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# REMARKS ON AN INTERPOLATION BETWEEN WILSON'S THEOREM AND GIUGA'S CONJECTURE

THOMAS SAUVAGET

ABSTRACT. A family of congruences interpolating between those of Wilson and Giuga is constructed. Several elementary results are presented, in order to present a possible approach to establishing Giuga's conjecture.

## 1. INTRODUCTION

Let  $p$  be a prime, and  $j$  a positive integer such that  $1 \leq j \leq p-1$ . Then by Fermat's Little Theorem we have  $j^{p-1} \equiv 1 \pmod{p}$ . And thus  $\sum_{j=1}^{p-1} j^{p-1} \equiv p-1 \equiv -1 \pmod{p}$ , an observation of Sierpiński [7].

In 1950, Giuga [5] asked whether the converse holds, i.e. if any integer  $n$  such that  $\sum_{j=1}^{n-1} j^{n-1} \equiv -1 \pmod{n}$  is necessarily prime. He provided support for that conjecture by showing that any counterexample, i.e. any composite integer satisfying the congruence, is greater than  $10^{1000}$ . This lower bound was later improved many times over the ensuing decades, by numerical means : to  $10^{1700}$  by Bedocchi [2], to  $10^{13887}$  by Borwein, Borwein, Borwein & Girgensohn [3], and finally to  $10^{19908}$  by Borwein, Maitland & Skerritt [4]. These works nevertheless have arithmetical limitations, in that the method used intrinsically cannot treat any number having more than 8,135 prime factors.

On the other hand, Tipu [12] proved by analytic means that for any real number  $x$ , the number of counterexamples to Giuga's conjecture  $G(x) := \#\{n < x : n \text{ is composite and } \sum_{a=1}^{n-1} a^{n-1} \equiv -1 \pmod{n}\}$ , if any, is at best of moderate growth:  $G(x) \ll \sqrt{x} \log x$ . This was later improved by Luca, Pomerance and Shparlinski [9] to  $G(x) = o(\sqrt{x})$ .

Agho [1] showed in 1995 that conjecturing that  $n$  is prime iff  $nB_{n-1} \equiv -1 \pmod{n}$  (where the  $(B_n)_{n \geq 0}$  are Bernoulli numbers, see e.g. Granville [6] and Sun [11]) is equivalent to Giuga's conjecture (see also Kellner [8]).

The paper is organized as follows. In the next section, we introduce a family of congruences that interpolates between Wilson's and Giuga's. Then, in section 3, we prove elementary results related to the passage from one member of the family to the next. Finally, in an appendix, we present an earlier incorrect attempt to prove Giuga's conjecture, using a combinatorial relation which might be of independent interest.

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**Notations.** Let  $\mathbb{N}_2 := \mathbb{N} - \{0, 1\}$ . The set of primes is denoted by  $\mathcal{P} := \{2, 3, 5, 7, 11, \dots\}$ . The double-brackets denote sets of consecutive integers:  $\llbracket a; b \rrbracket := \{i \in \mathbb{N} \mid a \leq i \leq b\}$ . Define also for each  $k \in \mathbb{N}$  the functions  $S_k : \mathbb{N} \rightarrow \mathbb{N}$  by  $S_k(n) := \sum_{j=1}^{n-1} j^k$ .

## 2. AN INTERPOLATION OF WILSON'S AND GIUGA'S CONGRUENCES

### 2.1. Basic definitions.

**Definition 2.1.** Let  $f_W : \mathbb{N}_2 \mapsto \mathbb{N}$  be defined by  $f_W(n) := 1 + (n - 1)!$

Then Wilson's theorem can be recast as:

**Theorem 1.**  $\forall n \in \mathbb{N}_2$  we have:  $n \in \mathcal{P} \Leftrightarrow n \mid f_W(n)$ .

**Remark 1.** *Wilson's theorem allows to change the definition of primality, which is negative in nature ( $n$  is prime iff it is not the product of strictly smaller integers), into a positive one ( $n$  is prime iff it divides some specific larger integer). This crucial reversal is at the heart of the philosophy of this paper.*

**Definition 2.2.** Let  $f_G : \mathbb{N}_2 \mapsto \mathbb{N}$  be defined by  $f_G(n) := 1 + S_{n-1}(n)$ .

Then Giuga's conjecture is:

**Conjecture 1.**  $\forall n \in \mathbb{N}_2$  we have:  $n \in \mathcal{P} \Leftrightarrow n \mid f_G(n)$ .

**2.2. An interpolation family.** We introduce the following integer-valued functions.

**Definition 2.3.** Let  $H_k : \mathbb{N}_2 \mapsto \mathbb{N}$  be defined by  $H_k(n) := 1 + \sum_{i=1}^{n-1} \frac{(n+i-1-k)!}{i!} i^k$  for  $k \in \llbracket 1; n-1 \rrbracket$

**Remark 2.** *For any given  $n \in \mathbb{N}_2$ , we have that for any  $k \in \llbracket 1; n-1 \rrbracket$  the function  $H_k$  is integer-valued, since for each  $i \in \llbracket 1; n-1 \rrbracket$  we have  $i \leq n + i - 1 - k$ .*

**Remark 3.** *We directly obtain  $H_{n-1} = f_G$ . The relationship between  $H_1$  and  $f_W$  is described in the proof of the next lemma.*

**Lemma 1.**  $\forall n \in \mathbb{N}_2$  we have:  $n \in \mathcal{P} \Leftrightarrow n \mid H_1(n)$ .

*Proof.* Since by definition  $H_1(n) = 1 + (n-1)! + \frac{n!}{2!} 2 + \frac{(n+1)!}{3!} 3 + \dots + \frac{(2n-3)!}{(n-1)!} (n-1) = 1 + (n-1)! + n \left[ \frac{(n-1)!}{2!} 2 + \frac{(n+1)!}{3!} 3 + \dots + \frac{(2n-3)!}{(n)!} (n-1) \right] \equiv 1 + (n-1)! \equiv f_W(n) \pmod{n}$ . The term in the bracket is necessarily an integer since for any  $j \in \llbracket 1, n-3 \rrbracket$  we have that  $(n+j)! = \underbrace{1 \times 2 \times \dots \times (j+2)}_{=(j+2)!} \times (j+3) \times \dots \times n \times (n+1) \times \dots \times (n+j)$ ,

so this is simultaneously divisible by  $(j+2)!$  and by  $n$ . The result then follows from Wilson's theorem.  $\square$

Thus  $H_1$  also characterizes the primes. To motivate the strategy employed in the next sections, consider the residues of the values of  $H_k(n)$  modulo  $n$ , for  $k \in \llbracket 1; 34 \rrbracket$  and  $n \in \llbracket 5; 35 \rrbracket$ , which are provided in the following table:



## 3. RESULTS RELATED TO CONJECTURE 2

**3.1. The case of  $n = p$  prime.** We start with the following elementary lemma, whose proof is taken almost verbatim from Nicolas [10].

**Lemma 3.** *Let  $p$  be an odd prime. Then for any  $k \in \mathbb{N}^*$  such that  $\gcd(p-1, k) = 1$  we have  $S_k(p) \equiv 0 \pmod{p}$ .*

*Proof.* Let  $b$  be any number relatively prime to  $p$ . Recall that the numbers  $b, 2b, 3b, 4b, \dots, (p-1)b$  are a reduced residue class modulo  $p$ .

It follows that  $b^k + (2b)^k + (3b)^k + \dots + ((p-1)b)^k \equiv 1^k + 2^k + 3^k + \dots + (p-1)^k \pmod{p}$  (\*). But  $(ib)^k = b^k i^k$ , so we can rewrite (\*) as  $(b^k - 1)(1^k + 2^k + 3^k + \dots + (p-1)^k) \equiv 0 \pmod{p}$  (\*\*).

Let  $b$  have order  $p-1$  modulo  $p$ . So we are letting  $b$  be a primitive root of  $p$ . Since  $p-1$  does not divide  $k$ , we have  $b^k \not\equiv 1 \pmod{p}$ . Then it follows from (\*) that  $1^k + 2^k + 3^k + \dots + (p-1)^k \equiv 0 \pmod{p}$ .  $\square$

This allows to prove the following result, which is one direction of conjecture 2.

**Theorem 2.** *For all odd prime  $p$ , and all  $k \in \llbracket 1; p-2 \rrbracket$  we have  $H_k(p) \equiv 0 \pmod{p}$ .*

*Proof.* Indeed, for any  $n$ , prime or composite, we can expand  $H_k(n)$  as function of  $H_{n-1}(n)$  and  $\{S_k(n), \dots, S_{n-2}(n)\}$ :

$$\begin{aligned} H_k(n) &= 1 + \sum_{i=1}^{n-1} \frac{(n+i-1-k)!}{i!} i^k \\ &= 1 + \sum_{i=1}^{n-1} (n+i-k)(n+i-k-1) \dots (1+i) i^k \\ &= H_{n-1}(n) + \alpha_{n-2}^{(k)} S_{n-2}(n) + \dots + \alpha_k^{(k)} S_k(n) \end{aligned}$$

where the  $\alpha_j^{(k)} \in \mathbb{N}^*$  are of the same form as Vieta's coefficients.

Now, for  $n = p$  prime we mentioned at the beginning of this paper that  $H_{p-1}(p) \equiv 0 \pmod{p}$ . Combining this with lemma 3 finishes the proof.  $\square$

**Remark 5.** *The following Sage code shows a computation of the coefficients  $\alpha_j^{(k)}$  for  $k = 5$  (i.e.  $\alpha_5^{(5)}$  is the leftmost coefficient 24) :*

```
In [4]: poly=1
        k=5
        for u in range(1,k):
            poly *= (x+u)
        b=poly.list()
        b
Out[4]: [24, 50, 35, 10, 1]
```

Now, to prove conjecture 2, one could try to show that, given some  $n \in \mathbb{N}_2$ , we have for any  $k \in \llbracket 1; n-2 \rrbracket$  :

$$\begin{cases} H_1(n) \equiv 0 \pmod{n} \\ \vdots \\ H_k(n) \equiv 0 \pmod{n} \end{cases} \Rightarrow H_{k+1}(n) \equiv 0 \pmod{n}$$

Given the previous expansions, this would be rewritten as :

$$\begin{cases} H_{n-1}(n) + \alpha_{n-2}^{(1)} S_{n-2}(n) + \dots + \alpha_1^{(1)} S_1(n) \equiv 0 \pmod{n} \\ \vdots \\ H_{n-1}(n) + \alpha_{n-2}^{(k)} S_{n-2}(n) + \dots + \alpha_k^{(k)} S_k(n) \equiv 0 \pmod{n} \end{cases}$$

$$\Rightarrow H_{n-1}(n) + \alpha_{n-2}^{(k+1)} S_{n-2}(n) + \cdots + \alpha_{k+1}^{(k+1)} S_{k+1}(n) \equiv 0 \pmod{n}$$

Unfortunately, the author does not see how at the present time. A more detailed analysis of the  $\alpha_j^{(k)}$  seems to be the way forward.

An alternative approach would be to show that there cannot be any false positives, i.e. there are no odd squarefree composites satisfying  $H_k(n) \equiv 0 \pmod{n}$  for some  $k$ . We present meagre results in that direction in the next section.

**3.2. The absence of composites satisfying the congruences of conjecture 2.** Recall from [3] that even, and non-squarefree, composite integers cannot satisfy Giuga's congruence. So in this section we consider only the case of odd squarefree integers, i.e.  $n = q_1 \cdots q_J$  with  $3 \leq q_1 < \cdots < q_J \leq \frac{n}{3}$ .

The following elementary lemma allows to reduce the problem to the prime factors of  $n$ .

**Lemma 4.** *Let  $n = q_1 \cdots q_J$  be odd composite squarefree. Then for any number  $M$  we have the equivalence:*

$$M \equiv 0 \pmod{q_1 \cdots q_J} \Leftrightarrow \begin{cases} M \equiv 0 \pmod{q_1} \\ \vdots \\ M \equiv 0 \pmod{q_J} \end{cases}$$

*Proof.* This follows immediately by the Chinese Remainder Theorem.  $\square$

We can apply this lemma to  $M = H_k(n)$ , so the condition  $H_k(n) \equiv 0 \pmod{n}$  is equivalent to having

$$\begin{cases} H_k(n) \equiv 0 \pmod{q_1} \\ \vdots \\ H_k(n) \equiv 0 \pmod{q_J} \end{cases}$$

So to show  $H_k(n) \not\equiv 0 \pmod{n}$  it would suffice to show that for at least one  $q_j$  we have  $H_k(n) \not\equiv 0 \pmod{q_j}$ .

Based on numerical data, the author makes the following conjecture :

**Conjecture 3.** *For any odd squarefree composite  $n = q_1 \cdots q_J$ , one has  $H_{n-q_J+1}(n) \equiv n+1-q_1 \pmod{n}$*

**Remark 6.** *This would be, for odd squarefree composites, the first "non-trivial value of  $k$ " i.e. for these  $n$  one could show like in lemma 2 that  $H_k(n) \equiv 1 \pmod{n}$  for  $k < n - q_J + 1$ , while for  $k = n - q_J + 1$  one would have  $H_k(n)$  different from either 0 or 1 modulo  $n$ .*

**Remark 7.** *In the numerical data of 2.2, one can observe for odd squarefree composites  $n$ , that  $H_{n-2}(n) \equiv H_{n-1}(n)$ . This presumably follows from  $S_{n-2}(n) \equiv 0 \pmod{n}$  in those cases (probably one could show that  $\gcd(n-2, q_j-1) = 1$  for all  $j$ , considering that the close-by number  $n$  is a multiple of  $q_j$ ).*

**Remark 8.** *It would of interest to at least find an infinite set of odd squarefree composites for which Giuga's conjecture can be proved. The case where the prime factors  $q_1-1, \dots, q_J-1$  form an arithmetic progression could be a case to consider, since lower bounds on  $\text{lcm}(q_1-1, \dots, q_J-1)$  might then allow to apply results akin to lemma 3.*

## 4. APPENDIX

In this appendix we present an earlier incorrect attempt of the author, stressing where the error lies. The author is very grateful for the input from multiple referees to finally help him understand the problem.

## 4.1. General properties.

**Lemma 5.** For each  $(n, k) \in \mathbb{N}_2^2$  with  $n \geq k + 1$ , we have :  $H_k(n) < H_{k-1}(n)$ .

*Proof.* With those  $n$  and  $k$ , since  $\forall i \in \llbracket 1; n-1 \rrbracket$  we have  $\frac{(n+i-1-k)!}{i!} i^k = \frac{i}{n+i-1-k} \times \frac{(n+i-k)!}{i!} i^{k-1}$ , we thus explicitly have  $H_k(n) = 1 + \sum_{i=1}^{n-1} \frac{(n+i-1-k)!}{i!} i^k < 1 + \sum_{i=1}^{n-1} \frac{(n+i-k)!}{i!} i^{k-1} = H_{k-1}(n)$   $\square$

**Lemma 6.** For each  $(n, k) \in \mathbb{N}_2^2$  with  $n \geq k + 1$  we have :

$$H_k(n) \equiv 1 + \sum_{i=1}^k \frac{(n+i-1-k)!}{i!} i^k \pmod{n}$$

*Proof.* Immediate.  $\square$

4.2. Passage from  $H_k$  to  $H_{k+1}$ . The following result can be established.

**Lemma 7.** For each  $n \in \mathbb{N}_2$  and any  $1 \leq k \leq n - 2$  we have :

$$H_{k+1}(n) = H_k(n) - (n-k-1) \sum_{i=1}^{n-1} \frac{(n+i-2-k)!}{i!} i^k$$

*Proof.* We have  $H_{k+1}(n) = 1 + \sum_{i=1}^{n-1} \frac{(n+i-(k+1)-1)!}{i!} i^{k+1} = 1 + \sum_{i=1}^{n-1} \frac{i}{n+i-k-1} \frac{(n+i-k-1)!}{i!} i^k$ , so adding and subtracting  $\sum_{i=1}^{n-1} \frac{n-k-1}{n+i-k-1} \frac{(n+i-k-1)!}{i!} i^k$  to that, one obtains the result.  $\square$

The following functions will be useful.

**Definition 4.1.** Let  $U_k : \mathbb{N}_2 \mapsto \mathbb{N}$  be defined by

$$U_k(n) := \sum_{i=1}^{k+1} \frac{(n+i-2-k)!}{i!} i^k \text{ if } k \in \llbracket 1; n-2 \rrbracket$$

**Remark 9.** For any given  $n \in \mathbb{N}_2$  we have that for any  $k \in \llbracket 1; n-2 \rrbracket$  the quantity  $U_k(n)$  is an integer, since for any  $i \in \llbracket 1; k+1 \rrbracket$  we have  $i \leq n+i-2-k$ .

**Remark 10.** Since  $\frac{(n+i-2-k)!}{i!} i^k \equiv 0 \pmod{n}$  for  $i \in \llbracket k+2; n-1 \rrbracket$ , we have  $H_{k+1}(n) - H_k(n) \equiv (k+1)U_k(n) \pmod{n}$ .

**Lemma 8.** For any  $n \in \mathbb{N}_2$  we have  $U_1(n) \equiv 0 \pmod{n}$ . In particular,  $H_2$  also characterizes  $\mathcal{P}$ .

*Proof.* We have to show that for  $n \geq 2$  we have  $\frac{(n-2)!}{1!} 1^1 + \frac{(n-1)!}{2!} 2^1 \equiv 0 \pmod{n}$ . This is true since it simplifies to showing  $(n-2)! + (n-1)! \equiv 0 \pmod{n}$ , i.e. showing that  $(n-2)! [1 + (n-1)] \equiv 0 \pmod{n}$ , which is true. So for  $n \geq 2$  we have  $H_2(n) \equiv H_1(n) \pmod{n}$ , so lemma 1 implies that  $H_2$  also characterizes the primes.  $\square$

**4.3. A combinatorial relationship.** Let us now introduce the following sums.

**Definition 4.2.** For each  $k \in \mathbb{N}^*$  let

$$V_k := \sum_{i=1}^k \left[ (-1)^{i-1} \binom{k+1}{i} i^k + (-1)^k \binom{k}{i} k^i \right]$$

**Theorem 3.** For each  $k \in \mathbb{N}^*$  we have  $V_k = (-1)^{k+1}$ .

This will be proved as a result of the following two lemmas. The author is very grateful to the referee of a previous version of this work for providing this proof. This result might be of independent interest.

**Lemma 9.** For each  $k \in \mathbb{N}^*$  we have  $\sum_{i=0}^k \binom{k}{i} k^i = (k+1)^k$ .

*Proof.* This follows immediately from the binomial theorem. □

**Lemma 10.** For each  $k \in \mathbb{N}^*$  we have  $\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} i^k = 0$ .

*Proof.* Defining for any function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  the forward difference operator  $\Delta$  by  $\Delta f(j) := f(j+1) - f(j)$ , we obtain by induction on  $n$  that the  $n$ -fold iteration of  $\Delta$  on  $f$  is  $\Delta^n f(j) = \sum_{i=0}^n (-1)^i \binom{n}{i} f(j+n-i)$ .

For  $n = k+1$  and the polynomial  $f(j) = j^k$ , since  $f$  is of degree  $k < n$  we thus have that  $\Delta^n f(j)$  is identically zero. That is, we have  $\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (j+k+1-i)^k = 0$ .

In particular, when evaluated at  $j = -(k+1)$  this gives  $\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (-i)^k = (-1)^k \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} i^k = 0$ , hence the result. □

We now prove theorem 3. Let

$$A := \sum_{i=1}^k \binom{k}{i} k^i$$

and

$$B := \sum_{i=1}^k (-1)^i \binom{k+1}{i} i^k$$

Then, given the slightly different summation ranges than in the previous lemmas, we get  $A = (k+1)^k - 1$  and  $B = (-1)^k (k+1)^k$ . So since  $(-1)^k A + B = (-1)^{k+1}$  we obtain  $V_k = (-1)^{k+1}$ .

**4.4. The error in the current attempt.** Our aim was then to prove the following:

**Conjecture 4.** With the previous notations, we have for any  $n \in \mathbb{N}_2$  and any  $1 \leq k \leq n-2$  that  $U_k(n) \equiv 0 \pmod{n}$ .

Unfortunately it is not true, as was explained to us by (we now realise) several referees. Our aim was to proceed as follows.

The  $i = 1$  term of  $U_k(n)$  is  $(n-1-k)!$ . Let us say that  $(n-1-k)! \equiv a \pmod{n}$  for some  $a \in \llbracket 0; n-1 \rrbracket$ . Then for any  $j \in \llbracket 1; k \rrbracket$  we have by induction  $(n-1-k+j)! \equiv a \times (n-k) \times \cdots \times (n-1-k+j) \equiv (-1)^j a \frac{k!}{(k-j)!} \pmod{n}$ . So  $a$  is in factor in each term of  $U_k(n)$ .

If  $a = 0$  then  $U_k(n) \equiv 0 \pmod{n}$ . If  $a \neq 0$ , then  $U_k(n)$  can be factored by  $a$ , but that is as far as one can proceed. In particular, one **cannot** write (using the previous paragraph with  $i = j+1$ ) that we have  $U_k(n) = a \times \sum_{i=1}^{k+1} \frac{(-1)^{i-1} \frac{k!}{(k+1-i)!}}{i!} i^k = a \times$

$\left(\sum_{i=1}^k \frac{1}{k+1} \binom{k+1}{i} (-1)^{i-1} i^k + (-1)^k \frac{(k+1)^k}{k+1}\right)$ , due to the division by  $i!$  (non-commutativity of two otherwise true congruences – this is nothing else than the fact that  $H_k(n) \not\equiv 0$  nor  $1 \pmod{n}$  as seen in the numerical data).

Had that been the case, recalling that the term in parenthesis is an integer, and multiplying by the quantity  $k+1$  (which is non-zero modulo  $n$  since  $1 \leq k \leq n-2$ ) we would have obtained  $(k+1)U_k(n) = a \times \left(\sum_{i=1}^k \binom{k+1}{i} (-1)^{i-1} i^k + (-1)^k (k+1)^k\right)$ . Expanding  $(k+1)^k$  with the binomial theorem and simplifying, we notice that the resulting expression is 0 by theorem 3. So we would have necessarily had  $U_k(n) \equiv 0 \pmod{n}$ ...

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