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► **To cite this version:**

Jacek Jendrej, Yvan Martel. Construction of multi-bubble solutions for the energy-critical wave equation in dimension 5. *Journal de Mathématiques Pures et Appliquées*, 2020, 139, pp.317-355. 10.1016/j.matpur.2020.02.007 . hal-02370549

**HAL Id: hal-02370549**

**<https://hal.science/hal-02370549v1>**

Submitted on 6 Jun 2022

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# Construction of multi-bubble solutions for the energy-critical wave equation in dimension 5

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## Abstract

We prove the existence of a global solution of the energy-critical focusing wave equation in dimension 5 blowing up in infinite time at any  $K$  given points  $z_k$  of  $\mathbb{R}^5$ , where  $K \geq 2$ . The concentration rate of each bubble is asymptotic to  $c_k t^{-2}$  as  $t \rightarrow \infty$ , where the  $c_k$  are positive constants depending on the distances between the blow-up points  $z_k$ . This result complements previous constructions of blow-up solutions and multi-solitons of the energy-critical wave equation in various dimensions  $N \geq 3$ .

## Résumé

Pour l'équation des ondes focalisante énergie critique en dimension 5, nous montrons l'existence d'une solution globale qui explose en temps infini en  $K$  points  $z_k$  donnés arbitrairement dans  $\mathbb{R}^5$ , où  $K \geq 2$ . Le taux de concentration de chaque bulle se comporte asymptotiquement comme  $c_k t^{-2}$  quand  $t \rightarrow \infty$ , où  $c_k$  sont des constantes strictement positives dépendant des distances entre les points d'explosion  $z_k$ . Ce résultat complète les constructions précédentes de solutions explosives ou de type multi-solitons pour l'équation des ondes focalisante énergie critique en toute dimension  $N \geq 3$ .

*Keywords:* ground state, multi-bubble, wave equation, energy-critical  
*2010 MSC:* 35L70, 35B44, 35B40

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## 1. Introduction

### 1.1. Main result

We consider the energy-critical focusing wave equation in dimension 5

$$\partial_t^2 u(t, x) = \Delta u(t, x) + f(u(t, x)), \quad t \in \mathbb{R}, x \in \mathbb{R}^5, \quad (1.1)$$

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<sup>1</sup>Partially supported by ANR-18-CE40-0028 project ESSED.

where  $f(u) := |u|^{\frac{4}{3}}u$ . Let  $F(u) := \frac{3}{10}|u|^{\frac{10}{3}}$ . The energy functional related to this equation

$$E(u, \partial_t u) := \int_{\mathbb{R}^5} \left( \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 - F(u) \right) dx$$

is well-defined for  $(u, \partial_t u) \in \dot{H}^1(\mathbb{R}^5) \times L^2(\mathbb{R}^5)$  by the Sobolev inequality

$$\|u\|_{L^{\frac{10}{3}}} \leq C \|\nabla u\|_{L^2}. \quad (1.2)$$

We equip the space of pairs of functions  $\vec{v} = (v, \dot{v})$  with the symplectic form

$$\omega(\vec{v}, \vec{w}) := \langle \dot{v}, w \rangle - \langle v, \dot{w} \rangle = \langle J\vec{v}, \vec{w} \rangle, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then (1.1) is the Hamiltonian system corresponding to the Hamiltonian function  $E$ . In other words for a solution  $u$  of (1.1),  $\vec{u} = (u, \partial_t u)$  satisfies

$$\omega(\vec{v}, \partial_t \vec{u}) = \langle DE(\vec{u}), \vec{v} \rangle, \quad \text{for all } \vec{v}. \quad (1.3)$$

We recall that this equation is locally well-posed in the energy space  $\dot{H}^1(\mathbb{R}^5) \times L^2(\mathbb{R}^5)$ , see [16, 26, 43, 44] and references therein. For such solutions, the energy  $E(u, \partial_t u)$  is constant in time.

Recall that the function

$$W(x) := \left( 1 + \frac{|x|^2}{15} \right)^{-\frac{3}{2}}, \quad x \in \mathbb{R}^5,$$

is the ground state solution of the elliptic equation

$$\Delta W = W^{\frac{7}{3}} \quad \text{on } \mathbb{R}^5. \quad (1.4)$$

Up to scaling and translation invariance,  $W$  is the unique positive solution of (1.4). In particular,  $\vec{u}(t, x) = (W(x), 0)$  is a stationary solution of (1.1) and other explicit solutions of (1.1) are deduced by the sign, scaling, translation and Lorentz invariances of the equation:

$$\vec{u}(t, x) = \pm (W_{\ell, \lambda}(x - \ell t - x_0), -(\ell \cdot \nabla)W_{\ell, \lambda}(x - \ell t - x_0)),$$

where for  $\lambda > 0$ ,  $x_0 \in \mathbb{R}^5$  and  $\ell \in \mathbb{R}^5$  with  $|\ell| < 1$ ,

$$W_{\lambda, \ell}(x) = W_{\lambda} \left( x + \sigma \frac{\ell(\ell \cdot x)}{|\ell|^2} \right), \quad \sigma = \frac{1}{\sqrt{1 - |\ell|^2}} - 1, \quad W_{\lambda}(x) = \lambda^{-\frac{3}{2}} W(\lambda^{-1}x).$$

It is well-known that the ground state  $W$  achieves the optimal constant in the critical Sobolev inequality (1.2), see [1, 45]. It is also characterized as the threshold element for global existence and scattering (asymptotic linear behavior) of solutions of (1.1), see [26]. Above this threshold, the study of the large time asymptotic behavior of solutions of (1.1) raises many questions like the following ones.

- (i) The classification of all possible long time behaviors of the solutions.
- (ii) The existence and properties of finite or infinite time bubbling solutions.
- (iii) The effect of the nonlinear interactions on the soliton dynamics.

Question (i) is strongly related to the soliton resolution conjecture, which predicts that any global bounded solution decomposes asymptotically as  $t \rightarrow \infty$  into a sum of a finite number  $K$  of decoupled energy bubbles plus a solution of the linear wave equation. Such a decomposition result is proved in [12] for radially symmetric solutions of the 3D energy-critical wave equation. In [12], a suitable variant of the decomposition result is also proved for finite time blow-up solutions of type-II, *i.e.* non ODE type. In the non radial case, a similar decomposition result (possibly involving excited states, *i.e.* solutions of (1.4) other than the ground state) is proved along a subsequence of time for dimensions 3, 4, 5 in [13, 10] and extended to any odd dimensions in [41]. These general results, valid for any initial data, do not specify the number of solitons nor the exact asymptotic behavior of the geometric parameters of each soliton, except a basic decoupling property of the various bubbles and the dispersive part.

In a different direction related to question (i), solutions with initial data close to  $(W, 0)$  have been studied by Krieger and Schlag [28], and Beceanu [2], who constructed, in various topologies, manifolds of initial data such that the solution converges to the soliton family in a suitable sense and provided a precise description of these solutions. Using a probabilistic approach, Kenig and Mendelson [25] have addressed similar questions in very weak topologies. Duyckaerts and Merle [14], and Krieger, Nakanishi and Schlag [27] classified the long time behavior of all the solutions with initial data close to  $(W, 0)$  in the energy space.

Concerning question (ii), several constructions of bubbling solutions with various explicit type-II blow-up rates are available: see [9, 29, 31] in dimension 3, [19] in dimension 4 and [20] in dimension 5. In complement to the above mentioned general decomposition results, it is also relevant to study the existence and properties of global solutions whose asymptotic behavior involves several decoupled solitons. For the energy-critical wave equation in dimension larger than 6, a global radial solution decomposing asymptotically as a concentrating bubble on the top of a standing soliton of same sign is constructed in [23]. Note that this behavior corresponds to a specific choice of sign and blow-up rate; see a nonexistence result in [21] and a classification result in a similar framework in [24]. In [34], a solution of (1.1) containing an arbitrary number  $K$  of bounded traveling solitons is constructed under some restrictions on the speeds  $\ell_k$  of the solitons. We also refer to [35] proving inelasticity of soliton interactions in the same context. Such works clearly relate questions (ii) and (iii) since the nonlinear interactions between the two solitons are responsible either for the blow up behavior or for the inelasticity property.

We state the main result of this paper.

**Theorem 1.** *Let  $K \geq 2$  and  $z_1, \dots, z_K$  be any  $K$  points of  $\mathbb{R}^5$  distinct two by two. There exist positive constants  $c_1, \dots, c_K$  and a solution  $(u, \partial_t u) : [0, \infty) \rightarrow$*

$\dot{H}^1(\mathbb{R}^5) \times L^2(\mathbb{R}^5)$  of (1.1) such that for all  $t > 0$ ,

$$\left\| u(t) - \sum_{k=1}^K \frac{1}{(c_k t^{-2})^{\frac{3}{2}}} W\left(\frac{\cdot - z_k}{c_k t^{-2}}\right) \right\|_{\dot{H}^1(\mathbb{R}^5)} + \|\partial_t u(t)\|_{L^2(\mathbb{R}^5)} \lesssim t^{-\frac{1}{3}}.$$

This result complements the above mentioned articles, providing an example of non radial infinite time multiple bubbling in dimension 5, in a context where radial multiple bubbling does not seem possible. Observe that the solutions constructed in Theorem 1 only contain bubbles, without any linear remainder, like in [23, 34]. Though we do not address uniqueness nor classification questions in this article, we conjecture that  $t^{-2}$  is the only possible infinite time blow-up rate for such distant blowing up multiple bubbles. Theorem 1 holds for any set of concentration points  $\{z_k\}$ , but the constants  $\{c_k\}$  then strongly depend on this choice. Indeed, in our proof, the determination of suitable constants  $\{c_k\}$  is related to the global minimum of some function depending on the distances between the solitons (see Lemma 3). Our method of proof should extend to higher space dimensions, however we do not address here the existence of suitable constants  $\{c_k\}$  for  $N \geq 6$ . We refer to Remark 4 for more comments on  $\{c_k\}$ .

Historically, for nonlinear dispersive equations, the construction of solutions blowing up in finite time at  $K$  given points using minimal bubbles was initiated in the case of the mass-critical nonlinear Schrödinger equation in [37]; see also [36] for multiple bubble infinite time blow-up. We refer to [3, 33] for recent analogous results for the mass-critical generalized Korteweg-de Vries equation.

Bubbling phenomena were also considered for other energy-critical dispersive or wave models, like the wave maps [23, 24, 30, 38] and the energy-critical nonlinear Schrödinger equation in [22]. In the parabolic setting, for the energy-critical heat equation in dimension 5, we mention some type-II finite time blow-up results [6, 8, 17, 42], and infinite time blow-up results [4, 7, 18]. See Remark 4 for a qualitative comparison between results in [4] and Theorem 1.

## 1.2. Notation

In this paper,  $\mathbb{S}^J$  denotes the unit sphere of  $\mathbb{R}^{J+1}$  and  $\bar{\mathcal{B}}_{\mathbb{R}^J}$  denotes the unit closed ball of  $\mathbb{R}^J$ . We denote by  $\mathcal{B}(z, r)$  the ball of  $\mathbb{R}^5$  of center  $z$  and radius  $r \geq 0$ .

The bracket  $\langle \cdot, \cdot \rangle$  denotes the distributional pairing and the scalar product in  $L^2$  and  $L^2 \times L^2$ .

We define a smooth radial cut-off function  $\chi$  satisfying  $\chi(x) = 0$  for  $|x| \geq \frac{2}{3}$  and  $\chi(x) = 1$  for  $|x| \leq \frac{1}{2}$  and  $0 \leq \chi(x) \leq 1$  for  $\frac{1}{2} \leq |x| \leq \frac{2}{3}$ .

For a function  $v : \mathbb{R}^5 \rightarrow \mathbb{R}$  and  $\lambda > 0$ , set

$$v_{\underline{\lambda}}(x) := \frac{1}{\lambda^{\frac{5}{2}}} v\left(\frac{x}{\lambda}\right), \quad v_{\lambda}(x) := \frac{1}{\lambda^{\frac{3}{2}}} v\left(\frac{x}{\lambda}\right).$$

Define

$$\underline{\Lambda} = \frac{5}{2} + x \cdot \nabla, \quad \Lambda = \frac{3}{2} + x \cdot \nabla.$$

For  $\vec{g} = (g, \dot{g})$ , we denote  $\|\vec{g}\|_\varepsilon = \|\vec{g}\|_{\dot{H}^1 \times L^2}$ . Let

$$X := (\dot{H}^1 \cap \dot{H}^2) \times (L^2 \cap \dot{H}^1).$$

### 1.3. Finite dimensional dynamics

Let  $z_1, \dots, z_K$  be  $K$  points of  $\mathbb{R}^5$  distinct two by two. In this formal discussion, we neglect possible translations of the bubbles and concentrate on the focusing behavior (this reduction will be justified by the control of translation parameters in the proof of Theorem 1).

For  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_K) \in (0, \infty)^K$  and  $\mathbf{b} = (b_1, \dots, b_K) \in \mathbb{R}^K$ , define

$$\vec{W}(\boldsymbol{\lambda}, \mathbf{b}) := \sum_k \left( W_{\lambda_k}(\cdot - z_k), b_k \lambda_k^{-1} \Lambda W_{\lambda_k}(\cdot - z_k) \right). \quad (1.5)$$

Here, and in what follows, unless otherwise indicated, sums  $\sum_k$  are for indices  $k \in \{1, \dots, K\}$ .

**Remark 2.** Note that  $(W, b\Lambda W)$  is the first-order asymptotic expansion of the self-similar blow-up profile  $\vec{W}_b$  for small  $b$ .

We take a small number  $\epsilon > 0$  and consider the manifold

$$\mathcal{M} := \{ \vec{W}(\boldsymbol{\lambda}, \mathbf{b}) : |\boldsymbol{\lambda}| + |\mathbf{b}| < \epsilon \}.$$

On this manifold,  $(\boldsymbol{\lambda}, \mathbf{b})$  is a natural system of coordinates. The associated basis of the tangent space is given by

$$\partial_{\lambda_k} = -(\lambda_k^{-1} \Lambda W_{\lambda_k}(\cdot - z_k), b_k \lambda_k^{-2} \underline{\Lambda} \Lambda W_{\lambda_k}(\cdot - z_k)), \quad \partial_{b_k} = (0, \lambda_k^{-1} \Lambda W_{\lambda_k}(\cdot - z_k)).$$

We wish to compute the restriction of the flow to  $\mathcal{M}$ . The Hamiltonian function is

$$E(\boldsymbol{\lambda}, \mathbf{b}) := E(\vec{W}(\boldsymbol{\lambda}, \mathbf{b})).$$

Let

$$\begin{pmatrix} \mathbf{M}(\boldsymbol{\lambda}, \mathbf{b}) & \mathbf{G}(\boldsymbol{\lambda}, \mathbf{b}) \\ -\mathbf{G}(\boldsymbol{\lambda}, \mathbf{b}) & \mathbf{N}(\boldsymbol{\lambda}, \mathbf{b}) \end{pmatrix} = \begin{pmatrix} (M_{jk})_{j,k=1}^K & (G_{jk})_{j,k=1}^K \\ (-G_{jk})_{j,k=1}^K & (N_{jk})_{j,k=1}^K \end{pmatrix}$$

be the matrix of the symplectic form  $\omega$  in this basis, in other words for  $j, k \in \{1, \dots, K\}$ ,

$$\begin{aligned} M_{jk} &= \omega(\partial_{\lambda_j}, \partial_{\lambda_k}) = \lambda_j^{-1} \lambda_k^{-1} (b_j \lambda_j^{-1} \langle \underline{\Lambda} \Lambda W_{\lambda_j}(\cdot - z_j), \Lambda W_{\lambda_k}(\cdot - z_k) \rangle \\ &\quad - b_k \lambda_k^{-1} \langle \Lambda W_{\lambda_j}(\cdot - z_j), \underline{\Lambda} \Lambda W_{\lambda_k}(\cdot - z_k) \rangle), \\ G_{j,k} &= \omega(\partial_{\lambda_j}, \partial_{b_k}) = \lambda_j^{-1} \lambda_k^{-1} \langle \Lambda W_{\lambda_j}(\cdot - z_j), \Lambda W_{\lambda_k}(\cdot - z_k) \rangle, \\ N_{j,k} &= \omega(\partial_{b_j}, \partial_{b_k}) = 0. \end{aligned}$$

The motion with constraints is given by the equation

$$\begin{pmatrix} \boldsymbol{\lambda}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} \phi(\boldsymbol{\lambda}, \mathbf{b}) \\ \psi(\boldsymbol{\lambda}, \mathbf{b}) \end{pmatrix} := \begin{pmatrix} \mathbf{M}(\boldsymbol{\lambda}, \mathbf{b}) & \mathbf{G}(\boldsymbol{\lambda}, \mathbf{b}) \\ -\mathbf{G}(\boldsymbol{\lambda}, \mathbf{b}) & \mathbf{N}(\boldsymbol{\lambda}, \mathbf{b}) \end{pmatrix}^{-1} \begin{pmatrix} \partial_{\boldsymbol{\lambda}} E(\boldsymbol{\lambda}, \mathbf{b}) \\ \partial_{\mathbf{b}} E(\boldsymbol{\lambda}, \mathbf{b}) \end{pmatrix}. \quad (1.6)$$

In a suitable regime for  $(\boldsymbol{\lambda}, \mathbf{b})$ , we claim

$$\partial_{b_k} E(\boldsymbol{\lambda}, \mathbf{b}) \simeq \|\Delta W\|_{L^2}^2 b_k \quad (1.7)$$

$$\partial_{\lambda_k} E(\boldsymbol{\lambda}, \mathbf{b}) \simeq -\|\Delta W\|_{L^2}^2 B_k(\boldsymbol{\lambda}) \quad (1.8)$$

where

$$B_k(\boldsymbol{\lambda}) = -\kappa \lambda_k^{\frac{1}{2}} \sum_{j \neq k} \left\{ \lambda_j^{\frac{3}{2}} |z_j - z_k|^{-3} \right\}, \quad \kappa = -\frac{7}{3} 15^{\frac{3}{2}} \frac{\langle \Delta W, W^{\frac{4}{3}} \rangle}{\|\Delta W\|_{L^2}^2} = \frac{128\sqrt{15}}{7\pi}. \quad (1.9)$$

We briefly justify (1.7)-(1.8). Using the equation  $\Delta W + f(W) = 0$ , we have

$$\begin{aligned} DE(\vec{W}(\boldsymbol{\lambda}, \mathbf{b})) &= \left( -f \left( \sum_k W_{\lambda_k}(\cdot - z_k) \right) + \sum_k f(W_{\lambda_k}(\cdot - z_k)) \right), \\ &\quad \sum_k b_k \lambda_k^{-1} \Delta W_{\lambda_k}(\cdot - z_k). \end{aligned}$$

We consider cases where  $\{\lambda_k\}$ , respectively  $\{b_k\}$ , are asymptotically of the size  $\lambda(t) > 0$ , respectively  $b(t)$ , up to fixed multiplicative constants, where  $\lambda(t) \rightarrow 0$  and  $(b/\lambda)(t) \rightarrow 0$  as  $t \rightarrow \infty$ . The first condition means concentration (or ‘‘grow up’’) of the solitons while the second condition is natural when searching polynomial regimes for  $\lambda$ , since  $b$  is related to the time derivative of  $\lambda$ . In such regime, we can easily bound cross terms. In particular, from computations similar to that of Lemma 14 below, we see that

$$\partial_{b_k} E(\boldsymbol{\lambda}, \mathbf{b}) = \|\Delta W\|_{L^2}^2 b_k + O(b\lambda),$$

which justifies (1.7).

To justify (1.8), we consider again the above expression of  $DE(\vec{W}(\boldsymbol{\lambda}, \mathbf{b}))$ . The inner product of the first components yields some constants times  $\lambda^2$ ; the second components yield a constant times  $b^2$ . Since we focus on the case  $b/\lambda \ll 1$ , this second contribution will be negligible with respect to the first. We thus focus on the first components. We expect the main contribution to come from

$$\sum_{j \neq k} \langle f'(W_{\lambda_k}(\cdot - z_k)) W_{\lambda_j}(\cdot - z_j), \lambda_k^{-1} \Delta W_{\lambda_k}(\cdot - z_k) \rangle.$$

Because of the asymptotics  $W(x) \simeq 15^{\frac{3}{2}} |x|^{-3}$  as  $|x| \rightarrow \infty$ , the factor  $W_{\lambda_j}(\cdot - z_j)$  can be replaced by the following expression independent of  $x$

$$W_{\lambda_j}(z_k - z_j) = \lambda_j^{-\frac{3}{2}} W(\lambda_j^{-1}(z_k - z_j)) \simeq 15^{\frac{3}{2}} \lambda_j^{\frac{3}{2}} |z_k - z_j|^{-3}.$$

Next, we have

$$\langle f'(W_{\lambda_k}(\cdot - z_k)), \lambda_k^{-1} \Delta W_{\lambda_k}(\cdot - z_k) \rangle = \lambda_k^{\frac{1}{2}} \langle f'(W), \Delta W \rangle = \frac{7}{3} \lambda_k^{\frac{1}{2}} \langle W^{\frac{4}{3}}, \Delta W \rangle,$$

so we obtain (1.8).

From (1.7)-(1.8), we compute the main order terms of  $\phi$  and  $\psi$ . Again, estimates of cross terms as in the proof of Lemma 14, yield

$$G(\boldsymbol{\lambda}, \mathbf{b}) = \|\Lambda W\|_{L^2}^2 \text{Id} + O(\lambda).$$

Thus, using also  $N(\boldsymbol{\lambda}, \mathbf{b}) = 0$  and the fact that  $M(\boldsymbol{\lambda}, \mathbf{b})$  is of size  $b/\lambda$ , we obtain

$$\begin{pmatrix} M(\boldsymbol{\lambda}, \mathbf{b}) & G(\boldsymbol{\lambda}, \mathbf{b}) \\ -G(\boldsymbol{\lambda}, \mathbf{b}) & N(\boldsymbol{\lambda}, \mathbf{b}) \end{pmatrix}^{-1} = \|\Lambda W\|_{L^2}^{-2} \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix} + O(\lambda) + O(b/\lambda).$$

Inserted in (1.6), these computations justify the introduction of the following formal system for the parameters  $(\boldsymbol{\lambda}, \mathbf{b})$ :

$$\begin{cases} \lambda'_k(t) = -b_k(t) \\ b'_k(t) = B_k(\boldsymbol{\lambda}(t)). \end{cases} \quad (1.10)$$

By analogy with the differential equation  $\lambda'' = \lambda^2$ , which admits the solution  $\lambda(t) = 6t^{-2}$ , we look for a solution of (1.10) of the form

$$\lambda_k(t) = c_k t^{-2}, \quad b_k(t) = 2c_k t^{-3}, \quad (1.11)$$

for positive constants  $c_k$ . We need to check that the system (1.10) is actually satisfied for some choice of constants  $\{c_k\}$ . The first equation is automatically satisfied by the above expression of  $(\lambda_k, b_k)$  and the second one is equivalent to

$$\mathbf{B}(\mathbf{c}) = -6\mathbf{c}$$

where we denote

$$\mathbf{c} = (c_1, \dots, c_K) \quad \text{and} \quad \mathbf{B} = (B_1, \dots, B_K).$$

We remark that this condition is related to the existence of a critical point for the following function  $V$ :

$$V : \boldsymbol{\theta} = (\theta_1, \dots, \theta_K) \in \mathbb{S}_+^{K-1} \mapsto V(\boldsymbol{\theta}) = -\frac{2}{3}\kappa \sum_k \sum_{j < k} \left\{ \theta_j^{\frac{3}{2}} \theta_k^{\frac{3}{2}} |z_j - z_k|^{-3} \right\},$$

where the notation  $\mathbb{S}_+^{K-1}$  means

$$\mathbb{S}_+^{K-1} = \left\{ \boldsymbol{\theta} = (\theta_1, \dots, \theta_K) \in [0, \infty)^K : \sum_{k=1}^K \theta_k^2 = 1 \right\}.$$

For later purposes (see Remark 4 below), we select a global minimum of the function  $V$ .

**Lemma 3.** *The following holds*

(i) *For any  $r \geq 0$  and  $\boldsymbol{\theta} \in \mathbb{S}_+^{K-1}$ ,*

$$\mathbf{B}(r\boldsymbol{\theta}) = r^2 \nabla V(\boldsymbol{\theta}).$$



- (ii) The function  $V$  has a global minimum on  $\mathbb{S}_+^{K-1}$ , reached at least at a point  $\underline{\theta} \in \mathbb{S}_+^{K-1}$  such that for all  $k = 1, \dots, K$ ,  $\theta_k \in (0, 1)$ . Moreover,

$$\underline{n} = -\underline{\theta} \cdot \nabla V(\underline{\theta}) > 0 \quad \text{satisfies} \quad -\nabla V(\underline{\theta}) = \underline{n} \underline{\theta}.$$

- (iii) For  $\underline{\theta} \in \mathbb{S}_+^{K-1}$  and  $\underline{n} > 0$  as in (ii), define

$$\mathbf{c} = \underline{r} \underline{\theta} \quad \text{where} \quad \underline{r} = \frac{6}{-\underline{\theta} \cdot \nabla V(\underline{\theta})} = \frac{6}{\underline{n}}. \quad (1.12)$$

Then, it holds  $\mathbf{B}(\mathbf{c}) = -6\mathbf{c}$ .

PROOF. (i) follows directly from the definitions of  $V$  and  $\mathbf{B}$ .

Proof of (ii). As a nonconstant nonpositive continuous function defined on the compact set  $\mathbb{S}_+^{K-1}$ , the function  $V$  has a negative global minimum. Let  $\underline{\theta} = (\theta_1, \dots, \theta_K) \in \mathbb{S}_+^{K-1}$  be such that  $\theta_k = 0$ , for some  $k = 1, \dots, K$ . For any  $a \in [0, 1)$ , set

$$\boldsymbol{\theta}(a) = ((1 - a^{\frac{4}{3}})^{\frac{1}{2}}\theta_1, \dots, a^{\frac{2}{3}}, \dots, (1 - a^{\frac{4}{3}})^{\frac{1}{2}}\theta_K), \quad v(a) = V(\boldsymbol{\theta}(a)),$$

where the  $a^{\frac{2}{3}}$  above is located at the  $k$ th row of the line vector  $\boldsymbol{\theta}(a)$ . Observe that  $\boldsymbol{\theta}(a) \in \mathbb{S}_+^{K-1}$  and

$$v(a) = -\frac{2}{3}\kappa \left[ (1 - a^{\frac{4}{3}})^{\frac{3}{2}} \sum_{\substack{j,l \neq k \\ j < l}} \left\{ \theta_j^{\frac{3}{2}} \theta_l^{\frac{3}{2}} |z_j - z_l|^{-3} \right\} + a(1 - a^{\frac{4}{3}})^{\frac{3}{4}} \sum_{j \neq k} \left\{ \theta_j^{\frac{3}{2}} |z_j - z_k|^{-3} \right\} \right].$$

A simple computation shows that  $v'(0) < 0$ , which proves that the global minimum of the function  $V$  on  $\mathbb{S}_+^{K-1}$  is not reached at such  $\underline{\theta}$ .

Consider  $\underline{\theta} \in \mathbb{S}_+^{K-1}$  any point of global minimum for  $V$ . It follows that there exists  $\underline{n} \in \mathbb{R}$  such that  $-\nabla V(\underline{\theta}) = \underline{n} \underline{\theta}$ . In particular, taking the scalar product by  $\underline{\theta}$ , we find  $-\underline{\theta} \cdot \nabla V(\underline{\theta}) = \underline{n}$ , and by (i) and the expression of  $\mathbf{B}$ , it holds  $\underline{n} > 0$ .

Proof of (iii). Let  $\mathbf{c} = \underline{r} \underline{\theta}$  where  $\underline{r}$  is defined as in (1.12). By (i), we have  $\mathbf{B}(\mathbf{c}) = \underline{r}^2 \nabla V(\underline{\theta})$ . Using also (ii), we obtain  $\mathbf{B}(\mathbf{c}) = -6\mathbf{c}$ .

**Remark 4.** The proof of Theorem 1 requires the fact that  $\mathbf{c}$  is related to a point of local minimum of  $V$  in the interior of  $\mathbb{S}_+^{K-1}$ . See Section 3.4. The same question in dimension  $N \geq 5$  involves the function

$$V(\boldsymbol{\theta}) = -C \sum_k \sum_{j < k} \left\{ \theta_j^{\frac{N-2}{2}} \theta_k^{\frac{N-2}{2}} |z_j - z_k|^{2-N} \right\},$$

where  $C > 0$ . In the proof of (ii) of Lemma 3, the dimension  $N = 6$  seems critical in some sense and the fact the global minimum of  $V$  is reached only at the interior of  $\mathbb{S}_+^{K-1}$  cannot be proved in the same way for  $N \geq 6$ . We do not pursue this issue here.

Though some configurations with changing signs seem possible, the proof also uses the fact that the bubbles all have the same sign. Indeed, only nonlinear interactions of bubbles of same sign have a focusing effect. See for instance the nonexistence result in [21].

It is interesting to compare the situation to that of the energy critical nonlinear heat equation considered in [4]. For the latter equation, the bubbling phenomenon involves the same function  $W$ . However, soliton-soliton interactions have opposite effects. In [4], the Dirichlet boundary condition has a focusing effect on the various positive bubbles, and the assumption on the locations of the concentration points ensures that the defocusing effect of the soliton-soliton interactions is lower than the focusing effect of the boundary condition. This is why the system obtained there (formula (2.19) in [4]) is different; in particular, dimension 4 seems critical and all dimensions higher than 5 can be treated in a unified way.

The strategy of the proof of Theorem 1 is to construct a solution of (1.1) converging as  $t \rightarrow \infty$  to the ansatz (1.5) with parameters  $(\lambda, \mathbf{b})$  as in (1.11) and  $\mathbf{c}$  given by Lemma 3.

In the next section, we recall coercivity results useful to apply the energy method in a neighborhood of the sum of decoupled solitons. In Section 3, we prove Theorem 1.

## 2. Coercivity results

### 2.1. Single potential

Linearizing the system (1.3) around  $\vec{W} = (W, 0)$ , one obtains

$$\partial_t \vec{g} = J \circ D^2 E(\vec{W}) \vec{g} = \begin{pmatrix} 0 & \text{Id} \\ -L & 0 \end{pmatrix} \vec{g},$$

where  $L$  is the following operator

$$Lg := -\Delta g - f'(W)g = -\Delta g - \frac{7}{3}W^{\frac{4}{3}}g.$$

For  $g \in \dot{H}^1(\mathbb{R}^5)$  we have the associated quadratic form

$$\langle g, Lg \rangle := \int_{\mathbb{R}^5} (|\nabla g|^2 - f'(W)g^2) dx.$$

**Lemma 5 ([40, Appendix D]).** *If  $0 \neq g \in \dot{H}^1(\mathbb{R}^5)$  satisfies the relations  $\langle \Delta W, g \rangle = \langle \Delta \Lambda W, g \rangle = \langle \Delta \nabla W, g \rangle = 0$  then  $\langle g, Lg \rangle > 0$ .*

Since  $\langle W, LW \rangle = -\frac{4}{3} \int_{\mathbb{R}^5} W^{7/3} dx < 0$ , the operator  $L$  has at least one negative eigenvalue. Denote the smallest eigenvalue  $-\nu^2$  ( $\nu > 0$ ) and the corresponding eigenfunction  $Y$ , normalized so that  $\|Y\|_{L^2} = 1$  and  $Y(x) > 0$  for all  $x \in \mathbb{R}^5$ .

The facts that  $Y(x) \neq 0$  for all  $x \in \mathbb{R}^5$  and that  $Y$  has exponential decay follow from the general theory of Schrödinger operators.

Denote

$$\tilde{L} := L + \nu^2 \langle Y, \cdot \rangle Y, \quad \langle g, \tilde{L}g \rangle := \int_{\mathbb{R}^5} (|\nabla g|^2 - f'(W)g^2) dx + \nu^2 \langle Y, g \rangle^2.$$

**Lemma 6.** *For all  $g \in \dot{H}^1(\mathbb{R}^5)$ , it holds  $\langle g, \tilde{L}g \rangle \geq 0$ . Moreover,  $\langle g, \tilde{L}g \rangle = 0$  if and only if  $g \in \text{span}(Y, \Lambda W, \partial_{x_1} W, \dots, \partial_{x_5} W)$ .*

PROOF. Let  $g \in \dot{H}^1(\mathbb{R}^5)$  and decompose  $g = g_1 + g_2$  so that

$$\begin{aligned} g_1 &\in \text{span}(Y, \Lambda W, \partial_{x_1} W, \dots, \partial_{x_5} W), \\ \langle \Delta W, g_2 \rangle &= \langle \Delta \Lambda W, g_2 \rangle = \langle \Delta \nabla W, g_2 \rangle = 0. \end{aligned}$$

In order to guarantee that such a decomposition exists, we need to check that the  $7 \times 7$  matrix

$$\begin{pmatrix} \langle \Delta W, Y \rangle & \langle \Delta W, \Lambda W \rangle & (\langle \Delta W, \partial_{x_j} W \rangle)_{j=1, \dots, 5} \\ \langle \Delta \Lambda W, Y \rangle & \langle \Delta \Lambda W, \Lambda W \rangle & (\langle \Delta \Lambda W, \partial_{x_j} W \rangle)_{j=1, \dots, 5} \\ (\langle \Delta \partial_{x_j} W, Y \rangle)_{j=1, \dots, 5} & (\langle \Delta \partial_{x_j} W, \Lambda W \rangle)_{j=1, \dots, 5} & (\langle \Delta \partial_{x_j} W, \partial_{x_k} W \rangle)_{j,k=1, \dots, 5} \end{pmatrix}$$

is non-singular. The upper left term is non-zero because  $\Delta W = -f(W) < 0$  and  $Y > 0$ . We also have  $\langle \Delta W, \Lambda W \rangle = 0$  and, using symmetry considerations, we obtain that the matrix is lower-triangular with non-zero entries on the diagonal.

Since  $\tilde{L}g_1 = 0$ , using Lemma 5 we obtain

$$\langle g, \tilde{L}g \rangle = \langle g_2, \tilde{L}g_2 \rangle \geq 0,$$

with equality if and only if  $g_2 = 0$ .

**Remark 7.** It follows that  $-\nu^2$  is the only negative eigenvalue of  $L$ .

**Lemma 8.** *There exists  $\eta > 0$  such that, for any  $g \in \dot{H}^1(\mathbb{R}^5)$ ,*

$$\int_{\mathbb{R}^5} (|\nabla g|^2 - f'(W)g^2) dx \geq \eta \|\nabla g\|_{L^2}^2 - ((\nu^2 + 1)\langle Y, g \rangle^2 + \langle \Delta \Lambda W, g \rangle^2 + |\langle \nabla W, g \rangle|^2).$$

PROOF. If this is false, then there exists a sequence  $g_n \in \dot{H}^1$  such that for  $n = 1, 2, \dots$

$$\int_{\mathbb{R}^5} f'(W)g_n^2 dx = 1$$

and

$$\begin{aligned} &\int_{\mathbb{R}^5} (|\nabla g_n|^2 - f'(W)g_n^2) dx \\ &\leq \frac{1}{n} \|\nabla g_n\|_{L^2}^2 - ((\nu^2 + 1)\langle Y, g_n \rangle^2 + \langle \Delta \Lambda W, g_n \rangle^2 + |\langle \nabla W, g_n \rangle|^2). \end{aligned}$$

These inequalities imply in particular that the sequence  $(g_n)$  is bounded in  $\dot{H}^1(\mathbb{R}^5)$ . Upon extracting a subsequence, we can assume  $g_n \rightharpoonup g$  in  $\dot{H}^1(\mathbb{R}^5)$ . By the Rellich theorem, we have  $\int_{\mathbb{R}^5} f'(W)g^2 dx = 1$  and thus  $g \neq 0$ . Moreover, it holds

$$\lim_{n \rightarrow \infty} \langle Y, g_n \rangle = \langle Y, g \rangle, \quad \lim_{n \rightarrow \infty} \langle \Delta \Lambda W, g_n \rangle = \langle \Delta \Lambda W, g \rangle, \quad \lim_{n \rightarrow \infty} \langle \nabla W, g_n \rangle = \langle \nabla W, g \rangle.$$

Hence, by the Fatou property,  $g$  satisfies

$$\langle g, \tilde{L}g \rangle + \langle Y, g \rangle^2 + \langle \Delta \Lambda W, g \rangle^2 + |\langle \nabla W, g \rangle|^2 \leq 0.$$

By Lemma 6, this implies

$$g \in \text{span}(Y, \Lambda W, \nabla W) \quad \text{and} \quad \langle Y, g \rangle = \langle \Delta \Lambda W, g \rangle = |\langle \nabla W, g \rangle| = 0.$$

This is impossible, since the  $7 \times 7$  matrix

$$\begin{pmatrix} \langle \Delta \Lambda W, \Lambda W \rangle & (\langle \Delta \Lambda W, \partial_{x_j} W \rangle)_{j=1, \dots, 5} & \langle \Delta \Lambda W, Y \rangle \\ ((\langle \partial_{x_j} W, \Lambda W \rangle)_{j=1, \dots, 5} & ((\langle \partial_{x_j} W, \partial_{x_k} W \rangle)_{j, k=1, \dots, 5} & (\langle \partial_{x_j} W, Y \rangle)_{j=1, \dots, 5} \\ \langle Y, \Lambda W \rangle & (\langle Y, \partial_{x_j} W \rangle)_{j=1, \dots, 5} & \langle Y, Y \rangle \end{pmatrix}$$

is non-singular (this matrix is upper-triangular with non-zero entries on its diagonal).

**Lemma 9.** *For any  $\eta > 0$  there exists  $R = R(\eta) > 0$  such that for all  $g \in \dot{H}^1(\mathbb{R}^5)$ ,*

$$\int_{|x| \leq R} |\nabla g|^2 dx - \int_{\mathbb{R}^5} f'(W)g^2 dx \geq -\eta \|\nabla g\|_{L^2}^2 - \nu^2 \langle Y, g \rangle^2.$$

**PROOF.** By contradiction, suppose there exists  $\eta > 0$  and a sequence  $g_n \in \dot{H}^1$  such that it holds  $\int_{\mathbb{R}^5} f'(W)g_n^2 dx = 1$  and

$$\int_{|x| \leq n} |\nabla g_n|^2 dx - \int_{\mathbb{R}^5} f'(W)g_n^2 dx \leq -\eta \|\nabla g_n\|_{L^2}^2 - \nu^2 \langle Y, g_n \rangle^2.$$

In particular,  $g_n$  is bounded in  $\dot{H}^1$ , and upon extracting a subsequence we can assume that  $g_n \rightharpoonup g \in \dot{H}^1$ . By Rellich's theorem,  $\int_{\mathbb{R}^5} f'(W)g^2 dx = 1$ , in particular  $g \neq 0$ . We also have  $\langle Y, g \rangle = \lim_{n \rightarrow \infty} \langle Y, g_n \rangle$ . Observe that  $\mathbf{1}_{\{|x| \leq n\}} \nabla g_n \rightharpoonup \nabla g$  in  $L^2(\mathbb{R}^5)$ , where  $\mathbf{1}$  denotes the indicator function. Thus, by the Fatou property, it holds  $\langle g, \tilde{L}g \rangle + \eta \|\nabla g\|_{L^2}^2 \leq 0$ , which contradicts Lemma 6.

## 2.2. Multiple potentials

For  $\lambda, \mu \in (0, \infty)$  and  $x, y \in \mathbb{R}^5$  we denote

$$\delta((\lambda, x), (\mu, y)) := \left| \log \left( \frac{\lambda}{\mu} \right) \right| + \frac{|x - y|}{\lambda}.$$

We say that two sequences  $(\lambda_n, x_n)$  and  $(\mu_n, y_n)$  are *orthogonal* if

$$\lim_{n \rightarrow \infty} \delta((\lambda_n, x_n), (\mu_n, y_n)) = \infty.$$

Let  $K \geq 1$ ; in what follows  $\sum_k$  denotes  $\sum_{k=1}^K$ . For  $(\lambda^{(k)}, x^{(k)}) \in (0, \infty) \times \mathbb{R}^5$ , we use the notation

$$W^{(k)}(x) := (\lambda^{(k)})^{-\frac{3}{2}} W((x - x^{(k)})/\lambda^{(k)})$$

and similarly for other functions.

**Lemma 10.** *There exist  $\eta > 0$  such that the following holds. Let  $(\lambda^{(k)}, x^{(k)}) \in (0, \infty) \times \mathbb{R}^5$  for  $k = 1, \dots, K$  satisfy  $\delta((\lambda^{(j)}, x^{(j)}), (\lambda^{(k)}, x^{(k)})) \geq \eta^{-1}$  for all  $j \neq k$ . Let  $U \in \dot{H}^1(\mathbb{R}^5)$  satisfy*

$$\left\| U - \sum_k W^{(k)} \right\|_{\dot{H}^1} \leq \eta.$$

Then for any  $g \in \dot{H}^1(\mathbb{R}^5)$

$$\begin{aligned} \int_{\mathbb{R}^5} (|\nabla g|^2 - f'(U)g^2) dx &\geq \eta \|\nabla g\|_{L^2}^2 \\ &\quad - \sum_k \left\{ (\nu^2 + 1) \langle (\lambda^{(k)})^{-2} Y^{(k)}, g \rangle^2 \right. \\ &\quad \left. + \langle (\lambda^{(k)})^{-2} (\Delta \Lambda W)^{(k)}, g \rangle^2 + |\langle (\lambda^{(k)})^{-2} (\nabla W)^{(k)}, g \rangle|^2 \right\}. \end{aligned}$$

PROOF. Assume that the conclusion fails. There exist sequences  $(\lambda_n^{(k)}, x_n^{(k)})$ ,  $U_n \in \dot{H}^1(\mathbb{R}^5)$  and  $g_n \in \dot{H}^1(\mathbb{R}^5)$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \delta((\lambda_n^{(j)}, x_n^{(j)}), (\lambda_n^{(k)}, x_n^{(k)})) &= \infty, \quad \text{for } j \neq k, \\ \lim_{n \rightarrow \infty} \left\| U_n - \sum_k W_n^{(k)} \right\|_{\dot{H}^1} &= 0, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^5} (|\nabla g_n|^2 - f'(U_n)g_n^2) dx &\leq \frac{1}{n} \|\nabla g_n\|_{L^2}^2 - \sum_k (\nu^2 + 1) \langle (\lambda^{(k)})^{-2} Y_n^{(k)}, g_n \rangle^2 \\ &\quad - \sum_k \langle (\lambda^{(k)})^{-2} (\Delta \Lambda W)_n^{(k)}, g_n \rangle^2 - \sum_k |\langle (\lambda^{(k)})^{-2} (\nabla W)_n^{(k)}, g_n \rangle|^2, \end{aligned}$$

with the normalization  $\int_{\mathbb{R}^5} f'(U_n)g_n^2 dx = 1$ . Here,  $Y_n^{(k)} = (\lambda_n^{(k)})^{-\frac{3}{2}} Y((x - x_n^{(k)})/\lambda_n^{(k)})$  and similarly for other functions.

The sequence  $g_n$  being bounded in  $\dot{H}^1(\mathbb{R}^5)$ , by [15, Théorème 1.1], upon extracting a subsequence, there exist pairwise orthogonal sequences  $(\mu_n^{(j)}, y_n^{(j)})$

and a sequence of profiles  $\psi^{(j)} \in \dot{H}^1$  such that

$$g_n = \sum_{j=1}^J (\mu_n^{(j)})^{-\frac{3}{2}} \psi^{(j)}((\cdot - y_n^{(j)})/\mu_n^{(j)}) + r_n^{(J)} \quad \text{with} \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|r_n^{(J)}\|_{L^{\frac{10}{3}}} = 0, \quad (2.1)$$

and

$$\|g_n\|_{\dot{H}^1}^2 = \sum_{j=1}^J \|\psi^{(j)}\|_{\dot{H}^1}^2 + \|r_n^{(J)}\|_{\dot{H}^1}^2 + o(1) \quad \text{as } n \rightarrow \infty. \quad (2.2)$$

Without loss of generality, we assume that  $(y_n^{(j)}, \mu_n^{(j)}) = (x_n^{(j)}, \lambda_n^{(j)})$  for  $j = 1, \dots, K$ . Indeed, if for some  $k \in \{1, \dots, K\}$  the sequence  $(x_n^{(k)}, \lambda_n^{(k)})$  is orthogonal to all the sequences  $(y_n^{(j)}, \mu_n^{(j)})$ , we can simply include it in the profile decomposition with identically zero corresponding profile. If, on the contrary, there exists  $j$  such that  $(x_n^{(k)}, \lambda_n^{(k)})$  is not orthogonal to  $(y_n^{(j)}, \mu_n^{(j)})$ , then, up to extracting a subsequence, we can assume that

$$\lim_{n \rightarrow \infty} \lambda_n^{(k)}/\mu_n^{(j)} = \lambda_0 \in (0, \infty) \quad \text{and} \quad \lim_{n \rightarrow \infty} x_n^{(k)} - y_n^{(j)} = x_0 \in \mathbb{R}^5.$$

Changing  $\psi^{(j)}$  if necessary, we can replace  $(y_n^{(j)}, \mu_n^{(j)})$  with  $(x_n^{(k)}, \lambda_n^{(k)})$ .

From  $\lim_{n \rightarrow \infty} \|U_n - \sum_k W_n^{(k)}\|_{\dot{H}^1} = 0$  and (2.1) we deduce

$$1 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^5} f'(U_n) g_n^2 dx = \sum_k \int_{\mathbb{R}^5} f'(W) (\psi^{(k)})^2 dx.$$

This shows that at least one of the profiles  $\psi^{(1)}, \dots, \psi^{(K)}$  is not identically zero. We also have

$$\begin{aligned} \lim_{n \rightarrow \infty} (\lambda^{(k)})^{-2} \langle Y_n^{(k)}, g_n \rangle &= \langle Y, \psi^{(k)} \rangle, \\ \lim_{n \rightarrow \infty} (\lambda^{(k)})^{-2} \langle (\Delta \Lambda W)_n^{(k)}, g_n \rangle &= \langle \Delta \Lambda W, \psi^{(k)} \rangle, \\ \lim_{n \rightarrow \infty} (\lambda^{(k)})^{-2} \langle (\nabla W)_n^{(k)}, g_n \rangle &= \langle \nabla W, \psi^{(k)} \rangle. \end{aligned}$$

The Pythagorean formula (2.2) thus yields

$$\sum_k \left\{ \langle \psi^{(k)}, \tilde{L}\psi^{(k)} \rangle + \langle Y, \psi^{(k)} \rangle^2 + \langle \Delta \Lambda W, \psi^{(k)} \rangle^2 + |\langle \nabla W, \psi^{(k)} \rangle|^2 \right\} \leq 0.$$

This contradicts Lemma 6, as in the proof of Lemma 8.

### 3. Construction of multi-bubble solutions

Let  $K \geq 2$  and  $z_1, \dots, z_K$  be  $K$  points of  $\mathbb{R}^5$  distinct two by two. Set

$$d := \frac{1}{2} \min_{j \neq k} |z_j - z_k| > 0 \quad \text{and} \quad \mathbf{z} = (z_1, \dots, z_K).$$

We consider  $\mathbf{c} \in (0, \infty)^K$  as given by (iii) of Lemma 3. Let  $T_0 > 1$  to be taken large enough.

### 3.1. Modulation and bootstrap

Let

$$\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_K) \in (0, \infty)^K, \quad \mathbf{b} = (b_1, \dots, b_K) \in \mathbb{R}^K, \quad \mathbf{y} = (y_1, \dots, y_K) \in (\mathbb{R}^5)^K,$$

and denote  $\boldsymbol{\Gamma} = (\lambda_1, b_1, y_1, \dots, \lambda_K, b_K, y_K)$ .

For all  $1 \leq k \leq K$ , we set

$$\begin{aligned} W_k &:= W_{\lambda_k}(\cdot - y_k) = \frac{1}{\lambda_k^{\frac{3}{2}}} W\left(\frac{\cdot - y_k}{\lambda_k}\right), \\ \nabla_k W_k &:= (\nabla W)_{\lambda_k}(\cdot - y_k) = \frac{1}{\lambda_k^{\frac{3}{2}}} \nabla W\left(\frac{\cdot - y_k}{\lambda_k}\right), \end{aligned}$$

and similarly,

$$\Lambda_k W_k := (\Lambda W)_{\lambda_k}(\cdot - y_k), \quad \Delta_k \Lambda_k W_k := (\Delta \Lambda W)_{\lambda_k}(\cdot - y_k), \quad Y_k := Y_{\lambda_k}(\cdot - y_k).$$

Note that the above functions all have the same scaling; in particular,  $\nabla_k W_k = \lambda_k \nabla W_k$ . We also define

$$\begin{aligned} \vec{Y}_k^\pm &= (\nu^{-1} Y_k, \pm \lambda_k^{-1} Y_k), \quad \vec{Z}_k^\pm = \frac{1}{2} \lambda_k^{-1} (\nu \lambda_k^{-1} Y_k, \pm Y_k), \\ \langle \vec{Y}_k^\pm, \vec{Z}_k^\pm \rangle &= 1, \quad \langle \vec{Y}_k^\pm, \vec{Z}_k^\mp \rangle = 0. \end{aligned} \tag{3.1}$$

Last, we set (recall that  $\sum_k$  means  $\sum_{k=1}^K$ )

$$W_\Gamma = \sum_k W_k, \quad \vec{W}_\Gamma = \sum_k \vec{W}_k, \quad \vec{W}_k = (W_k, b_k \lambda_k^{-1} \Lambda_k W_k).$$

The strategy of the proof is to construct solutions  $\vec{u}$  of (1.1) of the form

$$\vec{u}(t) = \vec{W}_{\Gamma(t)} + \vec{g}(t) \tag{3.2}$$

with  $\|\vec{g}(t)\|_\mathcal{E} \ll 1$  on intervals of time  $[T_0, T]$ , and where the choice of the time-dependent  $\mathcal{C}^1$  parameter vector  $\boldsymbol{\Gamma}(t)$  will ensure the orthogonality conditions

$$\langle \Delta_k \Lambda_k W_k, g \rangle = 0, \quad \langle \nabla_k W_k, g \rangle = 0, \quad \langle \Lambda_k W_k, \dot{g} \rangle = 0 \tag{3.3}$$

and will approximately follow the regime (1.11). We denote

$$a_k^\pm := \langle \vec{Z}_k^\pm, \vec{g} \rangle. \tag{3.4}$$

In the next lemma, we construct well-prepared initial conditions at  $t = T \geq T_0$  with sufficiently many free parameters  $(\alpha_0, \alpha_1, \dots, \alpha_K)$  related to instabilities (see Remark 12).

**Lemma 11.** For any  $T > T_0$  and any  $(\alpha_0, \alpha_1, \dots, \alpha_K) \in \bar{\mathcal{B}}_{\mathbb{R}^{K+1}}$ , there exists a data  $\vec{u}(T) = \vec{u}[T, (\alpha_0, \alpha_1, \dots, \alpha_K)] \in X$  such that

$$\vec{u}(T) = \vec{W}_{\Gamma(T)} + \vec{g}(T),$$

with  $\Gamma(T)$  defined by

$$\begin{aligned} r(T) &= |\mathbf{c}|T^{-2} + T^{-\frac{12}{5}}\alpha_0, & \boldsymbol{\lambda}(T) &= r(T)\frac{\mathbf{c}}{|\mathbf{c}|}, \\ \mathbf{b}(T) &= 2(r(T))^{\frac{3}{2}}\frac{\mathbf{c}}{|\mathbf{c}|^{\frac{3}{2}}}, & \mathbf{y}(T) &= \mathbf{z}, \end{aligned} \quad (3.5)$$

and  $\vec{g}(T)$  satisfies (3.3) and for all  $k = 1, \dots, K$ ,

$$\|\vec{g}(T)\|_{\mathcal{E}} \lesssim T^{-4}, \quad \langle \vec{Z}_k^+(T), \vec{g}(T) \rangle = 0, \quad \langle \vec{Z}_k^-(T), \vec{g}(T) \rangle = T^{-4}\alpha_k, \quad (3.6)$$

where  $\vec{Z}_k^{\pm}(T)$  are defined as in (3.1) for  $\Gamma = \Gamma(T)$ .

Moreover,  $\vec{u}(T)$  is continuous in  $X$  with respect to  $r_T$  and  $a_{k,T}^-$ .

PROOF. For  $\Gamma = \Gamma(T)$  fixed as in (3.5), we consider  $\vec{g} = \vec{g}(T) = (g, \dot{g})$  of the form

$$\vec{g} = \sum_k \left\{ b_k^+ \vec{Y}_k^+ + b_k^- \vec{Y}_k^- + \frac{((c_k \cdot \nabla_k)W_k, 0)}{\|\partial_{x_1} W\|_{L^2}^2} + \frac{d_k(\Lambda_k W_k, 0)}{\|\nabla \Lambda W\|_{L^2}^2} + \frac{e_k(0, \Lambda_k W_k)}{\lambda_k \|\Lambda W\|_{L^2}^2} \right\}.$$

Consider the linear map  $\Psi : (\mathbb{R}^9)^K \rightarrow (\mathbb{R}^9)^K$  defined as follows:

$$\begin{aligned} &\Psi((b_k^+, b_k^-, c_k, d_k, e_k)_{k=1, \dots, K}) \\ &= \left( \langle \vec{Z}_k^+, \vec{g} \rangle, \langle \vec{Z}_k^-, \vec{g} \rangle, \lambda_k^{-2} \langle \nabla_k W_k, g \rangle, \lambda_k^{-2} \langle \Delta_k \Lambda_k W_k, g \rangle, \lambda_k^{-1} \langle \Lambda_k W_k, \dot{g} \rangle \right)_{k=1, \dots, K}. \end{aligned}$$

It is easy to check that for  $T$  large enough the matrix of  $\Psi$  is a perturbation of the block matrix  $\text{diag}_K(A)$  where the  $9 \times 9$  matrix  $A$  is upper-triangular with entries 1 on the diagonal (the only nonzero entries off the diagonal are due to  $\langle Y, \Delta \Lambda W \rangle \neq 0$ ). Moreover,

$$\left| \Psi^{-1}((0, T^{-4}\alpha_k, 0, \dots, 0)_{k=1, \dots, K}) \right| \lesssim T^{-4},$$

and so  $\|\vec{g}\|_{\mathcal{E}} \lesssim |(b_k^+, b_k^-, c_k, d_k, e_k)_{k=1, \dots, K}| \lesssim T^{-4}$ . The continuity property is clear.

We introduce the following bootstrap estimates

$$\|\vec{g}\|_{\mathcal{E}} \leq t^{-\frac{11}{3}}, \quad (3.7)$$

$$|\boldsymbol{\lambda} - \mathbf{c}t^{-2}| \leq t^{-\frac{7}{3}}, \quad (3.8)$$

$$|\mathbf{b} - 2\mathbf{c}t^{-3}| \leq t^{-\frac{10}{3}}, \quad (3.9)$$

$$|\mathbf{y} - \mathbf{z}| \leq t^{-\frac{7}{3}}, \quad (3.10)$$

$$\sum_k (a_k^+)^2 \leq t^{-8}, \quad (3.11)$$



and

$$t^{\frac{24}{5}} (|\boldsymbol{\lambda}| - |\mathbf{c}|t^{-2})^2 + t^8 \sum_k (a_k^-)^2 \leq 1. \quad (3.12)$$

**Remark 12.** The parameters  $\alpha_0, \alpha_1, \dots, \alpha_K$  and the bootstrap estimate (3.12) are related to backwards instabilities to be controlled: the backward exponential instability of each soliton (controlled by  $(\alpha_k)_{k=1, \dots, K}$ ), and a one-dimensional instability related to the reduced system of ODE, controlled by  $\alpha_0$ .

Let  $\bar{u} \in \mathcal{C}(I_{\max}; \dot{H}^1 \times L^2)$  where  $I_{\max} \ni T$ , be the maximal solution of (1.1) corresponding to any data  $\bar{u}(T)$  as given by Lemma 11. Since  $\bar{u}(T) \in X$ , by persistence of regularity (see for instance Appendix B of [20]), we have  $\bar{u} \in \mathcal{C}(I_{\max}; X)$ . Such regularity will allow energy computations without density argument.

Define

$$T_\star := \inf\{t \in [T_0, T] : \text{on the interval } [t, T], \bar{u} \text{ is well-defined} \\ \text{and decomposes as (3.2) where } \Gamma \text{ and } \vec{g} \text{ satisfy (3.7)-(3.12)}\}.$$

**Lemma 13.** *It holds  $T_0 \leq T_\star \leq T$  and if  $T_\star > T_0$  then*

- (i) *Equality is reached at  $t = T_\star$  in at least one of the inequalities (3.7)-(3.12).*
- (ii) *On  $[T_\star, T]$ , it holds*

$$|\boldsymbol{\lambda}' + \mathbf{b}| \lesssim t^{-\frac{11}{3}}, \quad (3.13)$$

$$|\mathbf{y}'| \lesssim t^{-\frac{11}{3}}, \quad (3.14)$$

$$|\mathbf{b}' - \mathbf{B}(\boldsymbol{\lambda})| \lesssim t^{-\frac{14}{3}}, \quad (3.15)$$

$$\left| (a_k^\pm)' \mp \nu \lambda_k^{-1} a_k^\pm \right| \lesssim t^{-4}, \quad (3.16)$$

where  $\mathbf{B} = (B_1, \dots, B_K)$  is defined by (1.9).

We begin with a technical lemma.

**Lemma 14.** *Under the bootstrap estimates (3.7)-(3.12), the following bounds hold, for  $j \neq k$ ,*

$$\begin{aligned} \langle \lambda_j^{-1} W_j, \lambda_k^{-1} W_k \rangle &\lesssim t^{-2}, \quad \langle W_j^{\frac{5}{3}}, W_k^{\frac{5}{3}} \rangle \lesssim t^{-10} \log t, \\ \langle \lambda_j^{-1} |\nabla_j W_j|, \lambda_k^{-1} |\nabla_k W_k| \rangle &\lesssim t^{-6}, \quad \|W_k W_j^{\frac{4}{3}}\|_{L^{\frac{10}{7}}} \lesssim t^{-6}. \end{aligned} \quad (3.17)$$

PROOF (PROOF OF LEMMA 14). First, by change of variable

$$\langle \lambda_j^{-1} W_j, \lambda_k^{-1} W_k \rangle = \langle \tilde{\lambda}_j^{-1} W_{\tilde{\lambda}_j}(\cdot - t^2 z_j), \tilde{\lambda}_k^{-1} W_{\tilde{\lambda}_k}(\cdot - t^2 z_k) \rangle,$$

where  $\tilde{\lambda}_j := t^2 \lambda_j \sim 1$  and  $\tilde{\lambda}_k := t^2 \lambda_k \sim 1$ . The right-hand side term is estimated by dividing  $\mathbb{R}^5$  into three regions:  $D_j := \mathcal{B}(t^2 z_j, t^2 d)$ ,  $D_k := \mathcal{B}(t^2 z_k, t^2 d)$  and

$D_{j,k} = \mathbb{R}^5 \setminus (D_j \cup D_k)$ . In order to estimate the integral outside both balls, we use the bound  $|W(x)| \lesssim |x|^{-3}$  and the Cauchy-Schwarz inequality and obtain

$$\int_{D_{j,k}} |y - t^2 z_j|^{-3} \cdot |y - t^2 z_k|^{-3} dy \lesssim \int_{t^2 d}^{\infty} r^{-2} dr \lesssim t^{-2}.$$

For  $D_j$ , we observe

$$\int_{D_j} |y - t^2 z_j|^{-3} dy \lesssim \int_0^{t^2 d} r dr \lesssim t^4,$$

so using also the trivial  $L^\infty$  bound of order  $t^{-6}$  for the second factor on  $D_j$ , we obtain a bound of order  $t^{-2}$  for the contribution of  $D_j$ . This justifies the first bound in (3.17).

The other estimates in (3.17) are proved similarly, using  $|\nabla W(x)| \lesssim |x|^{-4}$ .

In the sequel we will make use of various pointwise estimates obtained from the Taylor expansion of the nonlinearity  $f$ . We claim that for all  $u, v \in \mathbb{R}$

$$|f(u+v) - f(u) - f(v) - f'(u)v| \lesssim |u|^{\frac{2}{3}} |v|^{\frac{5}{3}}. \quad (3.18)$$

To prove (3.18), we consider several cases. If  $|v| \leq \frac{1}{2}|u|$ , then by Taylor expansion, we have

$$|f(u+v) - f(u) - f(v) - f'(u)v| \lesssim |u|^{\frac{1}{3}} |v|^2 \lesssim |u|^{\frac{2}{3}} |v|^{\frac{5}{3}}.$$

If  $\frac{1}{2}|u| \leq |v| \leq 2|u|$ , then

$$|f(u+v)| + |f(u)| + |f(v)| + |f'(u)v| \lesssim |u|^{\frac{7}{3}} + |v|^{\frac{7}{3}} \lesssim |u|^{\frac{2}{3}} |v|^{\frac{5}{3}}.$$

Last, if  $2|u| \leq |v|$ , then

$$|f(u+v) - f(v)| + |f(u)| + |f'(u)v| \lesssim |v|^{\frac{4}{3}} |u| + |u|^{\frac{7}{3}} + |u|^{\frac{4}{3}} |v| \lesssim |u|^{\frac{2}{3}} |v|^{\frac{5}{3}}.$$

Next, it is easily checked by induction on  $J \geq 1$  that the following holds

$$\left| f\left(\sum_{j=1}^J v_j\right) - \sum_{j=1}^J f(v_j) \right| \leq \sum_{j \neq l} |v_j| |v_l|^{\frac{4}{3}}.$$

By the triangle inequality and (3.18), we deduce, for any  $u, v_j \in \mathbb{R}$ ,

$$\left| f\left(u + \sum_{j=1}^J v_j\right) - f(u) - \sum_{j=1}^J f(v_j) - f'(u) \sum_{j=1}^J v_j \right| \lesssim |u|^{\frac{2}{3}} \sum_{j=1}^J |v_j|^{\frac{5}{3}} + \sum_{j \neq l} |v_j| |v_l|^{\frac{4}{3}}. \quad (3.19)$$

PROOF (PROOF OF LEMMA 13). At  $t = T$ , Lemma 11 provides an initial data as in (3.2) with the estimates (3.7)-(3.12). Indeed,  $(\alpha_0, \alpha_1, \dots, \alpha_K) \in \tilde{\mathcal{B}}_{\mathbb{R}^{K+1}}$

implies that (3.12) holds at  $t = T$ . This gives (3.8)-(3.9). Moreover, (3.7), (3.10) and (3.11) are clear from Lemma 11.

By the local Cauchy theory for (1.1), it is clear that if a solution  $\vec{u}$  satisfies (3.2) with (3.7)-(3.12) on some interval  $[t, T]$ , then the solution  $\vec{u}$  also exists on  $[t - \tau, T]$ , for some  $\tau > 0$ .

To decompose  $\vec{u}(t)$  for  $t < T$ , the strategy is to express the orthogonality conditions (3.3) as a non-autonomous differential system  $\mathbf{D}\Gamma'(t) = \mathbf{F}(t, \Gamma(t))$ , where  $\mathbf{F}$  is continuous in  $t$  and locally Lipschitz in  $\Gamma$ , and the matrix  $\mathbf{D}$  is a perturbation of the block matrix  $\text{diag}_K(D_0) \in \mathbb{R}^{7K \times 7K}$ , where

$$D_0 = \text{diag} (\|\nabla \Lambda W\|_{L^2}^2, \|\Lambda W\|_{L^2}^2, (\|\partial_{x_j} W\|_{L^2}^2)_{j=1, \dots, 5}) \in \mathbb{R}^{7 \times 7}.$$

Then, (i) will follow from the Cauchy-Lipschitz theorem and continuity arguments. Moreover, estimates in (ii) will follow from similar computations combined with (3.7)-(3.12).

Formally, the evolution equation of  $\vec{g}(t) := (g, \dot{g})(t)$  is

$$\partial_t \vec{g} = JDE(\vec{W}_\Gamma + \vec{g}) - \lambda' \partial_\lambda \vec{W}_\Gamma - \mathbf{b}' \partial_{\mathbf{b}} \vec{W}_\Gamma - \mathbf{y}' \cdot \partial_{\mathbf{y}} \vec{W}_\Gamma \quad (3.20)$$

which rewrites as

$$\partial_t g = \dot{g} + \sum_{k=1}^K \lambda_k^{-1} (\lambda'_k + b_k) \Lambda_k W_k + \sum_{k=1}^K \lambda_k^{-1} y'_k \cdot \nabla_k W_k, \quad (3.21)$$

$$\begin{aligned} \partial_t \dot{g} &= \Delta g + f(W_\Gamma + g) - \sum_{k=1}^K f(W_k) \\ &+ \sum_{k=1}^K \lambda_k^{-2} \lambda'_k b_k \underline{\Delta}_k \Lambda_k W_k - \sum_{k=1}^K \lambda_k^{-1} b'_k \Lambda_k W_k + \sum_{k=1}^K \lambda_k^{-2} b_k (y'_k \cdot \nabla_k) \Lambda W_k. \end{aligned} \quad (3.22)$$

Proof of (3.13)-(3.14). We differentiate with respect to time the identity  $0 = \langle \lambda_k^{-1} \Delta_k \Lambda_k W_k, g \rangle$  which is the first orthogonality condition in (3.3) and we use (3.21)

$$\begin{aligned} 0 &= \frac{d}{dt} \langle \lambda_k^{-1} \Delta_k \Lambda_k W_k, g \rangle \\ &= \left\langle \lambda_k^{-1} \Delta_k \Lambda_k W_k, \dot{g} + \sum_j \lambda_j^{-1} (\lambda'_j + b_j) \Lambda_j W_j + \sum_j \lambda_j^{-1} (y'_j \cdot \nabla_j) W_j \right\rangle \\ &\quad - \langle \lambda'_k \lambda_k^{-2} \underline{\Delta}_k \Delta_k \Lambda_k W_k, g \rangle - \langle \lambda_k^{-2} (y'_k \cdot \nabla_k) \Delta_k \Lambda_k W_k, g \rangle. \end{aligned}$$

Rewrite the first term on the right-hand side as

$$\langle \lambda_k^{-1} \Delta_k \Lambda_k W_k, \dot{g} \rangle = \left\langle \lambda_k^{-1} \Delta_k \Lambda_k W_k, \partial_t u - \sum_k \lambda_k^{-1} b_k \Lambda_k W_k \right\rangle.$$

Note that  $\partial_t u$  is continuous in  $L^2$  as a function  $t$  and  $\lambda_k^{-1} \Delta_k \Lambda_k W_k$  is locally Lipschitz in  $L^2$  as a function of  $\Gamma$ . Thus,  $\langle \lambda_k^{-1} \Delta_k \Lambda_k W_k, \partial_t u \rangle$  is continuous

in  $t$  and locally Lipschitz in  $\Gamma$ . For the second term above, one checks the same properties. Regularity in  $t$  and  $\Gamma$  for all other terms appearing in the computations is proved similarly and omitted.

First, we estimate terms containing  $g$  and  $\dot{g}$ ,

$$|\langle \lambda_k^{-1} \Delta_k \Lambda_k W_k, \dot{g} \rangle| \lesssim \|\lambda_k^{-1} \Delta_k \Lambda_k W_k\|_{L^2} \|\dot{g}\|_{L^2} \lesssim \|\vec{g}\|_{\mathcal{E}},$$

next

$$\begin{aligned} |\langle \lambda'_k \lambda_k^{-2} \underline{\Delta}_k \Delta_k \Lambda_k W_k, g \rangle| &\lesssim |\lambda'_k| \|\lambda_k^{-2} \underline{\Delta}_k \Delta_k \Lambda_k W_k\|_{L^{\frac{10}{7}}} \|g\|_{L^{\frac{10}{3}}} \\ &\lesssim |\lambda'_k| \|g\|_{\dot{H}^1} \lesssim |\boldsymbol{\lambda}' + \mathbf{b}| \|\vec{g}\|_{\mathcal{E}} + |\mathbf{b}| \|\vec{g}\|_{\mathcal{E}}, \end{aligned}$$

and similarly

$$|\langle \lambda_k^{-2} (y'_k \cdot \nabla_k) \Delta_k \Lambda_k W_k, g \rangle| \lesssim |\mathbf{y}'| \|\vec{g}\|_{\mathcal{E}}.$$

Next, we claim that the matrix  $M^\lambda$  with coefficients

$$m_{jk}^\lambda := -\langle \lambda_k^{-1} \Delta_k \Lambda_k W_k, \lambda_j^{-1} \Lambda_j W_j \rangle$$

is diagonally dominant and that its inverse is uniformly bounded. Indeed, for  $j = k$ , it holds

$$m_{kk}^\lambda = \langle \lambda_k^{-1} \Delta_k \Lambda_k W_k, \lambda_k^{-1} \Lambda_k W_k \rangle = \|\nabla \Lambda W\|_{L^2}^2,$$

and for  $j \neq k$ , by (3.17)

$$|m_{jk}^\lambda| = \frac{\lambda_j}{\lambda_k} |\langle \lambda_k^{-1} \nabla_k \Lambda_k W_k, \lambda_j^{-1} \nabla_j \Lambda_j W_j \rangle| \lesssim t^{-6}.$$

Last, by symmetry  $\langle \Delta \Lambda W, \nabla W \rangle = 0$ , and so

$$\langle \lambda_k^{-1} \Delta_k \Lambda_k W_k, \lambda_k^{-1} (y'_k \cdot \nabla_k) W_k \rangle = 0;$$

for  $j \neq k$ , by (3.17)

$$|\langle \lambda_k^{-1} \Delta_k \Lambda_k W_k, \lambda_j^{-1} (y'_j \cdot \nabla_j) W_j \rangle| \lesssim t^{-6} |\mathbf{y}'|.$$

Collecting these estimates, using  $\|\vec{g}\|_{\mathcal{E}} \lesssim t^{-\frac{11}{3}}$  and  $|\mathbf{b}| \ll 1$ , we obtain

$$|\boldsymbol{\lambda}' + \mathbf{b}| \lesssim (1 + |\mathbf{y}'|) t^{-\frac{11}{3}}. \quad (3.23)$$

We differentiate with respect to time the identity  $0 = \langle \lambda_k^{-1} \nabla_k W_k, g \rangle$  which is the second orthogonality condition in (3.3) and we use (3.21)

$$\begin{aligned} 0 &= \frac{d}{dt} \langle \lambda_k^{-1} \nabla_k W_k, g \rangle \\ &= \left\langle \lambda_k^{-1} \nabla_k W_k, \dot{g} + \sum_j \lambda_j^{-1} (\lambda'_j + b_j) \Lambda_j W_j + \sum_j \lambda_j^{-1} (y'_j \cdot \nabla_j) W_j \right\rangle \\ &\quad - \langle \lambda_k^{-2} \lambda'_k \underline{\Delta}_k \nabla_k W_k, g \rangle - \langle \lambda_k^{-2} (y'_k \cdot \nabla_k) \nabla_k W_k, g \rangle. \end{aligned}$$

First, we have

$$|\langle \lambda_k^{-1} \nabla_k W_k, \dot{g} \rangle| \leq \|\lambda_k^{-1} \nabla_k W_k\|_{L^2} \|\dot{g}\|_{L^2} \lesssim \|\vec{g}\|_{\mathcal{E}}.$$

Second, by  $\langle \Lambda W, \nabla W \rangle = 0$  and (3.17), we obtain for any  $k, j$ ,

$$|\langle \lambda_k^{-1} \nabla_k W_k, \lambda_j^{-1} (\lambda'_j + b_j) \Lambda_j W_j \rangle| \lesssim t^{-2} |\boldsymbol{\lambda}' + \mathbf{b}|.$$

Then, for  $j = k$ , it holds

$$\langle \lambda_k^{-1} \nabla_k W_k, \lambda_k^{-1} (y'_k \cdot \nabla_k) W_k \rangle = y'_k \|\partial_{x_1} W\|_{L^2}^2,$$

and for  $j \neq k$ , by (3.17)

$$|\langle \lambda_k^{-1} \nabla_k W_k, \lambda_j^{-1} \nabla_j W_j \rangle| \lesssim t^{-6}.$$

Next, as before,

$$\begin{aligned} |\langle \lambda_k^{-2} \lambda'_k \underline{\Lambda}_k \nabla_k W_k, g \rangle| &\lesssim |\lambda'_k| \|\lambda_k^{-2} \underline{\Lambda}_k \nabla_k W_k\|_{L^{\frac{10}{9}}} \|g\|_{L^{\frac{10}{3}}} \\ &\lesssim |\lambda'_k| \|g\|_{\dot{H}^1} \lesssim |\boldsymbol{\lambda}' + \mathbf{b}| \|\vec{g}\|_{\mathcal{E}} + |\mathbf{b}| \|\vec{g}\|_{\mathcal{E}}, \end{aligned}$$

and

$$|\langle \lambda_k^{-2} (y'_k \cdot \nabla_k) \nabla_k W_k, g \rangle| \lesssim |\mathbf{y}'| \|\vec{g}\|_{\mathcal{E}}.$$

Collecting these estimates, using  $\|\vec{g}\|_{\mathcal{E}} \lesssim t^{-\frac{11}{3}}$  and  $|\mathbf{b}| \ll 1$ , we obtain

$$|\mathbf{y}'| \lesssim t^{-2} |\boldsymbol{\lambda}' + \mathbf{b}| + t^{-\frac{11}{3}}. \quad (3.24)$$

Combining (3.23) and (3.24), we have proved  $|\boldsymbol{\lambda}' + \mathbf{b}| + |\mathbf{y}'| \lesssim t^{-\frac{11}{3}}$ , which is (3.13)-(3.14).

Proof of (3.15). We differentiate the identity  $\langle \Lambda_k W_k, \dot{g} \rangle = 0$  (third orthogonality condition in (3.3)) with respect to time and we use (3.22)

$$\begin{aligned} 0 &= \frac{d}{dt} \langle \Lambda_k W_k, \dot{g} \rangle \\ &= -\langle \lambda'_k \lambda_k^{-1} \Lambda_k^2 W_k, \dot{g} \rangle - \langle \lambda_k^{-1} (y'_k \cdot \nabla_k) \Lambda_k W_k, \dot{g} \rangle \\ &\quad + \left\langle \Lambda_k W_k, \Delta g + f(W_{\Gamma} + g) - \sum_j f(W_j) \right\rangle \\ &\quad + \left\langle \Lambda_k W_k, \sum_j \lambda_j^{-2} \lambda'_j b_j \underline{\Lambda}_j \Lambda_j W_j \right\rangle - \left\langle \Lambda_k W_k, \sum_j \lambda_j^{-1} b'_j \Lambda_j W_j \right\rangle \\ &\quad + \left\langle \Lambda_k W_k, \sum_j \lambda_j^{-2} b_j (y'_j \cdot \nabla_j) \Lambda W_j \right\rangle. \end{aligned}$$

For the first two terms, we observe from (3.13),  $|\mathbf{b}| \lesssim t^{-3}$  and  $\|\vec{g}\|_{\mathcal{E}} \lesssim t^{-\frac{11}{3}}$

$$|\langle \lambda'_k \lambda_k^{-1} \Lambda_k^2 W_k, \dot{g} \rangle| \leq |\lambda'_k| \|\lambda_k^{-1} \Lambda_k^2 W_k\|_{L^2} \|\dot{g}\|_{L^2} \lesssim |\boldsymbol{\lambda}'| \|\vec{g}\|_{\mathcal{E}} \lesssim t^{-\frac{20}{3}},$$

and from (3.14),

$$|\langle \lambda_k^{-1}(y'_k \cdot \nabla_k) \Lambda_k W_k, \dot{g} \rangle| \leq \|\lambda_k^{-1}(y'_k \cdot \nabla_k) \Lambda_k W_k\|_{L^2} \|\dot{g}\|_{L^2} \lesssim |y'| \|\bar{g}\| \varepsilon \lesssim t^{-\frac{22}{3}}.$$

For the next line in the identity above, we set

$$I_1 = \langle \Lambda_k W_k, \Delta g + f(W_\Gamma + g) - f(W_\Gamma) \rangle, \quad I_2 = \left\langle \Lambda_k W_k, f(W_\Gamma) - \sum_j f(W_j) \right\rangle.$$

We first note that, using the cancellation  $L\Lambda W = 0$ ,

$$\begin{aligned} I_1 &= \langle \Lambda_k W_k, f(W_\Gamma + g) - f(W_\Gamma) - f'(W_k)g \rangle \\ &= \langle \Lambda_k W_k, f(W_\Gamma + g) - f(W_\Gamma) - f'(W_\Gamma)g \rangle \\ &\quad + \langle \Lambda_k W_k, (f'(W_\Gamma) - f'(W_k))g \rangle \\ &= I_3 + I_4. \end{aligned}$$

By the Taylor inequality,

$$|f(W_\Gamma + g) - f(W_\Gamma) - f'(W_\Gamma)g| \lesssim W_\Gamma^{\frac{1}{3}} |g|^2 + |g|^{\frac{7}{3}},$$

and so, by Holder and Sobolev inequalities

$$|I_3| \lesssim \int W_\Gamma^{\frac{4}{3}} |g|^2 + \int W_\Gamma |g|^{\frac{7}{3}} \lesssim \|W_\Gamma\|_{\dot{H}^1}^{\frac{4}{3}} \|g\|_{\dot{H}^1}^2 + \|W_\Gamma\|_{\dot{H}^1} \|g\|_{\dot{H}^1}^{\frac{7}{3}} \lesssim t^{-\frac{22}{3}}.$$

By the Taylor inequality,

$$|f'(W_\Gamma) - f'(W_k)| \lesssim \sum_{j \neq k} \left( W_j^{\frac{4}{3}} + W_k^{\frac{1}{3}} W_j \right),$$

and thus

$$\begin{aligned} |I_4| &\lesssim \sum_{j \neq k} \int \left( W_k W_j^{\frac{4}{3}} + W_k^{\frac{4}{3}} W_j \right) |g| \\ &\lesssim \|g\|_{L^{\frac{10}{3}}} \sum_{j \neq k} \left( \|W_k W_j^{\frac{4}{3}}\|_{L^{\frac{10}{7}}} + \|W_k^{\frac{4}{3}} W_j\|_{L^{\frac{10}{7}}} \right). \end{aligned}$$

For  $j \neq k$ , by (3.17) we have  $|I_4| \lesssim t^{-6} \|\bar{g}\| \varepsilon \lesssim t^{-\frac{29}{3}}$ . Therefore,  $|I_1| \lesssim t^{-\frac{22}{3}}$ .

We turn to  $I_2$  and set

$$\begin{aligned} I_2 &= \left\langle \Lambda_k W_k, f(W_\Gamma) - \sum_j f(W_j) - f'(W_k) \sum_{j \neq k} W_j \right\rangle \\ &\quad + \left\langle \Lambda_k W_k, f'(W_k) \sum_{j \neq k} W_j \right\rangle \\ &= I_5 + I_6. \end{aligned}$$

Using (3.19), we have

$$\begin{aligned}
& |\Lambda W_k| \left| f(W_{\mathbf{r}}) - \sum_j f(W_j) - f'(W_k) \sum_{j \neq k} W_j \right| \\
& \lesssim W_k^{\frac{5}{3}} \sum_{j \neq k} W_j^{\frac{5}{3}} + W_k \sum_{j \neq l, j \neq k, l \neq k} W_j W_l^{\frac{4}{3}} \\
& \lesssim \sum_{j \neq l} W_j^{\frac{5}{3}} W_l^{\frac{5}{3}}.
\end{aligned}$$

Thus, using (3.17), we obtain  $|I_5| \lesssim t^{-10} \log t$ .

Last, to estimate  $I_6$ , we only have to consider  $\langle \Lambda_k W_k, f'(W_k) W_j \rangle$  for all  $j \neq k$ . By change of variable,

$$\langle \Lambda_k W_k, f'(W_k) W_j \rangle = \frac{7}{3} \lambda_j^{-\frac{3}{2}} \lambda_k^{\frac{3}{2}} \int_{\mathbb{R}^5} W(x)^{\frac{4}{3}} \Lambda W(x) W \left( \frac{\lambda_k}{\lambda_j} x - \frac{z_j - z_k}{\lambda_j} \right) dx.$$

For  $|x| \geq \lambda_k^{-1} d$ , it holds by  $W(x) \leq |x|^{-3}$  and then Cauchy-Schwarz inequality

$$\begin{aligned}
& \int_{|x| \geq \lambda_k^{-1} d} W^{\frac{4}{3}}(x) |\Lambda W(x)| W \left( \frac{\lambda_k}{\lambda_j} x - \frac{z_j - z_k}{\lambda_j} \right) dx \\
& \lesssim \lambda_k^4 \int W(x) W \left( \frac{\lambda_k}{\lambda_j} x - \frac{z_j - z_k}{\lambda_j} \right) dx \\
& \lesssim \lambda_k^4 \lesssim t^{-8}.
\end{aligned}$$

For  $|x| \leq \lambda_k^{-1} d$ , it holds

$$\left| W \left( \frac{\lambda_k}{\lambda_j} x - \frac{z_j - z_k}{\lambda_j} \right) - W \left( \frac{z_j - z_k}{\lambda_j} \right) \right| \lesssim |x| \left| \frac{z_j - z_k}{\lambda_j} \right|^{-4} \lesssim t^{-8} |x|,$$

and by the explicit expression of  $W$ , for  $|y| \geq 1$ ,

$$\left| W(y) - 15^{\frac{3}{2}} |y|^{-3} \right| \leq |y|^{-5}.$$

We obtain for such  $x$ ,

$$\left| W \left( \frac{\lambda_k}{\lambda_j} x - \frac{z_j - z_k}{\lambda_j} \right) - 15^{\frac{3}{2}} \lambda_j^3 |z_j - z_k|^{-3} \right| \lesssim \left| \frac{z_j - z_k}{\lambda_j} \right|^{-5} + t^{-8} |x| \lesssim t^{-8} (1 + |x|).$$

We deduce from these estimates

$$\begin{aligned}
& \left| \int_{\mathbb{R}^5} W(x)^{\frac{4}{3}} \Lambda W(x) W \left( \frac{\lambda_k}{\lambda_j} x - \frac{z_j - z_k}{\lambda_j} \right) dx - 15^{\frac{3}{2}} \langle W^{\frac{4}{3}}, \Lambda W \rangle \lambda_j^3 |z_j - z_k|^{-3} \right| \\
& \lesssim t^{-8} + t^{-8} \int_{|x| \leq \lambda_k^{-1} d} (1 + |x|) W^{\frac{7}{3}}(x) dx \lesssim t^{-8}.
\end{aligned}$$

Therefore,

$$\left| \langle \Lambda_k W_k, f'(W_k) W_j \rangle - \frac{7}{3} 15^{\frac{3}{2}} \langle W^{\frac{4}{3}}, \Lambda W \rangle \lambda_j^{\frac{3}{2}} \lambda_k^{\frac{3}{2}} |z_j - z_k|^{-3} \right| \lesssim t^{-8},$$

and by the definition of  $B_k(\boldsymbol{\lambda})$  and  $\kappa$  in (1.9),

$$|I_6 - \lambda_k \|\Lambda W\|_{L^2}^2 B_k(\boldsymbol{\lambda})| \lesssim t^{-8}.$$

Next, for  $j \neq k$ , using (3.17),

$$|\langle \Lambda_k W_k, \lambda_j^{-2} \lambda'_j b_j \underline{\Lambda}_j \Lambda_j W_j \rangle| \lesssim |\boldsymbol{\lambda}'| |\mathbf{b}| t^{-2} \lesssim t^{-8},$$

while the identity  $\langle \Lambda W, \underline{\Lambda} \Lambda W \rangle = 0$  takes care of the corresponding term for  $j = k$ .

For the terms  $\langle \Lambda_k W_k, \lambda_j^{-1} b'_j \Lambda_j W_j \rangle$ , we observe if  $j = k$  that

$$\langle \Lambda_k W_k, \lambda_k^{-1} \Lambda_k W_k \rangle = \lambda_k \|\Lambda W\|_{L^2}^2,$$

and if  $j \neq k$ , by (3.17),  $|\langle \Lambda_k W_k, \lambda_j^{-1} \Lambda_j W_j \rangle| \lesssim t^{-4}$ .

Last, for any  $j, k$ ,

$$|\langle \Lambda_k W_k, \lambda_j^{-2} b_j (y'_j \cdot \nabla_j) \Lambda W_j \rangle| \lesssim |\mathbf{b}| |\mathbf{y}'| \lesssim t^{-\frac{20}{3}}.$$

Collecting these estimates, we have proved, for all  $k = 1, \dots, K$ ,

$$|b'_k - B_k(\boldsymbol{\lambda})| \lesssim t^{-\frac{14}{3}} + t^{-2} |\mathbf{b}'|,$$

and since  $|B_k(\boldsymbol{\lambda})| \lesssim t^{-4}$ , (3.15) follows.

Proof of (3.16). By the definition of  $a_k^\pm$  in (3.4), we compute  $\frac{d}{dt} a_k^\pm = \langle \partial_t \vec{Z}_k^\pm, \vec{g} \rangle + \langle \vec{Z}_k^\pm, \partial_t \vec{g} \rangle$ . First,

$$\partial_t \vec{Z}_k^\pm = \lambda'_k \partial_{\lambda_k} \vec{Z}_k^\pm + y'_k \cdot \partial_{y_k} \vec{Z}_k^\pm,$$

Since  $Y$  is exponentially decaying, we obtain from the definition of  $\vec{Z}_k^\pm$ , (3.13)-(3.14) and (3.7)-(3.11), the estimate

$$|\langle \partial_t \vec{Z}_k^\pm, \vec{g} \rangle| \lesssim \left( \left| \frac{\lambda'_k}{\lambda_k} \right| + \left| \frac{y'_k}{\lambda_k} \right| \right) \|\vec{g}\| \varepsilon \lesssim t^{-\frac{14}{3}}.$$

Second, using (3.20),

$$\begin{aligned} \langle \vec{Z}_k^\pm, \partial_t \vec{g} \rangle &= \langle \vec{Z}_k^\pm, JDE(\vec{W}_\Gamma + \vec{g}) \rangle - \langle \vec{Z}_k^\pm, \boldsymbol{\lambda}' \partial_\lambda \vec{W}_\Gamma \rangle \\ &\quad - \langle \vec{Z}_k^\pm, \mathbf{b}' \partial_b \vec{W}_\Gamma \rangle - \langle \vec{Z}_k^\pm, \mathbf{y}' \cdot \partial_y \vec{W}_\Gamma \rangle. \end{aligned}$$

Using

$$\boldsymbol{\lambda}' \partial_\lambda \vec{W}_\Gamma = \sum_j (\lambda_j^{-1} \lambda'_j \Lambda_j W_j, \lambda_j^{-2} \lambda'_j b_j \underline{\Lambda}_j \Lambda_j W_j),$$



$\langle Y_k, \Lambda_k W_k \rangle = 0$ ,  $|\langle \vec{Z}_k^\pm, \Lambda_j W_j \rangle| \lesssim t^{-6}$  for  $j \neq k$ , and estimates (3.9), (3.13), we obtain

$$|\langle \vec{Z}_k^\pm, \lambda' \partial_\lambda \vec{W}_\Gamma \rangle| \lesssim t^{-4}.$$

Similarly, using

$$\begin{aligned} \mathbf{b}' \partial_{\mathbf{b}} \vec{W}_\Gamma &= \sum_j (0, \lambda_j^{-1} b'_k \Lambda_j W_j), \\ \mathbf{y}' \cdot \partial_{\mathbf{y}} \vec{W}_\Gamma &= \sum_j (\lambda_j^{-1} (y'_j \cdot \nabla_j) W_j, \lambda_j^{-2} b_j (y'_j \cdot \nabla_j) \Lambda_j W_j), \\ \langle Y_k, \nabla_k W_k \rangle &= 0, \end{aligned}$$

and estimates (3.9), (3.14), (3.15), it holds

$$|\langle \vec{Z}_k^\pm, \mathbf{b}' \partial_{\mathbf{b}} \vec{W}_\Gamma \rangle| + |\langle \vec{Z}_k^\pm, \mathbf{y}' \cdot \partial_{\mathbf{y}} \vec{W}_\Gamma \rangle| \lesssim t^{-4}.$$

Now, we have

$$\begin{aligned} JDE(\vec{W}_\Gamma + \vec{g}) &= \\ &= \left( \sum_j \lambda_j^{-1} b_j \Lambda_j W_j, f(W_\Gamma + g) - \sum_j f(W_k) - f'(W_k)g \right) + JD^2E(W_k)\vec{g}. \end{aligned}$$

As before, for all  $j$ , it holds  $|\langle \vec{Z}_k^\pm, (\lambda_k^{-1} b_j \Lambda_j W_j, 0) \rangle| \lesssim t^{-4}$ , and arguing as in the proof of (3.15)

$$\left| \left\langle \lambda_k^{-1} Y_k, f(W_\Gamma + g) - \sum_j f(W_k) - f'(W_k)g \right\rangle \right| \lesssim t^{-4}.$$

Last, we check that  $\langle \vec{Z}_k^\pm, JD^2E(W_k)\vec{g} \rangle = \pm \nu \lambda_k^{-1} a_k^\pm$ , by direct computations and  $LY = -\nu^2 Y$ , which completes the proof of (3.16).

The following statement is the main part of the proof of Theorem 1.

**Proposition 15.** *For any  $T > T_0$ , there exist  $(\alpha_0, \alpha_1, \dots, \alpha_K) \in \bar{\mathcal{B}}_{\mathbb{R}^{K+1}}$  such that the solution  $\vec{u}$  of (1.1) with data  $\vec{u}(T)$  given by Lemma 11 satisfies  $T_\star = T_0$ .*

In Sections 3.2-3.5, devoted to the proof of Proposition 15, we tacitly use the following direct consequences of (3.7)-(3.12) and Lemma 13

$$\begin{aligned} \lambda_k(t) &\simeq t^{-2}, \quad b_k(t) \simeq t^{-3}, \quad |\lambda'_k(t)| \lesssim t^{-3}, \quad |b'_k(t)| \lesssim t^{-4}, \quad |y'_k(t)| \lesssim t^{-\frac{11}{3}}, \\ |\mathbf{B}(\lambda(t))| &\lesssim t^{-4}, \quad \left| \frac{d}{dt} \mathbf{B}(\lambda(t)) \right| \lesssim t^{-4}. \end{aligned} \tag{3.25}$$

### 3.2. Refined approximate solution

**Lemma 16.** *There exist smooth radially symmetric functions  $Q, S$  satisfying on  $\mathbb{R}^5$ , for all  $\beta \in \mathbb{N}^5$ ,*

$$\begin{aligned} LQ &= \frac{105\pi}{128} f'(W) + \Lambda W, \quad |\partial_x^\beta Q(x)| \lesssim |x|^{-1-|\beta|}, \\ LS &= \underline{\Lambda} \Lambda W, \quad |\partial_x^\beta S(x)| \lesssim |x|^{-1-|\beta|}. \end{aligned}$$

For a proof, see [20, Proposition 2.1]. Note that the explicit constant  $\frac{105\pi}{128}$  is related to the orthogonality condition

$$\left\langle \frac{105\pi}{128} f'(W) + \Lambda W, \Lambda W \right\rangle = 0.$$

In the framework of Proposition 15, we set

$$\begin{aligned} Q_k &:= Q_{\lambda_k}(\cdot - y_k), \quad S_k := S_{\lambda_k}(\cdot - y_k), \\ P &:= \sum_k \chi\left(\frac{\cdot - z_k}{d}\right) (\lambda_k B_k(\boldsymbol{\lambda}) Q_k + b_k^2 S_k), \end{aligned}$$

where  $\chi$  is defined in §1.2. We consider the following refined decomposition of  $u$

$$\phi := W_{\Gamma} + P, \quad h := g - P \quad \text{so that} \quad u = W_{\Gamma} + g = \phi + h.$$

**Lemma 17.** *Under the bootstrap estimates (3.7)-(3.12), it holds*

$$\|g - h\|_{\dot{H}^1} = \|P\|_{\dot{H}^1} \lesssim t^{-5}, \quad (3.26)$$

$$\|\partial_t g - \partial_t h\|_{\dot{H}^1} = \|\partial_t P\|_{\dot{H}^1} \lesssim t^{-6}, \quad (3.27)$$

and

$$\begin{aligned} &\left\| \partial_t \dot{g} - \left\{ \Delta h + f(\phi + h) - f(\phi) + \sum_k (\lambda'_k + b_k) b_k \lambda_k^{-2} \underline{\Lambda}_k \Lambda_k W_k \right. \right. \\ &\quad \left. \left. + \sum_k \lambda_k^{-2} b_k (y'_k \cdot \nabla_k) \Lambda_k W_k - \sum_k (b'_k - B_k(\boldsymbol{\lambda})) \lambda_k^{-1} \Lambda_k W_k \right\} \right\|_{L^2} \lesssim t^{-5}. \end{aligned} \quad (3.28)$$

PROOF. In order to prove (3.26), note first from (3.10) that  $|x - y_k| \geq d$  implies  $\chi((x - z_k)/d) = 0$ , and thus the Chain Rule yields

$$\|\chi((\cdot - z_k)/d) Q_k\|_{\dot{H}^1} \lesssim \|Q_{\lambda_k}\|_{L^2(|x| \leq d)} + \|\nabla Q_{\lambda_k}\|_{L^2(|x| \leq d)}.$$

Using  $\lambda_k \lesssim t^{-2}$ , we have

$$\|Q_{\lambda_k}\|_{L^2(|x| \leq d)} \lesssim \lambda_k \left( \int_0^{d/\lambda_k} r^{-2} r^4 dr \right)^{\frac{1}{2}} \lesssim \lambda_k^{-\frac{1}{2}} \lesssim t \quad (3.29)$$

and

$$\|\nabla Q_{\lambda_k}\|_{L^2(|x| \leq d)} \lesssim \left( \int_0^{d/\lambda_k} r^{-4} r^4 dr \right)^{\frac{1}{2}} \lesssim \lambda_k^{-\frac{1}{2}} \lesssim t. \quad (3.30)$$

Similar estimates involving  $S_{\lambda_k}$  hold. Using also  $|\lambda_k B_k(\boldsymbol{\lambda})| + |b_k|^2 \lesssim t^{-6}$ , we have proved (3.26).

In order to bound  $\partial_t P$ , we write

$$\partial_t (\lambda_k B_k(\boldsymbol{\lambda}) Q_k) = \left( \frac{d}{dt} (\lambda_k B_k(\boldsymbol{\lambda})) \right) Q_k - B_k(\boldsymbol{\lambda}) (y'_k \cdot \nabla_k) Q_k - B_k(\boldsymbol{\lambda}) \lambda'_k \Lambda_k Q_k.$$

Note that the cut-off  $\chi(\frac{\cdot - z_k}{d})$  is independent of  $t$ . For the first term on the right-hand side, the required bound follows from (3.29) and  $|\frac{d}{dt}(\lambda_k B_k(\boldsymbol{\lambda}))| \lesssim t^{-7}$ . For the second term, we use (3.30) (for these terms, we get a stronger bound  $\lesssim t^{-6-\frac{8}{3}}$ ). Finally, the last term is similar to the first one. Terms involving  $S_k$  are bounded similarly.

In view of (3.22), the refined bound (3.28) is equivalent to

$$\left\| \Delta P + f(W_{\Gamma} + P) - \sum_k f(W_k) - \sum_k \left( b_k^2 \lambda_k^{-2} \underline{\Lambda}_k \Lambda_k W_k + B_k(\boldsymbol{\lambda}) \lambda_k^{-1} \Lambda_k W_k \right) \right\|_{L^2} \lesssim t^{-5}.$$

First, consider the complement of the union of the balls  $\mathcal{B}(z_k, d/2)$ . In this region all the terms which do not involve  $P$  are controlled by  $t^{-5}$  in  $L^2$  norm (we call such terms negligible). Indeed, this follows from estimates in (3.25) and

$$\begin{aligned} \|f(W_k)\|_{L^2(|x-z_k| \geq d/2)} &= \lambda_k^{-1} \|f(W)\|_{L^2(|x| \geq d/(2\lambda_k))} \\ &\lesssim \lambda_k^{-1} \left( \int_{d/(2\lambda_k)}^{\infty} r^{-14} r^4 dr \right)^{\frac{1}{2}} \lesssim \lambda_k^{\frac{7}{2}}, \end{aligned} \quad (3.31)$$

$$\|\lambda_k^{-1} \underline{\Lambda}_k \Lambda_k W_k\|_{L^2(|x-z_k| \geq d/2)} = \|\underline{\Lambda} \Lambda W\|_{L^2(|x| \geq d/(2\lambda_k))} \lesssim \lambda_k^{\frac{1}{2}}, \quad (3.32)$$

$$\|\lambda_k^{-1} \Lambda_k W_k\|_{L^2(|x-z_k| \geq d/2)} = \|\Lambda W\|_{L^2(|x| \geq d/(2\lambda_k))} \lesssim \lambda_k^{\frac{1}{2}}. \quad (3.33)$$

Now fix  $k \in \{1, \dots, K\}$  and consider the ball  $\mathcal{B}(z_k, d)$ . We have just seen that in the sum  $\sum_{j=1}^K f(W_j)$  only  $j = k$  is significant. Next, we will prove that

$$\left\| f(W_{\Gamma} + P) - f(W_k) - f'(W_k) \sum_{j \neq k} W_j - f'(W_k) P \right\|_{L^2(\mathcal{B}(z_k, d))} \lesssim t^{-5}. \quad (3.34)$$

Note that in  $\mathcal{B}(z_k, d)$  we have  $W_k \gtrsim t^{-3}$ , whereas for  $j \neq k$  we have  $W_j \lesssim t^{-3}$  and  $|P| \lesssim t^{-3}$ . From (3.18), we have

$$|f(u+v) - f(u) - f'(u)v| \lesssim |u|^{\frac{1}{3}} |v|^2 + |v|^{\frac{7}{3}}.$$

Applying this estimate to  $u = W_k$  and  $v = \sum_{j \neq k} W_j + P$ , so that  $|u|^{\frac{1}{3}} |v|^2 + |v|^{\frac{7}{3}} \lesssim t^{-5}$ , and integrating over the ball  $\mathcal{B}(z_k, d)$  we get (3.34).

Next, we show that for all  $j \neq k$  we have

$$\|f'(W_k) W_j - (15)^{\frac{3}{2}} \lambda_j^{\frac{3}{2}} |z_k - z_j|^{-3} f'(W_k)\|_{L^2(\mathcal{B}(z_k, d))} \lesssim t^{-5}. \quad (3.35)$$

We consider separately  $x \in \mathcal{B}(y_k, \sqrt{\lambda_k})$  and  $x \notin \mathcal{B}(y_k, \sqrt{\lambda_k})$ . In the first case, (3.10) yields  $|x - z_k| \lesssim t^{-1}$ , which implies

$$\left| \frac{W_j(x)}{W_j(z_k)} - 1 \right| \lesssim t^{-1} \quad \text{and so} \quad W_j(x) = (15)^{\frac{3}{2}} \lambda_j^{\frac{3}{2}} |z_k - z_j|^{-3} + O(t^{-4}).$$

Since  $\|f'(W_k)\|_{L^2} \lesssim \lambda_k^{\frac{1}{2}} \lesssim t^{-1}$ , (3.35) is proved for the region  $\mathcal{B}(y_k, \sqrt{\lambda_k})$ .

Consider now the region  $\mathcal{B}(z_k, d) \setminus \mathcal{B}(y_k, \sqrt{\lambda_k})$ . We have

$$\begin{aligned} \|f'(W_k)\|_{L^2(|x-y_k| \geq \sqrt{\lambda_k})} &= \sqrt{\lambda_k} \|f'(W)\|_{L^2(|x| \geq \lambda_k^{-1/2})} \\ &\lesssim t^{-1} \left( \int_{1/\sqrt{\lambda_k}}^{\infty} r^{-8} r^4 dr \right)^{\frac{1}{2}} \lesssim t^{-1} \lambda_k^{\frac{3}{4}} \ll t^{-2}. \end{aligned}$$

Since in  $\mathcal{B}(z_k, d)$  we have  $W_j \lesssim t^{-3}$ , the proof of (3.34) is complete. Recalling the definition of  $B_k(\lambda)$  from (1.9), estimate (3.35) can be rewritten as

$$\left\| f'(W_k) W_j + \frac{105\pi}{128} B_k(\lambda) \lambda_k^{-\frac{1}{2}} f'(W_k) \right\|_{L^2(\mathcal{B}(z_k, d))} \lesssim t^{-5}.$$

Resuming, we have reduced the proof of (3.28) to showing that

$$\begin{aligned} \left\| \Delta P + f'(W_k) P - B_k(\lambda) \left( \frac{105\pi}{128} \lambda_k^{-\frac{1}{2}} f'(W_k) + \lambda_k^{-1} \Lambda_k W_k \right) \right. \\ \left. - b_k^2 \lambda_k^{-2} \underline{\Lambda}_k \Lambda_k W_k \right\|_{L^2(\mathcal{B}(z_k, d))} \lesssim t^{-5}. \end{aligned} \quad (3.36)$$

In the region  $|x - z_k| \leq \frac{d}{2}$ , it holds  $\chi(\frac{x-z_k}{d}) = 1$  and  $\chi(\frac{x-z_j}{d}) = 0$  for  $j \neq k$ ; thus the above expression equals 0 from the definition of  $P$  and Lemma 16. It remains to show that for the cut-off region  $\frac{d}{2} \leq |x - z_k| \leq d$ , this term is indeed negligible. By the estimates (3.31), (3.32), (3.33) dealing with the exterior of the balls  $\mathcal{B}(z_k, d/2)$ , the terms in (3.36) not involving  $P$  are negligible in this region. Thus it sufficient to show that

$$\| |\Delta Q_k| + |\nabla Q_k| + |Q_k| + f'(W_k) |Q_k| \|_{L^2(\frac{d}{2} \leq |x-z_k| \leq d)} \lesssim t$$

(the terms involving  $S_{\lambda_k}$  being bounded analogously). For the four terms above, the inequalities

$$\begin{aligned} t^2 \left( \int_{d/(2\lambda_k)}^{\infty} r^{-6} r^4 dr \right)^{\frac{1}{2}} \lesssim t, \quad \left( \int_0^{d/\lambda_k} r^{-4} r^4 dr \right)^{\frac{1}{2}} \lesssim t, \\ t^{-2} \left( \int_0^{d/\lambda_k} r^{-2} r^4 dr \right)^{\frac{1}{2}} \lesssim t, \quad t^2 \left( \int_{d/(2\lambda_k)}^{\infty} (r^{-4} r^{-1})^2 r^4 dr \right)^{\frac{1}{2}} \ll t, \end{aligned}$$

provide the desired estimate.

### 3.3. Energy estimates

**Lemma 18.** *Let any  $\epsilon > 0$  and  $R > 0$ . There exists a radially symmetric function  $q = q_{\epsilon, R} \in \mathcal{C}^{3,1}(\mathbb{R}^5)$  with the following properties*

- (i)  $q(x) = \frac{1}{2}|x|^2$  for  $|x| \leq R$ .
- (ii) There exists  $\tilde{R}$  (depending on  $\epsilon$  and  $R$ ) such that  $q$  is constant for  $|x| \geq \tilde{R}$ .
- (iii)  $|\nabla q(x)| \lesssim |x|$  and  $|\Delta q(x)| \lesssim 1$  for all  $x \in \mathbb{R}^5$ , with constants independent of  $\epsilon$  and  $R$ .

- (iv)  $\sum_{1 \leq j, l \leq 5} (\partial_{x_j x_l} q(x)) v_j v_l \geq -\epsilon \sum_{j=1}^5 |v_j|^2$ , for all  $x \in \mathbb{R}^5$ ,  $v \in \mathbb{R}^5$ .  
(v)  $\Delta^2 q(x) \leq \epsilon |x|^{-2}$ , for all  $x \in \mathbb{R}^5$ .

Such a function is constructed in Lemma 4.5 of [22] for dimensions  $N \geq 6$ , and the construction for  $N = 5$  follows from arguments in [20] and [22].

Fix a function  $q$  as in Lemma 18 and define the operators

$$\begin{aligned} [A_k h](x) &= \frac{3}{10} \frac{1}{\lambda_k} \Delta q \left( \frac{x - y_k}{\lambda_k} \right) h(x) + \nabla q \left( \frac{x - y_k}{\lambda_k} \right) \cdot \nabla h(x), \\ [\underline{A}_k h](x) &= \frac{1}{2} \frac{1}{\lambda_k} \Delta q \left( \frac{x - y_k}{\lambda_k} \right) h(x) + \nabla q \left( \frac{x - y_k}{\lambda_k} \right) \cdot \nabla h(x). \end{aligned}$$

**Lemma 19.** *For any  $k = 1, \dots, K$ , the operators  $A_k$  and  $\underline{A}_k$  satisfy the following properties.*

(i) *The families*

$$\begin{aligned} &\{A_k; \lambda_k > 0, y_k \in \mathbb{R}^5\}, \{\underline{A}_k; \lambda_k > 0, y_k \in \mathbb{R}^5\}, \\ &\{\lambda_k \partial_{\lambda_k} A_k; \lambda_k > 0, y_k \in \mathbb{R}^5\}, \{\lambda_k \partial_{\lambda_k} \underline{A}_k; \lambda_k > 0, y_k \in \mathbb{R}^5\}, \\ &\{\lambda_k \partial_{y_k} A_k; \lambda_k > 0, y_k \in \mathbb{R}^5\} \text{ and } \{\lambda_k \partial_{y_k} \underline{A}_k; \lambda_k > 0, y_k \in \mathbb{R}^5\} \end{aligned}$$

*are bounded in  $\mathcal{L}(\dot{H}^1, L^2)$ , with norms depending on  $q$ .*

(ii) *For any  $g, h \in \dot{H}^1 \cap \dot{H}^2$ ,*

$$\langle A_k h, f(h+g) - f(h) - f'(h)g \rangle = -\langle A_k g, f(h+g) - f(h) \rangle.$$

(iii) *For any  $\eta > 0$ , choosing  $\epsilon > 0$  small enough in Lemma 18, it holds for all  $g \in \dot{H}^1 \cap \dot{H}^2$ ,*

$$\langle \underline{A}_k g, \Delta g \rangle \leq \frac{\eta}{\lambda_k} \|g\|_{\dot{H}^1}^2 - \frac{1}{\lambda_k} \int_{|x-y_k| < R\lambda_k} |\nabla q(x)|^2 dx.$$

PROOF. (i) Denote

$$Ah = \frac{3}{10} (\Delta q)h + \nabla q \cdot \nabla h, \quad \underline{A}h = \frac{1}{2} (\Delta q)h + \nabla q \cdot \nabla h.$$

Since the functions  $\Delta q$  and  $\nabla q$  have compact supports, it is clear that  $A : \dot{H}^1 \rightarrow L^2$  is a bounded operator. For a function  $h$ , let  $h_k(x) = \lambda_k^{-\frac{3}{2}} h\left(\frac{x-y_k}{\lambda_k}\right)$ . Note that  $(A_k h_k)(x) = \lambda_k^{-\frac{5}{2}} (Ah)\left(\frac{x-y_k}{\lambda_k}\right)$ . Moreover,  $\|A_k h_k\|_{L^2} = \|Ah\|_{L^2}$  and  $\|h_k\|_{\dot{H}^1} = \|h\|_{\dot{H}^1}$ . Thus,  $A_k : \dot{H}^1 \rightarrow L^2$  is a bounded operator with the same norm as  $A$ . The same argument applies to  $\underline{A}_k$  and  $\underline{A}$ .

We compute

$$\begin{aligned} \lambda_k \partial_{\lambda_k} \underline{A}_k &= -\frac{1}{2\lambda_k} \Delta q \left( \frac{x - y_k}{\lambda_k} \right) - \frac{1}{2\lambda_k} \frac{x - y_k}{\lambda_k} \cdot \nabla \Delta q \left( \frac{x - y_k}{\lambda_k} \right) \\ &\quad - \frac{x - y_k}{\lambda_k} \cdot \nabla^2 q \left( \frac{x - y_k}{\lambda_k} \right) \cdot \nabla, \end{aligned}$$

and

$$\lambda_k \partial_{y_k} \underline{A}_k = -\frac{1}{2\lambda_k} \nabla \Delta q \left( \frac{x - y_k}{\lambda_k} \right) - \nabla^2 q \left( \frac{x - y_k}{\lambda_k} \right) \cdot \nabla.$$

Thus, the same arguments provide the desired results.

(ii) The relation  $\langle Ah, f(h+g) - f(h) - f'(h)g \rangle = -\langle Ag, f(h+g) - f(h) \rangle$  is proved in [23, Lemma 3.12], and the relation for  $A_k$  follows immediately by change of variable.

(iii) The estimate is proved for  $\underline{A}$  in [23, Lemma 3.12] and follows for  $\underline{A}_k$  by change of variable.

We establish energy estimates for the pair  $(h, \dot{g})$ . We define

$$\mathcal{I} := \int_{\mathbb{R}^5} \left\{ \frac{1}{2} (\dot{g})^2 + \frac{1}{2} |\nabla h|^2 - (F(\phi+h) - F(\phi) - f(\phi)h) \right\} dx,$$

and

$$\mathcal{J}_k := -b_k \langle \dot{g}, \underline{A}_k h \rangle.$$

Set

$$\mathcal{H} := \mathcal{I} + \sum_k \mathcal{J}_k.$$

**Lemma 20.** *For any  $\delta > 0$ , choosing  $\epsilon > 0$  small enough in Lemma 19, it holds*

$$\mathcal{H}' \geq -\delta t^{-\frac{25}{3}}. \quad (3.37)$$

PROOF. In this proof, the sign “ $\simeq$ ” means that equality holds up to error terms of order  $t^{-\frac{26}{3}}$ . We call such error terms “negligible”.

We start by computing  $\mathcal{I}'$ . We have by integration by parts,

$$\mathcal{I}' = \langle \partial_t \dot{g}, \dot{g} \rangle - \langle \partial_t h, \Delta h + f(\phi+h) - f(\phi) \rangle - \langle \partial_t \phi, f(\phi+h) - f(\phi) - f'(\phi)h \rangle. \quad (3.38)$$

By (3.28) in Lemma 17, the third orthogonality condition in (3.3) and (3.7), we have

$$\begin{aligned} \langle \partial_t \dot{g}, \dot{g} \rangle &\simeq \langle \dot{g}, \Delta h + f(\phi+h) - f(\phi) \rangle \\ &+ \sum_k (\lambda'_k + b_k) b_k \lambda_k^{-2} \langle \underline{\Delta}_k \Lambda_k W_k, \dot{g} \rangle + \sum_k b_k \lambda_k^{-2} \langle (y'_k \cdot \nabla_k) \Lambda_k W_k, \dot{g} \rangle. \end{aligned}$$

Moreover, by (3.26)-(3.27) in Lemma 17 and (3.7),

$$\langle \partial_t h, \Delta h + f(\phi+h) - f(\phi) \rangle \simeq \langle \partial_t g, \Delta h + f(\phi+h) - f(\phi) \rangle.$$

Now, we claim that

$$\langle \partial_t h, \Delta h + f(\phi+h) - f(\phi) \rangle \simeq \langle \dot{g}, \Delta h + f(\phi+h) - f(\phi) \rangle. \quad (3.39)$$

Note that (3.21) and (3.13)-(3.14) imply

$$\|\partial_t g - \dot{g}\|_{\dot{H}^1} \lesssim t^{-\frac{5}{3}}. \quad (3.40)$$

Since

$$\|f(\phi+h) - f(\phi) - f'(\phi)h\|_{\dot{H}^{-1}} \lesssim \|f(\phi+h) - f(\phi) - f'(\phi)h\|_{L^{\frac{10}{7}}} \lesssim \|h\|_{\dot{H}^1}^2 \lesssim t^{-\frac{22}{3}} \quad (3.41)$$

(the last bound follows from (3.7) and (3.26)), we have

$$\begin{aligned} & \langle \partial_t g, \Delta h + f(\phi+h) - f(\phi) \rangle \\ &= \langle \dot{g}, \Delta h + f(\phi+h) - f(\phi) \rangle + \langle \partial_t g - \dot{g}, \Delta h + f(\phi+h) - f(\phi) \rangle \\ &\simeq \langle \dot{g}, \Delta h + f(\phi+h) - f(\phi) \rangle + \langle \partial_t g - \dot{g}, \Delta h + f'(\phi)h \rangle. \end{aligned}$$

Now, we check that the last term is negligible. Fix  $k \in \{1, \dots, K\}$ . Using (3.21), then (3.13)-(3.14) and the cancellations  $L\Lambda W = 0$ ,  $L\nabla W = 0$ , it is sufficient to prove that

$$\begin{aligned} & |\langle \Lambda_k W_k, \Delta h + f'(\phi)h \rangle| = |\langle \Lambda_k W_k, (f'(\phi) - f'(W_k))h \rangle| \lesssim t^{-7}, \\ & |\langle \nabla_k W_k, \Delta h + f'(\phi)h \rangle| = |\langle \nabla_k W_k, (f'(\phi) - f'(W_k))h \rangle| \lesssim t^{-7}. \end{aligned}$$

Both inequalities will follow from

$$\int_{\mathbb{R}^5} W_k |f'(\phi) - f'(W_k)| |h| dx \lesssim t^{-7}. \quad (3.42)$$

In the exterior of all the balls  $\mathcal{B}(z_j, d)$  we have

$$W_k |f'(\phi) - f'(W_k)| \lesssim \sum_j f(W_j)$$

and

$$\|f(W_j)\|_{L^{\frac{10}{7}}(|x-z_j| \geq d)} \lesssim \left( \int_{d/(2\lambda_j)}^{\infty} r^{-10} r^4 dr \right)^{\frac{7}{10}} \lesssim \lambda_j^{\frac{7}{2}} \lesssim t^{-7},$$

which yields an estimate better than (3.42) for this region. In the ball  $\mathcal{B}(z_j, d)$  for  $j \neq k$  we have

$$|f'(\phi) - f'(W_k)| \lesssim f'(W_j) + f'(P).$$

Note that  $\|f'(W_j)\|_{L^2} = \sqrt{\lambda_j} \|f'(W)\|_{L^2} \lesssim t^{-1}$ . Also, since  $\|P\|_{L^\infty} \lesssim t^{-3}$ , we obtain

$$\|f'(\phi) - f'(W_k)\|_{L^2(\mathcal{B}(z_j, d))} \lesssim t^{-1},$$

hence Hölder inequality yields

$$\begin{aligned} \int_{\mathcal{B}(z_j, d)} W_k |f'(\phi) - f'(W_k)| |h| dx &\lesssim t^{-1} \|W_k\|_{L^5(\mathcal{B}(z_j, d))} \|h\|_{L^{\frac{10}{3}}(\mathcal{B}(z_j, d))} \\ &\lesssim t^{-4 - \frac{11}{3}} \ll t^{-7}, \end{aligned}$$

which proves (3.42) in the ball  $\mathcal{B}(z_j, d)$ . In  $\mathcal{B}(z_k, d)$  we write

$$|f'(\phi) - f'(W_k)| \lesssim |f''(W_k)| \left( |P| + \sum_{j \neq k} W_j \right) + f'(P) + \sum_{j \neq k} f'(W_j),$$

so that in particular

$$W_k |f'(\phi) - f'(W_k)| \lesssim t^{-3} (W_k + f'(W_k)).$$

We have  $\|f'(W_k)\|_{L^{\frac{10}{7}}} = \lambda_k^{\frac{3}{2}} \|f'(W)\|_{L^{\frac{10}{7}}} \lesssim t^{-3}$  and

$$\|W_k\|_{L^{\frac{10}{7}}(\mathcal{B}(z_k, d))} \lesssim \lambda_k^2 \left( \int_0^{2d/\lambda_k} r^{-\frac{30}{7}} r^4 dr \right)^{\frac{7}{10}} \lesssim t^{-3},$$

hence we obtain by Hölder inequality

$$\int_{\mathcal{B}(z_k, d)} W_k |f'(\phi) - f'(W_k)| |h| dx \lesssim t^{-6} t^{-\frac{11}{3}} \ll t^{-7}.$$

This finishes the proof of (3.42), which means we have proved (3.39).

Next, we consider the last term in (3.38). Using the equality  $\partial_t \phi = \partial_t u - \partial_t h = \sum_k b_k \lambda_k^{-1} \Lambda_k W_k + \dot{g} - \partial_t h$ , estimates (3.27) and (3.40) implies that

$$\left\| \partial_t \phi - \sum_k b_k \lambda_k^{-1} \Lambda_k W_k \right\|_{\dot{H}^1} \lesssim t^{-\frac{5}{3}}.$$

Thus, using also (3.41),

$$\langle \partial_t \phi, f(\phi + h) - f(\phi) - f'(\phi)h \rangle \simeq \sum_k b_k \lambda_k^{-1} \langle \Lambda_k W_k, f(\phi + h) - f(\phi) - f'(\phi)h \rangle.$$

We conclude that

$$\begin{aligned} \mathcal{I}' &\simeq \sum_k (\lambda'_k + b_k) b_k \lambda_k^{-2} \langle \underline{\Lambda}_k \Lambda_k W_k, \dot{g} \rangle + \sum_k b_k \lambda_k^{-2} \langle (y'_k \cdot \nabla_k) \Lambda_k W_k, \dot{g} \rangle \\ &\quad - \sum_k b_k \lambda_k^{-1} \langle \Lambda_k W_k, f(\phi + h) - f(\phi) - f'(\phi)h \rangle. \end{aligned} \quad (3.43)$$

These remaining terms can only be estimated by  $Ct^{-\frac{25}{3}}$ , which is the critical size for the energy method. Thus, they have to be cancelled by similar terms coming from the virial correction  $\mathcal{J}$ , see below (3.51). The original idea of such a virial correction in a blow-up context is due to [39] for the mass critical nonlinear Schrödinger equation, and was extended to the energy-critical wave and Schrödinger equations in [20, 22]. The presentation here follows closely the one in [20, 22, 23].

Let  $\eta > 0$  arbitrarily small. We compute  $\mathcal{J}'_k$  from its definition

$$\begin{aligned} \mathcal{J}'_k &= -b'_k \langle \dot{g}, \underline{A}_k h \rangle - b_k \lambda'_k \langle \dot{g}, (\partial_{\lambda_k} \underline{A}_k) h \rangle - b_k \langle \dot{g}, y'_k \cdot (\partial_{y_k} \underline{A}_k) h \rangle \\ &\quad - b_k \langle \dot{g}, \underline{A}_k \partial_t h \rangle - b_k \langle \partial_t \dot{g}, \underline{A}_k h \rangle. \end{aligned} \quad (3.44)$$

First, by (i) of Lemma 19, (3.7), (3.25) and (3.26), we have

$$\begin{aligned} |b'_k \langle \dot{g}, \underline{A}_k h \rangle| + |b_k \lambda'_k \langle \dot{g}, (\partial_{\lambda_k} \underline{A}_k) h \rangle| + |b_k y'_k \langle \dot{g}, (\partial_{y_k} \underline{A}_k) h \rangle| \\ \lesssim t^{-4} \|\dot{g}\|_{L^2} \|h\|_{\dot{H}^1} \lesssim t^{-\frac{28}{3}}. \end{aligned}$$



Next, by (i) of Lemma 19, (3.7) and (3.27), we have

$$|b_k \langle \dot{g}, \underline{A}_k(\partial_t h - \partial_t g) \rangle| \lesssim t^{-3} \|\dot{g}\|_{L^2} \|\partial_t h - \partial_t g\|_{\dot{H}^1} \lesssim t^{-\frac{38}{3}},$$

which implies  $b_k \langle \dot{g}, \underline{A}_k \partial_t h \rangle \simeq b_k \langle \dot{g}, \underline{A}_k \partial_t g \rangle$ . Using (3.21) and  $\langle \dot{g}, \underline{A}_k \dot{g} \rangle = 0$  (by integration by parts), we have

$$\langle \dot{g}, \underline{A}_k \partial_t g \rangle = \sum_j \left\{ \lambda_j^{-1} (\lambda'_j + b_j) \langle \dot{g}, \underline{A}_k \Lambda_j W_j \rangle + \lambda_j^{-1} \langle \dot{g}, \underline{A}_k (y'_j \cdot \nabla_j W_j) \rangle \right\}.$$

We first consider  $j = k$  in the above sum. We claim that for  $R$  large enough in the choice of  $q$  in Lemma 19, it holds

$$\|\underline{A}_k \Lambda_k W_k - \lambda_k^{-1} \underline{A}_k \Lambda_k W_k\|_{L^2} + \|\underline{A}_k \nabla_k W_k - \lambda_k^{-1} \underline{A}_k \nabla_k W_k\|_{L^2} \leq \eta. \quad (3.45)$$

Indeed, for  $|x| \leq R$ , we have  $\underline{A} \Lambda W(x) = \underline{A} \Lambda W(x)$ , and for  $|x| \geq R$ , using (iii) of Lemma 19 and the decay of  $W$ , we have  $|\underline{A} \Lambda W(x)| + |\underline{A} \nabla W(x)| \leq C|x|^{-3}$ . Thus,  $\|\underline{A} \Lambda W - \underline{A} \Lambda W\|_{L^2} \leq CR^{-\frac{1}{2}}$ , and estimate (3.45) for  $\Lambda_k W_k$  follows by change of variable. The estimate on  $\nabla_k W_k$  is proved similarly. For  $j \neq k$ , one checks that  $\|\underline{A}_k \Lambda_j W_j\|_{L^2} + \|\underline{A}_k \nabla_j W_j\|_{L^2} \lesssim t^{-1}$ .

Using also  $\underline{A} \nabla = \nabla \Lambda$ , it follows from what precedes and  $|\mathbf{b}| \|\dot{g}\|_{L^2} \lesssim t^{-\frac{25}{3}}$  that

$$\begin{aligned} & \left| b_k \langle \dot{g}, \underline{A}_k \partial_t h \rangle - \{ b_k \lambda_k^{-2} (\lambda'_k + b_k) \langle \dot{g}, \underline{A}_k \Lambda_k W_k \rangle \right. \\ & \quad \left. + b_k \lambda_k^{-2} \langle \dot{g}, (y'_k \cdot \nabla_k \Lambda_k W_k) \rangle \right\} \leq C \eta t^{-\frac{25}{3}}. \end{aligned}$$

Finally, we use (3.28),  $\underline{A} = \frac{\Delta q}{5} + A$  and (i)-(iii) of Lemma 19 to estimate the last term in (3.44) as follows

$$\begin{aligned} -b_k \langle \partial_t \dot{g}, \underline{A}_k h \rangle & \geq -\eta b_k \lambda_k^{-1} \|h\|_{\dot{H}^1}^2 \\ & + b_k \lambda_k^{-1} \left\{ \int_{|x-y_k| < R\lambda_k} |\nabla h(x)|^2 dx - \frac{1}{5} \left\langle \Delta q \left( \frac{\cdot - y_k}{\lambda_k} \right) h, f(\phi + h) - f(\phi) \right\rangle \right\} \\ & + b_k \langle \underline{A}_k \phi, f(\phi + h) - f(\phi) - f'(\phi)h \rangle \\ & - b_k \sum_{j=1}^K (\lambda'_j + b_j) b_j \lambda_j^{-2} \langle \underline{A}_j \Lambda_j W_j, \underline{A}_k h \rangle - b_k \sum_{j=1}^K b_j \lambda_j^{-2} y'_j \langle \nabla_j \Lambda_j W_j, \underline{A}_k h \rangle \\ & + b_k \sum_{j=1}^K (b'_j - B_j(\boldsymbol{\lambda})) \lambda_j^{-1} \langle \Lambda_j W_j, \underline{A}_k h \rangle - Ct^{-\frac{35}{3}}. \end{aligned} \quad (3.46)$$

The first line of (3.46) is lower bounded by  $-C\eta t^{-\frac{25}{3}}$ . For the second line, we first observe that since  $|\Delta q(x)| \lesssim 1$  for all  $x \in \mathbb{R}^5$ , using also (3.41), we have

$$\left| \left\langle \Delta q \left( \frac{\cdot - y_k}{\lambda_k} \right) h, f(\phi + h) - f(\phi) \right\rangle - \left\langle \Delta q \left( \frac{\cdot - y_k}{\lambda_k} \right) h, f'(\phi)h \right\rangle \right| \lesssim t^{-11}. \quad (3.47)$$

We claim that

$$\int_{|x-y_k| < \lambda_k \bar{R}} |f'(\phi) - f'(W_k)| h^2 \lesssim t^{-\frac{31}{3}}, \quad \int_{|x-y_k| > \lambda_k \bar{R}} |f'(W_k)| h^2 \lesssim R^{-2} t^{-\frac{22}{3}}. \quad (3.48)$$

Indeed, by Holder and Sobolev inequalities, and then Taylor expansion

$$\begin{aligned}
& \int_{|x-y_k| < \lambda_k \tilde{R}} |f'(\phi) - f'(W_k)| h^2 \\
& \lesssim \|h\|_{\dot{H}^1}^2 \|f'(\phi) - f'(W_k)\|_{L^{\frac{10}{3}}(|x-y_k| < \lambda_k \tilde{R})} \\
& \lesssim \|h\|_{\dot{H}^1}^2 \sum_{j \neq k} \left( \|W_j\|_{L^{\frac{10}{3}}(|x-y_k| < \lambda_k \tilde{R})}^{\frac{4}{3}} + \|W_j W_k^{\frac{1}{3}}\|_{L^{\frac{5}{2}}(|x-y_k| < \lambda_k \tilde{R})} \right) \\
& \lesssim \|h\|_{\dot{H}^1}^2 \sum_{j \neq k} \|W_j\|_{L^{\frac{10}{3}}(|x-y_k| < \lambda_k \tilde{R})} \lesssim \|h\|_{\dot{H}^1}^2 |\lambda|^{\frac{3}{2}} \lesssim t^{-\frac{31}{3}}.
\end{aligned}$$

Similar estimates give  $\int_{|x-y_k| > \lambda_k R} |f'(W_k)| h^2 \lesssim \|h\|_{\dot{H}^1}^2 R^{-2} \lesssim R^{-2} t^{-\frac{22}{3}}$  and thus (3.48) is proved.

From the definition of  $q$  in Lemma 19,  $\Delta q(x) = 5$  for  $|x| \leq R$  and  $\Delta q(x) = 0$  for  $|x| \geq \tilde{R}$ . Thus, (3.47) and (3.48) imply that

$$\left| \frac{1}{5} \left\langle \Delta q \left( \frac{\cdot - y_k}{\lambda_k} \right) h, f(\phi + h) - f(\phi) \right\rangle - \int_{|x-y_k| < \lambda_k R} f'(W_k(x)) h^2(x) dx \right| \lesssim R^{-2} t^{-\frac{25}{3}}.$$

Therefore, up to negligible terms, the second line of (3.46) is estimated by

$$b_k \lambda_k^{-1} \int_{|x-y_k| < R \lambda_k} \{ |\nabla h(x)|^2 - f'(W_k(x)) h^2(x) \} dx - C R^{-2} t^{-\frac{25}{3}}.$$

Using (3.26), (3.11)-(3.12) and the definitions of  $Z_k^\pm$ , it holds

$$|\langle \lambda_k^{-2} Y_k, h \rangle|^2 \lesssim \|h - g\|_{\dot{H}^1}^2 + |\langle \lambda_k^{-2} Y_k, g \rangle|^2 \lesssim t^{-10} + (a_k^-)^2 + (a_k^+)^2 \lesssim t^{-8}. \quad (3.49)$$

Thus, applying Lemma 9 to  $h$ , with  $R$  large enough, we have the lower bound

$$\int_{|x-y_k| < R \lambda_k} \{ |\nabla h(x)|^2 - f'(W_k(x)) h^2(x) \} dx \geq -\eta \|\nabla h\|_{L^2}^2 - C t^{-8} \geq -2\eta t^{-\frac{22}{3}}.$$

Next, we claim that

$$\|\lambda_k \underline{A}_k \phi - \underline{\Lambda}_k W_k\|_{L^{\frac{10}{3}}} \lesssim t^{-1} + R^{-2}, \quad (3.50)$$

which, combined with (3.41), implies that the third term in the right-hand side of (3.46) is equal to

$$b_k \lambda_k^{-1} \langle \underline{\Lambda}_k W_k, f(\phi + h) - f(\phi) - f'(\phi) h \rangle$$

up to negligible terms. To prove (3.50), we just observe that since  $\underline{A}_k W_k = \lambda_k^{-1} \underline{\Lambda}_k W_k$  for  $|x - y_k| \leq R \lambda_k$  and  $\underline{A}_k \phi = \underline{A}_k W_k = 0$  for  $|x - y_k| \geq \tilde{R} \lambda_k$ , it holds

$$\begin{aligned}
\|\lambda_k \underline{A}_k \phi - \underline{\Lambda}_k W_k\|_{L^{\frac{10}{3}}} & \leq \|\lambda_k \underline{A}_k (\phi - W_k)\|_{L^{\frac{10}{3}}(|x-y_k| < \tilde{R} \lambda_k)} \\
& \quad + \|\lambda_k \underline{A}_k W_k - \underline{\Lambda}_k W_k\|_{L^{\frac{10}{3}}(|x-y_k| > R \lambda_k)} \\
& \lesssim t^{-1} + R^{-2}.
\end{aligned}$$

Finally, we claim that the last three terms of (3.46) are negligible. Indeed, this is a consequence of Lemma 13, (i) of Lemma 19 and the bound  $\|h\|_{\dot{H}^1} \lesssim t^{-\frac{11}{3}}$ .

We conclude that

$$\begin{aligned} \mathcal{J}'_k &\leq -(\lambda'_k + b_k)b_k\lambda_k^{-2}\langle \underline{\Lambda}_k \Lambda_k W_k, \dot{g} \rangle - b_k\lambda_k^{-2}\langle (y'_k \cdot \nabla_k) \Lambda_k W_k, \dot{g} \rangle \\ &\quad + b_k\lambda_k^{-1}\langle \Lambda_k W_k, f(\phi + h) - f(\phi) - f'(\phi)h \rangle - C(\eta + R^{-2})t^{-\frac{25}{3}}. \end{aligned} \quad (3.51)$$

Combining (3.43) and (3.51), we obtain, with  $\delta > 0$  arbitrarily small and under the bootstrap assumptions, that  $\mathcal{H}' \geq -\delta t^{-\frac{25}{3}}$ , which is (3.37).

### 3.4. Control of the scaling parameters

In this subsection, we prove that for all  $t \in [T_*, T]$ ,

$$|\boldsymbol{\lambda} - ct^{-2}| \leq \frac{1}{2}t^{-\frac{7}{3}}, \quad (3.52)$$

$$|\mathbf{b} - 2ct^{-3}| \leq \frac{1}{2}t^{-\frac{10}{3}}. \quad (3.53)$$

The argument is one of the original aspects of this article compared to previous works on multi-solitons. Equations (3.13) and (3.15) are necessary but not sufficient to estimate  $\boldsymbol{\lambda}$  and  $\mathbf{b}$ . Indeed, to control a one-dimensional instability related to  $|\boldsymbol{\lambda}|$ , we need to use specific approximate Lyapunov functionals  $\mathcal{F}$  and  $\mathcal{G}$  and the following bound on  $|\boldsymbol{\lambda}|$  from (3.12)

$$||\boldsymbol{\lambda}| - |c|t^{-2}| \leq t^{-\frac{12}{5}}, \quad \text{for all } t \in [T_*, T]. \quad (3.54)$$

Recall that (3.12) gathers all terms for which a topological argument is required (see next subsection).

PROOF (PROOF OF (3.52)-(3.53)). For  $t \in [T_*, T]$  denote  $r := |\boldsymbol{\lambda}|$ , and define  $\boldsymbol{\theta} \in \mathbb{S}_+^{K-1}$ ,  $\rho \in \mathbb{R}$  and  $\mathbf{b}^\perp \in \mathbb{R}^K$  by the relations

$$\boldsymbol{\lambda} = r\boldsymbol{\theta}, \quad \mathbf{b} = \rho\boldsymbol{\theta} + \mathbf{b}^\perp, \quad \mathbf{b}^\perp \perp \boldsymbol{\theta}.$$

Note that from (3.5),  $\mathbf{b}^\perp(T) = 0$  and  $\boldsymbol{\theta}(T) = \mathbf{c}/|\mathbf{c}|$ . We will prove that for all  $t \in [T_*, T]$

$$|\mathbf{b}^\perp| \leq t^{-\frac{31}{9}}, \quad (3.55)$$

$$|\boldsymbol{\theta} - \mathbf{c}/|\mathbf{c}|| \leq t^{-\frac{4}{9}}. \quad (3.56)$$

Projecting (3.13) first on  $\boldsymbol{\theta}$ , and then on its orthogonal complement, we obtain, for all  $t \in [T_*, T]$ ,

$$|r' + \rho| \lesssim t^{-\frac{11}{3}}, \quad (3.57)$$

$$|\boldsymbol{\theta}' + \mathbf{b}^\perp/r| \lesssim t^{-\frac{5}{3}}. \quad (3.58)$$

From (3.15) and (i) of Lemma 3 we get

$$(\mathbf{b}^\perp)' = \mathbf{b}' - (\rho\boldsymbol{\theta})' = r^2\nabla V(\boldsymbol{\theta}) - \rho'\boldsymbol{\theta} - \rho\boldsymbol{\theta}' + O(t^{-\frac{14}{3}}). \quad (3.59)$$

Consider the following quantity

$$\mathcal{F} := \frac{1}{2}r^{-3}|\mathbf{b}^\perp|^2 + V(\boldsymbol{\theta}).$$

Let

$$\tau := \inf\{t \in [T_*, T] : (3.55) \text{ and } (3.56) \text{ hold on } [t, T]\}$$

and suppose that  $\tau > T_*$ . We check that for all  $t \in [\tau, T]$  we have

$$\mathcal{F}' \gtrsim -t^{-\frac{19}{9}}. \quad (3.60)$$

Indeed, we have

$$\mathcal{F}' = -\frac{3}{2}r'r^{-4}|\mathbf{b}^\perp|^2 + r^{-3}(\mathbf{b}^\perp)' \cdot \mathbf{b}^\perp + \boldsymbol{\theta}' \cdot \nabla V(\boldsymbol{\theta}). \quad (3.61)$$

From (3.55), (3.57) and (3.54) we obtain

$$\left| r'r^{-4}|\mathbf{b}^\perp|^2 + r^{-4}\rho|\mathbf{b}^\perp|^2 \right| \lesssim t^{-\frac{11}{3}+8-\frac{62}{9}} = t^{-\frac{23}{9}} \ll t^{-\frac{19}{9}}. \quad (3.62)$$

From (3.55), (3.54) and (3.59) we obtain

$$\left| r^{-3}(\mathbf{b}^\perp)' \cdot \mathbf{b}^\perp - r^{-3}(r^2\nabla V(\boldsymbol{\theta}) - \rho'\boldsymbol{\theta} - \rho\boldsymbol{\theta}') \cdot \mathbf{b}^\perp \right| \lesssim t^{6-\frac{14}{3}-\frac{31}{9}} = t^{-\frac{19}{9}}. \quad (3.63)$$

Using  $\boldsymbol{\theta} \cdot \mathbf{b}^\perp = 0$  and

$$\left| r^{-3}\rho\boldsymbol{\theta}' \cdot \mathbf{b}^\perp + r^{-4}\rho|\mathbf{b}^\perp|^2 \right| \leq r^{-3}\rho|\mathbf{b}^\perp| |\boldsymbol{\theta}' + \mathbf{b}^\perp/r| \lesssim t^{6-3-\frac{31}{9}-\frac{5}{3}} = t^{-\frac{19}{9}},$$

this yields

$$\left| r^{-3}(\mathbf{b}^\perp)' \cdot \mathbf{b}^\perp - (r^{-1}\nabla V(\boldsymbol{\theta}) \cdot \mathbf{b}^\perp + r^{-4}\rho|\mathbf{b}^\perp|^2) \right| \lesssim t^{-\frac{19}{9}}.$$

Since  $\mathbf{c}/|\mathbf{c}|^{-1}$  is a critical point of  $V|_{\mathbb{S}_+^{K-1}}$  and  $V$  is smooth in its neighborhood (see Lemma 3), (3.56) implies that the component of  $\nabla V(\boldsymbol{\theta})$  orthogonal to  $\boldsymbol{\theta}$  is  $O(t^{-\frac{4}{3}})$ . Thus (3.58) yields

$$\left| \boldsymbol{\theta}' \cdot \nabla V(\boldsymbol{\theta}) - (-\mathbf{b}^\perp/r) \cdot \nabla V(\boldsymbol{\theta}) \right| = \left| (\boldsymbol{\theta}' + \mathbf{b}^\perp/r) \cdot \nabla V(\boldsymbol{\theta}) \right| \lesssim t^{-\frac{4}{3}-\frac{5}{3}} = t^{-\frac{19}{9}}. \quad (3.64)$$

Formula (3.61) and the bounds (3.62), (3.63) and (3.64) yield

$$\mathcal{F}' = \frac{3}{2}r^{-4}\rho|\mathbf{b}^\perp|^2 + r^{-1}\nabla V(\boldsymbol{\theta}) \cdot \mathbf{b}^\perp + r^{-4}\rho|\mathbf{b}^\perp|^2 - r^{-1}\nabla V(\boldsymbol{\theta}) \cdot \mathbf{b}^\perp + O(t^{-\frac{19}{9}}),$$

which proves (3.60), because  $\rho > 0$ .

Integrating (3.60) between  $t \in [\tau, T]$  and  $T$  yields

$$\frac{1}{2}r^{-3}|\mathbf{b}^\perp|^2 + V(\boldsymbol{\theta}) - V(\mathbf{c}/|\mathbf{c}|) \lesssim t^{-\frac{10}{9}}.$$

Since  $V$  attains its global minimum at  $\mathbf{c}/|\mathbf{c}|$ , this implies

$$r^{-3}|\mathbf{b}^\perp|^2 \lesssim t^{-\frac{10}{9}} \quad \text{and so} \quad |\mathbf{b}^\perp| \lesssim t^{-3-\frac{5}{9}} = t^{-\frac{32}{9}}. \quad (3.65)$$

Thus (3.58) implies  $|\boldsymbol{\theta}'| \lesssim t^{-\frac{14}{9}}$ ; in particular, using  $\theta(T) = \frac{\mathbf{c}}{|\mathbf{c}|}$ , we obtain the following improved bound on  $[\tau, T]$

$$|\boldsymbol{\theta} - \mathbf{c}/|\mathbf{c}|| \lesssim t^{-\frac{5}{9}}. \quad (3.66)$$

Bounds (3.65) and (3.66) show that (3.55) and (3.56) cannot break down at  $t = \tau$ , thus proving that (3.55) and (3.56) indeed hold on  $[T_*, T]$ .

By the triangle inequality, (3.54) and (3.56) we have

$$|\boldsymbol{\lambda} - \mathbf{c}t^{-2}| = |r\boldsymbol{\theta} - \mathbf{c}t^{-2}| \leq |r - |\mathbf{c}|t^{-2}| + |\mathbf{c}|t^{-2}|\boldsymbol{\theta} - \mathbf{c}/|\mathbf{c}|| \lesssim t^{-\frac{12}{9}} + t^{-2}t^{-\frac{4}{9}} \ll t^{-\frac{7}{3}},$$

which proves (3.52).

Now, we analyse the evolution of  $(r, \rho)$ . For  $\boldsymbol{\theta} \in \mathbb{S}^{K-1}$  set

$$n(\boldsymbol{\theta}) := -\boldsymbol{\theta} \cdot \nabla V(\boldsymbol{\theta}).$$

Taking the inner product of (3.59) with  $\boldsymbol{\theta}$  gives

$$\rho' = -r^2 n(\boldsymbol{\theta}) - (\mathbf{b}^\perp)' \cdot \boldsymbol{\theta} + O(t^{-\frac{14}{3}}).$$

Note that (3.58) and (3.55) imply in particular  $|\boldsymbol{\theta}'| \lesssim t^{-\frac{31}{9}+2} = t^{-\frac{13}{9}}$ . Since  $\mathbf{b}^\perp \cdot \boldsymbol{\theta} = 0$  for all  $t$ , we have

$$|(\mathbf{b}^\perp)' \cdot \boldsymbol{\theta}| = |\mathbf{b}^\perp \cdot \boldsymbol{\theta}'| \leq |\mathbf{b}^\perp| |\boldsymbol{\theta}'| \lesssim t^{-\frac{31}{9}-\frac{13}{9}} \ll t^{-\frac{14}{3}},$$

and thus

$$\rho' = -r^2 n(\boldsymbol{\theta}) + O(t^{-\frac{14}{3}}).$$

Since  $n(\boldsymbol{\theta})$  is smooth in a neighborhood of  $\boldsymbol{\theta} = \mathbf{c}/|\mathbf{c}|$  (see Lemma 3), (3.56) yields the estimate  $|n(\boldsymbol{\theta}) - n(\mathbf{c}/|\mathbf{c}|)| \lesssim t^{-\frac{4}{9}}$ . Thus

$$\rho' = -r^2 n(\mathbf{c}/|\mathbf{c}|) + O(t^{-\frac{40}{9}}). \quad (3.67)$$

Consider

$$\mathcal{G} := \frac{1}{2}\rho^2 - 2|\mathbf{c}|^{-1}r^3.$$

Using  $r \lesssim t^{-2}$ ,  $\rho \lesssim t^{-3}$ , (3.57), (3.67) and the fact that  $|\mathbf{c}| = 6(n(\mathbf{c}/|\mathbf{c}|))^{-1}$  (see (1.12)), we compute

$$\mathcal{G}' = -r^2 \rho n(\mathbf{c}/|\mathbf{c}|) + 6|\mathbf{c}|^{-1}r^2 \rho + O(t^{-\frac{40}{9}}t^{-3} + t^{-4}t^{-\frac{11}{3}}) = O(t^{-\frac{67}{9}}).$$

Since  $\mathcal{G}(T) = 0$  by (3.5), we obtain by integration on  $[t, T]$ ,

$$(\rho - 2|\mathbf{c}|^{-\frac{1}{2}}r^{\frac{3}{2}})(\rho + 2|\mathbf{c}|^{-\frac{1}{2}}r^{\frac{3}{2}}) = 2\mathcal{G} \lesssim t^{-\frac{58}{9}};$$

thus (3.54) yields

$$|\rho - 2|\mathbf{c}|^{-\frac{1}{2}}r^{\frac{3}{2}}| \lesssim t^{-\frac{31}{9}}, \quad (3.68)$$

and last (3.57) implies

$$|r' + 2|\mathbf{c}|^{-\frac{1}{2}}r^{\frac{3}{2}}| \lesssim t^{-\frac{31}{9}}. \quad (3.69)$$

The bound (3.68) also implies, again using (3.54),

$$|\rho - 2|\mathbf{c}|t^{-3}| \lesssim t^{-\frac{31}{9}} + t^{-3}|(t^2r)^{\frac{3}{2}} - |\mathbf{c}|^{\frac{3}{2}}| \lesssim t^{-\frac{31}{9}} + t^{-3-\frac{2}{5}} \lesssim t^{-\frac{17}{5}}.$$

By the triangle inequality and previous estimates, we have

$$\begin{aligned} |\mathbf{b} - 2\mathbf{c}t^{-3}| &= |\rho\boldsymbol{\theta} + \mathbf{b}^\perp - 2\mathbf{c}t^{-3}| \leq |\mathbf{b}^\perp| + |\rho - 2|\mathbf{c}|t^{-3}| + 2|\mathbf{c}|t^{-3}|\boldsymbol{\theta} - \mathbf{c}/|\mathbf{c}|| \\ &\lesssim t^{-\frac{31}{9}} + t^{-\frac{17}{5}} + t^{-\frac{31}{9}} \ll t^{-\frac{10}{3}}, \end{aligned}$$

which proves (3.53).

### 3.5. Closing the bootstrap argument

Now, we prove that for all  $t \in [T_*, T]$ , it holds

$$\|\vec{g}\|_{\mathcal{E}} \leq \frac{1}{2}t^{-\frac{11}{3}}, \quad (3.70)$$

$$|\mathbf{y} - \mathbf{z}| \leq \frac{1}{2}t^{-\frac{7}{3}}, \quad (3.71)$$

$$\sum_{k=1}^K (a_k^+)^2 \leq \frac{1}{2}t^{-8}. \quad (3.72)$$

PROOF (PROOF OF (3.70)-(3.72)). First, from (3.6) and (3.26), it follows that  $\|h(T)\|_{\dot{H}^1} \leq \|g(T)\|_{\dot{H}^1} + \|h(T) - g(T)\|_{\dot{H}^1} \lesssim T^{-4}$ , and thus it holds

$$\mathcal{H}(T) \lesssim T^{-8}.$$

Hence, integration of (3.37) implies  $\mathcal{H}(t) \leq \delta t^{-\frac{22}{3}}$  and thus  $\mathcal{I}(t) \leq 2\delta t^{-\frac{22}{3}}$ , for all  $t \in [T_*, T]$ .

From (3.3) and (3.26), it holds

$$|\langle \lambda_k^{-2} \Delta_k \Lambda_k W_k, h \rangle| + |\langle \lambda_k^{-2} \nabla_k W_k, h \rangle| \lesssim t^{-5}.$$

Besides, from (3.49), we recall that  $|\langle \lambda_k^{-2} Y_k, h \rangle|^2 \lesssim t^{-8}$ . Therefore, applying Lemma 10 and standard arguments to estimate  $\int |F(\phi + h) - F(\phi) - f(\phi)h - f'(\phi)h^2| \ll t^{-\frac{22}{3}}$ , we obtain the following estimate, for  $\delta$  small enough,

$$\|\nabla h\|_{L^2}^2 + \|\dot{g}\|_{L^2}^2 \lesssim \mathcal{I} + t^{-8} \leq \frac{1}{3}t^{-\frac{22}{3}},$$

This yields (3.70) on  $\vec{g}$  using again the estimate on  $g - h$  from (3.26).

Bound (3.71) follows immediately from (3.14),  $\mathbf{y}(T) = \mathbf{z}$  (see (3.5)) and integration. In order to prove (3.72), we observe that (3.16) and (3.11) yields

$$\left| \frac{d}{dt} \sum_k (a_k^+)^2 - 2\nu \sum_k \frac{(a_k^+)^2}{\lambda_k} \right| \lesssim t^{-8},$$

hence, by (3.8) there is  $C > 0$  (independent of  $t$ ) such that

$$\frac{d}{dt} \sum_k (a_k^+)^2 \geq Ct^2 \sum_k (a_k^+)^2 + O(t^{-8}). \quad (3.73)$$

It is clear that (3.72) holds for  $t$  close to  $T$ . Supposing that (3.72) breaks down for the first time at some  $T_1 \in (T_*, T)$ , we would have on the one hand  $\frac{d}{dt} \sum_{k=1}^K |a_k^+(T_1)|^2 \leq 0$ ; on the other hand (3.73) yields  $\frac{d}{dt} \sum_{k=1}^K |a_k^+(T_1)|^2 > 0$ . This contradiction proves (3.72).

Finally, we complete the proof of Proposition 15, dealing with the remaining bootstrap estimate (3.12). For the sake of contradiction, suppose that for any  $(\alpha_0, \alpha_1, \dots, \alpha_K) \in \bar{\mathcal{B}}_{\mathbb{R}^{K+1}}$ , it holds  $T_* = T_*(\alpha_0, \alpha_1, \dots, \alpha_K) \in (T_0, T]$ . It follows from (3.52)-(3.53) and (3.70)-(3.72) that on  $[T_*, T]$ , equality is reached in none of the estimates (3.7)-(3.11). Therefore, from (i) of Lemma 13, equality has to be reached at  $t = T_*$  in estimate (3.12).

Recall that  $r := |\boldsymbol{\lambda}|$  and set also

$$\tilde{a}_0(t) := t^{\frac{42}{5}}(r(t) - |\mathbf{c}|t^{-2}), \quad \tilde{a}_k(t) := t^4 a_k^-(t)$$

so that from (3.5)-(3.6),

$$\tilde{a}_0(T) = \alpha_0 \quad \text{and} \quad \tilde{a}_k(T) = \alpha_k \quad \text{for } k = 1, \dots, K.$$

The contradiction assumption says that for any  $(\alpha_0, \alpha_1, \dots, \alpha_K) \in \bar{\mathcal{B}}_{\mathbb{R}^{K+1}}$ , for all  $t \in [T_*, T]$  it holds

$$(\tilde{a}_0(t), \tilde{a}_1(t), \dots, \tilde{a}_K(t)) \in \bar{\mathcal{B}}_{\mathbb{R}^{K+1}}, \quad (\tilde{a}_0(T_*), \tilde{a}_1(T_*), \dots, \tilde{a}_K(T_*)) \in \mathbb{S}^K.$$

Consider the application  $\Phi : \bar{\mathcal{B}}_{\mathbb{R}^{K+1}} \rightarrow \mathbb{S}^K$  defined by

$$\Phi(\alpha_0, \alpha_1, \dots, \alpha_K) := (\tilde{a}_0(T_*), \tilde{a}_1(T_*), \dots, \tilde{a}_K(T_*)).$$

To prove that  $\Phi$  is continuous, we only need to check that  $(\alpha_0, \alpha_1, \dots, \alpha_K) \mapsto T_*$  is continuous. This property is deduced from the following transversality condition: for any  $T_1 \in [T_*, T]$  such that  $(\tilde{a}_0(T_1), \tilde{a}_1(T_1), \dots, \tilde{a}_K(T_1)) \in \mathbb{S}^K$ , it holds

$$\sum_{k=0}^K \tilde{a}_k'(T_1) \tilde{a}_k(T_1) < 0. \quad (3.74)$$

Proof of (3.74). On the one hand, for  $k = 1, \dots, K$ , estimate (3.16) yields

$$(a_k^-)'(T_1)a_k^-(T_1) = -\frac{\nu}{\lambda_k(T_1)}(a_k^-(T_1))^2 + O(T_1^{-4}|a_k^-(T_1)|),$$

and so

$$\begin{aligned}\tilde{a}'_k(T_1)\tilde{a}_k(T_1) &= T_1^8(4T_1^{-1}a_k^-(T_1) + (a_k^-)'(T_1))a_k^-(T_1) \\ &= -\frac{\nu}{2\lambda_k(T_1)}\tilde{a}_k(T_1)^2 + O(|\tilde{a}_k(T_1)|).\end{aligned}$$

Using  $|O(|\tilde{a}_k(T_1)|)| \leq \frac{\nu}{4\lambda_k(T_1)}\tilde{a}_k(T_1)^2 + CT_1^{-2}$  for some constant  $C$  and taking the sum for  $k = 1, \dots, K$ , we obtain

$$\sum_{k=1}^K \tilde{a}'_k(T_1)\tilde{a}_k(T_1) \leq -\sum_{k=1}^K \frac{\nu}{4\lambda_k(T_1)}\tilde{a}_k(T_1)^2 + CT_1^{-2}. \quad (3.75)$$

On the other hand, using the definition of  $\tilde{a}_0$  and then (3.69), it holds

$$\begin{aligned}\tilde{a}'_0(T_1) &= \frac{12}{5}T_1^{\frac{7}{5}}(r(T_1) - |\mathbf{c}|T_1^{-2}) + T_1^{\frac{12}{5}}(r'(T_1) + 2|\mathbf{c}|T_1^{-3}) \\ &= \frac{12}{5}T_1^{\frac{7}{5}}(r(T_1) - |\mathbf{c}|T_1^{-2}) - 2|\mathbf{c}|^{-\frac{1}{2}}T_1^{\frac{12}{5}}(r(T_1)^{\frac{3}{2}} - |\mathbf{c}|^{\frac{3}{2}}T_1^{-3}) + O(T_1^{-\frac{47}{45}}).\end{aligned}$$

Observe that (3.54) implies

$$\begin{aligned}2|\mathbf{c}|^{-\frac{1}{2}}T_1 \frac{r(T_1)^{\frac{3}{2}} - |\mathbf{c}|^{\frac{3}{2}}T_1^{-3}}{r(T_1) - |\mathbf{c}|T_1^{-2}} &= 2|\mathbf{c}|^{-\frac{1}{2}} \frac{r(T_1)T_1^2 + (r(T_1)T_1^2)^{\frac{1}{2}}|\mathbf{c}|^{\frac{1}{2}} + |\mathbf{c}|^{\frac{3}{2}}}{(r(T_1)T_1^2)^{\frac{1}{2}} + |\mathbf{c}|} \\ &= 3 + O(T_1^{-\frac{2}{5}}),\end{aligned}$$

so that

$$\tilde{a}'_0(T_1) = -\frac{3}{5}T_1^{-1}a_0(T_1) + O(T_1^{-\frac{47}{45}}).$$

This estimate combined with (3.75) and  $\sum_{k=0}^K \tilde{a}_k(T_1)^2 = 1$  yield

$$\sum_{k=0}^K \tilde{a}'_k(T_1)\tilde{a}_k(T_1) \leq -\frac{3}{5}T_1^{-1} \sum_{k=0}^K \tilde{a}_k(T_1)^2 + O(T_1^{-\frac{47}{45}}) = -\frac{3}{5}T_1^{-1} + O(T_1^{-\frac{47}{45}}) < 0,$$

provided that  $T_1$  is large enough, which proves (3.74).

Therefore,  $\Phi$  is continuous on  $\tilde{\mathcal{B}}_{\mathbb{R}^{K+1}}$  and its restriction to  $\mathbb{S}^K$  is the identity. This is a contradiction with the no-retraction theorem.

### 3.6. Proof of Theorem 1 from Proposition 15

We follow the strategy by compactness from [5, 20, 23, 32, 37, 39], using the uniform estimates of Proposition 15 on a sequence of well-prepared solutions of (1.1).

Consider the solution  $\vec{u}_n$  given by Proposition 15 for  $T = T_n$  where  $T_n := n > T_0$ . On the interval  $[T_0, T_n]$ , this solution is well-defined and its decomposition



$(\Gamma_n, \vec{g}_n)$  satisfies the uniform estimates (3.7)-(3.10). In particular, from  $\vec{u}_n = \vec{W}_{\Gamma_n} + \vec{g}_n$ , we check that, for all  $t \in [T_0, T_n]$

$$\left\| u_n(t) - \sum_k \frac{1}{(c_k t^{-2})^{\frac{3}{2}}} W\left(\frac{\cdot - z_k}{c_k t^{-2}}\right) \right\|_{\dot{H}^1} + \|\partial_t u_n(t)\|_{L^2} \leq C t^{-\frac{1}{3}}. \quad (3.76)$$

We take a possibly larger  $T_0$  so that  $C T_0^{-\frac{1}{3}} < \eta$  where  $\eta > 0$  is the constant of Proposition 22.

Since the sequence  $(\vec{u}_n(T_0))_n$  is bounded in  $\dot{H}^1 \times L^2$ , after extraction of a subsequence, there exists  $\vec{u}_0$  in  $\dot{H}^1 \times L^2$  such that  $\vec{u}_n(T_0) \rightharpoonup \vec{u}_0$  weakly in  $\dot{H}^1 \times L^2$ . Fix  $T > T_0$ . From Proposition 22 applied to the compact set

$$\mathcal{K} = \left\{ \left( \sum_k \frac{1}{(c_k t^{-2})^{\frac{3}{2}}} W\left(\frac{\cdot - z_k}{c_k t^{-2}}\right), 0 \right), t \in [T_0, T] \right\},$$

the solution  $\vec{u}$  of (1.1) corresponding to  $\vec{u}(T_0) = u_0$  is well-defined and it holds  $\vec{u}_n(t) \rightharpoonup \vec{u}(t)$  weakly in  $\dot{H}^1 \times L^2$  on  $[T_0, T]$ . By (3.76) and the properties of weak convergence, the solution  $\vec{u}$  satisfies, for all  $t \in [T_0, T]$

$$\left\| u(t) - \sum_k \frac{1}{(c_k t^{-2})^{\frac{3}{2}}} W\left(\frac{\cdot - z_k}{c_k t^{-2}}\right) \right\|_{\dot{H}^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-\frac{1}{3}}.$$

Since  $T \geq T_0$  is arbitrary, the solution  $\vec{u}$  is defined and satisfies the conclusion of Theorem 1 on  $[T_0, \infty)$ . We obtain a solution defined on  $[0, \infty)$  with similar properties by time translation.

#### 4. Appendix. Weak continuity of the flow near a compact set

We reproduce two statements from Appendix A.2 of [23] with the only difference that they are given here for general solutions and not only for radially symmetric solutions. Using the result of profile decomposition stated in [11, Proposition 2.8], the proofs are similar up to dealing with additional position parameter.

**Proposition 21.** *There exists a constant  $\eta > 0$  such that the following holds. Let  $\vec{u} : [t_0, T_{\max}) \rightarrow \dot{H}^1 \times L^2$  be a maximal solution of (1.1) with  $T_{\max} < \infty$ . Then for any compact set  $\mathcal{K} \subset \dot{H}^1 \times L^2$  there exists  $\tau < T_{\max}$  such that  $\text{dist}(\vec{u}(t), \mathcal{K}) > \eta$  for all  $t \in [\tau, T_{\max})$ .*

**Proposition 22.** *There exists a constant  $\eta > 0$  such that the following holds. Let  $\mathcal{K} \subset \dot{H}^1 \times L^2$  be a compact set and let  $\vec{u} : [T_1, T_2] \rightarrow \dot{H}^1 \times L^2$  be a sequence of solutions of (1.1) such that*

$$\text{dist}(\vec{u}_n(t), \mathcal{K}) \leq \eta, \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [T_1, T_2].$$

*Suppose that  $u_n(T_1) \rightharpoonup \vec{u}_0$  weakly in  $\dot{H}^1 \times L^2$ . Then the solution  $\vec{u}(t)$  of (1.1) with the initial condition  $\vec{u}(T_1) = \vec{u}_0$  is defined for  $t \in [T_1, T_2]$  and*

$$\vec{u}_n(T_1) \rightharpoonup \vec{u}(t), \quad \text{weakly in } \dot{H}^1 \times L^2 \text{ for all } t \in [T_1, T_2].$$

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