sigma(A_p)$  does not depend on  $p$ , with  $1 \leq p \leq +\infty$ .

*Proof.* Take  $p \in [1, +\infty]$ . Let us factorize  $e^{-tA}$  in the following way (see [10, Theorem 2.1.5, p. 71]):

$$e^{-tA} : \mathbb{X}_1 \xrightarrow{\varphi_1 = e^{-\frac{t}{2}A}} \mathbb{X}_2 \xrightarrow{\varphi_2 = e^{-\frac{t}{2}A}} \mathbb{X}_2 \xrightarrow{\varphi_3 = \text{Id}} \mathbb{X}_1.$$

Here,  $\varphi_1$  is continuous by ultracontractivity;  $\varphi_2$  is compact from the foregoing;  $\varphi_3$  is continuous because  $\Omega$  is bounded. Consequently,  $e^{-tA} : \mathbb{X}_1 \rightarrow \mathbb{X}_1$  is compact. Furthermore,  $A$  is a linear real positive self-adjoint operator on  $\mathbb{X}_2$  and, for all  $t \geq 0$ ,  $e^{-tA}$  is positive and  $\mathbb{X}_\infty$ -contractive. Consequently, according to [10, Theorem 1.6.4, p. 36],  $e^{-tA} : \mathbb{X}_p \rightarrow \mathbb{X}_p$  is compact for all  $p \in [1, +\infty]$ . Moreover,

- $\forall p, q \in [1, +\infty], \sigma(A_p) = \sigma(A_q)$ ,
- every eigenfunction of  $A_2$  is also in  $\mathbb{X}_p$ , for all  $p \in [1, +\infty]$ .

$\square$

## 2.5. Exponential stability.

**Property 2.22.** For<sup>2</sup>  $1 < p < \infty$ ,  $\omega(T_p) = s(-A)$ . Consequently,  $(T_p(t))_{t \geq 0}$  is uniformly exponentially stable.

*Proof.* Let us start off by looking at the case  $p = 2$ . Clearly,  $\sigma_p(-A) \subset ]-\infty, 0]$ . Moreover,  $A$  is one-to-one. Indeed, let  $u \in D(A)$  such that  $Au = 0$ . Then  $\mathbf{a}(u, u) = 0$  and then

$$\sum_{j=1}^m \sum_{k=1}^N c_j^k \int_{\Omega} (u_j^k)'{}^2 = 0.$$

<sup>2</sup>See A.22 for the definition of growth bound and spectral bound.

It is deduced that  $(u_j^k)' = 0$  and that  $u_j^k$  is constant on  $\Omega$ . By continuity on the graph and connectedness of the network, every function  $u^k$  is constant. Finally, through Dirichlet condition at vertex  $\mathbf{v}_n$ , every  $u^k$  is equal to zero, i.e.  $u = 0$ . Thus,  $0 \notin \sigma_p(A)$ . Since  $A$  has compact resolvent,  $\sigma(A) = \sigma_p(A)$  and  $\sigma_p(A)$  consists of a sequence of eigenvalues converging to infinity (see Property A.20). It follows that  $s(-A) < 0$ . Since  $(T_2(t))_{t \geq 0}$  is an analytic semigroup, Property A.24 implies that  $s(-A) = \omega(T_2)$ . Consequently,  $\omega(T_2) < 0$  and  $T_2(t)$  is exponentially stable (see Remark A.23).

Now, fix  $p$  such that  $1 < p < +\infty$ . As seen in Property 2.21, one has<sup>3</sup>  $\sigma(A_p) = \sigma(A_2)$ . Thus  $s(-A_p) < 0$ . Since  $(T_p(t))_{t \geq 0}$  is analytic, we conclude as before<sup>4</sup>.  $\square$

**2.6. Local existence in  $\mathbb{X}_p$ ,  $1 < p < +\infty$ .** Throughout this section, we consider  $p \in (1, +\infty)$  (Note that we do not take account of the case  $p = +\infty$  because the semigroup  $(e^{-tA_\infty})_{t \geq 0}$  is not analytic). Recall that  $\mathbb{X}_p = L^p(\Omega, \mathbb{R})^{mN}$ . Let us consider the following abstract Cauchy problem

$$(6) \quad \frac{du}{dt} + A_p u = f(u) \quad , \quad u(0) = u^0 \in \mathbb{X}_p,$$

where  $A_p$  is the previously defined unbounded linear operator. We have seen that  $-A_p$  is the infinitesimal generator of an analytic semigroup of contractions. We also consider a real number  $T > 0$  which will be fixed later. We are looking for  $T$  sufficiently small such that Equation (6) has a solution on  $[0, T)$ . As usual,  $\Omega$  denotes the open interval  $(0, 1)$  and  $Q_T := (0, T) \times (0, 1)$ .

Following [27, definition 2.1, p. 105], we have the following definition:

**Definition 2.23.** *Let  $T > 0$ . A function  $u : [0, T) \rightarrow \mathbb{X}_p$  is a solution of (6) when  $u$  is continuous on  $[0, T)$ , has continuous derivative on  $(0, T)$ ,  $u(t) \in D(A)$  for all  $0 < t < T$  and (6) holds on  $(0, T)$ .*

We are particularly interested in solutions which are uniformly bounded. Hence the following definition.

**Definition 2.24.** *Let  $T > 0$ . A function  $u : [0, T) \rightarrow \mathbb{X}_p$  is a  $\mathbb{X}_\infty$ -solution of (6) when*

- for all  $p \in (1, +\infty)$ ,  $u$  is a solution of (6) in  $\mathbb{X}_p$ ,
- $u \in L_{loc}^\infty([0, T), \mathbb{X}_\infty)$ .

The following fundamental result seems to be rather classical. Since we were not able to find in the literature a detailed proof, and for the sake of exhaustiveness, we give a complete statement as well as a self-contained proof.

**Theorem 2.25.** *For  $T > 0$  sufficiently small, (6) has a unique  $\mathbb{X}_\infty$ -solution on  $[0, T)$ .*

*Proof.* (1) Take  $u^0 \in \mathbb{X}_\infty$ . Let  $\mathcal{B}$  be the closed ball in  $\mathbb{X}_{p,T} := L^p(Q_T, \mathbb{R}^{mN})$  centred at 0 of radius  $R > 0$  (the choice of  $R$  will be made later). For every  $\hat{u} \in \mathcal{B}$ , let us consider the new abstract Cauchy problem:

$$\frac{du}{dt}(t) + A_p u(t) = f(\hat{u}) =: g(t) \quad , \quad u(0) = u^0 \in \mathbb{X}_\infty,$$

<sup>3</sup> We can also note that  $A = A_2$  has compact resolvent and use [5, paragraph 7.2.2, p. 62].

<sup>4</sup> We can also use [11, Corollary 3.12, p. 281]: if  $A$  is the infinitesimal generator of an analytic semigroup, then  $s(A) = \omega(A)$ .



$g(t)$  denoting the function  $x \mapsto f(\hat{u}(t, x))$ . We would like to show that the map  $\hat{u} \mapsto u$  has a fixed point.

- (2) For  $\alpha > \|u^0\|_{\mathbb{X}_\infty} + 1$ , let  $T_\alpha$  be a  $C^\infty$ -regularization of the function

$$\mathbb{R} \rightarrow \mathbb{R}, x \mapsto \begin{cases} -\alpha & \text{if } x \leq -\alpha, \\ x & \text{if } x \in [-\alpha, \alpha], \\ \alpha & \text{if } x \geq \alpha. \end{cases}$$

More precisely, let  $T_\alpha : \mathbb{R} \rightarrow \mathbb{R}$  be an odd  $C^\infty$ -function such that

- $\forall t \in [0, \alpha - 1], T_\alpha(x) = x$ ,
- $\forall t > \alpha, T_\alpha(x) = \alpha$ ,
- $T_\alpha$  is concave on  $[0, +\infty)$ .

Let  $f_{j,\alpha}^k := f_j^k \circ (T_\alpha, \dots, T_\alpha)$ . Then, functions  $f_{j,\alpha}^k$  are globally Lipschitz continuous on  $\mathbb{R}^N$ . As before, denote  $f_\alpha := (f_{j,\alpha}^k)_{1 \leq j \leq m, 1 \leq k \leq N}$ . Then, we consider the new abstract Cauchy problem

$$\frac{d\underline{u}}{dt} + A_p \underline{u} = f_\alpha(\hat{u}) \quad , \quad \underline{u}(0) = u^0 \in \mathbb{X}_\infty.$$

Let  $\underline{u}$  be the mild-solution of (2). Then,  $\underline{u} \in C([0; T]; X_p)$  and, for all  $t \in [0; T)$ :

$$\underline{u}(t) = e^{-tA_p} u^0 + \int_0^t e^{-(t-s)A_p} f_\alpha(\hat{u})(s) ds = e^{-tA_p} u^0 + \int_0^t e^{-(t-s)A_p} g_\alpha(s) ds,$$

where  $g_\alpha(s) := f_\alpha(\hat{u})(s)$ . Let us stress that, since the semigroups  $(e^{-tA_p})_{t \geq 0}$  are consistent, the solution  $\underline{u}(t)$  does not depend on the choice of  $p$ .

- (3) In order to apply the Picard Theorem, we have to prove that the ball  $\mathcal{B}$  is invariant under the action of the function  $\hat{u} \mapsto \underline{u}$  and that the function  $\mathcal{B} \rightarrow \mathcal{B}, \hat{u} \mapsto \underline{u}$  is a contraction.

Let us start by studying the invariance of the ball  $\mathcal{B} := \mathcal{B}(0, R)$ , for sufficiently small  $T$ . For  $\hat{u} \in \mathcal{B}$ , we have (for  $t \in [0, T)$ )

$$\begin{aligned} \|\underline{u}(t)\|_{\mathbb{X}_p} &= \left\| e^{-tA_p} u^0 + \int_0^t e^{-(t-s)A_p} f_\alpha(\hat{u})(s) ds \right\|_{\mathbb{X}_p} \\ &\leq \|e^{-tA_p} u^0\|_{\mathbb{X}_p} + \int_0^t \left\| e^{-(t-s)A_p} f_\alpha(\hat{u})(s) \right\|_{\mathbb{X}_p} ds \\ &\leq \|u^0\|_{\mathbb{X}_p} + \int_0^t \|f_\alpha(\hat{u})(s)\|_{\mathbb{X}_p} ds \\ &\leq mN \|u^0\|_{\mathbb{X}_\infty} + \int_0^t mN \|f_\alpha(\hat{u})(s)\|_{\mathbb{X}_\infty} ds \end{aligned}$$

because  $e^{-tA_p}$  is a contraction and  $\|\cdot\|_{\mathbb{X}_p} \leq mN \|\cdot\|_{\mathbb{X}_\infty}$ : see Property 2.9. Consequently,

$$\begin{aligned} \|\underline{u}(t)\|_{\mathbb{X}_p} &\leq mN \|u^0\|_{\mathbb{X}_\infty} + mNT \sup_{s \in [0, T)} \|f_\alpha(\hat{u})(s)\|_{\mathbb{X}_\infty} \\ &\leq mN \|u^0\|_{\mathbb{X}_\infty} + mNT \|f_\alpha\|_\infty \end{aligned}$$

since  $f_\alpha$  is uniformly bounded. Integrating between 0 and  $T$  leads to

$$\|\underline{u}\|_{\mathbb{X}_p, T}^p = \int_0^T \|\underline{u}(t)\|_{\mathbb{X}_p}^p dt \leq T \left( mN \|u^0\|_{\mathbb{X}_\infty} + mNT \|f_\alpha\|_\infty \right)^p.$$

By imposing the limitation

$$(7) \quad T \leq 1,$$

we have

$$\|\underline{u}\|_{\mathbb{X}_p, T} \leq mN \|u^0\|_{\mathbb{X}_\infty} + mNT \|f_\alpha\|_\infty.$$

Then, let us take  $R$  such that  $R > mN \|u^0\|_{\mathbb{X}_\infty}$ . Choose, for example

$$R := mN \|u^0\|_{\mathbb{X}_\infty} + 1.$$

If  $T$  satisfies  $mNT \|f_\alpha\|_\infty \leq 1$ , we have

$$\|\underline{u}\|_{\mathbb{X}_p, T} \leq R \quad \text{i.e.} \quad \underline{u} \in \mathcal{B}.$$

Imposing the further limitation

$$(8) \quad T \leq \frac{1}{mN \|f_\alpha\|_\infty + 1},$$

we justify the desired invariance.

Now, let us prove that the function  $\mathcal{B} \rightarrow \mathcal{B}, \hat{u} \mapsto \underline{u}$  is a contraction. Let  $\hat{u}, \hat{v} \in \mathcal{B}$  and  $\underline{u}, \underline{v}$  be the mild-solutions of (2), i.e.

$$\frac{d\underline{u}}{dt} + A_p \underline{u} = f_\alpha(\hat{u}) \quad , \quad \underline{u}(0) = u^0 \in \mathbb{X}_\infty,$$

$$\frac{d\underline{v}}{dt} + A_p \underline{v} = f_\alpha(\hat{v}) \quad , \quad \underline{v}(0) = u^0 \in \mathbb{X}_\infty.$$

Thus, for all  $t \in [0, T)$ , we have

$$\underline{u}(t) = e^{-tA_p} u^0 + \int_0^t e^{-(t-s)A_p} f_\alpha(\hat{u})(s) ds,$$

$$\underline{v}(t) = e^{-tA_p} u^0 + \int_0^t e^{-(t-s)A_p} f_\alpha(\hat{v})(s) ds.$$

Taking the difference and using the triangle inequality, we have, with  $t \in [0, T]$ ,

$$\begin{aligned} D &:= \|\underline{v}(t) - \underline{u}(t)\|_{\mathbb{X}_p}^p = \left\| \int_0^t e^{-(t-s)A_p} (f_\alpha(\hat{v}) - f_\alpha(\hat{u}))(s) ds \right\|_{\mathbb{X}_p}^p \\ &\leq \left( \int_0^t \left\| e^{-(t-s)A_p} \right\|_{\mathcal{L}(\mathbb{X}_p)} \cdot \|f_\alpha(\hat{v})(s) - f_\alpha(\hat{u})(s)\|_{\mathbb{X}_p} ds \right)^p. \end{aligned}$$

Since  $(e^{-tA_p})_{t \geq 0}$  is a semigroup of contractions on  $\mathbb{X}_p$ , we have

$$D \leq \left( \int_0^t \|f_\alpha(\hat{v})(s) - f_\alpha(\hat{u})(s)\|_{\mathbb{X}_p} ds \right)^p.$$

Then, by Hölder inequality, with  $\frac{1}{p} + \frac{1}{q} = 1$ , it follows

$$\begin{aligned}
D &\leq t^{\frac{p}{q}} \int_0^t \|f_\alpha(\hat{v})(s) - f_\alpha(\hat{u})(s)\|_{\mathbb{X}_p}^p ds \leq T^{\frac{p}{q}} \int_0^T \|f_\alpha(\hat{v})(s) - f_\alpha(\hat{u})(s)\|_{\mathbb{X}_p}^p ds \\
&= T^{\frac{p}{q}} \sum_{j=1}^m \sum_{k=1}^N \int_0^T \int_\Omega |f_{j,\alpha}^k(\hat{v}(s,x)) - f_{j,\alpha}^k(\hat{u}(s,x))|^p dx ds \\
&\leq T^{\frac{p}{q}} \sum_{j=1}^m \sum_{k=1}^N \int_0^T \int_\Omega (Lip(f_{j,\alpha}^k))^p \sup_{1 \leq \ell \leq N} |\hat{v}_j^\ell(s,x) - \hat{u}_j^\ell(s,x)|^p dx ds \\
&\leq T^{\frac{p}{q}} \sum_{j=1}^m \sum_{k=1}^N \int_0^T \int_\Omega (Lip(f_{j,\alpha}^k))^p \sum_{\ell=1}^N |\hat{v}_j^\ell(s,x) - \hat{u}_j^\ell(s,x)|^p dx ds \\
&\leq T^{\frac{p}{q}} K_\alpha^p \sum_{j=1}^m \sum_{k=1}^N \sum_{\ell=1}^N \int_0^T \int_\Omega |\hat{v}_j^\ell(s,x) - \hat{u}_j^\ell(s,x)|^p dx ds \\
&= NK_\alpha^p T^{\frac{p}{q}} \sum_{j=1}^m \sum_{\ell=1}^N \int_0^T \int_\Omega |\hat{v}_j^\ell(s,x) - \hat{u}_j^\ell(s,x)|^p dx ds,
\end{aligned}$$

where  $K_\alpha := \max_{1 \leq j \leq N, 1 \leq k \leq N} Lip(f_{j,\alpha}^k)$  and where  $Lip(f_{j,\alpha}^k)$  denotes the Lipschitz constant of  $f_{j,\alpha}^k$ . Then,

$$D \leq NK_\alpha^p T^{\frac{p}{q}} \|\hat{v} - \hat{u}\|_{L^p(Q_T)^{mN}}^p.$$

Integrating between 0 and  $T$  leads to

$$\|\underline{v} - \underline{u}\|_{L^p(Q_T)^{mN}}^p \leq NK_\alpha^p T^{1+\frac{p}{q}} \|\hat{v} - \hat{u}\|_{L^p(Q_T)^{mN}}^p$$

and then to

$$\|\underline{v} - \underline{u}\|_{L^p(Q_T)^{mN}} \leq NK_\alpha T \|\hat{v} - \hat{u}\|_{L^p(Q_T)^{mN}}.$$

Now, all we need is to choose  $T$  such that

$$(9) \quad T \leq \frac{1}{NK_\alpha + 1}.$$

For such a  $T$ , the map  $\mathcal{B} \rightarrow \mathcal{B}, \hat{u} \mapsto \underline{u}$  is a contraction. We are now in position to use the Picard Theorem: there exists a unique function  $\underline{u} \in \mathcal{B}$  such that

$$\forall t \in [0, T], \underline{u}(t) = e^{-tA_p} u^0 + \int_0^t e^{-(t-s)A_p} f_\alpha(\underline{u})(s) ds \quad , \quad \underline{u}(0) = u^0.$$

Here, we stress that  $T$  does not depend on  $p$ .

- (4) According to [27, Corollary 3.3, p. 113], to prove that  $\underline{u}$  is a solution of the abstract Cauchy problem (2), it is enough to verify that

- $f_\alpha(\underline{u}) \in L^1(0, T; \mathbb{X}_p)$ ,
- $f_\alpha(\underline{u})$  is locally Lipschitz continuous on  $]0, T[$ .

The first condition is obvious since  $f_\alpha(\underline{u})$  is bounded.

Let  $t, t+h \in ]0; T]$ ,  $h > 0$ . We have

$$\begin{aligned} E &:= \|f_\alpha(\underline{u})(t+h) - f_\alpha(\underline{u})(t)\|_{\mathbb{X}_p}^p = \|f_\alpha(\underline{u}(t+h, \cdot)) - f_\alpha(\underline{u}(t, \cdot))\|_{\mathbb{X}_p}^p \\ &= \sum_{j=1}^m \sum_{k=1}^N \int_{\Omega} |f_{j,\alpha}^k(\underline{u}(t+h, x)) - f_{j,\alpha}^k(\underline{u}(t, x))|^p dx. \end{aligned}$$

But,  $f_\alpha$  is Lipschitz continuous and then

$$E \leq NK_\alpha^p \sum_{j=1}^m \sum_{k=1}^N \int_{\Omega} |\underline{u}_j^k(t+h, x) - \underline{u}_j^k(t, x)|^p dx = C \|\underline{u}(t+h) - \underline{u}(t)\|_{\mathbb{X}_p}^p$$

where  $C$  denotes a constant depending on the data,  $T$  and  $\alpha$ , but not  $h$ . Therefore,  $f_\alpha(\underline{u})$  is locally Lipschitz continuous. According to [27, Corollary 3.3, p. 113],  $\underline{u}$  is a solution of the abstract Cauchy problem (2).

(5) Now, let us prove that  $\underline{u}(t) \in \mathbb{X}_\infty$ . Let us start off with the functional equality

$$\underline{u}(t) = e^{-tA_p} u^0 + \int_0^t e^{-(t-s)A_p} f_\alpha(\underline{u})(s) ds,$$

valid in  $\mathbb{X}_p$  ( $p = 2$ , for instance). Then

$$(10) \quad \|\underline{u}(t)\|_{\mathbb{X}_\infty} \leq \|e^{-tA_p} u^0\|_{\mathbb{X}_\infty} + \int_0^t \|e^{-(t-s)A_p} f_\alpha(\underline{u})(s)\|_{\mathbb{X}_\infty} ds.$$

But,  $u_0 \in \mathbb{X}_\infty$ . Since semigroups  $e^{-tA_p}$  are consistent,  $e^{-tA_p} u^0 = e^{-tA_\infty} u^0$  and

$$(11) \quad \|e^{-tA_p} u^0\|_{\mathbb{X}_\infty} = \|e^{-tA_\infty} u^0\|_{\mathbb{X}_\infty} \leq \|u^0\|_{\mathbb{X}_\infty}$$

because  $e^{-tA_\infty}$  is contractive. For the same reason, since  $f_\alpha$  is bounded,  $f_\alpha(\underline{u})(s) \in \mathbb{X}_\infty$  and

$$(12) \quad \int_0^t \|e^{-(t-s)A_p} f_\alpha(\underline{u})(s)\|_{\mathbb{X}_\infty} ds \leq \int_0^t \|f_\alpha(\underline{u})(s)\|_{\mathbb{X}_\infty} ds \leq TM_\alpha,$$

where  $M_\alpha := \sup_{1 \leq j \leq m, 1 \leq k \leq N} \sup_{x \in [-\alpha, \alpha]} |f_j^k(x)|$ . Finally, (10), (11) and (12) show that

$$\|\underline{u}(t)\|_{\mathbb{X}_\infty} \leq \|u^0\|_{\mathbb{X}_\infty} + TM_\alpha.$$

Consequently,  $\underline{u}$  is uniformly bounded on  $[0; T)$ . Moreover, imposing the limitation

$$(13) \quad T \leq \frac{\alpha - 1 - \|u^0\|_{\mathbb{X}_\infty}}{M_\alpha + 1},$$

we obtain

$$\forall t \in (0, T), \|\underline{u}(t)\|_{\mathbb{X}_\infty} \leq \alpha - 1.$$

Consequently,  $f(\underline{u})$  and  $f_\alpha(\underline{u})$  are equal for  $t \in [0; T)$ , and  $\underline{u}$  is also solution of the abstract Cauchy problem

$$\frac{d\underline{u}}{dt} + A_p \underline{u} = f(\underline{u}) \quad , \quad \underline{u}(0) = u^0 \in \mathbb{X}_\infty.$$

Conditions (7), (8), (9) and (13) lead us to take

$$T := \min \left( 1, \frac{1}{mN \|f_\alpha\|_\infty + 1}, \frac{1}{NK_\alpha + 1}, \frac{\alpha - 1 - \|u^0\|_{\mathbb{X}_\infty}}{M_\alpha + 1} \right).$$

We have proved existence and uniqueness of a solution of (6) on  $[0, T)$ .

- (6) So, we have proved existence on  $[0, T)$  of a function  $u$  uniformly bounded on  $[0; T)$ , solution of the abstract Cauchy problems

$$\forall p \in (1, +\infty), \frac{du}{dt} + A_p u = f(u) \quad , \quad u(0) = u^0 \in \mathbb{X}_\infty.$$

Let us stress that since extrapolations  $(e^{-tA_p})_{t \geq 0}$  are consistent, solutions do not depend on  $p$ .  $\square$

**Remark 2.26.** *Note that the previous proof provides a "minimum lifetime"  $T$  for (6), and that  $T$  is substantially characterized by  $\|u^0\|_{\mathbb{X}_\infty}$ .*

**Remark 2.27.** *We turn now to the slightly modified system*

$$\frac{du}{dt} + A_p u = f(u, x) \quad , \quad u(0) = u^0 \in \mathbb{X}_\infty$$

where  $f = (f_j^k)_{1 \leq j \leq m, 1 \leq k \leq N}$ . Suppose an addition that, for  $(j, k)$  living in  $[[1, m] \times [[1, N]$ , the following properties hold:

- $f_j^k$  is locally Lipschitz continuous on  $\mathbb{R}^N \times \Omega$  with respect to  $u$ ,
- There exists a function  $\varphi_j^k$  bounded on every compact such that

$$\forall (u, x) \in \mathbb{R}^N \times \Omega, |f_j^k(u, x)| \leq \varphi_j^k(u)$$

- $f$  is locally Lipschitz continuous uniformly with respect to  $x$ , i.e. ,

$$\forall K \subset \mathbb{R}^N, K \text{ compact}, \exists \mu \geq 0 : \forall u, v \in \mathbb{R}^N, \forall x \in \Omega, |f_j^k(u, x) - f_j^k(v, x)| \leq \mu |u - v|.$$

Then, we readily check that the local existence Theorem is still valid.

To define a maximal  $\mathbb{X}_\infty$ -solution of (6), we proceed as follows. For  $1 < p < +\infty$ , consider the maximal solution  $(u, [0, T_p^*))$  of (6) in  $\mathbb{X}_p$  (note that  $u$  does not depend on  $p$ ). The previous theorem shows that  $\inf_{1 < p < +\infty} T_p^* > 0$ . Let  $\tilde{T} := \inf_{1 < p < +\infty} T_p^*$ . Using once again the above theorem, we find that there exists  $T \in (0, \tilde{T}]$  such that  $u(t)$  is uniformly bounded on  $[0, T)$ .

**Definition 2.28.** *Let  $T^* := \sup T, 0 < T \leq \tilde{T}$  such that  $u(t)$  is uniformly bounded on  $[0, T)$ . We say that  $(u, [0, T^*))$  is the maximal  $\mathbb{X}_\infty$ -solution of the abstract Cauchy problem (6).*

**Property 2.29.** *Let us keep notations and assumptions of the previous theorem. Then*

- either  $T^* = +\infty$ ,
- or  $T^* < +\infty$  and  $\|u(t)\|_{\mathbb{X}_\infty}$  is not bounded in the neighborhood of  $T^{*-}$ .

*Proof.* Let  $u(t)$  be a solution on  $[0; T)$  uniformly bounded on  $[0; T), T < +\infty$ . Denote  $S := \sup_{0 \leq t < T} \|u(t)\|_{\mathbb{X}_\infty}$ . Let us choose  $\alpha > S + 1$  and take

$$\tau = \min \left( 1, \frac{1}{mN \|f_\alpha\|_\infty + 1}, \frac{1}{NK_\alpha + 1}, \frac{\alpha - 1 - S}{M_\alpha + 1} \right).$$

According to the previous theorem, the equation

$$(14) \quad \frac{d\tilde{u}}{dt} + A_p \tilde{u} = f(\tilde{u}) \quad , \quad \tilde{u} \left( T - \frac{\tau}{2} \right) = u \left( T - \frac{\tau}{2} \right)$$

has a unique solution on  $(T - \frac{\tau}{2}, T + \frac{\tau}{2})$ . It remains to concatenate  $u$  and  $\tilde{u}$  to find a solution of (6) which extends  $u$  to a solution of (6) on  $[0, T + \frac{\tau}{2})$ .  $\square$

**Property 2.30** (Regularity).

$$\forall t \in (0; T^*), u(t) \in \bigcap_{\gamma < 1} (C^{1,\gamma}(\overline{\Omega}))^{mN} .$$

*Proof.* By Sobolev embedding (see [12, Theorem 6, p. 286]), we have

$$W^{k,p}(\Omega) \subset C^{k - [\frac{1}{p}] - 1, \gamma}(\overline{\Omega}) \quad \text{where} \quad \gamma := \begin{cases} [\frac{1}{p}] + 1 - \frac{1}{p}, & \text{if } \frac{1}{p} \notin \mathbb{N}, \\ \text{any positive number} < 1, & \text{if } \frac{1}{p} \in \mathbb{N}. \end{cases}$$

In particular, if  $k = 2$ :

$$W^{2,p}(\Omega) \subset C^{2 - [\frac{1}{p}] - 1, \gamma}(\overline{\Omega}).$$

Since this is true for  $p$  arbitrarily large, it follows that

$$\forall t \in (0; T^*), u(t) \in \bigcap_{\gamma < 1} (C^{1,\gamma}(\overline{\Omega}))^{mN} .$$

$\square$

**2.7. Positivity.** Let us assume that the quasi-positivity (**P**) Property (15)

$$\forall (j, k) \in \llbracket 1, m \rrbracket \times \llbracket 1, N \rrbracket, \forall (r^1, \dots, r^N) \in \mathbb{R}_+^N, f_j^k(r^1, \dots, r^{k-1}, 0, r^{k+1}, \dots, r^N) \geq 0$$

holds. Let us consider the modified system ( $\tilde{\mathbf{S}}$ ) obtained from (**S**) replacing the functions  $f_j^k$  by the functions  $\tilde{f}_j^k$ , where

$$\tilde{f}_j^k(r_j^1, \dots, r_j^N) = \begin{cases} f_j^k(r_j^1, \dots, r_j^N), & \text{if } \forall k \in \llbracket 1, N \rrbracket, r_j^k \geq 0, \\ f_j^{k+}(r_j^1, \dots, r_j^N), & \text{else .} \end{cases}$$

Denote  $(\tilde{u}_j^k)_{1 \leq k \leq N, 1 \leq j \leq m}$  the weak solution of ( $\tilde{\mathbf{S}}$ ), i.e.

$$\forall v \in H, \left\langle \frac{d\tilde{u}}{dt}, v \right\rangle + \mathbf{a}(\tilde{u}, v) = \left( \tilde{f} \middle| v \right),$$

where  $\langle \cdot, \cdot \rangle$  denotes the pairing between  $\mathbb{H}$  and its dual space  $\mathbb{H}'$ . Note that, since  $\frac{d\tilde{u}}{dt} \in \mathbb{X}_2$  (because it is a solution of the abstract Cauchy problem), the previous equation can be rewritten as

$$\forall v \in H, \left( \frac{d\tilde{u}}{dt} \middle| v \right) + \mathbf{a}(\tilde{u}, v) = \left( \tilde{f} \middle| v \right).$$

Take  $v = \tilde{u}^- := (\tilde{u}_j^{k-})_{j,k}$  as test function. We have

$$\left\langle \frac{d\tilde{u}}{dt}, \tilde{u}^- \right\rangle + \mathbf{a}(\tilde{u}, \tilde{u}^-) = \left( \tilde{f} \middle| \tilde{u}^- \right).$$

- According to the Mignot Lemma (see [14, p. 31]), we have

$$\left\langle \frac{d\tilde{u}}{dt}, \tilde{u}^- \right\rangle = \sum_{j=1}^m \sum_{k=1}^N \frac{d}{dt} \int_{\Omega} \left( \int_0^{\tilde{u}_j^k(\cdot, x)} r^- dr \right) dx = -\frac{1}{2} \sum_{j=1}^m \sum_{k=1}^N \frac{d}{dt} \int_{\Omega} (\tilde{u}_j^{k-})^2 dx$$

where we used that

$$\int_0^z r^- dr = \begin{cases} 0, & \text{if } z > 0, \\ -\frac{z^2}{2}, & \text{if } z < 0, \end{cases} = -\frac{z^2}{2} \mathbb{1}_{z < 0}.$$

- Moreover, we have

$$\mathbf{a}(\tilde{u}, \tilde{u}^-) = \sum_{j=1}^m \sum_{k=1}^N c_j^k \int_{\Omega} (\tilde{u}_j^k)' (\tilde{u}_j^{k-})' dx = -\sum_{j=1}^m \sum_{k=1}^N c_j^k \int_{\Omega} ((\tilde{u}_j^k)')^2 \mathbb{1}_{\tilde{u}_j^k < 0} dx \leq 0.$$

- Furthermore, by construction,  $\tilde{u}_j^{k-} \tilde{f}_j^k(\tilde{u}_j) \geq 0$ .

From the previous inequalities, we deduce that

$$\sum_{j=1}^m \sum_{k=1}^N \frac{d}{dt} \int_{\Omega} (\tilde{u}_j^{k-})^2 dx \leq 0$$

and the nonnegative function  $t \mapsto \sum_{j=1}^m \sum_{k=1}^N \int_{\Omega} (\tilde{u}_j^{k-})^2 dx$  is nonincreasing. Since  $u_j^k(t=0) \geq 0$ , it is inferred that  $\tilde{u}_j^{k-} = 0$  and that  $\tilde{u}_j^k \geq 0$ . Finally, it is clear that  $\tilde{u}_j^k$  is also solution of **(S)**. By uniqueness of the solution, it is deduced that  $u_j^k \geq 0$ .

**2.8. Maximal regularity on network.** Let us recall that

- $\Omega := (0, 1)$ .
- $A (= A_2)$  is the unbounded linear operator associated with the graph (see 2.5).
- As it will cause no confusion, we will use the same letter to denote  $A$  and  $A_p$ , where  $A_p$  is defined in 2.16.

A straightforward application of Theorem A.41 shows that we have

**Theorem 2.31.** *The operator  $A$  has the maximal  $L^p$ -regularity Property on  $\mathbb{X}_q$ , for  $p, q \in [1, +\infty]$  (see Definition A.37). Consequently, for  $p = q$  and  $u_0 = 0$ , we have*

$$\|u\|_{L^p(0, T, (W^{1,p}(0,1))^m)} + \left\| \frac{du}{dt} \right\|_{L^p(0, T, \mathbb{X}_p)} + \|Au\|_{L^p(0, T, \mathbb{X}_p)} \leq C \|f\|_{L^p(0, T, \mathbb{X}_p)}$$

where  $C$  denotes a constant depending only on the data,  $p$  and  $T$ . Coming back to the functions  $u_j^k$ , we obtain

$$\begin{aligned} \sum_{j=1}^m \sum_{k=1}^N \left( \int_0^T \int_{\Omega} |u_j^k|^p + \int_0^T \int_{\Omega} |(u_j^k)'|^p + \int_0^T \int_{\Omega} |\partial_t u_j^k|^p + \int_0^T \int_{\Omega} |(u_j^k)''|^p \right) \\ \leq C \|f\|_{L^p(0, T, \mathbb{X}_p)}. \end{aligned}$$

**Remark 2.32.** *Note that we may assume that  $C$  is a nondecreasing function with respect to  $T$ . Consequently, if  $T^* < +\infty$ , the previous theorem is valid for  $T = T^*$  and  $C$  may be extended to a nondecreasing function  $C : [0, +\infty) \rightarrow [0, +\infty)$ .*

**Remark 2.33.** *If  $f \in L^p(0, T; \mathbb{X}_p)$ , then  $u_j^k \in W^{1,p}([0, T] \times \Omega)$ .*

**Corollary 2.34.** *By continuity of the trace function, we have*

$$\begin{aligned} & \|\partial_t u_j^k\|_{L^p((0,T)\times\Omega)} + \|(u_j^k)''\|_{L^p((0,T)\times\Omega)} + \sup_{0\leq t\leq T} \|u_j^k(t)\|_{L^p(\Omega)} \\ & \quad + \|u_j^k(\cdot, 0)\|_{L^p(0,T)} + \|u_j^k(\cdot, 1)\|_{L^p(0,T)} \leq C\|f\|_{L^p(0,T;\mathbb{X}_p)}. \end{aligned}$$

**2.9. Poincaré inequality on network.** For  $p \in [1, +\infty]$ , let

$$H_p := \left\{ u = (u_1, \dots, u_m) \in (W^{1,p}(0,1))^m : \exists d^u \in \mathbb{R}^{n-1} \times \{0\}, \begin{cases} {}^t\phi^+ d = u(1), \\ {}^t\phi^- d = u(0) \end{cases} \right\},$$

where the condition:  $\exists d^u \in \mathbb{R}^{n-1} \times \{0\} / \begin{cases} {}^t\phi^+ d = u(1), \\ {}^t\phi^- d = u(0) \end{cases}$  means continuity on

network, as well as a Dirichlet condition at vertex  $\mathbf{v}_n$ . Likewise, let  $\mathbb{H}_p := (H_p)^N = H_p \times \dots \times H_p$ .

**Property 2.35** (Poincaré Inequality). *Let  $u \in H_p$ . Then*

$$\forall p \in [1, +\infty], \|u\|_{H_p} \leq m \|u'\|_{H_p}.$$

*Proof.* The proof of the case  $p = 1$  (see [25]) is easily generalized to every  $p \in [1, +\infty)$ . It is based on the connectedness of the graph and the usual Poincaré inequality on a interval. Passing to the limit as  $p \rightarrow +\infty$  yields the result for  $p = +\infty$ .  $\square$

### 3. THE CASE OF GLOBALLY CONSERVATIVE BOUNDARY CONDITIONS

**3.1. New problem.** In this paragraph, we replace the Dirichlet condition at  $\mathbf{v}_n$  with an assumption of continuity and a Kirchhoff condition at  $\mathbf{v}_n$ . Roughly speaking, this corresponds to the passage from  $H_0^1(\Omega)$  to  $H^1(\Omega)$  in the standard case of an open subset of  $\mathbb{R}^n$ . More specifically, we are now interested in the new problem

$$(\tilde{\mathbf{S}}) : \begin{cases} \partial_t u_j^k = c_j^k (u_j^k)'' + f_j^k(u_j^1, \dots, u_j^N), & 1 \leq j \leq m, 1 \leq k \leq N, 0 < x < 1, \\ u_j^k(t, \mathbf{v}_i) = u_\ell^k(t, \mathbf{v}_i) := d_i^k(t), & j, \ell \in \Gamma(\mathbf{v}_i), 1 \leq i \leq n, 1 \leq k \leq N, \\ \sum_{j=1}^m \phi_{i,j} c_j^k (u_j^k)'(t, \mathbf{v}_i) = 0, & 1 \leq i \leq n, 1 \leq k \leq N, \\ u_j^k(0, \cdot) = u_{0,j}^k, & 1 \leq j \leq m \end{cases}$$

subjected to the initial conditions

$$\begin{cases} u_{0,j}^k \geq 0, & 1 \leq j \leq m, 1 \leq k \leq N, \\ u_{0,j}^k \in L^\infty((0,1)), & 1 \leq j \leq m, 1 \leq k \leq N. \end{cases}$$

The weighted incidence matrices  $\Phi_w^{k,+} = (\omega_{i,j}^{k,+}) \in \mathcal{M}_{n,m}(\mathbb{R})$  and  $\Phi_w^{k,-} = (\omega_{i,j}^{k,-}) \in \mathcal{M}_{n,m}(\mathbb{R})$  are defined by

$$\omega_{i,j}^{k,+} := \begin{cases} c_j^k, & \text{if } \phi_{i,j}^{k,+} = 1, \\ 0, & \text{else} \end{cases} \quad \text{and} \quad \omega_{i,j}^{k,-} = \begin{cases} c_j^k, & \text{if } \phi_{i,j}^{k,-} = 1, \\ 0, & \text{else.} \end{cases}$$

With regard to weighted incidence matrices, continuity at vertices and Kirchhoff conditions become, in a more streamline form

$$\forall t, \forall k \in \llbracket 1, N \rrbracket, \exists d^{u^k}(t) \in \mathbb{R}^n : \begin{cases} {}^t\Phi_w^{k,+} d^{u^k}(t) = u^k(t, 1) \quad \text{and} \quad {}^t\Phi_w^{k,-} d^{u^k}(t) = u^k(t, 0), \\ \Phi_w^{k,-} (u^k)'(t, 0) = \Phi_w^{k,+} (u^k)'(t, 1). \end{cases}$$



**3.2. Similarities.** Some properties proved in the case of a Dirichlet condition at  $\mathbf{v}_n$  still hold. Let us summarize the main properties that persist. Let us define the new spaces  $H$  and  $\mathbb{H}$  by

$$H := \left\{ u^k \in (H^1(0, 1))^m : \exists d^{u^k} \in \mathbb{R}^n, \begin{cases} {}^t\phi^+ d^{u^k} = u^k(1), \\ {}^t\phi^- d^{u^k} = u^k(0) \end{cases} \right\},$$

$$\mathbb{H} := \left\{ u \in (H^1(0, 1))^{mN} : \forall k \in \llbracket 1, N \rrbracket, \exists d^{u^k} \in \mathbb{R}^n, \begin{cases} {}^t\phi^+ d^{u^k} = u^k(1), \\ {}^t\phi^- d^{u^k} = u^k(0) \end{cases} \right\}.$$

$H$  and  $\mathbb{H}$  are provided with their usual inner products:

$$\forall (u^k, v^k) \in H \times H, (u^k | v^k)_H := \sum_{j=1}^m \int_0^1 [u_j^k v_j^k + (u_j^k)' (v_j^k)'],$$

$$\forall (u, v) \in \mathbb{H} \times \mathbb{H}, (u | v)_{\mathbb{H}} = \sum_{k=1}^N (u^k | v^k)_H := \sum_{k=1}^N \sum_{j=1}^m \int_0^1 [u_j^k v_j^k + (u_j^k)' (v_j^k)'].$$

**Property 3.1.**  $H$  (respectively  $\mathbb{H}$ ) is densely and compactly embedded in  $X_2$  (respectively  $\mathbb{X}_2$ ). Moreover,  $H$  (respectively  $\mathbb{H}$ ) is a Hilbert space.

**Remark 3.2.** The Poincaré inequality being no more relevant,  $u \mapsto \sum_{k=1}^N \sum_{j=1}^m \int_0^1 ((u_j^k)')^2$  is not a norm anymore, but only a seminorm.

We have the following formula again:

**Property 3.3.** Let  $u^k \in H$  satisfying Kirchhoff conditions. Then

$$\forall w \in H, \sum_{j=1}^m c_j^k [(u_j^k)'(1)w_j^k(1) - (u_j^k)'(0)w_j^k(0)] = 0.$$

We also define the two bilinear forms  $\mathbf{a}^k$ ,  $\mathbf{a}$  on respectively  $H$ ,  $\mathbb{H}$  by

$$\mathbf{a}^k(u^k, v^k) := \sum_{j=1}^m c_j^k \int_0^1 (u_j^k)'(x) (v_j^k)'(x) dx$$

and

$$\mathbf{a}(u, v) := \sum_{j=1}^m \sum_{k=1}^N c_j^k \int_0^1 (u_j^k)'(x) (v_j^k)'(x) dx.$$

Then,  $\mathbf{a}$  is

- densely defined,
- continuous on  $\mathbb{H} \times \mathbb{H}$ :  $\exists M \geq 0 : \forall (u, v) \in \mathbb{H} \times \mathbb{H}, |\mathbf{a}(u, v)| \leq M \|u\|_{\mathbb{H}} \|v\|_{\mathbb{H}}$ ,
- symmetric,
- positive:  $\forall u \in \mathbb{H}, \mathbf{a}(u, u) \geq 0$ .

$A$  (or  $A_2$ ) will still denote the operator associated with the bilinear form  $\mathbf{a}$ .

**Property 3.4.** • The unbounded linear operator  $(A^k, D(A^k))$  associated with  $\mathbf{a}^k$  is equal to:

$$A^k = \left( -\text{diag} \left( c_j^k \frac{d^2}{dx^2} \right), 1 \leq j \leq m \right),$$

$$D(A^k) = \left\{ u^k \in (H^2(0, 1))^m : \exists d^{u^k} \in \mathbb{R}^n, \begin{cases} {}^t\phi^+ d^{u^k} = u^k(1), \\ {}^t\phi^- d^{u^k} = u^k(0), \\ \phi_w^{k,-}(u^k)'(0) = \phi_w^{k,+}(u^k)'(1). \end{cases} \right\}$$

- The unbounded linear operator associated with  $\mathfrak{a}$  is equal to  $(A, D(A))$ , where  $D(A) = \prod_{k=1}^N D(A^k)$  and  $A = \prod_{k=1}^N A^k$ .

By considering the particular case  $N = 1$ , we see that semigroup for on component enjoys the same properties of the semigroup for  $N$  components.

- Property 3.5.**
- $-A$  is the infinitesimal generator of an analytic semigroup of contractions on  $\mathbb{X}_2$ . Interchangeably,  $(T_2(t))_{t \geq 0}$  or  $(e^{-tA})_{t \geq 0}$  denotes this semigroup.
  - $A$  is a self-adjoint operator on  $\mathbb{X}_2$ ; for all  $t \geq 0$ ,  $T_2(t)$  is self-adjoint.
  - The semigroup  $(e^{-tA})_{t \geq 0}$  associated with the bilinear form  $\mathfrak{a}$  is sub-markovian, i.e.
    - positive,
    - $\mathbb{X}_\infty$ -contractive.

- Theorem 3.6.**
- $(T_2(t))_{t \geq 0}$  extrapolates to a family of contractive semigroups  $(T_p(t))_{t \geq 0}$  on  $\mathbb{X}_p$ ,  $1 \leq p \leq \infty$ .
  - $(T_p(t))_{t \geq 0}$  is real positive for  $p \in [1, +\infty]$ .
  - $(T_p(t))_{t \geq 0}$  is strongly continuous for  $1 \leq p < \infty$  and analytic for  $1 < p < \infty$ .

**Property 3.7.** Let  $p \in (1, +\infty)$ . Then

- $-A_p$  has compact resolvent.
- The semigroup  $(e^{-tA_p})_{t \geq 0}$  is compact.
- $\sigma(A_p)$  does not depend on  $p$ .

**Remark 3.8.** It should be noted that, unlike the case previously studied (with a Dirichlet condition at vertex  $\mathbf{v}_n$ ), values  $p = 1$  and  $p = +\infty$  are excluded. This is due to the fact that ultracontractivity (which doesn't hold anymore) is closely linked to the Dirichlet condition.

We have the following description of  $A_p$ .

**Property 3.9.** Let  $p \in [1, +\infty]$ .

- Let  $-A_p^k$  denote the infinitesimal generator of the semigroup  $(T_p^k(t))_{t \geq 0}$ . Then

$$D(A_p^k) = \left\{ u \in (W^{2,p}(\Omega))^m : \exists d^{u^k} \in \mathbb{R}^n, \begin{cases} {}^t\phi^+ d^{u^k} = u^k(1), \\ {}^t\phi^- d^{u^k} = u^k(0), \\ \phi_w^{k,-}(u^k)'(0) = \phi_w^{k,+}(u^k)'(1) \end{cases} \right\}$$

and the action of  $A_p^k$  on  $D(A_p^k)$  is given by:  $A_p^k = -\text{diag} \left( c_j^k \frac{d^2}{dx^2}, 1 \leq j \leq m \right)$ .

- $D(A_p) = \prod_{k=1}^N D(A_p^k)$  and  $A_p = \prod_{k=1}^N A_p^k$ .

Finally, note that the local existence Theorem still holds.

**3.3. But also some differences.** The bilinear form  $\mathbf{a}$  is not coercive anymore, but it is positive. Exponential stability doesn't hold anymore since  $0 \in \sigma(A)$  and so  $\omega(T) = s(A) = 0$ . Likewise, the Poincaré inequality is no longer valid. Ultracontractivity does not hold neither. Indeed, according to Theorem A.29, ultracontractivity is equivalent to (here,  $d = 1$ ):

$$\exists c \geq 0 : \forall u \in D(\mathbf{a}) \cap X_1, \|u\|_{X_2}^{2+\frac{4}{d}} \leq c\mathbf{a}(u, u) \|u\|_{X_1}^{\frac{4}{d}}.$$

This property is clearly not fulfilled (consider constant functions). Roughly speaking, the Poincaré inequality having disappeared, we can't estimate  $u$  according to  $u'$  anymore. Nevertheless, a Poincaré-Wirtinger type inequality can be established, which is our next goal. To do this, we will need a few preliminaries.

**Definition 3.10.** For  $p \in (1, +\infty)$ ,  $H_p$  will denote the vector space

$$H_p = \left\{ u^k \in (W^{1,p}(0,1))^m : \exists d^{u^k} \in \mathbb{R}^{n-1} \times \{0\}, \begin{cases} {}^t\phi^+ d^{u^k} = u^k(1), \\ {}^t\phi^- d^{u^k} = u^k(0) \end{cases} \right\}$$

provided with its usual norm

$$\forall u^k \in H_p, \|u^k\|_{H_p} = \left( \sum_{j=1}^m \int_0^1 \left[ (u_j^k)^p + |(u_j^k)'|^p \right] \right)^{\frac{1}{p}}.$$

Likewise, we define the vector space

$$\mathbb{H}_p := \left\{ u \in (W^{1,p}(0,1))^{mN} : \forall k \in \llbracket 1, N \rrbracket, \exists d^{u^k} \in \mathbb{R}^{n-1} \times \{0\}, \begin{cases} {}^t\phi^+ d^{u^k} = u^k(1), \\ {}^t\phi^- d^{u^k} = u^k(0) \end{cases} \right\}$$

provided with its usual norm

$$\forall u \in \mathbb{H}_p, \|u\|_{\mathbb{H}_p} := \left( \sum_{j=1}^m \sum_{k=1}^N \int_0^1 \left[ (u_j^k)^p + |(u_j^k)'|^p \right] \right)^{\frac{1}{p}}.$$

Note that, for  $p = 2$ , we have  $H_2 = H$  and  $\mathbb{H}_2 = \mathbb{H}$ .

**Property 3.11** (A Poincaré-Wirtinger inequality on network).

$$\forall p \in (1, +\infty), \exists C \geq 0 : \forall u \in H_p, \|u - (u)_G\|_{X_p} \leq C \|u'\|_{X_p},$$

where  $(u)_G := \frac{1}{m} \sum_{j=1}^m \int_0^1 u_j(x) dx$  denotes the average of  $u$  on the graph.

*Proof.* For the proof, we follow the guideline of [12, p. 292]. Nevertheless, we will have to pay a special attention to the continuity at vertices. To this end, we will use a weak convergence argument.

Let  $p > 1$ . Suppose by contradiction that the property is false. Therefore

$$(16) \quad \forall k \in \mathbb{N}^*, \exists u_k \in H_p : \|u_k - (u_k)_G\|_{X_p} > k \|Du_k\|_{X_p},$$

where  $u_k = (u_{k,1}, \dots, u_{k,m})$ . Let

$$\forall k \in \mathbb{N}^*, v_k = \frac{u_k - (u_k)_G}{\|u_k - (u_k)_G\|_{X_p}}.$$

Evidently, we have

$$(17) \quad \forall k \in \mathbb{N}^*, \|v_k\|_{X_p} = 1 \quad \text{and} \quad (v_k)_G = 0.$$

According to (16),

$$\forall k \in \mathbb{N}^*, 1 > k \|Dv_k\|_{X_p} \quad \text{i.e.} \quad \forall k \in \mathbb{N}^*, \|Dv_k\|_{X_p} < \frac{1}{k}$$

holds. As usual, let  $v_k = (v_{k,1}, \dots, v_{k,m})$ . For every  $j$  ( $1 \leq j \leq m$ ), the sequence  $(v_{k,j})_{k \geq 1}$  is bounded in  $W^{1,p}(\Omega)$  (since  $\|v_{k,j}\|_{X_p} \leq \|v_k\|_{X_p} = 1$  and  $\|Dv_{k,j}\|_{X_p} \leq \|Dv_k\|_{X_p} \leq \frac{1}{k}$ ). But, according to the Rellich-Kondrachov Theorem,  $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ , with compact embedding. Extracting  $m$  sub-sequences, we can assume that every sequence  $(v_{k,j})_{k \geq 1}$  is convergent in  $L^p(\Omega)$ . Denote  $V_j$  such that

$$\forall j \in \llbracket 1, m \rrbracket, v_{k,j} \xrightarrow[k \rightarrow +\infty]{} V_j \quad \text{in } L^p(\Omega)$$

i.e. , with  $V = (V_1, \dots, V_m)$

$$(18) \quad v_k \xrightarrow[k \rightarrow +\infty]{} V \quad \text{in } L^p(\Omega)^m.$$

We also know that, for  $1 < p < \infty$ ,  $W^{1,p}(0,1)$  is reflexive (more generally, for all  $1 < p < \infty$ ,  $W^{m,p}(\Omega)$  is reflexive, where  $\Omega$  denotes an open subset of  $\mathbb{R}^p$ : see [1, Theorem 3.5, p. 47]). Then so is the space  $(W^{1,p}(0,1))^m$ . Since the sequence  $(v_k)$  is bounded in the reflexive space  $(W^{1,p}(0,1))^m$ , extracting again  $m$  subsequences, we may assume that the sequence  $(v_k)$  is weakly convergent in  $(W^{1,p}(0,1))^m$ . Consequently, there exists  $\tilde{V} \in (W^{1,p}(0,1))^m$  such that

$$(19) \quad v_k \rightharpoonup \tilde{V} \quad \text{weakly in } (W^{1,p}(0,1))^m.$$

From (18) and (19), we deduce that  $\tilde{V} = V$ . Moreover,  $H_p$  is closed (because it is complete) and convex. It is deduced that  $H_p$  is weakly closed (see [9, Theorem III.7, p. 38]). Consequently,  $\tilde{V} \in H_p$  and then  $V = \tilde{V}$  is continuous on the graph. Then, (17) implies that

$$(20) \quad \|V\|_{X_p} = 1 \quad \text{and} \quad (V)_G = 0.$$

On the other hand, for  $j \in \llbracket 1, m \rrbracket$ , let us consider  $\varphi_j \in C_0^\infty(\Omega)$ . We have

$$\int_{\Omega} V_j \varphi_j' dx = \lim_{k \rightarrow +\infty} \int_{\Omega} v_{k,j} \varphi_j' dx = - \lim_{k \rightarrow +\infty} \int_{\Omega} v_{k,j}' \varphi_j dx = 0,$$

since  $\|Dv_k\|_{X_p} < \frac{1}{k}$ . It is deduced that  $V_j \in W^{1,p}(\Omega)$  and that  $V_j' = 0$  on  $\Omega$ . Consequently,  $V_j$  is constant on  $\Omega$ . Since  $V$  is continuous on the connected graph  $G$ ,  $V$  is constant on  $G$ . This contradicts (20).  $\square$

**3.4. The bilinear form of the globally conservative operator.** Later, we will need ultracontractivity to prove that solutions of the following abstract Cauchy problem

$$(\mathbf{ACP}) : \frac{du}{dt} + Au = f \quad , \quad u(0) = u_0$$

are global. Since, without a Dirichlet condition at vertex  $\mathbf{v}_n$ , ultracontractivity doesn't hold anymore, we rewrite  $(\mathbf{ACP})$  as

$$(\widetilde{\mathbf{ACP}}) : \frac{du}{dt} + (A + \text{Id})u = f + u \quad , \quad u(0) = u_0.$$

Let us introduce the new bilinear forms  $\tilde{\mathbf{a}}^k$  and  $\tilde{\mathbf{a}}$  respectively defined on  $H, \mathbb{H}$  by

$$\forall (u^k, v^k) \in H \times H, \tilde{\mathbf{a}}^k(u^k, v^k) = \sum_{j=1}^m c_j^k \int_0^1 (u_j^k)'(x) (v_j^k)'(x) dx + \sum_{j=1}^m \int_0^1 u_j^k(x) v_j^k(x) dx$$

and

$$\forall (u, v) \in \mathbb{H} \times \mathbb{H}, \tilde{\mathbf{a}}(u, v) = \sum_{j=1}^m \sum_{k=1}^N c_j^k \int_0^1 (u_j^k)'(x) (v_j^k)'(x) dx + \sum_{j=1}^m \sum_{k=1}^N \int_0^1 u_j^k(x) v_j^k(x) dx.$$

Clearly, the operator associated with  $\tilde{\mathbf{a}}$  is given by  $A + \text{Id}$ .

**Property 3.12.** *The bilinear forms  $\tilde{\mathbf{a}}^k$  and  $\tilde{\mathbf{a}}$  previously defined are*

- *closed,*
- *densely defined,*
- *continuous, i.e.*

$$\exists M^k \geq 0 : \forall (u^k, v^k) \in H \times H, |\tilde{\mathbf{a}}^k(u^k, v^k)| \leq M^k \|u^k\|_H \|v^k\|_H$$

and

$$\exists M \geq 0 : \forall (u, v) \in \mathbb{H} \times \mathbb{H}, |\tilde{\mathbf{a}}(u, v)| \leq M \|u\|_{\mathbb{H}} \|v\|_{\mathbb{H}},$$

- *symmetric,*
- *coercive i.e.*

$$\exists \alpha^k > 0 : \forall u^k \in H, \tilde{\mathbf{a}}^k(u^k, u^k) \geq \alpha^k \|u^k\|_H^2$$

and

$$\exists \alpha > 0 : \forall u \in \mathbb{H}, \tilde{\mathbf{a}}(u, u) \geq \alpha \|u\|_{\mathbb{H}}^2.$$

**Property 3.13.** *Let  $(\tilde{T}(t))_{t \geq 0}$  denote the semigroup generated by  $-(A + \text{Id})$ . Then,  $\tilde{T}(t) = e^{-t}T(t) = e^{-t}e^{-tA}$ . The main properties of  $\mathbf{a}$  and  $A$  are inherited by  $\tilde{\mathbf{a}}$  and  $\tilde{A} := A + \text{Id}$ . More precisely:*

- *$-\tilde{A}$  is the infinitesimal generator of an analytic semigroup of contractions on  $\mathbb{X}_2$ .*
- *$(\tilde{T}(t))_{t \geq 0}$  is positive,  $\mathbb{X}_\infty$ -contractive.*
- *$(\tilde{T}_2(t))_{t \geq 0}$  extrapolates to a family of real contractive semigroups  $(\tilde{T}_p(t))_{t \geq 0}$  on  $\mathbb{X}_p$ ,  $1 \leq p \leq \infty$ .*
  - *$(\tilde{T}_p(t))_{t \geq 0}$  is positive for  $p \in [1, +\infty]$ .*
  - *$(\tilde{T}_p(t))_{t \geq 0}$  is strongly continuous for  $1 \leq p < \infty$  and analytic for  $1 < p < \infty$ .*
- *$(\tilde{T}(t))_{t \geq 0}$  is ultracontractive; indeed, the proof of Property 2.15 is still valid. The important point to note here is that the bilinear form  $\tilde{\mathbf{a}}$  is coercive.*
- *Compactness, maximal regularity and positivity properties still hold.*
- *The local existence Theorem is still valid.*

**Property 3.14.** *Consider the following abstract Cauchy problem*

$$(\mathbf{ACP}) : \frac{du}{dt} + Au = f \quad , \quad u(0) = u^0 \in \mathbb{X}_\infty$$

where  $A$  is the unbounded linear operator previously defined (without Dirichlet condition at  $\mathbf{v}_n$ ). Let  $u$  be a solution of  $(\mathbf{ACP})$  on the interval  $[0, T)$ ,  $0 < T < +\infty$ . Assume that

$$\forall p \in (1, +\infty), f \in L^p(0, T; \mathbb{X}_p).$$

Then,  $u \in L^\infty(0, T; \mathbb{X}_\infty)$ .

*Proof.* We first observe that, since for all  $p \in (1, +\infty)$ ,  $f \in L^p(0, T; \mathbb{X}_p)$ , necessarily  $u \in L^p(0, T; \mathbb{X}_p)$  (see Property A.30). Consequently,

$$\forall p > 1, f + u \in L^p(0, T; \mathbb{X}_p).$$

Moreover, note that **(ACP)** is equivalent to the new abstract Cauchy problem

$$\widetilde{\text{(ACP)}} : \frac{du}{dt} + \tilde{A}u = f + u \quad , \quad u(0) = u^0 \in \mathbb{X}_\infty$$

where  $\tilde{A} = A + \text{Id}$ . Since  $\tilde{A}$  is ultracontractive (see Proposition 3.13), Lemma A.31 applies and  $u \in L^\infty(0, T; \mathbb{X}_\infty)$ .  $\square$

To sum up, we will remember that, with or without Dirichet condition at vertex  $\mathbf{v}_n$ ,  $L^p$  bounds of the right-hand term (for all  $p > 1$ ) implies  $L^\infty$  bound of the solution.

## Application to reaction-diffusion

In the two following sections, we propose to extend to one-dimensional networks two theorems concerning reaction-diffusion, the first proved by Martin and Pierre in [22], and the second by Haraux and Youkana in [17]. These results concerning global existence and asymptotic behavior of solutions are typical for the theories developed in the past 30 years.

### 4. APPLICATION I TO REACTION-DIFFUSION: A RESULT BY MARTIN AND PIERRE

Our goal is to extend to one-dimensional networks a theorem by Martin and Pierre (see [22, Theorem 4.2, p. 372]). We refer to paragraphs 4.4 and 4.5 for two practical cases.

For convenience of readers, we briefly recall the notations. Let  $\Omega$  denote the open interval  $(0, 1)$ . As usual, let  $\partial_t u$  denote the time derivative of  $u$  and  $u', u''$  the space derivatives of  $u$ . Recall that  $m$  denotes the number of edges on the graph  $G$ , and that  $Q_t$  denotes the open set  $(0, t) \times \Omega$ . Then, we are interested in the following problem

$$(21) \quad \begin{cases} \partial_t u_j - c_j^1 u_j'' = f_j(u_j, v_j), \\ \partial_t v_j - c_j^2 v_j'' = g_j(u_j, v_j) \end{cases}$$

with continuity on the graph, Kirchhoff conditions and Dirichlet condition at vertex  $\mathbf{v}_n$ . Functions  $f_j, g_j$  are supposed to have continuous derivative on  $[0, +\infty)^2$ ; assume that the quasi-positivity Property

$$\text{(P)} \quad : \quad \forall s \geq 0, f_j(0, s) \geq 0 \quad \text{and} \quad \forall r \geq 0, g_j(r, 0) \geq 0$$

holds. Assume also that the initial conditions  $u_0 = (u_{0,j})_{1 \leq j \leq m}$  and  $v_0 = (v_{0,j})_{1 \leq j \leq m}$  are nonnegative and belong to  $L^\infty(\Omega)^m$ . Thus, the solutions  $u$  and  $v$  of (21) are nonnegative.

Moreover, we will assume that the nonlinearities verify the following mass-control structure **(M)**

$$\text{(M)} \quad : \quad \exists L, M \geq 0 : \forall j \in [1, m], \forall r, s \geq 0, f_j(r, s) + g_j(r, s) \leq L(r + s) + M.$$

According to Theorem 2.25 and Property 2.29, System (21) has a unique maximal solution  $(u, v)$  on an interval  $[0, T^*)$ , with  $0 < T^* \leq +\infty$ . Our goal is to prove that  $T^* = +\infty$ .

Throughout the rest of this paragraph, and unless otherwise specified,  $C$  will denote a generic constant depending only on the data.

**4.1. Estimates in  $L^p$  ( $1 < p < +\infty$ ).** We recall that the method of Martin and Pierre is based on  $L^p$  estimates via maximal regularity.

**Lemma 4.1.** *For all  $p \in (1, +\infty)$  and all  $T < T^*$ , there exists a constant  $C$  depending only on the data,  $p$  and  $T$  such that*

$$(22) \quad \|v\|_{L^p(Q_T)^m} \leq C \left( 1 + \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} + \|u\|_{L^p(Q_T)^m} \right).$$

*Proof.* Adding the two equations of (21) leads to

$$(23) \quad \partial_t(u_j + v_j) - c_j^1 u_j'' - c_j^2 v_j'' = f_j(u_j, v_j) + g_j(u_j, v_j).$$

For nonnegative  $\chi_j \in C_0^\infty(Q_T)$ , let us consider the dual problem

$$(24) \quad -\partial_t \phi_j - c_j^2 \phi_j'' = \chi_j \quad \text{with } \phi_j(T, \cdot) = 0$$

with continuity on the graph, Kirchhoff conditions at vertices  $\mathbf{v}_i$ ,  $1 \leq i \leq n-1$ , and Dirichlet condition at vertex  $\mathbf{v}_n$  (see Remark 2.27 concerning the existence of such solutions). Since the right-hand side of Equation (24) is nonnegative, it follows that the functions  $\phi_j$  are also nonnegative. Let multiply (23) by  $e^{-tL} \phi_j$ :

$$\begin{aligned} \partial_t(u_j + v_j)e^{-tL} \phi_j - (c_j^1 u_j'' + c_j^2 v_j'')e^{-tL} \phi_j &= (f_j(u_j, v_j) + g_j(u_j, v_j))e^{-tL} \phi_j \\ &\leq L(u_j + v_j)e^{-tL} \phi_j + M e^{-tL} \phi_j \end{aligned}$$

i.e.

$$\partial_t(u_j + v_j)e^{-tL} \phi_j - L(u_j + v_j)e^{-tL} \phi_j \leq (c_j^1 u_j'' + c_j^2 v_j'')e^{-tL} \phi_j + M e^{-tL} \phi_j$$

i.e.

$$\frac{\partial}{\partial t} [(u_j + v_j)e^{-tL}] \phi_j \leq (c_j^1 u_j'' + c_j^2 v_j'')e^{-tL} \phi_j + M e^{-tL} \phi_j.$$

Integrate on  $[0, T]$ , and integrate by parts. Since  $\phi_j(T, \cdot) = 0$ , we have

$$\begin{aligned} -(u_{0,j} + v_{0,j})\phi_j(0, \cdot) - \int_0^T (u_j + v_j)e^{-tL} \partial_t \phi_j \, dt \\ \leq \int_0^T (c_j^1 u_j'' + c_j^2 v_j'')e^{-tL} \phi_j \, dt + M \int_0^T e^{-tL} \phi_j \, dt. \end{aligned}$$

Then, integrate on  $\Omega = (0, 1)$ . Perform an integration by parts two times with respect to  $x$ . We obtain

$$\begin{aligned}
 & - \int_{\Omega} (u_{0,j} + v_{0,j}) \phi_j(0, \cdot) \, dx - \int_0^T \int_{\Omega} (u_j + v_j) e^{-tL} \partial_t \phi_j \, dt \, dx \\
 & \leq \int_0^T \int_{\Omega} (c_j^1 u_j'' + c_j^2 v_j'') e^{-tL} \phi_j \, dx \, dt + M \int_0^T \int_{\Omega} e^{-tL} \phi_j \, dx \, dt \\
 & = \int_0^T [(c_j^1 u_j'(\cdot, 1) + c_j^2 v_j'(\cdot, 1)) e^{-tL} \phi_j(1) - (c_j^1 u_j'(\cdot, 0) + c_j^2 v_j'(\cdot, 0)) e^{-tL} \phi_j(\cdot, 0)] \, dt \\
 & - \int_0^T \int_{\Omega} (c_j^1 u_j' + c_j^2 v_j') e^{-tL} \phi_j' \, dx \, dt + M \int_0^T \int_{\Omega} e^{-tL} \phi_j \, dx \, dt \\
 & = \int_0^T [(c_j^1 u_j'(\cdot, 1) + c_j^2 v_j'(\cdot, 1)) e^{-tL} \phi_j(\cdot, 1) - (c_j^1 u_j'(\cdot, 0) + c_j^2 v_j'(\cdot, 0)) e^{-tL} \phi_j(\cdot, 0)] \, dt \\
 & - \int_0^T [(c_j^1 u_j(\cdot, 1) + c_j^2 v_j(\cdot, 1)) e^{-tL} \phi_j'(\cdot, 1) - (c_j^1 u_j(\cdot, 0) + c_j^2 v_j(\cdot, 0)) e^{-tL} \phi_j'(\cdot, 0)] \, dt \\
 & + \int_0^T \int_{\Omega} (c_j^1 u_j + c_j^2 v_j) e^{-tL} \phi_j'' \, dx \, dt + M \int_0^T \int_{\Omega} e^{-tL} \phi_j \, dx \, dt.
 \end{aligned}$$

In addition with the equality  $c_j^2 \phi_j'' = -\partial_t \phi_j - \chi_j$ , the previous inequality immediately leads to

$$\begin{aligned}
 & \int_{Q_T} (u_j + v_j) e^{-tL} \chi_j \\
 & \leq \int_{\Omega} (u_{0,j} + v_{0,j}) \phi_j(0, \cdot) + \int_{Q_T} [(c_j^1 - c_j^2) u_j e^{-tL} \phi_j'' + M e^{-tL} \phi_j] \\
 & + \int_0^T [(c_j^1 u_j'(\cdot, 1) + c_j^2 v_j'(\cdot, 1)) e^{-tL} \phi_j(\cdot, 1) - (c_j^1 u_j'(\cdot, 0) + c_j^2 v_j'(\cdot, 0)) e^{-tL} \phi_j(\cdot, 0)] \\
 & - \int_0^T [(c_j^1 u_j(\cdot, 1) + c_j^2 v_j(\cdot, 1)) e^{-tL} \phi_j'(\cdot, 1) - (c_j^1 u_j(\cdot, 0) + c_j^2 v_j(\cdot, 0)) e^{-tL} \phi_j'(\cdot, 0)] \\
 & =: I_j^1 + I_j^2 + I_j^3 + I_j^4.
 \end{aligned}$$

We estimate separately each of the above terms.

- Estimate of  $I_j^1$ .

$$\begin{aligned}
 |I_j^1| & = I_j^1 = \int_{\Omega} (u_{0,j} + v_{0,j}) \phi_j(0, \cdot) \leq \|u_{0,j} + v_{0,j}\|_{L^p(\Omega)} \|\phi_j(0, \cdot)\|_{L^q(\Omega)} \\
 & \leq (\|u_{0,j}\|_{L^p(\Omega)} + \|v_{0,j}\|_{L^p(\Omega)}) \|\phi_j(0, \cdot)\|_{L^q(\Omega)}.
 \end{aligned}$$

But Corollary 2.34 yields the estimate

$$\|\phi_j(0, \cdot)\|_{L^q(\Omega)} \leq C \|\chi\|_{L^q(Q_T)^m}.$$

Then, we obtain

$$|I_j^1| \leq C \left( \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} \right) \|\chi\|_{L^q(Q_T)^m}.$$



- Estimate of  $I_j^2$ .

$$\begin{aligned}
|I_j^2| &= \left| \int_{Q_T} [(c_j^1 - c_j^2)u_j e^{-tL} \phi_j'' + M e^{-tL} \phi_j] \right| \\
&\leq \int_{Q_T} [|c_j^1 - c_j^2| u_j |\phi_j''| + M \phi_j] e^{-tL} \leq \int_{Q_T} [|c_j^1 - c_j^2| u_j |\phi_j''| + M \phi_j] \\
&\leq |c_j^1 - c_j^2| \|u_j\|_{L^p(Q_T)} \|\phi_j''\|_{L^q(Q_T)} + M \underbrace{|\Omega|^{\frac{1}{p}}}_{\leq 1+|\Omega|} \|\phi_j\|_{L^q(Q_T)} \\
&\leq C \|u\|_{L^p(Q_T)^m} \|\phi''\|_{L^q(Q_T)^m} + C \|\phi\|_{L^q(Q_T)^m}.
\end{aligned}$$

Again, by maximal regularity (see Corollary 2.34),

$$\|\phi\|_{L^q(Q_T)^m} \leq C \|\chi\|_{L^q(Q_T)^m} \quad \text{and} \quad \|\phi''\|_{L^q(Q_T)^m} \leq C \|\chi\|_{L^q(Q_T)^m}.$$

Hence the following estimate:

$$|I_j^2| \leq C \left(1 + \|u\|_{L^p(Q_T)^m}\right) \|\chi\|_{L^q(Q_T)^m}.$$

- Estimate of  $I_j^3$ .

$$\begin{aligned}
\sum_{j=1}^m I_j^3 &= \int_0^T e^{-tL} \sum_{j=1}^m c_j^1 [u_j'(\cdot, 1) \phi_j(\cdot, 1) - u_j'(\cdot, 0) \phi_j(\cdot, 0)] \\
&\quad - \int_0^T e^{-tL} \sum_{j=1}^m c_j^2 [v_j'(\cdot, 1) \phi_j(\cdot, 1) - v_j'(\cdot, 0) \phi_j(\cdot, 0)] = 0,
\end{aligned}$$

each of the two sums being equal to zero by Kirchhoff conditions (see Property 2.4).

- Estimate of  $I_j^4$ . Now, let us underline that  $\sum_{j=1}^m I_j^4$  does not vanish by Kirchhoff conditions anymore. We have

$$\begin{aligned}
I_j^4 &= - \int_0^T [c_j^1 u_j(\cdot, 1) \phi_j'(\cdot, 1) - c_j^1 u_j(\cdot, 0) \phi_j'(\cdot, 0)] e^{-tL} \\
&\quad - \int_0^T [c_j^2 v_j(\cdot, 1) \phi_j'(\cdot, 1) - c_j^2 v_j(\cdot, 0) \phi_j'(\cdot, 0)] e^{-tL} =: I_j^5 + I_j^6
\end{aligned}$$

Here,  $\sum_{j=1}^m I_j^6 = 0$  by Kirchhoff conditions, and it remains to estimate  $\sum_{j=1}^m I_j^5$ . Naturally, it is sufficient to estimate  $\delta_j := \left| \int_0^T c_j^1 u_j(\cdot, 1) \phi_j'(\cdot, 1) \right|$ . We have, for all  $p \in (1, +\infty)$ :

$$\delta_j \leq C \int_0^T |u_j(\cdot, 1)| \cdot |\phi_j'(\cdot, 1)| \leq C \left( \int_0^T |u_j(\cdot, 1)|^p \right)^{\frac{1}{p}} \left( \int_0^T |\phi_j'(\cdot, 1)|^q \right)^{\frac{1}{q}}$$

where  $C$  denotes a constant depending only on the data. Moreover, since  $T < T^*$ , we have  $\|u_j(t)\|_{L^\infty(\Omega)} \leq \|u(t)\|_{L^\infty(\Omega)^m} \leq C = C(T)$ . Consequently,

$$\delta_j^q \leq C \int_0^T |\phi_j'(\cdot, 1)|^q.$$

But, by Sobolev embedding,  $W^{1,q}(\Omega) \hookrightarrow C(\bar{\Omega})$  (continuous embedding) and then

$$|\phi'_j(\cdot, 1)|^q \leq C \|\phi'_j\|_{W^{1,q}(\Omega)}^q = C \left( \int_{\Omega} |\phi'_j|^q + \int_{\Omega} |\phi''_j|^q \right)$$

which implies that

$$(25) \quad \delta_j^q \leq C \left( \int_0^T \int_{\Omega} |\phi'_j|^q + \int_0^T \int_{\Omega} |\phi''_j|^q \right).$$

But, according to Lemma B.1, we have

$$\int_{\Omega} |\phi'_j|^q \leq C \left( \int_{\Omega} |\phi_j|^q + \int_{\Omega} |\phi''_j|^q \right)$$

and inequality (25) becomes

$$\delta_j^q \leq C \left( \int_0^T \int_{\Omega} |\phi_j|^q + \int_0^T \int_{\Omega} |\phi''_j|^q \right).$$

Moreover, by maximal regularity (see Corollary 2.34),

$$\|\phi\|_{L^q(Q_T)^m} \leq C \|\chi\|_{L^q(Q_T)^m} \quad \text{and} \quad \|\phi''\|_{L^q(Q_T)^m} \leq C \|\chi\|_{L^q(Q_T)^m}.$$

Hence the following estimate:

$$\delta_j \leq C \|\chi\|_{L^q(Q_T)^m}.$$

It remains to concatenate the previous estimates to find

$$\sum_{j=1}^m \int_{Q_T} (u_j + v_j) e^{-tL} \chi_j \leq C \left[ 1 + \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} + \|u\|_{L^p(Q_T)^m} \right] \|\chi\|_{L^q(Q_T)^m},$$

and then

$$\sum_{j=1}^m \int_{Q_T} (u_j + v_j) \chi_j \leq C \left[ 1 + \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} + \|u\|_{L^p(Q_T)^m} \right] \|\chi\|_{L^q(Q_T)^m}.$$

Taking  $\chi_k = 0$  for  $k \neq j$ , we have

$$\int_{Q_T} (u_j + v_j) \chi_j \leq C \left[ 1 + \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} + \|u\|_{L^p(Q_T)^m} \right] \|\chi_j\|_{L^q(Q_T)},$$

which implies

$$\|u_j + v_j\|_{L^p(Q_T)} \leq C \left[ 1 + \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} + \|u\|_{L^p(Q_T)^m} \right].$$

Summing these inequalities for  $j$  from 1 to  $n$  yields

$$\|u + v\|_{L^p(Q_T)^m} \leq C \left[ 1 + \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} + \|u\|_{L^p(Q_T)^m} \right],$$

and finally, since  $u$  and  $v$  are nonnegative, we obtain:

$$\|v\|_{L^p(Q_T)^m} \leq \|u + v\|_{L^p(Q_T)^m} \leq C \left[ 1 + \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} + \|u\|_{L^p(Q_T)^m} \right].$$

□

**Remark 4.2.** *Interchanging  $u$  and  $v$ , we also have*

$$\|u\|_{L^p(Q_T)^m} \leq C \left( 1 + \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} + \|v\|_{L^p(Q_T)^m} \right).$$

**Remark 4.3.** *As in Remark 2.32, we may assume that  $C$  is a nondecreasing function with respect to  $T$  and, if  $T^* < +\infty$ ,  $C$  may be extended to a nondecreasing function  $C : [0, +\infty) \rightarrow [0, +\infty)$ .*

#### 4.2. Main statement.

**Theorem 4.4.** *Let us consider the maximal solution  $(u, v)$  introduced in the previous paragraph. Assume that there exists a continuous, nondecreasing function  $N_1 : [0, +\infty) \rightarrow [0, +\infty)$  such that*

$$(26) \quad \forall t \in [0, T^*), \|u(t, \cdot)\|_{L^\infty(\Omega)^m} \leq N_1(t)$$

*as well as a nonnegative real number  $\sigma$  and two nonnegative functions  $L_1, M_1 : [0, +\infty) \rightarrow [0, +\infty)$  such that*

$$(27) \quad \forall j \in \llbracket 1, m \rrbracket, \forall r, s \geq 0, [r \leq R \Rightarrow |g_j(r, s)| \leq L_1(R)s^\sigma + M_1(R)].$$

*Then,  $T^* = +\infty$  and the solution  $(u, v)$  is global.*

*Proof.* Suppose by contradiction that  $T^* < +\infty$ . By assumption, we have

$$\forall T < T^*, \|u(T, \cdot)\|_{L^\infty(\Omega)^m} \leq N_1(T) \leq N_1(T^*)$$

and then  $u \in L^\infty(Q_{T^*})^m$ . According to Property 4.1, for all  $p > 1$ , we have

$$\begin{aligned} \forall T < T^*, \|v\|_{L^p(Q_T)^m} &\leq C(T) \left(1 + \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} + \|u\|_{L^p(Q_T)^m}\right) \\ &\leq C(T^*) \left(1 + \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} + \|u\|_{L^p(Q_{T^*})^m}\right) \\ &\leq C(T^*) \left(1 + \|u_0\|_{L^p(\Omega)^m} + \|v_0\|_{L^p(\Omega)^m} + \|u\|_{L^\infty(Q_{T^*})^m}\right) < +\infty \end{aligned}$$

because, since  $Q_{T^*}$  is bounded,  $L^\infty(Q_{T^*}) \subset L^p(Q_{T^*})$  and because we may assume that  $T \mapsto C(T)$  is nondecreasing. Consequently,  $v \in L^p(Q_{T^*})^m$  for all  $p > 1$ . Since, by assumption

$$\forall p > 1, |g_j(u_j, v_j)| \leq L_1 \left(\|u_j\|_{L^\infty(Q_{T^*})}\right) v_j^\sigma + M_1 \left(\|u_j\|_{L^\infty(Q_{T^*})}\right),$$

we have

$$\forall p > 1, |g_j(u_j, v_j)| \in L^p(Q_{T^*}).$$

Proposition 2.17 (or 3.14 if we work without a Dirichlet condition at vertex  $\mathbf{v}_n$ ) claims that  $v \in L^\infty(Q_{T^*})^m$ . This leads to a contradiction and the solution is global.  $\square$

**4.3. Generalization to linearly growing source  $f$ .** Let us replace  $L^\infty$ -a priori estimate (26) by

$$\forall j \in \llbracket 1, m \rrbracket, \exists A_j, B_j, D_j \in [0, +\infty) : \forall u_j, v_j \geq 0, f_j(u_j, v_j) \leq A_j u_j + B_j v_j + D_j.$$

Note that we only estimate  $f_j$ , and not  $|f_j|$ . Without loss of generality, we can suppose that the constants  $A_j, B_j, D_j$  do not depend on  $j$ . Let  $A, B, D$  denote

these constants. For all  $p > 1$  we have

$$\begin{aligned}
 (28) \quad \frac{1}{p} \frac{d}{dt} \|u_j(t, \cdot)\|_{L^p(\Omega)}^p &= \frac{1}{p} \frac{d}{dt} \int_{\Omega} u_j^p = \int_{\Omega} \partial_t u_j \cdot u_j^{p-1} \\
 &= \int_{\Omega} u_j^{p-1} [c_j^1 u_j'' + f_j(u_j, v_j)] \leq \int_{\Omega} u_j^{p-1} [c_j^1 u_j'' + Au_j + Bv_j + D] \\
 &= c_j^1 \int_{\Omega} u_j^{p-1} u_j'' + A \int_{\Omega} u_j^p + B \int_{\Omega} u_j^{p-1} v_j + D \int_{\Omega} u_j^{p-1} \\
 &= c_j^1 \int_{\Omega} u^{p-1} u_j'' + A \|u_j\|_{L^p(\Omega)}^p + B \int_{\Omega} u_j^{p-1} v_j + D \int_{\Omega} u_j^{p-1} \\
 &=: I_j^1 + I_j^2 + I_j^3 + I_j^4.
 \end{aligned}$$

- Estimate of  $I_j^1$ : an integration by parts shows that

$$(29) \quad \sum_{j=1}^m I_j^1 = - \sum_{j=1}^m c_j^1 \int_{\Omega} (p-1) u^{p-1} (u_j')^2 \leq 0.$$

- Estimate of  $I_j^3$ : thanks to Hölder inequality, we have

$$\begin{aligned}
 (30) \quad \int_{\Omega} u_j^{p-1} v_j &\leq \left( \int_{\Omega} u_j^{(p-1)q} \right)^{\frac{1}{q}} \left( \int_{\Omega} v_j^p \right)^{\frac{1}{p}} = \left( \int_{\Omega} u_j^p \right)^{\frac{1}{q}} \left( \int_{\Omega} v_j^p \right)^{\frac{1}{p}} \\
 &= \|u_j\|_{L^p(\Omega)}^{\frac{p}{q}} \|v_j\|_{L^p(\Omega)} = \|u_j\|_{L^p(\Omega)}^{p-1} \|v_j\|_{L^p(\Omega)} \\
 &\leq \frac{1}{q} \|u_j\|_{L^p(\Omega)}^{(p-1)q} + \frac{1}{p} \|v_j\|_{L^p(\Omega)}^p = \frac{1}{q} \|u_j\|_{L^p(\Omega)}^p + \frac{1}{p} \|v_j\|_{L^p(\Omega)}^p
 \end{aligned}$$

where we used the Young inequality.

- Estimate of  $I_j^4$ : Same as previous with  $v_j$  replaced by 1, we have

$$(31) \quad \int_{\Omega} u_j^{p-1} \leq \frac{1}{q} \|u_j\|_{L^p(\Omega)}^p + \frac{1}{p}$$

Adding (28) for  $\leq j \leq m$ , and using (29), (30) and (31), we find

$$\begin{aligned}
 &\frac{1}{p} \frac{d}{dt} \|u(t, \cdot)\|_{L^p(\Omega)^m}^p \\
 &\leq A \|u\|_{L^p(\Omega)^m}^p + B \left[ \frac{1}{q} \|u\|_{L^p(\Omega)^m}^p + \frac{1}{p} \|v\|_{L^p(\Omega)^m}^p \right] + D \left[ \frac{1}{q} \|u_j\|_{L^p(\Omega)}^p + \frac{1}{p} \right] \\
 &\leq C(p) \left( \|u\|_{L^p(\Omega)^m}^p + \|v\|_{L^p(\Omega)^m}^p + 1 \right).
 \end{aligned}$$

Let us integrate this inequality between 0 and  $t$ ; we find

$$\begin{aligned}
 \|u(t, \cdot)\|_{L^p(\Omega)^m}^p &\leq \|u_0\|_{L^p(\Omega)^m}^p + C(p) \left[ \|u\|_{L^p(Q_t)^m}^p + \|v\|_{L^p(Q_t)^m}^p + t \right] \\
 &\leq C(p, t) \left[ 1 + \|u\|_{L^p(Q_t)^m}^p + \|v\|_{L^p(Q_t)^m}^p \right]
 \end{aligned}$$

where  $C(p, t)$  denotes a generic constant depending on the data,  $t$  and  $p$ . Replacing  $T$  by  $t$ , we may invoke Inequality (22) to find

$$\|u(t, \cdot)\|_{L^p(\Omega)^m}^p \leq C(p, t) \left[ 1 + \|u\|_{L^p(Q_t)^m}^p \right]$$

which is equivalent to

$$\|u(t, \cdot)\|_{L^p(\Omega)^m}^p \leq C(p, t) \left[ 1 + \int_0^t \|u(s, \cdot)\|_{L^p(\Omega)^m}^p ds \right]$$

This is a Gronwall-type inequality. This allows to estimate  $\|u(t, \cdot)\|_{L^p(\Omega)^m}$  for all  $t > 0$ . Indeed, we have

$$(32) \quad X'(t) \leq a(t)X(t) + a(t) \quad \text{with} \quad \begin{cases} X(t) := \int_0^t \|u(s, \cdot)\|_{L^p(\Omega)^m}^p ds, \\ a(t) := C(p, t) \end{cases}$$

where  $a \in L_{loc}^\infty([0, +\infty))$  (because  $a$  is nondecreasing with respect to  $t$ ). A classical computation shows that

$$X(t) \leq \int_0^t a(s) e^{\int_s^t a(z) dz} ds = - \int_0^t \frac{d}{ds} e^{\int_s^t a(z) dz} ds = e^{\int_0^t a(z) dz} - 1.$$

Suppose, by contradiction, that  $T^* < +\infty$ . Then,

$$\forall t \in [0, T^*), X(t) \leq e^{\int_0^{T^*} a(z) dz} - 1,$$

and  $X(t)$  is bounded on  $[0, T^*)$ , i.e.  $u \in L^p(Q_{T^*})^m$ . So, Lemma (4.1) claims that  $v \in L^p(Q_{T^*})^m$ . Now, let us consider the solution  $w = (w_1, \dots, w_n)$  of the system

$$\forall j \in \llbracket 1, m \rrbracket, \partial_t w_j - c_j^1 w_j'' = Au_j + Bv_j + D \quad \text{with} \quad w(0, \cdot) = u(0, \cdot).$$

From the foregoing,  $Au_j + Bv_j + D \in L^p(Q_{T^*})$ , and this for all  $p > 1$ . According to Proposition 2.17 (or Proposition 3.14 if we work without Dirichlet condition at vertex  $\mathbf{v}_n$ ),  $w_j \in L^\infty(Q_{T^*})$ . Since  $f_j(u_j, v_j) \leq Au_j + Bv_j + D$ , comparison Lemma B.6 yields

$$\forall j \in \llbracket 1, m \rrbracket, u_j \leq w_j.$$

Moreover, we know that  $u_j \geq 0$ . Therefore

$$\forall j \in \llbracket 1, m \rrbracket, 0 \leq u_j \leq w_j$$

and then  $u \in L^\infty(Q_{T^*})^m$ .

Then, the estimate (27) shows that  $g_j(u_j, v_j) \in L^p(Q_{T^*})$ ,  $\forall p > 1$ . Again,  $L^p$  bounds of  $g$  imply  $L^\infty$ -bound of  $v$ , i.e.  $v \in L^\infty(Q_{T^*})^m$ . This contradicts Property 2.29 and we deduce that  $T^* = +\infty$ : thus, the solution is global.

**4.4. Application to one-dimensional domain with piecewise constant diffusion coefficients.** Consider the reaction-diffusion system

$$(33) \quad \begin{cases} \partial_t u(t, x) - (c^1(x)u'(t, x))' = f(u, v), \\ \partial_t v(t, x) - (c^2(x)v'(t, x))' = g(u, v) \end{cases}$$

on an open interval  $I \subset \mathbb{R}$  and where  $c^1, c^2$  are piecewise constant diffusion coefficients. For the sake of simplicity, suppose that  $I = (0, L)$ , where  $L \in \mathbb{N}^*$  and suppose that  $c^1, c^2$  are constant on every subinterval  $(k-1, k)$ ,  $k \in \llbracket 1, L \rrbracket$ . Assume the following initial and homogenous Dirichlet boundary conditions:

$$u(0) = u^0 \in L^\infty(I), v(0) = v^0 \in L^\infty(I) \quad \text{and} \quad u(0) = v(0) = u(L) = v(L) = 0.$$

Thus, the standard weak formulation of the system (33) contains

- the continuity of  $u, v$ ,
- the fluxes conservation conditions at every node: these are the Kirchhoff conditions.

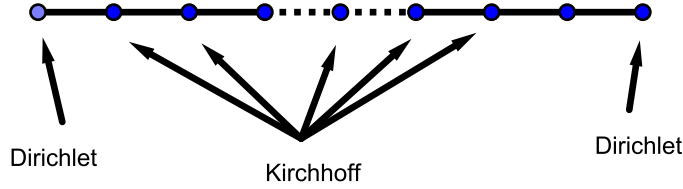


FIGURE 4.

This problem naturally enters the general framework previously described in the setting pictured in figure 4.

**4.5. Extension to  $N \times N$  systems; example of the bloodstream oxygenation.** The previous  $L^p$ -method can be extended to more general systems corresponding to  $N$  chemical components. As a concrete application which is naturally posed on a network domain, let us consider the following system which models bloodstream oxygenation (see [4], [13], [29]):

$$(34) \quad \begin{cases} \partial_t u^1 - c^1(u^1)'' = K_2 u^2 - K_1 u^1 u^5, \\ \partial_t u^2 - c^2(u^2)'' = -K_2 u^2 + K_1 u^1 u^5, \\ \partial_t u^3 - c^3(u^3)'' = K_4 u^4 - K_3 u^3 u^5, \\ \partial_t u^4 - c^4(u^4)'' = -K_4 u^4 + K_3 u^3 u^5, \\ \partial_t u^5 - c^5(u^5)'' = K_2 u^2 + K_4 u^4 - K_1 u^1 u^5 - K_3 u^3 u^5. \end{cases}$$

Here,  $u_1, u_2, u_3, u_4, u_5$  represent the concentrations of the species  $O_2, HbO_2, CO_2, HbCO_2, Hb$  and  $K_1, K_2, K_3, K_4$  are the reaction rates. Once again, we can exploit the  $L^p$ -method to extend [28, Theorem 3.5, p. 430] to show that the previous system has a global solution. To be more precise, it is sufficient to ensure the existence of  $\mathbf{b} = {}^t(b_1, b_2, b_3, b_4, b_5) \in \mathcal{M}_{5,1}(\mathbb{R})$  and a lower triangular invertible  $5 \times 5$  matrix  $P$  with non negative entries such that

$$\forall r := (r_1, r_2, r_3, r_4, r_5) \in [0, +\infty)^5, Pf(r) \leq \left[ 1 + \sum_{i=1}^5 r_i \right] \mathbf{b}$$

where

$$f(r) = \begin{pmatrix} K_2 r_2 - K_1 r_1 r_5 \\ -K_2 r_2 + K_1 r_1 r_5 \\ K_4 r_4 - K_3 r_3 r_5 \\ -K_4 r_4 + K_3 r_3 r_5 \\ K_2 r_2 + K_4 r_4 - K_1 r_1 r_5 - K_3 r_3 r_5 \end{pmatrix}.$$

But we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} f(r) = \begin{pmatrix} K_2 r_2 - K_1 r_1 r_5 \\ 0 \\ K_4 r_4 - K_3 r_3 r_5 \\ 0 \\ K_2 r_2 + K_4 r_4 - K_1 r_1 r_5 - K_3 r_3 r_5 \end{pmatrix} \leq \begin{pmatrix} K_2 r_2 \\ 0 \\ K_4 r_4 \\ 0 \\ K_2 r_2 + K_4 r_4 \end{pmatrix}$$

and the conclusion follows.

Given the tools developed in [4] concerning asymptotic behavior, as well as the contents of the section 2, we conclude that the results of [4] naturally extend to networks.

## 5. APPLICATION II TO REACTION-DIFFUSION: A RESULT BY HARAUX AND YOUKANA

Once again, let us consider a one-dimensional network with  $n$  vertices,  $m$  edges and 2 chemical components. Let  $u_j := u_j^1, v_j := u_j^2$  denote the respective concentrations of each chemical component on the edge  $j$ . Let  $\varphi_j \in C^1(\mathbb{R}_+, \mathbb{R})$  be  $m$  nonnegative functions. We assume that

$$\forall j \in \llbracket 1, m \rrbracket, \frac{\ln(1 + \varphi_j(r))}{r} \xrightarrow{r \rightarrow +\infty} 0.$$

Typically,  $\varphi_j$  satisfy  $0 \leq \varphi_j(v_j) \leq e^{\alpha v_j^{\beta_j}}$  with  $0 < \beta_j < 1$  and  $\alpha > 0$ . We are interested in the following reaction-diffusion system

$$(35) \quad \begin{cases} \partial_t u_j - c_j^1 u_j'' + u_j \varphi_j(v_j) = 0, \\ \partial_t v_j - c_j^2 v_j'' - u_j \varphi_j(v_j) = 0 \end{cases}$$

where  $u := (u_1, \dots, u_m)$ ,  $v := (v_1, \dots, v_m)$  are continuous on the graph and where Kirchhoff conditions and, optionally, Dirichlet condition at vertex  $\mathbf{v}_n$  hold. It is proposed to prove that the problem thus posed has global solutions. This will extend to networks the existence result by Haraux and Youkana ([17]). We will also include the analysis of the asymptotic behavior of these solutions.

We will write  $u_{j,0} := u_j(t=0)$  and  $v_{j,0} := v_j(t=0)$ . In the sequel, we assume that  $u_{j,0} \geq 0$  and  $v_{j,0} \geq 0$ . Recall that  $\Omega$  denotes the open interval  $(0, 1)$ . For  $T \geq 0$ , we recall that  $Q_T := (0, T) \times \Omega$ . Let  $C_B((0, +\infty) \times \bar{\Omega})$  denote the space of real, continuous and bounded functions on  $(0, +\infty) \times \bar{\Omega}$ .

**5.1. Global existence.** According to Theorem 2.25 and Proposition 2.29, we know that (35) has a maximal solution (in the sense of definition 2.23) on  $[0, T^*)$ . To prove that the solution  $(u, v)$  is global, we need to verify that  $(u, v)$  is uniformly bounded on every  $Q_T^m$ ,  $0 \leq T \leq T^*$ ,  $T < +\infty$ .

**Lemma 5.1.** *Let  $(u, v)$  be a solution of (35) on  $(0, T)$ . Then, for all  $j$ ,  $u_j \geq 0$ .*

*Proof.* The right-hand side (i.e. the nonlinearities) clearly satisfy the quasi-positivity Property **(P)** (see(15)). The conclusion follows.  $\square$

**Lemma 5.2.** *Let  $u = (u_1, \dots, u_m), v = (v_1, \dots, v_m)$  be a solution of (35) on  $(0, T)$ . Then, there exist two real positive numbers  $\varepsilon$  and  $\delta$ , depending only on the data*

such that

$$t \mapsto \sum_{j=1}^m \int_{\Omega} \{1 + \delta [u_j(t, x) + u_j^2(t, x)]\} e^{\varepsilon v_j(t, x)} dx \quad \text{is nonincreasing on } (0, T).$$

*Proof.* This is exactly the same proof as in [17, Theorem 1, p. 160], the vertices terms vanishing by Kirchhoff conditions.  $\square$

**Theorem 5.3.** *Let  $\varphi_j \in C^1(\mathbb{R}_+, \mathbb{R})$  be  $m$  nonnegative functions. Let us consider the reaction-diffusion system*

$$(36) \quad \begin{cases} \partial_t u_j - c_j^1 u_j'' + u_j \varphi_j(v_j) = 0, \\ \partial_t v_j - c_j^2 v_j'' - u_j \varphi_j(v_j) = 0 \end{cases}$$

where  $u = (u_1, \dots, u_m)$ ,  $v = (v_1, \dots, v_m)$  are continuous on the graph and satisfy Kirchhoff condition, with nonnegative initial conditions  $(u_{j,0})_{1 \leq j \leq m}$  in  $L^\infty(\Omega)$ .

Moreover, assume that

$$\forall j \in \llbracket 1, m \rrbracket, \frac{\ln(1 + \varphi_j(r))}{r} \xrightarrow{r \rightarrow +\infty} 0.$$

Then, the solutions of (36) are global.

If in addition we assume a Dirichlet condition at vertex  $\mathbf{v}_n$ , then the solution is uniformly bounded for  $t \geq 0$ .

*Proof.* Let  $T \leq T^*$ ,  $T < +\infty$ . Set any  $p > 1$ . Take  $\delta, \varepsilon > 0$  as in Lemma 5.2. Since

$$\forall j \in \llbracket 1, m \rrbracket, \frac{\ln(1 + \varphi_j(r))}{r} \xrightarrow{r \rightarrow +\infty} 0,$$

we have, for  $r \geq V$  sufficiently large:

$$\frac{\ln(1 + \varphi_j(r))}{r} \leq \frac{\varepsilon}{p} \quad \text{i.e.} \quad 1 + \varphi_j(r) \leq e^{\frac{\varepsilon}{p} r}.$$

Let  $K := \max_{1 \leq j \leq m} \sup_{0 \leq r \leq V} (1 + \varphi_j(r))$ . Then

$$\forall r \geq 0, 1 + \varphi_j(r) \leq K e^{\frac{\varepsilon}{p} r} \quad \text{and then} \quad \forall r \geq 0, \varphi_j(r) \leq K e^{\frac{\varepsilon}{p} r}.$$

It is deduced that

$$\Phi := \sum_{j=1}^m \int_{\Omega} (\varphi_j(v_j(t, x)))^p dx \leq K^p \sum_{j=1}^m \int_{\Omega} e^{\varepsilon v_j(t, x)}.$$

Since  $1 + \delta [u_j(t, x) + u_j^2(t, x)] \geq 1$ , we have

$$\Phi \leq K^p \sum_{j=1}^m \int_{\Omega} \{1 + \delta [u_j(t, x) + u_j^2(t, x)]\} e^{\varepsilon v_j(t, x)} dx.$$

Finally, since  $t \mapsto \sum_{j=1}^m \int_{\Omega} \{1 + \delta [u_j(t, x) + u_j^2(t, x)]\} e^{\varepsilon v_j(t, x)} dx$  is nonincreasing on  $(0, T)$ ,

$$\begin{aligned} \Phi &\leq K^p \sum_{j=1}^m \int_{\Omega} \{1 + \delta [u_j(0, x) + u_j^2(0, x)]\} e^{\varepsilon v_j(0, x)} dx \\ &= K^p \sum_{j=1}^m \int_{\Omega} \{1 + \delta [u_{j,0}(x) + u_{j,0}^2(x)]\} e^{\varepsilon v_{j,0}(x)} dx =: C, \end{aligned}$$



the constant  $C$  being independent of  $t$ . Since  $T$  is finite, an integration from 0 to  $T$  shows that the functions  $\varphi_j(v_j)$  belong to  $L^p(Q_T)$  (recall that  $Q_T = (0, T) \times \Omega$ ). Since the functions  $u_j$  are *uniformly bounded* (by maximum principle,  $\|u_j\|_{L^\infty(\Omega)} \leq \|u_{j,0}\|_{L^\infty(\Omega)}$ ), the functions  $u_j\varphi_j(v_j)$  belong to  $L^p(Q_T)$ . The previous computations being valid for all  $p > 1$ , Property 3.14 asserts that  $v \in L^\infty(Q_T)^m$ . Finally, the solution is global.

Assume in addition a Dirichlet condition at vertex  $\mathbf{v}_n$ . So, the semigroup  $(e^{-tA})_{t \geq 0}$  is ultracontractive. The previous computations show that  $\|u\varphi(v)\|_{X_p} \leq C$ , where the constant  $C$  does not depend on  $t$ . Consequently,  $u_j\varphi(v_j) \in L^\infty(0, +\infty; L^p(\Omega))$ . We are in position to apply Property A.34 which claims that  $v_j \in L^\infty(0, +\infty; L^\infty(\Omega))$ .  $\square$

## 5.2. Asymptotic behavior.

**Property 5.4.** *Let  $(u, v)$  be a nonnegative solution of (35), continuous on the graph and satisfying Kirchhoff conditions. Suppose, in addition, that for all  $j \in \llbracket 1, m \rrbracket$ , we have<sup>5</sup>  $u_j, v_j \in C_B((0, +\infty) \times \bar{\Omega})$ , where  $C_B((0, +\infty) \times \bar{\Omega})$  denotes the vector space of continuous functions  $\varphi : (0, +\infty) \times \bar{\Omega} \rightarrow \mathbb{R}$  bounded on  $(0, +\infty) \times \bar{\Omega}$ . Then*

$$\begin{aligned} \|u(t) - u^*\|_{X_\infty} &\xrightarrow[t \rightarrow +\infty]{} 0, \\ \|v(t) - v^*\|_{X_\infty} &\xrightarrow[t \rightarrow +\infty]{} 0, \end{aligned}$$

where  $u^*, v^*$  are two nonnegative real numbers such that  $\forall j \in \llbracket 1, m \rrbracket, u^*\varphi_j(v^*) = 0$ .

*Proof.* In the remainder of the proof, as a convenience,  $\|\cdot\|_p, 1 \leq p \leq \infty$ , will denote (if there is no possibility of confusion), interchangeably the norms  $\|\cdot\|_{L^p(\Omega)}, \|\cdot\|_{X_p}$ .

- Integrate the first equation of (35), namely  $\partial_t u_j - c_j^1 u_j'' + u_j \varphi_j(v_j) = 0$ . After summing with respect to  $j$ , we obtain

$$(37) \quad \frac{d}{dt} \sum_{j=1}^m \int_{\Omega} u_j \, dx = - \sum_{j=1}^m \int_{\Omega} u_j \varphi_j(v_j) \, dx \leq 0.$$

Then, integrate the equality (37) between 0 and  $t$ , getting

$$\sum_{j=1}^m \int_{\Omega} u_j(t, x) \, dx - \sum_{j=1}^m \int_{\Omega} u_{j,0}(x) \, dx = - \sum_{j=1}^m \int_0^t \int_{\Omega} u_j \varphi_j(v_j) \, dx \, ds$$

hence

$$\begin{aligned} 0 &\leq \sum_{j=1}^m \int_0^t \int_{\Omega} \underbrace{u_j \varphi_j(v_j)}_{\geq 0} \, dx \, ds \\ &= - \sum_{j=1}^m \int_{\Omega} \underbrace{u_j(t, x)}_{\geq 0} \, dx + \sum_{j=1}^m \int_{\Omega} u_{j,0}(x) \, dx \leq \sum_{j=1}^m \int_{\Omega} u_{j,0}(x) \, dx, \end{aligned}$$

which proves that for all  $j, t \mapsto \int_{\Omega} u_j \varphi_j(v_j) \, dx \in L^1(\mathbb{R}_+)$ .

Identify (37) also shows that the function  $t \mapsto \sum_{j=1}^m \int_{\Omega} u_j(t, x) \, dx$  is non-negative nonincreasing; so, it has a limit as  $t \rightarrow +\infty$ . Let  $u^*$  denote the

<sup>5</sup>This is the case, in particular, if in addition we assume a Dirichlet condition at vertex  $\mathbf{v}_n$ .

real number such that

$$mu^* = \lim_{t \rightarrow +\infty} \sum_{j=1}^m \int_{\Omega} u_j(t, x) dx.$$

• Moreover

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{j=1}^m \int_{\Omega} u_j^2 dx &= \sum_{j=1}^m \int_{\Omega} u_j \frac{\partial u_j}{\partial t} dx = \sum_{j=1}^m c_j^1 \int_{\Omega} u_j u_j'' dx - \int_{\Omega} u_j^2 \varphi_j(v_j) dx \\ &= - \sum_{j=1}^m c_j^1 \int_{\Omega} u_j'^2 dx - \int_{\Omega} u_j^2 \varphi_j(v_j) dx. \end{aligned}$$

Integrating between 0 and  $t$ , we get

$$\begin{aligned} &\sum_{j=1}^m c_j^1 \int_0^t \int_{\Omega} u_j'^2 dx ds + \sum_{j=1}^m \int_0^t \int_{\Omega} u_j^2 \varphi_j(v_j) dx ds \\ &= \frac{1}{2} \sum_{j=1}^m \int_{\Omega} u_{j,0}^2(t, x) dx - \frac{1}{2} \sum_{j=1}^m \int_{\Omega} u_j^2(t, x) dx \\ &\leq \frac{1}{2} \sum_{j=1}^m \int_{\Omega} u_{j,0}^2(t, x) dx. \end{aligned}$$

This shows that for all  $j$ , the function  $t \mapsto \int_{\Omega} u_j'^2 dx$  is integrable on  $\mathbb{R}_+$ .

• To study integrability of  $t \mapsto \int_{\Omega} v_j'^2 dx$ , we write

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{j=1}^m \int_{\Omega} v_j^2 dx &= \sum_{j=1}^m \int_{\Omega} v_j \frac{\partial v_j}{\partial t} dx = \sum_{j=1}^m c_j^2 \int_{\Omega} v_j v_j'' dx + \sum_{j=1}^m \int_{\Omega} u_j v_j \varphi_j(v_j) dx \\ &= - \sum_{j=1}^m c_j^2 \int_{\Omega} v_j'^2 dx + \sum_{j=1}^m \int_{\Omega} u_j v_j \varphi_j(v_j) dx. \end{aligned}$$

Integrate between 0 and  $t$  and then estimate:

$$\begin{aligned} &\sum_{j=1}^m c_j^2 \int_0^t \int_{\Omega} v_j'^2 dx ds \\ &= \frac{1}{2} \sum_{j=1}^m \int_{\Omega} v_{j,0}^2 dx - \frac{1}{2} \sum_{j=1}^m \int_{\Omega} v_j^2(t, x) dx \\ &+ \sum_{j=1}^m \int_0^t \int_{\Omega} u_j(s, x) v_j(s, x) \varphi_j(v_j(s, x)) dx ds \\ &\leq \frac{1}{2} \sum_{j=1}^m \int_{\Omega} v_{j,0}^2 dx + \sum_{j=1}^m \int_0^{+\infty} \int_{\Omega} u_j(s, x) v_j(s, x) \varphi_j(v_j(s, x)) dx ds \\ &\leq \frac{1}{2} \sum_{j=1}^m \int_{\Omega} v_{j,0}^2 dx + \sum_{j=1}^m \|v_j\|_{\infty} \underbrace{\int_0^{+\infty} \int_{\Omega} u_j(s, x) \varphi_j(v_j(s, x)) dx ds}_{<+\infty} < +\infty. \end{aligned}$$

Hence the integrability on  $\mathbb{R}_+$  of the function  $t \mapsto \int_{\Omega} v_j'^2 dx$ .

- By summation of the two equations of (35), integration on  $\Omega$  and summation with respect to  $j$ , we find

$$\frac{d}{dt} \sum_{j=1}^m \int_{\Omega} (u_j + v_j) dx = \sum_{j=1}^m c_j^1 \int_{\Omega} u_j'' dx + \sum_{j=1}^m c_j^2 \int_{\Omega} v_j'' dx = 0.$$

We lastly integrate between 0 and  $t$ , which gives

$$\sum_{j=1}^m \int_{\Omega} (u_j(t, x) + v_j(t, x)) dx = \sum_{j=1}^m \int_{\Omega} (u_{j,0} + v_{j,0}) dx.$$

Since  $\sum_{j=1}^m \int_{\Omega} u_j(t, x) dx$  has a finite limit as  $t \rightarrow +\infty$ , it is the same for  $\sum_{j=1}^m \int_{\Omega} v_j(t, x) dx$ . Let  $v^*$  be such that

$$mv^* = \lim_{t \rightarrow +\infty} \sum_{j=1}^m \int_{\Omega} v_j(t, x) dx$$

- In appendix C, we show that techniques used in [16] apply and then, for all  $\delta > 0$ , we have  $u_j, v_j \in C_B(\delta, +\infty; C^1(\bar{\Omega}))$ . According to the Ascoli Theorem, the family  $(u_j(t))_{t \geq 1}$  is relatively compact in  $C(\bar{\Omega})$ . Consequently, there exists a nondecreasing sequence  $(\tau_n)$  of  $[1, +\infty[$  converging to  $+\infty$  such that  $(u_j(\tau_n))_{n \geq 0}$  is *uniformly convergent* on  $\bar{\Omega}$ . Let  $\bar{u}_j := \lim_{n \rightarrow +\infty} u_j(\tau_n)$  and let  $\bar{u} := (\bar{u}_1, \dots, \bar{u}_m)$ .
- Let  $m_u(t)$  denote the average of  $u$  on the graph i.e.  $m_u(t) = \frac{1}{m} \sum_{j=1}^m \int_{\Omega} u_j(t, \cdot) dx$ . According to the Poincaré-Wirtinger inequality for networks (see Property 3.11), we have, for all  $p \in (1, +\infty)$ :

$$\|u(t) - m_u(t)\|_p \leq C \|Du\|_p.$$

Then, for  $p = 2$  and taking the square power, we get

$$(38) \quad \|u(t) - m_u(t)\|_2^2 \leq C \|Du(t)\|_2^2.$$

But

$$\begin{aligned} \|u(t) - m_u(t)\|_2^2 &= \sum_{j=1}^m \int_{\Omega} (u_j(t, \cdot) - m_u(t))^2 dx \\ &= \sum_{j=1}^m \int_{\Omega} u_j(t, \cdot)^2 dx + \sum_{j=1}^m \int_{\Omega} m_u^2(t) dx - 2m_u(t) \sum_{j=1}^m \int_{\Omega} u_j(t, \cdot) dx \\ &= \sum_{j=1}^m \int_{\Omega} u_j(t, \cdot)^2 dx + m \cdot m_u^2(t) - 2m \cdot m_u^2(t) \\ &= \sum_{j=1}^m \int_{\Omega} u_j(t, \cdot)^2 dx - m \cdot m_u^2(t). \end{aligned}$$

Now, the function  $t \mapsto \sum_{j=1}^m \int_{\Omega} u_j(t, \cdot)^2 dx$  is nonincreasing, and so it has a finite limit.

Likewise, the function  $t \mapsto m_u^2(t) = \frac{1}{m^2} \left( \sum_{j=1}^m \int_{\Omega} u_j(t, \cdot) dx \right)^2$  is nonincreasing, and so it has a finite limit. Consequently  $t \mapsto \|u(t) - m_u(t)\|_2^2$

has a finite limit as  $t \rightarrow +\infty$ . Let  $\ell := \lim_{t \rightarrow +\infty} \|u(t) - m_u(t)\|_2^2$ . According to (38)

$$\|u(t) - m_u(t)\|_2^2 \leq C \|Du\|_2^2$$

with  $t \mapsto \|Du(t)\|_2^2 = \sum_{j=1}^m \int_{\Omega} u_j'^2 dx$  integrable on  $\mathbb{R}_+$ . It is deduced that  $\ell = 0$  and  $\|u(t) - m_u(t)\|_2 \xrightarrow{t \rightarrow +\infty} 0$ . Since  $m_u(t) \xrightarrow{t \rightarrow +\infty} u^*$ , it follows that

$$u(t) \xrightarrow{t \rightarrow +\infty} u^* \quad \text{in } L^2(\Omega)^m.$$

- To obtain a contradiction, suppose that  $(u(t))_{t \geq 0}$  does not converge uniformly on  $\Omega$  toward  $u^*$  as  $t \rightarrow +\infty$ , i.e. that  $\|u(t) - u^*\|_{\infty} \not\xrightarrow{t \rightarrow +\infty} 0$ . In this case, there exists  $\varepsilon > 0$  and a nondecreasing sequence  $(\tau_n)$  of nonnegative numbers which converge to  $+\infty$  such that

$$\forall n, \|u(\tau_n) - u^*\|_{\infty} \geq \varepsilon.$$

By the relative compactness of the family  $(u(t))$ , and according to the Ascoli Theorem, there exists a subsequence  $(\tau_{\theta(n)})$  such that  $(u(\tau_{\theta(n)}))$  converge uniformly on  $\bar{\Omega}$ . Since uniform convergence on  $\bar{\Omega}$  imply  $L^2$ -convergence (because  $\bar{\Omega}$  is bounded), and by uniqueness of the limit,  $u(\tau_{\theta(n)}) \xrightarrow{n \rightarrow +\infty} u^*$  uniformly on  $\bar{\Omega}$ . Hence, we get a contradiction.

- Now, we have to deal with  $v$ . To this end, come back to the second equation, namely

$$\frac{\partial v_j}{\partial t} = c_j^2 v_j'' + u_j \varphi_j(v_j), \quad \forall j \in \llbracket 1, m \rrbracket.$$

Multiply by  $v_j$ , sum with respect to  $j$  and integrate by parts; we obtain:

$$\frac{1}{2} \frac{d}{dt} \sum_{j=1}^m \int_{\Omega} v_j^2 dx = - \sum_{j=1}^m c_j^2 \int_{\Omega} v_j'^2 dx + \sum_{j=1}^m \int_{\Omega} u_j v_j \varphi_j(v_j) dx.$$

Let us integrate between 0 and  $t$ :

$$\frac{1}{2} \sum_{j=1}^m \int_{\Omega} v_j(t)^2 dx - \frac{1}{2} \sum_{j=1}^m \int_{\Omega} v_{j,0}^2 dx = - \sum_{j=1}^m c_j^2 \int_0^t \int_{\Omega} v_j'^2 dx ds + \sum_{j=1}^m \int_0^t \int_{\Omega} u_j v_j \varphi_j(v_j) dx ds.$$

Now, by assumption,  $v$  is uniformly bounded on  $\mathbb{R}_+$ . Consequently

$$\int_0^t \int_{\Omega} |u_j v_j \varphi_j(v_j)| dx ds \leq \|v_j\|_{\infty} \int_0^t \int_{\Omega} u_j \varphi_j(v_j) dx ds.$$

In addition, we know that the function  $t \mapsto \int_{\Omega} u_j \varphi_j(v_j) dx$  is integrable on  $\mathbb{R}_+$ . Therefore, the function  $t \mapsto \sum_{j=1}^m \int_0^t \int_{\Omega} u_j v_j \varphi_j(v_j) dx ds$  has a finite limit as  $t \rightarrow +\infty$ .

Likewise, the function  $t \mapsto \int_{\Omega} v_j'^2 dx$  is integrable on  $\mathbb{R}_+$ . Consequently, the function  $t \mapsto \sum_{j=1}^m c_j^2 \int_0^t \int_{\Omega} v_j'^2 dx ds$  has a finite limit as  $t \rightarrow +\infty$ . It is deduced that the function  $t \mapsto \sum_{j=1}^m \int_{\Omega} v_j(t)^2 dx$  has a finite limit as  $t \rightarrow +\infty$ .

Moreover, we have seen that  $t \mapsto \sum_{j=1}^m \int_{\Omega} v_j(t, x) dx$  has a finite limit as  $t \rightarrow +\infty$ . So, we are allowed to reiterate the method used for  $u$ , which leads to

$$v(t) \xrightarrow{t \rightarrow +\infty} v^* \quad \text{in } L^2(\Omega)^m.$$

- The compactness argument used for  $u$  is valid also for  $v$  and it shows that  $v(t) \xrightarrow[t \rightarrow +\infty]{} v^*$  uniformly.
- For all  $j \in \llbracket 1, m \rrbracket$ ,  $u_j \xrightarrow[t \rightarrow +\infty]{} u^*$  and  $v_j \xrightarrow[t \rightarrow +\infty]{} v^*$  uniformly on  $\bar{\Omega}$ . Consequently, since  $\varphi_j$  is continuous,  $u_j \varphi_j(v_j) \xrightarrow[t \rightarrow +\infty]{} u^* \varphi_j(v^*)$  uniformly and then  $\int_{\Omega} u_j \varphi_j(v_j) dx \xrightarrow[t \rightarrow +\infty]{} \int_{\Omega} u^* \varphi_j(v^*) dx = u^* \varphi_j(v^*)$ . But, the function  $t \mapsto \int_{\Omega} u_j \varphi_j(v_j) dx$  is integrable on  $\mathbb{R}_+$ . Necessarily,  $\int_{\Omega} u_j \varphi_j(v_j) dx \xrightarrow[t \rightarrow +\infty]{} 0$ . Finally, we obtain

$$\forall j \in \llbracket 1, m \rrbracket, u^* \varphi_j(v^*) = 0,$$

which is the desired result.  $\square$

**Remark 5.5.** Assume we imposed a Dirichlet condition at vertex  $\mathbf{v}_n$ . Then, we necessarily have  $u^* = v^* = 0$ .

**5.3. A partial extension in the manner of Barabanova.** Following Barabanova (see [8]), we can partially generalize the result of Haraux-Youkana, assuming an exponential growth of  $\varphi_j$  (recall that the exponential growth is not attained in the hypothesis of Haraux-Youkana). In this case, and provided that the initial condition is small enough, the solutions are global. This additional assumption may seem rather surprising. The point is that the Lyapunov functional of Barabanova depends on the  $L^\infty$ -norm of the initial condition  $u^0$  whereas that of Haraux and Youkana does not. More precisely, the following properties hold.

**Property 5.6.** Assume that

$$\exists \alpha > 0 : \forall r \geq 0, \varphi_j(r) \leq e^{\alpha r}.$$

Let  $u = (u_1, \dots, u_m)$  and  $v = (v_1, \dots, v_m)$  be solutions of (21) on  $[0, T)$ . For  $C, \beta > 0$ , consider the function

$$g : u \mapsto g(u) = \left( \frac{C}{C - u} \right)^\beta \quad \text{where} \quad \beta = \min_{1 \leq j \leq m} \frac{4c_j^1 c_j^2}{(c_j^1 - c_j^2)^2}.$$

Then, there exist  $\beta > 0, C > \|u_0\|_{X_\infty}$  and  $p > 1$  such that the function

$$t \mapsto \sum_{j=1}^m \int_{\Omega} g(u_j) e^{\alpha p v_j} dx$$

is nonincreasing on  $[0, T)$ .

The previous property yields the following global existence result.

**Theorem 5.7.** Assume exists  $\alpha > 0$  such that

$$\forall j \in \llbracket 1, m \rrbracket, \forall r \geq 0, \varphi_j(r) \leq e^{\alpha r}.$$

Let  $(u, v)$  be a solution of (21), with  $u_0, v_0 \in X_\infty$  nonnegative satisfying

$$\|u_0\|_{X_\infty} < \frac{\beta}{\alpha} \quad \text{with} \quad \beta = \min_{1 \leq j \leq m} \frac{4c_j^1 c_j^2}{(c_j^1 - c_j^2)^2}.$$

Then, the solution  $(u, v)$  is global. Moreover, if in addition we assume a Dirichlet condition at vertex  $\mathbf{v}_n$ , then the solution is uniformly bounded.

We previously prove that,

$$\|u(t) - u^*\|_{X_\infty} \xrightarrow{t \rightarrow +\infty} 0 \quad \text{and} \quad \|v(t) - v^*\|_{X_\infty} \xrightarrow{t \rightarrow +\infty} 0$$

where  $u^*, v^*$  are two nonnegative numbers such that  $\forall j \in \llbracket 1, m \rrbracket, u^* \varphi_j(v^*) = 0$ . In the sequel, a Dirichlet condition at vertex  $\mathbf{v}_n$  is required. Consequently, as seen before,  $u^* = v^* = 0$ . It is also assumed that the functions  $\varphi_j$  are given by  $\varphi_j(r) = e^{\alpha_j r}$ , where  $\alpha_j$  denote nonnegative real numbers.

Let  $(T_p^1(t))_{t \geq 0}$  (respectively  $(T_p^2(t))_{t \geq 0}$ ) denote the semigroup on  $X_p$  governing the equation  $\partial_t u_j - c_j^1 u_j'' = 0$  (respectively  $\partial_t v_j - c_j^2 v_j'' = 0$ ) with continuity on the graph, Kirchhoff conditions and Dirichlet condition at vertex  $\mathbf{v}_n$ .

Let  $-A_p^k$  ( $k \in \{1, 2\}, p \in [1, +\infty]$ ) denote the infinitesimal generator of  $(T_p^k(t))_{t \geq 0}$ . Lastly, for  $1 < p < +\infty$ ,  $\lambda^k$  will denote the smallest eigenvalue of  $A^k$  (recall that  $\sigma(A_p^k)$  does not depend on  $p$  and that  $\lambda^k$  is nonnegative since  $A^k$  is accretive). The following property describes the asymptotic behavior of  $u(t)$  and  $v(t)$  as  $t$  converges to  $+\infty$ .

**Property 5.8.** *Let  $(u, v)$  be a global nonnegative solution of (21), continuous on the graph, with Kirchhoff conditions and Dirichlet condition at vertex  $\mathbf{v}_n$ . Then*

$$\forall t \geq 0, \|u(t)\|_{X_\infty} \leq C e^{-(1+\lambda^1)t}$$

and

$$\forall t \geq 0, \|v(t)\|_{X_\infty} \leq \begin{cases} C e^{-\min(\lambda^1+1, \lambda^2)t} & \text{if } \lambda^1 + 1 \neq \lambda^2, \\ C(1+t)e^{-\lambda^2 t} & \text{if } \lambda^1 + 1 = \lambda^2. \end{cases}$$

## APPENDIX A. THE ABSTRACT SETTING UNDERLYING THE ANALYSIS

**A.1. A few remainders about semigroups.** For the sake or readability, we include some essential facts about semigroups of linear operators on Banach spaces. We mainly refer to [27] (but also to [5],[11]) for details and further results. We pay particular attention to extrapolation of semigroups, to operator defined by bilinear forms (see [26]), to their ultracontractivity and  $L^\infty$ -bounds, and finally to maximal regularity (see [24]). Indeed, these are the key arguments of the analysis developed in sections 2, 3, 4 and 5.

**A.1.1. Semigroups; strongly continuous semigroups.**

**Definition A.1.** *Let  $X$  be a Banach space.*

- (1) *A semigroup  $(T(t))_{t \geq 0}$  is as family of bounded linear operators  $T(t) : X \rightarrow X$  such that*

$$T(0) = \text{Id}_X \quad \text{and} \quad \forall s, t \geq 0, T(s+T) = T(s)T(t).$$

- (2) *The semigroup  $(T(t))_{t \geq 0}$  is strongly continuous if*

$$\forall x \in X, T(t)x \xrightarrow{t \rightarrow 0^+} x.$$

*It is also said that  $(T(t))_{t \geq 0}$  is a  $C_0$ -semigroup.*

- (3)  *$(T(t))_{t \geq 0}$  is a semigroup of contractions if for all  $t \geq 0$ ,  $T(t)$  is a contraction, i.e.*

$$\forall x \in X, \|T(t)x\|_X \leq \|x\|_X.$$

**Definition A.2.** *Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on the Banach space  $X$ . We call infinitesimal generator of  $(T(t))_{t \geq 0}$  the unbounded operator  $A$  on  $X$  defined by*

- $D(A) := \left\{ x \in X : \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$ ,
- $\forall x \in D(A), Ax = \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}$ .

$T(t)$  is also denoted by  $e^{tA}$ .

**Remark A.3.** *The notion of semigroup is particularly powerful to study the existence and the uniqueness of solutions of abstract Cauchy problems. More precisely, let  $X$  be a Banach space and let  $A$  be an unbounded linear operator on  $X$ . Consider the following abstract Cauchy problem:*

$$(39) \quad (\mathbf{ACP}) \quad : \quad \begin{cases} \frac{du}{dt}(t) = Au(t), & t > 0 \\ u(0) = x \end{cases}.$$

We say that a function  $u : [0, T[ \rightarrow X$  is a classical solution of abstract evolution problem<sup>6</sup> **(ACP)** if

- $u$  is continuous on  $[0, T)$ , continuously differentiable on  $(0, T)$ ,
- $\forall t \in (0, T), u(t) \in D(A)$  and (39) is satisfied on  $(0, T)$ .

It is shown that (see [27, Theorem I.2.4, p. 4]) if  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $(e^{tA})_{t \geq 0}$ , then, for all  $x \in D(A)$ , **(ACP)** has a unique solution, given by  $u(t) = e^{tA}x$ . We say that the problem **(ACP)** is well posed.

**Property A.4.** Let  $(e^{tA})_{t \geq 0}$  be a  $C_0$ -semigroup. Then, there exists  $\omega \in \mathbb{R}$  and  $M \geq 1$  such that

$$\forall t \geq 0, \|T(t)\| \leq Me^{\omega t}.$$

**Definition A.5.** Let  $X$  be a Banach space and  $A$  an unbounded operator on  $X$ .

- The resolvent set of  $A$  is the set  $\rho(A)$  of all  $\lambda \in \mathbb{C}$  such that  $\lambda I - A$  is invertible, i.e.  $(\lambda I - A)^{-1}$  is a bounded operator on  $X$ .
- The spectrum of  $A$  is the set  $\sigma(A) = \mathbb{C} \setminus \rho(A)$ .
- The numbers  $\lambda \in \sigma(A)$  such that  $\lambda I - A$  is not one-to-one are the eigenvalues of  $A$ ; the collection, denoted  $\sigma_p(A)$ , of such elements is the point spectrum of  $A$ .

**Theorem A.6** (Hille-Yosida). *Let  $X$  a Banach space and  $A$  an unbounded linear operator on  $X$ . Then,  $A$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions if and only if*

- ◊  $A$  is closed, densely defined i.e.  $\overline{D(A)} = X$ ,
- ◊ The resolvent set  $\rho(A)$  of  $A$  contains  $(0, +\infty)$  and

$$\forall \lambda > 0, \|R(\lambda, A)\| \leq \frac{1}{\lambda}.$$

**Definition A.7.** Let  $X$  be a Banach space and let  $A$  be an unbounded linear operator on  $X$ . We say that  $A$  is accretive when

$$\forall x \in D(A), \forall \lambda > 0, \|(\lambda I + A)x\| \geq \lambda \|x\|.$$

**Remark A.8.** Let  $X$  be a Hilbert space. It is shown that  $A$  is accretive if and only if

$$\forall x \in D(A), (Ax|x) \geq 0,$$

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<sup>6</sup> Not to be confused with the classical solution in the meaning of the partial differential equations for the partial differential equation underlying **(ACP)**.

where  $(\cdot|\cdot)$  denotes the inner product on  $X$ . Roughly speaking,  $A$  is accretive means that for all  $x \in D(A)$ ,  $x$  and  $Ax$  point to the same direction.

**Theorem A.9** (Lumer-Phillips). *Let  $X$  be a Banach space and  $A$  an unbounded linear densely defined operator on  $X$ . Then, the following properties are equivalent:*

- ◊  $-A$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions in  $X$ ,
- ◊  $A$  is accretive and  $\exists \lambda_0 > 0 : R(\lambda_0 I + A) = X$ ,
- ◊  $A$  is accretive and  $\forall \lambda > 0, R(\lambda I + A) = X$ .

A.1.2. *Differentiable semigroups.*

**Definition A.10.** *Let  $X$  be a Banach space and  $(T(t))_{t \geq 0}$  a  $C_0$ -semigroup on  $X$ . We say that  $(T(t))_{t \geq 0}$  is differentiable when for all  $x \in X$ , the function  $t \mapsto T(t)x$  is differentiable on  $(0, +\infty)$ .*

**Property A.11.** *Let  $(T(t))_{t \geq 0}$  be a differentiable  $C_0$ -semigroup on the Banach space  $X$ . Then*

- $\forall x \in X, \forall t > 0, T(t)x \in \bigcap_{n \in \mathbb{N}} D(A^n)$ .
- The function  $t \mapsto T(t)$  is a  $C^\infty$ -function on  $(0, +\infty)$  in the uniform operator topology.

**Remark A.12.** *Let us consider once again the abstract Cauchy problem (ACP). Assume also that  $(e^{tA})_{t \geq 0}$  is differentiable on  $(0, +\infty)$ . Then, for all  $x \in X$ , (ACP) has a unique solution (see [27, p. 104]). Note that the differentiability removes the need  $x \in D(A)$ .*

**Property A.13.** *If  $(T(t))_{t \geq 0}$  is a differentiable  $C_0$ -semigroup then  $(T(t))_{t \geq 0}$  is infinitely many times differentiable in the norm operator topology, for  $t > 0$ .*

A.1.3. *Analytic semigroups.*

**Definition A.14.** *Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on the Banach space  $X$ .*

- For  $\psi \in (0, \frac{\pi}{2}]$ ,  $\Delta_\psi$  denotes the angular sector

$$\Delta_\psi := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \psi\}.$$

- We say that the semigroup  $(T(t))_{t \geq 0}$  is analytic if there exists  $\psi \in (0, \frac{\pi}{2}]$  and an extension  $(T(z))_{z \in \Delta_\psi}$  such that
  - The function  $z \mapsto T(z)$  is analytic on  $\Delta_\psi$ ,
  - $\forall (z_1, z_2) \in \Delta_\psi^2, T(z_1 + z_2) = T(z_1)T(z_2)$ ,
  - $\forall x \in X, \lim_{z \rightarrow 0, z \in \Delta} T(z)x = x$  : in other words,  $(T(z))_{z \in \Delta_\psi}$  is strongly continuous at 0.

**Definition A.15.** *Let us now consider the inhomogeneous abstract Cauchy problem*

$$(40) \quad (\mathbf{ACP}_i) \quad : \quad \begin{cases} \frac{du}{dt}(t) = Au(t) + f(t), & t > 0 \\ u(0) = x \end{cases}$$

where  $f : [0, T[ \rightarrow X$ . We say that the function  $u : [0, T[ \rightarrow X$  is a classical solution of  $(\mathbf{ACP}_i)$  when

- $u$  is continuous on  $[0, T)$ , continuously differentiable on  $(0, T)$ ,
- $\forall t \in (0, T), u(t) \in D(A)$  and (40) is satisfied on  $(0, T)$ .



**Remark A.16.** *The analyticity assumption allows, under certain conditions on  $f$ , to ensure the existence of solutions of the problem  $(\mathbf{ACP}_i)$ , and that, for all  $x \in X$ . For example (see [27, p. 113]), assume that  $A$  is the infinitesimal generator of an analytic semigroup  $(T(t))_{t \geq 0}$ . Let  $f \in L^1(0, T; X)$  be locally Hölder continuous on  $(0, T)$ . Then, for all  $x \in X$ , the problem  $(\mathbf{ACP}_i)$  has a unique solution.*

**A.2. Extrapolating semigroups.** We refer to [5] for more details about extrapolating semigroups. From now on, all the measures are supposed  $\sigma$ -finite.

Let  $(\Omega, \mathcal{A}, \mu)$  be a measure space. For  $S \in \mathcal{L}(L^2(\Omega))$  and  $1 \leq p, q \leq \infty$ , we define  $\|S\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))}$  by

$$\|S\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} := \sup \left\{ \|S(f)\|_{L^q(\Omega)} : f \in L^p(\Omega) \cap L^2(\Omega), \|f\|_{L^p(\Omega)} \leq 1 \right\}.$$

We will assume that for  $p \in \{1; \infty\}$ , we have

$$(41) \quad \exists M : \forall t \in [0, 1], \|T(t)\|_{\mathcal{L}(L^p(\Omega))} \leq M.$$

By interpolation, this inequality is always true for all<sup>7</sup>  $p \in [1, \infty]$ . From (41) we deduce the existence of a real number  $\omega$  such that (see [27, Theorem 2.2, p. 4])

$$\forall p \in [1, \infty], \forall t \geq 0, \|T(t)\|_{\mathcal{L}(L^p(\Omega))} \leq M e^{\omega t}.$$

Indeed, if  $0 \leq t \leq 1$ , we have

- $T(t) : L^1(\Omega) \cap L^2(\Omega) \rightarrow L^1(\Omega) \cap L^2(\Omega)$  is continuous with respect to the norm  $L^1$ , with

$$\forall t \in [0, 1], \|T(t)\|_{\mathcal{L}(L^1(\Omega))} \leq M.$$

- $T(t) : L^\infty(\Omega) \cap L^2(\Omega) \rightarrow L^\infty(\Omega) \cap L^2(\Omega)$  is continuous with respect to the norm  $L^\infty$ , with

$$\forall t \in [0, 1], \|T(t)\|_{\mathcal{L}(L^\infty(\Omega))} \leq M.$$

- By interpolation, for all  $p \in [1, \infty]$ ,  $T(t) : L^p(\Omega) \cap L^2(\Omega) \rightarrow L^p(\Omega) \cap L^2(\Omega)$  is continuous with respect to the norm  $L^p$ , with

$$\forall t \in [0, 1], \|T(t)\|_{\mathcal{L}(L^p(\Omega))} \leq M.$$

Now, for  $t \geq 0$ ,  $p \in [1, +\infty]$  and  $f \in L^p(\Omega)$ , we have, with  $n = [t]$  and  $t = n + \delta$ ,  $\delta \in [0, 1[$ :

$$\begin{aligned} \|T(t)f\|_{L^p(\Omega)} &= \|T(n + \delta)f\|_{L^p(\Omega)} = \|T(1)^n T(\delta)f\|_{L^p(\Omega)} \leq M^{n+1} \|f\|_{L^p(\Omega)} = M M^n \|f\|_{L^p(\Omega)} \\ &= M e^{n \ln M} \|f\|_{L^p(\Omega)} \leq M e^{t \ln M} \|f\|_{L^p(\Omega)} = M e^{\omega t} \|f\|_{L^p(\Omega)} \end{aligned}$$

and then  $\|T(t)\|_{\mathcal{L}(L^p(\Omega))} \leq M e^{\omega t}$  where  $\omega = \ln M$ .

For  $1 \leq p < \infty$ , and by density of  $L^p(\Omega) \cap L^2(\Omega)$  in  $L^p(\Omega)$  (hence the hypothesis  $p < \infty$ ), we deduce the existence of continuous linear operators  $T_p(t) \in \mathcal{L}(L^p(\Omega))$  which are *consistent* i.e. such that

$$\forall t \geq 0, \forall p, q \in [1, \infty[, \forall f \in L^p(\Omega) \cap L^q(\Omega), T_p(t)f = T_q(t)f$$

and such that

$$\forall t \geq 0, T_2(t) = T(t).$$

<sup>7</sup>Strictly speaking, this inequality is true only for complex-valued functions. For real-valued functions, we have to double the coefficient  $M$ . See [30]. Nevertheless, the operators we will consider will be associated to a real bilinear form. Consequently, we can get rid of this coefficient considering complex spaces; the sesquilinear forms, with real coefficients, will be in fact real.

**Property A.17.** [5, p. 61]

- For  $1 < p < \infty$ ,  $(T_p(t))_{t \geq 0}$  is a  $C_0$ -semigroup.
- If  $\Omega$  has finite measure, then  $(T_1(t))_{t \geq 0}$  is also a  $C_0$ -semigroup.

**A.3. Ultracontractivity.** In the following, we define ultracontractivity for general semigroups. In practice, for one-dimensional networks, we will always have  $d = 1$ . However, in the forthcoming work [3], we will use general  $d \in \mathbb{N}^*$ .

**Property A.18.** [5, p. 65]

Let  $d > 0$  be a positive number. Let  $(\Omega, \mathcal{A}, \mu)$  be a measure space and  $(T(t))_{t \geq 0}$  a  $C_0$ -semigroup on  $L^2(\Omega)$ . Then, the two following properties are equivalent:

- $\exists c > 0, 1 \leq p < q \leq \infty : \forall t \in (0; 1], \|T(t)\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \leq ct^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}$ .
- $\exists c > 0 / \forall 1 \leq p < q \leq \infty : \forall t \in (0; 1], \|T(t)\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \leq ct^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}$ .

**Definition A.19.** Let  $(\Omega, \mathcal{A}, \mu)$  a measure space and let  $(T(t))_{t \geq 0}$  be  $C_0$ -semigroup on  $L^2(\Omega)$ .  $(T(t))_{t \geq 0}$  is called ultracontractive when one of the two equivalent previous properties holds. The real

$$\dim(T) := \inf \left\{ d > 0 : \exists c > 0, 1 \leq p < q \leq \infty, \forall t \in (0; 1], \|T(t)\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \leq ct^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \right\}$$

is called the dimension of the semigroup  $(T(t))_{t \geq 0}$  (See [5, p. 65]).

**A.4. Compactness.**

**Property A.20.** [5, p. 15]

Let  $A$  be a unbounded linear operator on the Banach space  $X$  and assume that  $\rho(A) \neq \emptyset$ . We say that  $A$  has compact resolvent if one of the three following equivalent properties holds:

- For all  $\lambda \in \rho(A)$ ,  $R(\lambda, A)$  is compact,
- There exists  $\lambda \in \rho(A)$  such that  $R(\lambda, A)$  is compact,
- The embedding  $D(A) \hookrightarrow X$  is compact,  $D(A)$  provided with the graph norm.

In this case,  $\sigma(A) = \sigma_p(A)$  and  $\sigma_p(A)$  is a sequence converging to  $+\infty$  (or a finite sequence if  $\dim X < +\infty$ ).

**Property A.21.** [27, Theorem 3.3, p. 48]

Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup with infinitesimal generator  $A$ . Then,  $(T(t))_{t \geq 0}$  is a compact semigroup if and only if  $(T(t))_{t \geq 0}$  is continuous in the norm operator topology for  $t > 0$  and  $R(\lambda, A)$  is compact for  $\lambda \in \rho(A)$ .

**A.5. Spectral bound and growth bound; exponential stability.** Following [5, p. 12], we define the spectral bound and the growth bound as follows.

**Definition A.22.** Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup with infinitesimal generator  $A$ .

- We call spectral bound of  $A$

$$s(A) = \sup \{ \Re(\lambda), \lambda \in \sigma(A) \}.$$

- We say that  $(T(t))_{t \geq 0}$  is exponentially stable when

$$\exists \varepsilon > 0, \exists M \geq 1 : \forall t \geq 0, \|T(t)\| \leq Me^{-\varepsilon t}.$$

- We define the growth bound of  $(T(t))_{t \geq 0}$  by:

$$\omega(T) = \inf \left\{ w : \sup_{t > 0} e^{-wt} \|T(t)\| < +\infty \right\}.$$

**Remark A.23.**  $(T(t))_{t \geq 0}$  is exponentially stable if and only if  $\omega(T) < 0$ .

The inequality  $s(A) \leq \omega(T)$  always holds ([11, Proposition 2.2, p.251]). Sometimes, the previous inequality is in fact an equality. This is the object of the following proposition.

**Property A.24.** [11, Corollary 3.12, p. 281]

Let  $A$  be the infinitesimal generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$ . Then,  $\omega(T) = s(A)$  in the following cases:

- if  $(T(t))_{t \geq 0}$  is a compact semigroup,
- if  $(T(t))_{t \geq 0}$  is differentiable and, in particular, if  $(T(t))_{t \geq 0}$  is analytic.

**A.6. Bilinear forms; associated operators.** Let  $X$  be a real Hilbert space provided with its inner product  $(\cdot|\cdot)_X$  and its norm  $\|\cdot\|_X$ . Consider a bilinear form  $\mathfrak{a} : H \times H \rightarrow \mathbb{R}$  defined on a subspace  $H$  of  $X$ .  $H$  is called the *domain* of  $\mathfrak{a}$  and is denoted  $D(\mathfrak{a})$ . Throughout this paragraph, we suppose that  $\mathfrak{a}$  is

- densely defined, i.e.  $H$  is dense in  $X$ ,
- accretive, i.e.

$$\forall u \in H, \mathfrak{a}(u, u) \geq 0,$$

- continuous on  $H$ , i.e. such that

$$\exists M \geq 0 : \forall (u, v) \in H \times H, |\mathfrak{a}(u, v)| \leq M \|u\|_{\mathfrak{a}} \|v\|_{\mathfrak{a}},$$

with:  $\forall u \in D(\mathfrak{a}), \|u\|_{\mathfrak{a}} := \sqrt{\|u\|_X^2 + \mathfrak{a}(u, u)}$ ,

- closed, i.e.  $(H, \|\cdot\|_{\mathfrak{a}})$  is a complete space.

So, we can consider  $\mathfrak{a}$  as an unbounded bilinear form on  $X \times X$ , with domain  $D(\mathfrak{a}) = H$ .

With the bilinear form  $\mathfrak{a}$  is associated the unbounded linear operator  $A$  on  $X$  defined by

- $D(A) := \{u \in H : \exists \phi \in X, \forall v \in D(\mathfrak{a}), \mathfrak{a}(u, v) = (\phi|v)_H\}$ ,
- $Au := \phi$ .

Later, we will need the following properties. Recall that  $X$  is a real Hilbert space,  $\mathfrak{a}$  is a bilinear form defined on a subspace  $H \subset X$ . Let us assume that  $\mathfrak{a}$  is densely defined, accretive, continuous on  $H$  and closed.

**Property A.25.** [26, Proposition 1.51, Theorem 1.52, p. 29]

$-A$  is the infinitesimal generator of an analytic semigroup of contractions on  $X$ .

From now on,  $(e^{-tA})_{t \geq 0}$  or  $(T(t))_{t \geq 0}$  will denote indistinctly the semigroup generated by  $-A$ .

**Property A.26.** [26, Proposition 1.24, p. 15]

Suppose further that  $\mathfrak{a}$  is symmetric. Then,  $A$  is a self-adjoint operator.

**Definition A.27.** Let  $u : \Omega \rightarrow \mathbb{R}$ .

- sign  $(u)$  is defined by:  $\text{sign}(u) = \begin{cases} 1 & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) = 0, \\ -1 & \text{if } u(x) < 0. \end{cases}$

- Let  $Tu$  denote the function  $(1 \wedge |u|)\text{sign}(u)$ .  $Tu$  is a truncation of  $u$ . Indeed:

$$((1 \wedge |u|)\text{sign}(u))(x) = \begin{cases} u(x) & \text{if } |u(x)| \leq 1, \\ \frac{u(x)}{|u(x)|} & \text{if } |u(x)| \geq 1, \end{cases} = \begin{cases} u(x) & \text{if } |u(x)| < 1, \\ 1 & \text{if } u(x) \geq 1, \\ -1 & \text{if } u(x) \leq -1. \end{cases}$$

- Suppose now that  $u = (u_1, \dots, u_n) : \Omega \rightarrow \mathbb{R}^n$ . We define  $\text{sign}(u)$  and  $Tu$  as follows:  $\text{sign}(u) := (\text{sign}(u_1), \dots, \text{sign}(u_n))$  and  $Tu := (Tu_1, \dots, Tu_n)$ .

**Theorem A.28.** [26, Theorem 2.14 page 55]

Let  $\mathbf{a}$  be a bilinear form defined on the measure space  $X := L^2(\Omega, \mu, \mathbb{R})$  and assume that  $\mathbf{a}$  is symmetric. The two following properties are equivalent:

- The semigroup  $(e^{-tA})_{t \geq 0}$  is  $L^\infty$ -contractive,
- $\forall u \in D(\mathbf{a}), Tu \in D(\mathbf{a})$  and  $\mathbf{a}(Tu, Tu) \leq \mathbf{a}(u, u)$ .

**Theorem A.29.** [26, Theorem 6.3, p. 158]

Let  $\mathbf{a}$  defined on the measure space  $X := L^2(\Omega, \mu, \mathbb{R})$ . Assume, in addition, that  $\mathbf{a}$  is symmetric and that the semigroup  $(e^{-tA})_{t \geq 0}$  is  $L^\infty$ -contractive. The following properties are equivalent:

- The semigroup  $(e^{-tA})_{t \geq 0}$  is ultracontractive,
- $\exists c, d > 0 : \forall t > 0, \|e^{-tA}\|_{\mathcal{L}(L^1(\Omega), L^2(\Omega))} \leq ct^{-\frac{d}{4}}$ ,
- $\exists c', d > 0 : \forall u \in D(\mathbf{a}) \cap L^1(\Omega), \|u\|_{L^2(\Omega)}^{2+\frac{d}{4}} \leq c' \mathbf{a}(u, u) \|u\|_{L^1(\Omega)}^{\frac{d}{4}}$ .

**A.7.  $L^\infty$ -bounds and ultracontractivity.** Let  $(\Omega, \mathcal{A}, \mu)$  be a real measure space and let  $X := L^2(\Omega)$  provided with its natural inner product  $(\cdot | \cdot)_{L^2(\Omega)}$  and the associated norm  $\|\cdot\|_{L^2(\Omega)}$ . Consider a bilinear form  $\mathbf{a} : H \times H \rightarrow \mathbb{R}$  defined on a subspace  $H$  of  $X$ . Throughout this paragraph, we suppose that the form  $\mathbf{a}$  is densely defined, accretive, continuous on  $H$  and closed. Let  $A$  be the unbounded linear operator associated with  $\mathbf{a}$ . We are interested in the following abstract Cauchy problem

$$(\mathbf{ACP}) : \frac{du}{dt} + Au = f(t) \quad , \quad u(0) = u_0$$

where  $u_0 \in L^\infty(\Omega)$  and  $f \in L^1(0, T; L^2(\Omega))$ . Let  $u$  be the solution of  $(\mathbf{ACP})$  on  $(0, T)$  (for example, sufficient hypothesis to insure such an existence is that  $f$  is locally Hölder continuous on  $(0, T)$ : see [27]).

**Property A.30.** Assume that  $0 < T < +\infty$  and that

- $\forall q \in [1, +\infty], \forall t \in [0, T], \forall v \in L^2(\Omega) \cap L^q(\Omega), \|e^{-tA}v\|_{L^q(\Omega)} \leq \|v\|_{L^q(\Omega)}$ ,
- $\exists p > 1 : f \in L^p(0, T; L^p(\Omega))$ .

Then:  $u \in L^p(0, T; L^p(\Omega))$ .

*Proof.* According to Duhamel formula, we have

$$\forall t \in [0, T[, u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s) ds.$$

Then, for all  $t \in [0, T[$ , we have

$$\|u(t)\|_{L^p(\Omega)} \leq \|e^{-tA}u_0\|_{L^p(\Omega)} + \int_0^t \|e^{-(t-s)A} f(s)\|_{L^p(\Omega)} ds \leq \|u_0\|_{L^p(\Omega)} + \int_0^t \|f(s)\|_{L^p(\Omega)} ds$$

because  $(e^{-tA})_{L^q(\Omega)}$  is a semigroup of contractions. Consequently, thanks to Hölder inequality (here,  $p'$  denotes the conjugate of  $p$ ):

$$\|u(t)\|_{L^p(\Omega)} \leq \|u_0\|_{L^p(\Omega)} + \underbrace{\left( \int_0^t \|f(s)\|_{L^p(\Omega)}^p ds \right)^{\frac{1}{p}}}_{=\|f\|_{L^p(0,t;L^p(\Omega))}} t^{\frac{1}{p'}} \leq \|u_0\|_{L^p(\Omega)} + \|f\|_{L^p(0,T;L^p(\Omega))} T^{\frac{1}{p'}}.$$

It remains to integrate with respect to  $t$  between 0 and  $T$ .  $\square$

More interesting is to deduce  $L^\infty$ -bounds of  $u$  from  $L^p$ -bounds of  $f$ . The two following properties are very classical for open subsets  $\Omega \subset \mathbb{R}^n$ . Much more general results can be found in the literature (see [21, Theorem 7.1, p. 181]). Nevertheless, based on semigroups techniques, our proofs permit to expand much partially but sufficiently such results to networks, including multidimensional networks, studied in the forthcoming work [3].

**Property A.31.** *For  $T < +\infty$ , suppose in addition that the semigroup  $(e^{-tA})_{t \geq 0}$  is ultracontractive and that*

$$\forall p > 1, f \in L^p(0, T; L^p(\Omega)).$$

*Then,  $u \in L^\infty(0, T; L^\infty(\Omega))$ . The conclusion is not affected if we only suppose that  $f \in L^p(0, T; L^p(\Omega))$ , where  $p > \frac{d+2}{2}$ ,  $d$  denoting the dimension of the semigroup  $(e^{-tA})_{t \geq 0}$ .*

*Proof.* Duhamel formula asserts that  $u$  is given by

$$\forall t \in [0, T], u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(s) ds = e^{-tA}u_0 + \int_0^t e^{-sA}f(t-s) ds.$$

Since  $\|e^{-tA}u_0\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$  (because  $(e^{-tA})_{t \geq 0}$  is a semigroup of contractions), we only have to estimate  $\|e^{-sA}f(t-s)\|_{L^\infty(\Omega)}$ .

Recall that the semigroup  $(e^{-tA})_{t \geq 0}$  is ultracontractive, which means that

$$\exists C > 0 : \forall s \in (0, T), \forall 1 \leq p \leq q \leq +\infty, \|\|e^{-sA}\|\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \leq Cs^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}$$

Choosing  $q = +\infty$ , we have

$$\exists C > 0 : \forall s \in (0, T), \|\|e^{-sA}\|\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))} \leq Cs^{-\frac{d}{2p}}$$

and then

$$\|e^{-sA}f(t-s)\|_{L^\infty(\Omega)} \leq \|\|e^{-sA}\|\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))} \cdot \|f(t-s)\|_{L^p(\Omega)} \leq Cs^{-\frac{d}{2p}} \|f(t-s)\|_{L^p(\Omega)}$$

Then, integrate between 0 and  $t$ ; The Hölder inequality shows that

$$\int_0^t \|e^{-sA}f(t-s)\|_{L^\infty(\Omega)} ds \leq C \underbrace{\left( \int_0^t \|f(t-s)\|_{L^p(\Omega)}^p ds \right)^{\frac{1}{p}}}_{=\|f\|_{L^p(0,t;L^p(\Omega))}} \left( \int_0^t \frac{ds}{s^{\frac{dp'}{2p}}} \right)^{\frac{1}{p'}},$$

where  $p'$  denotes the conjugate of  $p$ , defined by:  $\frac{1}{p} + \frac{1}{p'} = 1$ . Choosing  $p > \frac{d+2}{2}$ , in such a way as  $\frac{dp'}{2p} < 1$ , and by triangle inequality, we obtain

$$\begin{aligned} \|u(t)\|_{L^\infty(\Omega)} &\leq \|u_0\|_{L^\infty(\Omega)} + C \|f\|_{L^p(0,t;L^p(\Omega))} \left( \int_0^t \frac{ds}{s^{\frac{dp'}{2p}}} \right)^{\frac{1}{p'}} \\ &\leq \|u_0\|_{L^\infty(\Omega)} + C \|f\|_{L^p(0,T;L^p(\Omega))} \left( \int_0^T \frac{ds}{s^{\frac{dp'}{2p}}} \right)^{\frac{1}{p'}} \\ &= \|u_0\|_{L^\infty(\Omega)} + CI \|f\|_{L^p(0,T;L^p(\Omega))} \end{aligned}$$

where  $I := \left( \int_0^T \frac{ds}{s^{\frac{dp'}{2p}}} \right)^{\frac{1}{p'}}$ . This completes the proof in the case  $T < +\infty$ .  $\square$

**Property A.32.** *Property A.31 is still valid for  $T = +\infty$ .*

*Proof.* Let  $(t_n)_{n \geq 0}$  denote an increasing sequence of real numbers such that  $t_0 = 0$ . For  $n \in \mathbb{N}$ , we define  $\delta_n := t_{n+1} - t_n$  in such a way as  $t_n = \delta_0 + \delta_1 + \dots + \delta_{n-1}$ . Moreover, we assume that the sequence  $(\delta_n)$  is bounded above by a positive real number. By Duhamel formula, we have

$$u(t_1) = e^{-t_1 A} u_0 + \int_0^{t_1} e^{-sA} f(t_1 - s) ds.$$

Consequently

$$(42) \quad \|u(t_1)\|_{L^\infty(\Omega)} \leq \|e^{-t_1 A} u_0\|_{L^\infty(\Omega)} + \int_0^{t_1} \|e^{-sA} f(t_1 - s)\|_{L^\infty(\Omega)} ds.$$

But we know that

$$\exists K \geq 0 : \forall t \geq 0, \|e^{-tA}\|_{\mathcal{L}^\infty(\Omega)} \leq K e^{-\lambda t}$$

where  $\lambda$  denotes the smallest eigenvalue of  $A$  ( $\lambda > 0$  by positivity and ultracontractivity). Moreover, by ultracontractivity (with  $q = +\infty$ ), we have

$$\exists C > 0 : \forall s \in [0, t_1], \|e^{-sA}\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))} \leq C s^{-\frac{d}{2p}}$$

Then, (42) becomes

$$\|u(t_1)\|_{L^\infty(\Omega)} \leq K e^{-\lambda t_1} \|u_0\|_{L^\infty(\Omega)} + C \int_0^{t_1} s^{-\frac{d}{2p}} \|f(t_1 - s)\|_{L^p(\Omega)} ds.$$

Hölder inequality shows that

$$\|u(t_1)\|_{L^\infty(\Omega)} \leq K e^{-\lambda t_1} \|u_0\|_{L^\infty(\Omega)} + C \underbrace{\left( \int_0^{t_1} \|f(t-s)\|_{L^p(\Omega)}^p ds \right)^{\frac{1}{p}}}_{=\|f\|_{L^p(0,t_1;L^p(\Omega))}} \left( \int_0^{t_1} \frac{ds}{s^{\frac{dp'}{2p}}} \right)^{\frac{1}{p'}}$$

where  $p'$  denotes the conjugate of  $p$ , defined by:  $\frac{1}{p} + \frac{1}{p'} = 1$ . Recall that  $t_1 = \delta_0$  and let  $I_1 := \int_0^{t_1} \frac{ds}{s^{\frac{dp'}{2p}}} = \int_0^{\delta_0} \frac{ds}{s^{\frac{dp'}{2p}}}$ . Then, we have

$$(43) \quad \|u(t_1)\|_{L^\infty(\Omega)} \leq K e^{-\delta_0 \lambda} \|u_0\|_{L^\infty(\Omega)} + CI_1^{\frac{1}{p'}} \|f\|_{L^p(0,t_1;L^p(\Omega))}.$$

Likewise,

$$\begin{aligned}
\|u(t_2)\|_{L^\infty(\Omega)} &= \left\| e^{-\delta_1 A} u(t_1) + \int_0^{\delta_1} e^{-sA} f(t_2 - s) \, ds \right\|_{L^\infty(\Omega)} \\
&\leq \|e^{-\delta_1 A} u(t_1)\|_{L^\infty(\Omega)} + \int_0^{\delta_1} \|e^{-sA} f(t_2 - s)\|_{L^\infty(\Omega)} \, ds \\
&\leq K e^{-\delta_1 \lambda} \|u(t_1)\|_{L^\infty(\Omega)} + C \int_0^{\delta_1} s^{-\frac{d}{2p}} \|f(t_2 - s)\|_{L^p(\Omega)} \, ds \\
&\leq K e^{-\delta_1 \lambda} \|u(t_1)\|_{L^\infty(\Omega)} + C I_2^{\frac{1}{p'}} \|f\|_{L^p(t_1, t_2; L^p(\Omega))}
\end{aligned}$$

where  $I_2 := \int_0^{\delta_1} \frac{ds}{s^{\frac{d}{2p}}}$ . Combining (43) and the previous inequality leads to

$$\begin{aligned}
\|u(t_2)\|_{L^\infty(\Omega)} &\leq K e^{-\delta_1 \lambda} \left[ K e^{-\delta_0 \lambda} \|u_0\|_{L^\infty(\Omega)} + C I_1^{\frac{1}{p'}} \|f\|_{L^p(0, t_1; L^p(\Omega))} \right] + C I_2^{\frac{1}{p'}} \|f\|_{L^p(t_1, t_2; L^p(\Omega))} \\
&= K^2 e^{-(\delta_0 + \delta_1) \lambda} \|u_0\|_{L^\infty(\Omega)} + C \left[ K e^{-\delta_1 \lambda} I_1^{\frac{1}{p'}} \|f\|_{L^p(t_0, t_1; L^p(\Omega))} + I_2^{\frac{1}{p'}} \|f\|_{L^p(t_1, t_2; L^p(\Omega))} \right]
\end{aligned}$$

Let  $I_n := \int_0^{\delta_{n-1}} \frac{ds}{s^{\frac{d}{2p}}}$ . By induction, we have

$$\|u(t_n)\|_{L^\infty(\Omega)} \leq K^n e^{-\lambda t_n} \|u_0\|_{L^\infty(\Omega)} + C \sum_{j=1}^n K^{n-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1})} I_j^{\frac{1}{p'}} \|f\|_{L^p(t_{j-1}, t_j; L^p(\Omega))}.$$

Indeed, assume that the previous equality is true. Then,

$$\begin{aligned}
&\|u(t_{n+1})\|_{L^\infty(\Omega)} \\
&= \left\| e^{-\delta_n A} u(t_n) + \int_0^{\delta_n} e^{-sA} f(t_{n+1} - s) \, ds \right\|_{L^\infty(\Omega)} \\
&\leq K e^{-\delta_n \lambda} \|u(t_n)\|_{L^\infty(\Omega)} + C I_{n+1}^{\frac{1}{p'}} \|f\|_{L^p(t_n, t_{n+1}; L^p(\Omega))} \\
&\leq K e^{-\delta_n \lambda} \left[ K^n e^{-\lambda t_n} \|u_0\|_{L^\infty(\Omega)} + C \sum_{j=1}^n K^{n-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1})} I_j^{\frac{1}{p'}} \|f\|_{L^p(t_{j-1}, t_j; L^p(\Omega))} \right] \\
&\quad + C I_{n+1}^{\frac{1}{p'}} \|f\|_{L^p(t_n, t_{n+1}; L^p(\Omega))} \\
&= K^{n+1} e^{-\lambda(t_n + \delta_n)} \|u_0\|_{L^\infty(\Omega)} + C \sum_{j=1}^n K^{n+1-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1} + \delta_n)} I_j^{\frac{1}{p'}} \|f\|_{L^p(t_{j-1}, t_j; L^p(\Omega))} \\
&\quad + C I_{n+1}^{\frac{1}{p'}} \|f\|_{L^p(t_n, t_{n+1}; L^p(\Omega))} \\
&= K^{n+1} e^{-\lambda t_{n+1}} \|u_0\|_{L^\infty(\Omega)} + C \sum_{j=1}^{n+1} K^{n+1-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1} + \delta_n)} I_j^{\frac{1}{p'}} \|f\|_{L^p(t_{j-1}, t_j; L^p(\Omega))}.
\end{aligned}$$

Then

(44)

$$\|u(t_n)\|_{L^\infty(\Omega)} \leq K^n e^{-\lambda t_n} \|u_0\|_{L^\infty(\Omega)} + C I \sum_{j=1}^n K^{n-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1})} \|f\|_{L^p(t_{j-1}, t_j; L^p(\Omega))}$$

where  $I := \sup_{j \geq 1} I_j^{\frac{1}{p'}}$  (recall that the sequence  $(\delta_n)$  is bounded above, and so is the sequence  $(I_j^{\frac{1}{p'}})$ ). By the Hölder inequality, we have

$$(45) \quad \sum_{j=1}^n K^{n-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1})} \|f\|_{L^p(t_{j-1}, t_j; L^p(\Omega))} \\ \leq \left( \sum_{j=1}^n K^{n-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1}) p'} \right)^{\frac{1}{p'}} \left( \sum_{j=1}^n \|f\|_{L^p(t_{j-1}, t_j; L^p(\Omega))}^p \right)^{\frac{1}{p}}.$$

Now, assume that  $\delta_j = r$ , where  $r$  denotes a positive integer. (44) combined with (45) becomes

$$\|u(t_n)\|_{L^\infty(\Omega)} \leq K^n e^{-\lambda r n} \|u_0\|_{L^\infty(\Omega)} + CI \left( \sum_{j=1}^n K^{n-j} e^{-\lambda(n-j)r p'} \right)^{\frac{1}{p'}} \|f\|_{L^p(t_0, t_n; L^p(\Omega))},$$

i.e.

$$\|u(t_n)\|_{L^\infty(\Omega)} \leq K^n e^{-\lambda r n} \|u_0\|_{L^\infty(\Omega)} + CI \left( \sum_{j=0}^{n-1} K^j e^{-\lambda j r p'} \right)^{\frac{1}{p'}} \|f\|_{L^p(t_0, t_n; L^p(\Omega))},$$

and then

$$\|u(t_n)\|_{L^\infty(\Omega)} \leq (K e^{-\lambda r})^n \|u_0\|_{L^\infty(\Omega)} + CI \left( \sum_{j=0}^{n-1} (K e^{-\lambda r p'})^j \right)^{\frac{1}{p'}} \|f\|_{L^p(0, +\infty; L^p(\Omega))}.$$

Now, choose  $r$  sufficiently large such that  $K e^{-\lambda r} < 1$  and  $K e^{-\lambda r p'} < 1$ . From the previous inequality, we deduce that

$$\forall n \in \mathbb{N}, \|u(t_n)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + CI \left( \sum_{j=0}^{+\infty} (K e^{-\lambda r p'})^j \right)^{\frac{1}{p'}} \|f\|_{L^p(0, +\infty; L^p(\Omega))}$$

Considering  $t \in [t_n, t_{n+1}]$  and applying once again Duhamel formula, it follows that

$$\sup_{t \geq 0} \|u(t)\|_{L^\infty(\Omega)} < +\infty,$$

which ends the proof.  $\square$

In the same way, we can derive  $L^\infty$  bounds from uniform  $L^p$  bounds.

**Property A.33.** For  $0 < T < +\infty$ , assume that the semigroup  $(e^{-tA})_{t \geq 0}$  is ultracontractive and that

$$\exists p > \frac{d}{2} : f \in L^\infty(0, T; L^p(\Omega)),$$

$d$  denoting the dimension of the semigroup  $(e^{-tA})_{t \geq 0}$ . Then,  $u \in L^\infty(0, T; L^\infty(\Omega))$ .

*Proof.* Duhamel formula asserts that  $u$  is given by

$$\forall t \in [0, T], u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} f(s) ds = e^{-tA} u_0 + \int_0^t e^{-sA} f(t-s) ds.$$



Since  $\|e^{-tA}u_0\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$  (because  $(e^{-tA})_{t \geq 0}$  is a semigroup of contractions), we only have to estimate  $\|e^{-sA}f(t-s)\|_{L^\infty(\Omega)}$ .

Recall that the semigroup  $(e^{-tA})_{t \geq 0}$  is ultracontractive, which means that

$$\exists C > 0 : \forall s \in (0, T), \forall 1 \leq p \leq q \leq +\infty, \left\| e^{-sA} \right\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \leq C s^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}.$$

Choosing  $q = +\infty$ , we have

$$\exists C > 0 : \forall s \in (0, T), \left\| e^{-sA} \right\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))} \leq C s^{-\frac{d}{2p}},$$

and then

$$\|e^{-sA}f(t-s)\|_{L^\infty(\Omega)} \leq \left\| e^{-sA} \right\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))} \cdot \|f(t-s)\|_{L^p(\Omega)} \leq C s^{-\frac{d}{2p}} \|f(t-s)\|_{L^p(\Omega)}.$$

Then, integrate between 0 and  $t$  and obtain

$$\int_0^t \|e^{-sA}f(t-s)\|_{L^\infty(\Omega)} ds \leq C \|f\|_{L^\infty(0, T; L^p(\Omega))} \left( \int_0^t \frac{ds}{s^{\frac{d}{2p}}} \right).$$

By triangle inequality, we have

$$\begin{aligned} \|u(t)\|_{L^\infty(\Omega)} &\leq \|u_0\|_{L^\infty(\Omega)} + C \|f\|_{L^\infty(0, T; L^p(\Omega))} \left( \int_0^t \frac{ds}{s^{\frac{d}{2p}}} \right) \\ &\leq \|u_0\|_{L^\infty(\Omega)} + C \|f\|_{L^\infty(0, T; L^p(\Omega))} \left( \int_0^T \frac{ds}{s^{\frac{d}{2p}}} \right) \\ &= \|u_0\|_{L^\infty(\Omega)} + CI \|f\|_{L^\infty(0, T; L^p(\Omega))} \end{aligned}$$

where  $I := \int_0^T \frac{ds}{s^{\frac{d}{2p}}}$ . This completes the proof in the case  $T < +\infty$ .  $\square$

**Property A.34.** *Property A.33 is still valid for  $T = +\infty$ .*

*Proof.* Let  $(t_n)_{n \geq 0}$  denote an increasing sequence of real numbers such that  $t_0 = 0$ . For  $n \in \mathbb{N}$ , we define  $\delta_n := t_{n+1} - t_n$  in such a way that  $t_n = \delta_0 + \delta_1 + \dots + \delta_{n-1}$ . Moreover, we assume that the sequence  $(\delta_n)$  is bounded above by a positive real number. By Duhamel formula, we have

$$u(t_1) = e^{-t_1 A} u_0 + \int_0^{t_1} e^{-sA} f(t_1 - s) ds.$$

Consequently

$$(46) \quad \|u(t_1)\|_{L^\infty(\Omega)} \leq \|e^{-t_1 A} u_0\|_{L^\infty(\Omega)} + \int_0^{t_1} \|e^{-sA} f(t_1 - s)\|_{L^\infty(\Omega)} ds.$$

We also have

$$\exists K \geq 0 : \forall t \geq 0, \left\| e^{-tA} \right\|_{\mathcal{L}^\infty(\Omega)} \leq K e^{-\lambda t}$$

where  $\lambda$  denotes the smallest eigenvalue of  $A$  ( $\lambda > 0$  by positivity and ultracontractivity). Moreover, by ultracontractivity (with  $q = +\infty$ ), we have

$$\exists C > 0 : \forall s \in [0, t_1], \left\| e^{-sA} \right\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))} \leq C s^{-\frac{d}{2p}}.$$

Then, (46) becomes

$$\|u(t_1)\|_{L^\infty(\Omega)} \leq K e^{-\lambda t_1} \|u_0\|_{L^\infty(\Omega)} + C \int_0^{t_1} s^{-\frac{d}{2p}} \|f(t_1 - s)\|_{L^p(\Omega)} ds.$$

Recall that  $t_1 = \delta_0$  and let  $I_1 := \int_0^{t_1} \frac{ds}{s^{\frac{d}{2p}}} = \int_0^{\delta_0} \frac{ds}{s^{\frac{d}{2p}}}$ . Then, we have

$$(47) \quad \|u(t_1)\|_{L^\infty(\Omega)} \leq K e^{-\delta_0 \lambda} \|u_0\|_{L^\infty(\Omega)} + C I_1 \|f\|_{L^\infty(0, \infty; L^p(\Omega))}.$$

Likewise,

$$\begin{aligned} \|u(t_2)\|_{L^\infty(\Omega)} &= \left\| e^{-\delta_1 A} u(t_1) + \int_0^{\delta_1} e^{-sA} f(t_2 - s) ds \right\|_{L^\infty(\Omega)} \\ &\leq \|e^{-\delta_1 A} u(t_1)\|_{L^\infty(\Omega)} + \int_0^{\delta_1} \|e^{-sA} f(t_2 - s)\|_{L^\infty(\Omega)} ds \\ &\leq K e^{-\delta_1 \lambda} \|u(t_1)\|_{L^\infty(\Omega)} + C \int_0^{\delta_1} s^{-\frac{d}{2p}} \|f(t_2 - s)\|_{L^p(\Omega)} ds \\ &\leq K e^{-\delta_1 \lambda} \|u(t_1)\|_{L^\infty(\Omega)} + C I_2 \|f\|_{L^\infty(0, \infty; L^p(\Omega))} \end{aligned}$$

where  $I_2 := \int_0^{\delta_1} \frac{ds}{s^{\frac{d}{2p}}}$ . Combining (47) and the previous inequality leads to

$$\begin{aligned} \|u(t_2)\|_{L^\infty(\Omega)} &\leq K e^{-\delta_1 \lambda} \left[ K e^{-\delta_0 \lambda} \|u_0\|_{L^\infty(\Omega)} + C I_1 \|f\|_{L^\infty(0, \infty; L^p(\Omega))} \right] + C I_2 \|f\|_{L^p(0, \infty; L^p(\Omega))} \\ &= K^2 e^{-(\delta_0 + \delta_1) \lambda} \|u_0\|_{L^\infty(\Omega)} + C [K e^{-\delta_1 \lambda} I_1 + I_2] \|f\|_{L^\infty(0, \infty; L^p(\Omega))}. \end{aligned}$$

Let  $I_n := \int_0^{\delta_{n-1}} \frac{ds}{s^{\frac{d}{2p}}}$ . By induction, we have

$$\|u(t_n)\|_{L^\infty(\Omega)} \leq K^n e^{-\lambda t_n} \|u_0\|_{L^\infty(\Omega)} + C \left( \sum_{j=1}^n K^{n-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1})} I_j \right) \|f\|_{L^\infty(0, \infty; L^p(\Omega))}.$$

Indeed, assume that the previous equality is true. Then,

$$\begin{aligned} \|u(t_{n+1})\|_{L^\infty(\Omega)} &= \left\| e^{-\delta_n A} u(t_n) + \int_0^{\delta_n} e^{-sA} f(t_{n+1} - s) ds \right\|_{L^\infty(\Omega)} \\ &\leq K e^{-\delta_n \lambda} \|u(t_n)\|_{L^\infty(\Omega)} + C I_{n+1} \|f\|_{L^\infty(0, \infty; L^p(\Omega))} \\ &\leq K e^{-\delta_n \lambda} \left[ K^n e^{-\lambda t_n} \|u_0\|_{L^\infty(\Omega)} + C \left( \sum_{j=1}^n K^{n-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1})} I_j \right) \|f\|_{L^\infty(0, \infty; L^p(\Omega))} \right] \\ &\quad + C I_{n+1} \|f\|_{L^\infty(0, \infty; L^p(\Omega))} \\ &= K^{n+1} e^{-\lambda(t_n + \delta_n)} \|u_0\|_{L^\infty(\Omega)} + C \left( \sum_{j=1}^n K^{n+1-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1} + \delta_n)} I_j \right) \|f\|_{L^\infty(0, \infty; L^p(\Omega))} \\ &\quad + C I_{n+1} \|f\|_{L^\infty(0, \infty; L^p(\Omega))} \\ &= K^{n+1} e^{-\lambda t_{n+1}} \|u_0\|_{L^\infty(\Omega)} + C \left( \sum_{j=1}^{n+1} K^{n+1-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1} + \delta_n)} I_j \right) \|f\|_{L^\infty(0, \infty; L^p(\Omega))}. \end{aligned}$$

Then

$$(48) \quad \|u(t_n)\|_{L^\infty(\Omega)} \leq K^n e^{-\lambda t_n} \|u_0\|_{L^\infty(\Omega)} + C I \left( \sum_{j=1}^n K^{n-j} e^{-\lambda(\delta_j + \dots + \delta_{n-1})} I_j \right) \|f\|_{L^\infty(0, \infty; L^p(\Omega))}$$

where  $I := \sup_{j \geq 1} I_j$  (recall that the sequence  $(\delta_n)$  is bounded above, and so is the sequence  $(I_j)$ ). Now, assume that  $\delta_j = r$ , where  $r$  denotes a positive integer. (48) becomes

$$\|u(t_n)\|_{L^\infty(\Omega)} \leq K^n e^{-\lambda r n} \|u_0\|_{L^\infty(\Omega)} + CI \left( \sum_{j=1}^n K^{n-j} e^{-\lambda(n-j)r} \right) \|f\|_{L^\infty(0, \infty; L^p(\Omega))},$$

i.e.

$$\|u(t_n)\|_{L^\infty(\Omega)} \leq K^n e^{-\lambda r n} \|u_0\|_{L^\infty(\Omega)} + CI \left( \sum_{j=0}^{n-1} K^j e^{-\lambda j r} \right) \|f\|_{L^\infty(0, \infty; L^p(\Omega))},$$

and then

$$\|u(t_n)\|_{L^\infty(\Omega)} \leq (K e^{-\lambda r})^n \|u_0\|_{L^\infty(\Omega)} + CI \left( \sum_{j=0}^{n-1} (K e^{-\lambda r})^j \right) \|f\|_{L^\infty(0, +\infty; L^p(\Omega))}.$$

Now, choose  $r$  sufficiently large such that  $K e^{-\lambda r} < 1$ . From the previous inequality, we deduce that

$$\forall n \in \mathbb{N}, \|u(t_n)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + CI \left( \sum_{j=0}^{+\infty} (K e^{-\lambda r})^j \right) \|f\|_{L^\infty(0, +\infty; L^p(\Omega))}.$$

Considering  $t \in [t_n, t_{n+1}]$  and applying once again Duhamel formula, it follows that

$$\sup_{t \geq 0} \|u(t)\|_{L^\infty(\Omega)} < +\infty,$$

and the proof is complete.  $\square$

**A.8. Positive semigroups.** As in [5, p. 13], we have the following definition.

**Definition A.35.** Let  $(\Omega, \mathcal{A}, \mu)$  be a measure space,  $p \in [1, +\infty)$  and let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on  $X := L^p(\Omega, \mathcal{A}, \mu)$ .

- We call positive cone of  $X$  the set

$$X^+ := \{u \in X : u(x) \geq 0 \quad \mu - a.e. \quad x \in \Omega\}.$$

- We say that the semigroup  $(T(t))_{t \geq 0}$  is positive when

$$\forall t \geq 0, e^{-tA} X^+ \subset X^+.$$

**Theorem A.36.** [26, Theorem 2.6, p. 50]

Let us assume that the bilinear form  $\mathfrak{a}$  is defined on the measure space  $L^2(\Omega, \mu, \mathbb{R})$ . The two following properties are equivalent:

- The semigroup  $(e^{-tA})_{t \geq 0}$  is positive,
- $\forall u \in D(\mathfrak{a}), u^+ \in D(\mathfrak{a})$  and  $\mathfrak{a}(u^+, u^-) \leq 0$ .

**A.9. Maximal regularity.** Let  $X$  be a Banach space and let  $A$  be an unbounded, closed, and densely defined linear operator on  $X$ . Let  $f : [0, +\infty) \rightarrow X$  be a measurable function. We consider the existence and regularity problem:

$$\forall t \geq 0, u'(t) + Au(t) = f(t) \quad , \quad u(0) = 0.$$

**Definition A.37.** Let  $p \in (1, +\infty)$ . We say that  $A$  has the maximal  $L^p$ -regularity Property (see [24]) if there exists a constant  $C > 0$  such that for all  $f \in L^p(0, +\infty; X)$ , there exists a unique function  $u \in L^p(0, +\infty; D(A))$  with  $u' \in L^p(0, +\infty; X)$  satisfying the previous equation for almost every  $t > 0$  and

$$\|u\|_{L^p(0, +\infty; X)} + \|u'\|_{L^p(0, +\infty; X)} + \|Au\|_{L^p(0, +\infty; X)} \leq C\|f\|_{L^p(0, +\infty; X)}.$$

**Remark A.38.** We can naturally consider  $(0, T)$ ,  $T > 0$ , instead of  $(0, +\infty)$ . In this case, we say that  $A$  has the maximal  $L^p$ -regularity Property on  $(0, T)$ .

**Property A.39.** [24, Proposition 2.4, p. 6]

Let  $A$  be an unbounded operator on a Banach space  $X$  and assume that there exists  $p \in (1, +\infty)$  such that  $A$  has the maximal  $L^p$ -regularity Property. Then,  $A$  has the maximal  $L^q$ -regularity Property for all  $q \in (1, +\infty)$ .

**Property A.40.** [24, Theorem 2.6, p. 9]

Let  $-A$  be the infinitesimal generator of an analytic semigroup on an Hilbert space  $H$ . Then,  $A$  has the maximal  $L^p$ -regularity Property for all  $p \in (0, +\infty)$ .

**Theorem A.41.** [24, Theorem 3.1, p. 14]

Let  $(\Omega, \mu)$  be a measure space and let  $-A$  be the infinitesimal generator of an analytic semigroup of contractions  $(T(t))_{t \geq 0}$  on  $L^2(\Omega, \mu)$ . Assume that for all  $q \in [1, +\infty]$ , the inequality

$$\forall t \geq 0, \forall u \in L^2(\Omega) \cap L^q(\Omega), \|T(t)u\|_q \leq \|u\|_q$$

holds. Then, for all  $p \in (1, +\infty)$ , the operator  $A$  has the maximal  $L^p$ -regularity Property on  $L^q(\Omega)$ ,  $A$  being extrapolated to  $L^q(\Omega)$  in the way of paragraph 2.

## APPENDIX B. AUXILIARY RESULTS

### B.1. An interpolation inequality for intermediate derivatives.

**Lemma B.1.** [1, Lemma 4.10, p. 70]

Let  $-\infty < a < b < +\infty$ ,  $1 < p < \infty$ , and  $0 < \varepsilon_0 < +\infty$ . Then, there exists  $K = K(\varepsilon_0, p, b - a)$  such that for every  $\varepsilon$  satisfying  $0 < \varepsilon \leq \varepsilon_0$ , and for every function  $f$  twice continuously differentiable on the open interval  $(a, b)$ , we have

$$\int_a^b |f'(t)|^p dt \leq K\varepsilon \int_a^b |f''(t)|^p dt + \frac{K}{\varepsilon} \int_a^b |f(t)|^p dt.$$

### B.2. Lipschitz continuous functions and Sobolev spaces.

**Theorem B.2.** [31, Theorem 2.1.11, p. 48]

Let  $\Omega \subset \mathbb{R}^n$  be an open subset of  $\mathbb{R}^n$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a Lipschitz continuous function and  $u \in W^{1,p}(\Omega)$ ,  $p > 1$ . If  $f \circ u \in L^p(\Omega)$ , then  $f \circ u \in W^{1,p}(\Omega)$  and for almost all  $x \in \Omega$ , we have:

$$D(f \circ u)(x) = f'(u(x)) \cdot Du(x).$$

**Corollary B.3.** Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and let  $u \in H^1(\Omega)$  (with  $u$  real valued). Then,  $u^+ := u \vee 0 \in H^1(\Omega)$  and

$$\forall i \in \llbracket 1, n \rrbracket, \partial_{x_i} u^+ = \mathbf{1}_{u > 0} \partial_{x_i} u \quad \text{almost everywhere on } \Omega.$$

Consequently,  $u^- := (-u)^+ \in H^1(\Omega)$  and  $|u| = u^+ + u^- \in H^1(\Omega)$  and

$$\forall i \in \llbracket 1, n \rrbracket, \partial_{x_i} |u| = \text{sign}(u) \partial_{x_i} u \quad \text{almost everywhere on } \Omega$$

where  $\text{sign}(u)$  is defined by

$$\text{sign}(u) = \begin{cases} 1 & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) = 0, \\ -1 & \text{if } u(x) < 0. \end{cases}$$

Moreover,

$$\forall i \in \llbracket 1, n \rrbracket, \partial_{x_i} u = 0 \quad \text{almost everywhere on } \{x \in \Omega : u(x) = 0\}.$$

*Proof.* It is a straightforward consequence of the previous theorem. See also [7, Lemma 2.2, p. 91] and [15, Lemma 7.6, p. 152].  $\square$

**Corollary B.4.** *Let  $\Omega$  be an open subset (eventually unbounded) of  $\mathbb{R}^n$  and  $u \in H^1(\Omega)$  (real valued). Recall that  $Tu = (1 \wedge |u|)\text{sign}(u)$ . Then, we have  $Tu \in H^1(\Omega)$ . Furthermore*

$$\forall i \in \llbracket 1, n \rrbracket, \partial_{x_i} Tu = \mathbb{1}_{-1 < u < 1} \partial_{x_i} u.$$

*Proof.* It is enough to recall that  $Tu$  is obtained from  $u$  truncating the values of  $u$  greater than 1 or smaller than -1, i.e. that

$$Tu(x) = \begin{cases} u(x) & \text{if } |u(x)| \leq 1, \\ \frac{u(x)}{|u(x)|} & \text{if } |u(x)| \geq 1 \end{cases} = \begin{cases} u(x) & \text{if } |u(x)| < 1, \\ 1 & \text{if } u(x) \geq 1, \\ -1 & \text{if } u(x) \leq -1. \end{cases}$$

Note that this property holds true by replacing  $H^1(\Omega)$  by  $W^{1,p}(\Omega)$ , for all  $p \in [1, +\infty[$ .  $\square$

**B.3. Maximum principle on networks.** According to the Mignot Lemma, and doing exactly the same computations as in subsection 2.7, we obtain the following property:

**Property B.5** (Maximum principle). *Let  $u = (u_1, \dots, u_m)$  continuous on the graph, satisfying Kirchhoff conditions and such that*

$$(49) \quad \forall t \in (0, T), \forall j \in \llbracket 1, m \rrbracket, \partial_t u_j - c_j u_j'' = f_j$$

Assume that

- $\forall j \in \llbracket 1, m \rrbracket, f_j \leq 0$ ,
- $u(0) = u_0 \geq 0$ .

Then

$$\forall p \in [1, +\infty], \forall t, \|u(t)\|_{X_p} \leq \|u(0)\|_{X_p}.$$

**B.4. Comparison Lemma.**

**Lemma B.6** (Comparison Lemma). *Let  $u = (u_1, \dots, u_m)$  and  $v = (v_1, \dots, v_m)$  continuous on the graph, satisfying Kirchhoff conditions and such that*

$$\forall j \in \llbracket 1, m \rrbracket, \partial_t u_j - c_j u_j'' = f_j \quad \text{and} \quad \partial_t v_j - c_j v_j'' = g_j.$$

Assume in addition that

- $\forall j \in \llbracket 1, m \rrbracket, f_j \leq g_j$ ,
- $u_j(t=0) = v_j(t=0)$ .

Then, for all  $j$ ,  $u_j \leq v_j$ .

Once again, the same computations as in subsection 2.7 yield the result.

## APPENDIX C. A SUMMARY OF HARAUX-KIRANE TECHNIQUES

In [16], A. Haraux and M. Kirane developed a method which yields estimates in the  $C^1$ -norm of solutions of a semi-linear evolution equation on open subsets  $\Omega \subset \mathbb{R}^n$ . We need such estimates in subsection 5.2 to study the asymptotic behavior of the solution of a reaction-diffusion system on networks. So, the purpose of this appendix is to make sure that these techniques transfer to networks. There are two key points. The first important result is given by [16, Theorem 1.1, p. 15]. In terms of networks, this is Theorem C.1. The proof involves the Gagliardo-Nirenberg inequalities and the passage to networks consists in a straightforward adaptation of the Haraux-Kirane computations. The second key point is given by Theorem C.2. This is a general result concerning semigroups and the analysis of Section C.3 ensures that it applies to networks. Finally, combining Theorems C.1 and C.2, we obtain the desired estimates.

**C.1. Introductory remarks.** For the sake of readability, we now recall some general results which will be useful in the sequel.

- As usual,  $\Omega$  denotes the open interval  $(0, 1)$ .
- Recall that we are interested in the reaction-diffusion system (35);  $\mathbb{X}_p$  denotes the space  $(L^p(\Omega))^{mN}$  ( $1 \leq p \leq +\infty$ ) and for all  $p \in [1; +\infty[$ ,  $-A_p$  denotes the infinitesimal generator of the semigroup  $(T_p(t))_{t \geq 0}$ .  $(T_p(t))_{t \geq 0}$  is also denoted by  $(e^{-tA_p})_{t \geq 0}$ .
- Since there is no possibility of confusion, let  $A$  denote  $A_p$ .
- $\forall p \geq 2, D(A_p) \hookrightarrow (W^{2,p}(\Omega))^{mN}$ ,  $D(A_p)$  provided with the graph norm defined by

$$\forall u \in D(A_p), \|u\|_{D(A_p)} = \|u\|_p + \|Au\|_p.$$

The embedding  $D(A_p) \hookrightarrow (W^{2,p}(\Omega))^{mN}$  is continuous, due to Lemma B.1.

- In the proof of Property 2.30, we saw that

$$\forall p > 1, D(A_p) \subset W^{2,p}(\Omega) \subset C^1(\overline{\Omega}).$$

Consequently, we have

$$D_\infty(A) := \bigcap_{p \geq 1} D(A_p) \subset (C^1(\overline{\Omega}))^{mN}.$$

**C.2. A linear estimate.**

**Theorem C.1.** *For all  $u_0 \in \mathbb{X}_\infty$  and all  $t > 0$ , the following estimates*

$$\forall \varepsilon \geq \frac{1}{2}, \exists C \geq 0 : \forall t \in (0, 1], \|e^{-tA}u_0\|_{\mathbb{X}_\infty} \leq Ct^{-\frac{1}{1+\varepsilon}} \|u_0\|_{\mathbb{X}_{\frac{1}{2}+\varepsilon}}$$

and

$$\forall \varepsilon \in \left(0, \frac{1}{2}\right), \exists D \geq 0 : \forall t \in (0, 1], \|e^{-tA}u_0\|_{(W^{1,\infty}(\Omega))^{mN}} \leq Dt^{-\frac{1}{2}-\varepsilon} \|u_0\|_{\mathbb{X}_\infty}$$

hold.

*Proof.* We closely follow [16, Theorem 1.1].

- Let  $u_0 \in \mathbb{X}_\infty$ . Since  $\Omega$  is bounded, we have

$$\forall p \geq 1, u_0 \in \mathbb{X}_p.$$

Then

$$\forall p \geq 1, \forall t > 0, e^{-tA_p} u_0 \in D(A_p)$$

and:

$$\forall t > 0, e^{-tA_p} u_0 \in D_\infty(A) = \bigcap_{p \geq 1} D(A_p) \subset (C^1(\bar{\Omega}))^{mN}.$$

- According to [27, p. 62]

$$(50) \quad \forall p \geq 2, \exists C(p) \geq 0 : \forall t > 0, \|A_p e^{-tA_p} u_0\|_{\mathbb{X}_p} \leq \frac{C(p)}{t} \|u_0\|_{\mathbb{X}_p}.$$

Indeed, we need to verify that

- $(e^{-tA_p})$  is a uniformly bounded semigroup: this is the case because  $(e^{-tA_p})_{t \geq 0}$  is a semigroup of contractions (see Theorem 2.13).
- $0 \in \rho(A)$ . Indeed, since  $A$  has compact resolvent, the spectrum of  $A$  contains only eigenvalues. So, we are reduced to verify that 0 is not an eigenvalue of  $A$ . Let  $u$  such that  $Au = 0$ . Then,  $\mathbf{a}(u, u) = 0$ , i.e.

$$\sum_{j=1}^m \sum_{k=1}^N c_j^k \int_{\Omega} (u_j^k)' = 0.$$

By continuity of  $u$  and connectedness of the graph,  $u$  is constant. By Dirichlet condition at  $\mathbf{v}_n$ ,  $u = 0$ .

Moreover, since the embedding  $D(A_p) \hookrightarrow (W^{2,p}(\Omega))^{mN}$  is continuous, we have

$$\forall u \in D(A_p), \|u\|_{(W^{2,p}(\Omega))^{mN}} \leq C \|u\|_{D(A_p)} = C(\|u\|_{\mathbb{X}_p} + \|Au\|_{\mathbb{X}_p}),$$

where  $C$  denotes a constant depending only on the data. Now, we substitute  $e^{-tA_p} u_0$  for  $u$ , which gives

$$\begin{aligned} \|e^{-tA_p} u_0\|_{(W^{2,p}(\Omega))^{mN}} &\leq C \left[ \|e^{-tA_p} u_0\|_{\mathbb{X}_p} + \|A_p e^{-tA_p} u_0\|_{\mathbb{X}_p} \right] \\ &\leq C \left[ \|u_0\|_{\mathbb{X}_p} + \|A_p e^{-tA_p} u_0\|_{\mathbb{X}_p} \right] \quad \text{because } e^{-tA_p} \text{ is a contraction} \\ &\leq C \left[ \|u_0\|_{\mathbb{X}_p} + \frac{C(p)}{t} \|u_0\|_{\mathbb{X}_p} \right] \quad \text{according to (50)} \\ &= C \frac{1 + C(p)}{t} \|u_0\|_{\mathbb{X}_p} \leq \frac{C_1(p)}{t} \|u_0\|_{\mathbb{X}_p} \end{aligned}$$

for  $t \in (0, 1]$ . Hence the estimate

$$(51) \quad \forall t \in (0, 1], \|e^{-tA_p} u_0\|_{(W^{2,p}(\Omega))^{mN}} \leq \frac{C_1(p)}{t} \|u_0\|_{\mathbb{X}_p}.$$

- Recall the Gagliardo-Nirenberg inequalities (see, for example, [18, p. 37]). Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^n$  with Lipschitz continuous boundary<sup>8</sup>. Let  $m \in \mathbb{N}$  and  $p, q \geq 1$  two real numbers. If  $u \in W^{m,p}(\Omega)$  and if

<sup>8</sup> See, for example, [1, Theorem 4.17, p.79], [18, p. 37]

$\nu \geq 0$  satisfies  $\nu < m - \frac{n}{p}$ , then<sup>9</sup>  $u \in C^\nu(\overline{\Omega})$ . Moreover, if  $\theta \in [0, 1]$  satisfies

$$\nu < \theta \left( m - \frac{n}{p} \right) - (1 - \theta) \frac{n}{r},$$

we have

$$\|u\|_{W^{\nu, \infty}(\Omega)} \leq C(\theta) \|u\|_{W^{m, p}(\Omega)}^\theta \|u\|_{L^r(\Omega)}^{1-\theta}.$$

Concerning networks, applying these inequalities at every function  $u_j^k$ , we find

$$\|u_j^k\|_{W^{\nu, \infty}(\Omega)} \leq C_j^k(\theta) \|u_j^k\|_{W^{m, p}(\Omega)}^\theta \|u_j^k\|_{L^r(\Omega)}^{1-\theta} \leq C_j^k(\theta) \|u\|_{(W^{m, p}(\Omega))^{mN}}^\theta \|u\|_{(L^r(\Omega))^{mN}}^{1-\theta},$$

which gives, after summing

$$(52) \quad \|u\|_{(W^{\nu, \infty}(\Omega))^{mN}} \leq C(\theta) \|u\|_{(W^{m, p}(\Omega))^{mN}}^\theta \|u\|_{(L^r(\Omega))^{mN}}^{1-\theta}.$$

- Let apply (52) with  $\nu = 0$ ,  $r = p$  and  $m = 2$ . Substituting  $e^{-tA}u_0$  for  $u$  yields:

$$\begin{aligned} \|e^{-tA}u_0\|_{\mathbb{X}_\infty} &\leq C(\theta) \|e^{-tA}u_0\|_{(W^{2, p}(\Omega))^{mN}}^\theta \|e^{-tA}u_0\|_{\mathbb{X}_p}^{1-\theta} \\ &\leq C(\theta) \|e^{-tA}u_0\|_{(W^{2, p}(\Omega))^{mN}}^\theta \|u_0\|_{\mathbb{X}_p}^{1-\theta} \quad \text{because } e^{-tA_p} \text{ is a contraction} \\ &\leq C(\theta) \frac{C_1^\theta(p)}{t^\theta} \|u_0\|_{\mathbb{X}_p}^\theta \|u_0\|_{\mathbb{X}_p}^{1-\theta} = C(\theta) \frac{C_1^\theta(p)}{t^\theta} \|u_0\|_{\mathbb{X}_p} \quad \text{according to (51)}. \end{aligned}$$

Taking  $r = p = \frac{1}{2} + \varepsilon$  (recall that  $\varepsilon \geq \frac{1}{2}$ , in such a way that  $p = r \geq 1$ ), we have

$$\forall t \in (0, 1], \|e^{-tA}u_0\|_{\mathbb{X}_\infty} \leq C(\theta) \frac{C_1^\theta(p)}{t^\theta} \|u_0\|_{\mathbb{X}_{\frac{1}{2} + \varepsilon}}.$$

Here,  $\theta$  must satisfy

$$0 = \nu < \theta \left( m - \frac{1}{p} \right) - (1 - \theta) \frac{1}{r}.$$

Since  $\nu = 0$  and  $p = r = \frac{1}{2} + \varepsilon$ , this inequality is equivalent to  $\theta > \frac{1}{1+2\varepsilon}$ . Taking

$$\theta > \frac{1}{1 + \varepsilon},$$

we have

$$\forall t \in (0, 1], \|e^{-tA}u_0\|_{\mathbb{X}_\infty} \leq \frac{C(\varepsilon)}{t^{\frac{1}{1+\varepsilon}}} \|u_0\|_{\mathbb{X}_{\frac{1}{2} + \varepsilon}}.$$

- We now turn to the second property. Choose  $\nu = 1$ ,  $m = 2$  ( $p$  and  $r$  will be later determined) and apply (52), substituting  $e^{-tA}u_0$  for  $u$ :

$$\begin{aligned} \|e^{-tA}u_0\|_{(W^{1, \infty}(\Omega))^{mN}} &\leq C(\theta) \|e^{-tA}u_0\|_{(W^{2, p}(\Omega))^{mN}}^\theta \|e^{-tA}u_0\|_{(L^r(\Omega))^{mN}}^{1-\theta} \\ &\leq C(\theta) \|e^{-tA}u_0\|_{(W^{2, p}(\Omega))^{mN}}^\theta \|u_0\|_{(L^r(\Omega))^{mN}}^{1-\theta} \quad \text{because } e^{-tA_p} \text{ is a contraction} \\ &\leq C(\theta) \frac{C_1^\theta(p)}{t^\theta} \|u_0\|_{\mathbb{X}_p}^\theta \|u_0\|_{\mathbb{X}_r}^{1-\theta} \quad \text{according to (51)}. \end{aligned}$$

<sup>9</sup> Following [18, p. 7], if  $\nu \geq 0$  is not an integer,  $C^\nu(\overline{\Omega})$  denotes the space of  $[\nu]$  times continuously differentiable functions on  $\overline{\Omega}$  where the  $[\nu]$ -order derivative satisfies a Hölder condition with exponent  $\nu - [\nu]$ .



Now, we need to choose  $p, r$  and  $\theta$  such that:

$$\begin{cases} \nu < m - \frac{1}{p} & \mathbf{I} \\ \nu < \theta \left( m - \frac{1}{p} \right) - (1 - \theta) \frac{1}{r} & \mathbf{II} \end{cases}$$

Taking account of  $\nu = 1$  and  $m = 2$ , the inequality **I** is equivalent to  $p > 1$ . Then, inequality **II** amounts to

$$\theta > \frac{1 + \frac{1}{r}}{2 - \frac{1}{p} + \frac{1}{r}}.$$

Since  $\frac{1}{p} < 1$ , we have

$$\frac{1 + \frac{1}{r}}{2 - \frac{1}{p} + \frac{1}{r}} < \frac{1 + \frac{1}{r}}{2 - 1 + \frac{1}{r}} = \frac{1 + \frac{1}{r}}{1 + \frac{1}{r}} = 1.$$

Then, for every choice of  $p > 1$ , we can determine a suitable  $\theta$ . Since  $\Omega$  is bounded, we have

$$\|u_0\|_{\mathbb{X}_p}, \|u_0\|_{\mathbb{X}_r} \leq C \|u_0\|_{\mathbb{X}_\infty}$$

and the inequality

$$\|e^{-tA}u_0\|_{(W^{1,\infty}(\Omega))^{mN}} \leq C(\theta) \frac{C_1^\theta(p)}{t^\theta} \|u_0\|_{\mathbb{X}_p}^\theta \|u_0\|_{\mathbb{X}_r}^{1-\theta}$$

leads to

$$\|e^{-tA}u_0\|_{(W^{1,\infty}(\Omega))^{mN}} \leq C(p, r, \theta) \frac{1}{t^\theta} \|u_0\|_{\mathbb{X}_\infty}$$

and this, for every  $\theta \in \left( \frac{1 + \frac{1}{r}}{2 - \frac{1}{p} + \frac{1}{r}}, 1 \right]$ . A quick study on  $[0, 1]$  of the function  $f : x \mapsto f(x) = \frac{1+x}{2 - \frac{1}{p} + x}$  shows that  $f$  is nondecreasing on  $[0, 1]$  and that  $f(0) = \frac{1}{2 - \frac{1}{p}} < 1$ .

Let  $\theta \in \left( \frac{1}{2}, 1 \right)$  fixed. Since  $f(0) = \frac{1}{2 - \frac{1}{p}} \xrightarrow{p \rightarrow +\infty} \frac{1}{2}$ , we can take  $p$  sufficiently large so that  $f(0) < \theta$ . Then, we choose  $r$  such that  $\theta > f\left(\frac{1}{r}\right) = \frac{1 + \frac{1}{r}}{2 - \frac{1}{p} + \frac{1}{r}}$ . So we have found  $p, r$  and  $\theta = \frac{1}{2} + \varepsilon$  with  $\varepsilon \in \left(0, \frac{1}{2}\right)$  such that

$$\forall t \in (0, 1], \|e^{-tA}u_0\|_{(W^{1,\infty}(\Omega))^{mN}} \leq C(\varepsilon) \frac{1}{t^{\frac{1}{2} + \varepsilon}} \|u_0\|_{\mathbb{X}_\infty},$$

and the proof is complete.  $\square$

**C.3. A result about semi-linear evolution equations.** Let  $E$  be a Banach space and let  $a \in \mathbb{R}$ . Let  $C_B([a, +\infty), E)$  denote the vector space of continuous functions  $\varphi : [a, +\infty[ \rightarrow E$  bounded on  $[a, +\infty)$ . For  $p \in [1, +\infty[$ , let  $S^p(\mathbb{R}_+, E)$  denote the vector space of functions  $f \in L_{loc}^p(\mathbb{R}_+, E)$  such that

$$\sup_{t \geq 0} \int_t^{t+1} \|f(s)\|_E^p ds < +\infty.$$

**Theorem C.2.** [16, Theorem 2.1] *Let  $E, F$  two real Banach spaces, respectively provided with the norms  $\|\cdot\|_E$  and  $\|\cdot\|_F$ . Assume that  $F \hookrightarrow E$ . We consider an unbounded linear operator  $L$  on  $E$  which generates a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on  $E$  such that*

- $\forall t > 0, T(t)E \subset F$ ,
- $\exists \alpha \in [0, 1[, \exists C \geq 0 : \forall t \in (0, 1], \forall x \in E, \|T(t)x\|_F \leq \frac{C}{t^\alpha} \|x\|_E$ .

Lastly, let  $p > \frac{1}{1-\alpha}$ ,  $f \in S^p(\mathbb{R}^+, E)$  and let  $u$  be a solution on  $\mathbb{R}_+$  of the equation

$$(53) \quad \frac{du}{dt}(t) = Lu(t) + f(t),$$

in the sense of Definition (2.23), i.e.  $u$  is continuous on  $[0, +\infty)$ , has continuous derivative on  $(0, +\infty)$ ,  $u(t) \in D(L)$  for all  $t > 0$  and (53) holds on  $(0, +\infty)$ . Then, if  $u : [0, +\infty) \rightarrow E$  is bounded, actually

- $\forall t > 0, u(t) \in F$ ,
- $\forall \delta > 0, u \in C_B([\delta, +\infty); F)$ .

**C.4. Application to networks.** Let  $(u, v)$  be a global nonnegative solution of the reaction-diffusion system

$$\begin{cases} \partial_t u_j - c_j^1 u_j'' + u_j \varphi_j(v_j) = 0, \\ \partial_t v_j - c_j^2 v_j'' - u_j \varphi_j(v_j) = 0 \end{cases}$$

where  $u = (u_1, \dots, u_m)$ ,  $v = (v_1, \dots, v_m)$  are continuous on the graph and satisfy the Kirchhoff conditions. Assume moreover that for all  $j \in \llbracket 1, m \rrbracket$ ,  $u_j, v_j \in C_B((0, +\infty) \times \bar{\Omega})$ . Our intention is to show that for all  $j \in \llbracket 1, m \rrbracket$ ,  $u_j, v_j \in C_B([1, +\infty), C^1(\bar{\Omega}))$ . This is precisely the purpose of Theorem C.2.

The initial system becomes an abstract Cauchy problem, namely

$$(\mathbf{ACP}) \quad : \quad \begin{cases} \frac{dX}{dt}(t) + AX(t) = f(t), & t > 0, \\ X(0) = X_0 \end{cases}$$

with  $X = (u, v) = (u_1, \dots, u_m, v_1, \dots, v_m)$ ,  $X_0 = (u_0, v_0)$  and

$$f(t) = (-u_1(t, \cdot) \varphi_1(v_1(t, \cdot)), \dots, -u_m(t, \cdot) \varphi_m(v_m(t, \cdot)), u_1(t, \cdot) \varphi_1(v_1(t, \cdot)), \dots, u_m(t, \cdot) \varphi_m(v_m(t, \cdot))).$$

In view to apply Theorem C.2, take  $E = \mathbb{X}_\infty$  and  $F = (C^1(\bar{\Omega}))^{mN}$  equipped with their natural norms. Let us verify assumptions of Theorem C.2.

- $\forall t > 0, T(t)E \subset F$ : see Theorem C.1.
- $\exists \alpha \in [0, 1[, \exists C \geq 0 : \forall t \in (0, 1], \forall X \in E, \|T(t)X\|_{1, \infty} \leq \frac{C}{t^\alpha} \|X\|_\infty$ : this is precisely the purpose of Theorem C.1, taking  $\varepsilon < \frac{1}{2}$ .
- $f \in S^p(\mathbb{R}_+, E)$ , for a particular  $p > \frac{1}{1-\alpha}$ . Indeed, consider such a  $p$ . We want to make sure that  $\sup_{t \geq 0} \|f\|_{L^p(t, t+1; E)} < +\infty$ , i.e. that for all  $j, 1 \leq j \leq m$

$$\sup_{t \geq 0} \int_t^{t+1} \|u_j(s, \cdot) \varphi_j(u_j(s, \cdot))\|_\infty^p ds < +\infty.$$

But, by assumption

$$\exists M \geq 0 : \forall t \geq 0, \forall x \in \Omega, 0 \leq u_j(t, x), v_j(t, x) \leq M.$$

Since the functions  $\varphi_j$  continuous, we have similar estimates for the functions  $u_j \varphi_j(v_j)$ . Hence

$$\exists C \geq 0 : \forall t \geq 0, \forall x \in \Omega, 0 \leq u_j(t, x) \varphi_j(v_j(t, x)) \leq C,$$

and

$$\int_t^{t+1} \sup_{s \geq 0} \|f\|_{L^p(s, s+1; E)} \leq C,$$

where  $C$  denotes a generic constant depending only on the data. So, Theorem C.2 applies and  $u, v \in C_B([\delta, +\infty[, F)$  for all  $\delta > 0$ .

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