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1           **AN ASYMPTOTIC-PRESERVING ALL-SPEED SCHEME FOR**  
2           **FLUID DYNAMICS AND NONLINEAR ELASTICITY \***

3           EMANUELA ABBATE<sup>†</sup>, ANGELO IOLLO<sup>‡</sup>, AND GABRIELLA PUPPO<sup>§</sup>

4           **Abstract.** An implicit relaxation scheme is derived for the simulation of multi-dimensional flows  
5 at all Mach numbers, ranging from very small to order unity. An analytical proof of the asymptotic  
6 preserving property is proposed and the divergence-free condition on the velocity in the incompress-  
7 ible regime is respected. The scheme possesses a general structure, which is independent of the  
8 considered state law and thus can be adopted to solve gas and fluid flows, but also deformations of  
9 elastic solids. This is achieved by adopting the Jin-Xin relaxation technique in order to get a linear  
10 transport operator. The spatial derivatives are thus independent of the EOS and an easy implemen-  
11 tation of fully implicit time discretizations is possible. Several validations on multi-dimensional tests  
12 are presented, showing that the correct numerical viscosity is recovered in both the fully compress-  
13 ible and the low Mach regimes. An algorithm to perform grid adaptivity is also proposed, via the  
14 computation of the entropy residual of the scheme.

15           **Key words.** low Mach limit, all-speed schemes, asymptotic-preserving property, relaxation,  
16 non-linear elasticity, entropy production

17           **AMS subject classifications.** 65M08, 65M99, 76T15

18           **1. Introduction.** In the propagation of flows of gases and of other compressible  
19 materials, complex and non stationary phenomena are generated. The complexity  
20 is often related to the local stiffness of the involved media, to the geometry of the  
21 physical problem or to boundary and initial conditions. All these aspects can produce  
22 waves propagating at very different speeds inside the considered materials, giving rise  
23 to specific numerical problems.

24           The accurate simulation of these phenomena requires the construction of numeri-  
25 cal schemes that are able to deal with different regimes. The Mach number  $M$  is given  
26 by the ratio between the flow velocity and the speed of sound, namely it measures  
27 how much the acoustic waves are faster with respect to the flow velocity. In gas and  
28 fluid dynamics,  $M$  is used to identify the regime of the considered flow, ranging from  
29 fully compressible (local Mach number of order unity or higher) to incompressible  
30 (very small local Mach number).

31           Standard explicit-upwind codes developed for the simulation of compressible flows  
32 usually fail in approximating fluid flows or solid deformations at low speed. This is  
33 mainly due to the excessive numerical viscosity introduced on the slow waves by  
34 an upwind Godunov-like spatial discretization when the Mach number of the flow  
35 becomes small, as it has been proved in [23, 22, 15]. Specifically, an upwind scheme on  
36 a Cartesian grid leads to pressure fluctuations of order  $\mathcal{O}(M)$ , while in the continuous  
37 case the pressure fluctuations are of order  $\mathcal{O}(M^2)$ .

38           Moreover, when explicit methods are used to solve low Mach number flows, the  
39 time step  $\Delta t$  becomes extremely small due to the enforcement of the CFL stabil-  
40 ity condition. By imposing this constraint,  $\Delta t$  is roughly proportional to the Mach

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<sup>†</sup>Università degli Studi dell'Insubria, via Valleggio, Como 22100, Italy; Université Bordeaux, IMB, UMR 5251, F-33400 Talence, France; Equipe-projet Memphis, Inria Bordeaux-Sud Ouest, F-33400 Talence, France ([eabbate@uninsubria.it](mailto:eabbate@uninsubria.it), [emanuela.abbate@inria.fr](mailto:emanuela.abbate@inria.fr)).

<sup>‡</sup>Université Bordeaux, IMB, UMR 5251, F-33400 Talence, France; Equipe-projet Memphis, Inria Bordeaux-Sud Ouest, F-33400 Talence, France ([angelo.iollo@inria.fr](mailto:angelo.iollo@inria.fr)).

<sup>§</sup>La Sapienza Università di Roma, Piazzale Aldo Moro, Roma 00185, Italy ([gabriella.puppo@uniroma1.it](mailto:gabriella.puppo@uniroma1.it)).

41 number  $M$ :

42 (1.1) 
$$\Delta t \leq \frac{\Delta x}{\lambda_{max}} = \frac{\Delta x}{max|u \pm c|} = M \frac{\Delta x}{max|u(M \pm 1)|},$$

43  $\Delta x$  being the space step and  $\lambda_{max}$  the fastest characteristic speed. For the Euler  
 44 system we have that  $\lambda_{max} = u \pm c$ ,  $c$  being the sound speed and  $u$  the velocity.  
 45 Consequently, compressible codes require an increasingly large computational time as  
 46 the incompressible regime is reached.

47 The derivation of *all-speed* solvers is motivated by all the above mentioned reasons.  
 48 In general, the purpose of an all-speed scheme is to handle both the compressible  
 49 regime and the incompressible one. To this end, the derivation of preconditioning  
 50 methodologies has been triggered by Chorin [11] and by Turkel [42, 43], proposing  
 51 a modification of the Roe matrix that moderates the numerical diffusion of upwind  
 52 schemes, inside a fully implicit time discretization [45]. More recent methods falling  
 53 in this category have been proposed in [32, 6, 44]. However, the main problem of  
 54 these techniques is related to the difficulty in handling the non-linearities of classical  
 55 upwind discretizations (e.g. approximate Riemann solvers) within the fully implicit  
 56 time integration.

57 Numerical schemes based on low Mach asymptotics have been proposed by Klainerman  
 58 and Majda [27, 28]. Klein adapted this technique to derive an operator splitting  
 59 into convection and pressure waves [29]. These methods have been derived for the low  
 60 Mach number regime, but they also have been the starting point for the development  
 61 of different all-speeds schemes. These are mainly based on the separation of the fast  
 62 and slow scales, that are respectively integrated in time explicitly and implicitly  
 63 in an IMEX logic, producing stability conditions on  $\Delta t$  that are independent of the  
 64 Mach number. Some examples may be found in [8, 9, 14, 12, 36, 16].

65 Klainerman and Majda have also shown that solutions of the compressible Euler  
 66 equations converge to the solutions of the incompressible Euler equations as the Mach  
 67 number tends to zero [27]. The *asymptotic preserving* (AP) property is a consistency  
 68 criterion for numerical schemes that have to deal with these two regimes. This prop-  
 69 erty was firstly introduced by Jin in [25]: a scheme is AP, if its lowest order multiscale  
 70 expansion is a consistent discretization of the incompressible limit. Thus, an AP  
 71 scheme for the Euler equations should provide a consistent and stable discretization  
 72 independently of the Mach number. In the fully compressible regime, it should pos-  
 73 sess the desirable features of a compressible solver (e. g. good resolution of shocks  
 74 and discontinuities and non-oscillatory solution profiles). Moreover, it should provide  
 75 a consistent discretization of the incompressible equations when  $M \rightarrow 0$ .

76 In the present work, our interest is the numerical simulation of flows with Mach  
 77 numbers ranging from the incompressible limit to compressible regimes with  $M \simeq$   
 78  $\mathcal{O}(1)$ . To this end, we have derived in [2] an implicit all-speed scheme, which is  
 79 here extended for the accurate solution of multi-dimensional flows. Moreover, its AP  
 80 property is proved analytically and numerically. The scheme has a general formula-  
 81 tion that can be adopted without any structural modification for the simulation of  
 82 waves propagating inside materials with different behavior, such as fluids and elastic  
 83 solids. The ability of the scheme in dealing with very general equations of state (EOS)  
 84 is achieved by adopting the Jin-Xin relaxation technique [26]. With this method, the  
 85 fluxes are relaxed at the continuous level and a linear transport operator is obtained:  
 86 this avoids a direct dependence of the spatial derivatives on the specific EOS. The  
 87 second advantage consists in the fact that Riemann solvers are not necessary and  
 88 fully implicit time integrators are easily implemented. This allows to get rid of the

89 acoustic stability constraint (1.1), avoiding the requirement of a time stepping de-  
 90 creasing as  $M \rightarrow 0$ , which would enormously increase the computational effort. The  
 91 spatial discretization is built by a suitable combination of upwind and centered fluxes,  
 92 in order to recover the correct viscosity at all speeds. The ability of the scheme in  
 93 moderating the numerical viscosity with respect to standard explicit-upwind methods  
 94 is here investigated by solving the Gresho vortex test and also different 2D Riemann  
 95 problems involving material waves. Moreover, we propose an adaptive mesh refine-  
 96 ment (AMR) algorithm, specifically designed for this all-speed scheme. We build a  
 97 numerical estimate of the scheme entropy production and use it as an error indicator  
 98 to drive the mesh refinement and coarsening. This estimate is also useful to assess the  
 99 fact that the all-speed discretization is superior in approximating the different waves  
 100 with respect to standard upwind-like schemes.

101 The structure of the paper is the following: in Section 2 we briefly revise the  
 102 asymptotic analysis of the Euler equations for fluid dynamics. The 2D all-speed  
 103 relaxation scheme is presented in Section 3. Then, the AP property is analytically  
 104 proved in Section 4. The scheme is validated on fluid dynamics tests in Section 5,  
 105 where we also describe the adaptive mesh refinement algorithm based on the entropy  
 106 residual for the all-speed scheme. Then, the scheme is extended to solve non-linear  
 107 elasticity problems in Section 6, where two different limits occurring in elastic solids  
 108 are analyzed. Conclusions are drawn in Section 7.

109 **2. Asymptotic analysis of the continuous Euler equations.** Let us con-  
 110 sider the two dimensional Euler system:

$$111 \quad (2.1) \quad \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho \mathbf{u}) = 0 \\ \partial_t (\rho \mathbf{u}) + \nabla_x \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I}) = 0 \\ \partial_t (\rho e) + \nabla_x \cdot ((\rho e + p) \mathbf{u}) = 0, \end{cases}$$

112  $\rho$  being the density,  $\mathbf{u}$  the velocity field,  $p$  the pressure,  $\mathbf{I}$  the identity matrix and  $e$   
 113 the total energy per unit mass, which is given by the sum of the kinetic energy and  
 114 the internal energy per unit mass  $\epsilon$ :

$$115 \quad (2.2) \quad e = \frac{1}{2} |\mathbf{u}|^2 + \epsilon.$$

116 For fluid dynamics problems, we consider the following state law, which extends the  
 117 applicability of the ideal gas EOS:

$$118 \quad (2.3) \quad \epsilon(\rho, s) = \frac{\kappa(s)}{\gamma - 1} \rho^{\gamma-1} + \frac{p_\infty}{\rho} = \frac{p + \gamma p_\infty}{\rho(\gamma - 1)}.$$

119 The internal energy is written as a function of the density and the entropy  $s$  and we  
 120 have used the definition of the pressure for the second equality (see [46, 31]). Here,  
 121  $\gamma = c_p/c_v$  is the polytropic gas constant,  $\kappa(s) = exp((s - s_0)/c_v)$  with  $s_0$  reference  
 122 entropy and  $p_\infty$  is a constant describing the intermolecular interaction, typical of  
 123 stiffened gases, liquids and also some solids. The perfect gas EOS is recovered by  
 124 setting  $p_\infty = 0$ . The general formulation for the speed of sound reads as follows

$$125 \quad (2.4) \quad c = \sqrt{\left. \frac{\partial p}{\partial \rho} \right|_{s=const}} = \sqrt{\frac{\gamma(p + p_\infty)}{\rho}}.$$

126 We now briefly revise the analysis of the low Mach number regime in fluid dy-  
 127 namics. For simplicity of notation, we consider a perfect gas, i. e.  $p_\infty = 0$ .

128 **2.1. Non-dimensional Euler system.** To write the non-dimensional Euler  
 129 system, we follow Klainerman and Majda's works [27, 28], by decomposing each  
 130 quantity into a product of a reference value (denoted with the subscript  $*$ ) and a  
 131 dimensionless number (denoted with  $\hat{\cdot}$ ), e.g.  $\rho = \rho^* \hat{\rho}$  for the density. The reference  
 132 value is a "scaling" factor and it should be chosen in such a way that the dimensionless  
 133 value is of order one. In this perspective, the reference pressure  $p^*$  is defined with the  
 134 reference sound speed  $c^* = \sqrt{p^*/\rho^*}$  (see definition (2.4)), which is also used in the  
 135 scaling of the energy.

After some simple algebraic manipulations, most of the reference quantities cancel out, leading to the non-dimensional Euler system (we omit the hat notation  $\hat{\cdot}$  for the sake of simplicity):

$$(2.5) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{M^2} \nabla p = 0 \\ \partial_t(\rho e) + \operatorname{div}((\rho e + p) \mathbf{u}) = 0. \end{cases}$$

136 The non-dimensional Euler system depends only on a single non-dimensional refer-  
 137 ence quantity. This is the reference Mach number, defined as the ratio between the  
 138 reference flow velocity and the reference sound speed as follows:

$$(2.8) \quad M = \frac{u^*}{c^*}.$$

140 The non-dimensional state law takes the following formulation:

$$(2.9) \quad p = (\gamma - 1) \left[ \rho e - \frac{M^2}{2} \rho |\mathbf{u}|^2 \right].$$

142 **2.2. Low Mach number asymptotics.** In order to analyze the asymptotic  
 143 behaviour of the Euler equations in the zero Mach number limit, we perform an  
 144 expansion of the scaled variables in terms of the Mach number [23]. The density is  
 145 expanded as follows:

$$(2.10) \quad \rho = \rho_0 + M \rho_1 + M^2 \rho_2 + \mathcal{O}(M^3),$$

147 and all other variables are developed in the same way. The terms of zeroth order  
 148 (subscript  $_0$ ) represent the zero Mach number limit. By inserting this expansion in  
 149 the scaled system (2.5)-(2.6)-(2.7) and collecting terms with equal powers of  $M$ , one  
 150 obtains:

- order  $\mathcal{O}(1/M^2)$ :

$$(2.11) \quad \nabla p_0 = 0$$

- order  $\mathcal{O}(1/M)$ :

$$(2.12) \quad \nabla p_1 = 0$$

- order  $\mathcal{O}(1)$ :

$$(2.13) \quad \begin{cases} \partial_t \rho_0 + \nabla \cdot (\rho_0 \mathbf{u}_0) = 0 \\ \partial_t(\rho_0 \mathbf{u}_0) + \nabla \cdot (\rho_0 \mathbf{u}_0 \otimes \mathbf{u}_0) + \nabla p_2 = 0 \\ \partial_t(\rho e)_0 + \nabla \cdot ((\rho e)_0 \mathbf{u}_0 + p_0 \mathbf{u}_0) = 0, \end{cases}$$

$$(2.14)$$

$$(2.15)$$

155 with the first order of the state law

$$(2.16) \quad p_0 = (\gamma - 1) (\rho e)_0.$$

157 Relations (2.11) and (2.12) prove that the pressure is constant in space up to  
 158 fluctuations of order  $M^2$ . Hence, we can write the following pressure asymptotic:

$$159 \quad (2.17) \quad p(x, t) = P_0(t) + M^2 p_2(x, t),$$

160 where  $P_0(t)$  is a thermodynamic pressure constant in space. In presence of open  
 161 boundaries, the thermodynamic pressure  $P_0$  is imposed to be equal to the exterior  
 162 pressure  $P_{ext}$ . For the sake of simplicity, we assume that the exterior pressure does  
 163 not change in time, i. e.  $P_0$  is constant in both space and time. From (2.16), we  
 164 deduce that also the energy is constant in space and time and the incompressibility  
 165 constraint  $\nabla \cdot \mathbf{u}_0 = 0$  is easily derived.

166 Introducing this constraint into the continuity equation (2.13), the material deriva-  
 167 tive of the density is zero  $\frac{D\rho}{Dt} = 0$ . This means that the density is constant along a  
 168 trajectory of any fluid element. Therefore, when the incompressibility constraint is  
 169 respected, the density of the fluid is constant in time and space, i. e.  $\rho_0 = \text{const}$ , in  
 170 the case where the initial density of the fluid is constant in space.

171 With the study of the asymptotics carried out above, system (2.13)-(2.14)-(2.15)  
 172 reduces to the *incompressible* Euler system in its non-dimensional form. This system  
 173 is the zero Mach number limit of the compressible Euler system and reads as follows

$$174 \quad (2.18) \quad \begin{cases} \rho_0 = \text{const} \\ \rho_0 (\partial_t \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0) + \nabla p_2 = 0 \\ \nabla \cdot \mathbf{u}_0 = 0. \end{cases}$$

175 **3. Implicit relaxation all-speed scheme.** To allow for an efficient and robust  
 176 numerical procedure, we adopt the multi-dimensional Jin-Xin relaxation approach  
 177 [26]: a linear hyperbolic relaxation system is built to approximate the original system  
 178 (here the Euler system (2.1)) with a small dissipative correction. Thanks to the  
 179 linearity of the advection terms, the spatial derivatives lose their dependence on the  
 180 state law. Numerical schemes are then derived in a general formulation that is not  
 181 related to the EOS of a given material. Nonlinear terms appear only in the right hand  
 182 side of the relaxation system, thus only diagonal terms are interested by the need of  
 183 a linearization in the discretization process.

184 A fully implicit time integration of the relaxation system is proposed. Thus,  
 185 acoustic CFL constraints are not required and a centered spatial discretization of the  
 186 stiff parts when the  $M \rightarrow 0$  can be adopted, without stability issues. This is crucial  
 187 to get the correct numerical viscosity in the low Mach number regime.

188 **3.1. The Jin-Xin relaxation.** Letting  $x = (x_1, x_2)$  be the coordinates in the  
 189 canonical basis of  $\mathbb{R}^2$ ,  $\mathbf{u} = (u_1, u_2)$  the velocity components, system (2.1) may be  
 190 rewritten in a general compact form in the following way:

$$191 \quad (3.1) \quad \partial_t \boldsymbol{\psi} + \partial_{x_1} \mathbf{F}(\boldsymbol{\psi}) + \partial_{x_2} \mathbf{G}(\boldsymbol{\psi}) = 0,$$

192 where we have adopted the directional splitting of the flux function, with  $\boldsymbol{\psi} \in \mathbb{R}^n$  and  
 193  $\mathbf{F}(\boldsymbol{\psi}), \mathbf{G}(\boldsymbol{\psi}) \in \mathbb{R}^n$ . This is a system of  $n$  equations, with  $(x, t) \in (\mathbb{R}^d, \mathbb{R}^+)$ , with  
 194  $d = 2$ . In our case, we have  $n = 4$  and the conservation variables and the fluxes along  
 195 the two directions read as follows:

$$196 \quad (3.2) \quad \boldsymbol{\psi} = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho e \end{bmatrix}, \quad \mathbf{F}(\boldsymbol{\psi}) = \begin{bmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1 u_2 \\ (\rho e + p) u_1 \end{bmatrix}, \quad \mathbf{G}(\boldsymbol{\psi}) = \begin{bmatrix} \rho u_2 \\ \rho u_1 u_2 \\ \rho u_2^2 + p \\ (\rho e + p) u_2 \end{bmatrix}.$$

197 The corresponding Jin-Xin relaxation system is constructed by introducing two  
 198 vectors containing the relaxation variables  $\mathbf{v} \in \mathbb{R}^n$  and  $\mathbf{w} \in \mathbb{R}^n$  in the two directions,  
 199 approximating the fluxes  $\mathbf{F}(\boldsymbol{\psi})$  and  $\mathbf{G}(\boldsymbol{\psi})$  respectively. The relaxation system of  
 200 dimension  $n \times (d + 1)$  takes the following formulation:

$$201 \quad (3.3) \quad \begin{cases} \partial_t \boldsymbol{\psi} + \partial_{x_1} \mathbf{v} + \partial_{x_2} \mathbf{w} = 0 \\ \partial_t \mathbf{v} + \mathbf{A}_1 \partial_{x_1} \boldsymbol{\psi} = \frac{1}{\eta} (\mathbf{F}(\boldsymbol{\psi}) - \mathbf{v}) \\ \partial_t \mathbf{w} + \mathbf{A}_2 \partial_{x_2} \boldsymbol{\psi} = \frac{1}{\eta} (\mathbf{G}(\boldsymbol{\psi}) - \mathbf{w}). \end{cases}$$

202 The small positive parameter  $\eta$  is called relaxation rate. The right hand sides of the  
 203 second and third equations are stiff source terms and are the only non-linear parts of  
 204 the system. The state law dependence is given by the fluxes in the sources, thus the  
 205 EOS expression does not enter in the resulting linear transport operator.

206 The relaxation matrices  $\mathbf{A}_1$  and  $\mathbf{A}_2$  are positive diagonal matrices and are chosen  
 207 by enforcing the subcharacteristic condition [34, 46]:

$$208 \quad (3.4) \quad \mathbf{A}_1 - \mathbf{F}'(\boldsymbol{\psi})^2 \geq 0 \text{ and } \mathbf{A}_2 - \mathbf{G}'(\boldsymbol{\psi})^2 \geq 0 \quad \forall \boldsymbol{\psi}.$$

209 Specifically, we construct these matrices by a-priori estimating the wave speeds  $\lambda_i$  of  
 210 the original system (2.1). In the simplified case of  $\mathbf{A} = \mathbf{A}_1 = \mathbf{A}_2 = \text{diag}\{a_i\}$ , the  
 211 eigenvalues of the relaxation system are computed as follows

$$212 \quad (3.5) \quad \mu_j = \pm \sqrt{a_i}, \quad i = 1, \dots, n, \quad j = 1, \dots, 2n,$$

213 thus having  $\mu_1 \geq \max|u - c|$ ,  $\mu_2 = \mu_3 \geq \max|u|$  and  $\mu_4 \geq \max|u + c|$  for the Euler  
 214 system (for more details on the construction of the relaxation matrices see [2]). We  
 215 remark that the relaxation is performed *direction by direction*, namely the equations  
 216 on the relaxation variables are one dimensional problems.

217 The relaxation system (3.3) approximates the original system (3.1). This can be  
 218 shown by applying the Chapman-Enskog expansion of the variables for small  $\eta$  [10]:

$$219 \quad (3.6) \quad \begin{cases} \mathbf{v} = \mathbf{v}^0 + \eta \mathbf{v}^1 + \mathcal{O}(\eta^2) \\ \mathbf{w} = \mathbf{w}^0 + \eta \mathbf{w}^1 + \mathcal{O}(\eta^2) \end{cases}$$

220 At leading order  $\mathcal{O}_\eta(1)$  (zero relaxation limit  $\eta \rightarrow 0$ ), the original system is easily  
 221 recovered, with the relaxation variables identically equal to the fluxes:

$$222 \quad (3.7) \quad \begin{cases} \mathbf{v}^0 = \mathbf{F}(\boldsymbol{\psi}) \\ \mathbf{w}^0 = \mathbf{G}(\boldsymbol{\psi}) \\ \partial_t \boldsymbol{\psi} + \partial_{x_1} \mathbf{F}(\boldsymbol{\psi}) + \partial_{x_2} \mathbf{G}(\boldsymbol{\psi}) = 0. \end{cases}$$

223 The state satisfying (3.7) is called local equilibrium. Then, after some manipulations,  
 224 the following first order approximation in expansion (3.6) is obtained

$$225 \quad (3.8) \quad \begin{cases} \mathbf{v}^1 = - \left( \mathbf{A}_1 - \mathbf{F}'(\boldsymbol{\psi})^2 \right) \partial_{x_1} \boldsymbol{\psi} \\ \mathbf{w}^1 = - \left( \mathbf{A}_2 - \mathbf{G}'(\boldsymbol{\psi})^2 \right) \partial_{x_2} \boldsymbol{\psi} \\ \partial_t \boldsymbol{\psi} + \partial_{x_1} \mathbf{F}(\boldsymbol{\psi}) + \partial_{x_2} \mathbf{G}(\boldsymbol{\psi}) = \eta \left[ \partial_{x_1} \left( \left( \mathbf{A}_1 - \mathbf{F}'(\boldsymbol{\psi})^2 \right) \partial_{x_1} \boldsymbol{\psi} \right) + \partial_{x_2} \left( \left( \mathbf{A}_2 - \mathbf{G}'(\boldsymbol{\psi})^2 \right) \partial_{x_2} \boldsymbol{\psi} \right) \right], \end{cases}$$

226 where  $\mathbf{F}'(\boldsymbol{\psi})$  and  $\mathbf{G}'(\boldsymbol{\psi})$  are the Jacobian matrices of the flux functions. It is ev-  
 227 ident that satisfying the subcharacteristic condition (3.4) amounts to ensuring the  
 228 dissipative nature of system (3.8).

229 **3.2. Numerical discretization.** The derivation of the scheme is here extended  
 230 to solve two-dimensional problems. System (3.3) is discretized with finite volumes on a  
 231 Cartesian mesh. The first reason of this choice is due to the fact that the parallelization  
 232 is very efficient on this kind of grids. Moreover, on a Cartesian grid, the direction-by-  
 233 direction relaxation approximation (3.3) is easily discretized. As pointed out above,  
 234 this produces one dimensional problems on the relaxation variables.

235 **3.2.1. Implicit time discretization.** We adopt a fully implicit integration in  
 236 time with the aim of avoiding the acoustic CFL constraint (1.1). The resulting scheme  
 237 possesses a general structure, with a time integrator that is not based on the specific  
 238 EOS. Letting  $\Delta t = t_{n+1} - t_n$  be the time stepping, the implicit discretization at first  
 239 order is a simple backward Euler scheme and reads:

$$240 \quad (3.9) \quad \begin{cases} \frac{\psi^{n+1} - \psi^n}{\Delta t} + \partial_{x_1} \mathbf{v}^{n+1} + \partial_{x_2} \mathbf{w}^{n+1} = 0 \\ \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} + \mathbf{A}_1 \partial_{x_1} \psi^{n+1} = \frac{1}{\eta} (\mathbf{F}(\psi^{n+1}) - \mathbf{v}^{n+1}) \\ \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} + \mathbf{A}_2 \partial_{x_2} \psi^{n+1} = \frac{1}{\eta} (\mathbf{G}(\psi^{n+1}) - \mathbf{w}^{n+1}) \end{cases}$$

241 The non-linear fluxes in the right hand sides are solved with one iteration of the  
 242 Newton's method, which is enough to obtain convergence for the proposed first order  
 243 scheme. The resulting approximation consists in a Taylor expansion:

$$244 \quad (3.10) \quad \mathbf{F}(\psi^{n+1}) \simeq \mathbf{F}(\psi^n) + \mathbf{F}'(\psi^n)(\psi^{n+1} - \psi^n)$$

245 and the same holds for  $\mathbf{G}(\psi)$ .  $\mathbf{F}'(\psi^n)$  and  $\mathbf{G}'(\psi^n)$  are the Jacobians of the fluxes in  
 246 the two directions and can be computed analytically. Linearization (3.10) introduces  
 247 a coupling among all the equations and the following linear system is obtained:

$$248 \quad (3.11) \quad \begin{cases} \mathbf{L}\Psi^{n+1} + \mathbf{M}\mathbf{V}^{n+1} + \mathbf{N}\mathbf{W}^{n+1} = \mathbf{r} \\ \mathbf{P}\Psi^{n+1} + \mathbf{Q}\mathbf{V}^{n+1} = \mathbf{s}_1 \\ \mathbf{T}\Psi^{n+1} + \mathbf{U}\mathbf{W}^{n+1} = \mathbf{s}_2 \end{cases}$$

249 where  $\Psi^{n+1}$ ,  $\mathbf{V}^{n+1}$  and  $\mathbf{W}^{n+1}$  are the vectors containing the grid point values of  
 250 the conservative and relaxation variables and the matrices structure is given by the  
 251 spatial discretization (see Sec. 3.2.2). Relation (3.10) implies that the Jacobians enter  
 252 only on the diagonal terms of the sub-matrices of  $\mathbf{P}$  and  $\mathbf{T}$ . The full linear system is  
 253 solved with the GMRES iterative solver implemented in the PETSc library [5].

254 Scheme (3.9) is fully implicit, thus unconditionally stable (see [2]). Nevertheless,  
 255  $\Delta t$  has to be chosen accordingly to the desired accuracy. If the aim is the accurate  
 256 resolution of material waves, the enforcement a material CFL condition is recom-  
 257 mended. The material Courant number is defined on the speed of the material wave  
 258 as  $\nu_{mat} = \mu_{mat} \Delta t / \Delta x$ , where we use the ‘‘material’’ eigenvalue of the relaxation sys-  
 259 tem  $\mu_{mat} = \sqrt{a_{mat}} \geq \max|u|$ , with  $\mu_j$  defined in (3.5). A material CFL condition is  
 260 enforced by setting  $\nu_{mat} \leq 1$ . This way the problem of a Mach-dependent  $\Delta t$  shown  
 261 by relation (1.1) is avoided. On the other hand, since the material CFL does not  
 262 depend on the speed of the fast waves, these waves are not captured in the low Mach  
 263 limit. If an accurate resolution also of the acoustic waves is required, a standard  
 264 acoustic CFL has to be enforced, adapting relation (1.1) to the relaxation system.  
 265 This gives  $\nu_{ac} = \mu_{max} \Delta t / \Delta x$ , where  $\mu_{max} = \sqrt{a_{max}} \geq \max|u + c|$ .



266 **3.2.2. All-speed spatial discretization.** Let  $\Delta x_l$  be the grid spacing in the  
 267  $x_l$  direction and  $\Omega_{ij}$  the control volume centered in the node  $(i\Delta x_1, j\Delta x_2)$ . For a  
 268 generic variable  $h$ ,  $h_{ij}$  denotes the approximate cell average of the variable in the cell  
 269  $C_{ij} = [x_{i-1/2,j}, x_{i+1/2,j}] \times [x_{i,j-1/2}, x_{i,j+1/2}]$  and  $h_{i+1/2,j}$  denotes the approximate  
 270 point value at the cell interface  $x_{i+1/2,j}$ . The spatial discretization of (3.3) reads

$$271 \quad (3.12) \quad \begin{cases} \partial_t \psi_{ij} + \frac{\mathbf{v}_{i+1/2,j} - \mathbf{v}_{i-1/2,j}}{\Delta x_1} + \frac{\mathbf{w}_{i,j+1/2} - \mathbf{w}_{i,j-1/2}}{\Delta x_2} = 0 \\ \partial_t \mathbf{v}_{ij} + \mathbf{A}_1 \frac{\psi_{i+1/2,j} - \psi_{i-1/2,j}}{\Delta x_1} = \frac{1}{\eta} (\mathbf{F}(\psi_{ij}) - \mathbf{v}_{ij}) \\ \partial_t \mathbf{w}_{ij} + \mathbf{A}_2 \frac{\psi_{i,j+1/2} - \psi_{i,j-1/2}}{\Delta x_2} = \frac{1}{\eta} (\mathbf{G}(\psi_{ij}) - \mathbf{w}_{ij}). \end{cases}$$

272 The numerical fluxes in (3.12) are computed by constructing a convex combination  
 273 of upwind and centered schemes. This is based on the local Mach number of the flow  
 274  $M_{loc}$ . For a generic variable  $h$  this yields

$$275 \quad (3.13) \quad h_{i+1/2,j} = f(M_{loc}) (h_{i+1/2,j})_{upw} + (1 - f(M_{loc})) (h_{i+1/2,j})_{cent}.$$

276 The same holds for direction  $x_2$ . The function  $f(M_{loc})$  has to satisfy  $0 \leq f(M_{loc}) \leq$   
 277  $1$  (in the numerical experiments we choose  $f(M_{loc}) = \min\{1, M_{loc}\}$ ) and  $M_{loc}$  is  
 278 computed at the previous time step at the numerical interface  $x_{i+1/2,j}$ . This allows  
 279 to recover the correct numerical viscosity for each considered regime [2, 3].

280 The numerical fluxes with the centered scheme are computed as follows:

$$281 \quad (3.14) \quad (h_{i+1/2,j})_{cent} = \frac{1}{2} (h_{i+1,j} + h_{ij}).$$

282 This discretization is stable inside the adopted fully implicit framework. Since we  
 283 are centering all conservative and relaxation variables, the pressure gradients result  
 284 to be discretized with central differencing, thus providing the correct Mach scaling  
 285 [15]. The upwind scheme is built along the characteristic variables of the relaxation  
 286 system, as in [26], getting the following numerical fluxes along direction  $x_1$ :

$$287 \quad (3.15) \quad \begin{cases} (\psi_{i+1/2,j})_{upw} = \frac{1}{2} (\psi_{i+1,j} + \psi_{ij}) - \frac{1}{2} \mathbf{A}_1^{-1/2} (\mathbf{v}_{i+1,j} - \mathbf{v}_{ij}) \\ (\mathbf{v}_{i+1/2,j})_{upw} = \frac{1}{2} (\mathbf{v}_{i+1,j} + \mathbf{v}_{ij}) - \frac{1}{2} \mathbf{A}_1^{1/2} (\psi_{i+1,j} - \psi_{ij}). \end{cases}$$

288 The full 2D implicit all-speed scheme then reads:

$$289 \quad (3.16) \quad \begin{cases} \frac{\psi_{ij}^{n+1} - \psi_{ij}^n}{\Delta t} + \frac{1}{2\Delta x_1} (\mathbf{v}_{i+1,j}^{n+1} - \mathbf{v}_{i-1,j}^{n+1}) - \frac{f(M_{loc}) \mathbf{A}_1^{1/2}}{2\Delta x_1} (\psi_{i+1,j}^{n+1} - 2\psi_{ij}^{n+1} + \psi_{i-1,j}^{n+1}) + \\ \frac{1}{2\Delta x_2} (\mathbf{w}_{i,j+1}^{n+1} - \mathbf{w}_{i,j-1}^{n+1}) - \frac{f(M_{loc}) \mathbf{A}_2^{1/2}}{2\Delta x_2} (\psi_{i,j+1}^{n+1} - 2\psi_{ij}^{n+1} + \psi_{i,j-1}^{n+1}) = 0 \\ \frac{\mathbf{v}_{ij}^{n+1} - \mathbf{v}_{ij}^n}{\Delta t} + \frac{\mathbf{A}_1}{2\Delta x_1} (\psi_{i+1,j}^{n+1} - \psi_{i-1,j}^{n+1}) - \frac{f(M_{loc}) \mathbf{A}_1^{1/2}}{2\Delta x_1} (\mathbf{v}_{i+1,j}^{n+1} - 2\mathbf{v}_{ij}^{n+1} + \mathbf{v}_{i-1,j}^{n+1}) = \\ \frac{1}{\eta} (\mathbf{F}(\psi_{ij}^{n+1}) - \mathbf{v}_{ij}^{n+1}) \\ \frac{\mathbf{w}_{ij}^{n+1} - \mathbf{w}_{ij}^n}{\Delta t} + \frac{\mathbf{A}_2}{2\Delta x_2} (\psi_{i,j+1}^{n+1} - \psi_{i,j-1}^{n+1}) - \frac{f(M_{loc}) \mathbf{A}_2^{1/2}}{2\Delta x_2} (\mathbf{w}_{i,j+1}^{n+1} - 2\mathbf{w}_{ij}^{n+1} + \mathbf{w}_{i,j-1}^{n+1}) = \\ \frac{1}{\eta} (\mathbf{G}(\psi_{ij}^{n+1}) - \mathbf{w}_{ij}^{n+1}). \end{cases}$$

290 **4. The asymptotic preserving property.** The Asymptotic-Preserving (AP)  
 291 property is defined considering a continuous physical model  $\mathcal{S}_M$  (in our case the com-  
 292 pressible Euler system (2.1)) that involves a perturbation parameter  $M$  (the acoustic  
 293 Mach number). The perturbation parameter can range from  $M \simeq 1$  to  $M \ll 1$  values.  
 294 In the Euler case, there exists a reduced system  $\mathcal{S}_0$ , which is the limit system of  $\mathcal{S}_M$   
 295 as  $M \rightarrow 0$ , i. e.  $\mathcal{S}_0$  is the incompressible system (2.18). Then, let  $\mathcal{S}_M^\Delta$  be a numerical  
 296 scheme providing a consistent discretization of  $\mathcal{S}_M$ , with discrete time and space steps  
 297  $\Delta = (\Delta t, \Delta x)$ . The scheme  $\mathcal{S}_M^\Delta$  is said to be AP if the two following properties are  
 298 verified [25]:

- 299 1. its stability condition is independent of  $M$ , namely the time step  $\Delta t$  does not  
 300 depend on the Mach number of the flow;
- 301 2. as  $M$  goes to zero, there exists the limit discrete  $\mathcal{S}_0^\Delta$ , which provides a con-  
 302 sistent discretization of the continuous limit system  $\mathcal{S}_0$ .

303 Here we show that scheme (3.16) is AP. Property 1 is satisfied, since the scheme  
 304 is fully implicit, thus unconditionally stable. In order to prove that property 2 is  
 305 respected, we write the limit discrete scheme  $\mathcal{S}_0^\Delta$  as  $M \rightarrow 0$  of the implicit relaxation  
 306 scheme and show that it is consistent with the continuous limit model  $\mathcal{S}_0$ .

307 In what follows, we begin with the non-dimensionalization of the scheme and then  
 308 we carry out the analysis of its asymptotics. We expose the reasoning on the time  
 309 semi-discrete scheme (3.9) for readability. The extension to the full time and space  
 310 discretization is straightforward.

311 **4.1. Non-dimensional implicit relaxation scheme.** In deriving the non-  
 312 dimensional implicit relaxation scheme, we adopt the same notation of Section 2.1.  
 313 Here, we also have to scale the relaxation variables  $\mathbf{v}$  and  $\mathbf{w}$ : due to the relaxation  
 314 leading order (3.7), these variables have the same physical dimensions of the fluxes  
 315  $\mathbf{F}(\boldsymbol{\psi})$  and  $\mathbf{G}(\boldsymbol{\psi})$  defined in (3.2). By considering the pressure as predominant, we  
 316 choose to scale  $v_2$  and  $w_3$  using the speed of sound. Thus, we obtain the following  
 317 non-dimensional formulation of the semi-discrete scheme (3.9) (we omit from now on  
 318 the subscript  $\hat{\cdot}$  for simplicity of notation):

- 319 1. the non-dimensional conservation of mass, setting  $\mathbf{z}_1 = [v_1, w_1]^T$ , reads

$$320 \quad (4.1) \quad \begin{cases} \rho^{n+1} - \rho^n + \Delta t \nabla \cdot \mathbf{z}_1^{n+1} = 0 \\ \mathbf{z}_1^{n+1} - \mathbf{z}_1^n + a_1 \Delta t \nabla \rho^{n+1} = \frac{\Delta t}{\eta} \left( (\rho \mathbf{u})^{n+1} - \mathbf{z}_1^{n+1} \right). \end{cases}$$

- 321 2. the non-dimensional conservation of momentum is given by the two parts:

$$322 \quad (4.2) \quad \begin{cases} (\rho u_1)^{n+1} - (\rho u_1)^n + \Delta t \left( \frac{\partial_{x_1} v_2^{n+1}}{M^2} + \partial_{x_2} w_2^{n+1} \right) = 0 \\ \frac{v_2^{n+1} - v_2^n}{M^2} + a_2 \Delta t \partial_{x_1} (\rho u_1)^{n+1} = \frac{\Delta t}{\eta} \left( (\rho u_1^2)^{n+1} + \frac{p^{n+1}}{M^2} - \frac{v_2^{n+1}}{M^2} \right) \\ w_2^{n+1} - w_2^n + a_2 \Delta t \partial_{x_2} (\rho u_1)^{n+1} = \frac{\Delta t}{\eta} \left( (\rho u_1 u_2)^{n+1} - w_2^{n+1} \right) \end{cases}$$

323

$$(4.3) \quad \begin{cases} (\rho u_2)^{n+1} - (\rho u_2)^n + \Delta t \left( \partial_{x_1} v_3^{n+1} + \frac{\partial_{x_2} w_3^{n+1}}{M^2} \right) = 0 \\ v_3^{n+1} - v_3^n + a_3 \Delta t \partial_{x_1} (\rho u_2)^{n+1} = \frac{\Delta t}{\eta} \left( (\rho u_1 u_2)^{n+1} - v_3^{n+1} \right) \\ \frac{w_3^{n+1} - w_3^n}{M^2} + a_3 \Delta t \partial_{x_2} (\rho u_2)^{n+1} = \frac{\Delta t}{\eta} \left( (\rho u_2^2)^{n+1} + \frac{p^{n+1}}{M^2} - \frac{w_3^{n+1}}{M^2} \right). \end{cases}$$

324

325

3. setting  $\mathbf{z}_4 = [v_4, w_4]^T$ , the non-dimensional conservation of energy is given by

(4.4)

326

$$\begin{cases} (\rho e)^{n+1} - (\rho e)^n + \Delta t \nabla \cdot \mathbf{z}_4^{n+1} = 0 \\ \mathbf{z}_4^{n+1} - \mathbf{z}_4^n + a_4 \Delta t \nabla (\rho e)^{n+1} = \frac{\Delta t}{\eta} \left( ((\rho e)^{n+1} + p^{n+1}) \mathbf{u}^{n+1} - \mathbf{z}_4^{n+1} \right), \end{cases}$$

327

with the scaled state law

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$$(4.5) \quad p^{n+1} = (\gamma - 1) \left( (\rho e)^{n+1} - \frac{M^2}{2} \rho^{n+1} |\mathbf{u}^{n+1}| \right).$$

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**4.2. Asymptotics of the implicit relaxation scheme.** In the spirit of studying the low Mach number asymptotics, we develop all scaled variables, i. e. both conservative and relaxation variables, in powers of the Mach number, as done in Eq. (2.10). The expansion of the scaled relaxation variable  $v_1$  reads as follows

333

$$(4.6) \quad (v_1)^{n+1} = (v_1^{0,\bar{\eta}})^{n+1} + M (v_1^{1,\bar{\eta}})^{n+1} + M^2 (v_1^{2,\bar{\eta}})^{n+1} + \mathcal{O}(M^3),$$

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where we have introduced the notation with two superscripts, the first one indicating the order in the power of  $M$  and the second one indicating the order in the power of  $\eta$ . Here, we are keeping  $\eta = \bar{\eta}$  fixed. Terms of zero-th order (superscript  $\cdot^{0,\eta=0}$ ) represent the zero Mach number limit in the zero relaxation limit. Relaxation variables may be expanded also in powers of  $\eta$ , as in a Chapman-Enskog expansion (3.6). One could then combine expansions (4.6) and (3.6) and write a full expansion for  $\mathbf{v}$  and  $\mathbf{w}$  in powers of both  $M$  and  $\eta$ . The expansion for  $v_1$  reads

341

$$(4.7) \quad \begin{aligned} (v_1)^{n+1} &= (v_1^{0,0})^{n+1} + \eta (v_1^{0,1})^{n+1} + M \left( (v_1^{1,0})^{n+1} + \eta (v_1^{1,1})^{n+1} \right) \\ &+ M^2 \left( (v_1^{2,0})^{n+1} + \eta (v_1^{2,1})^{n+1} \right) + \mathcal{O}(M^3) + \mathcal{O}(\eta^2). \end{aligned}$$

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By setting  $M = 0$  and  $\eta = 0$ , the incompressible Euler system can be recovered.

Since we need to preserve the low Mach number behaviour in the expansions introduced above, we require that  $\eta \ll M$ . This is necessary to recover the correct zero relaxation limit (i. e.  $\mathbf{v} = \mathbf{F}(\boldsymbol{\psi}) + \mathcal{O}(\eta)$  and  $\mathbf{w} = \mathbf{G}(\boldsymbol{\psi}) + \mathcal{O}(\eta)$  as presented in Section 3.1) also in the case of  $M \rightarrow 0$ . More precisely, we require that  $\eta < M^2$ , in order to preserve the terms of order  $\mathcal{O}(M^2)$  and  $\mathcal{O}(M)$  when substituting expansion (4.7) in the non dimensional scheme. These terms cannot be canceled if we want to analyze the low Mach number regime.

We begin by substituting expansion (4.6) in powers of  $M$  in the scaled relaxation scheme (4.1)-(4.2)-(4.3)-(4.4) and we collect terms of equal power of  $M$ :

352 1. **order**  $\mathcal{O}_M(1/M^2)$ :

$$353 \quad (4.8) \quad \begin{cases} \partial_{x_1} \left( v_2^{0,\bar{\eta}} \right)^{n+1} = 0 \\ \left( v_2^{0,\bar{\eta}} \right)^{n+1} - \left( v_2^{0,\bar{\eta}} \right)^n = \frac{\Delta t}{\eta} \left( p_0^{n+1} - \left( v_2^{0,\bar{\eta}} \right)^{n+1} \right) \end{cases}$$

354 and the same holds for  $w_3^{0,\bar{\eta}}$ . At this point, we substitute expansion (3.6) in  
355 powers of  $\eta$  in (4.8) (once again, the same holds for variable  $w_3$ ):

$$356 \quad \begin{cases} \partial_x \left( \left( v_2^{0,0} \right)^{n+1} + \eta \left( v_2^{0,1} \right)^{n+1} \right) = 0 \\ \left( v_2^{0,0} \right)^{n+1} + \eta \left( v_2^{0,1} \right)^{n+1} - \left( v_2^{0,0} \right)^n - \eta \left( v_2^{0,1} \right)^n = \frac{\Delta t}{\eta} \left( p_0^{n+1} - \left( v_2^{0,0} \right)^{n+1} - \eta \left( v_2^{0,1} \right)^{n+1} \right). \end{cases}$$

357 We are interested in the zero relaxation limit  $\eta \rightarrow 0$ , hence we collect the terms  
358  $\mathcal{O}_\eta(1/\eta)$  in the last equation, obtaining  $\left( v_2^{0,0} \right)^{n+1} = p_0^{n+1}$  and  $\left( w_3^{0,0} \right)^{n+1} =$   
359  $p_0^{n+1}$ . This is plugged in (4.8), obtaining:

$$360 \quad \begin{cases} \left( v_2^{0,0} \right)^{n+1} = p_0^{n+1} & \left( w_3^{0,0} \right)^{n+1} = p_0^{n+1} \\ \partial_{x_1} p_0^{n+1} = 0 & \partial_{x_2} p_0^{n+1} = 0. \end{cases}$$

361 It is clear then that  $\nabla p_0^{n+1} = 0$  is respected.

362 2. **order**  $\mathcal{O}_M(1/M)$ :

$$363 \quad (4.9) \quad \begin{cases} \partial_{x_1} \left( v_2^{1,\bar{\eta}} \right)^{n+1} = 0 \\ \left( v_2^{1,\bar{\eta}} \right)^{n+1} - \left( v_2^{1,\bar{\eta}} \right)^n = \frac{\Delta t}{\eta} \left( p_1^{n+1} - \left( v_2^{1,\bar{\eta}} \right)^{n+1} \right) \end{cases}$$

364 and the same holds for  $w_3^{1,\bar{\eta}}$ . After inserting the expansion in powers of  $\eta$ ,  
365 taking the zero relaxation limit  $\eta \rightarrow 0$  and collecting the terms  $\mathcal{O}_\eta(1/\eta)$ , we  
366 obtain the following relations:

$$367 \quad \begin{cases} \left( v_2^{1,0} \right)^{n+1} = p_1^{n+1} & \left( w_3^{1,0} \right)^{n+1} = p_1^{n+1} \\ \partial_{x_1} p_1^{n+1} = 0 & \partial_{x_2} p_1^{n+1} = 0. \end{cases}$$

368 This means that also  $\nabla p_1^{n+1} = 0$  is respected.

369 3. **order**  $\mathcal{O}_M(1)$ :

370 • for the conservation of mass we have:

$$371 \quad (4.10) \quad \begin{cases} \rho_0^{n+1} - \rho_0^n + \Delta t \nabla \cdot \left( \mathbf{z}_1^{0,\bar{\eta}} \right)^{n+1} = 0 \\ \left( \mathbf{z}_1^{0,\bar{\eta}} \right)^{n+1} - \left( \mathbf{z}_1^{0,\bar{\eta}} \right)^n + a_1 \Delta t \nabla \rho_0^{n+1} = \frac{\Delta t}{\eta} \left( (\rho \mathbf{u})_0^{n+1} - \left( \mathbf{z}_1^{0,\bar{\eta}} \right)^{n+1} \right). \end{cases}$$

372 Once again, we expand in powers of  $\eta$  and take the zero relaxation limit  
373 by collecting terms of order  $\mathcal{O}_\eta(1/\eta)$ , obtaining

$$374 \quad (4.11) \quad \begin{cases} \left( \mathbf{z}_1^{0,0} \right)^{n+1} = (\rho \mathbf{u})_0^{n+1} \\ \rho_0^{n+1} - \rho_0^n + \Delta t \nabla \cdot (\rho \mathbf{u})_0^{n+1} = 0. \end{cases}$$

- for the conservation of momentum we have:

(4.12)

$$\begin{cases} (\rho u_1)_0^{n+1} - (\rho u_1)_0^n + \Delta t \left( \partial_{x_1} \left( v_2^{2,\bar{\eta}} \right)^{n+1} + \partial_{x_2} \left( w_2^{0,\bar{\eta}} \right)^{n+1} \right) = 0 \\ \left( v_2^{2,\bar{\eta}} \right)^{n+1} - \left( v_2^{2,\bar{\eta}} \right)^n + a_2 \Delta t \partial_{x_1} (\rho u_1)_0^{n+1} = \frac{\Delta t}{\eta} \left( (\rho u_1^2)_0^{n+1} + p_2^{n+1} - \left( v_2^{2,\bar{\eta}} \right)^{n+1} \right) \\ \left( w_2^{0,\bar{\eta}} \right)^{n+1} - \left( w_2^{0,\bar{\eta}} \right)^n + a_2 \Delta t \partial_{x_2} (\rho u_1)_0^{n+1} = \frac{\Delta t}{\eta} \left( (\rho u_1 u_2)_0^{n+1} - \left( w_2^{0,\bar{\eta}} \right)^{n+1} \right), \end{cases}$$

(4.13)

$$\begin{cases} (\rho u_2)_0^{n+1} - (\rho u_2)_0^n + \Delta t \left( \partial_{x_1} \left( v_3^{0,\bar{\eta}} \right)^{n+1} + \partial_{x_2} \left( w_3^{2,\bar{\eta}} \right)^{n+1} \right) = 0 \\ \left( v_3^{0,\bar{\eta}} \right)^{n+1} - \left( v_3^{0,\bar{\eta}} \right)^n + a_3 \Delta t \partial_{x_1} (\rho u_2)_0^{n+1} = \frac{\Delta t}{\eta} \left( (\rho u_1 u_2)_0^{n+1} - \left( v_3^{0,\bar{\eta}} \right)^{n+1} \right) \\ \left( w_3^{2,\bar{\eta}} \right)^{n+1} - \left( w_3^{2,\bar{\eta}} \right)^n + a_3 \Delta t \partial_{x_2} (\rho u_2)_0^{n+1} = \frac{\Delta t}{\eta} \left( (\rho u_2^2)_0^{n+1} + p_2^{n+1} - \left( w_3^{2,\bar{\eta}} \right)^{n+1} \right) \end{cases}$$

In the expansion in powers of  $\eta$ , terms of order  $\mathcal{O}_\eta(1/\eta)$  give expressions for  $(v_2^{2,0})^{n+1}$ ,  $(v_3^{0,0})^{n+1}$ ,  $(w_2^{0,0})^{n+1}$  and  $(w_3^{2,0})^{n+1}$ . This yields

(4.14)

$$\begin{cases} \left( v_2^{2,0} \right)^{n+1} = (\rho u_1^2)_0^{n+1} + p_2^{n+1} \\ \left( w_2^{0,0} \right)^{n+1} = (\rho u_1 u_2)_0^{n+1} \\ \left( v_3^{0,0} \right)^{n+1} = (\rho u_1 u_2)_0^{n+1}, \\ \left( w_3^{2,0} \right)^{n+1} = (\rho u_2^2)_0^{n+1} + p_2^{n+1} \\ (\rho \mathbf{u})_0^{n+1} - (\rho \mathbf{u})_0^n + \Delta t \nabla \cdot (\rho_0 \mathbf{u}_0 \otimes \mathbf{u}_0)^{n+1} + \Delta t \nabla p_2^{n+1} = 0. \end{cases}$$

- for the conservation of energy we adopt the same procedure of the conservation of mass, getting

$$(4.15) \quad \begin{cases} \left( \mathbf{z}_4^{0,0} \right)^{n+1} = \left( (\rho e)_0^{n+1} + p_0^{n+1} \right) \mathbf{u}_0^{n+1} \\ \left( \rho e \right)_0^{n+1} - \left( \rho e \right)_0^n + \Delta t \nabla \cdot \left( (\rho e)_0^{n+1} + p_0^{n+1} \right) \mathbf{u}_0^{n+1} = 0. \end{cases}$$

This goes with the  $\mathcal{O}_M(1)$  state law

$$(4.16) \quad p_0^{n+1} = (\gamma - 1) (\rho e)_0^{n+1}.$$

Scheme (4.11)-(4.14)-(4.15) is clearly a consistent discretization of the Euler system in its incompressible limit, derived in (2.13)-(2.14)-(2.15). This means that the scheme is asymptotic preserving. Nevertheless, in what follows, we show that the incompressibility constraint  $\nabla \cdot \mathbf{u}_0^{n+1} = 0$  is respected.

Relations  $\nabla p_0^{n+1} = 0$  and  $\nabla p_1^{n+1} = 0$  imply that  $p^{n+1}$  is constant in space up to fluctuations of order  $M^2$ . From the state law (4.16), we get that also  $(\rho e)_0^{n+1}$  is independent of space. We can rewrite the conservation of energy (4.15) as follows:

$$(\rho e)_0^{n+1} - (\rho e)_0^n + \Delta t \left( \mathbf{u}_0^{n+1} \nabla (\rho e)_0^{n+1} + (\rho e)_0^{n+1} \nabla \cdot \mathbf{u}_0^{n+1} + \mathbf{u}_0^{n+1} \nabla p_0^{n+1} + p_0^{n+1} \nabla \cdot \mathbf{u}_0^{n+1} \right) = 0$$

which becomes, due to the previous considerations on  $p_0^{n+1}$  and  $(\rho e)_0^{n+1}$ :

$$(4.17) \quad (\rho e)_0^{n+1} - (\rho e)_0^n + \Delta t \left( (\rho e)_0^{n+1} + p_0^{n+1} \right) \nabla \cdot \mathbf{u}_0^{n+1} = 0.$$

397 We now assume that the boundary conditions are such that  $p_0^{n+1}$  is independent of  $n$   
 398 too, i. e.  $p_0^{n+1} = p_0^n = \dots = p_0^1 = p_0^0$  and the same for  $p_1^{n+1}$ . Of course, this means that  
 399 also the energy is independent of  $n$ , namely  $(\rho e)_0^{n+1} - (\rho e)_0^n = 0$ . Inserting this inside  
 400 equation (4.17), one obtains directly the incompressibility constraint  $\nabla \cdot \mathbf{u}_0^{n+1} = 0$ .

401 The proof of the AP property for the fully discrete scheme (3.16) easily follows by  
 402 introducing discretization (3.13) into the spatial derivatives of (4.11)-(4.14)-(4.15), in  
 403 the limit  $M \rightarrow 0$ . This implies that a fully centered discretization is adopted in the  
 404 incompressible limit, i. e. the numerical viscosity is consistent with the incompressible  
 405 regime.

406 **5. Numerical validations on fluid dynamics problems.** At first, the Gresho  
 407 vortex test case is analyzed, in order to verify that the scheme possesses the correct  
 408 numerical viscosity in the low Mach number regime. This test verifies that the pro-  
 409 posed scheme is able to preserve the incompressible regime at all times, after setting  
 410 an incompressible initial flow. The vortex is solved for both perfect and stiffened  
 411 gases. Then, we introduce an estimate of the numerical entropy production of the  
 412 scheme. We use this indicator to show that the all-speed discretization is more pre-  
 413 cise with respect to an upwind one and also to perform grid adaptivity. Due to the  
 414 AP estimates derived in Sec. 4.2, we set  $\eta = 10^{-8}$  in all the proposed computational  
 415 experiments.

416 **5.1. The Gresho vortex.** We test the relaxation all-speed scheme (3.16) on  
 417 the classical Gresho vortex test case [21, 33]. This vortex is a stationary solution  
 418 of the incompressible Euler equations, where centrifugal forces are exactly balanced  
 419 by pressure gradients. A rotating vortex is positioned at the center (0.5, 0.5) of the  
 420 computational domain  $[0, 1] \times [0, 1]$ . The initial conditions are specified in terms of  
 421 the radial distance  $r = \sqrt{(x - 0.5)^2 + (y - 0.5)^2}$  in the form

$$422 \quad \begin{cases} \rho(x, y, 0) = 1 \\ u_1(x, y, 0) = -u_\phi(r) \sin\phi \\ u_2(x, y, 0) = u_\phi(r) \cos\phi \end{cases}$$

423 The rotation is initiated by imposing a simple angular velocity distribution of

$$424 \quad u_\phi(x, y, 0) = \begin{cases} 5r & 0 \leq r \leq 0.2 \\ 2 - 5r & 0.2 \leq r \leq 0.4 \\ 0 & r \geq 0.4, \end{cases} \quad p(x, y, 0) = \begin{cases} p_0 + \frac{25}{2}r^2 & 0 \leq r \leq 0.2 \\ p_0 + \frac{25}{2}r^2 + 4(1 - 5r - \ln 0.2 + \ln r) & 0.2 \leq r \leq 0.4 \\ p_0 - 2 + 4 \ln 2 & r \geq 0.4. \end{cases}$$

425 and the background pressure is adjusted such that it matches the maximum Mach  
 426 number  $M_{max}$ :

$$427 \quad (5.1) \quad p_0 = \frac{\rho (u_\phi)_{max}^2}{(\gamma M_{max}^2)} = \frac{\rho}{(\gamma M_{max}^2)}$$

428 since the maximum velocity  $(u_\phi)_{max} = 1$ , which is reached in  $r = 0.2$

429 We follow the flow over one full rotation of the vortex with different  $M_{max}$  in a  
 430 perfect gas, which is completed at time  $t = 1$ . We compare the results obtained with  
 431 the relaxation all-speed scheme (3.16) with the results of a standard explicit-upwind  
 432 discretization of the relaxation system. All the Gresho vortex tests are performed on  
 433 a uniform grid of  $128 \times 128$  cells and a material CFL constraint of  $\nu_{mat} = 0.2$ . In *test 1*  
 434 we set  $M_{max} = 10^{-1}$ : the initial condition in  $t = 0$  is presented in Fig. 1(a). In Figs.

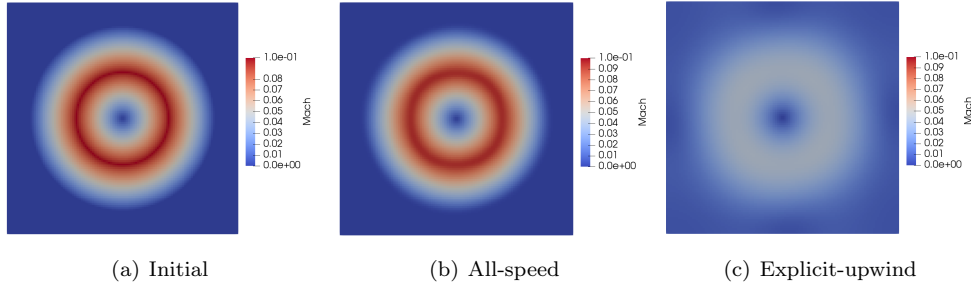


FIG. 1. Test 1: *Gresho vortex* with  $M_{max} = 10^{-1}$  (initial condition and results at time  $t = 1$  of the relaxation all-speed scheme and with an explicit-upwind scheme).

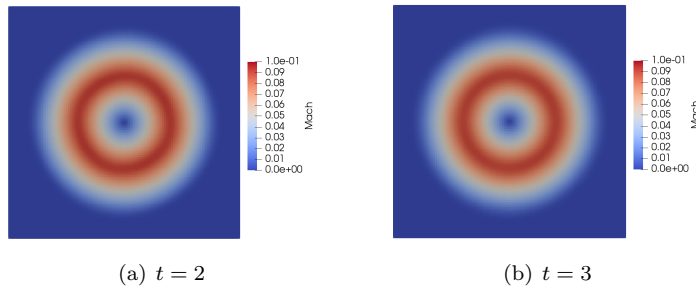


FIG. 2. Test 1: *Gresho vortex* with  $M_{max} = 10^{-1}$ , results at time  $t = 2$  and  $t = 3$  obtained with the relaxation all-speed scheme.

435 1(b) and 1(c), we compare the results of the relaxation all-speed scheme (3.16)  
 436 and of the explicit-upwind scheme of [26]. A comparison with the initial Mach number  
 437 distribution shows that the all-speed spatial discretization accurately preserves the  
 438 shape of the vortex, due to the convex combination of upwind and centered fluxes.  
 439 Instead, an upwind scheme is too diffusive for the targeted regime and thus the shape  
 440 of the vortex is lost. We also plot the results obtained after 2 and 3 rotations of the  
 441 vortex in Fig. 2, confirming that the all-speed scheme is able to recover the correct  
 442 weakly incompressible solution also after long times.

443 For tests 2 and 3 we set  $M_{max} = 10^{-2}$  and  $M_{max} = 10^{-3}$  respectively. The results  
 444 after one full rotation are reported in Fig. 3, for the relaxation all-speed scheme  
 445 and the explicit-upwind scheme [26]. The shape of the two vortices is completely  
 446 diffused when adopting an upwind flux discretization, whereas the all-speed convex  
 447 combination accurately preserves the initial vortex shape for both cases, besides a  
 448 small noise probably due to directional splitting.

449 In Table 1, we report the total kinetic energy in the simulation domain at time  
 450  $t = 1$  relative to the total kinetic energy at time  $t = 0$  for the three tests. With the  
 451 all-speed scheme, the kinetic energy reduces by about 1.5 per cent over one rotation of  
 452 the vortex. However, this loss is clearly independent of the Mach number of the flow.  
 453 On the contrary, when adopting conventional upwind discretizations the dissipation  
 454 rate of kinetic energy consistently increases as the Mach number decreases.

455 We also perform a study on the pressure fluctuations  $p_{fl} = (p_{max} - p_{min}) / p_{max}$ ,  
 456 computed on the same grid at time  $t = 1$  for the three considered Mach numbers.

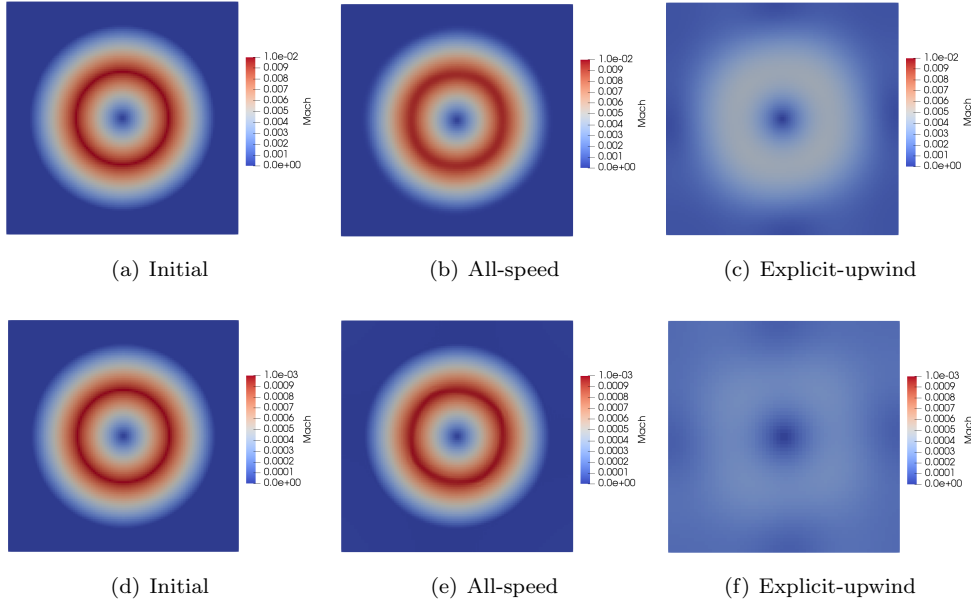


FIG. 3. Tests 2 and 3: *Gresho vortex* with  $M_{max} = 10^{-2}$  (first row) and with  $M_{max} = 10^{-3}$  (second row). Initial condition and results at time  $t = 1$  with the relaxation all-speed scheme and with an explicit-upwind scheme.

	$M_{max} = 10^{-1}$	$M_{max} = 10^{-2}$	$M_{max} = 10^{-3}$
all-speed	0.985	0.987	0.984
explicit-upwind	0.652	0.355	0.273

TABLE 1

Total kinetic energy  $E_{kin,tot}(t=1)$  after one full rotation of the *Gresho vortex* relative to its initial value  $E_{kin,tot}(t=0)$  for different maximum Mach numbers.

457 These results are reported in Table 2 and they show that pressure fluctuations scale  
 458 exactly with  $M^2$  when adopting the all-speed convex combination. We can thus infer  
 459 that the simulated flow is kept in the incompressible regime by the proposed numerical  
 460 scheme. This is clearly not the case with an upwind spatial discretization. The scaling  
 461 of  $p_{fl}$  as  $M$ , theoretically expected for this latter scheme, can be asymptotically  
 462 reached for small Mach numbers (it can be seen by comparing the simulations with  
 $M_{max} = 10^{-2}$  and  $M_{max} = 10^{-3}$ ).

	$M_{max} = 10^{-1}$	$M_{max} = 10^{-2}$	$M_{max} = 10^{-3}$
all-speed	$1.02 \cdot 10^{-2}$	$1.06 \cdot 10^{-4}$	$1.15 \cdot 10^{-6}$
explicit-upwind	$3.37 \cdot 10^{-3}$	$3.43 \cdot 10^{-5}$	$1.86 \cdot 10^{-6}$

TABLE 2

Global pressure fluctuations  $p_{fl}$  after one full rotation of the *Gresho vortex* for different maximum Mach numbers.

463  
 464

We here adapt the standard *Gresho vortex* test case to a water flow. It suffices



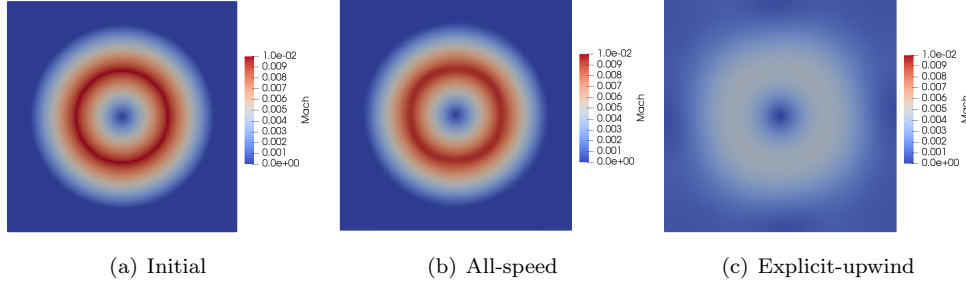


FIG. 4. *Water Gresho vortex with  $M_{max} = 10^{-2}$  (initial condition and results at time  $t = 1$  of the relaxation all-speed scheme and with the explicit-upwind relaxation scheme).*

465 to adjust the background pressure (5.1) with the stiffened gas state law as follows:

$$466 \quad p_0 = \frac{\rho (u_\phi)_{max}^2}{(\gamma M_{max}^2)} - p_\infty.$$

467 The initial density is set to  $\rho = 1000 \text{Kg/m}^3$  and the water parameters  $\gamma = 4.4$  and  
 468  $p_\infty = 6.8 \cdot 10^8$  are given with state law (2.3). The results for  $M_{max} = 10^{-2}$  obtained  
 469 with the two schemes are reported in Figs. 4 at time  $t = 1$ . The all-speed scheme is  
 470 preserving the vortex shape also in the case of water.

471 **5.2. Numerical entropy production.** The behaviour of the all-speed dis-  
 472 cretization on the different waves can be assessed by studying an a-posteriori error  
 473 indicator. We adopt the numerical entropy production introduced in [38, 39], since  
 474 the entropy is naturally available for any system of conservation laws with an entropy  
 475 inequality and has a well defined physical meaning. Moreover, the entropy produc-  
 476 tion of a scheme scales as the truncation error in the regular regions and its behavior  
 477 allows to distinguish between contact discontinuities and shocks. In [39] it has been  
 478 shown that, if the solution is locally smooth,  $S_{ij}^{n+1} = \mathcal{O}(h^r)$  with  $r$  equal to the order  
 479 of the scheme. On the other hand,  $S_{ij}^{n+1} = \mathcal{O}(1)$  if there is a contact discontinuity  
 480 and  $S_{ij}^{n+1} = \mathcal{O}(1/h)$  if there is a shock in the considered cell.

481 Let us consider the entropy pair  $(\eta, \zeta)$ . The entropy inequality  $\partial_t \eta + \nabla_x \cdot \zeta \leq 0$  has  
 482 to be integrated with the finite volume scheme used to integrate the hyperbolic system  
 483 that we are interested in solving. Coherently with discretizations (3.9)-(3.12), we  
 484 employ a first order implicit scheme for time integration in a finite volume framework.  
 485 We get the following numerical entropy production in every cell:

$$486 \quad S_{ij}^{n+1} = \eta(\psi_{ij}^{n+1}) - \eta(\psi_{ij}^n) + \frac{\Delta t}{\Delta x_1} (\zeta_{1;i+1/2,j}^{n+1} - \zeta_{1;i-1/2,j}^{n+1}) + \frac{\Delta t}{\Delta x_2} (\zeta_{2;i,j+1/2}^{n+1} - \zeta_{2;i,j-1/2}^{n+1}),$$

487 where we are considering the spatial discretization of the two components of the nu-  
 488 merical entropy flux  $\zeta = (\zeta_1, \zeta_2)$ . The interface values have to be computed according  
 489 to the all-speed convex combination (3.13). This is equivalent to a Lax-Friedrichs  
 490 scheme with a numerical viscosity modulated by the local Mach number, namely:

$$491 \quad (5.3) \quad \zeta_{1;i+1/2,j}^{n+1} = \frac{1}{2} (\zeta_{1;i+1,j}^{n+1} + \zeta_{1;i,j}^{n+1}) - \frac{\sqrt{a_{max} f(M_{loc})}}{2} (\eta_{i+1,j}^{n+1} - \eta_{i,j}^{n+1}).$$

492 In the numerical experiments, we build the entropy pair on the physical entropy of

493 the Euler system, as follows:

$$494 \quad \eta(\boldsymbol{\psi}) = -\rho \log\left(\frac{p+p_\infty}{\rho^\gamma}\right), \quad \zeta(\boldsymbol{\psi}) = \eta(\boldsymbol{\psi}) \mathbf{u}.$$

495 The proposed estimate can also be adopted as a criterion that is able to pivot  
 496 mesh adaptivity. An effective adaptive algorithm has to be driven by an indicator  
 497 able to provide a robust a-posteriori measure of the local error and also to recognize  
 498 the qualitative structure of the flow. This will be the focus of Section 5.2.2.

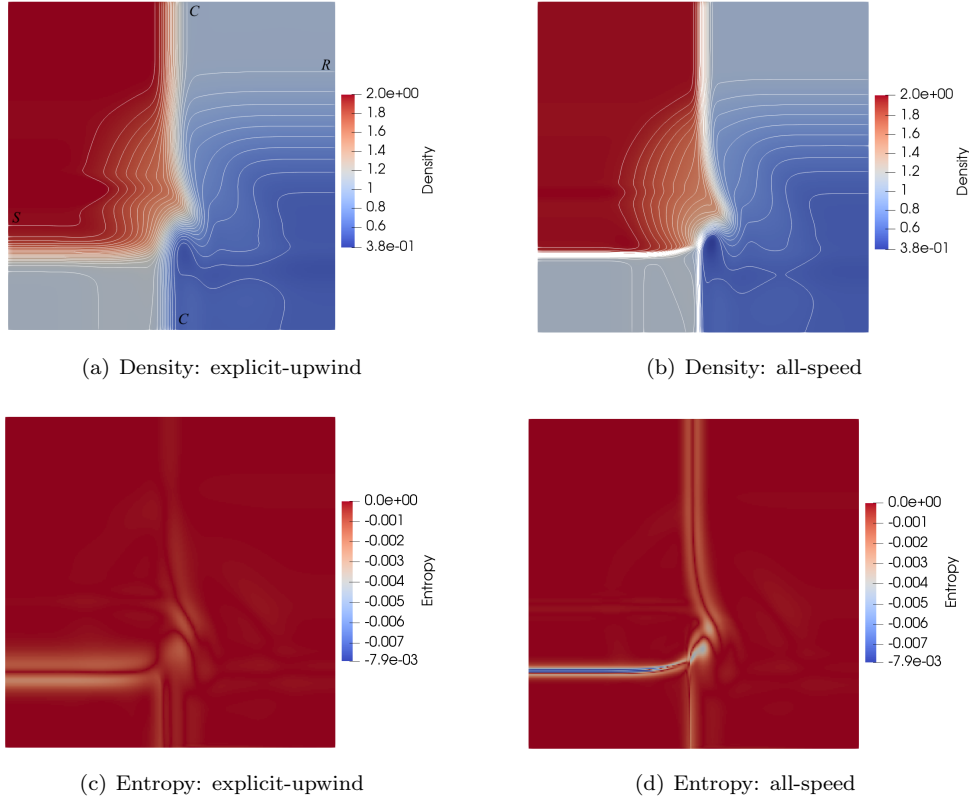


FIG. 5. 2D Riemann problem at time  $t = 0.3$ : comparison between the explicit-upwind relaxation scheme and the relaxation all-speed scheme. Density plots (30 contours: from 0.53 to 1.98) and numerical entropy production. In panel (a)  $R$  indicates a rarefaction,  $S$  a shock and  $C$  a contact wave.

499 **5.2.1. 2D Riemann problem.** We use the entropy production to compare the  
 500 behaviour of the all-speed and of the upwind discretizations on contact waves. We  
 501 study a Riemann problem in the squared domain  $[0, 1] \times [0, 1]$ , with the following  
 502 initial conditions [30, 41]:

$$503 \quad (p, \rho, u_1, u_2)(x_1, x_2, t = 0) = \begin{cases} (1, 1, 0, -0.4), & \Omega_1 \\ (1, 2, 0, -0.3), & \Omega_2 \\ (0.4, 1.0625, 0, 0.2145), & \Omega_3 \\ (0.4, 0.5197, 0, -1.1259), & \Omega_4, \end{cases}$$

504 where we have introduced the subdomains  $\Omega_1 = [0.5, 1] \times [0.5, 1]$ ,  $\Omega_2 = [0, 0.5] \times [0.5, 1]$ ,  
 505  $\Omega_3 = [0, 0.5] \times [0, 0.5]$  and  $\Omega_4 = [0.5, 1] \times [0, 0.5]$ . This test is characterized by the  
 506 following wave pattern: two contact waves, one left moving shock and one right moving  
 507 rarefaction:  $C_{21}$ ,  $\overleftarrow{S}_{32}$ ,  $C_{34}$ ,  $\overrightarrow{R}_{41}$ .

508 We adopt a grid of  $256 \times 256$  cells and we enforce an acoustic constraint on the  
 509 time step  $\nu_{ac} = 0.9$ , in order to have stability for the explicit-upwind scheme and in  
 510 order to have a good resolution of all the propagating waves. In Fig. 5 we compare  
 511 the density contours and the numerical entropy production for the two schemes. It is  
 512 evident from the density contours that the all-speed property helps in keeping sharp  
 513 the two contact waves and the shock. The contact wave at the bottom of the domain  
 514 is in the low Mach number regime, since  $M \simeq 8 \cdot 10^{-3}$ . Also the small vortex in the  
 515 center is more accurately captured by the all-speed scheme. The entropy plots confirm  
 516 that the all-speed discretization is superior in capturing the solution structure. For  
 517 both schemes, the entropy production is higher in the shock wave region, as expected  
 518 by the theory of [39]. The all-speed discretization produces more entropy than the  
 519 explicit-upwind one, confirming that this region is more accurately resolved by the  
 520 proposed scheme.

521 **5.2.2. Adaptive mesh refinement.** As anticipated above, the numerical en-  
 522 tropy production can be exploited to decide where to locally refine or coarsen the  
 523 mesh. In order to reduce the computational costs, the discretization of the solution  
 524 can be done by using non-conforming hierarchical meshes. Specifically, we use the  
 adaptive mesh refinement (AMR) technique [7].

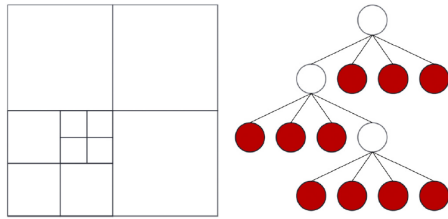


FIG. 6. *Decomposition of a squared domain and corresponding quadtree [40].*

525 Octree-meshes allow for a strong reduction of the number of degrees of freedom  
 526 where the problem exhibits smooth behavior and also a strongly localized increase of  
 527 information in areas needing more accuracy. Octrees are a hierarchical data structure  
 528 based on the principle of recursive decomposition of space. Each internal node has  
 529 exactly four children (quadtree) for 2D problems, and eight children (octree) for 3D  
 530 problems. Here we focus on quadtree meshes, that are non-conforming hierarchical  
 531 meshes defined in a square as shown in Fig. 6. We use the Bitpit library [17] for  
 532 the efficient implementation of our computational grid. The hierarchical nature of  
 533 the grid makes mesh generation, adaptivity and partitioning very efficient and with a  
 534 low-memory footprint. The data structure is based on a linear quadtree [18], namely  
 535 only the leafs of the tree structure are stored. This structure is easily dispatched to a  
 536 distributed memory architecture and parallel communications are limited to only the  
 537 first layer of neighboring cells. This constraint is perfectly in line with the stencil of  
 538 the proposed numerical scheme (3.16). A Z-order index is assigned to every cell [35].  
 539 More details on the library can be found in [17, 40].

541 We design an AMR algorithm based on the entropy production (5.2). We start

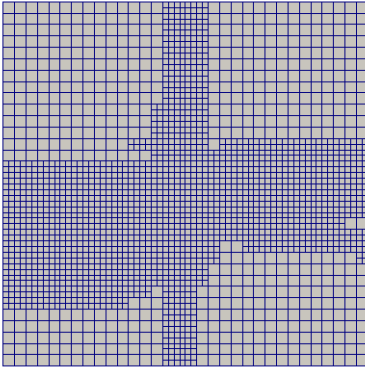
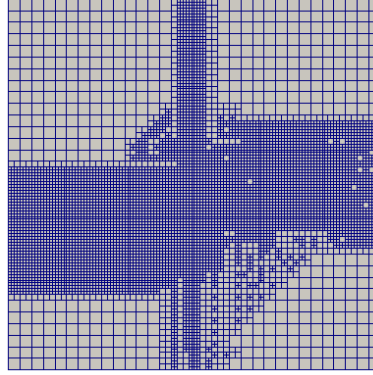
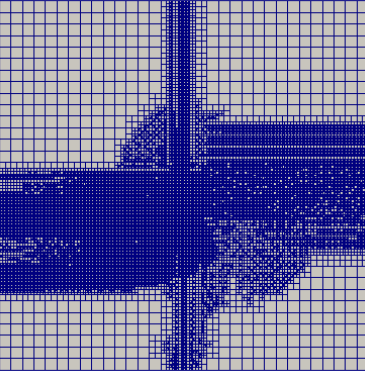
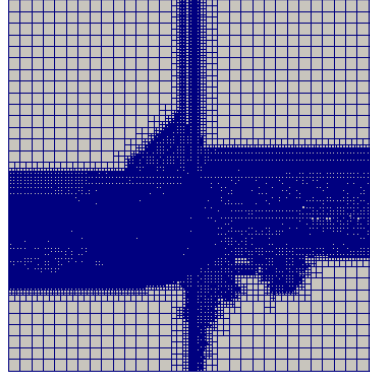
(a)  $L_{max} = 6$ (b)  $L_{max} = 7$ (c)  $L_{max} = 8$ (d)  $L_{max} = 9$ 

FIG. 7. Grids for the 2D Riemann problem at time  $t = 0.3$ , obtained with a  $L_{min} = 5$  and with different  $L_{max}$  ( $L_{max} = 6$ : 3916 cells,  $L_{max} = 7$ : 10030 cells,  $L_{max} = 8$ : 22918 cells,  $L_{max} = 9$ : 68251 cells).

542 from a uniform coarse grid of  $2^d \times 2^d$  grid points. This grid is associated to the  
 543 minimum level  $L_{min} = d$  of the quadtree data structure. At every refinement, each cell  
 544 of the grid may be replaced by four children. Let  $L_{max}$  be the maximum refinement  
 545 level allowed for a grid. At the end of every time step, the following procedure is  
 546 implemented:

- 547 1. the quantity  $S_{ij}^n$  is computed in every cell with (5.2);
- 548 2. if  $|S_{ij}^n| > S_{ref}$  and if the level of refinement of cell  $C_{ij}$  does not equal  $L_{max}$ ,  
 549 then the cell is marked for refinement; this cell is thus split into four children  
 550 and cell averages in the newly created cells are set by taking the cell average  
 551 of the “ancestor”;
- 552 3. if  $|S_{ij}^n| < S_{coa}$  for all four children and if the level of refinement of cell  $C_{ij}$   
 553 does not equal  $L_{min}$  then the cell is marked for coarsening; the four children  
 554 are replaced by the ancestor cell and the cell average in the ancestor is set by  
 555 taking the mean of the cell averages of the four children;
- 556 4. the time step  $\Delta t$  is computed with the chosen CFL constraint using the  
 557 smallest cell size of the grid.

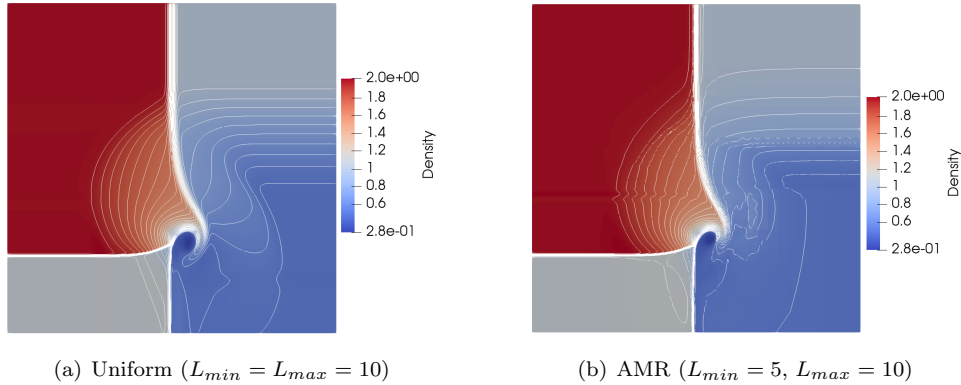


FIG. 8. Density for the 2D Riemann problem at time  $t = 0.3$ . Comparison between uniform (number of cells: 1,048,576) and AMR grids (number of cells: 146,578).

558 We test the AMR algorithm on the previous 2D Riemann problem (Section 5.2.1).  
 559 We set  $L_{min} = 5$ ,  $S_{ref} = 0.002$  and  $S_{coa} = 0.0001$  and we enforce a material CFL  
 560 constraint  $\nu_{mat} = 0.3$ .  $L_{max}$  is varied from 6 to 9 and these grids at time  $t = 0.3$   
 561 are reported in Fig. 7. We observe that the entropy criterion is pivoting the AMR in  
 562 the correct way, since the refinement levels are introduced where the 4 waves occur.  
 563 By comparing the grids of Fig. 7 and the solution structure in Fig. 8, we can see  
 564 that the postshock region is refined also after the solution has become smooth. For  
 565 this reason, the refinement needs to be further optimized. In Fig. 8 we compare the  
 566 density profile obtained with a uniform grid built with  $L_{min} = L_{max} = 10$ , which  
 567 gives 1,048,576 cells and the density obtained with the AMR by setting  $L_{min} = 5$   
 568 and  $L_{max} = 10$ , which gives a total number of cells of 146,578 at the end of the  
 569 simulation. The solution structure, the different waves and the small vortex in the  
 570 center are accurately approximated for both grids, gaining a very similar precision  
 571 (the small oscillations that can be seen on the contours of Fig. 8(b) are due to the  
 572 visualization software that is not able to fully handle quadtree grids). However, it  
 573 is evident that with the AMR technique the computational effort is reduced, since  
 574 the number of degrees of freedom is consistently reduced (of about 10 times). The  
 575 CPU times required with the uniform grid and with the AMR grid, using 32 cores,  
 576 are respectively  $9.25 \cdot 10^3$  and  $1.61 \cdot 10^3$  seconds.

577 **6. An extension to non-linear elasticity.** As anticipated above, the relax-  
 578 ation all-speed scheme (3.16) can be adopted also for the simulation of flows with  
 579 more complex EOS. Specifically, the accurate approximation of the deformation of  
 580 elastic solids can be addressed. To this end, we adopt a monolithic Eulerian model  
 581 describing different materials with the same system of conservation laws. The model  
 582 derivation has been extensively discussed in several previous works [19, 37, 20, 2].  
 583 Here we briefly report the 2D system and the chosen hyperelastic state law.

584 Let  $\Omega_0 \in \mathbb{R}^2$  the initial (or reference) configuration of a continuum and  $\Omega_t \in \mathbb{R}^2$   
 585 the deformed configuration at time  $t$ . The considered system of conservation laws  
 586 models fluids and hyperelastic solids in the Eulerian framework, thus it is written in  
 587 the deformed configuration  $\Omega_t$ . As for fluids, the equations describing the evolution  
 588 of elastic solids are the conservation of mass, of momentum and energy. In addition, a  
 589 law for the description of the deformation is needed: this is done with the introduction

590 of equations of transport of the *backward characteristics*. These functions describe the  
 591 continuum in the Eulerian framework: for a time  $t$  and a point  $x$  in the deformed  
 592 configuration, the corresponding initial point  $\xi$  in the initial configuration is given,  
 593 i. e.  $Y : \Omega_t \times [0, T] \rightarrow \Omega_0$ ,  $(x, t) \rightarrow Y(x, t)$ . One can also introduce the forward  
 594 characteristics  $X(\xi, t)$ , defined as the image at time  $t$  in the deformed configuration  
 595 of a material point  $\xi$  belonging to the initial configuration, i. e.  $X : \Omega_0 \times [0, T] \rightarrow$   
 596  $\Omega_t$ ,  $(\xi, t) \rightarrow X(\xi, t)$ . The corresponding Eulerian velocity field  $\mathbf{u}$  is defined as  $\mathbf{u} :$   
 597  $\Omega_t \times [0, T] \rightarrow \mathbb{R}^2$ ,  $(x, t) \rightarrow \mathbf{u}(x, t)$ . Forward and backward characteristics are related  
 598 as follows:  $Y(X(\xi, t)) = \xi$ . By differentiating this latter relation, one gets

$$599 \quad (6.1) \quad \begin{cases} \partial_t Y + \mathbf{u} \cdot \nabla Y = 0 \\ Y(x, 0) = x, \quad x \in \Omega_t. \end{cases}$$

600 Since the stress tensors have a direct dependence on  $[\nabla Y]$ , the gradient of (6.1) is  
 601 taken as a governing equation (see for details [2, 13]). The full monolithic Eulerian  
 602 model thus is written as follows:

$$603 \quad (6.2) \quad \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho \mathbf{u}) = 0 \\ \partial_t (\rho \mathbf{u}) + \nabla_x \cdot (\rho \mathbf{u} \otimes \mathbf{u} - \sigma) = 0 \\ \partial_t ([\nabla_x Y]) + \nabla_x (\mathbf{u} \cdot [\nabla_x Y]) = 0 \\ \partial_t (\rho e) + \nabla_x \cdot (\rho e \mathbf{u} - \sigma^T \mathbf{u}) = 0. \end{cases}$$

604 Here  $\sigma$  is the Cauchy stress tensor, which is derived through the chosen EOS. The  
 605 total energy  $e$  is still given by expression (2.2), but now the internal energy  $\epsilon$  has to  
 606 account for the behaviour of different materials, including gases, liquids and solids.  
 607 Therefore, we adopt the following general constitutive law [24, 20]:

$$608 \quad (6.3) \quad \epsilon(\rho, s, [\nabla_x Y]) = \underbrace{\frac{\kappa(s)}{\gamma-1} \rho^{\gamma-1} + \frac{p_\infty}{\rho}}_{\text{general gas}} + \underbrace{\frac{\chi}{\rho} (\text{tr} \bar{B} - 2)}_{\text{neohookean solid}},$$

609 where  $\bar{B}$  is the normalized Cauchy stress tensor:

$$610 \quad \bar{B} = \frac{B}{J} = \frac{[\nabla_x Y]^{-1} [\nabla_x Y]^{-T}}{J}, \quad J = \det([\nabla_x Y])^{-1}.$$

611 The energy function (6.3) includes different physical behaviors. The first part is  
 612 exactly the internal energy (2.3) for gases and liquids. The last term describes the  
 613 variation of energy in a neohookean elastic solid due to elastic deformations ( $\chi$  is the  
 614 shear elastic modulus). As shown in Table 3, classical models are obtained by specific  
 choices of the coefficients.

Material	$\gamma$	$p_\infty$ [Pa]	$\chi$ [Pa]
Perfect biatomic gas	1.4	0	0
Stiffened gas (water)	4.4	$6.8 \cdot 10^8$	0
Elastic solid (copper)	4.22	$3.42 \cdot 10^{10}$	$5 \cdot 10^{10}$

TABLE 3  
 Typical parameters for different materials.

615

616 The general expression of the Cauchy stress tensor  $\sigma$  is easily obtained [13, 1]

$$617 \quad (6.4) \quad \begin{cases} \sigma(\rho, s, [\nabla_x Y]) = -p(\rho, s) \mathbf{I} + 2\chi J^{-1} \left( \overline{B} - \frac{\text{tr} \overline{B}}{2} \mathbf{I} \right) \\ p(\rho, s) = -p_\infty + k(s) \rho^\gamma. \end{cases}$$

618 We rewrite the Eulerian model (6.2) in the compact formulation with the direc-  
619 tional splitting (3.1), having  $\psi, \mathbf{F}(\psi), \mathbf{G}(\psi) \in \mathbb{R}^n$  with  $n = 8$ :

$$620 \quad (6.5) \quad \psi = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ Y_{,1}^1 \\ Y_{,2}^1 \\ Y_{,1}^2 \\ Y_{,2}^2 \\ \rho e \end{bmatrix}, \quad \mathbf{F}(\psi) = \begin{bmatrix} \rho u_1 \\ \rho u_1^2 - \sigma^{11} \\ \rho u_1 u_2 - \sigma^{21} \\ u_1 Y_{,1}^1 + u_2 Y_{,2}^1 \\ u_1 Y_{,1}^2 + u_2 Y_{,2}^2 \\ 0 \\ 0 \\ (\rho e - \sigma^{11}) u_1 - \sigma^{21} u_2 \end{bmatrix}, \quad \mathbf{G}(\psi) = \begin{bmatrix} \rho u_2 \\ \rho u_1 u_2 - \sigma^{12} \\ \rho u_2^2 - \sigma^{22} \\ 0 \\ 0 \\ u_1 Y_{,1}^1 + u_2 Y_{,2}^1 \\ u_1 Y_{,1}^2 + u_2 Y_{,2}^2 \\ (\rho e - \sigma^{22}) u_2 - \sigma^{12} u_2 \end{bmatrix}.$$

621 In this notation, the superscript  $i$  indicates the component of  $Y$  and the subscript  $,j$   
622 stands for the direction along which the derivative is calculated. With this formu-  
623 lation, the Jin-Xin relaxation system (3.3) is easily constructed, by introducing the  
624 relaxation variables vectors. Then, the implicit time integration and the all-speed  
625 spatial discretization are adopted just as in the case of fluid dynamics, with the an-  
626 alytical derivation of the jacobians of the fluxes  $\mathbf{F}(\psi)$  and  $\mathbf{G}(\psi)$  to be used inside  
627 linearization (3.10) (for the jacobians derivation see Appendix A). Thus, the scheme  
628 has exactly the same formulation derived for the Euler equation case: this means that  
629 the relaxation all-speed scheme introduced in Section 3 can solve flows with more  
630 complex EOS, including elastic deformations, without any substantial modification.

631 **6.1. Low Mach number limits in elastic solids.** We now analyze the low  
632 Mach number regime in compressible elastic solids. This regime can be seen as a  
633 small deformation or a deformation that is slow with respect to other propagating  
634 waves. Alongside with the sound speed, which is computed as for fluid dynamics with  
635 expression (2.4), we define an “*elastic speed*” in the following way:

$$636 \quad (6.6) \quad u_{el} = \sqrt{\frac{2\chi}{\rho}}.$$

637 We can thus define an “isochoric Mach number” on this elastic speed, as the ratio  
638 between the isochoric speed and the advective velocity [2]:

$$639 \quad (6.7) \quad M_\chi = \frac{u}{u_{el}} = \sqrt{\frac{\rho u^2}{2\chi}}.$$

640 Therefore, two different scales can be distinguished. They can be identified when  
641 the Eulerian system is non-dimensionalized. As done for the standard case of the Euler  
642 system in Section 2.1, every variable is decomposed into a product of a reference value  
643 and a dimensionless number. By using the definitions of the two Mach numbers (2.8)  
644 and (6.7) and by using the speed of sound to scale pressure and total energy, we get  
645 the following formulation of the non-dimensional Eulerian system (for simplicity of

646 notation we loose the hat  $\hat{\cdot}$  on the non-dimensional variables):

$$647 \quad (6.8) \quad \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho \mathbf{u}) = 0 \\ \partial_t (\rho \mathbf{u}) + \nabla_x \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\nabla_x p}{M^2} - \frac{2\chi}{M_\chi^2} \nabla_x \left( J^{-1} \left( \bar{B} - \frac{\text{tr} \bar{B}}{2} \mathbf{I} \right) \right) = 0 \\ \partial_t ([\nabla_x Y]) + \nabla_x (\mathbf{u} \cdot [\nabla_x Y]) = 0 \\ \partial_t \frac{\rho e}{M^2} + \nabla_x \cdot \left( \frac{\rho e}{M^2} \mathbf{u} + \left( \frac{p}{M^2} - \frac{2\chi}{M_\chi^2} \nabla_x \left( J^{-1} \left( \bar{B} - \frac{\text{tr} \bar{B}}{2} \mathbf{I} \right) \right) \right) \mathbf{u} \right) = 0. \end{cases}$$

648 Here we have substituted the expression for the stress tensor (6.4) inside the conser-  
649 vation equations, in order to separate terms scaling with  $M$  and with  $M_\chi$ . The scaled  
650 state law reads as follows

$$651 \quad (6.9) \quad \frac{\rho e}{M^2} = \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{p + \gamma p_\infty}{M^2 (\gamma - 1)} + \frac{\chi (\text{tr} \bar{B} - 2)}{2M_\chi^2},$$

652 where again both the acoustic and the elastic Mach numbers are present.

653 Two different low Mach number regimes can be observed:

- 654 1. *acoustic and elastic low Mach regime*, namely  $M \ll 1$  and  $M_\chi \ll 1$ . This case  
655 can be verified only if the parameters of the considered material are such that  
656  $\mathcal{O}(p_\infty) \simeq \mathcal{O}(\chi)$ . We thus get that the two Mach numbers are of the same  
657 order, i. e.  $\mathcal{O}(M) \simeq \mathcal{O}(M_\chi)$ . Copper can be representative of this limit,  
658 since  $p_\infty = 3.42 \cdot 10^{10} \text{Pa}$  and  $\chi = 5 \cdot 10^{10} \text{Pa}$ . In this regime, the gradient of  
659 stress tensor  $\sigma$  fully scales as  $\mathcal{O}(M^2)$ , i. e. the entire stress tensor gradient  
660 introduces stiffness in system (6.8). All propagating waves are consistently  
661 faster with respect to the material waves, since both the acoustic and elastic  
662 terms of the EOS are large and of the same order of magnitude.
- 663 2. *acoustic low Mach regime*, namely  $M \ll 1$  and  $M \ll M_\chi$ . This is verified  
664 if the material parameters are such that  $p_\infty \gg \chi$ , giving an acoustic Mach  
665 number consistently smaller with respect to the elastic Mach number. This  
666 limit can be observed in rubber-like materials. In this regime, the gradient  
667 of stress tensor  $\sigma$  can be split into two different scales: the acoustic scale,  
668 which is represented by the pressure gradient and the elastic one, which is  
669 represented by the gradient of the elastic deformation. The pressure gradient  
670 is the only responsible for the stiffness of system (6.8) and in the EOS only  
671 the acoustic part is consistently large. Thus, only the longitudinal acoustic  
672 waves are consistently faster with respect to the deformation, whereas the  
673 shear elastic waves have a speed which is similar to the material deformation.

674 Physically,  $M \leq M_\chi$  is always verified. Therefore, with the aim of recovering the  
675 correct numerical viscosity in both low Mach regimes, the proposed convex combina-  
676 tion of upwind and centered fluxes (3.13) still holds for the all-speed scheme. For a  
677 detailed study of the low Mach regime in elastic solids and of the behaviour of the  
678 different waves, we refer the reader to [2, 1]. In the following section we will solve two  
679 different Riemann problems representing the two limits inside two elastic solids.

680 **6.2. Numerical validations in elastic solids.** We test the relaxation all-speed  
681 scheme on two 2D Riemann problems on the squared domain  $[0, 2] \times [0, 2]$ . Simulations  
682 are performed on a uniform grid of  $256 \times 256$  cells. For the first test, the domain is  
683 filled with copper (for the EOS parameters see Table 3), i. e.  $\chi \simeq p_\infty$ . The initial



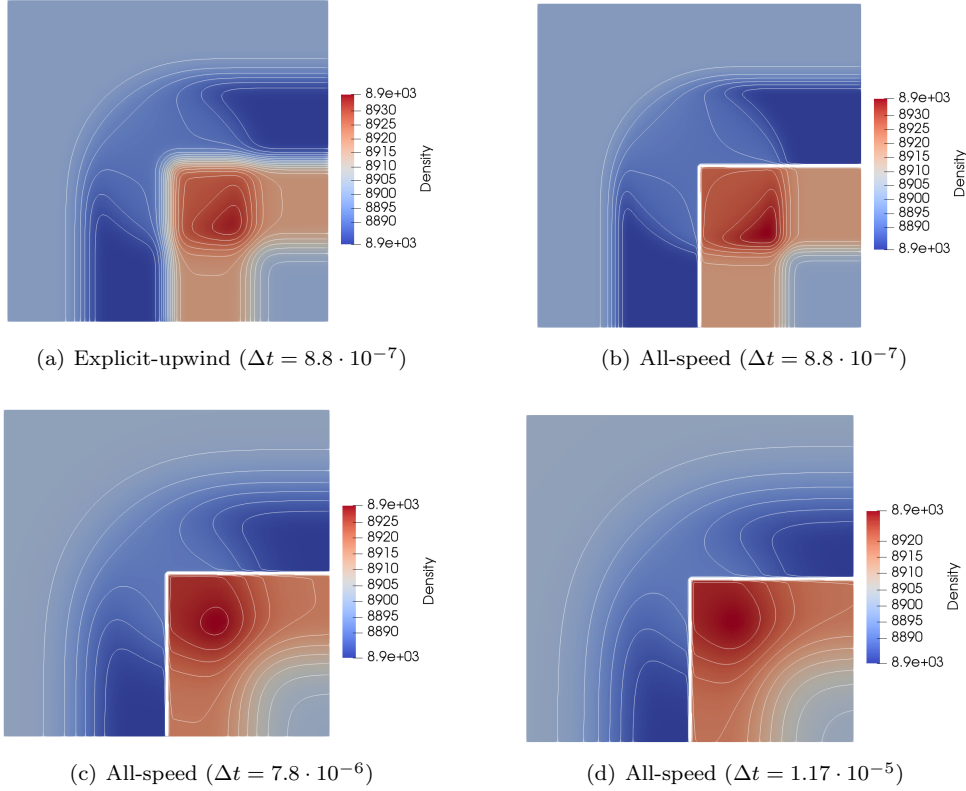


FIG. 9. *Copper Riemann problem: density profiles at time  $t = 10^{-4}$  s (20 contours: from 8882 to 8935). Panel (a): explicit-upwind scheme with  $\Delta t = 8.8 \cdot 10^{-7}$ , given by  $\nu_{ac} = 0.45$ . Panels (b), (c) and (d): all-speed relaxation scheme with  $\Delta t = 8.8 \cdot 10^{-7}$ ,  $\Delta t = 7.8 \cdot 10^{-6}$  (given by  $\nu_{mat} = 0.2$ ) and  $\Delta t = 1.17 \cdot 10^{-5}$  (given by  $\nu_{mat} = 0.3$ ) respectively.*

684 condition is the following

$$685 \quad (p, \rho, u_1, u_2)(x_1, x_2, t = 0) = \begin{cases} (10^9, 8900, 0, 0), & \Omega_1 \\ (10^9, 8900, 0, 0), & \Omega_2 \\ (10^9, 8900, 0, 0), & \Omega_3 \\ (10^5, 8900, 0, 0), & \Omega_4, \end{cases}$$

686 where we have introduced the subdomains  $\Omega_1 = [1, 2] \times [1, 2]$ ,  $\Omega_2 = [0, 1] \times [1, 2]$ ,  
 687  $\Omega_3 = [0, 1] \times [0, 1]$  and  $\Omega_4 = [1, 2] \times [0, 1]$ . Copper is at rest and at high pressure  
 688 everywhere, except for the right-bottom part of the domain  $\Omega_4$ , which is at a pressure  
 689 of  $10^5$  Pa and hence it is compressed by the surrounding copper. Two material waves  
 690 are present due to the initial pressure discontinuity: they represent the deformation  
 691 of copper in  $\Omega_4$  and they are slow if compared to all the other propagating waves.  
 692 With this initial condition, we have  $M \simeq M_\chi \simeq \mathcal{O}(10^{-3})$  on the two material waves  
 693 at the edges of  $\Omega_4$ , i.e. this test is representative of the *acoustic and elastic low Mach*  
 694 *regime*.

695 In Figs. 9(a)-9(b), we show the density profiles at final time  $t = 10^{-4}$  s obtained  
 696 with the standard explicit-upwind relaxation scheme and with the all-speed relaxation

697 scheme, by enforcing the same acoustic CFL  $\nu_{ac} = 0.45$ , which gives  $\Delta t = 8.8 \cdot 10^{-7}$ .  
 698 It is evident that the all-speed spatial discretization accurately approximates the two  
 699 material waves, which are instead diffused by an upwind-like discretization. Then,  
 700 in Figs. 9(c)-9(d), we solve the same problem with the all-speed relaxation scheme  
 701 by enforcing a material CFL constraint  $\nu_{mat} = 0.2$ , which gives  $\Delta t = 7.8 \cdot 10^{-6}$ , and  
 702  $\nu_{mat} = 0.3$ , which gives  $\Delta t = 1.17 \cdot 10^{-5}$ : the accuracy on the material waves is  
 703 maintained also with large material time steps. Of course the fast waves are diffused,  
 704 since their speed is too high to be followed by a material  $\Delta t$ , as explained in Sec.  
 705 3.2.1. In Fig. 10 we report the components  $\sigma^{11}$  and  $\sigma^{22}$  of the stress tensor, the  
 706 pressure and the velocity field  $\mathbf{u}$ , which is of course continuous on the two material  
 707 waves. We report these profiles obtained with the acoustic CFL condition, in order  
 to show all the propagating waves with a good resolution.

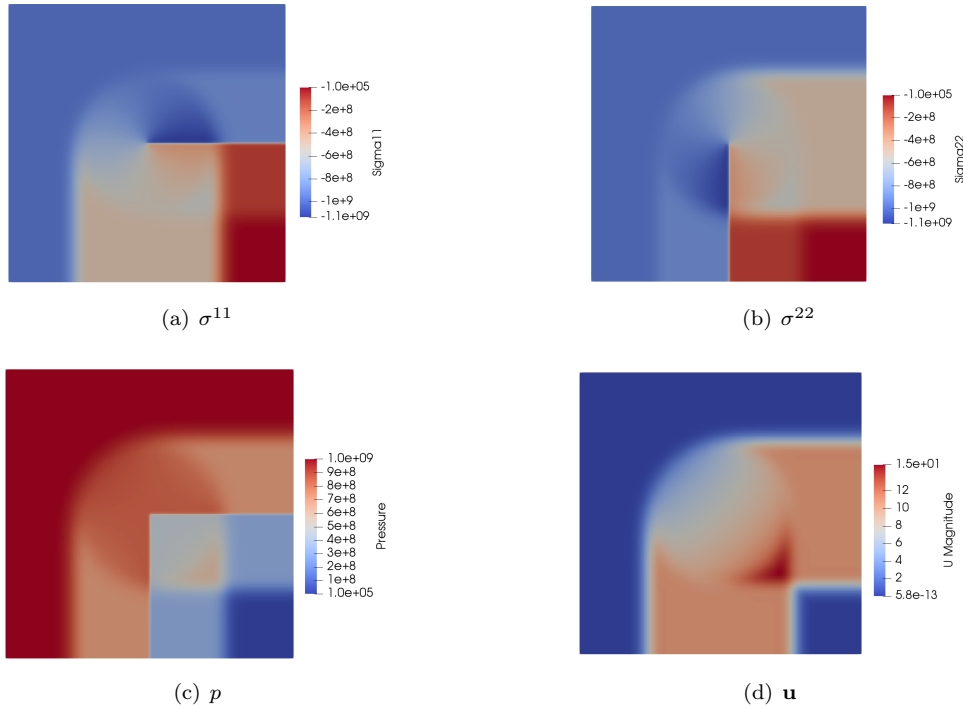


FIG. 10. Copper Riemann problem: normal components of the stress tensor  $\sigma$ , pressure  $p$  and velocity field  $\mathbf{u}$  at time  $t = 10^{-4}$  s ( $\nu_{ac} = 0.45$ ).

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The second test is instead representative of the *acoustic low Mach regime*. In order to simulate a rubber-like material, we adopt the EOS parameters  $\gamma = 4.4$ ,  $\chi = 8 \cdot 10^5$  Pa and  $p_\infty = 6.8 \cdot 10^8$  Pa and the following initial conditions:

712

$$(p, \rho, u_1, u_2)(x_1, x_2, t = 0) = \begin{cases} (10^8, 1000, 0, 0), & \Omega_1 \\ (10^8, 1000, 0, 0), & \Omega_2 \\ (10^8, 1000, 0, 0), & \Omega_3 \\ (9.8 \cdot 10^7, 1000, 20, 20), & \Omega_4 \end{cases}$$

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We impose a small initial pressure discontinuity and also an initial velocity field in  $\Omega_4$ . The velocity field is imposed in order to analyze the propagation of slow

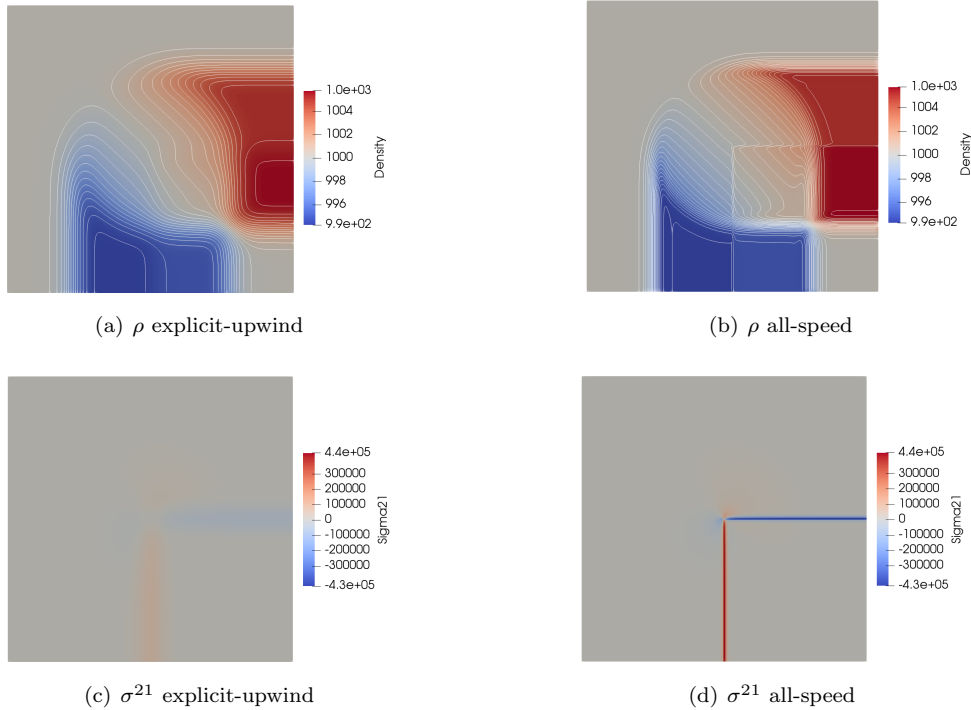


FIG. 11. Rubber-like Riemann problem: density (30 contours: from 994 to 1005) and tangential stress  $\sigma^{21}$  at time  $t = 3 \cdot 10^{-4}$  s ( $\nu_{ac} = 0.45$ ).

715 shear waves. In this framework, we get  $M \simeq 2.5 \cdot 10^{-3}$  and  $M_\chi \simeq 0.125$  on the  
 716 two material waves at the edges of  $\Omega_4$ . Thus, the longitudinal acoustic waves are  
 717 consistently faster with respect to all other waves, including the material waves and  
 718 the shear elastic waves. In fact, these latter waves propagate with a speed similar to  
 719 the deformation velocity (this wave pattern is briefly presented in the previous section  
 720 and analysed in detail in [2]). In Figs. 11(a)-11(b) we show the density contours at  
 721 final time  $t = 3 \cdot 10^{-4}$  s: we compare the results obtained with the all-speed relaxation  
 722 scheme and with the explicit-upwind relaxation scheme. For both schemes we enforce  
 723  $\nu_{ac} = 0.45$ , which gives  $\Delta t = 5.6 \cdot 10^{-6}$ , in order to have a good resolution of all waves.  
 724 It is evident that the all-speed scheme is providing a more accurate approximation of  
 725 the deformation waves also for this specific low Mach number limit. In Figs. 11(c)-  
 726 11(d) the stress tensor component  $\sigma^{21}$  is plotted and we can focus on the shear elastic  
 727 waves approximation. At time  $t = 3 \cdot 10^{-4}$  s these slow waves are still very close to the  
 728 material waves. We can observe that the sharpness and the correct magnitude of the  
 729 shear waves are accurately approximated by the all-speed discretization, whereas the  
 730 upwind scheme is once again diffusive. These results are consistent with the results  
 731 of a similar 1D test proposed in [2].

732 **7. Conclusions and future developments.** In this work we have proposed an  
 733 asymptotic preserving scheme to solve flows of compressible materials at all speeds.  
 734 The scheme exploits the relaxation method proposed by Jin and Xin, in order to deal  
 735 with an advective operator that is independent of the EOS. Thanks to this, we have  
 736 been able to simulate fluid flows and elastic deformations without any modification

737 of the scheme structure.

738 A proof of the asymptotic preserving property has been proposed, showing that  
 739 the incompressibility condition is respected by the scheme. The Gresho vortex tests  
 740 have shown that, by setting an incompressible initial flow, the scheme is able to pre-  
 741 serve the incompressible regime at all times, with pressure fluctuations of order  $M^2$ .  
 742 The scheme has also been tested on the solution of Riemann problems, accurately  
 743 approximating material waves propagating in both fluids and elastic materials. The  
 744 material waves are kept sharp also with large material time steps. Moreover, an adap-  
 745 tive mesh refinement algorithm has been designed for the proposed scheme, providing  
 746 a consistent reduction of the computational effort. The algorithm is based on the  
 747 estimate of the scheme entropy production, which has also been used to study the  
 748 accuracy of the proposed spatial discretization.

749 Different improvements will be proposed in the future, including the extension of  
 750 the scheme to higher orders and the derivation of suitable preconditioners that can  
 751 provide a more efficient solution of the linear system. The scheme will also be applied  
 752 to the simulation of multi-material problems such as low speed-impacts. This can be  
 753 done with the introduction of a level set function to track the physical interfaces and  
 754 by extending the implicit multi-material scheme proposed in [4] to multi-dimensional  
 755 problems. Moreover, the AMR technique will be extensively analyzed and improved.

756 **Appendix A. Jacobian of the fluxes in elasticity.** The jacobians of the  
 757 fluxes  $\mathbf{F}(\boldsymbol{\psi})$  and  $\mathbf{G}(\boldsymbol{\psi})$  inside the Eulerian model (6.2) can be computed analytically.  
 758 We introduce the notation  $\sigma_{,\psi_i}^{jk}$ , which stands for the derivative of the  $j^k$ ,  $j, k = 1, 2$   
 759 component of the tensor  $\sigma$  with respect to the conservative variable  $\psi_i$ ,  $i = 1, \dots, 8$ .  
 760 These derivatives are computed using definition (6.4) of  $\sigma$ , where it can be useful to  
 761 rewrite the pressure as function of the energy as follows

$$762 \quad p = -\gamma p_\infty + (\gamma - 1) \left( \rho e - \frac{1}{2} \rho |\mathbf{u}|^2 - \chi (\text{tr} \bar{B} - 2) \right).$$

763 We remind that tensor  $\sigma$  is symmetric, i.e.  $\sigma^{12} = \sigma^{21}$ . The expression for  $\mathbf{F}'(\boldsymbol{\psi})$  is

764 here reported:

$$\mathbf{F}'(\psi) = \begin{bmatrix}
 0 & 1 & 0 & 0 \\
 -u_1^2 - \sigma_{,\psi_1}^{11} & 2u_1 - \sigma_{,\psi_2}^{11} & -\sigma_{,\psi_3}^{11} & -\sigma_{,\psi_4}^{11} \\
 -u_1 u_2 - \sigma_{,\psi_1}^{21} & u_2 - \sigma_{,\psi_2}^{21} & u_1 - \sigma_{,\psi_3}^{21} & -\sigma_{,\psi_4}^{21} \\
 \frac{u_1 Y_{,1}^1 + u_2 Y_{,2}^1}{\rho} & \frac{Y_{,1}^1}{\rho} & \frac{Y_{,2}^1}{\rho} & u_1 \\
 \frac{u_1 Y_{,1}^2 + u_2 Y_{,2}^2}{\rho} & \frac{Y_{,1}^2}{\rho} & \frac{Y_{,2}^2}{\rho} & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 \frac{-Eu_1 + \sigma^{11} u_1}{\rho} - u_1 \sigma_{,\psi_1}^{11} + \frac{\sigma^{21} u_2}{\rho} & \frac{E - \sigma^{11}}{\rho} - u_1 \sigma_{,\psi_2}^{11} & -u_1 \sigma_{,\psi_3}^{11} - \frac{\sigma^{21}}{\rho} & -u_1 \sigma_{,\psi_4}^{11} - u_2 \sigma_{,\psi_4}^{21} \\
 0 & 0 & 0 & 0 \\
 -\sigma_{,\psi_5}^{11} & -\sigma_{,\psi_6}^{11} & -\sigma_{,\psi_7}^{11} & -\sigma_{,\psi_8}^{11} \\
 -\sigma_{,\psi_5}^{21} & -\sigma_{,\psi_6}^{21} & -\sigma_{,\psi_7}^{21} & 0 \\
 0 & u_2 & 0 & 0 \\
 u_1 & 0 & u_2 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 -u_1 \sigma_{,\psi_5}^{11} - u_2 \sigma_{,\psi_5}^{21} & -u_1 \sigma_{,\psi_6}^{11} - u_2 \sigma_{,\psi_6}^{21} & -u_1 \sigma_{,\psi_7}^{11} - u_2 \sigma_{,\psi_7}^{21} & (1 - \sigma_{,\psi_8}^{11}) u_1
 \end{bmatrix}$$

766 and a similar expression is derived for  $\mathbf{G}'(\psi)$  in the same way.

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