



Several explicit and recurrent formulas for determinants of tridiagonal matrices via generalized continued fractions

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SEVERAL EXPLICIT AND RECURRENT FORMULAS FOR DETERMINANTS OF TRIDIAGONAL MATRICES VIA GENERALIZED CONTINUED FRACTIONS

FENG QI, WEN WANG, BAI-NI GUO*, AND DONGKYU LIM*

ABSTRACT. In the paper, by the aid of mathematical induction and some properties of determinants, the authors present several explicit and recurrent formulas of evaluations for determinants of general tridiagonal matrices in terms of finite generalized continued fractions and apply these newly-established formulas to evaluations for determinants of the Sylvester matrix and two Sylvester type matrices.

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1. INTRODUCTION

A finite generalized continued fraction is of the form

$$p_0 + \frac{q_1}{p_1 + \frac{q_2}{p_2 + \frac{q_3}{\ddots \frac{q_{m-1}}{p_{m-2} + \frac{q_m}{p_{m-1} + \frac{q_m}{p_m}}}}}},$$

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where p_0, p_1, \dots, p_m and q_1, q_2, \dots, q_m can be any complex numbers or functions. It can also be written equivalently as

$$p_0 + \sum_{\ell=1}^m \frac{q_\ell}{p_\ell} = p_0 + \sum_{\ell=1}^m \frac{|q_\ell|}{|p_\ell|} = p_0 + \frac{q_1}{p_1 + p_2 + \dots + p_{m-1} + p_m}.$$

In this paper, we will use the second compact form above. For more information on the theory of continued fractions, please refer to the papers [7, 14] and closely related references therein.

In general, a tridiagonal matrix of order n is defined for $n \in \mathbb{N}$ by

$$D_n = (e_{i,j})_{1 \leq i,j \leq n} = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \gamma_1 & \alpha_2 & \beta_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & \alpha_3 & \beta_3 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & \alpha_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_{n-3} & \beta_{n-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \gamma_{n-3} & \alpha_{n-2} & \beta_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \gamma_{n-1} & \alpha_n \end{pmatrix}, \quad (1.1)$$

where

$$e_{i,j} = \begin{cases} \alpha_i, & 1 \leq i = j \leq n; \\ \beta_i, & 1 \leq i = j - 1 \leq n - 1; \\ \gamma_j, & 1 \leq j = i - 1 \leq n - 1; \\ 0, & \text{otherwise.} \end{cases}$$

In the papers [15, 16, 18], the determinant $|D_n|$ and some special cases were discussed, computed, and applied to several problems in analytic combinatorics and analytic number theory. In the papers [2, 5, 6, 9, 15, 16, 18], there are some computation of the inverse and determinant of the general tridiagonal matrix D_n . For more information about this topic, please refer to the papers [4, 8, 12, 13] and closely related references therein.

Let $n \geq 2$ and

$$P_n = (p_{i,j})_{1 \leq i,j \leq n} = \begin{pmatrix} a_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_3 & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & b_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{n-3} & 0 & 0 & 0 & \cdots & b_{n-3} & c_{n-3} & 0 & 0 \\ a_{n-2} & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & c_{n-2} & 0 \\ a_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-1} & c_{n-1} \\ a_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_n \end{pmatrix}, \quad (1.2)$$

where

$$p_{i,j} = \begin{cases} a_i, & 1 \leq i \leq n, j = 1; \\ b_i, & 2 \leq i = j \leq n; \\ c_i, & 1 \leq i = j - 1 \leq n - 1; \\ 0, & \text{otherwise.} \end{cases}$$

In this paper, by the help of mathematical induction and some properties of determinants, we will present several explicit and recurrent formulas for evaluations of two determinants $|P_n|$ and $|D_n|$ and will apply these newly-established formulas to evaluations for determinants of the Sylvester matrix and two Sylvester type matrices.

2. EXPLICIT AND RECURRENT FORMULAS FOR $|P_n|$

Right now we start off to present explicit and recurrent formulas for $|P_n|$.

Theorem 2.1. *Let $n \geq 2$ and $b_k \neq 0$ for $2 \leq k \leq n$. Then the determinant $|P_n|$ can be computed recurrently by*

$$|P_n| = \lambda_{1,n} \prod_{k=2}^n b_k, \quad (2.1)$$

where

$$\lambda_{k,n} = a_k - \frac{c_k}{b_{k+1}} \lambda_{k+1,n}, \quad 1 \leq k \leq n-1 \quad (2.2)$$

and $\lambda_{n,n} = a_n$.

Proof. When $n = 2$, it is easy to see that

$$|P_2| = \begin{vmatrix} a_1 & c_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 c_1$$

and

$$\lambda_{1,2} \prod_{k=2}^2 b_k = \lambda_{1,2} b_2 = \left(a_1 - \frac{c_1 \lambda_{2,2}}{b_2} \right) b_2 = \left(a_1 - \frac{c_1 a_2}{b_2} \right) b_2 = a_1 b_2 - a_2 c_1 = |P_2|.$$

This means that the formula (2.1) is valid for $n = 2$.

Assume that the formula (2.1) validates for $n = m - 1$, equivalently speaking,

$$|P_{m-1}| = \lambda_{1,m-1} \prod_{k=2}^{m-1} b_k.$$

When $n = m$, expanding the determinant $|P_m|$ according to the first rank and utilizing the assumption for $n = m - 1$ give

$$\begin{aligned}
 |P_m| &= a_1 \prod_{k=2}^m b_k - c_1 \begin{vmatrix} a_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ a_4 & 0 & b_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{m-3} & 0 & 0 & \cdots & b_{m-3} & c_{m-3} & 0 & 0 \\ a_{m-2} & 0 & 0 & \cdots & 0 & b_{m-2} & c_{m-2} & 0 \\ a_{m-1} & 0 & 0 & \cdots & 0 & 0 & b_{m-1} & c_{m-1} \\ a_m & 0 & 0 & \cdots & 0 & 0 & 0 & b_m \end{vmatrix} \\
 &= a_1 \prod_{k=2}^m b_k - c_1 \lambda_{2,m} \prod_{k=3}^m b_k = \left(a_1 - \frac{c_1}{b_2} \lambda_{2,m} \right) \prod_{k=2}^m b_k = \lambda_{1,m} \prod_{k=2}^m b_k.
 \end{aligned}$$

By mathematical induction, we derive the formula (2.1). The proof of Theorem 2.1 is complete. \square

Theorem 2.2. For $n \geq 2$, the determinant $|P_n|$ can be computed explicitly by

$$|P_n| = a_1 \prod_{k=2}^n b_k - \sum_{k=2}^n (-1)^k \left(\prod_{\ell=1}^{k-1} c_\ell \prod_{m=k+1}^n b_m \right) a_k. \quad (2.3)$$

Proof. From the recurrent relation (2.2), it follows that

$$\begin{aligned}
 \lambda_{1,n} &= a_1 - \frac{c_1}{b_2} \lambda_{2,n} \\
 &= a_1 - \frac{c_1}{b_2} \left(a_2 - \frac{c_2}{b_3} \lambda_{3,n} \right) \\
 &= a_1 - \frac{c_1}{b_2} \left[a_2 - \frac{c_2}{b_3} \left(a_3 - \frac{c_3}{b_4} \lambda_{4,n} \right) \right] \\
 &= \cdots \\
 &= a_1 - \frac{c_1}{b_2} \left[a_2 - \frac{c_2}{b_3} \left(a_3 - \frac{c_3}{b_4} \left[a_4 - \cdots - \frac{c_{\ell-1}}{b_\ell} \left(a_\ell - \frac{c_\ell}{b_{\ell+1}} \lambda_{\ell+1,n} \right) \right] \right) \right] \\
 &= \cdots \\
 &= a_1 - \frac{c_1}{b_2} \left[a_2 - \frac{c_2}{b_3} \left(a_3 - \frac{c_3}{b_4} \left[a_4 - \cdots - \frac{c_{\ell-1}}{b_\ell} \left(a_\ell - \cdots \right. \right. \right. \right. \\
 &\quad \left. \left. \left. - \frac{c_{n-3}}{b_{n-2}} \left[a_{n-2} - \frac{c_{n-2}}{b_{n-1}} \left(a_{n-1} - \frac{c_{n-1}}{b_n} \lambda_{n,n} \right) \right] \right) \right] \right) \right] \\
 &= a_1 - \frac{c_1}{b_2} \left[a_2 - \frac{c_2}{b_3} \left(a_3 - \frac{c_3}{b_4} \left[a_4 - \cdots - \frac{c_{\ell-1}}{b_\ell} \left(a_\ell - \cdots \right. \right. \right. \right. \right. \\
 &\quad \left. \left. \left. - \frac{c_{n-3}}{b_{n-2}} \left[a_{n-2} - \frac{c_{n-2}}{b_{n-1}} \left(a_{n-1} - \frac{c_{n-1}}{b_n} a_n \right) \right] \right) \right] \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= a_1 - \frac{c_1}{b_2} \left(a_2 - \frac{c_2}{b_3} \left[a_3 - \frac{c_3}{b_4} \left(a_4 - \cdots - \frac{c_{\ell-1}}{b_\ell} \left[a_\ell - \cdots \right. \right. \right. \right. \\
 &\quad \left. \left. \left. - \frac{c_{n-3}}{b_{n-2}} \left(a_{n-2} - \frac{c_{n-2}}{b_{n-1}} a_{n-1} + \frac{c_{n-2}c_{n-1}}{b_{n-1}b_n} a_n \right) \right] \right] \right] \right) \\
 &= a_1 - \frac{c_1}{b_2} \left(a_2 - \frac{c_2}{b_3} \left[a_3 - \frac{c_3}{b_4} \left(a_4 - \cdots - \frac{c_{\ell-1}}{b_\ell} \left[a_\ell - \cdots \right. \right. \right. \right. \\
 &\quad \left. \left. \left. - \left(\frac{c_{n-3}}{b_{n-2}} a_{n-2} - \frac{c_{n-3}c_{n-2}}{b_{n-2}b_{n-1}} a_{n-1} + \frac{c_{n-3}c_{n-2}c_{n-1}}{b_{n-2}b_{n-1}b_n} a_n \right) \right] \right] \right] \right) \\
 &= \cdots \\
 &= a_1 - \sum_{k=2}^n (-1)^k \left(\prod_{\ell=2}^k \frac{c_{\ell-1}}{b_\ell} \right) a_k
 \end{aligned}$$

for $n \geq 2$. Substituting this result into (2.1) and simplifying lead to (2.3). The proof of Theorem 2.2 is complete. \square

Remark 2.1. Applying $a_k = k$, $b_k = k$, and $c_k = k$ to the explicit formula (2.3) in Theorem 2.2 reveals

$$\begin{vmatrix}
 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 2 & 2 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 3 & 0 & 3 & 3 & \cdots & 0 & 0 & 0 & 0 \\
 4 & 0 & 0 & 4 & \cdots & 0 & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 n-3 & 0 & 0 & 0 & \cdots & n-3 & n-3 & 0 & 0 \\
 n-2 & 0 & 0 & 0 & \cdots & 0 & n-2 & n-2 & 0 \\
 n-1 & 0 & 0 & 0 & \cdots & 0 & 0 & n-1 & n-1 \\
 n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & n
 \end{vmatrix} = \frac{1 - (-1)^n}{2} n!.$$

3. EXPLICIT AND RECURRENT FORMULAS FOR $|D_n|$

Now we are in a position to present explicit and recurrent formulas for $|D_n|$.

Theorem 3.1. For $n \in \mathbb{N}$, the determinant $|D_n|$ can be explicitly and recurrently computed by

$$\begin{aligned}
 |D_n| &= \alpha_1 \alpha_2 + (\alpha_1 - \beta_1 \gamma_1) \prod_{m=3}^n \left[\alpha_m + \frac{K^{m-2}}{\ell=1} \frac{(-\beta_{m-\ell} \gamma_{m-\ell})}{\alpha_{m-\ell}} \right] \\
 &\quad - \sum_{k=3}^n \left[\prod_{\ell=1}^{k-1} (\beta_\ell \gamma_\ell) \right] \frac{\prod_{m=k+1}^n \left[\alpha_m + K_{\ell=1}^{m-2} \frac{(-\beta_{m-\ell} \gamma_{m-\ell})}{\alpha_{m-\ell}} \right]}{\prod_{m=2}^{k-1} \left[\alpha_m + K_{\ell=1}^{m-2} \frac{(-\beta_{m-\ell} \gamma_{m-\ell})}{\alpha_{m-\ell}} \right]} \quad (3.1)
 \end{aligned}$$

and

$$|D_n| = \eta_{1,n} \left(\alpha_2 + \prod_{k=3}^n \left[\alpha_k + \frac{K^{k-2}}{\ell=1} \frac{(-\beta_{k-\ell} \gamma_{k-\ell})}{\alpha_{k-\ell}} \right] \right), \quad (3.2)$$

where $K_{\ell=q}^p$ for $p < q$ is understood to be zero,

$$\eta_{1,n} = -1 - \frac{\beta_1}{\alpha_2} \eta_{2,n}, \quad \eta_{2,n} = \gamma_1 - \frac{\beta_2}{\alpha_3 - \frac{\beta_2 \gamma_2}{\alpha_2}} \eta_{3,n}, \quad \eta_{3,n} = -\frac{\gamma_1 \gamma_2}{\alpha_2} - \frac{\beta_3}{\alpha_4 - \frac{\beta_3 \gamma_3}{\alpha_3 - \frac{\beta_2 \gamma_2}{\alpha_2}}} \eta_{4,n},$$

$$\eta_{k,n} = (-1)^k \frac{\prod_{\ell=1}^{k-1} \gamma_\ell}{\alpha_2 + \prod_{\ell=3}^{k-1} \left[\alpha_\ell + K_{m=1}^{\ell-2} \frac{(-\beta_{\ell-m} \gamma_{\ell-\ell})}{\alpha_{\ell-m}} \right]} - \frac{\beta_k}{\alpha_{k+1} + K_{\ell=1}^{k-1} \frac{(-\beta_{k-\ell+1} \gamma_{k-\ell+1})}{\alpha_{k-\ell+1}}} \eta_{k+1,n}$$

for $4 \leq k \leq n-1$, and

$$\eta_{n,n} = (-1)^n \frac{\prod_{\ell=1}^{n-1} \gamma_\ell}{\alpha_2 + \prod_{k=3}^{n-1} \left[\alpha_k + K_{\ell=1}^{k-2} \frac{(-\beta_{k-\ell} \gamma_{k-\ell})}{\alpha_{k-\ell}} \right]}.$$

Proof. The determinant $|D_n|$ of the tridiagonal matrix D_n in (1.1) can be rewritten as

$$\begin{aligned} |D_n| &= \begin{vmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \gamma_1 & \alpha_2 & \beta_2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -\frac{\gamma_1 \gamma_2}{\alpha_2} & 0 & \alpha_3 - \frac{\beta_2 \gamma_2}{\alpha_2} & \beta_3 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & \alpha_4 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_{n-3} & \beta_{n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \gamma_{n-3} & \alpha_{n-2} & \beta_{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{n-2} & \alpha_{n-1} & \beta_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \gamma_{n-1} & \alpha_n & 0 \end{vmatrix} \\ &= \begin{vmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \gamma_1 & \alpha_2 & \beta_2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -\frac{\gamma_1 \gamma_2}{\alpha_2} & 0 & \alpha_3 - \frac{\beta_2 \gamma_2}{\alpha_2} & \beta_3 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \frac{\gamma_1 \gamma_2 \gamma_3}{\alpha_2 \alpha_3 - \beta_2 \gamma_2} & 0 & 0 & \alpha_4 - \frac{\alpha_2 \beta_3 \gamma_3}{\alpha_2 \alpha_3 - \beta_2 \gamma_2} & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_{n-3} & \beta_{n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \gamma_{n-3} & \alpha_{n-2} & \beta_{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{n-2} & \alpha_{n-1} & \beta_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \gamma_{n-1} & \alpha_n & 0 \end{vmatrix} \\ &= \cdots \end{aligned}$$

$$= \begin{vmatrix} a_1 & \beta_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & \beta_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_3 & 0 & b_3 & \beta_3 & \cdots & 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & b_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{n-3} & 0 & 0 & 0 & \cdots & b_{n-3} & \beta_{n-3} & 0 & 0 \\ a_{n-2} & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & \beta_{n-2} & 0 \\ a_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-1} & \beta_{n-1} \\ a_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_n \end{vmatrix},$$

where

$$b_2 = \alpha_2, \quad b_3 = \alpha_3 - \frac{\beta_2 \gamma_2}{b_2}, \quad b_4 = \alpha_4 - \frac{\beta_3 \gamma_3}{b_3}, \quad \dots, \quad b_{n-3} = \alpha_{n-3} - \frac{\beta_{n-4} \gamma_{n-4}}{b_{n-4}},$$

$$b_{n-2} = \alpha_{n-2} - \frac{\beta_{n-3} \gamma_{n-3}}{b_{n-3}}, \quad b_{n-1} = \alpha_{n-1} - \frac{\beta_{n-2} \gamma_{n-2}}{b_{n-2}}, \quad b_n = \alpha_n - \frac{\beta_{n-1} \gamma_{n-1}}{b_{n-1}}$$

and

$$a_1 = \alpha_1, \quad a_2 = \gamma_1, \quad a_3 = -\frac{\gamma_2}{b_2} a_2, \quad a_4 = -\frac{\gamma_3}{b_3} a_3, \quad \dots, \quad a_{n-3} = -\frac{\gamma_{n-4}}{b_{n-4}} a_{n-4},$$

$$a_{n-2} = -\frac{\gamma_{n-3}}{b_{n-3}} a_{n-3}, \quad a_{n-1} = -\frac{\gamma_{n-2}}{b_{n-2}} a_{n-2}, \quad a_n = -\frac{\gamma_{n-1}}{b_{n-1}} a_{n-1}.$$

The sequences b_k and a_k for $k \geq 3$ can be formulated by finite generalized continued fractions

$$b_k = \alpha_k - \frac{\beta_{k-1} \gamma_{k-1}}{\alpha_{k-1} - \frac{\beta_{k-2} \gamma_{k-2}}{\alpha_{k-2} - \frac{\beta_{k-3} \gamma_{k-3}}{\alpha_{k-3} - \frac{\beta_{k-4} \gamma_{k-4}}{\ddots \frac{\beta_3 \gamma_3}{\alpha_3 - \frac{\beta_2 \gamma_2}{\alpha_2}}}}}} = \alpha_k + \frac{K \frac{(-\beta_{k-\ell} \gamma_{k-\ell})}{\alpha_{k-\ell}}}{\ell=1}^{k-2}$$

and

$$a_k = (-1)^k \frac{\prod_{\ell=1}^{k-1} \gamma_\ell}{\prod_{\ell=2}^{k-1} b_\ell}.$$

Using (2.3) results in

$$|D_n| = \left(b_2 + \prod_{k=3}^n B_k \right) a_1 - \left(\beta_1 \prod_{m=3}^n B_m \right) a_2 - \sum_{k=3}^n (-1)^k \left(\prod_{\ell=1}^{k-1} \beta_\ell \prod_{m=k+1}^n B_m \right) a_k$$

$$= \alpha_1 \left(\alpha_2 + \prod_{k=3}^n \left[\alpha_k + \frac{K \frac{(-\beta_{k-\ell} \gamma_{k-\ell})}{\alpha_{k-\ell}}}{\ell=1}^{k-2} \right] \right) - \beta_1 \gamma_1 \prod_{m=3}^n \left[\alpha_m + \frac{K \frac{(-\beta_{m-\ell} \gamma_{m-\ell})}{\alpha_{m-\ell}}}{\ell=1}^{m-2} \right]$$

$$- \sum_{k=3}^n \prod_{\ell=1}^{k-1} \beta_\ell \prod_{m=k+1}^n \left[\alpha_m + \frac{K \frac{(-\beta_{m-\ell} \gamma_{m-\ell})}{\alpha_{m-\ell}}}{\ell=1}^{m-2} \right] \frac{\prod_{\ell=1}^{k-1} \gamma_\ell}{\prod_{\ell=2}^{k-1} [\alpha_\ell + K \frac{(-\beta_{\ell-i} \gamma_{\ell-i})}{\alpha_{\ell-i}}]}$$

which can be rearranged as (3.1).

Making use of (2.1) and (2.2) yields (3.2). The proof of Theorem 3.1 is complete. \square

4. DISCUSSIONS

In this section, we discuss our main results and related ones by several remarks.

Remark 4.1. In [3, p. 1018], it was stated that J. J. Sylvester found in 1854 that

$$|M_n(s)| = \begin{vmatrix} s & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ n & s & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & n-1 & s & 3 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & n-2 & s & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & s & n-2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 3 & s & n-1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2 & s & n \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & s \end{vmatrix} = \prod_{k=0}^n (s + n - 2k).$$

An application of (3.1) to $|M_n(s)|$ yields

$$\begin{aligned} |M_n(s)| &= s^2 + (s-n) \prod_{m=3}^n \left[s + K_{\ell=1}^{m-2} \frac{-(m-\ell)(n-m+\ell+1)}{s} \right] \\ &\quad - \sum_{k=3}^n \left[\prod_{\ell=1}^{k-1} \ell(n-\ell+1) \right] \frac{\prod_{m=k+1}^n \left[s + K_{\ell=1}^{m-2} \frac{-(m-\ell)(n-m+\ell+1)}{s} \right]}{\prod_{m=2}^{k-1} \left[s + K_{\ell=1}^{m-2} \frac{-(m-\ell)(n-m+\ell+1)}{s} \right]} \\ &= s^2 + (s-n) \prod_{m=3}^n \left[s + K_{\ell=1}^{m-2} \frac{-(m-\ell)(n-m+\ell+1)}{s} \right] \\ &\quad - n! \sum_{k=3}^n \frac{(k-1)!}{(n-k+1)!} \frac{\prod_{m=k+1}^n \left[s + K_{\ell=1}^{m-2} \frac{-(m-\ell)(n-m+\ell+1)}{s} \right]}{\prod_{m=2}^{k-1} \left[s + K_{\ell=1}^{m-2} \frac{-(m-\ell)(n-m+\ell+1)}{s} \right]} \\ &\triangleq s^2 + (s-n) \prod_{m=3}^n S(s; m, n) - n! \sum_{k=3}^n \frac{(k-1)!}{(n-k+1)!} \frac{\prod_{m=k+1}^n S(s; m, n)}{\prod_{m=2}^{k-1} S(s; m, n)}. \end{aligned}$$

Now we try to explicitly compute

$$S(s; m, n) = s + K_{\ell=1}^{m-2} \frac{-(m-\ell)(n-m+\ell+1)}{s}.$$

When $m = 3$, it is easy to obtain that

$$S(s; 3, n) = \frac{s^2 - 2(n-1)}{s} \triangleq \frac{\beta_1}{\alpha_1}.$$

When $m = 4$, employing the above result for $S(s; 3, n)$, we can acquire

$$S(s; 4, n) = \frac{\beta_1 s - 3(n-2)\alpha_1}{\beta_1} = \frac{s(s^2 - 5n + 8)}{s^2 - 2n + 2} \triangleq \frac{\beta_2}{\alpha_2}.$$

If assuming $S(s; k+1, n) = \frac{\beta_{k-1}}{\alpha_{k-1}}$, then, by mathematical induction, we have

$$S(s; k+2, n) = \frac{\beta_k}{\alpha_k} = \frac{\beta_{k-1}s - \alpha_{k-1}(k+1)(n-k)}{\beta_{k-1}}.$$

Note that $\alpha_{k-1} = \beta_{k-2}$. Then

$$\beta_k - \beta_{k-1}s + \beta_{k-2}(k+1)(n-k) = 0.$$

Further replacing k by $k+2$ results in

$$\beta_{k+2} - \beta_{k+1}s + (k+3)(n-k-2)\beta_k = 0.$$

By the approach utilized in [15, Theorem 3.1], the characteristic equation is

$$t^2 - st + (k+3)(n-k-2) = 0$$

which has solutions

$$t = \frac{s \pm \sqrt{s^2 - 4(k+3)(n-k-2)}}{2}.$$

Consequently, it follows that

$$\beta_k = A \left(\frac{s + \sqrt{s^2 - 4(k+3)(n-k-2)}}{2} \right)^{k-1} + B \left(\frac{s - \sqrt{s^2 - 4(k+3)(n-k-2)}}{2} \right)^{k-1},$$

where

$$A = -\frac{2s^3 - 2(5n-8) - (s^2 - 2n+2)(s + \sqrt{s^2 - 20n+80})}{2\sqrt{s^2 - 20n+80}}$$

and

$$B = \frac{2s^3 - 2(5n-8) - (s^2 - 2n+2)(s - \sqrt{s^2 - 20n+80})}{2\sqrt{s^2 - 20n+80}}.$$

In a word, we provide an alternative expression for the Sylvester determinant $|M_n(s)|$.

Remark 4.2. In [3], by virtue of left eigenvector method, the determinants

$$\begin{aligned} |M_n(s, t)| &= \begin{vmatrix} s & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ n & s+t & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & n-1 & s+2t & 3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & n-2 & s+3t & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & s+(n-2)t & n-1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2 & s+(n-1)t & n \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & s+nt \end{vmatrix} \\ &= \prod_{k=0}^n \left(s + \frac{nt}{2} + \frac{n-2k}{2} \sqrt{t^4 + 4} \right), \end{aligned}$$

and

$$\begin{aligned}
 |M_n(s, t; x, y)| &= \begin{vmatrix} s & x & 0 & 0 & \cdots & 0 & 0 & 0 \\ nv & s+t & 2x & 0 & \cdots & 0 & 0 & 0 \\ 0 & (n-1)y & s+2t & 3x & \cdots & 0 & 0 & 0 \\ 0 & 0 & (n-2)y & s+3t & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & s+(n-2)t & (n-1)x & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2y & s+(n-1)t & nx \\ 0 & 0 & 0 & 0 & \cdots & 0 & y & s+nt \end{vmatrix} \\
 &= \prod_{k=0}^n \left(s + \frac{nt}{2} + \frac{n-2k}{2} \sqrt{t^4 + 4xy} \right)
 \end{aligned}$$

of tridiagonal matrices similar to the Sylvester matrix were collected and calculated. These evaluations can be computed alternatively by Theorem 3.1.

Remark 4.3. The condition $b_k \neq 0$ for $2 \leq k \leq n$ in Theorem 2.1 is removed off in Theorem 2.2. Therefore, the explicit formula (2.3) is better than the recurrent formulas (2.1) and (2.2).

Remark 4.4. The explicit formula (2.3) can be simply reformulated as

$$|P_n| = \sum_{k=1}^n (-1)^{k+1} \left(\prod_{\ell=1}^{k-1} c_\ell \prod_{m=k+1}^n b_m \right) a_k,$$

where the empty product is understood to be 1 as usual.

Remark 4.5. Let

$$U_n = (u_{i,j})_{1 \leq i,j \leq n} = \begin{pmatrix} \alpha_1 & \gamma_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \tau_1 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots & 0 & 0 & 0 & \tau_2 \\ \alpha_3 & 0 & \beta_3 & \gamma_3 & \cdots & 0 & 0 & 0 & \tau_3 \\ \alpha_4 & 0 & 0 & \beta_4 & \cdots & 0 & 0 & 0 & \tau_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-3} & 0 & 0 & 0 & \cdots & \beta_{n-3} & \gamma_{n-3} & 0 & \tau_{n-3} \\ \alpha_{n-2} & 0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & \gamma_{n-2} & \tau_{n-2} \\ \alpha_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-1} & \gamma_{n-1} \\ \alpha_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_n \end{pmatrix},$$

where

$$u_{i,j} = \begin{cases} \alpha_i, & 1 \leq i \leq n, j = 1; \\ \beta_i, & 2 \leq i = j \leq n; \\ \gamma_i, & 1 \leq i = j - 1 \leq n - 1; \\ \tau_i, & 1 \leq i \leq n - 2, j = n; \\ 0, & \text{otherwise.} \end{cases}$$

The determinant $|U_n|$ can be rewritten as

$$\begin{aligned}
 |U_n| &= \begin{vmatrix} \alpha_1 & \gamma_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \tau_1 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots & 0 & 0 & 0 & \tau_2 \\ \alpha_3 & 0 & \beta_3 & \gamma_3 & \cdots & 0 & 0 & 0 & \tau_3 \\ \alpha_4 & 0 & 0 & \beta_4 & \cdots & 0 & 0 & 0 & \tau_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-3} & 0 & 0 & 0 & \cdots & \beta_{n-3} & \gamma_{n-3} & 0 & \tau_{n-3} \\ \alpha_{n-2} & 0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & \gamma_{n-2} & \tau_{n-2} \\ \alpha_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-1} & \gamma_{n-1} \\ \alpha_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_n \end{vmatrix} \\
 &= \begin{vmatrix} \alpha_1 - \frac{\alpha_n \tau_1}{\beta_n} & \gamma_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_2 - \frac{\alpha_n \tau_2}{\beta_n} & \beta_2 & \gamma_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_3 - \frac{\alpha_n \tau_3}{\beta_n} & 0 & \beta_3 & \gamma_3 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_4 - \frac{\alpha_n \tau_4}{\beta_n} & 0 & 0 & \beta_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-3} - \frac{\alpha_n \tau_{n-3}}{\beta_n} & 0 & 0 & 0 & \cdots & \beta_{n-3} & \gamma_{n-3} & 0 & 0 \\ \alpha_{n-2} - \frac{\alpha_n \tau_{n-2}}{\beta_n} & 0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & \gamma_{n-2} & 0 \\ \alpha_{n-1} - \frac{\alpha_n \gamma_{n-1}}{\beta_n} & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-1} & 0 \\ \alpha_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_n \end{vmatrix} \\
 &= \beta_n \begin{vmatrix} \alpha_1 - \frac{\alpha_n \tau_1}{\beta_n} & \gamma_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_2 - \frac{\alpha_n \tau_2}{\beta_n} & \beta_2 & \gamma_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_3 - \frac{\alpha_n \tau_3}{\beta_n} & 0 & \beta_3 & \gamma_3 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_4 - \frac{\alpha_n \tau_4}{\beta_n} & 0 & 0 & \beta_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-3} - \frac{\alpha_n \tau_{n-3}}{\beta_n} & 0 & 0 & 0 & \cdots & \beta_{n-3} & \gamma_{n-3} & 0 & 0 \\ \alpha_{n-2} - \frac{\alpha_n \tau_{n-2}}{\beta_n} & 0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & \gamma_{n-2} & 0 \\ \alpha_{n-1} - \frac{\alpha_n \gamma_{n-1}}{\beta_n} & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-1} & 0 \end{vmatrix}.
 \end{aligned}$$

An application of Theorem 2.1 and 2.2 and Remark 4.4 straightforwardly yields

$$|U_n| = \sum_{k=1}^{n-2} (-1)^{k+1} (\alpha_k \beta_n - \tau_k \alpha_n) \prod_{\ell=1}^{k-1} \gamma_\ell \prod_{m=k+1}^{n-1} \beta_m + (-1)^n (\alpha_{n-1} \beta_n - \tau_{n-1} \alpha_n) \prod_{\ell=1}^{n-2} \gamma_\ell$$

and

$$|U_n| = \Lambda_{1,n-1} \prod_{k=2}^n \beta_k,$$

where

$$\Lambda_{k,n-1} = \alpha_k - \frac{\alpha_n}{\beta_n} \tau_k - \frac{\gamma_k}{\beta_{k+1}} \Lambda_{k+1,n-1}, \quad 1 \leq k \leq n-2$$

and $\Lambda_{n-1,n-1} = \alpha_{n-1} - \frac{\alpha_n}{\beta_n} \gamma_{n-1}$.

Remark 4.6. The determinant $|P_n|$ of P_n in (1.2) can be rearranged as

$$\begin{aligned}
 |P_n| &= \begin{vmatrix} a_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_3 & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & b_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{n-3} & 0 & 0 & 0 & \cdots & b_{n-3} & c_{n-3} & 0 & 0 \\ a_{n-2} & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & c_{n-2} & 0 \\ a_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{a_n b_{n-1}}{a_{n-1}} & b_n - \frac{a_n c_{n-1}}{a_{n-1}} \end{vmatrix} \\
 &= \cdots \\
 &= \begin{vmatrix} a_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b_2 - \frac{a_2 c_1}{a_1} & c_2 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{a_3 b_2}{a_2} & b_3 - \frac{a_3 c_2}{a_2} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{a_4 b_3}{a_3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-3} & 0 & 0 \\ 0 & 0 & 0 & \cdots & b_{n-2} - \frac{a_{n-2} c_{n-3}}{a_{n-3}} & c_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{a_{n-1} b_{n-2}}{a_{n-2}} & b_{n-1} - \frac{a_{n-1} c_{n-2}}{a_{n-2}} & c_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{a_n b_{n-1}}{a_{n-1}} & b_n - \frac{a_n c_{n-1}}{a_{n-1}} \end{vmatrix} \\
 &= a_1 \begin{vmatrix} b_2 - \frac{a_2 c_1}{a_1} & c_2 & \cdots & 0 & 0 & 0 \\ -\frac{a_3 b_2}{a_2} & b_3 - \frac{a_3 c_2}{a_2} & \cdots & 0 & 0 & 0 \\ 0 & -\frac{a_4 b_3}{a_3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{n-3} & 0 & 0 \\ 0 & 0 & \cdots & b_{n-2} - \frac{a_{n-2} c_{n-3}}{a_{n-3}} & c_{n-2} & 0 \\ 0 & 0 & \cdots & -\frac{a_{n-1} b_{n-2}}{a_{n-2}} & b_{n-1} - \frac{a_{n-1} c_{n-2}}{a_{n-2}} & c_{n-1} \\ 0 & 0 & \cdots & 0 & -\frac{a_n b_{n-1}}{a_{n-1}} & b_n - \frac{a_n c_{n-1}}{a_{n-1}} \end{vmatrix} \\
 &\triangleq a_1 |Q_{n-1}|.
 \end{aligned}$$

Therefore, by virtue of Theorems 2.1 and 2.2, we derive that the determinant $|Q_{n-1}|$ satisfies

$$|Q_{n-1}| = \frac{|P_n|}{a_1} = \frac{\lambda_{1,n}}{a_1} \prod_{k=2}^n b_k \quad (4.1)$$

and

$$|Q_{n-1}| = \frac{|P_n|}{a_1} = \prod_{k=2}^n b_k - \frac{1}{a_1} \sum_{k=2}^n (-1)^k \left(\prod_{\ell=1}^{k-1} c_\ell \prod_{m=k+1}^n b_m \right) a_k. \quad (4.2)$$

Further letting

$$\begin{cases} \alpha_k = b_{k+1} - \frac{a_{k+1}c_k}{a_k}, & 1 \leq k \leq n-1 \\ \beta_k = c_{k+1}, & 1 \leq k \leq n-2 \\ \gamma_k = -\frac{a_{k+2}b_{k+1}}{a_{k+1}}, & 1 \leq k \leq n-2 \end{cases} \quad (4.3)$$

in equations (4.1) and (4.2) reveals

$$|D_{n-1}| = \frac{\lambda_{1,n}}{a_1} \prod_{k=2}^n b_k \quad (4.4)$$

and

$$|D_{n-1}| = \prod_{k=2}^n b_k - \frac{1}{a_1} \sum_{k=2}^n (-1)^k \left(\prod_{\ell=1}^{k-1} c_\ell \prod_{m=k+1}^n b_m \right) a_k. \quad (4.5)$$

From the second equality in (4.3), it is not difficult to see that $c_k = \beta_{k-1}$ for $2 \leq k \leq n-1$. If we can derive another relations from (4.3) to express a_k for $1 \leq k \leq n$ and b_k for $2 \leq k \leq n$ in terms of α_k for $1 \leq k \leq n-1$, β_k for $1 \leq k \leq n-2$, and γ_k for $1 \leq k \leq n-2$, then, by substituting these relations into (4.4) and (4.5), an alternative and explicit expression for evaluation of $|D_n|$ would be concluded. This is an open problem and we leave it to the interested readers.

Remark 4.7. In [1, Lemma 1.1] and [11, Lemma 2.1], it was acquired that

$$\begin{vmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 & \cdots & \tau_{n-2} & \tau_{n-1} & \tau_n \\ \alpha & \beta & 0 & 0 & \cdots & 0 & 0 & 0 \\ \gamma & \alpha & \beta & 0 & \cdots & 0 & 0 & 0 \\ 0 & \gamma & \alpha & \beta & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \alpha & \beta & 0 \\ 0 & 0 & 0 & 0 & \cdots & \gamma & \alpha & \beta \end{vmatrix} = \sum_{k=1}^n (-1)^{k-1} \tau_k b^{n-k} (\beta\gamma)^{(k-1)/2} U_{k-1} \left(\frac{\alpha}{2\sqrt{\beta\gamma}} \right), \quad (4.6)$$

where $U_k(s)$ is the k th Chebyshev polynomials of the second kind, which can be generated [19, 20, 22] by

$$\frac{1}{1-2st+t^2} = \sum_{k=0}^{\infty} U_k(s)t^k, \quad |s| < 1, \quad |t| < 1.$$

Taking $\tau_1 = \tau_2 = \cdots = \tau_{n-1} = 0$ and $\tau_n = 1$ and reformulating, the formula (4.6) becomes

$$\begin{vmatrix} \alpha & \beta & 0 & \cdots & 0 & 0 \\ \gamma & \alpha & \beta & \cdots & 0 & 0 \\ 0 & \gamma & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha & \beta \\ 0 & 0 & 0 & \cdots & \gamma & \alpha \end{vmatrix}_{n \times n} = (\beta\gamma)^{n/2} U_n\left(\frac{\alpha}{2\sqrt{\beta\gamma}}\right). \quad (4.7)$$

This is different from

$$\begin{vmatrix} \alpha & \beta & 0 & \cdots & 0 & 0 \\ \gamma & \alpha & \beta & \cdots & 0 & 0 \\ 0 & \gamma & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha & \beta \\ 0 & 0 & 0 & \cdots & \gamma & \alpha \end{vmatrix}_{n \times n} = \begin{cases} \frac{(\alpha + \sqrt{\alpha^2 - 4\beta\gamma})^{n+1} - (\alpha - \sqrt{\alpha^2 - 4\beta\gamma})^{n+1}}{2^{n+1}\sqrt{\alpha^2 - 4\beta\gamma}}, & \alpha^2 \neq 4\beta\gamma \\ (n+1)\left(\frac{\alpha}{2}\right)^n, & \alpha^2 = 4\beta\gamma \end{cases} \quad (4.8)$$

and

$$\begin{vmatrix} \alpha & \beta & 0 & \cdots & 0 & 0 \\ \gamma & \alpha & \beta & \cdots & 0 & 0 \\ 0 & \gamma & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha & \beta \\ 0 & 0 & 0 & \cdots & \gamma & \alpha \end{vmatrix}_{n \times n} = \prod_{j=1}^n \left(\beta + 2\alpha\sqrt{\frac{\gamma}{\alpha}} \cos \frac{j\pi}{n+1} \right) \quad (4.9)$$

which are established and collected in [15, pp. 130] and [18, Theorem 4].

Comparing (4.7) with (4.8) and (4.9), taking $\beta = \gamma = 1$ and $\alpha = 2s$, and simplifying yield

$$\begin{aligned} U_n(s) &= \prod_{j=1}^n \left(1 + 2\sqrt{2s} \cos \frac{j\pi}{n+1} \right) \\ &= \begin{cases} \frac{(s + \sqrt{s^2 - 1})^{n+1} - (s - \sqrt{s^2 - 1})^{n+1}}{2^{n+1}\sqrt{s^2 - 1}}, & s^2 \neq 1 \\ (n+1)s^n, & s^2 = 1 \end{cases} \end{aligned}$$

which are alternative explicit formulas for the Chebyshev polynomials of the second kind $U_n(s)$.

Remark 4.8. On 21 September 2019, we were reminded of the paper [10] in which an alternative explicit formula for elements of the inverse of a tridiagonal matrix and an efficient and fast computing method to obtain elements of the inverse of a tridiagonal matrix by backward continued fractions were investigated.

Remark 4.9. Theorem 2.2 in this paper has been applied in the proof of [17, Theorem 3.3].

Remark 4.10. This paper is a revised version of the preprint [21].

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