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# EXAMPLES OF QUANTUM ALGEBRA IN POSITIVE CHARACTERISTIC

PIERRE GUILLOT

**ABSTRACT.** There have been few examples of computations of Sweedler cohomology, or its generalization in low degrees known as lazy cohomology, for Hopf algebras of positive characteristic. In this paper we first provide a detailed calculation of the Sweedler cohomology of the algebra of functions on  $(\mathbb{Z}/2)^r$ , in all degrees, over a field of characteristic 2. Here the result is strikingly different from the characteristic zero analog.

Then we show that there is a variant in characteristic  $p$  of the result obtained by Kassel and the author in characteristic zero, which provides a near-complete calculation of the second lazy cohomology group in the case of function algebras over a finite group; in positive characteristic the statement is, rather surprisingly, simpler.

**Keywords:** Hopf algebras; Sweedler cohomology; lazy cohomology; Drinfeld twists;  $R$ -matrices.

## 1. INTRODUCTION

Sweedler cohomology was defined in [Swe68]. Given a field  $k$ , a *cocommutative* Hopf algebra  $\mathcal{H}$ , and an  $\mathcal{H}$ -module algebra  $A$ , Sweedler defines cohomology groups which we will write  $H_{sw}^n(\mathcal{H}, A)$ , for  $n \geq 1$ . In fact we shall only discuss the case where  $A = k$ , the base field, viewed as an  $\mathcal{H}$ -module via the augmentation, and write simply  $H_{sw}^n(\mathcal{H})$ .

When  $\mathcal{H} = k[G]$ , the group algebra of the finite group  $G$ , one has  $H_{sw}^n(\mathcal{H}) = H^n(G, k^\times)$ ; and if  $\mathcal{H} = U(\mathfrak{g})$ , the universal enveloping algebra of the Lie algebra  $\mathfrak{g}$ , one has  $H_{sw}^n(U(\mathfrak{g})) = H^n(\mathfrak{g}, k)$ . The first virtue of Sweedler cohomology is thus to unify these two classical cohomology theories.

There are few other examples of Hopf algebras for which the Sweedler cohomology is known. An easy way to construct Hopf algebras is, of course, to consider algebras of functions on groups. In the simplest case thus, we may take a finite group  $G$  and consider the algebra  $\mathcal{H} = \mathcal{O}_k(G)$  of  $k$ -valued functions on  $G$ . However,  $\mathcal{H}$  is only cocommutative when  $G$  is abelian, which severely restricts the choices. What is worse, in the familiar case when  $k$  is the field of complex numbers  $\mathbb{C}$ , one can use the discrete Fourier transform to get an isomorphism  $\mathcal{O}_k(G) \cong k[\widehat{G}]$ , where  $\widehat{G}$  is the Pontryagin dual of  $G$ ; we are thus reduced to the group algebra case and will not get anything new.

However in positive characteristic, the Fourier transform is not available, and the Hopf algebra  $\mathcal{O}_k(G)$  is genuinely different from a group algebra. Our main result in this paper involves the elementary abelian 2-groups, that is groups of the form  $G = (\mathbb{Z}/2)^r$ . For these, over any field  $k$  of characteristic zero or  $p > 2$ , we have  $\mathcal{O}_k(G) \cong k[G]$ , so  $H_{sw}^n(\mathcal{O}_k(G)) = H^n(G, k^\times)$ . For example for  $G = \mathbb{Z}/2$ , the latter is  $k^\times/(k^\times)^2$  when  $n$  is even, and  $\{\pm 1\}$  when  $n$  is odd. When  $k$  has characteristic 2, by contrast, we obtain the following result (Theorem 5.10 in the text):

**THEOREM 1.1** – *Let  $k$  be a field of characteristic 2. The Sweedler cohomology of  $\mathcal{O}_k((\mathbb{Z}/2)^r)$  is given by*

$$H_{sw}^n(\mathcal{O}_k((\mathbb{Z}/2)^r)) = \begin{cases} 0 & \text{for } n \geq 3 \text{ or } n = 0, \\ (\mathbb{Z}/2)^r & \text{for } n = 1, \\ (k/\{x + x^2 : x \in k\})^{\oplus r} & \text{for } n = 2. \end{cases}$$

When  $k$  is an algebraically closed field, we have in particular  $H_{sw}^2(\mathcal{O}_k((\mathbb{Z}/2)^r)) = 0$ .

In fact for any ring of characteristic 2, the cohomology groups vanish in degrees greater than 2.

As far as non-cocommutative Hopf algebras go, there is at least a definition in low-degrees of the so-called “lazy cohomology groups”  $H_\ell^n(\mathcal{H})$ , for  $n = 1, 2$ , with no restriction on the Hopf algebra  $\mathcal{H}$ ; of course these agree with Sweedler’s cohomology groups when  $\mathcal{H}$  happens to be cocommutative. Lazy cohomology was defined originally by Schauenburg, and systematically explored in [BC06]. This opens up the exploration of  $H_\ell^n(\mathcal{O}_k(G))$  for any finite group  $G$ , mostly for  $n = 2$ : there is not much mystery held in the case  $n = 1$ , since one has  $H_\ell^1(\mathcal{O}_k(G)) = \mathcal{Z}(G)$ , the centre of  $G$ .

The groups  $H_\ell^2(\mathcal{O}_k(G))$  were investigated by Kassel and the author in [GK10] in characteristic 0. Our main result is as follows. Let  $\mathcal{B}(G)$  denote the set of all pairs  $(A, b)$  where  $A$  is an abelian, normal subgroup of  $G$ , and  $b$  is an alternating, non-degenerate,  $G$ -invariant bilinear form  $\widehat{A} \times \widehat{A} \rightarrow k^\times$ . The point is perhaps that  $\mathcal{B}(G)$  is easy to describe in finite time. We have then constructed a map of sets

$$\Theta: H_\ell^2(\mathcal{O}_k(G)) \longrightarrow \mathcal{B}(G)$$

with good properties. In particular, when  $k$  is algebraically closed, the fibres of  $\Theta$  are finite (and explicitly described), which proves that  $H_\ell^2(\mathcal{O}_k(G))$  is finite in this situation. It is easy to use the map  $\Theta$  to compute  $H_\ell^2(\mathcal{O}_k(G))$  in many cases, and we have thus been able to show that this group can be arbitrary large, and possibly even non-commutative.

In this paper, we extend the results of *loc. cit.* to positive characteristic. It turns out that the result is easier in this case. The following is made precise in Theorem 6.3 in the text.

**THEOREM 1.2** – *When  $k$  has characteristic  $p$ , the map*

$$\Theta: H_\ell^2(\mathcal{O}_k(G)) \longrightarrow \mathcal{B}(G)$$

*exists with the same formal properties as in characteristic 0, except that the subgroups  $A$  of order divisible by  $p$  have been excluded from the definition of  $\mathcal{B}(G)$ .*

*In particular, this map is surjective when  $k$  has characteristic 2.*

*Organization of the paper.* Section 2 recalls all the relevant definitions. The strategy for the computations of Theorem 1.1 is given in Section 3, together with a few tools from homological algebra. The computation itself is carried out in Section 4 for the case  $r = 1$ , which is much less technical, and in Section 5 for the general case. The last Section is devoted to Theorem 1.2.

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## 2. DEFINITIONS

*Notation.* Throughout the paper, we write  $R_\times$  for the multiplicative group of units in the ring  $R$ .

We recall all the relevant definitions. The Appendix contains some material on cosimplicial objects, should the reader feel the need to review this topic.

**2.1. Sweedler cohomology.** (See [Swe68].) Let  $\mathcal{H}$  be a Hopf algebra over the field  $k$ . For each integer  $n \geq 1$ , we form the coalgebra  $\mathcal{H}^{\otimes n}$  and define faces and degeneracies by the following formulae:

$$d_i(x_0 \otimes \cdots \otimes x_n) = \begin{cases} x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n & \text{for } i < n, \\ x_0 \otimes \cdots \otimes x_{n-1} \varepsilon(x_n) & \text{for } i = n, \end{cases}$$

and

$$s_i(x_0 \otimes \cdots \otimes x_n) = x_0 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_n.$$

We are thus in the presence of a simplicial coalgebra. The monoid  $\text{Hom}(\mathcal{H}^{\otimes n}, k)$ , equipped with the ‘‘convolution product’’, contains the group  $R^n(\mathcal{H}) = \text{Reg}(\mathcal{H}^{\otimes n}, k)$  (comprised of all the invertible elements in  $\text{Hom}(\mathcal{H}^{\otimes n}, k)$ ). Since  $\text{Reg}(-, k)$  is a functor, we obtain a cosimplicial group  $R^*(\mathcal{H})$  (sometimes written  $R^*$  for short in what follows).

Whenever  $\mathcal{H}$  is cocommutative,  $R^*$  is a cosimplicial abelian group. Thus it gives rise to a cochain complex  $(R^*, d)$  whose differential is

$$d = \sum_{i=0}^{n+1} (-1)^i d^i$$

in additive notation, or (as we shall also encounter it)

$$d = \prod_{i=0}^{n+1} (d^i)^{(-1)^i} = d^0 (d^1)^{-1} d^2 (d^3)^{-1} \dots$$

in multiplicative notation.

The cohomology  $H^*(R^*, d)$  is by definition the Sweedler cohomology of the cocommutative Hopf algebra  $\mathcal{H}$ , denoted by  $H_{sw}^*(\mathcal{H})$ .

**2.2. Twist cohomology & Finite dimensional algebras.** Now suppose that  $\mathcal{H}$  is a finite-dimensional Hopf algebra. Then its dual  $\mathcal{K} = \mathcal{H}^*$  is again a Hopf algebra. In this situation, the cosimplicial group associated to  $\mathcal{H}$  by Sweedler’s method may be described purely in terms of  $\mathcal{K}$ , and is sometimes easier to understand when we do so.

In fact, let us start with any Hopf algebra  $\mathcal{K}$  at all. We may construct a cosimplicial group directly as follows. Let  $A_\times^n(\mathcal{K}) = (\mathcal{K}^{\otimes n})_\times$  and let the cofaces and codegeneracies be defined by

$$d^i = \begin{cases} 1 \otimes id^{\otimes n} & \text{for } i = 0, \\ id^{\otimes(i-1)} \otimes \Delta \otimes id^{\otimes(n-i)} & \text{for } i = 1, \dots, n-1, \\ id^{\otimes n} \otimes 1 & \text{for } i = n, \end{cases}$$

and

$$s^i = \begin{cases} \varepsilon \otimes id^{\otimes(n-1)} & \text{for } i = 0, \\ id^{\otimes(i-1)} \otimes \varepsilon \otimes id^{\otimes(n-i)} & \text{for } i = 1, \dots, n-1, \\ id^{\otimes(n-1)} \otimes \varepsilon & \text{for } i = n. \end{cases}$$

When  $\mathcal{K}$  is commutative, then  $A_\times^*(\mathcal{K})$  is a cosimplicial abelian group, giving rise to a cochain complex  $(A_\times^*, d)$  in the usual way. Its cohomology  $H^*(A_\times^*, d)$  is what we call the *twist cohomology* of  $\mathcal{K}$ , written  $H_{tw}^*(\mathcal{K})$ . This terminology comes from

the fact, easily checked, that an element of  $A_{\times}^2(\mathcal{K}) = (\mathcal{K} \otimes \mathcal{K})_{\times}$  is in the kernel of  $d$  if and only if it is a twist in the sense of Drinfeld (see equation (†) below).

Coming back to the case when  $\mathcal{K} = \mathcal{H}^*$  for a finite-dimensional Hopf algebra  $\mathcal{H}$ , it is straightforward to check that  $R^*(\mathcal{H})$  can be identified with  $A_{\times}^*(\mathcal{K})$  (Theorem 1.10 and its proof in [GK10] may help).

In this paper we are chiefly interested in computing with  $R^*(\mathcal{O}_k(G))$ , where  $\mathcal{O}_k(G)$  denotes the algebra of functions on the finite group  $G$ . By the above, this is the same as  $A_{\times}^*(k[G])$ , writing  $k[G]$  for the group algebra of  $G$ . It turns out to be easier to work with the latter.

The following remarks will be useful in the sequel. As the notation suggest, it is possible to define another cosimplicial abelian group  $A^*(\mathcal{K})$  by simply taking  $A^n(\mathcal{K}) = \mathcal{K}^{\otimes n}$ , the vector space underlying  $\mathcal{K}^{\otimes n}$ . The cofaces and codegeneracies are exactly the same as above (they are really maps of algebras), although the differential of the corresponding cochain complex is now the alternating *sum* of the cofaces rather than the alternating *product*. Still, we denote it by  $d$ . Let us explain why the cohomology of  $(A^*, d)$  must in fact be very simple.

We consider the case  $\mathcal{K} = k[G]$  for a finite group  $G$ , and write  $A^*$  for  $A^*(\mathcal{K})$ . Note that  $(A^*, d)$  does not depend on the group structure on  $G$ , and we expect its cohomology to be trivial. Indeed, let  $X$  denote  $G$  viewed as a pointed set only. There is an obvious cosimplicial set which in degree  $n$  is  $X^n$  (cartesian product of  $n$  copies of  $X$ ), and such that the cosimplicial group  $A^*$  is obtained by applying the functor “free  $k$ -vector space” to  $X^*$ . In this sort of situation we may apply the following Lemma, which we prove in the Appendix.

LEMMA 2.1 – *Let  $X^*$  be any cosimplicial set, let  $k$  be any ring, and let  $k[X]^*$  be the cosimplicial  $k$ -module obtained by taking  $k[X]^n$  to be the free  $k$ -module on  $X^n$ . Then  $H^n(k[X]^*, d) = 0$  for  $n > 0$ .*

In particular, the cohomology of  $A^*$  indeed vanishes (even in degree 0, in this case).

**2.3. Lazy cohomology.** When  $\mathcal{H}$  is not cocommutative, Sweedler’s cohomology is not defined. However, there is a general definition of low-dimensional groups  $H_{\ell}^i(\mathcal{H})$  for  $i = 1, 2$ , called the *lazy* cohomology groups of  $\mathcal{H}$ , for any Hopf algebra  $\mathcal{H}$ : this definition is originally due to Schauenburg and is systematically explored in [BC06]. Of course when  $\mathcal{H}$  happens to be cocommutative, then  $H_{\ell}^i(\mathcal{H}) = H_{sw}^i(\mathcal{H})$ . This is perfectly analogous to the construction of the non-abelian  $H^1$  in Galois cohomology – note that  $H_{\ell}^2(\mathcal{H})$  may be non-commutative (cf [GK10]).

When  $\mathcal{H}$  is finite-dimensional, there is again a description of  $H_{\ell}^i(\mathcal{H})$  in terms of the dual Hopf algebra  $\mathcal{K}$ . Since this is the case of interest for us, we only give the details of the definition in this particular situation (using results from [GK10], §1). Quite simply,  $H_{\ell}^1(\mathcal{H})$  is the (multiplicative) group of central group-like elements in  $\mathcal{K}$ . The group  $H_{\ell}^2(\mathcal{H})$  is defined as a quotient. Consider first the group  $Z^2$  of all invertible elements  $F \in \mathcal{K} \otimes \mathcal{K}$  satisfying

$$\Delta(a)F = F\Delta(a)$$

(here  $\Delta$  is the diagonal of  $\mathcal{K}$  – one says that  $F$  is invariant), and

$$(†) \quad (F \otimes 1)(\Delta \otimes id)(F) = (1 \otimes F)(id \otimes \Delta)(F)$$

(which says that  $F$  is a Drinfeld twist). The group  $Z^2$  contains the group  $B^2$  of so-called trivial twists, that is elements of the form  $F = (a \otimes a)\Delta(a^{-1})$  for  $a$  central in  $\mathcal{K}$ . Then  $H_{\ell}^2(\mathcal{H}) = Z^2/B^2$ .

### 3. HOMOLOGICAL PRELIMINARIES

This section prepares the ground for the next two.

**3.1. Strategy.** It is classical that the cohomology of any abelian cosimplicial group  $A^*$  can be computed by restricting attention to the “normalized” cocycles, that is those cocycles for which all the codegeneracies  $s^*$  vanish. In this paper we shall focus our attention on the last codegeneracy ( $s^{n-1}$  on  $A^n$ ). As we shall explain at length in this section, the cocycles for which  $s^{n-1}$  vanish can also be used to compute the cohomology.

The reason for paying special attention to this map is the following. Put  $A^n = k[G^n]$ , where  $G$  is a finite group. The cosimplicial group we are interested in is  $A_\times^n = k[G^n]_\times$  as in §2.2. Thus  $A_\times^n = R[G]_\times$  for  $R = k[G^{n-1}]$ , and the map  $s^{n-1}: R[G]_\times \rightarrow R_\times$  is simply the augmentation. So the “unit spheres”  $S(A^n)$ , comprised of those elements of augmentation 1, can be taken to compute the cohomology. In turn, this is useful because in positive characteristic  $S(R[G])$  can sometimes be very simple, and indeed isomorphic to a sum of several copies of the abelian group underlying  $R$ . In the sequel we shall prove a much more precise statement (Theorem 5.2), but for the present discussion let us be content with the following.

**LEMMA 3.1** – *Let  $G = (\mathbb{Z}/2)^r$ , and let  $R$  be any commutative ring of characteristic 2. Put*

$$S(R[G]) = \{x \in R[G] : \varepsilon(x) = 1\},$$

*where  $\varepsilon$  is the augmentation. Then there is an isomorphism*

$$S(R[G]) \cong R^{\oplus 2^r - 1}.$$

(In order to avoid confusion with other upperscripts, we write  $R^{\oplus 2^r - 1}$  for a direct sum of  $2^r - 1$  copies of  $R$ .)

*Proof.* By induction on  $r$ . For  $r = 1$  it is immediate that, writing  $\mathbb{Z}/2 = \{1, \sigma\}$ , the map

$$a \mapsto 1 + a + a\sigma$$

is the required isomorphism between  $R$  and  $S(R[\mathbb{Z}/2])$ .

Now suppose  $G = H \times \mathbb{Z}/2$  with  $H = (\mathbb{Z}/2)^r$ . It is clear that

$$S(R[G]) \cong S(R[H]) \times K$$

where  $K$  is the group of those  $x \in S(R[G])$  mapping to 1 under the map  $S(R[G]) \rightarrow S(R[H])$ , itself induced by the projection  $G \rightarrow H$ . However we can view  $R[G]$  as  $R'[\mathbb{Z}/2]$  with  $R' = k[H]$ , and under this identification,  $K$  is simply  $S(R'[\mathbb{Z}/2])$ .

By induction,  $S(R[H])$  is isomorphic to  $R^{\oplus 2^r - 1}$ , while the case  $r = 1$  just treated shows that  $S(R'[\mathbb{Z}/2])$  is isomorphic to  $R'$ , itself clearly isomorphic to  $R^{\oplus 2^r}$  as abelian group. The result follows.  $\square$

So at least in the case of the group  $G = (\mathbb{Z}/2)^r$ , we are led to study a cochain complex which in degree  $n$  is made of  $2^r - 1$  copies of the additive abelian group  $k[G^{n-1}] = A^{n-1}$ .

It is of course tempting to compare this cochain complex to another one having the same underlying abelian groups: indeed in §2.2 we proved that the “additive” cosimplicial group  $A^*$  had zero cohomology. In principle, the differential on  $S(A^*)$  must be related to that on  $A^*$ , but in order to make this statement precise we need a much more explicit description of the isomorphism in the Lemma.

Eventually we shall do just that, proving that the cohomology of  $S(A^*)$  vanishes in degree  $\geq 3$  (though not below).

In the rest of this section, we explain in detail a two-step reduction process for the computation of the cohomology of cosimplicial groups: first the restriction to the kernel of the last codegeneracy, and then in good cases a second, very similar restriction. In brief, expressive terms, when this two-step reduction is performed for both  $A^*$  and  $A_\times^*$ , the complexes we get are almost the same.

**3.2. Basic fact.** We shall elaborate on the following trivial lemma in homological algebra.

LEMMA 3.2 – *Let  $(C^*, d)$  be a cochain complex of abelian groups. Assume that*

$$d = \sum_{i=0}^{n+1} (-1)^i d^i,$$

where  $d^i: C^n \rightarrow C^{n+1}$  is a homomorphism (we do not assume the cosimplicial identities!).

Assume that there are maps  $\varepsilon_n: C^n \rightarrow C^{n-1}$  satisfying

$$\varepsilon_{n+1}(d^i(x)) = d^i(\varepsilon_n(x)) \quad \text{for } 0 \leq i < n,$$

and

$$\varepsilon_{n+1}(d^n(x)) = \varepsilon_{n+1}(d^{n+1}(x)) = x.$$

Put  $K^n = \ker \varepsilon_n$ . Then  $d$  carries  $K^n$  into  $K^{n+1}$ , and we have

$$H^n(C^*, d) = H^n(K^*, d).$$

*Proof.* The fact that  $d$  carries  $K^n$  into  $K^{n+1}$  is trivial.

Observe the following: if  $x \in C^n$  is any element such that  $\varepsilon(d(x)) = 0$ , then  $d(\varepsilon(x)) = \pm d^n(\varepsilon(x))$ , and in particular  $d^n(\varepsilon(x))$  is a coboundary.

Applying this to an  $x \in C^n$  such that  $d(x) = 0$ , we see that the cohomology class of  $x$  is the same as that of  $x' = x - d^n(\varepsilon(x))$ . However  $\varepsilon_n(x') = \varepsilon_n(x) - \varepsilon_n d^n \varepsilon_n(x) = \varepsilon_n(x) - \varepsilon_n(x) = 0$ , that is  $x' \in K^n$ . Thus the natural map

$$H^n(K^*, d) \rightarrow H^n(C^*, d)$$

is surjective.

To see that it is injective, too, pick  $x \in K^n$  such that  $x = d(y)$  for some  $y \in C^{n-1}$ . Since  $\varepsilon(x) = 0 = \varepsilon(d(y))$ , the observation above applied to  $y$  shows that  $d^{n-1}(\varepsilon_{n-1}(y))$  is a coboundary. Therefore if we put  $y' = y - d^{n-1}(\varepsilon_{n-1}(y))$ , we have  $d(y') = d(y) = x$ . However  $\varepsilon_{n-1}(y') = \varepsilon_{n-1}(y) - \varepsilon_{n-1} d^{n-1}(\varepsilon_{n-1}(y)) = 0$ , so  $y' \in K^{n-1}$ , and  $y$  is a coboundary in the complex  $K^*$ .  $\square$

This lemma allows us to replace the cochain complex  $(C^*, d)$  by a smaller complex, without losing the cohomological information. The purpose of the next subsection is to show that, if  $(A^*, d)$  is the complex associated to a cosimplicial group, then in the vein of the above lemma we may produce a subcomplex  $(B^*, d)$  which computes the cohomology of  $(A^*, d)$ . However this time there is finer information available on the coboundary of  $B^*$  (see Lemma 3.6). What is more, the shifted complex  $(C^*, d) := (B^{*+1}, d)$  retains enough of the original cosimplicial structure for us to apply Lemma 3.2 in good cases. We are thus capable of making a two-step reduction from  $A^*$  to  $C^*$  to  $K^*$ .

**3.3. Reduction of cosimplicial abelian groups.** We assume that  $A^*$  is a cosimplicial abelian group, written additively for now. We write  $\varepsilon_n: A^n \rightarrow A^{n-1}$  for the degeneracy map  $s^{n-1}$ , in order to make a parallel with Lemma 3.2. We recall that  $\varepsilon_{n+1}(d^i(x)) = d^i(\varepsilon_n(x))$  for  $0 \leq i < n$ , while  $\varepsilon_{n+1}(d^n(x)) = \varepsilon_{n+1}(d^{n+1}(x)) = x$ . These follow from the cosimplicial identities (which are recalled in the Appendix). In particular, the map  $d^{n+1}: A^n \rightarrow A^{n+1}$  is injective.

We define  $\beta^i: A^{n-1} \rightarrow A^n$  by  $\beta^i = d^i$  for  $0 \leq i \leq n$ , and  $\beta^{n+1} = \beta^n = d^n$ . Note that in this way we have created  $n+2$  maps out of  $A^{n-1}$ , and for each of them

there is a commutative diagram

$$\begin{array}{ccc} A^{n-1} & \xrightarrow{\beta^i} & A^n \\ d^n \downarrow & & \downarrow d^{n+1} \\ A^n & \xrightarrow{d^i} & A^n. \end{array}$$

Indeed, for  $0 \leq i \leq n$  the commutativity follows from the cosimplicial relations, while for  $i = n + 1$  it is tautological. Let us write  $\iota_{n+1} = d^{n+1}: A^n \rightarrow A^{n+1}$ .

We now further define a map  $\beta = \beta_n: A^{n-1} \rightarrow A^n$  by

$$\beta = \sum_{i=0}^{n+1} (-1)^i \beta^i = \sum_{i=0}^{n-1} (-1)^i \beta^i,$$

the equality following from  $\beta^n = \beta^{n+1}$ . We have  $d \circ \iota = \iota \circ \beta$ .

Since  $\iota_n$  is injective, for all  $n$ , we deduce from  $d^2 = 0$  that  $\beta^2 = 0$ .

Thus we have produced a new cochain complex  $(m(A)^*, \beta)$ , where  $m(A)^n = A^{n-1}$ . Moreover  $\iota$  is a cochain map  $m(A)^* \rightarrow A^*$ . As a result there is an induced map in cohomology

$$H^n(m(A)^*, \beta) \rightarrow H^n(A^*, d).$$

LEMMA 3.3 – *Under these conditions,  $\varepsilon_*: (A^*, d) \rightarrow (m(A)^*, \beta)$  is a map of cochain complexes.*

*Proof.*

$$\varepsilon_{n+1} \left( \sum_{i=0}^{n+1} (-1)^i d^i(x) \right) = \sum_{i=0}^{n-1} (-1)^i d^i(\varepsilon_n(x)) = \beta(\varepsilon_n(x)).$$

This also works for  $n = 0$ . □

Now for each  $n \geq 1$  we have  $\varepsilon_n \circ \iota_n = id$ . It follows that  $p_n = \iota_n \circ \varepsilon_n$  is a projector that commutes with the coboundary maps; we may write  $A^n = A^{n-1} \oplus B^n$  with  $B^n = \ker(p_n)$ , and  $d$  carries  $B^n$  into  $B^{n+1}$ .

In other words there is a direct sum of cochain complexes  $A^* = m(A)^* \oplus B^*$ , and in cohomology we get

$$H^n(A^*, d) = H^n(m(A)^*, \beta) \oplus H^n(B^*, d).$$

However under these assumptions we can also show:

LEMMA 3.4 – *For all  $n$  we have  $H^n(m(A)^*, \beta) = 0$ .*

*Proof.* Let  $x \in A_{n-1}$  be a cycle in degree  $n$ . The condition  $\beta(x) = 0$  reads

$$\sum_{i=0}^{n-1} (-1)^i d^i(x) = 0.$$

By adding  $(-1)^n d^n(x) = (-1)^n \iota_n(x)$  on each side, we find that  $d(x) = (-1)^n \iota_n(x)$ . Now applying  $\varepsilon_n$  to this equality yields

$$\varepsilon(d(x)) = \beta(\varepsilon(x)) = (-1)^n \varepsilon(\iota(x)) = (-1)^n x.$$

Hence  $x$  is the coboundary of  $(-1)^n \varepsilon(x)$ . □

Hence:

PROPOSITION 3.5 – *When  $A^*$  is a cosimplicial abelian group, then for all  $n \geq 0$  there is an isomorphism*

$$H^n(A^*, d) \cong H^n(B^*, d),$$

where  $B^n$  is the kernel of  $\varepsilon_n$ .

For computational purposes, the following expressions will help dealing with the differential on  $B^*$ . As observed, the differential  $d$  of the complex  $A^*$  carries  $B^*$  into itself, but the same *cannot* be said of the individual coface maps  $d^i$ . Instead, we have the following formulae. Let  $q = q_n = id - p_n$  be the projector orthogonal to  $p_n$ , which is a projector onto  $B^n$ , and let  $\bar{d}^i = q \circ d^i: B^n \rightarrow B^{n+1}$ ; from the relation  $q \circ d(x) = d(x)$  for  $x \in B^n$  we certainly have

$$d(x) = \sum_{i=0}^{n+1} (-1)^i \bar{d}^i(x) \quad \text{for } x \in B^n.$$

This relation will also be clear from the following more precise equations.

LEMMA 3.6 – *We have*

$$\bar{d}^i = d^i \text{ for } 0 \leq i < n,$$

while

$$\bar{d}^n = d^n - d^{n+1},$$

and

$$\bar{d}^{n+1} = 0.$$

*Proof.* For  $x \in B^n$  we have

$$\bar{d}^i(x) = d^i(x) - \iota_{n+1} \varepsilon_{n+1}(d^i(x)).$$

We have  $\varepsilon_n(x) = 0$  by definition of  $B^n$ , so from assumption two we get the formula in the case  $0 \leq i < n$ .

For  $i = n$ , we use  $\varepsilon_{n+1}(d^n(x)) = \varepsilon_{n+1}(d^{n+1}(x)) = x$  from assumptions two and three. Since  $\iota_{n+1}(x) = d^{n+1}(x)$ , we do have  $\bar{d}^n = d^n - d^{n+1}$ .

The case  $i = n + 1$  is similar.  $\square$

The fact that  $\bar{d}^{n+1}$  is the zero map encourages us to consider  $B^n$  as being in degree  $n - 1$ , that is, to consider the complex  $(C^*, d) = (B^{*+1}, d)$ . As announced, in practice we will be able to apply Lemma 3.2 to  $(C^*, d)$ , though we will not try to look for axioms on  $(A^*, d)$  for this to hold in general. Let us give an example at once.

**3.4. First application.** Let  $G = (\mathbb{Z}/2)^r$ , and let  $A^n = k[G^n]$  as in §2.2; these comprise a cosimplicial abelian group whose differential will be denoted by  $d$ . Its cohomology is zero by Lemma 2.1. We shall apply Proposition 3.5 and deduce the existence of certain cochain complexes with zero cohomology.

Here and elsewhere, we shall use the following notation: for  $\sigma \in G$ , we write  $\sigma_n$  for the element

$$(1, 1, \dots, 1, \sigma, 1, \dots, 1) \in G^N$$

with  $\sigma$  in the  $n$ -th position, for some  $N \geq n$  which is always clear from the context. (Usually  $N = n$ .)

PROPOSITION 3.7 – *Let  $\sigma \in G$ . Define a cochain complex  $(A^*, \delta_\sigma)$  with  $A^*$  as above and  $\delta: A^{n-1} \rightarrow A^n$  given by*

$$\delta_\sigma(a) = d(a) + a(1 + \sigma_n).$$

*Then  $\delta_\sigma \circ \delta_\sigma = 0$  and  $H^n(A^*, \delta_\sigma) = 0$  for  $n \geq 0$ . Moreover, the subcomplex  $\bar{A}^*$  of elements of having zero augmentation is preserved by  $\delta_\sigma$  and we also have  $H^n(\bar{A}^*, \delta_\sigma) = 0$  for  $n \geq 0$ .*

*Proof.* We apply Proposition 3.5. We have a decomposition  $A^n = A^{n-1} \oplus B^n$  where  $B^n$  is the kernel of the augmentation  $A^n \rightarrow A^{n-1}$ , and the cohomology of  $B^n$  vanishes.

For  $\sigma \in G$  such that  $\sigma \neq 1$ , let  $A_\sigma^n$  be a copy of  $A^n$ . For  $n \geq 1$  we use the identification

$$\begin{aligned} \psi: \bigoplus_{\sigma} A_\sigma^{n-1} &\xrightarrow{\cong} B^n \\ (a_\sigma)_\sigma &\mapsto \left( \sum_{\sigma} a_\sigma \right) + \sum_{\sigma} a_\sigma \sigma_n. \end{aligned}$$

This isomorphism defines a differential  $\delta = \psi^{-1} \circ d \circ \psi$  on  $C^* = \bigoplus_{\sigma} A_\sigma^*$  which must then satisfy  $H^n(C^*, \delta) = H^n(B^{*+1}, d) = 0$  at least for  $n \geq 1$ .

Now checking the definitions, we see that  $\delta$  splits as the direct sum of the differentials  $\delta_\sigma$  given in the statement of the Proposition. It is immediate that the cohomology of  $\delta_\sigma$  is also zero in degree 0. We have proved the first statement.

Finally, since  $\varepsilon_n(1 + \sigma_n) = 0$ , we can apply Lemma 3.2 to  $(A^*, \delta_\sigma)$ . The second statement follows.  $\square$

#### 4. THE CASE OF $\mathbb{Z}/2$

In this section we compute completely the Sweedler cohomology of  $\mathcal{O}_k(\mathbb{Z}/2)$ , or equivalently the twist cohomology of  $k[\mathbb{Z}/2]$ . The group with two elements is so simple that other approaches than the one below are possible, which could be easier (looking at the normalized cocycles is a good idea). However, we choose to apply the strategy described in §3.1 as an illustration which is much less technical than the general case.

**4.1. The unit sphere.** Let  $R$  be a commutative ring of characteristic 2. The elements of the group  $\mathbb{Z}/2$  will be written 1 and  $\sigma$ . The group algebra  $A = R[\mathbb{Z}/2]$  consists, of course, of the elements  $z = x + y\sigma$  with  $x, y \in R$ .

We define the *modulus* of  $z$  to be

$$|z| = x + y \in R \quad (= \sqrt{(x+y)^2} = \sqrt{x^2 + y^2}),$$

or in other words we shall write  $|z|$  for the augmentation of  $z$ . We note that  $z \mapsto |z|$  is a map of algebras  $A \rightarrow R$ .

We have the relation

$$z^2 = |z|^2,$$

from which it follows that  $z$  is invertible in  $A$  if and only if  $|z|$  is invertible in  $R$  (and then  $z^{-1} = |z|^{-2}z$ ). As a result the elements in the *unit sphere*

$$S(A) = \{z \in A : |z| = 1\}$$

are all invertible in  $A$ .

There is an isomorphism

$$A_\times \xrightarrow{\cong} R_\times \times S(A),$$

given by  $z \mapsto (|z|, \frac{z}{|z|})$ .

An element in  $S(A)$  is of the form  $(1+x) + x\sigma$ . For any  $x \in R$  we define its *exponential* to be precisely

$$e^x = (1+x) + x\sigma \in A_\times.$$

There is the usual formula

$$e^{a+b} = e^a e^b.$$

The exponential gives an isomorphism  $R \rightarrow S(A)$ , whose inverse we call the logarithm and write  $\log: S(A) \rightarrow R$ . We end up with an isomorphism

$$A_\times \xrightarrow{\cong} R_\times \times R,$$

given by  $z \mapsto (|z|, \log(\frac{z}{|z|}))$ .

**4.2. Higher group algebras.** Let  $k$  be a ring of characteristic 2, let  $A^0 = k$ , and for  $n \geq 1$  let  $A^n = k[(\mathbb{Z}/2)^n]$ . We always see  $A^n$  as a subring of  $A^{n+1}$ . The evident generators for  $(\mathbb{Z}/2)^n$  will be written  $\sigma_1, \dots, \sigma_n$ , so that for example  $\sigma_1\sigma_2$  is an element of  $A^2$ . Of course it may also be considered as an element of  $A^3$ , but in practice the ambiguities created are of no consequence.

It is fundamental that  $A^{n+1} = A^n[\mathbb{Z}/2]$ .

**PROPOSITION 4.1** – *Let  $A^*$  and  $d$  be as above. Define a cochain complex  $(A^*, \partial)$  with  $\partial: A^{n-1} \rightarrow A^n$  given by*

$$\begin{aligned} \partial(a) &= d(a) + d^n(a)(1 + e^{1+a}) \\ &= d(a) + (a + a^2)(1 + \sigma_n). \end{aligned}$$

*Then  $\partial \circ \partial = 0$  and  $H^n(A_\times^*, d) = H^{n-1}(A^*, \partial)$  for  $n \geq 2$ . Moreover, the cochain complex  $(A^*, \partial)$  satisfies the hypotheses of Lemma 3.2.*

*Proof.* We apply Proposition 3.5. The decomposition  $A_\times^n = A_\times^{n-1} \times B^n$  as in §3.3 can be identified for  $n \geq 1$  with the decomposition  $A_\times^n = A_\times^{n-1} \times S(A^n)$ . Using the logarithm, we deduce the decomposition  $A_\times^n = A_\times^{n-1} \times A^{n-1}$ , and Proposition 3.5 now states that there is an isomorphism for  $n \geq 2$

$$H^n(A_\times^*, d) = H^n(A^{*-1}, \partial),$$

where  $\partial$  needs to be explicitly described.

This is based on Lemma 3.6, and we write  $\bar{d}^i$  for the maps described there – keeping in mind that we need to use multiplicative notation now. We point out that the elements of  $B^n = S(A^n) \cong A^{n-1}$  are of order 2, so we may ignore the inverses, just like we can ignore the signs in additive notation.

We write  $\bar{\partial}^i(a) = \log(\bar{d}^i(e^a))$ , so that

$$\partial(a) = \sum_{i=0}^{n+1} (-1)^i \bar{\partial}^i(a).$$

(Again the signs are here for decoration.)

For  $0 \leq i < n$ , and  $a \in A^{n-1}$ , we check readily that  $d^i(1 + a + a\sigma_n) = 1 + d^i(a) + d^i(a)\sigma_n$ , which reads  $d^i(e^a) = e^{d^i(a)}$ . It follows that  $\bar{\partial}^i(a) = d^i(a)$  in these cases.

For  $i = n$ , we first compute

$$d^n(e^a) = 1 + a + a\sigma_n\sigma_{n+1},$$

and

$$d^{n+1}(e^a) = 1 + a + a\sigma_n.$$

The product (=quotient) of these is

$$\begin{aligned} 1 + a^2 + (a + a^2)\sigma_n + a^2\sigma_{n+1} + (a + a^2)\sigma_n\sigma_{n+1} \\ = e^{a^2+(a+a^2)\sigma_n} = e^{a(a+(1+a)\sigma_n)} = e^{d^n(a)e^{1+a}}. \end{aligned}$$

Thus  $\bar{\partial}^n(a) = d^n(a)e^{1+a} = d^n(a) + d^n(a)(1 + e^{1+a})$ . And  $\bar{d}^{n+1} = 0$  implies  $\bar{\partial}^{n+1} = 0$ , of course.

This gives the expression for  $\partial(a)$ . To show that the hypotheses of lemma 3.2 are satisfied, we note that, this time,  $(A^*, \partial)$  is obtained from  $(A^*, d)$  by replacing the last coface  $d^{n+1}(a) = a \otimes 1$  by  $d^{n+1}(a)(e^{1+a})$ ; however  $\varepsilon(e^{1+a}) = 1$  so  $\varepsilon(d^{n+1}(a)(e^{1+a})) = \varepsilon(d^{n+1}(a)) = a$ , as we wanted.  $\square$

Comparing the Propositions 3.7 and 4.1 shows how close the differentials  $\delta_\sigma$  and  $\partial$  are: the expression  $a + a^2$  simply replaces  $a$ , so that they are the same “at first order”.

### 4.3. Sweedler cohomology of $\mathcal{O}_k(\mathbb{Z}/2)$ .

**THEOREM 4.2** – *Let  $k$  be a ring of characteristic 2. The twist cohomology of  $k[\mathbb{Z}/2]$ , or the Sweedler cohomology of  $\mathcal{O}_k(\mathbb{Z}/2)$ , is given by*

$$H_{sw}^n(\mathcal{O}(\mathbb{Z}/2)) = \begin{cases} 0 & \text{for } n \geq 3 \text{ or } n = 0, \\ k/\{x + x^2 \mid x \in k\} & \text{for } n = 2, \\ \mathbb{Z}/2 & \text{for } n = 1. \end{cases}$$

*In particular when  $k$  is algebraically closed then these groups vanish in degrees  $\geq 2$ .*

*Proof.* The statements for  $n = 0$  or  $n = 1$  are (easy) general facts. We first prove that  $H^n(A_\times^*, d) = 0$  for  $n \geq 3$ , which is the first case above.

We have seen (Proposition 3.7) that for  $n \geq 2$

$$0 = H^n(A^*, d) = H^{n-1}(A^*, \delta_\sigma).$$

Moreover, by applying Lemma 3.2 we have

$$H^{n-1}(A^*, \delta) = H^{n-1}(K^*, \delta),$$

where  $K^n$  is the subgroup of elements  $a \in A^n$  such that  $\varepsilon_n(a) = 0$ . (This is also part of the conclusion of Proposition 3.7 where  $K^*$  is denoted  $\bar{A}^*$ .)

On the other hand we have also (Proposition 4.1) for  $n \geq 2$

$$H^n(A_\times^*, d) = H^{n-1}(A^*, \partial).$$

By applying Lemma 3.2 we have

$$H^{n-1}(A^*, \partial) = H^{n-1}(K^*, \partial),$$

where  $K^n$  is precisely the same as above.

Now the fundamental observation is that  $a^2 = \varepsilon_n(a)^2$  for  $n \geq 1$ . So for an element  $a \in K^n$ , we have  $a^2 = 0$ . As a result, the differentials  $\delta$  and  $\partial$  agree on  $K^*$ , from  $K^1$  and above. Thus  $H^r(K^*, \delta) = H^r(K^*, \partial)$  for  $r \geq 2$ .

It follows that

$$H^{n-1}(A^*, \delta) \cong H^{n-1}(A^*, \partial),$$

for  $n \geq 3$ , whence the result.

Now we turn to the computation of  $H^2(A_\times^*, d) = H^1(A^*, \partial)$ . For any element  $a + b\sigma_1 \in A^1$ , we compute that

$$\partial(a + b\sigma_1) = (a^2 + b^2) + (a + b + a^2 + b^2)\sigma_2 \in A^2,$$

so the kernel of  $\partial$  in degree 1 is isomorphic to  $k$ , and is comprised of those elements of the form  $a + a\sigma_1$ . On the other hand for  $x \in k = A^0$ , we have

$$\partial(x) = (x + x^2) + (x + x^2)\sigma_1.$$

This shows the announced result for  $n = 2$ . When  $k$  is algebraically closed, note that the equation  $x^2 + x = a$  always has a solution regardless of the parameter  $a \in k$ , so  $H^2$  vanishes, too.  $\square$

EXAMPLE 4.3 – Let us illustrate the theorem with a simple example. Let  $q = 2^r$ , and take  $k = \mathbb{F}_q$ , the field with  $q$  elements. The map  $\mathbb{F}_q \rightarrow \mathbb{F}_q$  sending  $x$  to  $x + x^2$  has kernel  $\mathbb{F}_2$ , so its cokernel has dimension 1 over  $\mathbb{F}_2$ . Thus  $H_{tw}^2(\mathbb{F}_q[\mathbb{Z}/2]) = \mathbb{F}_2$ .

The non-trivial element is described as follows. There is a non-zero element  $a \in \mathbb{F}_q$  which is not of the form  $x + x^2$ , and  $a + a\sigma_1$  is a representative of the non-zero class in  $H^1(A^*, \partial)$ . Via the isomorphism with  $H_{tw}^2(\mathbb{F}_q[\mathbb{Z}/2])$ , we obtain the twist

$$(*) \quad F = e^{a+a\sigma_1} = (1+a)1 \otimes 1 + a\sigma_1 \otimes 1 + a1 \otimes \sigma_2 + a\sigma_1 \otimes \sigma_2 \in \mathbb{F}_q[\mathbb{Z}/2]^{\otimes 2}.$$

It is symmetric, that is  $F = F_{21}$  in common Hopf-algebraic notation, so that the element  $R_F = F_{21}F^{-1} = 1 \otimes 1$ . (Whenever  $F$  is a twist, the element  $R_F$  is always an “ $R$ -matrix”, of which much more in the rest of this paper, and it normally holds important information about  $F$ .)

There is a simple way to see  $F$  in action. Whenever  $\mathcal{A}$  is an  $\mathbb{F}_q$ -algebra endowed with a  $\mathbb{Z}/2$ -action, we can twist it using  $F$  into a new algebra  $\mathcal{A}_F$ . A lot of information about this is presented in [GKM], but we will keep things elementary and only state that if  $\mu: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$  is the original multiplication, then it is twisted to

$$x * y = \mu(x \otimes y F).$$

That this new multiplication is associative is equivalent to  $F$  being a twist. It also follows from *loc. cit.* that a fundamental example is  $\mathcal{A} = \mathcal{O}(\mathbb{Z}/2)$ , the algebra of functions on  $\mathbb{Z}/2$ , so let us only look at this case.

This algebra is 2-dimensional over  $\mathbb{F}_q$ , with a basis given by the constant function 1 and the Dirac function  $\delta$  at the neutral element of  $\mathbb{Z}/2$ . In  $\mathcal{A}_F$  the unit will be unchanged (easy check), and there remains to compute

$$\delta * \delta = \mu(\delta \otimes \delta F) = a + \delta.$$

So in  $\mathcal{A}_F$ , we have a solution of  $x^2 + x = a$ , namely  $x = \delta$ . It follows that  $\mathcal{A}_F$  is simply  $\mathbb{F}_{q^2}$ .

In fact, if  $F$  corresponds to *any*  $a \in k$  by formula (\*), we will have  $\mathcal{O}(\mathbb{Z}/2)_F = \mathbb{F}_q[x]/(x^2 + x + a)$  (which is two copies of  $\mathbb{F}_q$  when  $a$  is already of the form  $x^2 + x$  in  $\mathbb{F}_q$ ).

We recall that in characteristic 0, we have  $H_{tw}^2(k[\mathbb{Z}/2]) = H^1(k, \mathbb{Z}/2) = k^\times / (k^\times)^2$ . If  $F$  is the twist corresponding to the class of  $a$  modulo squares, then  $\mathcal{O}(\mathbb{Z}/2)_F$  is isomorphic to  $k[\sqrt{a}] = k[x]/(x^2 - a)$ .

## 5. THE GENERAL CASE

We now let  $G = (\mathbb{Z}/2)^r$  be any elementary abelian 2-group, and we let  $k$  be any ring of characteristic 2. In this section we prove that the twist cohomology of  $k[G]$  vanishes in degrees  $\geq 3$ . We also give information about the low-dimensional cohomology groups.

**5.1. The unit sphere.** Let  $R$  be a ring of characteristic 2. In this paragraph we seek a description of

$$S(R[G]) = \{x \in R[G] : \varepsilon(x) = 1\}.$$

Here  $\varepsilon: R[G] \rightarrow R$  is the augmentation. It will sometimes be convenient to write  $S_R(R[G])$  for emphasis; consider for example the group algebra  $R[G \times G] = R'[G]$  for  $R' = R[G]$ , for which the notation  $S_{R[G]}(R[G \times G])$  refers to the augmentation  $\varepsilon: R[G \times G] \rightarrow R[G]$ . In fact, the following decomposition will be useful in the sequel:

$$(\dagger) \quad S_R(R[G \times G]) = S_R(R[G]) \times S_{R[G]}(R[G \times G]).$$

The proof is immediate.

A useful device for the study of  $S(R[G])$  is the *exponential*. Namely, whenever  $u \in R[G]$  satisfies  $\varepsilon(u) = 0$ , and thus  $u^2 = \varepsilon(u)^2 = 0$ , we put

$$\exp_u(a) = 1 + au \quad (a \in R).$$

Of course  $\exp_u(a)$  is an element of  $S(R[G])$ .

LEMMA 5.1 – *The exponential enjoys the following properties:*

- (1)  $\exp_u(a + b) = \exp_u(a) \exp_u(b)$ .
- (2)  $\exp_u(a) \exp_v(a) = \exp_{u+v}(a) \exp_{uv}(a^2)$ .

Fix once and for all a basis  $\Sigma \subset G$  for  $G$  as an  $\mathbb{F}_2$ -vector space. For a subset  $X \subset \Sigma$ , we put

$$u_X = \prod_{\sigma \in X} (1 + \sigma).$$

THEOREM 5.2 – *For each non-empty  $X \subset \Sigma$ , let  $R_X$  be a copy of the abelian group underlying  $R$ . Then the map*

$$\begin{aligned} \text{EXP} : \bigoplus_{\emptyset \neq X \subset \Sigma} R_X &\longrightarrow S(R[G]) \\ (a_X)_X &\mapsto \prod_X \exp_{u_X}(a_X) \end{aligned}$$

*is an isomorphism.*

EXAMPLE 5.3 – For  $G = \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle \sigma, \tau \rangle$ , the Theorem asserts that there is an isomorphism of abelian groups

$$\begin{aligned} R^{\oplus 3} = R_{\{\sigma, \tau\}} \oplus R_{\{\sigma\}} \oplus R_{\{\tau\}} &\longrightarrow S(R[G]) \\ (\lambda, \mu, \nu) &\mapsto \exp_{1+\sigma+\tau+\sigma\tau}(\lambda) \exp_{1+\sigma}(\mu) \exp_{1+\tau}(\nu). \end{aligned}$$

In fact the generic element

$$1 + (a + b + c) + a\sigma + b\tau + c\sigma\tau \in S(R[G])$$

is of the form above with  $\lambda = 1 + c + (a + c)(b + c)$ ,  $\mu = a + c$  and  $\nu = b + c$ .

Before we turn to the proof, we need some notation and a Lemma. For  $X \subset \Sigma$ , we let  $G_X$  denote the subgroup of  $G$  generated by  $X$ . We can see  $G_X$  as a quotient of  $G$  as well, the evident map  $G \rightarrow G_X$  having kernel  $G_{\Sigma \setminus X}$ . Thus we can speak of the image of an element  $x \in R[G]$  in  $R[G_X]$ .

LEMMA 5.4 – *Let  $x \in S(R[G])$ . Then  $x$  is of the form  $\exp_{u_\Sigma}(a)$  for  $a \in R$  if and only if its image in all the rings  $R[G_X]$  for  $\emptyset \neq X \subsetneq \Sigma$  is 1.*

*Proof.* The condition is clearly necessary, as  $u_\Sigma$  maps to 0 in all the rings  $R[G_K]$ . To prove that it is sufficient, we proceed by induction on the rank of  $G$ . For  $G = \mathbb{Z}/2 = \{1, \sigma\}$ , it is certainly true that any  $x \in S(R[\mathbb{Z}/2])$  must be of the form  $x = 1 + a + a\sigma = \exp_{1+\sigma}(a)$ .

Now for the general case, write  $\Sigma = \{\sigma\} \cup \Sigma_0$  and  $H = G_{\Sigma_0}$ , so that  $G = \mathbb{Z}/2 \times H$ . We have  $R[G] = R'[H]$  for  $R' = R[\mathbb{Z}/2]$ , and we know that the result of the Lemma holds for  $R'[H]$ .

Let  $x$  be as in the Lemma. As an element of  $R'[H]$ , the augmentation of  $x$  (which is also its image under the map  $G \rightarrow G_{\{\sigma\}}$ ) must be 1 by hypothesis; further, when viewed in  $S(R'[H])$  the element  $x$  still satisfies the hypotheses of the Lemma, so  $x = \exp_{u_{\Sigma_0}}(a') = 1 + a'u_{\Sigma_0}$  for some  $a' \in R' = R[\mathbb{Z}/2]$ . By looking at the image of  $x$  under  $G \rightarrow H$ , which must be 1, we see that  $a' = a + a\sigma$  for some  $a \in R$ . We observe that

$$(1 + \sigma)u_{\Sigma_0} = u_\Sigma,$$

so in the end  $x = \exp_{u_\Sigma}(a)$ , as we wished to prove.  $\square$

*Proof of Theorem 5.2.* We prove that  $\text{EXP}$  is injective first. Let  $(a_X)_X$  be such that  $\text{EXP}(a_X)_X = 1$ . Let  $Z$  be such that  $a_Z \neq 0$ , if there is such a  $Z$ , and assume that  $Z$  has minimal cardinality with respect to this property.

We consider the images in  $R[G_Z]$  of various elements. Of course the image of  $\text{EXP}(a_X)_X$  is 1. Let  $Y \subset \Sigma$ . If  $Y$  is not a subset of  $Z$ , then  $u_Y$  maps to 0 in  $R[G_Z]$ , so  $\exp_{u_Y} a_Y$  maps to 1. If  $Y \subsetneq Z$ , then  $a_Y = 0$  by hypothesis, so again  $\exp_{u_Y} a_Y$  maps to 1. Finally, for  $Y = Z$  the image of  $\exp_{u_Z}(a_Z)$  is itself, so in the end  $\text{EXP}(a_X)_X$  restrict to  $\exp_{u_Z}(a_Z) = 1$ . This implies (easily) that  $a_Z = 0$ , a contradiction showing that  $\text{EXP}$  is injective.

We turn to the surjectivity. Let  $x \in S(R[G])$ . If the image of  $x$  in  $R[G_X]$  is 1 for all the proper subsets  $X$  of  $\Sigma$ , then  $x$  is in the image of  $\text{EXP}$  by the Lemma.

Now let  $Z$  be a proper subset of  $\Sigma$  such that the image of  $x$  in  $R[G_Z]$  is not equal to 1, and assume that  $Z$  has minimal cardinality with respect to this property. By the Lemma again, the image of  $x$  in  $R[G_Z]$  is of the form  $\exp_{u_Z}(a)$  for some  $a \in R$ . Let  $x' = \exp_{u_Z}(a)$ , viewed as an element of  $S(R[G])$ , and consider  $x_1 = xx'$ . Its image in  $R[G_Z]$  is 1 by construction. What is more, if the image of  $x$  is 1 in  $R[G_X]$  for some  $X$ , then  $Z$  is certainly not a subset of  $X$ ; so  $u_Z$  maps to 0 in  $R[G_X]$  and  $x'$  maps to 1 there, as does  $x_1$ .

Continuing, we form  $x_2, x_3$ , etc, such that  $x_{i+1} = x_i x^{(i+1)}$  with  $x^{(i+1)}$  belonging to the image of  $\text{EXP}$ , and such that  $x_{i+1}$  maps to 1 in  $R[G_X]$  whenever  $x_i$  does *and* for one extra subset. This process stops when some  $x_i$  maps to 1 in all the rings corresponding to all the proper subsets of  $\Sigma$ , in which case  $x_i$  is in the image of  $\text{EXP}$  as already observed. We conclude that  $x$  is in the image of  $\text{EXP}$ , and this map is surjective.  $\square$

**5.2. The map  $\phi$ .** We shall now study a certain map  $\phi$  from  $R[G]$  to  $R[G \times G]$ . It is defined as the product of the inclusion and the diagonal, that is

$$\phi(x) = x \Delta(x) \in R[G \times G].$$

Whenever  $x \in S(R[G])$ , it is clear that  $\phi(x)$  lies in the subgroup  $S_{R'}(R'[G])$  where  $R' = R[G]$ , as in  $(\dagger)$ . Thus we shall consider  $\phi$  as a map

$$\phi: S(R[G]) \longrightarrow S(R'[G]).$$

We can apply Theorem 5.2 to both  $S(R[G])$  and  $S(R'[G])$ , of course. The map  $\phi$  induces, via the  $\text{EXP}$  isomorphisms, a map  $\tilde{\phi}$ . The latter does not preserve the direct sum decompositions, but it is compatible with certain filtrations.

For each  $\ell \geq 1$ , let  $S(R[G])_\ell$  denote the image of the sum of all the  $R_X$  with  $X$  of cardinality  $\leq \ell$ , under the map of the Theorem. Also, let  $S(R'[G])_X$  denote the image of the sum of all the  $R_Y$  with  $Y \subset X$ .

LEMMA 5.5 – *We have*

$$\phi(S(R[G])_\ell) \subset S(R'[G])_\ell$$

and

$$\phi(S(R[G])_X) \subset S(R'[G])_X.$$

*Proof.* The first statement follows from the second. There is nothing to prove if  $X = \Sigma$ . If not, consider the subgroup  $G_X \subset G$  spanned by  $X$ , and appeal to the naturality of all the maps in sight with respect to the inclusion  $G_X \rightarrow G$ .  $\square$

There is a canonical isomorphism, for  $X$  of cardinality  $\ell$ ,

$$R_X \cong \frac{S(R[G])_X}{S(R[G])_X \cap S(R[G])_{\ell-1}},$$

induced by  $\exp_{u_X}$ . Thus  $\phi$  induces a map

$$\phi_X: R_X \rightarrow R'_X$$

which we wish to describe explicitly. Let  $m_X$  be the product of the elements of  $X$ .

PROPOSITION 5.6 – *Let  $a \in R_X$  be such that  $a^2 = 0$ . Then*

$$\phi_X(a) = a m_X .$$

Note that  $m_X \in G$  and  $a m_X$  is indeed an element of  $R' = R[G]$ .

*Remark 5.7.* It is very likely that  $\phi_X(a) = a m_X$  for all  $a \in R_X$  without restriction, as soon as the cardinality of  $X$  is at least 2 (though definitely not when  $X$  is reduced to one element). Computer calculations have confirmed this when  $2 \leq |X| \leq 7$ .

*Proof.* The case when  $X = \{\sigma\}$  is both simple and important for the general case. In this situation we have  $u = u_X = 1 + \sigma$ , and we need to consider  $\phi(x)$  for

$$x = \exp_u(a) = 1 + a + a\sigma .$$

Direct calculation yields then

$$\phi(x) = 1 + (a\sigma_1 + a^2\sigma_1 + a^2)(1 + \sigma_2) = \exp_{1+\sigma_2}(a\sigma_1 + a^2\sigma_1 + a^2),$$

were the elements of  $G \times G$  are decorated with indices. So if we assume that  $a^2 = 0$  we have indeed

$$(*) \quad \phi(\exp_{u_X}(a)) = \exp_{u_X}(a\sigma)$$

where  $u_X$  is interpreted (slightly) differently on either side of this equation.

We turn to the general case. Let  $X = \{x_1, \dots, x_\ell\}$ , and let  $s_i$  be the  $i$ -th symmetric function in the  $x_j$ 's (so that  $s_\ell = m_X$ ). We have  $u = u_X = 1 + s_1 + \dots + s_\ell$ . The idea is to replace  $u$  by  $1 + s_\ell$  and reduce to the case  $\ell = 1$ .

To see this, start by observing that

$$\exp_u(a) \exp_{1+s_\ell}(a) = \exp_{s_1+\dots+s_{\ell-1}}(a),$$

from Lemma 5.1 (2) (either since  $a^2 = 0$  or since  $u(1 + s_\ell) = 0$ ). Call  $y$  the right hand side. By a repeated use of Lemma 5.1 (2), we see that we may write  $y$  as a product of terms of the form  $\exp_v(a)$  with  $v$  in the  $\mathbb{F}_2$ -subalgebra of  $R[G]$  generated by some of the  $x_i$ 's, but always less than  $\ell$  of them, so  $y$  lies in  $S(R[G])_{\ell-1}$ . Finally

$$(**) \quad \exp_u(a) = \exp_{1+s_\ell}(a) \text{ mod } S(R[G])_{\ell-1} .$$

On the other hand we can compute the value of  $\phi(\exp_{1+s_\ell}(a))$  by (\*) (applied to the case  $X = \{s_\ell\}$ ):

$$\phi(\exp_{1+s_\ell}(a)) = \exp_{1+s_\ell}(a s_\ell) .$$

(Here the cautious reader can rewrite this with indices  $s_{\ell,1}$  and  $s_{\ell,2}$  if she wishes to distinguish between the two.) Working with (\*\*) backwards yields the result.  $\square$

We conclude with some remarks about the compatibility of  $\phi$  with augmentation maps. We write  $\varepsilon: R[G] \rightarrow R$  for the usual augmentation. Keeping the notation  $R' = R[G]$ , we point out that the construction of  $R[G]$  is natural in  $R$ , so that  $\varepsilon: R' \rightarrow R$  induces a map  $e: R'[G] \rightarrow R[G]$ . (If we think of  $R'[G]$  as  $R[G] \otimes_R R[G]$ , then we have  $e(\sigma \otimes \tau) = \tau$ .) It is immediate that

LEMMA 5.8 – *For all  $x \in R[G]$ , one has  $e \circ \phi(x) = x$ .*

We summarize the notation of this section in a commutative diagram.

$$\begin{array}{ccc}
\bigoplus_X R_X & \xrightarrow{\text{EXP}} & S(R[G]) \\
\tilde{\phi} \downarrow & & \downarrow \phi \\
\bigoplus_X R'_X & \xrightarrow{\text{EXP}} & S(R'[G]) \\
\oplus \varepsilon \downarrow & & \downarrow e \\
\bigoplus_X R_X & \xrightarrow{\text{EXP}} & S(R[G])
\end{array}$$

The horizontal maps are isomorphisms, and the compositions of the vertical maps are identities.

### 5.3. Vanishing of the twist cohomology.

**THEOREM 5.9** – *For  $n \geq 3$ , we have  $H^n(A_X^*, d) = 0$ .*

*Proof.* We apply Proposition 3.5. It follows that  $H^n(A_X^*, d) = H^n(B^*, d)$  where  $B^* = S_{A^{*-1}}(A^*)$  in the notation above. Put

$$C^* = \bigoplus_X A_X^*,$$

where  $A_X^*$  is a copy of  $A^*$ , so that the EXP isomorphism of Theorem 5.2 provides us with an isomorphism  $C^* \cong B^{*+1}$ . We set  $\partial = \text{EXP}^{-1} \circ d \circ \text{EXP}$ , so that our goal is to prove that  $H^n(C^*, \partial) = 0$  for  $n \geq 2$ .

The differential  $\partial$  on  $C^{n-1} \cong B^n$  is the sum of maps  $\partial^i = \text{EXP}^{-1} \circ d^i \circ \text{EXP}$  for  $0 \leq i \leq n$  which are given by Lemma 3.6. For  $0 \leq i < n$ , it turns out that  $\partial^i$  coincides with  $d^i$ , or rather a direct sum of copies of  $d^i$  indexed by the subsets  $X$ . For  $i = n$ , we have  $\partial^n = \text{EXP}^{-1} \circ \phi \circ \text{EXP} = \tilde{\phi}$  in the notation of §5.2, where  $R = A^{n-1}$ .

From Lemma 5.5, it follows that  $C^*$  has a filtration by subcomplexes (preserved by  $\partial$ ), and the subquotients are cochain complexes of the form  $(A_X^*, \partial_X)$ . In degree  $n$  the underlying abelian group is  $A_X^n$ , a copy of  $A^n$ , and the differential  $\partial_X : A_X^{n-1} \rightarrow A_X^n$  is given by

$$(*) \quad \partial_X = \sum_{i=0}^{n-1} d^i + \phi_X,$$

where the notation  $\phi_X$  is as in Proposition 5.6 (again for  $R = A^{*-1}$ ). We will prove that  $H^n(A_X^*, \partial_X) = 0$  for  $n \geq 2$  (and for each  $X$ ), which implies the Theorem from the long exact sequences in cohomology.

From Lemma 5.8, we see that the complex  $(A_X^*, \partial_X)$  satisfies the hypotheses of Lemma 3.2. As a result, in order to compute its cohomology, we may restrict to the subgroup  $\bar{A}_X^*$  of those elements  $a$  with  $\varepsilon(a) = 0$ . In degree  $n \geq 1$ , we have  $a^2 = \varepsilon(a)^2$ , so these elements satisfy  $a^2 = 0$ . (Recall that in degree 0 we have  $A_X^0 = k$  and the “augmentation” is the zero map, so we can draw no such conclusion).

We can thus use Proposition 5.6. Together with (\*), it implies for  $a \in \bar{A}_X^n$  and  $n \geq 1$  that

$$\partial_X(a) = d(a) + a(1 + m_X).$$

In other words, in degrees  $n \geq 1$ , the differential  $\partial_X$  on  $\bar{A}_X^*$  coincides with  $\delta_{m_X}$  considered in Proposition 3.7. By that Proposition, the cohomology does vanish in degrees  $\geq 2$ .  $\square$

**5.4. The Sweedler cohomology groups.** In this section we prove the following result.

**THEOREM 5.10** – *Let  $k$  be a field of characteristic 2. The twist cohomology of  $k[(\mathbb{Z}/2)^r]$ , or the Sweedler cohomology of  $\mathcal{O}_k((\mathbb{Z}/2)^r)$ , is given by*

$$H_{sw}^n(\mathcal{O}_k((\mathbb{Z}/2)^r)) = \begin{cases} 0 & \text{for } n \geq 3 \text{ or } n = 0, \\ (\mathbb{Z}/2)^r & \text{for } n = 1, \\ (k/\{x + x^2 : x \in k\})^{\oplus r} & \text{for } n = 2. \end{cases}$$

When  $k$  is an algebraically closed field, we have in particular  $H_{sw}^2(\mathcal{O}_k((\mathbb{Z}/2)^r)) = 0$ .

The statement for  $n \geq 3$  is Theorem 5.9, while the statements for  $n = 0$  or 1 are classical and easy (they hold for any finite group  $G$ ). What remains is the result for  $n = 2$ , which we have established in the case  $r = 1$  with Theorem 4.2.

Fortunately there is a Künneth-type theorem for  $H_{sw}^2$ , established in the context of “lazy cohomology” by Bichon and Carnovale: see Theorem 4.8 in [BC06] which stipulates that

$$H_{\ell}^2(A \otimes B) \cong H_{\ell}^2(A) \times H_{\ell}^2(B) \times \mathcal{ZP}(A \otimes B),$$

for any two Hopf algebras  $A$  and  $B$ . Recall that  $H_{\ell}^2(A) = H_{sw}^2(A)$  when  $A$  is co-commutative. Moreover, Lemma 4.9 in *loc. cit.* describes  $\mathcal{ZP}(A \otimes B)$  as a group of Hopf algebra homomorphisms  $A \rightarrow B^*$  (the dual of  $B$ ), satisfying certain conditions. However, in our situation the following must be noticed.

**LEMMA 5.11** – *Let  $G$  be any finite group, let  $P$  be a finite  $p$ -group, and let  $k$  be a field of characteristic  $p$ . Then there is only one homomorphism of Hopf algebras*

$$\mathcal{O}_k(G) \longrightarrow k[P],$$

namely the “augmentation”  $f \mapsto f(1)1$ .

*Proof.* Let  $K$  be any algebra at all, and let  $\phi: \mathcal{O}_k(G) \rightarrow K$  be any algebra homomorphism. Letting  $\delta_g$  denote the Dirac function at  $g \in G$ , we see that  $\phi$  is entirely determined by the elements  $x_g = \phi(\delta_g) \in K$ , which must be idempotents summing to 1 and satisfying  $x_g x_h = 0$  whenever  $g \neq h$ . If we assume that the only idempotents in  $K$  are 0 and 1, then it follows that there is one and only one  $g$  such that  $x_g = 1$  and all other  $x_h$  are zero. Thus  $\phi(f) = f(g)1$ .

Assume further that  $K$  is a Hopf algebra and that  $\phi$  is a Hopf algebra homomorphism. Examination of the relation  $\Delta(\phi(\delta_g)) = \phi \otimes \phi(\Delta(\delta_g))$  reveals that  $g = 1$ .

The key point is then the fact that this argument applies to  $K = k[P]$ , since the group algebra of a  $p$ -group, in characteristic  $p$ , is indecomposable and thus has no other idempotents beside 0 and 1.  $\square$

The Lemma implies that  $\mathcal{ZP}(A \otimes B)$  is the trivial group when  $A = \mathcal{O}_k(\mathbb{Z}/2)$  and  $B = \mathcal{O}_k((\mathbb{Z}/2)^r)$ . Thus what remains to be proved in Theorem 5.10 follows from Theorem 4.2 by induction.

## 6. LAZY COHOMOLOGY OF FUNCTION ALGEBRAS

We now turn our attention to the result obtained by Kassel and the author in [GK10], and seek to adapt it to positive characteristic. So now  $k$  is any field of characteristic  $p$ , we consider an arbitrary finite group  $G$ , and we consider the second lazy cohomology group  $H_{\ell}^2(\mathcal{O}(G))$  which was described at the end of section 2.

**6.1. Twists and  $R$ -matrices.** Let  $F$  be a Drinfeld twist on the Hopf algebra  $\mathcal{H}$ . If we put

$$R_F = F_{21}F^{-1},$$

then  $R_F \in \mathcal{H} \otimes \mathcal{H}$  is an  $R$ -matrix. In [GK10], we have exploited the fact that, for  $\mathcal{H} = k[G]$  with  $G$  a finite group, the  $R$ -matrix  $R_F$  essentially determines  $F$  up to equivalence (a more precise statement follows). What is more, a result of Radford ([Rad93]) shows that any  $R$ -matrix at all for  $k[G]$  lives in fact in  $k[A] \otimes k[A]$  where  $A$  is an abelian, normal subgroup of  $G$ .

These results are valid regardless of the characteristic of  $k$ , and in order to extend the main theorem in [GK10] we are thus led to investigate  $R$ -matrices for Hopf algebras of the form  $k[A]$  where  $k$  has positive characteristic.

**6.2.  $R$ -matrices on abelian  $p$ -group algebras.** We wish to prove the following result.

**PROPOSITION 6.1** – *Let  $k$  be a field of characteristic  $p$ . If  $A$  is a finite abelian  $p$ -group, then the only  $R$ -matrix on the Hopf algebra  $k[A]$  is the trivial one  $R = 1 \otimes 1$ . More generally, if  $A$  is a finite abelian group, and if we write  $A = A_p \times A'$  where  $A_p$  is the  $p$ -Sylow subgroup of  $A$ , then any  $R$ -matrix on  $k[A]$  belongs to  $k[A'] \otimes k[A']$ .*

*Proof.* Writing  $R = \sum_{a,b} \lambda_{ab} a \otimes b$ , where  $\lambda_{ab} \in k$  and  $a, b \in A$ , we define a map  $\phi_R: \mathcal{O}(A) \rightarrow k[A]$  by the formula

$$(*) \quad \phi_R(f) = \sum_{a,b} \lambda_{ab} f(a)b.$$

The axioms for  $R$ -matrices imply that  $\phi_R$  is a homomorphism of Hopf algebras, as the reader will check. Thus Lemma 5.11 implies that  $\phi_R(f) = f(1)1$ , for all  $f \in \mathcal{O}(A)$ . It follows that  $R = 1 \otimes 1$ .

For the general statement, one establishes that (\*) gives in fact a bijection  $R \mapsto \phi_R$  between  $R$ -matrices for  $k[A]$  and homomorphisms of Hopf algebras  $\mathcal{O}(A) \rightarrow k[A]$ ; moreover this bijection is natural in  $A$ . Once this is granted, one starts with an  $R$ -matrix  $R$  for  $k[A]$  and composes  $\phi_R$  with the projection  $k[A] \rightarrow k[A_p]$ ; this composition  $\mathcal{O}(A) \rightarrow k[A_p]$  must be the trivial (augmentation) homomorphism, by Lemma 5.11. It follows that  $\phi(f) \in k[A']$ , for all  $f$ , so that precomposing with  $\mathcal{O}(A') \rightarrow \mathcal{O}(A)$  gives a homomorphism of Hopf algebras  $\phi_{R'}: \mathcal{O}(A') \rightarrow k[A']$  corresponding to an  $R$ -matrix  $R'$  for  $k[A']$ . By inspection, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}(A) & \xrightarrow{\phi_R} & k[A] \\ \downarrow & & \uparrow \\ \mathcal{O}(A') & \xrightarrow{\phi_{R'}} & k[A'] \end{array}$$

where the vertical maps are induced by the inclusion  $A' \rightarrow A$ . It follows that  $R$  is the image of  $R'$  under the map  $k[A'] \rightarrow k[A]$ .  $\square$

This explains the relation  $F_{21}F^{-1} = 1 \otimes 1$  which we had observed in example 4.3.

In order to complete the picture, at least when  $k$  is algebraically closed, there remains only to state the following.

**PROPOSITION 6.2** – *Let  $k$  be algebraically closed of characteristic  $p$ , and let  $A$  be a finite abelian group of order prime to  $p$ . Then there is a bijection between the set of  $R$ -matrices on  $k[A]$  and the bilinear forms on the Pontryagin dual of  $A$  with values in  $k^\times$ .*

*Moreover, if  $R = \sum_i \lambda_i a_i \otimes b_i$  with  $\lambda_i \in k$  and  $a_i, b_i \in A$ , then the bilinear form corresponding to  $R$  is alternating if and only if*

$$u_R := \sum_i \lambda_i a_i^{-1} b_i = 1.$$

(In the proof we recall the relevant definitions. The element  $u_R$  is called the Drinfeld element of  $R$ .)

*Proof.* Let  $\widehat{A} = \text{Hom}(A, k^\times)$  be the Pontryagin dual of  $A$ . The discrete Fourier transform is the homomorphism

$$k[A] \longrightarrow \mathcal{O}(\widehat{A})$$

defined by  $g \mapsto \widehat{g}$ , where  $\widehat{g}(\chi) = \chi(g)$  for  $\chi \in \widehat{A}$ . The hypotheses on  $k$  guarantee that the discrete Fourier transform is an isomorphism of Hopf algebras.

As a consequence of this result, applied in fact to  $A \times A$ , we have a dictionary between  $k[A] \otimes k[A]$  and  $\mathcal{O}(\widehat{A} \times \widehat{A})$ , that is the algebra of functions  $\widehat{A} \times \widehat{A} \rightarrow k$ . An  $R$ -matrix for  $k[A]$  thus defines (and can be defined by) a map

$$b: \widehat{A} \times \widehat{A} \longrightarrow k^\times,$$

such that  $x \mapsto b(x, y)$  is a homomorphism for fixed  $y$ , and  $y \mapsto b(x, y)$  is a homomorphism for fixed  $x$ . It is also immediate that  $u_R = 1$  if and only if  $b(x^{-1}, x) = 1$  for all  $x \in \widehat{A}$ . This is the conclusion of the Proposition.  $\square$

It is instructive to see how this proof compares with the previous one. The reader who is so inclined will check that, letting  $\mathbf{A} = \text{Spec}(k[A])$  denote the affine group scheme associated to  $k[A]$ , then  $R$ -matrices on  $k[A]$  are in bijection with bilinear maps  $\mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{G}_m$ . One can prove both Proposition 6.1 and Proposition 6.2 using this language; in the former case, the correspondence between  $R$  and  $\phi_R$  is elucidated, while in the latter case the bilinear maps  $\mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{G}_m$  turn out to be equivalent to bilinear maps  $\widehat{A} \times \widehat{A} \longrightarrow k^\times$  via the Fourier transform.

**6.3. The main theorem.** Let  $G$  be a finite group, and  $k$  an algebraically closed field of characteristic  $p$ . We let  $\mathcal{B}(G)$  denote the set of pairs  $(A, b)$  where  $A$  is an abelian, normal subgroup of  $G$  of order prime to  $p$ , and  $b$  is an alternating bilinear form  $\widehat{A} \times \widehat{A} \rightarrow k^\times$  which is  $G$ -invariant, and non-degenerate.

Moreover, let  $\text{Int}_k(G)$  denote the group of automorphisms of  $G$  induced by conjugation by elements of  $k[G]$ , while  $\text{Inn}(G)$  is the group of inner automorphisms of  $G$ ; the quotient  $\text{Int}_k(G)/\text{Inn}(G)$  is a subgroup (which is often trivial in practice) of  $\text{Out}(G)$ .

**THEOREM 6.3** – *There is a map  $\Theta: H_\ell^2(G) \rightarrow \mathcal{B}(G)$  such that*

- (a) *The subset  $\Theta^{-1}(1)$  is a subgroup of  $H_\ell^2(G)$  isomorphic to  $\text{Int}_k(G)/\text{Inn}(G)$ ;*
- (b) *The fibres of  $\Theta$  are the left cosets of  $\Theta^{-1}(1)$ ;*
- (c)  *$\Theta$  is surjective if all the subgroups  $A$  involved in the definition of  $\mathcal{B}(G)$  have odd order. In particular,  $\Theta$  is surjective if  $k$  has characteristic 2.*

*Proof.* The proof of Theorem 4.5 in [GK10] goes through with only one simple change, emphasized below. The details of the following argument can all be found in *loc. cit.*

To construct  $\Theta$ , consider a twist  $F$  and the  $R$ -matrix  $R_F = F_{21}F^{-1}$ . There is a unique minimal, abelian, normal subgroup  $A$  of  $G$  such that  $R_F \in k[A] \otimes k[A]$ , and by Proposition 6.1, we know that the order of  $A$  is prime to  $p$ . By Proposition 6.2, the  $R$ -matrix  $R_F$  gives rise to a bilinear form  $b$  on  $\widehat{A}$ . One can prove that the Drinfeld element of  $R_F$  is 1 so that  $b$  is alternating, and the minimality of  $A$  shows that  $b$  is non-degenerate; the fact that  $F$  is assumed to be  $G$ -invariant shows that  $b$  is  $G$ -invariant. Thus it makes sense to put  $\Theta(F) = (A, b)$ .

The study of the fibres of the map  $\Theta$  so constructed is identical to that carried out in [GK10]. Likewise for the surjectivity of  $\Theta$  in good cases.  $\square$

EXAMPLE 6.4 – Let  $G$  be a  $p$ -group, and let  $k$  have characteristic  $p$ . Then  $\mathcal{B}(G)$  has only one element, by construction, so we conclude from the Theorem that

$$H_\ell^2(\mathcal{O}(G)) = \text{Int}_k(G)/\text{Inn}(G)$$

in this case. If moreover  $G$  is abelian, it follows that  $H_\ell^2((G)) = 0$ , which we had observed with  $G = \mathbb{Z}/2$  earlier. This example also shows that the condition that  $k$  be algebraically closed cannot be removed.

#### APPENDIX A. COSIMPLICIAL GROUPS OBTAINED FROM COSIMPLICIAL SETS

In this Appendix we aim to prove Lemma 2.1. In passing we recall the basic definitions of cosimplicial sets. The material below grew out of an exchange on MathOverflow which the author had with Tom Goodwillie and Fernando Muro.

Let  $\Delta$  be the simplex category, whose objects are  $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots$  where  $\mathbf{n}$  is the ordered set  $\{0, 1, 2, \dots, n\}$ , and whose morphisms are the non-decreasing maps. A cosimplicial set is simply a functor from  $\Delta$  to the category of sets. For the convenience of the reader we recall that the morphisms in  $\Delta$  are compositions of certain maps  $d^i$  and  $s^j$ , satisfying

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \text{ for } i < j \\ s^j d^i &= \begin{cases} d^i s^{j-1} \text{ for } i < j, \\ Id \text{ for } i = j, j+1, \\ d^{i-1} s^j \text{ for } i > j+1, \end{cases} \\ s^j s^i &= s^i s^{j+1} \text{ for } i \leq j. \end{aligned}$$

Moreover these “are enough”; that is, one can show that a cosimplicial set  $X^*$  is precisely defined by a set  $X^n$  for each integer  $n$  (we say that  $X^n$  is in “degree  $n$ ”), together with maps  $d^i = d_n^i: X^n \rightarrow X^{n+1}$  and  $s^j = s_n^j: X^n \rightarrow X^{n-1}$  (with  $0 \leq i \leq n+1$  and  $0 \leq j \leq n-1$ ) satisfying the relations above.

Given an integer  $m \geq 0$ , there is a cosimplicial set which can be called the “free cosimplicial set on one point in degree  $m$ ”, and which is given by  $\text{Hom}_\Delta(\mathbf{m}, \mathbf{n})$  in degree  $n$ . However, we will instead consider the *semi*-cosimplicial set  $F_m$  which in degree  $n$  consists of all *injective* maps  $\mathbf{m} \rightarrow \mathbf{n}$  in  $\Delta$ . Recall that “semi-cosimplicial” means that that  $F_m$  is endowed with cofaces, but no codegeneracies. Note also that  $F_m^n$  is empty for  $n < m$ . We shall also need to speak of the cosimplicial set which is reduced to a point in every degree; we call it “the cosimplicial point”.

The next Lemma says that any cosimplicial set is almost free as a semi-cosimplicial set, except for the presence of cosimplicial points.

LEMMA A.1 (GOODWILLIE) – *Any cosimplicial set is a disjoint union of cosimplicial points and copies of  $F_m$  (for various values of  $m$ ), as semi-cosimplicial set.*

*Proof.* The *dual* of this statement is probably more familiar to the reader. Namely in a simplicial set  $S_*$ , if we call non-degenerate the simplices which are not in the image of any degeneracy map, then any element  $x \in S_*$  can be written uniquely  $x = s_{i_q} \cdots s_{i_1} y$  with  $i_1 \leq i_2 \leq \cdots \leq i_q$  and  $y$  non-degenerate.

Dually, in a cosimplicial set  $X$ , call an element a *root* of  $X$  if it is not in the image of any coface map. Then any  $x \in X^*$  can (almost tautologically) be written  $x = d^{i_1} \cdots d^{i_q} y$  where  $y$  is a root and  $i_1 \leq i_2 \leq \cdots \leq i_q$ ; more importantly, if  $y$  can be taken in degree  $> 0$ , then this writing is unique; as for roots in degree 0, they generate either a cosimplicial point or a copy of  $F_0$ . We let the proof of this fact as a (not entirely painless) exercise (start by proving that the relation  $d^i(x) = d^j(y)$ , when  $x$  and  $y$  are roots, implies that  $x = y$  and either  $i = j$  or the degree of  $x$

is 0). We point out however that the presence of codegeneracies is crucial here (for example the relations  $s^i d^i = Id$  guarantee that the cofaces are injective).

The lemma follows immediately from this. The various copies of  $F_m$  are indexed by the set of roots of  $X$ ; from now on the word “root” will exclude the elements of degree 0 which generate a cosimplicial point.  $\square$

COROLLARY A.2 – *Let  $k$  be any ring, and let  $k[X]^*$  be the cosimplicial  $k$ -module obtained by taking in degree  $n$  the free  $k$ -module on  $X^n$ . Then for  $n > 0$*

$$H^n(k[X]^*) = \bigoplus_r H^n(k[F_{m_r}]^*).$$

where  $r$  runs through the roots of  $X$ , and  $m_r$  is the degree of  $r$ .

We have used that the cohomology of a cosimplicial point is 0 in degrees  $> 0$ . Let us now consider a specific cosimplicial set  $S^*$ .

LEMMA A.3 (MURO) – *Let  $S$  be any pointed set, and let  $S^n$  be the cartesian product of  $n$  copies of  $S$ . Define a cosimplicial set structure on  $S^*$  by*

$$\begin{aligned} d^0(x_1, \dots, x_n) &= (*, x_1, \dots, x_n), \\ d^i(x_1, \dots, x_n) &= (x_1, \dots, x_i, x_i, \dots, x_n), \\ d^n(x_1, \dots, x_n) &= (x_1, \dots, x_n, *), \end{aligned}$$

while the codegeneracy  $s^i$  omits the  $i$ -th entry. (Here  $*$  is the base-point of  $S$ ).

Then for any field  $k$  the cohomology  $H^n(k[S]^*)$  vanishes for  $n > 0$ .

*Proof.* The trick is to consider the dual chain complex. Let  $V = k[S]^1$ , so that  $k[S]^n = V^{\otimes n}$ , and let  $R = Hom_k(V, k)$ . Then  $R$  can be seen as the vector space of  $k$ -valued functions on  $S$ , and as such is a ring. We have  $Hom_k(V^{\otimes n}, k) = R^{\otimes n}$ .

If we now inspect the chain complex  $Hom_k(k[S]^*, k)$ , we recognize the Hochschild complex of the ring  $R$  (with values in the  $R$ -module  $k$ , the module structure being given by evaluation at the base point of  $S$ ). Since  $R$  is a product of  $N$  copies of  $k$ , where  $N$  is the cardinal of  $S$ , the Künneth formula shows then that  $H_n(Hom_k(k[S]^*, k)) = 0$  for  $n > 0$ . Therefore, we also have  $H^n(k[S]^*) = 0$  for  $n > 0$ .  $\square$

COROLLARY A.4 – *Let  $X^*$  be any cosimplicial set. Then the group  $H^n(k[X]^*)$  vanishes for  $n > 0$ .*

*Proof.* Since this holds for the example  $S^*$  of the Lemma, we gather from the previous Corollary that  $H^n(k[F_m]^*) = 0$  for  $n > 0$  whenever  $m$  is one of those integers such that  $F_m$  shows up in the decomposition of  $S^*$ . However, whatever the integer  $m$ , it suffices to take  $S$  with  $m + 1$  elements  $x_0 = *, x_1, \dots, x_m$  to obtain a root  $(x_1, \dots, x_m)$  in degree  $m$  for the cosimplicial set  $S^*$ .

We conclude that  $H^n(k[F_m]^*) = 0$  for all  $m \geq 0$  and all  $n > 0$ . Thus from the previous Corollary,  $H^n(k[X]^*) = 0$  for any  $X^*$ .  $\square$

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