



**HAL**  
open science

## Notes on the Chernoff Product Formula

Valentin Zagrebnov

► **To cite this version:**

| Valentin Zagrebnov. Notes on the Chernoff Product Formula. 2019. hal-02373231

**HAL Id: hal-02373231**

**<https://hal.science/hal-02373231>**

Preprint submitted on 20 Nov 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Notes on the Chernoff Product Formula

Valentin A. Zagrebnov

Aix-Marseille Université, CNRS, Centrale Marseille  
Institut de Mathématiques de Marseille (UMR 7373)  
Centre de Mathématiques et Informatique - Technopôle Château-Gombert  
39, rue F. Joliot Curie, 13453 Marseille Cedex 13, France  
e-mail Valentin.Zagrebnov@univ-amu.fr

ABSTRACT. We revise the strong convergent Chernoff product formula and extend it, in a Hilbert space, to convergence in the operator-norm topology. Main results deal with the self-adjoint Chernoff product formula. The nonself-adjoint case concerns the quasi-sectorial contractions.

## Contents

1	The Chernoff product formula	1
2	Lifting the Chernoff product formula to operator-norm topology	4
3	Chernoff product formula: rate of the operator-norm convergence	8
4	Nonself-adjoint operator-norm Chernoff product formula	14
5	Comments: Trotter-Kato product formulae	18

## 1 The Chernoff product formula

In paper [Che68], Chernoff has proved the following Theorem:

**Proposition 1.1.** *Let  $F(t)$  be a strongly continuous function from  $[0, \infty)$  to the linear contractions on a Banach space  $\mathfrak{X}$  such that  $F(0) = \mathbb{1}$ . Suppose that the closure  $C$  of the strong derivative  $F'(+0)$  is the generator of a contraction semigroup. Then  $F(t/n)^n$ ,  $n \in \mathbb{N}$ , converges for  $t \geq 0$  and  $n \rightarrow \infty$  to contraction  $C_0$ -semigroup  $\{e^{tC}\}_{t \geq 0}$  in the strong operator topology.*

Note, that by condition of Proposition 1.1 the operator  $C(\tau) := (F(\tau) - \mathbb{1})/\tau$  for  $\tau > 0$  is dissipative and  $F'(+0)$  is derivative  $\lim_{\tau \rightarrow +0} C(\tau)$  in the strong resolvent sense. We also recall another version of this proposition, see [Che74] (Theorem 1.1).

---

*Mathematics Subject Classification.*

47D05, 47A55, 81Q15, 47B65.

*Keywords.* Strongly continuous semigroups, semigroup approximations, Chernoff product formula, quasi-sectorial contractions, Trotter-Kato product formulae.

**Proposition 1.2.** *Let  $t \mapsto F(t)$  be a measurable operator-valued function from  $[0, \infty)$  to the linear contractions on a Banach space  $\mathfrak{X}$  such that  $F(0) = \mathbb{1}$ . If*

$$\text{s-}\lim_{\tau \rightarrow +0} (\mathbb{1} - C(\tau))^{-1} = (\mathbb{1} - C)^{-1}, \quad (1.1)$$

*in strong operator topology, then*

$$\text{s-}\lim_{n \rightarrow +\infty} F(t/n)^n = e^{tC}, \quad (1.2)$$

*for all  $t \geq 0$  uniformly on bounded  $t$ -intervals.*

*Proof.* The proof needs two ingredients. The first is the *Trotter-Neveu-Kato* theorem ([Kat80], Ch.IX, Theorem 2.16): the strong convergence in (1.1) yields for contractions  $\{e^{tC(\tau)}\}_{t \geq 0}$

$$\lim_{\tau \rightarrow +0} e^{tC(\tau)} u = e^{tC} u, \quad (1.3)$$

for all  $u \in \mathfrak{X}$ , locally uniformly on closed intervals  $\mathcal{I} \subset \mathbb{R}_0^+$ .

Now, for any  $t > 0$  and  $u \in \mathfrak{X}$  we define  $\{u_n := [\mathbb{1} - C(t/n)/\sqrt{n}]^{-1} u\}_{n \geq 1}$ . Then by (1.3)

$$\lim_{n \rightarrow \infty} u_n := \lim_{n \rightarrow \infty} \sqrt{n} \int_0^\infty ds e^{-\sqrt{n}s} e^{sC(t/n)} u = u. \quad (1.4)$$

The second ingredient comes from the, so-called,  $\sqrt{n}$ -Lemma ([Che68] Lemma 2). It yields the estimate

$$\|e^{tC(t/n)} w - F(t/n)^n w\| \leq \sqrt{n} \|(F(t/n) - \mathbb{1}) w\|, \quad (1.5)$$

for any  $w \in \mathfrak{X}$ . Then (1.4) and (1.5) imply

$$\begin{aligned} \lim_{n \rightarrow \infty} \|e^{tC(t/n)} u - F(t/n)^n u\| &= \lim_{n \rightarrow \infty} \|e^{tC(t/n)} u_n - F(t/n)^n u_n\| \\ &\leq \lim_{n \rightarrow \infty} t/\sqrt{n} \|C(t/n) [\mathbb{1} - C(t/n)/\sqrt{n}]^{-1} u\| = \lim_{n \rightarrow \infty} t \|u_n - u\| = 0. \end{aligned}$$

Hence, by (1.3) together with estimate

$$\|e^{tC} u - F(t/n)^n u\| \leq \|e^{tC} u - e^{tC(t/n)} u\| + \|e^{tC(t/n)} u - F(t/n)^n u\|,$$

for any  $u \in \mathfrak{X}$ , we obtain (1.2). □

**Remark 1.3.** (a) Equivalence of condition (1.1) to the existence of the strong derivative  $F'(+0)$  is a part of the *Trotter-Neveu-Kato* theorem, [EN00] Ch.III, Sects.4.8 and 4.9.

(b) For analysis of optimality, generalisation and improvement of the  $\sqrt{n}$ -Lemma, see [Zag17].

**Definition 1.4.** Equation (1.2) is called the *Chernoff product formula*, or the Chernoff approximation formula, in the strong operator topology for contraction  $C_0$ -semigroup  $\{e^{tC}\}_{t \geq 0}$ .

The aim of the present Notes is *lifting* the strongly convergent Chernoff product formula to formula convergent in the operator norm topology, whereas a majority of results concerns only strong convergence, see, e.g., a detailed review [But19]. Our main results are focused on analysis of the *operator-norm* convergence of the Chernoff product formula in a Hilbert

space  $\mathfrak{H}$ , see Section 2. In Section 3 we establish estimates of the operator-norm *rate* of convergence for *self-adjoint* Chernoff product formula. The case of Chernoff product formula for nonself-adjoint *quasi-sectorial* contractions is the subject of Section 4. The Trotter-Kato product formula is a direct application of the Chernoff product formula, see Section 5.

We conclude this section by the proof of our first *Note*: for strongly convergent Chernoff product formula the *self-adjointness* allows to relax condition (1.1) to the weak operator convergence. To this aim we recall a *lifting* topology assertion in  $\mathfrak{H}$ :

**Proposition 1.5.** *Let  $\{u_n\}_{n \geq 1}$  be a weakly convergent sequence of vectors,  $w\text{-}\lim_{n \rightarrow \infty} u_n = u$ , in a Hilbert space  $\mathfrak{H}$ . If, in addition,  $\lim_{n \rightarrow \infty} \|u_n\| = \|u\|$ , then  $\lim_{n \rightarrow \infty} \|u_n - u\| = 0$ .*

This assertion implies the following statement.

**Lemma 1.6.** *Let  $S : \mathbb{R}^+ \rightarrow \mathcal{L}(\mathfrak{H})$  be a measurable family of non-negative self-adjoint operators and let  $H$  be non-negative self-adjoint operator. If the weak operator limit*

$$w\text{-}\lim_{\tau \rightarrow +0} (\lambda \mathbb{1} + S(\tau))^{-1} = (\lambda \mathbb{1} + H)^{-1}, \quad (1.6)$$

for each  $\lambda > 0$ , then it is also true in the strong operator topology :

$$s\text{-}\lim_{\tau \rightarrow +0} (\lambda \mathbb{1} + S(\tau))^{-1} = (\lambda \mathbb{1} + H)^{-1}. \quad (1.7)$$

*Proof.* From (1.6) we get

$$\lim_{\tau \rightarrow +0} \|(\lambda \mathbb{1} + S(\tau))^{-1/2} u\|^2 = \|(\lambda \mathbb{1} + H)^{-1/2} u\|^2, \quad u \in \mathfrak{H}, \quad (1.8)$$

for  $\lambda \geq 1$ . Since

$$(\mathbb{1} + S(\tau))^{-1/2} = \frac{1}{\pi} \int_0^\infty d\eta \frac{1}{\sqrt{\eta}} (\eta \mathbb{1} + \mathbb{1} + S(\tau))^{-1},$$

(1.6) also yields  $w\text{-}\lim_{\tau \rightarrow +0} (\mathbb{1} + S(\tau))^{-1/2} = (\mathbb{1} + H)^{-1/2}$ . This, together with (1.8) and the lifting Proposition 1.5, implies  $s\text{-}\lim_{\tau \rightarrow +0} (\mathbb{1} + S(\tau))^{-1/2} = (\mathbb{1} + H)^{-1/2}$ . Then strong convergence of the product of operators yields (1.7).  $\square$

By Lemma 1.6, the conditions of Proposition 1.2 for the family of operators

$$S(\tau) := \frac{\mathbb{1} - F(\tau)}{\tau}, \quad \tau > 0, \quad (1.9)$$

can be reformulated in a Hilbert space  $\mathfrak{H}$ . Then we get a stronger assertion:

**Theorem 1.7.** *Let  $F : \mathbb{R}^+ \rightarrow \mathcal{L}(\mathfrak{H})$  be a measurable family of non-negative self-adjoint contractions such that  $F(0) = \mathbb{1}$  and let  $H \geq 0$  be a self-adjoint operator in  $\mathfrak{H}$ . Then*

$$s\text{-}\lim_{n \rightarrow +\infty} F(t/n)^n = e^{-tH}, \quad (1.10)$$

if and only if for each  $\lambda > 0$  the condition

$$w\text{-}\lim_{\tau \rightarrow +0} (\lambda \mathbb{1} + S(\tau))^{-1} = (\lambda \mathbb{1} + H)^{-1}, \quad (1.11)$$

is satisfied.

*Proof.* If (1.11) is valid, then applying Lemma 1.6 we verify (1.7). Using Proposition 1.2 in a Hilbert space  $\mathfrak{H}$  for  $C(\tau) = -S(\tau)$ ,  $C = -H$  and (1.7) for  $\lambda = 1$ , we prove the self-adjoint Chernoff product formula (1.10).

To prove the converse we use the representation

$$e^{-tS(t/n)} - e^{-tH} = F(t/n)^n - e^{-tH} + e^{-tS(t/n)} - F(t/n)^n. \quad (1.12)$$

Then  $\lim_{n \rightarrow +\infty} \|(F(t/n)^n - e^{-tH})u\| = 0$  by condition (1.10), whereas by the spectral functional calculus for self-adjoint operator  $F(\tau)$  we obtain the *operator norm* estimate

$$\|F(t/n)^n - e^{-n(\mathbb{1} - F(t/n))}\| = \left\| \int_{[0,1]} dE_{F(t/n)}(\lambda) \left( \lambda^n - e^{-n(1-\lambda)} \right) \right\| \leq \frac{1}{n}. \quad (1.13)$$

Therefore, (1.12) yields for  $\tau = t/n$  the limit

$$\text{s-}\lim_{\tau \rightarrow +0} e^{-tS(\tau)} = e^{-tH}, \quad (1.14)$$

for  $t \in \mathbb{R}^+$ . Note that by the self-adjoint *Trotter-Neveu-Kato* convergence theorem the limit (1.14) is equivalent to

$$\text{s-}\lim_{\tau \rightarrow +0} (\mathbb{1} + S(\tau))^{-1} = (\mathbb{1} + H)^{-1}. \quad (1.15)$$

Since (1.15) is, in turn, equivalent to

$$\text{s-}\lim_{\tau \rightarrow +0} (\lambda \mathbb{1} + S(\tau))^{-1} = (\lambda \mathbb{1} + H)^{-1},$$

for  $\lambda > 0$ , the latter yields the weak limit (1.11).  $\square$

**Remark 1.8.** (a) We note that due to self-adjointness of the family  $\{F(t)\}_{t \geq 0}$  the estimate (1.13) is stronger than the  $\sqrt{n}$ -estimate (1.5) for general contractions.

(b) By definition of  $\{F(t)\}_{t \geq 0}$  and by condition (1.11) the  $C_0$ -semigroup property of  $\{e^{-tH}\}_{t \geq 0}$  ensures that strong limits  $\text{s-}\lim_{t \rightarrow +0}$  of the left- and the right-hand sides of (1.10) are well-defined and coincide with  $\mathbb{1}$ . Similarly, since by (1.14) the limits in  $\lim_{t \rightarrow +0} \text{s-}\lim_{\tau \rightarrow +0} e^{-tS(\tau)}$  commute, one gets that  $t \in \mathbb{R}_0^+$ .

## 2 Lifting the Chernoff product formula to operator-norm topology

The first result about lifting the Chernoff product formula to the operator-norm topology is due to [NZ99a], Theorem 2.2. Our next *Note* will be about improvement of this result by extension to any closed  $t$ -interval  $\mathcal{I} \subset \mathbb{R}_0^+$ .

Similarly to the case of the strong operator topology our strategy includes the estimate of two ingredients (1.3) and (1.5) involved into Proposition 1.2, but now in the *operator norm* topology. Since the second ingredient has estimate (1.13), it rests to find conditions for *lifting* to operator norm convergence of the limit (1.14) in the Trotter-Neveu-Kato convergence theorem. We proceed with the following lemma.

**Lemma 2.1.** *Let  $K$  and  $L$  be non-negative self-adjoint operators in a Hilbert space  $\mathfrak{H}$ . Then*

$$\|e^{-K} - e^{-L}\| \leq c \|(\mathbb{1} + K)^{-1} - (\mathbb{1} + L)^{-1}\| \quad (2.1)$$

with a constant  $c > 0$  independent of operators  $K$  and  $L$ .

By the Riesz-Dunford functional calculus one obtains representation:

$$e^{-K} - e^{-L} = \frac{1}{2\pi i} \int_{\Gamma} dz e^z ((z + K)^{-1} - (z + L)^{-1}). \quad (2.2)$$

Essentially, the line of reasoning is based on straightforward estimates that use (2.2) for a given contour  $\Gamma$ . Then arguments show that constant  $c$  is only  $\Gamma$ -dependent. We skip the proof.

By virtue of (1.13) and Lemma 2.1 the *operator-norm* Trotter-Neveu-Kato theorem needs *lifting* of the strong convergence in (1.15) to the operator-norm convergence.

**Lemma 2.2.** *Let  $F : \mathbb{R}_0^+ \rightarrow \mathcal{L}(\mathfrak{H})$  be a measurable family of non-negative self-adjoint contractions such that  $F(0) = \mathbb{1}$ . Let self-adjoint family  $\{S(\tau)\}_{\tau>0}$  be defined by (1.9) and (1.15) for non-negative self-adjoint operator  $H$  in  $\mathfrak{H}$ . Then condition*

$$\lim_{\tau \rightarrow +0} \|(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1}\| = 0, \quad (2.3)$$

is satisfied if and only if

$$\lim_{\tau \rightarrow +0} \sup_{t \in \mathcal{I}} \|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| = 0, \quad (2.4)$$

for any closed interval  $\mathcal{I} \subset \mathbb{R}^+$ .

*Proof.* A straightforward computation shows that

$$\begin{aligned} & (\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1} \\ &= t(\mathbb{1} + S(\tau))(\mathbb{1} + tS(\tau))^{-1} [(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1}] (\mathbb{1} + H)(\mathbb{1} + tH)^{-1}. \end{aligned} \quad (2.5)$$

Here we used that if  $t > 0$  and  $\tau > 0$  then for self-adjoint operator  $S(\tau)$  the closure

$$\overline{(\mathbb{1} + tS(\tau))^{-1}(\mathbb{1} + S(\tau))} = (\mathbb{1} + S(\tau))(\mathbb{1} + tS(\tau))^{-1}.$$

For these values of arguments  $t$  and  $\tau$  we obtain estimates:

$$\begin{aligned} \|(\mathbb{1} + S(\tau))(\mathbb{1} + tS(\tau))^{-1}\| &\leq (1 + 2/t), \\ \|(\mathbb{1} + H)(\mathbb{1} + tH)^{-1}\| &\leq (1 + 2/t). \end{aligned}$$

If  $\mathcal{I}$  is a closed interval of  $\mathbb{R}^+$ , e.g.,  $\mathcal{I} := [a, b]$  for  $0 < a < b < \infty$ , then by (2.5)

$$\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \leq b(1 + 2/a)^2 \|(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1}\|, \quad (2.6)$$

for  $t \in [a, b]$  and  $\tau > 0$ . By (2.3) the estimate (2.6) yields (2.4). The converse is obvious.  $\square$

**Theorem 2.3.** Let  $F : \mathbb{R}_0^+ \rightarrow \mathcal{L}(\mathfrak{H})$  be a measurable family of non-negative self-adjoint contractions such that  $F(0) = \mathbb{1}$ . Let self-adjoint family  $\{S(\tau)\}_{\tau>0}$  be defined by (1.9) and (1.15) for self-adjoint operator  $H \geq 0$  in  $\mathfrak{H}$ . Then we have

$$\limsup_{n \rightarrow \infty} \sup_{t \in \mathcal{I}} \|F(t/n)^n - e^{-tH}\| = 0, \quad (2.7)$$

for any closed interval  $\mathcal{I} \subset \mathbb{R}^+$ , if and only if the family  $\{S(\tau)\}_{\tau>0}$  satisfies condition (2.3).

*Proof.* For  $t > 0$  and  $n \geq 1$  we get estimate

$$\|F(t/n)^n - e^{-tH}\| \leq \|F(t/n)^n - e^{-tS(t/n)}\| + \|e^{-tS(t/n)} - e^{-tH}\|. \quad (2.8)$$

Then (1.13) and (2.8) imply

$$\|F(t/n)^n - e^{-tH}\| \leq \frac{1}{n} + \|e^{-tS(t/n)} - e^{-tH}\|, \quad t > 0, \quad n \geq 1. \quad (2.9)$$

Note that by Lemma 2.1 there is a constant  $c > 0$  such that for  $t > 0$

$$\|e^{-tS(t/n)} - e^{-tH}\| \leq c \|(\mathbb{1} + tS(t/n))^{-1} - (\mathbb{1} + tH)^{-1}\|. \quad (2.10)$$

Inserting the estimate (2.10) into (2.9) we obtain

$$\|F(t/n)^n - e^{-tH}\| \leq \frac{1}{n} + c \|(\mathbb{1} + tS(t/n))^{-1} - (\mathbb{1} + tH)^{-1}\|, \quad (2.11)$$

Then (2.11) and Lemma 2.2 yield (2.7).

Conversely, let us assume (2.7). For  $t > 0$  and  $n \geq 1$  we have estimate

$$\|e^{-tS(t/n)} - e^{-tH}\| \leq \|F(t/n)^n - e^{-tH}\| + \|F(t/n)^n - e^{-tS(t/n)}\|,$$

and by (1.13)

$$\|e^{-tS(t/n)} - e^{-tH}\| \leq \|F(t/n)^n - e^{-tH}\| + \frac{1}{n}. \quad (2.12)$$

Then by assumption (2.7) for any closed interval  $\mathcal{I} \subset \mathbb{R}^+$  the estimate (2.12) yields

$$\limsup_{n \rightarrow \infty} \sup_{t \in \mathcal{I}} \|e^{-tS(t/n)} - e^{-tH}\| = 0.$$

Hence,  $\lim_{n \rightarrow \infty} \|e^{-tS(t/n)} - e^{-tH}\| = 0$  implies  $\lim_{\tau \rightarrow +0} \|e^{-tS(\tau)} - e^{-tH}\| = 0$  for any  $t > 0$ .

Now, using representation:

$$(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1} = \int_0^\infty ds e^{-s} (e^{-sS(\tau)} - e^{-sH}), \quad (2.13)$$

we obtain the estimate

$$\|(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1}\| \leq \int_0^\infty ds e^{-s} \|e^{-sS(\tau)} - e^{-sH}\|. \quad (2.14)$$

Let  $\Phi_\tau(s) := e^{-s} \|e^{-sS(\tau)} - e^{-sH}\|$ . Since  $S(\tau) \geq 0$  and  $H \geq 0$ , one gets  $\Phi_\tau(s) \leq 2e^{-s} \in L^1(\mathbb{R}^+)$  and  $\lim_{\tau \rightarrow +0} \Phi_\tau(s) = 0$ . Then  $\lim_{\tau \rightarrow +0}$  in the right-hand side of (2.14) is zero by the Lebesgue dominated convergence theorem, that yields (2.3).  $\square$

Extension of this statement to *any* bounded interval  $\mathcal{J} \subset \mathbb{R}_0^+$  (cf. Remark 1.8(b)) needs a *uniform* operator-norm extension of the Trotter-Neveu-Kato theorem, that we present below.

**Theorem 2.4.** *Let  $F : \mathbb{R}_0^+ \rightarrow \mathcal{L}(\mathfrak{H})$  be a measurable family of non-negative self-adjoint contractions such that  $F(0) = \mathbb{1}$ . Let self-adjoint family  $\{S(\tau)\}_{\tau>0}$  be defined by (1.9) and (1.15), for self-adjoint operator  $H \geq 0$  in  $\mathfrak{H}$ . Then the convergence*

$$\lim_{\tau \rightarrow +0} \sup_{t \in \mathcal{J}} \|e^{-tS(\tau)} - e^{-tH}\| = 0, \quad (2.15)$$

holds for any bounded interval  $\mathcal{J} \subset \mathbb{R}_0^+$  if and only if the condition

$$\lim_{\tau \rightarrow +0} \sup_{t \in \mathcal{J}} \|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| = 0, \quad (2.16)$$

is valid for any bounded interval  $\mathcal{J} \subset \mathbb{R}_0^+$ .

*Proof.* By conditions of theorem and by Lemma 2.1 we obtain from (2.10) the estimate

$$\sup_{t \in \mathcal{J}} \|e^{-tS(\tau)} - e^{-tH}\| \leq c \sup_{t \in \mathcal{J}} \|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\|,$$

for  $\tau > 0$  and for any bounded interval  $\mathcal{J} \subset \mathbb{R}_0^+$ . This estimate and condition (2.16) imply the convergence in (2.15).

Conversely, assume (2.15). Note that by representation (2.13) one gets for  $t \geq 0$ :

$$(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1} = \int_0^\infty ds e^{-s} (e^{-stS(\tau)} - e^{-stH}).$$

This yields the estimate

$$\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \leq \int_0^\infty ds e^{-s} \|e^{-stS(\tau)} - e^{-stH}\|,$$

for  $\tau > 0$  and  $t \geq 0$ .

Now, let  $0 < \varepsilon < 1$  and let  $N_\varepsilon := -\ln(\varepsilon/2)$ . Then

$$\int_{N_\varepsilon}^\infty ds e^{-s} \|e^{-stS(\tau)} - e^{-stH}\| \leq \varepsilon,$$

for  $\tau > 0$  and  $t \geq 0$ . Hence,

$$\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \leq \int_0^{N_\varepsilon} ds e^{-s} \|e^{-stS(\tau)} - e^{-stH}\| + \varepsilon,$$

that for any bounded interval  $\mathcal{J} \subset \mathbb{R}_0^+$  and  $\tau > 0$  yields

$$\sup_{t \in \mathcal{J}} \|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \leq \sup_{t \in \mathcal{J} \wedge s \in [0, N_\varepsilon]} \|e^{-stS(\tau)} - e^{-stH}\| + \varepsilon.$$

Applying now (2.15) we obtain

$$\lim_{\tau \rightarrow +0} \sup_{t \in \mathcal{J}} \|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \leq \varepsilon,$$

for any  $\varepsilon > 0$ . This completes the proof of (2.16).  $\square$



Now we are in position to improve Theorem 2.3. We relax the restriction to closed intervals  $\mathcal{I} \subset \mathbb{R}^+$  to condition on any bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$ .

**Theorem 2.5.** *Let  $F : \mathbb{R}_0^+ \rightarrow \mathcal{L}(\mathfrak{H})$  be a measurable family of non-negative self-adjoint contractions such that  $F(0) = \mathbb{1}$ . Let self-adjoint family  $\{S(\tau)\}_{\tau>0}$  be defined by (1.9) and (1.15) for self-adjoint operator  $H \geq 0$  in  $\mathfrak{H}$ . Then*

$$\limsup_{n \rightarrow \infty} \sup_{t \in \mathcal{I}} \|F(t/n)^n - e^{-tH}\| = 0, \quad (2.17)$$

for any bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$  if and only if

$$\limsup_{n \rightarrow \infty} \sup_{t \in \mathcal{I}} \|(\mathbb{1} + tS(t/n))^{-1} - (\mathbb{1} + tH)^{-1}\| = 0, \quad (2.18)$$

is satisfied for any bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$ .

*Proof.* By (2.11) and by assumption (2.18) we obtain the limit (2.17).

Conversely, using (2.12) and assumption (2.17) one gets (2.15) for  $\tau = t/n$  and for any bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$ . Then application of Theorem 2.4 yields (2.18).  $\square$

### 3 Chernoff product formula: rate of the operator-norm convergence

Theorem 2.4 admits further improvements. They allow to establish estimates for the *rate* of operator-norm convergence in (2.15) under certain conditions in (2.16). Our next *Note* concerns the estimates of the convergence rate in Theorem 2.5. Recall that the first result in this direction was due to [IT01] (Lemma 2.1).

**Lemma 3.1.** *Let  $F : \mathbb{R}_0^+ \rightarrow \mathcal{L}(\mathfrak{H})$  be a measurable family of non-negative self-adjoint contractions such that  $F(0) = \mathbb{1}$ . Let self-adjoint family  $\{S(\tau)\}_{\tau>0}$  be defined by (1.9) and (1.15) for self-adjoint operator  $H \geq 0$  in  $\mathfrak{H}$ .*

(i) *If  $\rho \in (0, 1]$  and there is a constant  $M_\rho > 0$  such that the estimate*

$$\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \leq M_\rho \left(\frac{\tau}{t}\right)^\rho, \quad (3.1)$$

*holds for  $\tau, t \in (0, 1]$  and  $0 < \tau \leq t$ , then there is a constant  $c_\rho > 0$  such that the estimate*

$$\|F(\tau)^{t/\tau} - e^{-tH}\| \leq c_\rho \left(\frac{\tau}{t}\right)^\rho, \quad (3.2)$$

*is valid for  $\tau, t \in (0, 1]$  with  $0 < \tau \leq t$ .*

(ii) *If  $\rho \in (0, 1)$  and there is a constant  $c_\rho$  such that (3.2) holds, then there is a constant  $M_\rho > 0$  such that the estimate (3.1) is valid.*

*Proof.* (i) By Lemma 2.1 there is a constant  $c > 0$  such that

$$\|e^{-tS(\tau)} - e^{-tH}\| \leq c \|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\|, \quad (3.3)$$

for  $\tau, t > 0$ . Then (3.1), for  $\tau, t \in (0, 1]$  with  $0 < \tau \leq t$ , yields, cf. Theorem 2.4,

$$\|e^{-tS(\tau)} - e^{-tH}\| \leq c M_\rho \left(\frac{\tau}{t}\right)^\rho.$$

By definition (1.9) and inequality (1.13)

$$\|F(\tau)^{t/\tau} - e^{-tS(\tau)}\| \leq \frac{\tau}{t}. \quad (3.4)$$

Therefore, by estimate

$$\|F(\tau)^{t/\tau} - e^{-tH}\| \leq \|F(\tau)^{t/\tau} - e^{-tS(\tau)}\| + \|e^{-tS(\tau)} - e^{-tH}\|, \quad (3.5)$$

we obtain for  $\tau, t \in (0, 1]$  with  $0 < \tau \leq t$

$$\|F(\tau)^{t/\tau} - e^{-tH}\| \leq \frac{\tau}{t} + c M_\rho \left(\frac{\tau}{t}\right)^\rho.$$

Since for  $\rho \in (0, 1]$  one has  $\tau/t \leq (\tau/t)^\rho$ ,

$$\|F(\tau)^{t/\tau} - e^{-tH}\| \leq (1 + c M_\rho) \left(\frac{\tau}{t}\right)^\rho. \quad (3.6)$$

Setting  $c_\rho := 1 + c M_\rho$ , we prove (3.2) for  $\rho \in (0, 1]$ .

(ii) To prove (3.1) we use the identity:

$$(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1} = \sum_{n=0}^{\infty} \int_n^{n+1} dx e^{-x} (e^{-xtS(\tau)} - e^{-xtH}),$$

here  $\tau, t > 0$ . Substitution  $x = y + n$  yields

$$\begin{aligned} (\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1} = \\ \sum_{n=0}^{\infty} e^{-n} \int_0^1 dy e^{-y} (e^{-(y+n)tS(\tau)} - e^{-(y+n)tH}), \end{aligned}$$

that gives the representation

$$\begin{aligned} (\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1} = \\ \sum_{n=0}^{\infty} e^{-n} \left\{ \left( \sum_{k=0}^{n-1} e^{-ktS(\tau)} (e^{-tS(\tau)} - e^{-tH}) e^{-(n-k-1)tH} \right) \int_0^1 dy e^{-y} e^{-ytS(\tau)} + \right. \\ \left. e^{-ntH} \int_0^1 dy e^{-y} (e^{-ytS(\tau)} - e^{-ytH}) \right\}. \end{aligned}$$

Hence, we obtain the estimate

$$\begin{aligned} \|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \leq \\ \sum_{n=0}^{\infty} e^{-n} \left\{ n \|e^{-tS(\tau)} - e^{-tH}\| + \int_0^1 dy e^{-y} \|e^{-ytS(\tau)} - e^{-ytH}\| \right\}. \end{aligned} \quad (3.7)$$

Note that assumption (3.2) and estimate (3.4) yield for  $\tau, t \in (0, 1]$ , with  $0 < \tau \leq t$ ,

$$\|e^{-tS(\tau)} - e^{-tH}\| \leq (1 + c_\rho) \left(\frac{\tau}{t}\right)^\rho. \quad (3.8)$$

To treat the last term in (3.7) we use decomposition

$$\begin{aligned} \int_0^1 dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\| = \\ \int_{\tau/t}^1 dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\| + \int_0^{\tau/t} dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\|. \end{aligned} \quad (3.9)$$

Since by (3.8) for  $\tau, t, y \in (0, 1]$  and  $\tau/t \leq y$

$$\|e^{-yS(\tau)} - e^{-yH}\| \leq (1 + c_\rho) \left(\frac{\tau}{ty}\right)^\rho,$$

we obtain for  $0 < \tau \leq t$  the estimate

$$\int_{\tau/t}^1 dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\| \leq (1 + c_\rho) \int_0^1 dy e^{-y} y^{-\rho} \left(\frac{\tau}{t}\right)^\rho. \quad (3.10)$$

Moreover, for  $\rho < 1$  one obviously gets

$$\int_0^{\tau/t} dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\| \leq 2 \left(\frac{\tau}{t}\right)^\rho. \quad (3.11)$$

Taking into account (3.10) and (3.11) we obtain from (3.9):

$$\int_0^1 dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\| \leq \left[ (1 + c_\rho) \int_0^1 dy e^{-y} y^{-\rho} + 2 \right] \left(\frac{\tau}{t}\right)^\rho, \quad (3.12)$$

for  $\tau, t \in (0, 1]$ , with  $0 < \tau \leq t$ .

Finally, by virtue of (3.8) and (3.12) we get for (3.7)

$$\begin{aligned} \|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \leq \\ \sum_{n=0}^{\infty} e^{-n} \left\{ n(1 + c_\rho) + (1 + c_\rho) \int_0^1 dy e^{-y} y^{-\rho} + 2 \right\} \left(\frac{\tau}{t}\right)^\rho. \end{aligned}$$

Now, setting

$$M_\rho := \sum_{n=0}^{\infty} e^{-n} \left\{ n(1 + c_\rho) + (1 + c_\rho) \int_0^1 dy e^{-y} y^{-\rho} + 2 \right\}$$

we obtain estimate (3.1). □

**Remark 3.2.** In Lemma 3.1(i) it is shown that for  $\rho = 1$  the condition (3.1) implies (3.2). But it is *unclear* for converse since Lemma 3.1(ii) does not cover this case.

The next assertion extends the result of Lemma 3.1(i) to *any* bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$ .

**Theorem 3.3.** Let  $F : \mathbb{R}_0^+ \rightarrow \mathcal{L}(\mathfrak{H})$  be a measurable family of non-negative self-adjoint contractions such that  $F(0) = \mathbf{1}$ . Let self-adjoint family  $\{S(\tau)\}_{\tau>0}$  be defined by (1.9) and (1.15) for self-adjoint operator  $H \geq 0$  in  $\mathfrak{H}$ .

If for some  $\rho \in (0, 1]$  there is a constant  $M_\rho > 0$  such that the estimate (3.1) holds for  $\tau, t \in (0, 1]$  and  $0 < \tau \leq t$ , then for any bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$  there is a constant  $c_\rho^\mathcal{I} > 0$  such that the estimate

$$\sup_{t \in \mathcal{I}} \|F(t/n)^n - e^{-tH}\| \leq c_\rho^\mathcal{I} \frac{1}{n^\rho}, \quad (3.13)$$

holds for  $n \geq 1$ .

*Proof.* Let  $N \in \mathbb{N}$  such that  $\mathcal{I} \subseteq [0, N]$ . Then representation

$$\begin{aligned} F(t/Nn)^{Nn} - e^{-tH} \\ = \sum_{k=0}^{N-1} e^{-k tH/N} (F(t/Nn)^n - e^{-tH/N}) F(t/Nn)^{(N-1-k)n}, \end{aligned}$$

for  $n \geq 1$ , yields the estimate

$$\|F(t/Nn)^{Nn} - e^{-tH}\| \leq N \|F(t/Nn)^n - e^{-tH/N}\|. \quad (3.14)$$

Let  $t' := t/N \leq 1$  and  $\tau' := t'/n \leq 1$ . Then  $t \leq N$  and  $0 < \tau' \leq t' \leq 1$ . By Lemma 3.1(i) we obtain

$$\|F(t/Nn)^n - e^{-tH/N}\| = \|F(\tau')^{t'/\tau'} - e^{-t'H}\| \leq c_\rho \left(\frac{\tau'}{t'}\right)^\rho,$$

and by (3.14) the estimate

$$\|F(t/Nn)^{Nn} - e^{-tH}\| \leq c_\rho N \left(\frac{\tau'}{t'}\right)^\rho.$$

Let  $n' := Nn \geq 1$ . Then

$$\|F(t/n')^{n'} - e^{-tH}\| \leq c_\rho N^{1+\rho} \left(\frac{1}{n'}\right)^\rho, \quad t \in [0, N].$$

Setting  $c_\rho^{[0, N]} := c_\rho N^{1+\rho}$ , we prove the theorem for  $\mathcal{I} = [0, N]$ . Since for any bounded interval  $\mathcal{I}$  one can always find a  $N \in \mathbb{N}$  such that  $\mathcal{I} \subseteq [0, N]$ , this completes the proof.  $\square$

To extend Theorem 3.3 to  $\mathcal{I} = \mathbb{R}_0^+$  one needs conditions when the values of  $\tau, t$  are allowed to be unbounded.

**Theorem 3.4.** Let  $F : \mathbb{R}_0^+ \rightarrow \mathcal{L}(\mathfrak{H})$  be a measurable family of non-negative self-adjoint contractions such that  $F(0) = \mathbf{1}$ . Let self-adjoint family  $\{S(\tau)\}_{\tau>0}$  be defined by (1.9) and (1.15) for self-adjoint operator  $H \geq 0$  in  $\mathfrak{H}$ .

If for some  $\rho \in (0, 1]$  there is a constant  $M_\rho > 0$  such that the estimate (3.1) holds for  $0 < \tau \leq t < \infty$ , then there exists a constant  $c_\rho^{\mathbb{R}_0^+} > 0$  such that for  $\tau = t/n$  the estimate

$$\sup_{t \in \mathbb{R}_0^+} \|F(t/n)^n - e^{-tH}\| \leq c_\rho^{\mathbb{R}_0^+} \frac{1}{n^\rho}, \quad (3.15)$$

holds for  $n \geq 1$ .

*Proof.* The arguments ensuring that (3.3) yields estimate (3.6) go through verbatim if we assume (3.1) for  $0 < \tau \leq t < \infty$ . Then setting  $\tau := t/n$ ,  $n \in \mathbb{N}$  we deduce from (3.6)

$$\|F(t/n)^n - e^{-tH}\| \leq c_\rho^{\mathbb{R}^+} \frac{1}{n^\rho}, \quad n \geq 1,$$

for  $t \in \mathbb{R}_0^+$ , where  $c_\rho^{\mathbb{R}^+} := 1 + cM_\rho$ . So, this proves (3.15).  $\square$

**Remark 3.5.** We *Note* that in Theorem 2.5 we established the self-adjoint operator-norm convergent *extension* of the Chernoff product formula for any bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$  under *t-dependent* condition (2.18), which is necessary and sufficient.

We *Note* that Theorem 3.3 and Theorem 3.4 prove self-adjoint operator-norm Chernoff product formula in  $\mathbb{R}_0^+$  with *estimate* of the *rate* of convergence. They are also based on *t-dependent* fractional power condition (3.1), which is necessary and sufficient for  $\rho \in (0, 1)$ .

Our next *Note* is that *t-dependence* in assumption (3.1) for  $\rho = 1$  can be relaxed. The assertion below extends to  $\mathbb{R}_0^+$  (with convergence rate) the established in Theorem 2.3 operator-norm convergence of the self-adjoint Chernoff product formula for bounded interval  $\mathcal{I} \subset \mathbb{R}^+$ . It yields an operator-norm version of original Chernoff product formula, see Proposition 1.2.

**Theorem 3.6.** *Let  $F : \mathbb{R}_0^+ \rightarrow \mathcal{L}(\mathfrak{H})$  be a measurable family of non-negative self-adjoint contractions such that  $F(0) = \mathbb{1}$ . Let self-adjoint family  $\{S(\tau)\}_{\tau>0}$  be defined by (1.9) and (1.15) for self-adjoint operator  $H \geq 0$  in  $\mathfrak{H}$ .*

*If there is a constant  $M_1 > 0$  such that the estimate*

$$\|(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1}\| \leq M_1 \tau, \quad (3.16)$$

*holds for  $\tau \in (0, 1]$ , then for any bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$  there is a constant  $c_1^{\mathcal{I}} > 0$  such that the estimate*

$$\sup_{t \in \mathcal{I}} \|F(t/n)^n - e^{-tH}\| \leq c_1^{\mathcal{I}} \frac{1}{n}, \quad (3.17)$$

*holds for  $n \geq 1$ .*

*Proof.* By virtue of (2.5) and (3.16) we obtain the estimate

$$\begin{aligned} & \|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \\ & \leq M_1 \tau t \|(\mathbb{1} + S(\tau))(\mathbb{1} + tS(\tau))^{-1}\| \|(\mathbb{1} + H)(\mathbb{1} + tH)^{-1}\|. \end{aligned} \quad (3.18)$$

Then (3.18), together with estimates  $\|(\mathbb{1} + S(\tau))(\mathbb{1} + tS(\tau))^{-1}\| \leq 1/t$  and  $\|(\mathbb{1} + H)(\mathbb{1} + tH)^{-1}\| \leq 1/t$ , for self-adjoint  $S(\tau)$  and  $H$ , imply for  $0 < \tau \leq t \leq 1$  and  $\rho = 1$  the estimate (3.1) and thus (3.2). Finally, applying Theorem 3.3 we extend the proof to any bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$ .  $\square$

Now we extend Theorem 3.6 for condition (3.16), to infinite interval  $\mathcal{I} = \mathbb{R}_0^+$ . To this end, similar to Theorem 3.4, one needs additional assumption valid on infinite *t-intervals*.

**Theorem 3.7.** *Let in addition to conditions of Theorem 3.6 operator  $H \geq \mu \mathbb{1}$ ,  $\mu > 0$  and for any  $\varepsilon > 0$  there exists a  $\delta_\varepsilon \in (0, 1)$  such that*

$$0 \leq F(\tau) \leq (1 - \delta_\varepsilon) \mathbb{1}, \quad (3.19)$$

*is valid for  $\tau \geq \varepsilon$ . If there is a constant  $M_1 > 0$  such that (3.16) holds for  $\tau \in (0, \varepsilon \leq 1)$ , then there exists a constant  $c_1^{\mathbb{R}^+} > 0$  such that (3.17) is valid for infinite interval  $\mathcal{I} = \mathbb{R}_0^+$ .*

*Proof.* Since (3.16) implies the resolvent-norm convergence of  $\{S(\tau)\}_{\tau>0}$ , when  $\tau \rightarrow +0$ , and since  $H \geq \mu \mathbb{1}$ , there exists  $0 < \mu_\varepsilon \leq \mu$  such that  $S(\tau) \geq \mu_\varepsilon \mathbb{1}$  for  $\tau \in (0, \varepsilon)$ , where  $\varepsilon$  is sufficiently *small*.

On the other hand, (3.18) for self-adjoint  $S(\tau)$  and  $H$ , yields

$$\begin{aligned} & \|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| = \\ & \leq M_1 \frac{\tau}{t} \|(\mathbb{1} + S(\tau))(\mathbb{1}/t + S(\tau))^{-1}\| \|(\mathbb{1} + H)(\mathbb{1}/t + H)^{-1}\|, \end{aligned} \quad (3.20)$$

for  $t > 0$ . Since  $S(\tau) \geq \mu_\varepsilon \mathbb{1}$ ,  $\tau \in (0, \varepsilon)$ , and  $H \geq \mu \mathbb{1}$ , we obtain estimates

$$\begin{aligned} & \|(\mathbb{1} + S(\tau))(\mathbb{1}/t + S(\tau))^{-1}\| \leq \frac{1 + \mu_\varepsilon}{\mu_\varepsilon}, \\ & \|(\mathbb{1} + H)(\mathbb{1}/t + H)^{-1}\| \leq \frac{1 + \mu}{\mu}. \end{aligned}$$

By (3.20) these estimates give

$$\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \leq M_1^{\mathbb{R}^+} \frac{\tau}{t}, \quad (3.21)$$

for  $\tau \in (0, \varepsilon)$  and  $t > 0$ . Here  $M_1^{\mathbb{R}^+} := M_1(1 + \mu_\varepsilon)(1 + \mu)/\mu_\varepsilon \mu$ .

Note that if  $\tau/t \leq 1$ , then by (3.4)

$$\|F(\tau)^{t/\tau} - e^{-tS(\tau)}\| \leq \frac{\tau}{t}, \quad 0 < \tau \leq t < \infty.$$

Therefore, (3.3), (3.5) and (3.21), which are valid for  $\tau \in (0, \varepsilon \leq 1)$ , allow to use the result (3.17) of Theorem 3.6 for the case  $0 < \tau \leq t < \infty$ , and  $\tau = t/n$ ,  $n \geq 1$ . This yields

$$\|F(t/n)^n - e^{-tH}\| \leq \widehat{c}_1^{\mathbb{R}^+} \frac{1}{n}, \quad (3.22)$$

for bounded interval:  $t \in [0, \varepsilon n)$ , and for  $\widehat{c}_1^{\mathbb{R}^+} := 1 + cM_1^{\mathbb{R}^+}$ .

Now let  $t \geq \varepsilon n$ . Then by assumption (3.19) we have

$$\|F(t/n)^n\| \leq (1 - \delta_\varepsilon)^n = e^{n \ln(1 - \delta_\varepsilon)}, \quad t \geq n\varepsilon. \quad (3.23)$$

Note that  $H \geq \mu I$  implies  $\|e^{-tH}\| \leq e^{-n\varepsilon\mu}$  for  $t \geq n\varepsilon$ . This together with (3.22) and (3.23) yield for a small  $\varepsilon > 0$ , cf. (3.21), the estimate

$$\|F(t/n)^n - e^{-tH}\| \leq \widehat{c}_1^{\mathbb{R}^+} \frac{1}{n} + e^{n \ln(1 - \delta_\varepsilon)} + e^{-n\varepsilon\mu},$$

valid for *any*  $t \geq 0$ .

Since  $\widetilde{c}_1 := \sup_{n \geq 1} n(e^{n \ln(1 - \delta_\varepsilon)} + e^{-n\varepsilon\mu}) < \infty$ , there exists constant  $c_1^{\mathbb{R}^+} := \widehat{c}_1^{\mathbb{R}^+} + \widetilde{c}_1$  such that (3.17) is valid for  $n \geq 1$  and infinite interval  $\mathcal{I} = \mathbb{R}_0^+$   $\square$

## 4 Nonsself-adjoint operator-norm Chernoff product formula

The results on the *nonsself-adjoint* Chernoff product formula in operator-norm topology are more restricted. The most of them concern the *quasi-sectorial* contractions [CZ01].

**Definition 4.1.** A contraction  $F$  on the Hilbert space  $\mathfrak{H}$  is called *quasi-sectorial* with semi-angle  $\alpha \in [0, \pi/2)$  with respect to the vertex at  $z = 1$ , if its numerical range  $W(F) \subseteq D_\alpha$ , where closed domain

$$D_\alpha := \{z \in \mathbb{C} : |z| \leq \sin \alpha\} \cup \{z \in \mathbb{C} : |\arg(1-z)| \leq \alpha \text{ and } |z-1| \leq \cos \alpha\}. \quad (4.1)$$

The limits:  $\alpha = 0$  and  $\alpha = \pi/2 - 0$ , correspond, respectively, to non-negative self-adjoint contractions and to general contractions.

A characterisation of quasi-sectorial contraction *semigroups* is due to [Zag08], [ArZ10].

**Proposition 4.2.**  $C_0$ -semigroup  $\{e^{-tH}\}_{t \geq 0}$  is, for  $t > 0$ , a family of quasi-sectorial contractions with  $W(e^{-tH}) \subseteq D_\alpha$ , if and only if generator  $H$  is an  $m$ -sectorial operator with  $W(H) \subset S_\alpha$ , the open sector with semi-angle  $\alpha \in [0, \pi/2)$  and vertex at  $z = 0$ .

Note that if operator  $F$  is a quasi-sectorial contraction and  $W(F) \subseteq D_\alpha$ , then  $\mathbb{1} - F$  is also  $m$ -sectorial operator with vertex  $z = 0$  and semi-angle  $\alpha$ . Using the Riesz-Dunford functional calculus one obtains estimate

$$\|F^n(\mathbb{1} - F)\| \leq \frac{K}{n+1}, \quad n \in \mathbb{N}. \quad (4.2)$$

The estimate (4.2) allows to go beyond the Chernoff  $\sqrt{n}$ -Lemma (1.5) and to establish the  $(1/\sqrt[3]{n})$ -Theorem [Zag17].

**Proposition 4.3.** Let  $F$  be a quasi-sectorial contraction on  $\mathfrak{H}$  with numerical range  $W(F) \subseteq D_\alpha$  for  $\alpha \in [0, \pi/2)$ . Then

$$\left\| F^n - e^{n(F-\mathbb{1})} \right\| \leq \frac{M}{n^{1/3}}, \quad n \in \mathbb{N}, \quad (4.3)$$

where  $M = 2K + 2$  and  $K$  is defined by (4.2).

Next we recall nonsself-adjoint operator-norm extension of the Trotter-Neveu-Kato convergence theorem for *quasi-sectorial* contraction semigroups [CZ01] (Lemma 4.1):

**Proposition 4.4.** Let  $\{S(\tau)\}_{\tau > 0}$  be a family of  $m$ -sectorial operators with  $W(S(\tau)) \subseteq S_\alpha$  for some  $\alpha \in [0, \pi/2)$  and for all  $\tau > 0$ . Let  $H$  be an  $m$ -sectorial operator with  $W(H) \subset S_\alpha$ . Then the following conditions are equivalent:

- (a)  $\lim_{\tau \rightarrow +0} \|(\zeta \mathbb{1} + S(\tau))^{-1} - (\zeta \mathbb{1} + H)^{-1}\| = 0$ , for some  $\zeta \in S_{\pi-\alpha}$ ;
- (b)  $\lim_{\tau \rightarrow +0} \|e^{-tS(\tau)} - e^{-tH}\| = 0$ , for  $t$  in a subset of  $\mathbb{R}^+$  having a limit point.

Therefore, the estimate (4.3) together with Proposition 4.4 and inequality

$$\|F(t/n)^n - e^{-tH}\| \leq \|F(t/n)^n - e^{-tS(t/n)}\| + \|e^{-tS(t/n)} - e^{-tH}\|, \quad (4.4)$$

yield *nonsel-adjoint* operator-norm version of the Chernoff product formula for quasi-sectorial contractions (cf. Proposition 1.2 in [CZ01]):

**Proposition 4.5.** *Let  $\{F(\tau)\}_{\tau \geq 0}$  be a family of uniformly quasi-sectorial contractions on a Hilbert space  $\mathfrak{H}$ , that is,  $W(F(\tau)) \subseteq D_\alpha$  (4.1), for all  $\tau > 0$ . Let family  $\{S(\tau)\}_{\tau > 0}$  be defined by (1.9) and (1.15). If  $H$  is  $m$ -sectorial operator with  $W(H) \subseteq S_\alpha$ , then*

$$\lim_{n \rightarrow \infty} \|F(t/n)^n - e^{-tH}\| = 0, \quad \text{for } t > 0, \quad (4.5)$$

if and only if

$$\lim_{\tau \rightarrow +0} \|(\zeta \mathbb{1} + S(\tau))^{-1} - (\zeta \mathbb{1} + H)^{-1}\| = 0, \quad \text{for some } \zeta \in S_{\pi-\alpha}. \quad (4.6)$$

**Remark 4.6.** Since semigroups  $\{e^{-zS(\tau)}\}_{z \in S_{\pi/2-\alpha}}$ ,  $\tau > 0$ , and  $\{e^{-zH}\}_{z \in S_{\pi/2-\alpha}}$  are holomorphic in sector  $S_{\pi/2-\alpha}$ , proof of the Chernoff product formula (4.5) for nonself-adjoint quasi-sectorial contractions is based on the Riesz-Dunford functional calculus. To establish the operator-norm convergence *without* rate it successfully (although *not* completely, since  $t > 0$ ) substitutes the self-adjointness in the proofs in Sections 2 and 3.

We use this calculus to prove the operator-norm Chernoff product formula for quasi-sectorial contractions *with* estimate of the rate of convergence. We *Note* that this is a generalisation of the Chernoff product formulae proven respectively in Theorem 3.6 (for self-adjoint case) and in Proposition 4.5 (without rate of convergence).

**Theorem 4.7.** *Let  $\{F(\tau)\}_{\tau \geq 0}$  be a family of uniformly quasi-sectorial contractions on a Hilbert space  $\mathfrak{H}$  and let  $\{S(\tau)\}_{\tau > 0}$  be a family of  $m$ -sectorial operators defined by (1.9) and (1.15) for  $m$ -sectorial operator  $H$  with  $W(H) \subseteq S_\alpha$ . If there is  $L > 0$  such that estimate*

$$\|(\zeta \mathbb{1} + S(\tau))^{-1} - (\zeta \mathbb{1} + H)^{-1}\| \leq \frac{L \tau}{\text{dist}(\zeta, -S_\alpha)}, \quad (4.7)$$

holds for  $\zeta \in S_{\pi-\alpha}$ , then for any bounded interval  $\mathcal{I} \subset \mathbb{R}^+$  there is a constant  $C^{\mathcal{I}} > 0$  such that estimate

$$\sup_{t \in \mathcal{I}} \|F(t/n)^n - e^{-tH}\| \leq C^{\mathcal{I}} \frac{1}{n^{1/3}}, \quad (4.8)$$

holds for  $n \geq 1$ .

*Proof.* Estimation of the last term in inequality (4.4). Since by (1.9) and conditions on  $\{F(\tau)\}_{\tau \geq 0}$  operators  $\{S(\tau)\}_{\tau > 0}$  are  $m$ -sectorial with  $W(S(\tau)) \subseteq S_\alpha$ , the Riesz-Dunford formula

$$e^{-tS(\tau)} = \frac{1}{2\pi i} \int_{\Gamma} d\zeta \frac{e^{t\zeta}}{\zeta \mathbb{1} + S(\tau)},$$



defines for  $\tau > 0$  a family of holomorphic semigroups  $\tau \mapsto \{e^{-tS(\tau)}\}_{t \in S_{\pi/2-\alpha}}$ . Here  $\Gamma \subset S_{\pi-\alpha}$  is a positively-oriented closed (at infinity) contour in  $\mathbb{C}$  around  $-S_\alpha$ . The same is true for  $m$ -sectorial operator  $H$  since  $W(H) \subseteq S_\alpha$ :

$$e^{-tH} = \frac{1}{2\pi i} \int_{\Gamma} d\zeta \frac{e^{t\zeta}}{\zeta \mathbb{1} + H}.$$

We define  $\Gamma := \overline{\Gamma_\varepsilon} \cup \Gamma_\delta \cup \Gamma_\varepsilon$ , where the arc  $\Gamma_\delta = \{z \in \mathbb{C} : |z| = \delta > 0, |\arg z| \leq \pi - \alpha - \varepsilon\}$  (for  $0 < \varepsilon < \pi/2 - \alpha$ ) and  $\Gamma_\varepsilon, \overline{\Gamma_\varepsilon}$  are two conjugate radial rays with  $\Gamma_\varepsilon = \{z \in \mathbb{C} : \arg z = \pi - \alpha - \varepsilon, |z| \geq \delta\}$ . Then for  $t > 0$  one gets

$$\|e^{-tS(\tau)} - e^{-tH}\| \leq \frac{1}{2\pi} \int_{\Gamma} |d\zeta| |e^{t\zeta}| \|(\zeta \mathbb{1} + S(\tau))^{-1} - (\zeta \mathbb{1} + H)^{-1}\|. \quad (4.9)$$

Since operators  $\{S(\tau)\}_{\tau > 0}$  and  $H$  are  $m$ -sectorial with numerical ranges in open sector  $S_\alpha$ , by condition (4.7) we obtain:

$$\|(\zeta \mathbb{1} + S(\tau))^{-1} - (\zeta \mathbb{1} + H)^{-1}\| \leq \frac{L\tau}{\delta \sin \varepsilon}, \quad \text{for } \zeta \in \Gamma_\delta, \quad (4.10)$$

$$\|(\zeta \mathbb{1} + S(\tau))^{-1} - (\zeta \mathbb{1} + H)^{-1}\| \leq \frac{L\tau}{|\zeta| \sin \varepsilon}, \quad \text{for } \zeta \in \Gamma_\varepsilon \vee \overline{\Gamma_\varepsilon}. \quad (4.11)$$

Then for  $t > 0$  the estimates (4.9) and (4.10), (4.11) yield

$$\|e^{-tS(\tau)} - e^{-tH}\| \leq \frac{L\tau}{\sin \varepsilon} \left\{ e^{t\delta} + \frac{e^{-t\delta \cos(\alpha+\varepsilon)}}{\pi t \delta \cos(\alpha+\varepsilon)} \right\}. \quad (4.12)$$

Hence, inequality (4.12) proves the existence of  $N^{\mathcal{J}}$  such that for  $\tau = t/n$

$$\sup_{t \in \mathcal{J}} \|e^{-tS(t/n)} - e^{-tH}\| \leq N^{\mathcal{J}} \frac{1}{n}. \quad (4.13)$$

To estimate the first term in the right-hand side of (4.4) we use Proposition 4.3. Then

$$\|F(t/n)^n - e^{-tS(t/n)}\| \leq M \frac{1}{n^{1/3}}. \quad (4.14)$$

The inequalities (4.4) and (4.13), (4.14) prove the estimate (4.8).  $\square$

**Remark 4.8.** Note that the rate (4.8) of the operator-norm convergence of the Chernoff product formula for quasi-sectorial contractions is slower than for the self-adjoint case (3.17). This rate is limited by *non-optimal* estimate due to Proposition 4.3.

**Corollary 4.9.** Let  $\{S(\tau)\}_{\tau > 0}$  be a family of  $m$ -sectorial operators defined by (1.9) and (1.15) with  $W(S(\tau)) \subseteq S_\alpha$  for some  $\alpha \in [0, \pi/2)$ . Let  $H$  be an  $m$ -sectorial operator with  $W(H) \subseteq S_\alpha$ . Then there exist  $M' > 0$  and  $\delta' > 0$  such that

$$\|e^{-tS(\tau)} - e^{-tH}\| \leq \frac{M'}{t} e^{t\delta'} \|(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1}\|, \quad (4.15)$$

holds for positive  $\tau$  and  $t$ .

For extension of Theorem 4.7 to  $\mathbb{R}_0^+$  the estimate (4.15) suggests a weaker form of the operator-norm Trotter-Neveu-Kato theorem. We *Note* that, under *stronger* than (4.7) (the  $t$ -dependent resolvent condition, see (3.1) for  $\rho = 1$ ) we obtain a new version of Proposition 4.4.

**Theorem 4.10.** *Let  $\{F(\tau)\}_{\tau \geq 0}$  be a family of uniformly quasi-sectorial contractions on a Hilbert space  $\mathfrak{H}$ . Let  $\{S(\tau)\}_{\tau > 0}$  be a family of  $m$ -sectorial operators defined by (1.9), (1.15) with  $W(S(\tau)) \subseteq S_\alpha$  for some  $\alpha \in [0, \pi/2)$  and for  $m$ -sectorial operator  $H$  with  $W(H) \subseteq S_\alpha$ . Then estimate*

$$\sup_{t \in \mathcal{I}} \|(\zeta \mathbb{1} + tS(\tau))^{-1} - (\zeta \mathbb{1} + tH)^{-1}\| \leq \frac{L^{\mathcal{I}} \tau}{\text{dist}(\zeta, -S_\alpha)}, \quad \zeta \in S_{\pi-\alpha}, \quad (4.16)$$

holds for any interval  $\mathcal{I} \subseteq \mathbb{R}_0^+$  if and only if the condition

$$\sup_{t \in \mathcal{I}} \|e^{-tS(\tau)} - e^{-tH}\| \leq K^{\mathcal{I}} \tau, \quad (4.17)$$

is valid for any interval  $\mathcal{I} \subseteq \mathbb{R}_0^+$ .

*Proof.* Necessity of (4.17): As in the proof of Theorem 4.7 we use the Riesz-Dunford functional calculus for holomorphic semigroups to obtain estimate

$$\|e^{-tS(\tau)} - e^{-tH}\| \leq \frac{1}{2\pi} \int_{\Gamma} |dz| |e^z| \| (z\mathbb{1} + tS(\tau))^{-1} - (z\mathbb{1} + tH)^{-1} \|, \quad (4.18)$$

for the same contour  $\Gamma \subset S_{\pi-\alpha}$ . After change of variable:  $z = t\zeta$ , the right-hand side of estimate (4.18) gets the same expression as (4.12), but for  $\delta$  substituted by  $\delta/t$ . This yields (4.17) for any bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$ .

Sufficiency of (4.17): First, using the Laplace transform, we estimate:

$$\|(\zeta \mathbb{1} + tS(\tau))^{-1} - (\zeta \mathbb{1} + tH)^{-1}\| \leq \int_0^\infty ds e^{-s\text{Re} \zeta} \|e^{-stS(\tau)} - e^{-stH}\|, \quad (4.19)$$

in the half-plane  $\mathbb{C}_+ = S_{\pi/2}$ . To this aim we let  $\varepsilon \in (0, 1)$  and  $N_\varepsilon := -\ln(\varepsilon/2)$  such that for  $\zeta \in S_{\pi/2}$  and  $\tau > 0, t \geq 0$

$$\int_{N_\varepsilon}^\infty ds e^{-s\text{Re} \zeta} \|e^{-stS(\tau)} - e^{-stH}\| \leq \varepsilon.$$

Therefore, one gets

$$\|(\zeta \mathbb{1} + tS(\tau))^{-1} - (\zeta \mathbb{1} + tH)^{-1}\| \leq \int_0^{N_\varepsilon} ds e^{-s\text{Re} \zeta} \|e^{-stS(\tau)} - e^{-stH}\| + \varepsilon,$$

that by condition (4.17) for any interval  $\mathcal{I} \subseteq \mathbb{R}_0^+$  and  $\varepsilon \in (0, 1)$  yields

$$\begin{aligned} \sup_{t \in \mathcal{I}} \|(\zeta \mathbb{1} + tS(\tau))^{-1} - (\zeta \mathbb{1} + tH)^{-1}\| &\leq \\ \frac{1}{\text{Re} \zeta} \sup_{t \in \mathcal{I} \wedge s \in [0, N_\varepsilon]} \|e^{-stS(\tau)} - e^{-stH}\| + \varepsilon &\leq \frac{1}{\text{Re} \zeta} K^{\mathbb{R}_0^+} \tau + \varepsilon. \end{aligned} \quad (4.20)$$

Since  $\varepsilon$  may be arbitrary small, we obtain the estimate (4.16) for any  $\zeta \in S_{\pi/2}$ .

For extension of  $\zeta$  to sector  $S_{\pi-\alpha}$  we note that semigroups involved into estimate (4.19) are holomorphic in sector  $S_{\pi/2-\alpha}$ . Therefore, the Laplace transform is also valid for integration along the radial rays:  $s e^{i\varphi} \in S_{\pi/2-\alpha}$ . Then conditions for convergence of Laplace integrals take the form:

$$-\pi/2 < \varphi + \arg \zeta < \pi/2 \quad \wedge \quad -(\pi/2 - \alpha) < \varphi < (\pi/2 - \alpha), \quad (4.21)$$

that yields  $\arg \zeta \in S_{\pi-\alpha}$  and makes  $\operatorname{Re}(e^{i\varphi} \zeta)$  proportional to  $\operatorname{dist}(\zeta, -S_\alpha)$ .  $\square$

**Corollary 4.11.** *Linear  $\tau$ -estimate (4.17) implies*

$$\sup_{t \in \mathcal{I}} \|e^{-tS(t/n)} - e^{-tH}\| \leq K_1^{\mathcal{I}} \frac{1}{n}, \quad (4.22)$$

for any  $n \in \mathbb{N}$  and any bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$ . Then the same line of reasoning as in Theorem 4.7 yields under  $t$ -dependent resolvent condition (4.16) the estimate

$$\sup_{t \in \mathcal{I}} \|F(t/n)^n - e^{-tH}\| \leq C_1^{\mathcal{I}} \frac{1}{n^{1/3}}, \quad (4.23)$$

for  $n \geq 1$  and any bounded interval  $\mathcal{I} \subset \mathbb{R}_0^+$ .

We Note that extension (4.23) of the operator-norm Chernoff product formula for quasi-sectorial contractions on  $\mathbb{R}_0^+$  inherits the estimate of the rate of convergence established in Proposition 4.3.

## 5 Comments: Trotter-Kato product formulae

The first application of the Chernoff product formula (Proposition 1.2) was the proof of the strongly convergent Trotter product formula, see [Che68]:

$$s\text{-}\lim_{n \rightarrow \infty} (e^{-tA/n} e^{-tB/n})^n = e^{-tH}, \quad t \geq 0. \quad (5.1)$$

Here  $A$  and  $B$  are positive self-adjoint operators in a Hilbert space  $\mathfrak{H}$  with domains  $\operatorname{dom} A$  and  $\operatorname{dom} B$  such that  $\operatorname{dom} A \cap \operatorname{dom} B = \operatorname{core} H$  of the self-adjoint  $H$ . Note that operator family  $\{F(t) := e^{-tA} e^{-tB}\}_{t \geq 0}$  is not self-adjoint.

Later the operator-norm Chernoff product formula from Sections 2 and 3 was used to lift (5.1) to operator-norm topology, as well as to extend it from the *exponential* Trotter product formula to the *Trotter-Kato* product formulae for Kato functions  $\mathcal{K}$ , see [NZ98].

Recall that if a real-valued Borel measurable function  $f : [0, \infty) \rightarrow [0, 1]$  satisfying

$$0 \leq f(s) \leq 1, \quad f(0) = 1, \quad f'(+0) = -1, \quad (5.2)$$

then  $f \in \mathcal{K}$ . Different conditions on local continuity at  $t = +0$  and global behaviour on  $\mathbb{R}^+$  select subclasses of the Kato functions, see Appendix C in [Zag19]. If functions  $f, g \in \mathcal{K}$  substitute exponents in formula (5.1) then it is called the Trotter-Kato product formula.

To apply a full power of the self-adjoint Chernoff product formula (Sections 2 and 3) we symmetrise and produce *self-adjoint* family  $\{F(t) := g(tB)^{1/2}f(tA)g(tB)^{1/2}\}_{t \geq 0}$ . Let positive operators  $A$  and  $B$  be such that operator  $A + B =: H \geq \mu \mathbb{1}$  is self-adjoint. If  $F(t)$  is sufficiently smooth at  $t = +0$  and satisfy (3.19) (see [IT01] (1.2)), then (3.16) holds for  $\tau = t/n$  and Theorem 3.7 proves the operator-norm convergent symmetrised Trotter-Kato product formula:

$$\|\cdot\| - \lim_{n \rightarrow \infty} (g(tB/n)^{1/2}f(tA/n)g(tB/n)^{1/2})^n = e^{-tH}, \quad (5.3)$$

uniformly on  $\mathbb{R}_0^+$  with  $O(1/n)$  as the rate of convergence, see [IT01], [ITTZ01]. There it was also shown that this rate is *optimal*.

For proving convergence of the *nonself-adjoint* Trotter-Kato approximants, for example the simplest:  $\{(f(tA/n)g(tB/n))^n\}_{n \geq 1}$ , note that for  $n \in \mathbb{N}$  and  $t \geq 0$ :

$$(f(tA/n)g(tB/n))^n = f(tA/n)g(tB/n)^{1/2}F(t/n)^{n-1}g(tB/n)^{1/2}.$$

This representation yields:

$$\begin{aligned} \|(f(tA/n)g(tB/n))^n - e^{-tH}\| &\leq \|F(t/n)^{n-1} - e^{-tH}\| \\ &\quad + 2\|(\mathbb{1} - g(tB/n))e^{-tH}\| + \|(\mathbb{1} - f(tA/n))e^{-tH}\|. \end{aligned}$$

Then by estimates (3.22) and (4.2) we get for  $n - 1 \geq 1$

$$\|F(t/n)^{n-1} - e^{-tH}\| \leq (\widehat{c}_1^{\mathbb{R}^+} + K) \frac{1}{n}, \quad t \geq 0. \quad (5.4)$$

On the other hand, since  $f, g \in \mathcal{K}$  and  $H = A + B$ :

$$\|(\mathbb{1} - f(tA/n))e^{-tH}\| \leq C_C^A \gamma[f] \frac{1}{n} \quad \text{and} \quad \|(\mathbb{1} - g(tB/n))e^{-tH}\| \leq C_C^B \gamma[g] \frac{1}{n}, \quad (5.5)$$

where  $\gamma[f] := \sup_{x>0} (1 - f(x))/x$  and similar for  $g$ . Inequalities (5.4) and (5.5) yield for some  $\Gamma > 0$  the estimate

$$\|(f(tA/n)g(tB/n))^n - e^{-tH}\| \leq \Gamma \frac{1}{n},$$

which proves the asymptotic  $O(1/n)$  for  $n \rightarrow \infty$ .

*This paper is dedicated to the memory of Hagen Neidhardt passed away on 23 March 2019. I am deeply grateful to Hagen for valuable discussions on the subjects of this and of many others of my projects.*

## References

- [ArZ10] Yu. Arlinskiĭ and V. Zagrebnoy, *Numerical range and quasi-sectorial contractions*, J. Math. Anal. Appl. **366** (2010), 33-43.

- [But19] Ya. A. Butko, *The method of Chernoff approximation*, arXiv:submit/2697593 [math.FA], 21 May 2019, 1–22.
- [CZ01] V. Cachia and V. A. Zagrebnov, *Operator-norm approximation of semigroups by quasi-sectorial contractions*, J. Funct. Anal. **180** (2001), 176–194.
- [Che68] P. R. Chernoff, *Note on product formulas for operator semigroups*, J. Funct. Anal. **2** (1968), 238–242.
- [Che74] P. R. Chernoff, *Product formulas, nonlinear semigroups and addition of unbounded operators*, Memoirs Amer. Math. Soc. **140** (1974), 1–121.
- [EN00] K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, Berlin, 2000.
- [IT01] T. Ichinose, H. Tamura, *The norm convergence of the Trotter-Kato product formula with error bound*, Commun. Math. Phys. **217** (2001), 489–502.
- [ITTZ01] T. Ichinose, Hideo Tamura, Hiroshi Tamura, and V. A. Zagrebnov, *Note on the paper “The norm convergence of the Trotter-Kato product formula with error bound” by Ichinose and Tamura*, Commun. Math. Phys. **221** (2001), 499–510.
- [Kat80] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1980, Corrected Printing of the Second Edition.
- [NZ98] H. Neidhardt and V. A. Zagrebnov, *On error estimates for the Trotter-Kato product formula*, Lett. Math. Phys. **44** (1998), 169–186.
- [NZ99a] H. Neidhardt and V. A. Zagrebnov, *Trotter-Kato product formula and operator-norm convergence*, Commun. Math. Phys. **205** (1999), 129–159.
- [NZ99] H. Neidhardt and V. A. Zagrebnov, *Trotter-Kato product formula and symmetrically normed ideals*, J. Funct. Anal. **167** (1999), 113–167.
- [Zag08] V. A. Zagrebnov, *Quasi-sectorial contractions*, J. Funct. Anal. **254** (2008), 2503–2511.
- [Zag17] V. A. Zagrebnov, *Comments on the Chernoff  $\sqrt{n}$ -lemma*, Functional Analysis and Operator Theory for Quantum Physics (The Pavel Exner Anniversary Volume), European Mathematical Society, Zürich, 2017, pp. 565–573.
- [Zag19] V. A. Zagrebnov, *Gibbs Semigroups*, Operator Theory Series: Advances and Applications, Vol. 273, Birkhäuser - Springer, Basel 2019.