



**HAL**  
open science

## Structures in Concrete Categories

Henri Bourlès

► **To cite this version:**

| Henri Bourlès. Structures in Concrete Categories. 2019. hal-02372397

**HAL Id: hal-02372397**

**<https://hal.science/hal-02372397>**

Preprint submitted on 20 Nov 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Structures in Concrete Categories

Henri Bourlès\*

October 1, 2015

## Abstract

The monumental treatise "Éléments de mathématique" of N. Bourbaki is based on the notion of structure and on the theory of sets. On the other hand, the theory of categories is based on the notions of morphism and functor. An appropriate definition of a structure in a concrete category is given in this paper. This definition makes it possible to bridge the gap between Bourbaki's structural approach and the categorical language.

## 1 Introduction

The monumental treatise "Éléments de mathématique" of N. Bourbaki is based on the notion of structure and on the theory of sets [2]. On the other hand, the theory of categories is based on the notions of morphism and functor. The notion of "concrete category", as developed in [1], did not bridge the gap between the categorical approach and Bourbaki's structures. According to ([1], Remark 5.12), a structure in a concrete category  $(\mathcal{C}, |\cdot|, \mathcal{X})$ , where  $\mathcal{X}$  is the base category and  $|\cdot| : \mathcal{C} \rightarrow \mathcal{X}$  is the forgetful functor, is an object  $A$  of  $\mathcal{C}$ . If  $A, A'$  are two  $\mathcal{C}$ -isomorphic objects such that  $|A| = |A'|$ , these objects are not equal in general. Nevertheless, if  $\mathcal{C}$  is, for example, the category **Top** of topological spaces and  $\mathcal{X}$  is the category **Set** of sets,  $A$  and  $A'$  are two equal sets endowed with the same topology.

**Definition 1** *A structure in  $(\mathcal{C}, |\cdot|, \mathcal{X})$  is an equivalence class of objects of  $\mathcal{C}$  with respect to the following equivalence relation: two objects  $A$  and  $A'$  of  $\mathcal{C}$  are equivalent if  $|A| = |A'|$  and there is a  $\mathcal{C}$ -isomorphism  $A \rightarrow A'$ .*

Some consequences of this definition are studied the sequel.

---

\*Satie, ENS de Cachan/CNAM, 61 Avenue Président Wilson, F-94230 Cachan, France (henri.bourles@satie.ens-cachan.fr)

## 2 Structures

### 2.1 Isomorphisms of structures

Let  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$  be two structures in  $(\mathcal{C}, |\cdot|, \mathcal{X})$ . Assume that there exist representatives  $A_1$  and  $A_2$  of  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$ , respectively, which are isomorphic in  $\mathcal{C}$ . Then  $|A_1|$  and  $|A_2|$  are not necessarily equal, but are isomorphic in  $\mathcal{X}$ . If  $A'_1$  and  $A'_2$  are other representatives of  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$ , they are isomorphic in  $\mathcal{C}$ .

**Definition 2** *The structures  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$  are called isomorphic if the above condition holds.*

This definition determines an equivalence relation on structures in  $(\mathcal{C}, |\cdot|, \mathcal{X})$ .

### 2.2 Comparison of structures

Obviously, if  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$  are two structures in  $(\mathcal{C}, |\cdot|, \mathcal{X})$ , and  $A_1$  and  $A_2$  are representatives of  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$ , respectively, such that  $|A_1| = |A_2|$  and there exists a  $\mathcal{C}$ -morphism  $A_1 \rightarrow A_2$  such that  $|A_1 \rightarrow A_2| = \text{id}_{|A_1|}$ , then for any representatives  $A'_1$  and  $A'_2$  of  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$ , respectively, there exists a  $\mathcal{C}$ -morphism  $A'_1 \rightarrow A'_2$  such that  $|A'_1 \rightarrow A'_2| = \text{id}_{|A'_1|}$ .

**Definition 3** *If the above condition holds, then the structure  $\mathfrak{S}_1$  is called finer than  $\mathfrak{S}_2$  (written  $\mathfrak{S}_1 \geq \mathfrak{S}_2$ ) and  $\mathfrak{S}_2$  is called coarser than  $\mathfrak{S}_1$  (written  $\mathfrak{S}_2 \leq \mathfrak{S}_1$ ).*

**Lemma 4** *The relation  $\geq$  on structures in  $(\mathcal{C}, |\cdot|, \mathcal{X})$  is an order relation.*

**Proof.** The reflexivity and the transitivity are obvious. Assume that  $\mathfrak{S}_1 \geq \mathfrak{S}_2$  and  $\mathfrak{S}_2 \geq \mathfrak{S}_1$ . Let  $A_1, A_2$  be representatives of  $\mathfrak{S}_1, \mathfrak{S}_2$  respectively. There exist  $\mathcal{C}$ -morphisms  $A_1 \rightarrow A_2$  and  $A_2 \rightarrow A_1$  such that  $|A_1 \rightarrow A_2| = \text{id}_{|A_1|}$  and  $|A_2 \rightarrow A_1| = \text{id}_{|A_2|}$ . Thus  $A_1 \rightarrow A_2 \rightarrow A_1$  and  $A_2 \rightarrow A_1 \rightarrow A_2$  are  $\mathcal{C}$ -morphisms such that  $|A_1 \rightarrow A_2 \rightarrow A_1| = \text{id}_{|A_1|}$  and  $|A_2 \rightarrow A_1 \rightarrow A_2| = \text{id}_{|A_2|}$ . Since the forgetful functor is faithful, it follows that  $A_1 \rightarrow A_2 \rightarrow A_1 = \text{id}_{A_1}$  and  $A_2 \rightarrow A_1 \rightarrow A_2 = \text{id}_{A_2}$ . Therefore,  $A_1 \rightarrow A_2$  and  $A_2 \rightarrow A_1$  are isomorphisms, and each of them is the inverse of the other. As a result,  $\mathfrak{S}_1 = \mathfrak{S}_2$ . ■

**Example 5** *A structure in the concrete category  $(\mathbf{Top}, |\cdot|, \mathbf{Set})$  is a topology. The structure  $\mathfrak{S}_1$  is finer than  $\mathfrak{S}_2$  if, and only if the topology  $\mathfrak{S}_1$  is finer than the topology  $\mathfrak{S}_2$ .*

### 2.3 Initial structures and final structures

Let  $A, B$  be objects of  $\mathcal{C}$ . Recall that the relation "  $f : |B| \rightarrow |A|$  is a  $\mathcal{C}$ -morphism " means that there exists a morphism  $f : B \rightarrow A$  (necessarily unique) such that  $|B \rightarrow A| = |B| \rightarrow |A|$  ([1], Remark 6.22).

Let  $(f_i : A \rightarrow A_i)_{i \in I}$  be a source in  $\mathcal{C}$ , i.e. a family of  $\mathcal{C}$ -morphisms. This source is called initial if for each object  $B$  of  $\mathcal{C}$ , the relation

” $f : |B| \rightarrow |A|$  is a  $\mathcal{C}$ -morphism”  
 is equivalent to the relation

”for all  $i \in I$ ,  $f_i \circ f : |B| \rightarrow |A_i|$  is a  $\mathcal{C}$ -morphism”.

Obviously, the source  $(f_i : A \rightarrow A_i)_{i \in I}$  is initial if, and only if for every object  $A'$  with the same structure as  $A$ , i.e. for which there exists a  $\mathcal{C}$ -isomorphism  $\varphi : A' \rightarrow A$  with  $|\varphi| = \text{id}_{|A|}$ , the source  $(f_i \circ \varphi : A' \rightarrow A_i)_{i \in I}$  is initial.

**Definition 6** *If the source  $(f_i : A \rightarrow A_i)_{i \in I}$  is initial, then the structure of  $A$  is called initial with respect to the family  $(f_i : |A| \rightarrow |A_i|)_{i \in I}$ .*

**Theorem 7** *If the structure  $\mathfrak{S}$  of  $A$  is initial for the family  $(f_i : |A| \rightarrow |A_i|)_{i \in I}$ ,  $\mathfrak{S}$  is the coarsest of all structures  $\mathfrak{S}'$  of objects  $A'$  of  $\mathcal{C}$  such that all  $\mathcal{X}$ -morphisms  $f_i : |A'| \rightarrow |A_i|$  ( $i \in I$ ) are  $\mathcal{C}$ -morphisms, and consequently is unique.*

**Proof.** Assume that the structure  $\mathfrak{S}$  of  $A$  is initial for the family  $(f_i : |A| \rightarrow |A_i|)_{i \in I}$ . Then all  $f_i : |A| \rightarrow |A_i|$  ( $i \in I$ ) are  $\mathcal{C}$ -morphisms. Let  $A'$  be such all  $f_i : |A'| \rightarrow |A_i|$  are  $\mathcal{C}$ -morphisms. Since  $f_i$  is an  $\mathcal{X}$ -morphism,  $|A'| = |A|$  and  $f = \text{id}_{|A|}$  is an  $\mathcal{X}$ -morphism and for each  $i \in I$ ,  $f_i \circ f : |A'| \rightarrow |A_i| = f_i$  is a  $\mathcal{C}$ -morphism. Thus  $f : |A'| \rightarrow |A|$  is a  $\mathcal{C}$ -morphism and  $\mathfrak{S}$  is coarser than the structure of  $A'$ . This determines  $\mathfrak{S}$  in a unique way by Lemma 4. ■

As shown by ([2], p. IV.30, exerc. 30), it may happen that a structure  $\mathfrak{S}$  in a concrete category  $(\mathcal{C}, |\cdot|, \mathcal{X})$  be the coarsest of all structures  $\mathfrak{S}'$  of objects  $A'$  of  $\mathcal{C}$  such all  $\mathcal{X}$ -morphisms  $f_i : |A'| \rightarrow |A_i|$  are  $\mathcal{C}$ -morphisms, and that this structure be not initial with respect to the family  $(f_i : |A| \rightarrow |A_i|)_{i \in I}$ .

The opposite of the concrete category  $(\mathcal{C}, |\cdot|, \mathcal{X})$  is  $(\mathcal{C}^{\text{op}}, |\cdot|^{\text{op}}, \mathcal{X}^{\text{op}})$  where for each pair of objects  $A, B$  of  $\mathcal{C}$ ,  $|A \rightarrow B|^{\text{op}} = |A \leftarrow B|$ . The structure of  $A$  is called final with respect to the family  $(f_i : |A| \leftarrow |A_i|)_{i \in I}$  in  $(\mathcal{C}, |\cdot|, \mathcal{X})$  if it is initial with respect to this family in  $(\mathcal{C}, |\cdot|, \mathcal{X})$ .

### 3 Applications

Examples of initial structures ([2], § 2.4): inverse image of a structure, induced structure, concrete product (see Remark 8).

Examples of final structures ([2], § 2.6): direct image of a structure, quotient structure, concrete coproduct (see Remark 8).

**Remark 8** *Let  $\mathcal{P} = (\text{pr}_i : A \rightarrow A_i)_{i \in I}$  be a product in  $\mathcal{C}$ . This product is called concrete in  $(\mathcal{C}, |\cdot|, \mathcal{X})$  if  $|\mathcal{P}| = (|\text{pr}_i| : |A| \rightarrow |A_i|)_{i \in I}$  is a product in  $\mathcal{X}$  ([1], §10.52). All products considered in ([2], § 2.4) are concrete but there exist concrete categories with non-concrete products ([1], §10.55(2)). Dually, there exist concrete categories with non-concrete coproducts ([1], §10.67).*

## 4 References

### References

- [1] J. Adámek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories*, John Wiley, 1990 (new edition: J. Adámek, Horst Herrlich, and George E. Strecker, 2004).
- [2] N. Bourbaki, *Éléments de mathématique. Livre I : Théorie des ensembles*, Hermann, 1970.