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Resurgence and holonomy of the ϕ^{2k} model in zero dimension

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We describe the resurgence properties of some partition functions corresponding to field theories in dimension 0. We show that these functions satisfy linear differential equations with polynomial coefficients and then use elementary stability results for holonomic functions to prove resurgence properties, enhancing previously known results on growth estimates for the formal series involved, which had been obtained through a delicate combinatorics.

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I. INTRODUCTION

The theory of resurgent functions and alien differential calculus has been introduced in the late 70's, and then developed single-handedly by Jean Ecalle, in the field of singularities of dynamical systems with complex analytic data. It was a major discovery that the divergent series appearing in the formal solutions at a singular point, when expressed as expansions in a suitable variable z close to ∞ , admit Borel transforms with isolated singularities which can be analyzed with operators of a new kind: the alien derivations.

Resurgent functions and alien calculus have made it possible to solve difficult problems of classification of dynamical systems and to tackle singular perturbation theory, in particular for the Schrödinger equation, which display features of resurgence with respect to \hbar . Recently alien calculus has been applied in a number of domains of Theoretical Physics: Quantum Mechanics, perturbative Quantum Field Theory, matrix models, topological strings, etc¹.

In the present text, we focus on the *integrals* discussed by Rivasseau et al²⁻⁴ that correspond to scalar quantum field theories in dimension 0. That is, we consider scalar fields $\phi \in \mathbb{R}$ and a Lagrangian

$$\mathcal{L}_k(\phi) = -\frac{1}{2}\phi^2 - \lambda\phi^{2k}, \quad (k \geq 2)$$

with “potential” $V(\phi) = \lambda\phi^{2k}$ and study the associated “partition function”:

$$Z_0(\lambda) = \int_{\mathbb{R}} e^{-\frac{1}{2}\phi^2 - \lambda\phi^{2k}} \frac{d\phi}{\sqrt{2\pi}}.$$

We shall consider its related moments

$$Z_{2j}(\lambda) = \int_{\mathbb{R}} \phi^{2j} e^{-\frac{1}{2}\phi^2 - \lambda\phi^{2k}} \frac{d\phi}{\sqrt{2\pi}}, \quad (j \geq 1)$$

and the “Free energy” $W(\lambda) = \log Z_0(\lambda)$.

Classical results, reminded in section II for a wider class of potentials, that is polynomials $V \in \mathbb{R}_{2k}[\phi]$ of degree $2k$ with positive dominant coefficient, already allow to prove the following:

Proposition 1. *For a fixed $k \geq 1$, the integrals $Z_{2j}(\lambda)$ are convergent for $\Re\lambda \geq 0$ and define continuous functions on the half-plane $\bar{S} = \{\lambda \in \mathbb{C}; \Re\lambda \geq 0\}$ that are analytic in $S = \{\lambda \in \mathbb{C}; \Re\lambda > 0\}$. Moreover these integrals have asymptotic expansions in \bar{S} : there exists $(a_n^j)_{n \geq 0}$ such that, for all $N \geq 0$,*

$$\lim_{\substack{\lambda \rightarrow 0 \\ \lambda \in S}} \lambda^{-N} \left[Z_{2j}(\lambda) - \sum_{n=0}^N a_n^j \lambda^n \right] = 0.$$

This is the classical definition of an asymptotic expansion, following Ref. 5 we note \widetilde{Z}_{2j} the formal series $\sum_{n=0}^{\infty} a_n^j \lambda^n$ that corresponds to the “perturbative expansion” of $Z_{2j}(\lambda)$. These are results already obtained in Refs. 2 and 3 but it happens that the partition function, as well as its asymptotic expansion, are solutions of a differential equation, so that resurgence theory for the case of linear differential equations will in the end provide more precise results on the partition function and its logarithm.

As we shall see in section III, we have the following:

Theorem 1. *For the potential $V(\phi) = \phi^{2k}$, $k \geq 1$, the partition function Z_0 , as well as its asymptotic expansion, do satisfy the differential equation (E_k)*

$$\left[\left(\prod_{j=0}^{k-1} (2k\lambda \partial_\lambda + 2j + 1) \right) + \partial_\lambda \right] . Z_0 = 0. \quad (1)$$

This equation completely determines the resurgence of the partition function and consequently also the resurgence properties of the free energy $W = \log Z_0$. The *parameter* λ now plays the role of a *variable* in a linear ordinary differential equation with polynomial coefficients and we can make use of the general theory of holonomic functions to derive results for the function $\widetilde{Z}_0(\lambda)$.

Thus, we shall see below how to obtain without any combinatorics the Gevrey–1 growth for the coefficients of the series $\widetilde{Z}_0(\lambda)$, but also beyond that analyze the singularities of its Borel transform and reach resurgence properties.

Remark 1. The derivation of linear ODEs for some families of integrals depending on parameters is of course the topic of Picard–Fuchs theory, with its far reaching generalizations (Gauss–Manin connection); the present text implements essentially elementary techniques, focussing on the resurgence properties, for the family of integrals under consideration.

We remind in section II some analytic properties of partition functions in dimension 0 that will be used in section III, devoted to the different recursive relations between the partition function and its moments so that we can give the proof of theorem 1. Sections IV and V recall basic results on Borel–Laplace summation of divergent series and resurgence theory respectively. In sections VI and VII we perform an analysis of the equations (E_k) with the tools of alien calculus and then investigate the resurgence properties of the free energy functions. In a last section we show how the same techniques might apply to treat some particular cases of singular perturbation problems for the Schrödinger equation and we implement in detail the method for the easy “Airy case”.

Data Availability Statement: The data that supports the findings of this study are available within the article.

II. PROPERTIES OF PARTITION FUNCTIONS: ESTIMATES

We give here some general results for the partition function for a potential $V(\phi) = \sum_{i=0}^{2k} v_i \phi^i$ with $k \geq 1$ and $v_{2k} > 0$ and its related moments, noticing that all these functions share the same integral shape

$$\int_{\mathbb{R}} P(\phi) e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \frac{d\phi}{\sqrt{2\pi}},$$

where P is a polynomial.

A. Analytic properties

Theorem 2. Let $V(\phi) = \sum_{i=0}^{2k} v_i \phi^i$ with $k \geq 1$ and $v_{2k} > 0$ and $P \in \mathbb{R}[\phi]$, the integral

$$\int_{\mathbb{R}} P(\phi) e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \frac{d\phi}{\sqrt{2\pi}}$$

is convergent for any λ in the closed sector $\bar{S} = \{\lambda \in \mathbb{C}; \Re \lambda \geq 0\}$ and defines a function $f_{P,V}$ continuous on \bar{S} and analytic in $S = \{\lambda \in \mathbb{C}; \Re \lambda > 0\}$. Moreover, for any $n \in \mathbb{N}$ and $\lambda \in S$,

$$f_{P,V}^{(n)}(\lambda) = \int_{\mathbb{R}} P(\phi) (-V(\phi))^n e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \frac{d\phi}{\sqrt{2\pi}}.$$

Proof. The proof is based on variations on Lebesgue's dominated convergence theorem. We can first observe that V has a minimum m on \mathbb{R} so that:

$$\forall \phi \in \mathbb{R}, \forall \lambda \in \bar{S}, \quad \left| P(\phi) e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \right| \leq |P(\phi)| e^{-\Re \lambda \cdot m} e^{-\frac{1}{2}\phi^2}$$

and $f_{P,V}$ is well defined. In order to prove the continuity and the analyticity of $f_{P,V}$, since $f_{P,V} = e^{-\lambda m} f_{P,V-m}$, we can assume without loss of generality that V is non negative on the real axis.

Following Ref. 6, chapter XIII, section 8, the map

$$\begin{aligned} F &: \mathbb{R} \times \bar{S} \rightarrow \mathbb{C} \\ (\phi, \lambda) &\mapsto P(\phi) e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \end{aligned}$$

is such that:

1. For any $\lambda \in \bar{S}$, the function $\phi \mapsto F(\phi, \lambda)$ is integrable on \mathbb{R} .

2. For any $\phi \in \mathbb{R}$, the function $\lambda \mapsto F(\phi, \lambda)$ is continuous on \bar{S} and analytic on S .

3. For any $\lambda \in \bar{S}$, and any $\phi \in \mathbb{R}$

$$\left| P(\phi) e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \right| \leq |P(\phi)| e^{-\frac{1}{2}\phi^2} = g(\phi),$$

where g is integrable on \mathbb{R} .

Thanks to Lebesgue's dominated convergence theorem, it automatically ensures that $f_{P,V}$ is continuous on \bar{S} , analytic on S with the attempted formulas for its derivatives. \square

One can also observe that $f_{P,V}$ is indeed \mathcal{C}^∞ on $[0, +\infty[$ and its derivatives at $\lambda = 0$ are given, for $n \geq 0$, by

$$f_{P,V}^{(n)}(0) = \int_{\mathbb{R}} P(\phi) (-V(\phi))^n e^{-\frac{1}{2}\phi^2} \frac{d\phi}{\sqrt{2\pi}},$$

that are linear combination of gaussian moments:

$$a_j = \int_{\mathbb{R}} \phi^j e^{-\frac{1}{2}\phi^2} \frac{d\phi}{\sqrt{2\pi}} = \begin{cases} 0 & \text{if } j \text{ is odd} \\ \frac{j!}{2^{j/2}(j/2)!} & \text{if } j \text{ is even} \end{cases}.$$

This suggests that $f_{P,V}$ has an asymptotic expansion when $\lambda \rightarrow 0$ in S (see for example Ref. 5, chapter III).

B. Asymptotics expansions

Theorem 3. Let $V(\phi) = \sum_{i=0}^{2k} v_i \phi^i$ with $k \geq 1$ and $v_{2k} > 0$, $P \in \mathbb{R}[\phi]$, and for $\Re \lambda \geq 0$ and $n \geq 0$

$$f_{P,V}(\lambda) = \int_{\mathbb{R}} P(\phi) e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \frac{d\phi}{\sqrt{2\pi}},$$

$$\alpha_n = \frac{f_{P,V}^{(n)}(0)}{n!} = \int_{\mathbb{R}} \frac{1}{n!} P(\phi) (-V(\phi))^n e^{-\frac{1}{2}\phi^2} \frac{d\phi}{\sqrt{2\pi}}.$$

For any positive integer N

$$\lim_{\substack{\lambda \rightarrow 0 \\ \lambda \in S}} \lambda^{-N} \left[f_{P,V}(\lambda) - \sum_{n=0}^N \alpha_n \lambda^n \right] = 0.$$

We note $\tilde{f}_{P,V}(\lambda)$ the formal series $\sum_{n=0}^{\infty} \alpha_n \lambda^n$ which is the asymptotic expansion of $f_{P,V}$ when $\lambda \in S$, $\lambda \rightarrow 0$ and write

$$f_{P,V}(\lambda) \sim \sum_{n=0}^{\infty} \alpha_n \lambda^n, \quad \lambda \in S, \quad \lambda \rightarrow 0.$$

Since, for $n \geq 0$,

$$f_{P,V}^{(n)}(\lambda) = \int_{\mathbb{R}} P(\phi) (-V(\phi))^n e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \frac{d\phi}{\sqrt{2\pi}},$$

it is a matter of fact to check that

$$f_{P,V}^{(n)}(\lambda) \sim \tilde{f}_{P,V}^{(n)}(\lambda), \quad \lambda \in S, \quad \lambda \rightarrow 0,$$

where $\tilde{f}_{P,V}^{(n)}$ is the n -th formal derivative of the formal series $\tilde{f}_{P,V}(\lambda)$.

Proof. Since the asymptotic expansion of a product of functions is the product of their asymptotic expansions⁵, we can assume as in the previous proof that V is non negative on \mathbb{R} . Otherwise, if the minimum m of V is negative, we write $f_{P,V} = e^{-\lambda m} f_{P,V-m}$, where $e^{-\lambda m}$ has an obvious asymptotic expansion as an entire function of λ , this last result can be obtained with the help of the Taylor formula that will be useful to achieve this proof:

$$\forall N \geq 0, \quad \forall \zeta \in \mathbb{C}, \quad R_N(\zeta) = e^\zeta - \sum_{n=0}^N \frac{\zeta^n}{n!} = \zeta^{N+1} \int_0^1 \frac{(1-t)^N}{N!} e^{t\zeta} dt$$

that leads to

$$\forall N \geq 0, \quad \forall \zeta \in \mathbb{C}, \quad |R_N(\zeta)| = \left| e^\zeta - \sum_{n=0}^N \frac{\zeta^n}{n!} \right| \leq \frac{|\zeta|^{N+1}}{(N+1)!} \max\{1, e^{\Re \zeta}\}.$$

For $\lambda \in S$, we have

$$Q_N(\lambda) = \lambda^{-N} \left[f_{P,V}(\lambda) - \sum_{n=0}^N \alpha_n \lambda^n \right] = \lambda^{-N} \int_{\mathbb{R}} P(\phi) R_N(-\lambda V(\phi)) e^{-\frac{1}{2}\phi^2} \frac{d\phi}{\sqrt{2\pi}}$$

thus

$$|Q_N(\lambda)| \leq \frac{|\lambda|}{(N+1)!} \int_{\mathbb{R}} |P(\phi) \cdot V(\phi)^{N+1}| e^{-\frac{1}{2}\phi^2} \frac{d\phi}{\sqrt{2\pi}}$$

and this ends the proof of theorem. □

For such potentials V , the asymptotic expansion $\tilde{f}_{1,V}$ is, in physicists' terms, the perturbative expansion of the partition function $f_{1,V}$. A crucial remark for the sequel is that the asymptotic expansion of a function is unique so that, if we have functions satisfying a linear recursive equation or a linear differential equation, so do the formal series corresponding to their respective asymptotic expansions (see Ref. 5, chapter III, for more properties on asymptotic expansions).

III. GOVERNING EQUATIONS FOR PARTITION FUNCTIONS

A. Recursive and differential relations for the moments

Let $V(\phi) = \sum_{i=0}^{2k} v_i \phi^i$ with $k \geq 1$ and $v_{2k} > 0$ a given potential and its relative moments,

$$Z_j(\lambda) = \int_{\mathbb{R}} \phi^j e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \frac{d\phi}{\sqrt{2\pi}}$$

whose asymptotics expansions are noted $\tilde{Z}_j(\lambda)$. The key properties of these moments are the following:

Proposition 2. *The family of functions $(Z_j)_{j \geq 0}$ as well as the family of formal series $(\tilde{Z}_j)_{j \geq 0}$ satisfy the following relations:*

$$\forall j \geq 0, \quad Z_j' = - \sum_{i=0}^{2k} v_i Z_{j+i}, \quad (2)$$

$$\forall j \geq 0, \quad (j+1)Z_j = Z_{j+2} + \lambda \sum_{i=1}^{2k} i v_i Z_{i+j}. \quad (3)$$

We call these relations the governing equations associated to the potential V .

Proof. The proof is quite elementary: the first equation is a simple differentiation under the integral whereas the second one is a simple integration by part. For all $j \geq 0$, we have

$$\begin{aligned} (j+1)Z_j &= \int_{\mathbb{R}} (j+1)\phi^j e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \frac{d\phi}{\sqrt{2\pi}} = \int_{\mathbb{R}} (\phi^{j+1})' e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \frac{d\phi}{\sqrt{2\pi}} \\ &= \left[\phi^{j+1} e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \right]_{-\infty}^{+\infty} + \int_{\mathbb{R}} \phi^{j+1} (\phi + \lambda V'(\phi)) e^{-\frac{1}{2}\phi^2} e^{-\lambda V(\phi)} \frac{d\phi}{\sqrt{2\pi}} \\ &= Z_{j+2} + \lambda \sum_{i=1}^{2k} i v_i Z_{i+j} \end{aligned}$$

The uniqueness of asymptotic expansion ensures that the formal series $(\tilde{Z}_j)_{j \geq 0}$ have the same properties. \square

Let us focus now on the case $V(\phi) = \phi^{2k}$ for which these recursive relations will provide us a fundamental differential equation for the partition function.

B. The case $V(\phi) = \phi^{2k}$, $k \geq 1$

In this particular case, for any odd j , $Z_j \equiv 0$ so that one can focus on the integrals

$$\forall j \in \mathbb{N}, \quad \forall \lambda \in \mathcal{S}, \quad U_j(\lambda) = \int_{\mathbb{R}} \phi^{2j} e^{-\frac{1}{2}\phi^2} e^{-\lambda\phi^{2k}} \frac{d\phi}{\sqrt{2\pi}}$$

and the above recursive relations, (2) and (3), read:

$$\forall j \geq 0, \quad \begin{cases} \partial_\lambda U_j = -U_{j+k} \\ (2j+1)U_j = U_{j+1} + 2k\lambda U_{j+k} \end{cases} \quad (4)$$

In the above set (4) the second line can be recognized to be the Schwinger-Dyson equations written on the unnormalized even green functions with $U_j = Z_{2j} = Z_0 G_{2j}$ (e.g. see Refs. 7 and 8 where those are addressed by truncation for $k = 2$). The first equation in (4), however, is of a kind akin to a Step- k equation as seen in Ref. 9.

Combining both equations from (4), namely plugging the right hand side of the first into the second, brings about

$$\forall j \geq 0, \quad (2k\lambda \partial_\lambda + 2j + 1)U_j = U_{j+1}$$

that leads to

Theorem 4. *For the potential $V(\phi) = \phi^{2k}$, $k \geq 1$, the partition function U_0 , as well as its asymptotic expansion, do satisfy the differential equation (E_k)*

$$\left[\left(\prod_{j=0}^{k-1} (2k\lambda \partial_\lambda + 2j + 1) \right) + \partial_\lambda \right] .U_0 = 0$$

Proof. The proof is straightforward using (2) and (3):

$$\left(\prod_{j=0}^{k-1} (2k\lambda \partial_\lambda + 2j + 1) \right) .U_0 = U_k = -\partial_\lambda U_0. \quad \square$$

This equation (E_k) will provide all the information on the resurgence of the partition function.

IV. DIVERGENT SERIES AND ODES

A. Singularities of linear ODEs

In the present article, we shall be concerned with linear ordinary differential equations with polynomial or rational coefficients; it is in this context that we now briefly recall some of the basic elements of the theory of irregular singularities of ODEs with analytic data.

Let us consider such a linear ODE, for a complex variable x :

$$a_n(x)y^{(n)}(x) + \dots + a_1(x)y'(x) + a_0(x)y(x) = 0, \quad (5)$$

where the coefficient functions $a_i(x)$ belong to $\mathbb{C}[x]$.

Each point x_0 in the complex plane which is not a root of the leading coefficient $a_n(x)$ is *regular*: at such a point, Cauchy–Lipschitz theorem applies and we have a basis of the vector space of solutions of (5) composed of functions which are analytic near x_0 .

Moreover, it is the specificity of linear equations that all the local solutions can be continued along any path γ starting at x_0 that avoids the *singular points* of the equation – namely the roots of a_n and this global property of the solutions will be of constant use later on. Singular points of these linear ODEs can be of 2 types:

– *Regular–singular*, around which a basis of solution will only involve analytic functions, possibly ramified at the singular point, together with logarithms, that is, when $x_0 = 0$, solutions:

$$y(x) = x^\beta \left(\sum_{i=0}^m \Phi_i(x) \log^i x \right), \quad \beta \in \mathbb{C}, \quad \Phi_i \in \mathbb{C}\{x\}.$$

– *Irregular–singular*, for which, when $x_0 = 0$, local solutions will concomitantly involve exponentials of so-called determining factors $e^{q(x^{-\nu})}$ (with $q(u) \in \mathbb{C}[u]$) and divergent series, that is solutions:

$$y(x) = x^\beta e^{q(x^{-\nu})} \left(\sum_{i=0}^m \Phi_i(x) \log^i x \right), \quad \beta \in \mathbb{C}, \quad \Phi_i \in \mathbb{C}[[x]],$$

with a growth order of Gevrey–type for the formal series Φ_i in the variable $z = x^{-1}$:

Definition 1. A formal series $\sum_{n \geq 0} a_n z^{-n} \in \mathbb{C}[[z^{-1}]]$ is said to be Gevrey of order s , where s is a positive real number (Gevrey– s for short, or of level $q = \frac{1}{s}$) iff:

$$\exists C, A > 0, \text{ such that } \forall n \geq 0, |a_n| \leq CA^n (n!)^s.$$

Formal series with such factorial growth estimates have been systematically studied by Maurice Gevrey during the 1910’ for PDEs. In the context of linear ODEs a number of results involving these “Gevrey series” have notably been obtained by Maillet and Perron; they are ubiquitous in solutions of linear and non linear differential equations with meromorphic data, as shown in many works since the seminal work of Ramis¹⁰, which marked a revival of the topic.

Generically, such Gevrey series that are formal solutions of an ODE at an irregular–singular point will be divergent, yet there exists a standard process to express with them analytic solutions *in sectors* at the singular point: Borel–Laplace summation.

B. Borel, Laplace and Stokes

The paradigmatic example¹¹ of an ODE with an irregular singularity is the so-called Euler equation, which can be given in the following form

$$x^2 f'(x) = -f(x) + x. \quad (6)$$

This equation has a singularity at the origin $x = 0$ and we find there a unique formal series solution

$$\tilde{f}_0(x) = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1}.$$

This formal series is Gevrey-1 and divergent. The general formal solution of (6) is

$$\Phi(x) = \tilde{f}_0(x) + \sigma e^{\frac{1}{x}}, \quad (\sigma \in \mathbb{C}).$$

On this simple example we indeed notice the simultaneous presence of Gevrey-1 divergence and of an exponential factor $e^{\frac{1}{x}}$. Equation (6) can be given in linear homogeneous form: after division by x , derivation and finally multiplication by x^2 we obtain the following linear second order ODE

$$x^3 f''(x) + (x^2 + x)f'(x) - f(x) = 0 \quad (7)$$

and the vector space of solutions of (7) is spanned by $\tilde{f}_0(x)$ and $e^{\frac{1}{x}}$.

As a function of the variable x close to the origin, $e^{\frac{1}{x}}$ will have a very different behaviour according to the direction along which $x \rightarrow 0$: it will be explosive when $\Re x$ is positive, will vanish when it is negative and oscillate when x is purely imaginary.

This completely elementary observation is in fact crucial: for matters of resummation of divergent series, the situation is *polarized*. Polarizability is built-in the general solution of the problem and any sensible summation process will have to take care of this; thus, the consideration of sectors of opening π in the case of Gevrey-1 series is relevant precisely because of the concomitant $e^{\frac{\alpha}{x}}$ occurrence of exponential factors, for solutions close to the singular point.

We are now ready to introduce the Borel-Laplace mechanism; it is convenient to change from $x \sim 0$ to $z \sim \infty$.

Definition 2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{-(n+1)}$, we define its Borel transform by

$$\mathcal{B}(f)(\zeta) := \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n = \hat{f}(\zeta).$$

The shift by one unit in the exponents simplifies the formulas and changing the name of the variable is most useful. For the moment, we thus suppose in the definition of the Borel transform that the constant term of $f(z)$ is null; we shall in fact see later how to enhance this transformation to more general formal series and in the applications we have in view it will always be possible to deal with series with $a_0 = 0$.

Thus, for the formal series solution of Euler's equation, $\tilde{f}_0(z) = \sum_{n=0}^{\infty} (-1)^n n! z^{-(n+1)}$, we have

$$\mathcal{B}(\tilde{f}_0)(\zeta) = \sum_{n=0}^{\infty} (-1)^n \zeta^n = \frac{1}{1 + \zeta}.$$

On any direction $d_\theta = e^{i\theta} \mathbb{R}_>$ in the Borel plane, except $\mathbb{R}_<$, $\hat{f}_0(\zeta)$ can be analytically continued and it decays at ∞ ; thus we can consider its Laplace transforms on any non-singular direction

$$\mathcal{L}_{d_\theta} \hat{f}(z) := \int_0^{e^{i\theta} \infty} e^{-z\zeta} \hat{f}_0(\zeta) d\zeta.$$

For any $d_\theta \neq \mathbb{R}_<$ this integral yields an analytic function in a half plane $\{\Re(z e^{i\theta}) > \text{constant}\}$ bisected by $e^{-i\theta}$, which translates in an analytic function in a sector of opening π for $x \sim 0$.

As both Borel and Laplace transforms are *morphisms of differential algebras*, on the spaces we are considering, it automatically ensures that the analytic functions obtained by this process are solutions of Euler's equation: we have performed a *resummation* of the divergent series $f_0(x)$ in sectors at the origin¹².

When we move the direction d_θ without crossing the singular direction, the Laplace sums coincide on the overlapping sectors (by Cauchy's formula and Lebesgue dominated convergence at ∞) but it is not so when we perform the integration slightly above and slightly under the singular direction $\mathbb{R}_<$: the 2 integrals differ – this is *Stokes phenomenon*.

In this case, the difference can be explicitly calculated because we have a closed-form formula for $\hat{f}_0(\zeta)$, that has as single singularity $\omega = -1$, which is a simple pole with residue equal to 1 and we get, for $\varepsilon > 0$, $\mathcal{L}_{d_{\pi-\varepsilon}} \hat{f}_0(z) - \mathcal{L}_{d_{\pi+\varepsilon}} \hat{f}_0(z) = 2\pi i e^z = 2\pi i e^{\frac{1}{x}}$.

For Euler's equation, everything is explicit but this simple case displays features which will be quite general for solutions of ODEs with analytic data: in generic cases, when expressed in a **suitable variable** $z \sim \infty$, these series $\tilde{f}(z)$ have Borel transforms \hat{f} which are convergent and a Laplace transform of these \hat{f} , when justified, yields “sectorial sums” of the $\tilde{f}(z)$.

Stokes phenomenon is precisely the fact that, on some overlapping sectors, these sums may differ.

In practice, the functions $\hat{f}(\zeta)$ have analytic continuations with **isolated singularities** ω and the very presence of the singularities of \hat{f} accounts for the divergent character of \tilde{f} ; the analysis of these singularities is achieved by alien calculus and Stokes phenomenon can eventually be expressed through the use of alien derivations on the formal solutions.

C. The Newton polygon

Newton polygons (NP for short) are pervasive for the study of singularities; in the context of singularities of linear ordinary differential equations, they were introduced by J.-P. Ramis in the seminal Ref. 10 as a crucial tool for Gevrey asymptotics.

The NP of a linear ODE encodes many properties of the formal solutions of this equation; we recall now the definition and main properties of the Newton polygon for a linear differential operator with polynomial coefficients, see Refs. 11 and 13 for further information and proofs of the classical results we shall use. As such equations arise naturally in the sequel, we consider, near $x = 0$ linear ODEs such as

$$H(f)(x) = a_n(x)f^{(n)}(x) + \dots + a_1(x)f'(x) + a_0(x)f(x) = 0 \quad (8)$$

where the coefficients functions $a_i(x)$ belong to $\mathbb{C}(x)$. Up to a polynomial factor, such equations do not differ from equation (5), but in this case we will deal with rational fractions $h \in \mathbb{C}(x)$ that can be written as a Laurent series

$$h(x) = \sum_{q \geq -N} h_q x^q.$$

Definition 3. We consider such a linear differential equation $H(f) = 0$, where $H \in \mathbb{C}(x)[d/dx]$ and we express such an operator H of order n in the following way:

$$H = H_n(x)\theta^n + \dots + H_1(x)\theta + H_0(x) \quad \text{where} \quad \theta = x \frac{d}{dx}.$$

We consider the set \mathcal{S} of pairs (i, q) , with $0 \leq i \leq n$ and $q \in \mathbb{Q}$ such that the monomial x^q appears with a non-vanishing coefficient in the Laurent series of $H_i(x)$.

The NP at 0 of equation (8) is the convex hull of the set $\{(u, v), (i, q) \in \mathcal{S}, 0 \leq u \leq i, q \leq v\}$ in $\mathbb{R}^+ \times \mathbb{R}$. Its boundary is the union of 2 vertical lines (corresponding to $i = 0$ and $i = n$) and a finite number of segments, with slopes q_i , with $0 \leq q_1 < q_2 \dots < q_r$.

In the same way, the NP at ∞ of equation (8) is the convex hull of the set $\{(u, v), (i, q) \in \mathcal{S}, 0 \leq u \leq i, v \leq q\}$ in $\mathbb{R}^+ \times \mathbb{R}$.

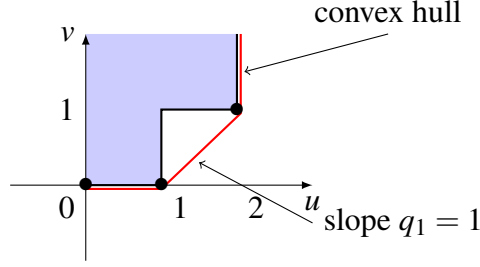


FIG. 1. Newton Polygon at 0 for Euler's equation (7).

It is readily checked¹¹ that the NP at ∞ of equation (8) is the symmetric with respect to the horizontal axis of the NP at 0 of the equation which is obtained from (8) by the change of variable $x \rightarrow 1/x$.

In the case of Euler's equation in homogeneous form (7), the corresponding operator is

$$x\theta^2 + \theta - 1$$

and its NP has 3 points $(0,0)$, $(1,0)$ and $(2,1)$ so that we get the NP depicted in Fig. 1 and the slopes are thus 0 and 1.

We shall notably use in the present article the so-called “main theorem for Gevrey asymptotics”, which can be stated in the following way:

Theorem 5. *Let $H \in \mathbb{C}(x)[d/dx]$. The equation $H(f) = 0$ will have a formal series solution f if and only if its NP at 0 has a slope which is horizontal. In that case, if there is no positive slope, then the formal series solution is analytic; else, if we denote by $q_1 < \dots < q_r$ the positive slopes of the Newton polygon of the equation at the origin, the degrees of the determining factors appearing in a basis of formal solutions belong to the set $\{q_1, \dots, q_r\}$.*

Moreover, in any direction d , there is a sector S at the origin, bisected by d , such that $H(f) = 0$ admits in S a fundamental system of analytic solutions and the maximal growth rate of a solution approaching the origin is of type $|x|^\beta e^{\frac{\alpha}{|x|^{q_r}}}$ ($\alpha \in \mathbb{R}^+$).

Whenever the Borel transform \widehat{f} of a formal series satisfies a linear ODE in the variable ζ , that is $\widehat{H}(\widehat{f}) = 0$ (as this will be the case in our examples), this theorem is useful to derive the growth at infinity of this function. If the NP at $\zeta = \infty$ of \widehat{H} has slopes $0 \geq q'_1 > \dots > q'_r$ then, in any nonsingular direction, the growth of \widehat{f} is of type $|\zeta|^\gamma e^{\alpha|\zeta|^{-q'_r}}$ ($\gamma \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$).

The general algorithm that computes a basis of formal solutions is very clearly exposed in Ref. 14 and it turns out that, for the Euler equation, as well as for the equations (E_k) , the algorithm simplifies drastically. For these equations we can luckily apply the following procedure.

Suppose the Newton polygon of the differential operator

$$H = H_n(x)\theta^n + \dots + H_1(x)\theta + H_0(x)$$

has one horizontal slope from $(0, d_0)$ to $(1, d_0)$ and then one positive slope q from $(1, d_0)$ to (n, d_1) ($q = \frac{d_1 - d_0}{n - 1}$) then

Step 1 The equation $H(f)$ has exactly one-dimensional family of solutions generated by some y_0 in $x^\beta \mathbb{C}[[x]]$ and the exponent β is determined by the “indicial equation”

$$H_{1,d_0}\beta + H_{0,d_0} = 0,$$

where H_{1,d_0} (resp. H_{0,d_0}) is the coefficient of degree d_0 in H_1 (resp. H_0). More generally, the indicial equation $Q(\beta) = 0$ where the polynomial $Q(\beta)$ corresponds to the coefficient of the monomial of lowest degree in x in $x^{-\beta} H x^\beta$. Note that the $H x^\beta$ is the operator that associates to any test function f the function $H(x^\beta f)$ and not the action of H on the function x^β that would be noted $H(x^\beta)$.

Step 2 For $u \in \mathbb{C}$ one compute the operator

$$\begin{aligned} H_u &= e^{-\frac{u}{x^q}} H e^{\frac{u}{x^q}} \\ &= H_n(x)(\theta - qux^{-q})^n + \dots + H_1(x)(\theta - qux^{-q}) + H_0(x) \\ &= H_n(x, u)\theta^n + \dots + H_1(x, u)\theta + H_0(x, u). \end{aligned}$$

The term of lowest degree in x in $H_0(x, u)$ is a polynomial $P(u)$ of degree n , with 0 as a root.

Step 3 If this polynomial has exactly $n - 1$ non zero distinct roots u_1, \dots, u_{n-1} , for each u_i , the newton polygon of H_{u_i} has the same shape as for H and one can apply Step 1 to get a formal solution y_i of H_{u_i} .

Once this procedure is finished, we obtain a basis of formal solutions:

$$y_0(x), e^{\frac{u_1}{x^q}} y_1(x), \dots, e^{\frac{u_{n-1}}{x^q}} y_{n-1}(x).$$

Note that the situation is less simple whenever the polynomial $P(u)$ has multiple roots¹⁴.

D. An exercise: the homogeneous Euler equation

For $H = x\theta^2 + \theta - 1$ (see (7)), Step 1 in the algorithm ensures that we have a nontrivial solution $\tilde{f}_0 \in x^\beta \mathbb{C}[[x]]$ where the indicial equation is in this case $\beta - 1 = 0$, thus $\beta = 1$ and $\tilde{f}_0 \in x\mathbb{C}[[x]]$. Step 2, with slope $q = 1$, gives

$$H_u = e^{-\frac{u}{x}} H e^{\frac{u}{x}} = x\theta^2 + (1 - 2u)\theta + [(u - 1) + (u^2 - u)x^{-1}]$$

and $P(u) = u^2 - u$ so that we can move to Step 3 with $u = 1$ and $H_1 = x\theta^2 - \theta$. The algorithm ensures that there exists a formal solution to $H_1(f) = 0$, here $y_1(x) = 1$ for instance. We have a basis of solution $\tilde{f}_0, e^{\frac{1}{x}}$.

As we have already noticed, Newton's polygons are also informative on the Borel transform of a solution and its growth at infinity. Toward a process described in section V C (see also Ref. 15), the Borel transform of \tilde{f}_0 is itself a solution of an equation $\widehat{H}(\widehat{f}_0) = 0$ where

$$\widehat{H} = (1 + \zeta)\theta + \zeta \quad \text{with} \quad \theta = \zeta \frac{d}{d\zeta}$$

and without any computation of the solutions, the Newton's polygon of this equation gives once again many informations:

1. The NP at $\zeta = 0$ has no positive slope: this equation has analytic solutions at $\zeta = 0$ (the indicial equation gives $\beta = 0$) and the Cauchy-Lipschitz theorem ensure that the solutions can be continued along any path that avoids $\zeta = -1$.
2. The NP at $\zeta = \infty$ has no negative slope, that is to say that its symmetric with respect to the horizontal axis, that corresponds to the Newton's polygon at 0 in the variable $\xi = 1/\zeta$ has only a zero slope: in the variable ξ , solutions near $\xi = 0$ are in $\xi^\beta \mathbb{C}\{\xi\}$ for some $\beta \in \mathbb{R}$ and thus the solutions of $\widehat{H}(\widehat{f}_0) = 0$ are $O(|\zeta|^{-\beta})$ near $\zeta = \infty$. This ensures that the Laplace Transform exists in any direction avoiding $\zeta = -1$.

This interplay between the Borel Transform and Newton's polygon is very useful to understand, with few computations, the resurgence of solutions of ODEs.

V. RESURGENT FUNCTIONS

A. Algebras of resurgent functions and resurgence

We recall now the main features of resurgent functions theory, *first in an informal way*; special cases shall then be introduced next, with the precise definitions for the restricted spaces of resurgent functions which are relevant in the present article.

At its core, resurgence involves analytic functions of one complex variable ζ which have isolated singular points and families of operators which “measure” these isolated singularities. There are spaces of resurgent functions of various levels of complexity, the definitions of which shall depend of the complexity of the problems they are involved in. In the present article we shall only need to consider resurgent functions of a very elementary type, introduced below.

We first consider the linear space \mathcal{E} of germs $\varphi(\zeta)$ at the origin of \mathbb{C} which can be continued along any broken line Γ starting at 0, possibly circumventing a finite number of isolated singular points ω_i met on Γ . This definition presupposes that the continuation of φ close to such a point ω is defined for paths going around it, following a small half-circle on the right or on the left of the direction d and, in general, this process will entail multivaluedness.

With the convolution product $\varphi * \psi(\zeta) := \int_0^\zeta \varphi(s)\psi(\zeta - s)ds$ (for ζ close to 0), \mathcal{E} is in fact an algebra (a highly non trivial fact^{16–18}); multiplication by $-\zeta$ is a derivation with respect to the convolution product and \mathcal{E} has a structure of differential algebra, with these operations.

We shall however very soon need to consider “germs that are in fact also singular at the origin”, namely functions defined in the lift of a pointed disk at the origin of \mathbb{C} on the Riemann surface of the logarithm \mathbb{C}_\bullet .

By germ at the origin of \mathbb{C}_\bullet , we shall mean in this article a function defined on the lift on \mathbb{C}_\bullet of a pointed disk $D^*(0, r)$ on \mathbb{C} . A regular germ $\varphi(\zeta)$ can be seen as the (class of the) singular germ $\check{\varphi}(\zeta) = \frac{\log(\zeta)}{2i\pi} \varphi(\zeta)$

Definition 4. Let \mathcal{F} be the set of classes $\varphi^\nabla(\zeta) \pmod{\mathbb{C}\{\zeta\}}$ of germs $\check{\varphi}(\zeta)$ on \mathbb{C}_\bullet such that the germ $\hat{\varphi}(\zeta)$ has isolated singularities, where

$$\hat{\varphi}(\zeta) = \check{\varphi}(\zeta) - \check{\varphi}(e^{-2\pi i} \zeta)$$

$\check{\varphi}(\zeta)$ is called a **major** of φ and $\hat{\varphi}(\zeta)$ is called **the minor** of φ . Elements of \mathcal{F} will be called resurgent functions and will simply be denoted by φ when the abuse of language is innocuous.

The set \mathcal{F} carries a natural structure of vector space but in fact it is an algebra, with a suitable definition of convolution^{17,18}.

Elements φ of \mathcal{F} for which a major $\check{\varphi}$ satisfies $\zeta \check{\varphi}(\zeta) \rightarrow 0$ when $\zeta \rightarrow 0$ uniformly in sectors of bounded opening and such that $\hat{\varphi}$ is integrable at 0 are called *integrable resurgent functions* and their set will be denoted by \mathcal{F}_0 . These resurgent functions are characterized by their minors and the convolution alluded to above boils down to the convolution of minors: we can identify such a function φ with its minor $\hat{\varphi}$ and accordingly we shall have, for any φ, ψ in \mathcal{F} :

$$\varphi * \psi(\zeta) := \int_0^\zeta \hat{\varphi}(s) \hat{\psi}(\zeta - s) ds, \quad (\zeta \sim 0).$$

In the applications considered by the present paper, it will always be possible to work with integrable resurgent functions, possibly after some simple transformation (essentially, for formal series, premultiplication of $\varphi(z)$ by some suitable power z^n ($n \in \mathbb{N}$), which corresponds to performing a finite number of integrations in the Borel plane.

Generally, (see e.g. below or Refs. 17 and 18), resurgent functions which appear as Borel transforms of divergent series which are solution to some complex analytic dynamical system at a singular point will display exponential growth in directions of the Borel plane which don't contain singularities. As such, they will be amenable to the Borel–Laplace summation mechanism and we thus have, in Ecalle's terminology three models for a resurgent function φ :

1. space of formal series $\widetilde{\varphi}(z)$;
2. space of classes of singular germs in the Borel plane $\varphi^\nabla(\zeta)$;
3. space(s) of analytic functions $\varphi(z)$ in sector(s).

We have given above the definition of Borel transform for formal series without constant terms first, by:

$$\mathcal{B} \left(\sum_{n=0}^{\infty} a_n z^{-n-1} \right) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+1)} \zeta^n$$

and then extended it to any formal series by defining $\mathcal{B}(1) = \delta$.

As we have seen, the Major/Minor formalism very naturally incorporates the convolution unit by expressing δ as the class determined by the major $\frac{1}{2\pi i \zeta}$, without making it necessary to consider spaces of distributions – and this is particularly valuable for questions of convergence of sequences of resurgent functions.

Now, for the applications, it is necessary to enhance the definitions of Borel and Laplace transforms to more general classes of formal objects: indeed, even for the case of linear differential equations considered in the present work, we meet in the formal solutions expansions involving ramified (Puiseux) formal series, together with integer powers of logarithms.

A first extension consists thus in considering power functions, of any exponent; we have to define the corresponding classes φ^∇ of $\tilde{\varphi} = z^u$ in such a way as to respect the properties of morphisms of differential algebras, which are absolutely crucial. In the following definition, it is necessary to make a distinction between integer and non integer exponents:

Definition 5. 1. For $\sigma \notin \mathbb{Z}$, we define $\mathcal{B}(z^{-\sigma}) = \varphi^\nabla$ with, as pair (natural major, minor):

$$\left(\check{\varphi} = (1 - e^{-2\pi i \sigma}) \frac{\zeta^{\sigma-1}}{\Gamma(\sigma)}, \hat{\varphi} = \frac{\zeta^{\sigma-1}}{\Gamma(\sigma)} \right).$$

2. For $n \in \mathbb{N}$, we define $\mathcal{B}(z^n) = \varphi^\nabla$ with, as pair (natural major, minor):

$$\left(\check{\varphi} = \frac{(-1)^n}{2\pi i} \zeta^{-n-1} \Gamma(n+1), \hat{\varphi} = 0 \right).$$

For any $\sigma \in \mathbb{C}$, with $\Re \sigma > 0$, the Borel transform of $z^{-\sigma}$ is characterized by its minor (it is an integrable singularity) and we can safely denote in this case $\mathcal{B}(z^{-\sigma}) = \frac{\zeta^{\sigma-1}}{\Gamma(\sigma)}$, using the same abuse of notations as for formal series with integer exponents. This can be extended formally to:

$$\mathcal{B} \left(\sum_{\sigma \in \mathbb{Q}^>} c_\sigma z^{-\sigma} \right) = \sum_{\sigma \in \mathbb{Q}^>} c_\sigma \frac{\zeta^{\sigma-1}}{\Gamma(\sigma)}.$$

In expansions using Puiseux series, if we have geometrical growth estimates ($|c_\sigma/\Gamma(\sigma)| \leq AB^\sigma$; $A, B > 0$, for exponents σ which are multiples of a given rational number, say), then the right hand side defines a germ (on \mathbb{C}_∞) and the Borel transform indeed extends as a morphism of differential algebras from the space of Puiseux formal series in z^{-1} with the ordinary product and derivation $\partial_z = \frac{d}{dz}$ to the algebra of local integrable resurgent, with convolution of minors and as derivation the multiplication (of minors) by $-\zeta$.

More generally, for any $\sigma \in \mathbb{C}$, with $\Re \sigma > 0$ and $r \in \mathbb{N}$, the Borel transform of $z^{-\sigma}(\log z)^r$ is the integrable resurgent function characterized by its minor, which is:

$$\mathcal{B}(z^{-\sigma}(\log z)^r) = \zeta^{\sigma-1} \sum_{i=0}^r \binom{r}{i} \left(\frac{1}{\Gamma(\zeta)} \right)^{(r)} (\log \zeta)^{r-i}.$$

B. Alien operators

Let us consider a resurgent function φ^∇ , as in the previous subsection, which is characterized by its minor $\hat{\varphi}$; we will moreover suppose first that $\hat{\varphi}$ is regular at the origin of \mathbb{C} (this will be the case for the applications to the equations E_k below), and denote by Ω its set of singularities in \mathbb{C} . We shall also suppose that is a fixed discrete set of \mathbb{C} , which will be enough to introduce all the necessary concepts; in the applications below, Ω will in fact be finite.

For any ω in \mathbb{C} and any path γ from the origin to ω there is an operator Δ_ω^γ that measures the singularity at ω of the analytic continuation along γ of any element of \mathcal{E} :

$$\Delta_\omega^\gamma \varphi(\zeta) := \varphi_\omega(\omega + \zeta) \pmod{\mathbb{C}\{\zeta\}}.$$

In this formula, φ_ω is the analytic continuation of φ along γ , which is defined on the lift on \mathcal{S} of a pointed disk $D(\omega, r)$.

It is important to observe that this definition involves 2 operations: “extraction of singularity” at ω and translation to the origin. The operator Δ_ω^γ associates to any given regular germ at 0 (element of \mathcal{E}) a class of singular germs. The Δ_ω^γ are linear and we will obtain endomorphisms if we enhance the following definition to the vector space \mathcal{F} : if φ^∇ is an element of \mathcal{F} , $\Delta_\omega^\gamma \varphi$ is given by the same formula above, in which φ_ω is by definition now the analytic continuation of *the minor* of φ along γ .

Averages of some of these Δ_ω^γ , with coefficients which satisfy relevant symmetry properties^{16–18} will next yield the alien derivations (there are several families of them; we shall only need the standard one). When ω is the first singularity met in the continuation in a given direction d (“polarization”), we define: $\Delta_\omega \varphi(\zeta) := \Delta_\omega^\gamma \varphi(\zeta)$, where γ is the segment on d from the origin to ω .

When there are several singularities $\omega_1, \dots, \omega_{r-1}$ on d before reaching $\omega_r = \omega$, we define

$$\Delta_\omega^+ \varphi(\zeta) := \Delta_\omega^\gamma \varphi(\zeta),$$

where γ is a path from the origin to ω closely following d from the right.

Finally, there are families of numbers $A^{(\omega_1, \dots, \omega_r)}$ such that the operators defined by

$$\Delta_\omega \varphi(\zeta) := \sum_{\|(\omega_1, \dots, \omega_r)\| = \omega} A^{(\omega_1, \dots, \omega_r)} \Delta_{\omega_r}^+ \dots \Delta_{\omega_1}^+ \quad \text{where } \|(\omega_1, \dots, \omega_r)\| = \omega_1 + \dots + \omega_r$$

are indeed derivations of the algebra \mathcal{F} . The symmetry property for the $A^{(\omega_1, \dots, \omega_r)}$ which will ensure that the associated Δ_ω will satisfy Leibniz rule is called alternality but we won’t need to

go into these considerations in the present text because we shall only consider the simplest of these families which is given by $A^{(\omega_1, \dots, \omega_r)} = \frac{(-1)^{r-1}}{r}$, which yields the so-called standard alien derivations.

When the resurgent functions have corresponding expressions $\tilde{\varphi}(z)$ “in the formal model” (it is e.g. the case for majors of the type $\zeta^\sigma (\log \zeta)^k$, as we have seen above, then pullbacks by \mathcal{B}^{-1} then gives alien operators acting on the formal objects $\tilde{\varphi}(z)$, for which we keep the same notation Δ_ω .

Finally, we introduce the exponential-carrying alien operators $\mathbf{\Delta}_\omega$ (denoted in bold, or also as pointed alien derivations in earlier texts) by the following expression in the formal model

$$\mathbf{\Delta}_\omega \tilde{\varphi}(z) := e^{-\omega z} \Delta_\omega \tilde{\varphi}(z).$$

These $\mathbf{\Delta}_\omega$ act on formal expressions involving not only resurgent functions but also exponentials $e^{-\omega z}$; there is a thorough theory of these so-called *transseries* but in the present work we shall only cope with finite sums of terms $e^{-\omega z} \tilde{\psi}(z)$, typically involving a basis of the finite dimensional vector spaces of the solutions of the linear ODEs we are dealing with. The $\mathbf{\Delta}_\omega$ commute with $\partial = \frac{\partial}{\partial z}$, which makes them particularly convenient in formal calculations, as we shall see in practice below.

C. Resurgence for holonomic functions with one critical time

Definition 6. A function of one complex variable x or a (Puiseux) formal series $f(x)$ is called holonomic if it is solution to a linear ODE

$$H(f) = 0 \quad \text{where} \quad H \in \mathbb{C}[x] \left[\frac{d}{dx} \right].$$

An equivalent definition would be to require the existence of $H \in \mathbb{C}(x) \frac{d}{dx}$ (which can then chosen to be monic) such that $H(f) = 0$ and a convenient characterization of a holonomic function is that $\mathbb{C} \left[\frac{d}{dx} \right] (f)$ be of finite type over $\mathbb{C}(x)$.

A function which is a solution of an equation “of affine type”, say $H(f) = g$ (with $H \in \mathbb{C}(x) \left[\frac{d}{dx} \right]$ and $g \in \mathbb{C}(x)$) is holonomic, as seen by dividing the equation by g and applying $\frac{d}{dx}$. Thus, solutions of Euler’s equation above are holonomic.

We recall the very well known following stability properties, for holonomic functions:

Proposition 3. *The sum and the product of two holonomic functions are holonomic. The postcomposition of a holonomic function by an algebraic function is holonomic*

These properties are in fact true for holonomic functions of several variables and rely on the characterization of holonomy mentioned above (for algorithmic aspects see e.g. Ref. 15). In particular, if $f(x)$ is holonomic, then, for any rational number r , $g(x) := f(x^r)$ is holonomic and this simple result will be crucial for questions of resurgence in our context.

Proposition 4. *The Borel transform of a holonomic function $f \in x\mathbb{C}[[x^{\mathbb{Q}^+}]]$ with vanishing constant coefficient is holonomic. As a consequence it is resurgent, with a finite number of singularities, by applying to the Borel plane the global theory of linear ODEs.*

Proof. If $H(f)(x) = a_n(x)f^{(n)}(x) + \dots + a_1f'(x) + a_0(x)f(x) = 0$, we can rewrite this equation using the derivation D defined by $D(f)(x) = x^2f'(x)$, possibly after having multiplied it by a suitable integer power of x to get an equation $K(f) = 0$, and then use the fact that the Borel transform of $D(f)(x)$ is $\zeta\hat{f}(\zeta)$ and that for any series $f \in x\mathbb{C}[[x^{\mathbb{Q}^+}]]$ of valuation > 1 , $\mathcal{B}(x^{-1}f)(\zeta) = \hat{f}'(\zeta)$.

As a consequence, it is resurgent with a finite number of singularities, by applying to the Borel plane the global theory of linear ODEs (for these matters, see also Ref. 19). \square

Thus, from equation (E_2) expressed in the variable x , we obtain successively

$$\begin{aligned}
16x^2f''(x) + 32x^3f'(x) + 3f(x) + f'(x) &= 0 \\
16D^2(f)(x) + 3x^2f(x) + D(f)(x) &= 0 \\
x^{-2}(16D^2(f)(x) + D(f)(x)) + 3f(x) &= 0 \\
((16\zeta^2 + \zeta)\hat{f}(\zeta))'' + 3\hat{f}(\zeta) &= 0 \\
(16\zeta^2 + \zeta)\hat{f}''(\zeta) + 2(32\zeta + 1)\hat{f}'(\zeta) + 35\hat{f}(\zeta) &= 0
\end{aligned} \tag{9}$$

The last equation is singular at 0 and $-\frac{1}{16}$ and this entails that \hat{f} can only have a singularity at $-\frac{1}{16}$.

Theorem 6. *Let f be a formal series which is solution to $H(f) = 0$, where $H \in \mathbb{C}[x](d/dx)$ has a Newton polygon at 0 with a single non zero slope, equal to q .*

Then f is resurgent with respect to the critical time $z = 1/x^q$, with a finite number of singularities in the Borel plane and exponential growth of order 1 in any non-singular direction

Proof. By premultiplying f by a suitable power of x with a positive exponent, we can suppose that $\text{val}(f) > q$; this operation doesn't change the holonomic character, nor the slopes of the Newton polygon.

Let $g(x) := f\left(x^{\frac{1}{q}}\right)$. Then g is holonomic and the Newton polygon at 0 of the differential equation $U(g) = 0$, with $U \in \mathbb{C}[x](d/dx)$ obtained from $H(f) = 0$ by the action of ramification $x \rightarrow x^{\frac{1}{q}}$ has a single slope, equal to one.

The series g is Gevrey-1 and its Borel transform is holonomic, by the previous proposition ($\text{val}(g) > 1$). Moreover, the NP of the Borel transform of U has a single slope equal to one at ∞ (see Ref. 11), which entails that $\hat{g}(\zeta)$ has at most exponential growth of order 1 at ∞ and thus \hat{g} is Laplace-summable in every non singular direction d ($d \cap S = \emptyset$, where S designates the finite set of singularities of \hat{g}).

By reverting to the original variable, we get sectorial solutions on sectors of opening π/q for $f(x)$. □

All the formal series considered in the present text will satisfy the hypotheses of this theorem and we shall see how the singularities of their Borel transforms can be analyzed by the action of alien operators introduced in the previous subsection, eventually providing a control of Stokes phenomenon for the sectorial sums of the series.

The algebra of resurgent functions which we need for the present article is simply the algebra \mathcal{R}_h of holonomic functions f : for any $q \in \mathbb{Q}^>$, the function $g(x) := f\left(x^{\frac{1}{q}}\right)$ has a Borel transform which is characterized by its minor, which has isolated singularities and g is thus resurgent and amenable to alien calculus.

VI. RESURGENT STUDY OF THE FAMILY (E_k)

A. The first equation

All the equations in the family $(E_k)_{k \geq 2}$ will be tractable by the same techniques, yet it is worthwhile to start by an exhaustive study of E_2 because the resurgence properties will be accessible in a straightforward way.

For $k = 2$, we have thus obtained above the following linear ODE, with $\partial_\lambda = \frac{d}{d\lambda}$:

$$[(4\lambda \partial_\lambda + 1)(4\lambda \partial_\lambda + 3) + \partial_\lambda]f(\lambda) = 0 \quad (E_2)$$

For any given a_0 this equation has a unique solution in $\mathbb{C}[[\lambda]]$, namely $\tilde{f}(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$, with

$$a_{n+1} = -\frac{(4n+1)(4n+3)}{n+1}a_n$$

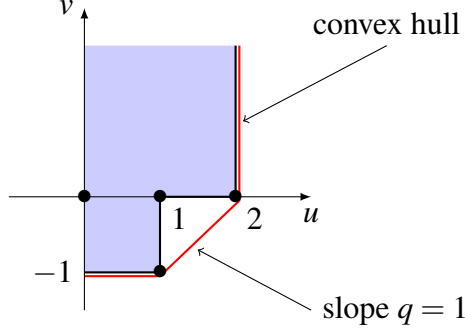


FIG. 2. NP corresponding to $H_2 = 16\theta^2 + (16 + x^{-1})\theta + 3$.

and $\tilde{Z}_0(\lambda)$ is the solution of (E_2) in $\mathbb{C}[[\lambda]]$ with $a_0 = 1$.

This recurrence relation immediately entails (for any $a_0 \neq 0$) the *exact* Gevrey-1 rate of growth for the coefficients a_n , with:

$$|a_n| \sim 16^n n! |a_0|.$$

From this estimate, we can already deduce that $\sum_{n=0}^{\infty} a_n \lambda^n$ is indeed divergent; it has $z = \frac{1}{\lambda}$ as critical time and the distance to the origin of the closest singularity in the Borel plane of $\mathcal{B}(\tilde{Z}_0)$ is $\frac{1}{16}$ but considerations on the Newton's polygon and the *Borel transform of equation* (E_2) will easily yield a more precise results.

If $x = \lambda$ and $\theta = x \frac{d}{dx}$ the operator associated to (E_2) is

$$H_2 = 16\theta^2 + (16 + x^{-1})\theta + 3$$

that corresponds to the Newton's polygon drawn in Fig. 2.

The algorithm in section IV C applies here :

Step 1 The indicial equation gives $\beta = 0$: (E_2) as a nontrivial formal solution in $\mathbb{C}[[x]]$, for example $\tilde{Z}_0(x)$.

Step 2 The unique positive slope is $q = 1$ and

$$\begin{aligned} H_{2,u} &= e^{-\frac{u}{x}} H_2 e^{\frac{u}{x}} \\ &= 16(\theta - ux^{-1})^2 + (16 + x^{-1})(\theta - ux^{-1}) + 3 \\ &= 16\theta^2 + (16 + (1 - 32u)x^{-1})\theta + 3 + u(16u - 1)x^{-2}, \end{aligned}$$

thus $P(u) = u(16u - 1)$ with $u_1 = \frac{1}{16}$ as a root.

Step 3 We can complete the basis of formal solutions by $e^{\frac{1}{16x}}\tilde{Z}_1$ where \tilde{Z}_1 cancels out the operator:

$$H_{2, \frac{1}{16}} = 16\theta^2 + (16 - x^{-1})\theta + 3.$$

For this latter equation, we can also find an explicit series as a solution: $\tilde{Z}_1(\lambda) = \sum_{n=0}^{\infty} b_n \lambda^n$ with the following recurrence relation:

$$b_{n+1} = \frac{(4n+1)(4n+3)}{n+1} b_n$$

and of course the same growth estimates for the b_n as for the a_n .

We have seen above (see equation (9)) that the Borel transform of (E_2) can be written in the following form:

$$(16\zeta^2 + \zeta)\hat{f}''(\zeta) + 2(32\zeta + 1)\hat{f}'(\zeta) + 35\hat{f}(\zeta) = 0.$$

By the general properties of the Borel transform, we have that $\hat{Z}_0(\zeta)$ cancels out the corresponding operator (up to a left multiplication by ζ)

$$\hat{H}_2 = (16\zeta + 1)\theta^2 + (48\zeta + 1)\theta + 35\zeta$$

that has no singular point at $\zeta = 0$: its solutions are analytically continuable in \mathbb{C}_ζ with as only singularity the point $\omega = -\frac{1}{16}$.

As in section IV D, the Newton polygon of \hat{H}_2 at ∞ has only a zero slope which entails that $\hat{Z}_0(\zeta)$ has at most a polynomial growth at ∞ , along any direction but the real negative one. As a consequence, \tilde{Z}_0 is 1-summable in every direction except $\mathbb{R}_<$ and there is only one alien derivation that can act non trivially on it, namely $\Delta_{-\frac{1}{16}}$.

For the solution corresponding to $H_{2, \frac{1}{16}}$, one can check that this is the same NP as for H_2 and that the Borel transform corresponds to the equation

$$(16\zeta^2 - \zeta)\hat{f}''(\zeta) + 2(32\zeta - 1)\hat{f}'(\zeta) + 35\hat{f}(\zeta) = 0$$

and we obtain that \hat{Z}_1 is resurgent with a single singularity at $\omega = \frac{1}{16}$ and exponential growth at infinity.

The general solution $\Phi(\lambda)$ of (E_2) is thus:

$$\Phi(\lambda) = \sigma_0 \tilde{Z}_0(\lambda) + \sigma_1 e^{\frac{1}{16\lambda}} \tilde{Z}_1(\lambda),$$

where $\sigma_i \in \mathbb{C}$. We already know that the only alien derivations Δ_ω which act non trivially on $\tilde{Z}_0(\lambda)$ and $\tilde{Z}_1(\lambda)$ are respectively Δ_u and Δ_{-u} , with $u = -\frac{1}{16}$ but a key remark is that we can also *deduce that from the shape of the general solution above*.

Indeed, for any $\omega \in \mathbb{C}$, we have:

$$\begin{aligned} e^{-\frac{\omega}{\lambda}} \Delta_{\omega} \Phi(\lambda) &= \sigma_0 e^{-\frac{\omega}{\lambda}} \Delta_{\omega} \tilde{Z}_0(\lambda) + \sigma_1 e^{-\frac{\omega}{\lambda}} \Delta_{\omega} e^{\frac{1}{16\lambda}} \tilde{Z}_1(\lambda) \\ &= \sigma_0 e^{-\frac{\omega}{\lambda}} \Delta_{\omega} \tilde{Z}_0(\lambda) + \sigma_1 e^{\frac{1}{16\lambda} - \frac{\omega}{\lambda}} \Delta_{\omega} \tilde{Z}_1(\lambda) \end{aligned}$$

but $e^{-\frac{\omega}{\lambda}} \Delta_{\omega} \Phi(\lambda)$ is also a solution of (E_2) , because $e^{-\frac{\omega}{\lambda}} \Delta_{\omega}$ commutes with ∂_{λ} and vanishes on the polynomial coefficients of the equation.

Thus, there exist 2 complex constants c and d such that:

$$e^{-\frac{\omega}{\lambda}} \Delta_{\omega} \Phi(\lambda) = c \tilde{Z}_0(\lambda) + d e^{\frac{1}{16\lambda}} \tilde{Z}_1(\lambda)$$

and this *entails* that the only alien derivations which can act non trivially correspond to the 2 following indices:

$$\omega = -u, \text{ with } \Delta_{-u} \tilde{Z}_0(\lambda) = A_{-u} \tilde{Z}_1(\lambda) \ (A_{-u} \in \mathbb{C}) \text{ and } \Delta_{-u} \tilde{Z}_1(\lambda) = 0$$

$$\omega = u, \text{ with } \Delta_u \tilde{Z}_1(\lambda) = A_u \tilde{Z}_1(\lambda) \ (A_u \in \mathbb{C}) \text{ and } \Delta_u \tilde{Z}_0(\lambda) = 0$$

This is an illustration, *in an elementary situation*, of the power of alien calculus: once we know that the series we are dealing with are resurgent, we can easily get highly non trivial relations just by taking into account formal rules and relations of homogeneity when applying alien derivations to them.

The previous 2 resurgence relations can be expressed in the following more compact form:

$$\Delta_{\omega} \Phi = \left(A_0 \sigma_0 \frac{\partial}{\partial \sigma_0} + A_1 \sigma_1 \frac{\partial}{\partial \sigma_1} \right) \Phi \quad (10)$$

We see here that alien derivations act on the formal integral (the general solution of E_2) as ordinary differential operators.

We thus have a bridge between alien and ordinary calculus and (10) is an elementary instance of the *bridge equation*, which is a very general fact: for all the known cases of analysis of irregular singularities of functional (differential, difference, etc) with the apparatus of resurgence and alien calculus, some form of the bridge equation, depending upon the class of equation under study, can be explicitated and this fact has numerous important consequences.

Remark 2. The partition function for $k = 2$ is of course to be found in many articles and textbooks, for various illustrative purposes. *Notably*, a very thorough resurgent study of it is thus done in section 2 of Ref. 1 – with a slightly different normalization; as shown in this reference, E_2 belongs to the hypergeometric family (it is a modified Bessel eq.) and so does its Borel transform, which gives access to exact values of the coefficients A_{ω} .

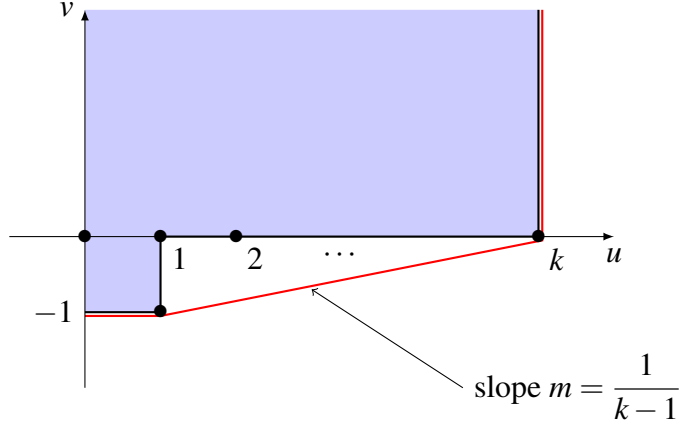


FIG. 3. The NP for H_k with λ as variable

It is however more important in the present work, rather than relying on properties of the hypergeometric family to stress the efficiency of the general formal arguments used above, which will work in a similar way for higher dimensional equations and also in non linear situations, where as a rule no exact values will exist for the coefficients of the bridge equation.

B. Formal results

We start by the Newton polygon at 0 of equation (E_k) that corresponds to the operator

$$H_k = \left(\prod_{j=0}^{k-1} (2k\theta + 2j + 1) \right) + \lambda^{-1}\theta$$

which is given in Fig. 3. It has one horizontal slope, of length one, and only one non-zero slope, of value $m = \frac{1}{k-1}$

Accordingly, (E_k) has one formal series solution $\tilde{Z}_0(\lambda)$ with a constant coefficient equal to 1 (the indicial equation gives $\beta = 0$, which is Gevrey-1 with respect to the variable $z_k = \frac{1}{\lambda^m}$, as a consequence of the propositions above). This unique slope suggests to study the equation in the variable $x = \lambda^m$ ($\lambda = x^{k-1}$) and since

$$\theta_\lambda = \lambda \frac{\partial}{\partial \lambda} = mx \frac{\partial}{\partial x} = m\theta_x$$

we will focus on the operator in the variable x :

$$H_k = \left(\prod_{j=0}^{k-1} (2km\theta_x + 2j + 1) \right) + mx^{1-k}\theta_x$$

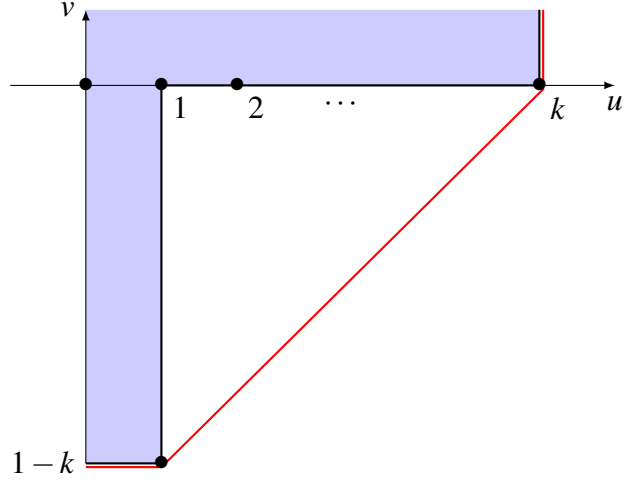


FIG. 4. The NP for H_k after choosing $x = \lambda^{\frac{1}{k-1}}$ as the new variable, notice its slope $q = 1$.

whose Newton's polygon is given in Fig. 4 and has a single non-zero slope $q = 1$.

In order to compute the operator $H_{k,u} = e^{-u/x} H_k e^{u/x}$ let us introduce some notations and combinatorial coefficients. We first note, for $k \geq 1$,

$$\prod_{j=0}^{k-1} (X + 2j + 1) = \sum_{i=0}^k a_{k,i} X^i$$

and one can check that for any k , $a_{k,k} = 1$ and $a_{k,k-1} = k^2$. Second, let us define the Exponential (or Touchard) polynomials recursively by $T_0(X) = 1$ and, for $n \geq 1$,

$$T_n(X) = \left(X + X \frac{d}{dX} \right) T_{n-1}(X)$$

the polynomial T_n is of degree n and, if $n \geq 1$ its coefficients are the stirling numbers of second kind (see for instance Ref. 20):

$$T_n(X) = \sum_{k=1}^n s_{n,k} X^k$$

for which we can notice that $s_{n,n} = 1$ and $s_{n,n-1} = \frac{n(n-1)}{2}$. These polynomials will appear naturally in the sequel since:

$$\forall n \geq 0, \quad e^{-u/x} [\theta_x^n \cdot e^{u/x}] = (-1)^n T_n \left(\frac{u}{x} \right).$$

We perform now the change of unknown function $f(x) = e^{u/x} g(x)$, and use the previous coef-

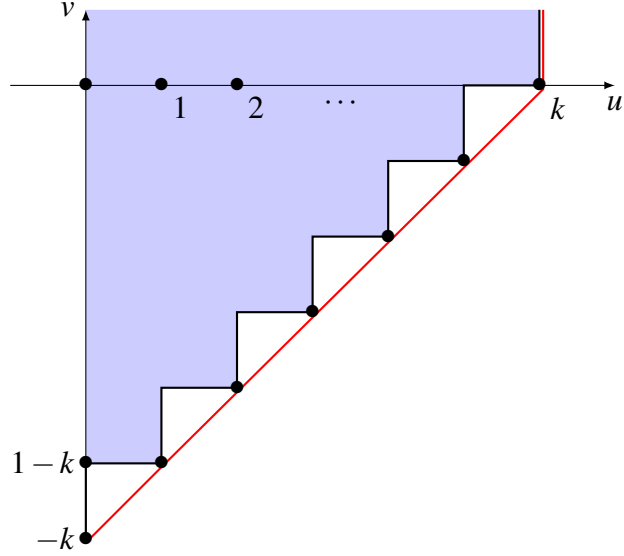


FIG. 5. NP at 0 of $H_{k,u} = e^{-\frac{u}{x}} H_k e^{\frac{u}{x}}$ for a “generic” $u \in \mathbb{C}$.

ficients:

$$\begin{aligned}
e^{-\frac{u}{x}} H_k(e^{\frac{u}{x}} g(x)) &= \sum_{i=0}^k a_{k,i} e^{-\frac{u}{x}} (2km)^i \theta_x^i(e^{\frac{u}{x}} g(x)) + e^{-\frac{u}{x}} m x^{1-k} \theta_x(e^{\frac{u}{x}} g(x)) \\
&= \sum_{i=0}^k a_{k,i} e^{-\frac{u}{x}} (2km)^i \sum_{j=0}^i \binom{i}{j} (\theta_x^{i-j}(e^{\frac{u}{x}})) (\theta_x^j g(x)) \\
&\quad + m x^{1-k} \theta_x g(x) - m u x^{-k} g(x) \\
&= \sum_{i=0}^k a_{k,i} (2km)^i \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} T_{i-j} \left(\frac{u}{x} \right) \theta_x^j g(x) \\
&\quad + m x^{1-k} \theta_x g(x) - m u x^{-k} g(x) \\
&= \sum_{j=0}^k \sum_{i=j}^k a_{k,i} (2km)^i \binom{i}{j} (-1)^{i-j} T_{i-j} \left(\frac{u}{x} \right) \theta_x^j g(x) \\
&\quad + m x^{1-k} \theta_x g(x) - m u x^{-k} g(x) \\
H_{k,u} g(x) &= \sum_{j=0}^k \left(\sum_{i=0}^{k-j} a_{k,j+i} (2km)^{i+j} \binom{i+j}{j} (-1)^i T_i \left(\frac{u}{x} \right) \right) \theta_x^j g(x) \\
&\quad + m x^{1-k} \theta_x g(x) - m u x^{-k} g(x)
\end{aligned}$$

For a “generic” value of u , the operator $H_{k,u}$ has the Newton polygon shown in Fig. 5 and we can specify three coefficients in the operator:

- The coefficient of $x^{-k}\theta_x^0$ is

$$P_k(u) = a_{k,k}(-1)^k(2km)^k s_{k,k}u^k - mu = -mu + (-2kmu)^k,$$

that has only simple roots and for each of its roots, the point $(0, -k)$ disappears.

- The coefficient of $x^{1-k}\theta_x$ is

$$\begin{aligned} Q_k(u) &= m + a_{k,k}(2km)^k \binom{k}{1} (-1)^{k-1} s_{k-1,k-1} u^{k-1} \\ &= m + 2k^2 m (-2kmu)^{k-1} \end{aligned}$$

so that, whenever $P_k(u)$ vanishes, $Q_k(u)$ does not: for each root of $P_k(u)$, the operator $H_{k,u}$ has the same NP as H_k so there exists a Gevrey-1 series (up to a factor x^β) that cancels out this operator.

- The coefficient of $x^{1-k}\theta_x^0$ is

$$\begin{aligned} a_{k,k-1}(-2km)^{k-1} s_{k-1,k-1} u^{k-1} + a_{k,k}(-2km)^k s_{k,k-1} u^{k-1} \\ = (-2kmu)^{k-1} \left(k^2 - 2km \frac{k(k-1)}{2} \right) = 0, \end{aligned}$$

since $m = \frac{1}{k-1}$. This means that, for a root of $P_k(u)$ the indicial equation (that corresponds to the coefficient of lowest degree in $H_{k,u} \cdot x^\beta$) is simply $\beta Q_k(u) = 0$ thus $\beta = 0$.

We can now state the following:

Proposition 5. *The equation (E_k) , in the variable $x = \lambda^m$ has as general solution:*

$$\Phi(x) = \sigma_0 \widetilde{\Phi}_0(x) + \sigma_1 e^{\frac{u_1}{x}} \widetilde{\Phi}_1(x) + \dots + \sigma_{k-1} e^{\frac{u_{k-1}}{x}} \widetilde{\Phi}_k(x), \quad \sigma_0, \dots, \sigma_{k-1} \in \mathbb{C}$$

where:

1. $m = \frac{1}{k-1}$.
2. $u_0 = 0, u_1, \dots, u_{k-1}$ are the k roots of the polynomial

$$P(u) = -mu[(-1)^{k-1}(2k)^k(mu)^{k-1} + 1].$$

3. The formal series $\widetilde{\Phi}_i(x)$ are Gevrey of order 1.
4. The formal series $\widetilde{Z}_0(\lambda)$ is proportionnal to $\widetilde{\Phi}_0(\lambda^m)$ and thus Gevrey of order $s = \frac{1}{m} = k-1$.

C. Resurgence and the bridge equation

The results concerning the general formal solutions of equation E_k , together with the general properties of resurgence for holonomic functions at an irregular singularity with one-level enable us to state the following:

Proposition 6. *With the notations of the last proposition, the formal series $\widetilde{\Phi}_i(x)$ are resurgent functions for the critical time $z = \frac{1}{x}$; each has a finite number of singularities in the Borel plane, which are regular-singular.*

Moreover, the only alien derivations which may act non trivially on Φ are the ${}^z\Delta_{u_i-u_j}$ where $0 \leq i, j \leq k-1$ and $i \neq j$ and we have:

$${}^z\Delta_{u_i-u_j}(\widetilde{\Phi}_i(x)) = A_{i,j}\widetilde{\Phi}_j(x), \quad A_{i,j} \in \mathbb{C},$$

which can be summarized in the following bridge equation

$${}^z\Delta_{u_i-u_j}\Phi = \sum A_{i,j}u_j \frac{\partial}{\partial u_j}\Phi.$$

Proof. Each $\widetilde{\Phi}_i(x)$ is holonomic and Gevrey-1 by the formal constructions above and we have seen that $\widetilde{\Phi}_i(x)$ has a single critical time $z = \frac{1}{x}$.

Accordingly, the Borel transform of $\widetilde{\Phi}_i(x)$ with respect to the critical time $z = \frac{1}{x}$ is holonomic and as such it is resurgent, with a finite number of singularities which belong to the set of singular points of the differential equation in the Borel plane, and exponential growth of order 1 at ∞ .

The formal integral $\Phi(\lambda)$ is thus resurgent with respect to the single critical time z . It is then licit to apply to it the operators ${}^z\Delta_\omega$ and the same formal argument used for (E_2) works:

for any $\omega \in \mathbb{C}$ and any $i \in \{0, \dots, k-1\}$, $\Delta_\omega(\widetilde{\Phi}_i(x))$ is a solution of (E_k) and this forces $\omega \in \{u_i - u_j; j = 0, \dots, k-1; j \neq i\}$, with ${}^z\Delta_{u_i-u_j}(\widetilde{\Phi}_i(x)) = \text{constant} \cdot \widetilde{\Phi}_j(x)$.

Finally, the resurgence relations entail that all the singularities of the functions $\widehat{\Phi}_i$ are logarithmic and thus of the regular-singular type. □

At this stage, we have access to all the information for $\widetilde{\Phi}_0(x)$ and all the $\widetilde{\Phi}_j(x)$, through the coefficients of the bridge equation: we have a control on the Stokes phenomenon for these functions, which for example can be crucial for questions of *real resummation*^{21,22}.

Let us finally remark that, in the present case, the location of the singularities could have been directly deduced from the Borel transform of the operator H_{k,u_l} .

VII. NON-LINEAR OPERATIONS

Let us recall the following general result, for resurgent functions^{16,17}:

Proposition 7. *Let φ be a resurgent function, with $\tilde{\varphi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ and χ an analytic function at the origin; then $\chi \circ \varphi$ is resurgent. Moreover, for any $\omega \in \mathbb{C}^*$*

$$\Delta_{\omega}(\chi \circ \varphi) = (\partial\chi \circ \varphi)\Delta_{\omega}\varphi, \quad (\text{with } \partial = d/dz).$$

Remark 3. This proposition is a particular case of a much more general result on composition of resurgent functions, possibly involving in particular Puiseux series $\tilde{\varphi}(z)$. We shall only need the present version and refer to the foundational papers by Ecalle for a treatment involving a systematic use of majors, for resurgent functions which are non necessarily integrable.

As $\tilde{Z}_0(\lambda) - 1 \in \lambda\mathbb{C}[[\lambda]]$, we can thus state that the free energy $W(\lambda) = \log \tilde{Z}_0(\lambda)$ is a resurgent function (see Ref. 23 for a recent proof of the resurgent character of the logarithm of a resurgent function, under hypotheses which are satisfied in our rather elementary situation). The function W will have an infinite set of singularities in the Borel plane, that is the additive semigroup generated by the singularities of $\tilde{Z}_0(\lambda)$ in the Borel plane, yet only a finite number of alien derivations will act non-trivially on it: the same ones as for $\tilde{Z}_0(\lambda)$. There is no contradiction here, the singularities of the Borel transform of $\tilde{Z}_0(\lambda)$ are in various sheets of its Riemann surface and they can be reached by *compositions* of alien derivations, which give access to all the sheets of the Riemann surface of $\widehat{W}(\lambda)$.

For $k = 2$, $G(\lambda) := W'(\lambda)$ is solution of a Riccati equation, as it is well known; there are only 2 acting alien derivations $\Delta_{\pm u}$ with $u = -1/16$ and the singularities of the Borel transform of G are on $\frac{1}{16}\mathbb{Z}$: there is an infinite number of singularities but their set is discrete.

For $k > 2$, the situation is not as simple: the function $W(\lambda)$ has an infinity of singularities which might project on a dense subset in the Borel plane (see proposition below). The subtle point, however, is that it does not prevent \widehat{W} from being resurgent: in the first sheet (the star of holomorphy), standard results (see Ref. 18, section 5.13) indeed ensure that it has a finite set of singular directions, with isolated singularities on these directions and exponential growth at infinity on the non-singular ones. The other singularities are in the other sheets and accessible by composition of alien derivations:

$$\Delta_{\omega_r} \dots \Delta_{\omega_1}$$

and ultimately, the highly ramified structure of the Riemann surface \mathcal{S} of \widehat{W} and the behaviour of W when reaching its singularities on \mathcal{S} is totally encoded by the resurgent structure of the partition function which is as we have seen above quite explicit. It is in such a non-linear context that the alien derivations show all their efficiency, to explore the surface \mathcal{S} and describe the singularities on all its sheets, which eventually govern the analytic properties of the function W .

Let us end this section with a more precise statement on the sets of singularities:

Proposition 8. *For $k > 2$ the set of singularities of W is an additive subgroup of \mathbb{C} which is discrete if $k \in \{3, 4, 5, 7\}$ and dense otherwise.*

Proof. Let us recall that the singularities in the Borel plane of $\widetilde{Z}_0(\lambda)$ are the roots u_1, \dots, u_{k-1} of the polynomial :

$$Q(u) = (-1)^{k-1} (2k)^k (mu)^{k-1} + 1.$$

The powers of $\widetilde{Z}_0(\lambda)_1$ in $W(\lambda) = \log \widetilde{Z}_0(\lambda)$ will generate in the Borel plane singularities for W in the set

$$\Omega = u_1\mathbb{N} + \dots + u_{k-1}\mathbb{N}$$

which is a priori an additive semigroup in \mathbb{C} . This set coincides with the additive subgroup L of \mathbb{C} generated by u_1, \dots, u_{k-1} : $\Omega \subset L$ but, since $u_1 + \dots + u_{k-1} = 0$ any element of L can be written as a linear combination of u_1, \dots, u_{k-1} with coefficients in \mathbb{N} .

The roots of Q are invariant by the rotation of angle $\frac{2\pi}{k-1}$ so that, whenever L is discrete, it is a lattice in \mathbb{C} invariant by this rotation. A classical result on lattices²⁴, sometimes called the Crystallographic restriction theorem, ensures that the only possible angles are:

$$\frac{2\pi}{k-1} = \pi, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3},$$

thus $k = 3, 4, 5, 7$. □

VIII. AIRY'S EQUATION AND COEQUATIONAL RESURGENCE.

The stationary Schrödinger equation in one dimension, with a polynomial potential W is:

$$\hbar^2 \psi''(q) - V(q)\psi(q) = 0. \tag{11}$$

Around 1980, A. Voros had discovered that the expansions in \hbar of the solutions of (11) display a resurgent behavior, with a specific pattern and beautiful algebraic properties (“Voros coefficients,

Voros algebra”); this was enhanced by Pham and his school and has lately been the object of numerous articles (e.g. Ref. 25).

However, all these works admittedly^{26–28} presupposed the resurgence of the series – a fact that for which there was convincing numerical evidence. J. Ecalle had given a clear method to establish the resurgent character with respect to the critical time $z = 1/\hbar$ of the series (which he dubbed “coequational resurgence”, to stress on a kind of duality with resurgence properties for the dynamical variable deduced from q), by systematic expansions involving iterated integrals in the Borel plane yet it was never implemented in details for practical cases.

Using the same ideas as in the previous sections, we show below that, for the solutions to Airy’s equation (which corresponds to $V(q) = q$), the dependence in the parameter \hbar is also governed by a linear ODE with polynomial coefficients, the variable q being considered now as a parameter.

Let us consider Airy’s equation

$$\hbar^2 \psi''(q) - q\psi(q) = 0 \quad (12)$$

and let $\lambda = \hbar^2$. The solutions to (12) have integral solutions

$$\psi_\gamma(q, \lambda) = \int_\gamma e^{q\phi - \frac{\lambda}{3}\phi^3} d\phi,$$

where $\gamma: \mathbb{R} \rightarrow \mathbb{C}$ is an infinite path such that $\Re(\lambda \gamma(t)^3) \xrightarrow{|t| \rightarrow \infty} +\infty$. For the same kind of reasons as in theorem 2, such an integral defines an analytic function in some half plane H_γ and we have:

$$\lambda \partial_q^2 \psi_\gamma(q, \lambda) - q\psi_\gamma(q, \lambda) = \int_\gamma (\lambda \phi^2 - q) e^{q\phi - \frac{\lambda}{3}\phi^3} d\phi = \left[-e^{q\phi - \frac{\lambda}{3}\phi^3} \right]_\gamma = 0.$$

Once γ is fixed, we can define analogs of “moments” (we view now q as a parameter that we omit in the notation):

$$\forall j \in \mathbb{N}, \forall \lambda \in H_\gamma, \quad Z_j(\lambda) = \int_\gamma \phi^j e^{q\phi - \frac{\lambda}{3}\phi^3} d\phi$$

and, as in proposition 2, we get

$$\forall j \geq 0, \quad \partial_\lambda Z_j = Z'_j = -\frac{1}{3} Z_{j+3}$$

and

$$\forall j \geq 0, \quad jZ_{j-1} = \lambda Z_{j+2} - qZ_j,$$

the latter equations being obtained by integration by parts, with the convention $jZ_{j-1} = 0$ for $j = 0$ so that $\lambda Z_2 = qZ_0$. Combining the previous equations, with $\theta_\lambda = \lambda \partial_\lambda$, yields:

$$\forall j \geq 0, \quad (3\theta_\lambda + j + 1)Z_j = 3\lambda Z'_j + (j + 1)Z_j = -\lambda Z_{j+3} + (j + 1)Z_j = -qZ_{j+1}$$

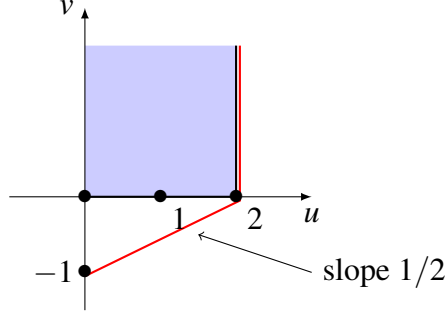


FIG. 6. NP at 0 for eq. (13), provided $q \neq 0$, with $\lambda = \hbar^2$ as variable.

so that we get once again a governing equation for Z_0 :

$$(3\theta_\lambda + 2)(3\theta_\lambda + 1)Z_0 = -q(3\theta_\lambda + 2)Z_1 = q^2Z_2 = \frac{q^3}{\lambda}Z_0. \quad (13)$$

As a function of the parameter $\lambda = \hbar^2$, the solutions of Airy's equation satisfy a differential equation for which the previous machinery works. From the formal point of view, assuming that $q \neq 0$, we get the Newton polygon depicted in Fig. 6. This suggests the change of variable $x = \lambda^{1/2}$ (that is $x = \hbar$) so that the function $f(x) = Z_0(x^2)$ satisfies the equation

$$\left(\frac{3}{2}\theta_x + 2\right) \left(\frac{3}{2}\theta_x + 1\right) f(x) = \frac{q^3}{x^2} f(x)$$

since $\theta_\lambda = \frac{1}{2}\theta_x$. We multiply by 4 and observe that now, the NP of the equation

$$(3\theta_x + 4)(3\theta_x + 2)f = \frac{4q^3}{x^2} f(x) \quad (14)$$

has a unique slope equal to 1. We don't get immediately formal solutions but we perform the change of unknown function $f(x) = e^{u/x}g(x)$ so that

$$\left(\left(3\theta_x + 4 - 3\frac{u}{x}\right) \left(3\theta_x + 2 - 3\frac{u}{x}\right) - \frac{4q^3}{x^2} \right) g(x) = 0$$

that also reads

$$\left(9\theta_x^2 + \left(18 - \frac{18u}{x}\right) \theta_x + \left(8 - \frac{9u}{x} + \frac{9u^2 - 4q^3}{x^2}\right) \right) g(x) = 0. \quad (15)$$

For $u = u_\pm = \pm \frac{2}{3}q^{3/2}$ the NP is given in Fig. 7 and the indicial equation gives $\beta = -1/2$ so that the general formal solution of equation (14) is

$$f(x) = c_+ x^{-1/2} e^{\frac{u_+}{x}} h_+(x) + c_- x^{-1/2} e^{\frac{u_-}{x}} h_-(x) \quad (c_+, c_- \in \mathbb{C}) \quad (16)$$

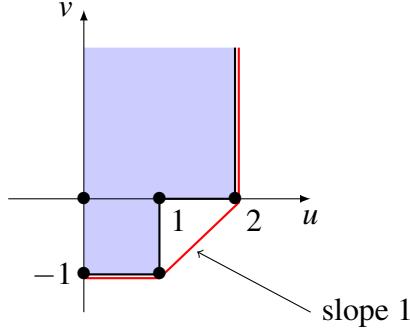


FIG. 7. NP at 0 for eq. (15) for $u = u_{\pm} = \pm \frac{2}{3}q^{3/2}$, notice its horizontal slope followed by its positive slope 1.

where h_+, h_- are formal Gevrey-1 series that are solutions of the equations:

$$\left(9\theta_x^2 + \left(9 - \frac{18u_{\pm}}{x}\right)\theta_x + \frac{5}{4}\right)h_{\pm}(x) = 0.$$

If $D_x = x^2\partial_x$, $\theta_x = x^{-1}D_x$ so that equation also read:

$$\begin{aligned} H_{\pm}.h_{\pm} &= \left(9\theta_x^2 + \left(9 - \frac{18u_{\pm}}{x}\right)\theta_x + \frac{5}{4}\right)h_{\pm} \\ &= \left(\frac{9}{x^2}D_x^2 - \frac{9}{x}D_x + \frac{1}{x}\left(9 - \frac{18u_{\pm}}{x}\right)D_x + \frac{5}{4}\right)h_{\pm} \\ &= \left(\frac{9}{x^2}D_x^2 - \frac{18u_{\pm}}{x^2}D_x + \frac{5}{4}\right)h_{\pm}. \end{aligned}$$

The Borel transform of this equation is

$$\begin{aligned} \widehat{H}_{\pm}\widehat{h}_{\pm} &= 9(\zeta^2\widehat{h}_{\pm})'' - 18u_{\pm}(\zeta\widehat{h}_{\pm})'' + \frac{5}{4}\widehat{h}_{\pm} \\ &= (9\zeta^2 - 18u_{\pm}\zeta)\widehat{h}_{\pm}'' + 36(\zeta - u_{\pm})\widehat{h}_{\pm}' + \frac{77}{4}\widehat{h}_{\pm}. \end{aligned}$$

The general theory of linear differential equations, together with the NP at $\zeta = 0$ ensures that the Borel transform of the series h_{\pm} are analytic with $\zeta = 2u_{\pm}$ as unique singularity. The NP at $\zeta = \infty$ also ensures that, in any direction avoiding u_{\pm} , the Borel transform has a polynomial growth. As in the previous sections, the location of the singularities could also be deduced from the associated bridge equation.

In fact, for this equation, all these properties are very well known; the 2 variables z and x are directly coupled and the resurgence in x is thus also a consequence of the ‘‘equational resurgence’’ in the dynamical variable q , through a rescaling. For other polynomial potentials, the challenge

consists in the working out of relevant x -dependent integrals in order to apply the techniques developed in the present paper.

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REFERENCES

- ¹I. Aniceto, G. Başar, and R. Schiappa, “A Primer on Resurgent Transseries and Their Asymptotics,” *Phys. Rept.* **809**, 1–135 (2019), arXiv:1802.10441 [hep-th].
- ²V. Rivasseau and Z. Wang, “Loop Vertex Expansion for ϕ^{2k} Theory in Zero Dimension,” *J. Math. Phys.* **51**, 092304 (2010), arXiv:1003.1037 [math-ph].
- ³L. Lionni and V. Rivasseau, “Note on the intermediate field representation of ϕ^{2k} theory in zero dimension,” *Math. Phys. Anal. Geom.* **21**, 23 (2018).
- ⁴V. Rivasseau, “Loop Vertex Expansion for Higher Order Interactions,” *Lett. Math. Phys.* **108**, 1147–1162 (2018), arXiv:1702.07602 [math-ph].
- ⁵W. Wasow, *Asymptotic expansions for ordinary differential equations* (Dover Publications, Inc., New York, 1987) pp. x+374, reprint of the 1976 edition.
- ⁶J. Dieudonné, *Treatise on analysis. Vol. II* (Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976) pp. xv+453, enlarged and corrected printing, Translated by I. G. Macdonald, With a loose erratum, Pure and Applied Mathematics, 10-II.
- ⁷C. M. Bender, F. Cooper, and L. M. Simmons, “Nonunique Solution to the Schwinger-Dyson Equations,” *Phys. Rev.* **D39**, 2343–2349 (1989).
- ⁸A. Okopinska, “Solving Schwinger-Dyson equations by truncation in zero-dimensional scalar quantum field theory,” *Phys. Rev.* **D43**, 3561–3564 (1991).
- ⁹E. N. Argyres, A. F. W. van Hameren, R. H. P. Kleiss, and C. G. Papadopoulos, “Zero-dimensional field theory,” *Eur. Phys. J.* **C19**, 567–582 (2001), arXiv:hep-ph/0101346 [hep-ph].
- ¹⁰J.-P. Ramis, “Dévissage Gevrey,” in *Journées Singulières de Dijon (Univ. Dijon, Dijon, 1978)*, Astérisque, Vol. 59 (Soc. Math. France, Paris, 1978) pp. 4, 173–204.

- ¹¹M. Loday-Richaud, *Divergent series, summability and resurgence. II*, Lecture Notes in Mathematics, Vol. 2154 (Springer, [Cham], 2016) pp. xxiii+272, simple and multiple summability, With prefaces by Jean-Pierre Ramis, Éric Delabaere, Claude Mitschi and David Sauzin.
- ¹²M. Loday-Richaud, “Stokes phenomenon, multisummability and differential Galois groups,” *Ann. Inst. Fourier (Grenoble)* **44**, 849–906 (1994).
- ¹³M. van der Put and M. F. Singer, *Galois theory of linear differential equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 328 (Springer-Verlag, Berlin, 2003) pp. xviii+438.
- ¹⁴J. Della Dora, C. Dicrescenzo, and E. Tournier, “An algorithm to obtain formal solutions of a linear homogeneous differential equation at an irregular singular point,” in *Computer algebra (Marseille, 1982)*, Lecture Notes in Comput. Sci., Vol. 144 (Springer, Berlin-New York, 1982) pp. 273–280.
- ¹⁵J. van der Hoeven, “Efficient accelero-summation of holonomic functions,” *J. Symbolic Comput.* **42**, 389–428 (2007).
- ¹⁶J. Écalle, *Les fonctions résurgentes. Tome I*, Publications Mathématiques d’Orsay 81 [Mathematical Publications of Orsay 81], Vol. 5 (Université de Paris-Sud, Département de Mathématique, Orsay, 1981) p. 247, les algèbres de fonctions résurgentes. [The algebras of resurgent functions], With an English foreword.
- ¹⁷J. Écalle, *Les fonctions résurgentes. Tome III*, Publications Mathématiques d’Orsay [Mathematical Publications of Orsay], Vol. 85 (Université de Paris-Sud, Département de Mathématiques, Orsay, 1985) p. 587, l’équation du pont et la classification analytique des objets locaux. [The bridge equation and analytic classification of local objects].
- ¹⁸C. Mitschi and D. Sauzin, *Divergent series, summability and resurgence. I*, Lecture Notes in Mathematics, Vol. 2153 (Springer, [Cham], 2016) pp. xxi+298, monodromy and resurgence, With a foreword by Jean-Pierre Ramis and a preface by Éric Delabaere, Michèle Loday-Richaud, Claude Mitschi and David Sauzin.
- ¹⁹Y. André, “Séries Gevrey de type arithmétique. I. Théorèmes de pureté et de dualité,” *Ann. of Math. (2)* **151**, 705–740 (2000), arXiv:math/0003238 [math.NT].
- ²⁰K. N. Boyadzhiev, “Exponential polynomials, Stirling numbers, and evaluation of some gamma integrals,” *Abstr. Appl. Anal.*, Art. ID 168672, 18 (2009), arXiv:0909.0979 [math.CA].
- ²¹F. Menous, “The well-behaved Catalan and Brownian averages and their applications to real resummation,” in *Proceedings of the Symposium on Planar Vector Fields (Lleida, 1996)*, Vol. 41

- (1997) pp. 209–222.
- ²²J. Ecalle and F. Menous, “Well-behaved convolution averages and the non-accumulation theorem for limit-cycles,” in *The Stokes phenomenon and Hilbert’s 16th problem (Groningen, 1995)* (World Sci. Publ., River Edge, NJ, 1996) pp. 71–101.
- ²³S. Kamimoto and D. Sauzin, “Nonlinear analysis with endlessly continuable functions,” RIMS Kôkyûroku Bessatsu Proceedings of the RIMS conference "Several aspects of microlocal analysis", **B57**, 235–247 (2016), arXiv:1509.01473 [math.DS].
- ²⁴H. Coxeter, *Introduction to Geometry* (Wiley, 1989).
- ²⁵K. Iwaki and T. Nakanishi, “Exact WKB analysis and cluster algebras,” J. Phys. A **47**, 474009, 98 (2014), arXiv:1401.7094 [math.CA].
- ²⁶A. Voros, “Problème spectral de Sturm-Liouville: le cas de l’oscillateur quartique,” in *Bourbaki seminar, Vol. 1982/83*, Astérisque, Vol. 105 (Soc. Math. France, Paris, 1983) pp. 95–104.
- ²⁷A. Voros, “The return of the quartic oscillator: the complex WKB method,” Ann. Inst. H. Poincaré Sect. A (N.S.) **39**, 211–338 (1983).
- ²⁸E. Delabaere, H. Dillinger, and F. Pham, “Résurgence de Voros et périodes des courbes hyper-elliptiques,” Ann. Inst. Fourier (Grenoble) **43**, 163–199 (1993).