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# Error estimates for phase recovering from phaseless scattering data <sup>\*</sup>

R.G. Novikov <sup>†</sup>      V.N. Sivkin <sup>‡</sup>

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## Abstract

We study the simplest explicit formulas for approximate finding the complex scattering amplitude from modulus of the scattering wave function. We obtain detailed error estimates for these formulas in dimensions  $d = 3$  and  $d = 2$ .

## 1 Introduction

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in R^d, \quad d \geq 1, \quad E > 0, \quad (1)$$

$$v \in L_\infty(R^d), \quad \text{supp } v \subset D, \quad (2)$$

$D$  is an open bounded domain in  $R^d$ .

Here  $\Delta$  is the standart Laplacian in  $x$ ,  $v$  is a scalar potential.

For equation (1) we consider the classical scattering solutions  $\psi^+$  specified by the following asymptotics as  $|x| \rightarrow \infty$ :

$$\psi^+(x, k) = e^{ikx} + c(d, |k|) \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k, |k| \frac{x}{|x|}) + O\left(\frac{1}{|x|^{(d+1)/2}}\right), \quad (3)$$

$$c(d, |k|) = -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}, \quad \text{for } \sqrt{-2\pi i} = \sqrt{2\pi} e^{-i\pi/4}, \quad x, k \in R^d, \quad k^2 = E,$$

where a priori unknown function  $f = f(k, l)$ ,  $k, l \in R^d, k^2 = l^2 = E$ , arising in (3) is the classical scattering amplitude for (1).

In order to study  $\psi^+$  and  $f$  one can use the Lippmann-Schwinger integral equation (4) and formula (6):

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<sup>†</sup>CMAP, Ecole Polytechnique, CNRS, Université Paris-Saclay, 91128 Palaiseau, France; and IEPT RAS, 117997 Moscow, Russia (novikov@cmap.polytechnique.fr)

<sup>‡</sup>Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Moscow, Russia (sivkin96@yandex.ru)

$$\psi^+(x, k) = e^{ikx} + \int_{R^d} G^+(x - y, k)v(y)\psi^+(y, k)dy, \quad (4)$$

$$G^+(x, k) := -\frac{1}{(2\pi)^d} \int_{R^d} \frac{e^{i\xi x} d\xi}{\xi^2 - k^2 - i0} = G_0^+(|x|, |k|), \quad (5)$$

$$f(k, l) = \frac{1}{(2\pi)^d} \int_{R^d} e^{-ilx}v(x)\psi^+(x, k)dx, \quad (6)$$

where  $x, k, l \in R^d, k^2 = l^2 = E$ , and  $G_0^+$  also depends on  $d$ . Note that:

$$G^+(x, k) = -\frac{i}{4}H_0^1(|x||k|) \text{ for } d = 2, \quad G^+(x, k) = -\frac{e^{i|k||x|}}{4\pi|x|} \text{ for } d = 3, \quad (7)$$

where  $H_0^1$  is the Hankel function of the first type.

In the present work we also assume that

$$\text{equation (4) is uniquely solvable for } \psi^+(\cdot, k) \in L^\infty(R^d) \text{ for fixed } E > 0, \quad (8)$$

where  $k \in R^d, k^2 = E$ . For example, for real-valued  $v$  satisfying (2) it holds true.

We recall that  $\psi^+$  describes scattering of the incident plane waves described by  $e^{ikx}$  on the scatterer described by  $v$ . In addition, the second term on the right-hand side of (3) describes the leading scattered spherical waves.

We also recall that in quantum mechanics the values of scattering functions  $\psi^+(x, k)$  and  $f(k, l)$  have no direct physical sense, whereas the phaseless values  $|\psi^+(x, k)|^2$  and  $|f|^2$  have probabilistic interpretation (the Born's principle) and can be obtained in experiments; see [B, FM].

We consider the following problems:

**Problem 1.** Find  $v$  on  $R^d$  from  $f = f(k, l)$  given for appropriate pairs  $(k, l)$ .

**Problem 2.** Find  $f(k, l)$  from  $|\psi^+(x, k)|^2$  at appropriate points  $x$  such that  $x \in R^d \setminus D$  and  $x/|x| = l/|l|$ .

**Problem 3.** Find  $v$  on  $R^d$  from  $|\psi^+|^2$  appropriately given outside of  $D$ .

Problem 1 is the classical inverse scattering problem. This problem was studied in many works; see, for example, [ChS], [E], [M], [N1] and references therein.

Problem 2 is a problem of phase recovering. Note that finding  $f$  considered in this problem and formula (3) also yield approximate finding  $\psi^+$  for large  $|x|$ . In connection with known results on Problem 2, see [N2]-[N5].

Problem 3 is a problem of inverse scattering without phase information. In connection with known results on this problem, see [JL], [N2]-[N5], [K2].

Actually, in the present work we continue studies on Problem 2 in dimensions  $d = 3$  and  $d = 2$ . Problem 2 for  $d = 1$  was solved in [N3]. In addition, results on Problem 1 and Problem 2 admit direct applications to Problem 3.

Note that Problem 2 is one of possible problems of phase recovering and Problem 3 is one of possible problems of inverse scattering without phase information. In

connection with results given on other inverse wave propagation problems without phase information, see [ChS], [IK], [K1], [N2], [N4], [KR], [R], [AHN], [HN], [P] and references therein.

We recall that Problem 2 can be solved approximately by the following explicit formulas of [N2], [N3]:

$$\begin{pmatrix} \operatorname{Re} c(d, |k|) f(k, l) \\ \operatorname{Im} c(d, |k|) f(k, l) \end{pmatrix} = M \left( \begin{pmatrix} a(x_1, k) \\ a(x_2, k) \end{pmatrix} - \begin{pmatrix} \delta a(x_1, k) \\ \delta a(x_2, k) \end{pmatrix} \right), \quad (9)$$

$$\text{where} \quad (10)$$

$$a(x, k) = |x|^{(d-1)/2} (|\psi^+(x, k)|^2 - 1), \quad (11)$$

$$(12)$$

$$M = \frac{1}{2 \sin(\phi_2 - \phi_1)} \begin{pmatrix} \sin(\phi_2) & -\sin(\phi_1) \\ -\cos(\phi_1) & \cos(\phi_2) \end{pmatrix}, \quad (13)$$

$$x_1 = s\hat{l}, \quad x_2 = (s + \tau)\hat{l}, \quad \hat{l} = l/|l|, \quad (14)$$

$$\varphi_j = |k||x_j| - kx_j, \quad j = 1, 2, \quad (15)$$

$$\varphi_2 - \varphi_1 = \tau(|k| - k\hat{l}), \quad (16)$$

$$\delta a(x_1, k) = O(s^{-\sigma}), \quad \delta a(x_2, k) = O(s^{-\sigma}) \quad \text{as } s \rightarrow +\infty, \quad (17)$$

uniformly in  $\hat{k} = k/|k|$ ,  $\hat{l} = l/|l|$  and  $\tau$  at fixed  $E > 0$ ,

$$\sigma = 1/2 \text{ for } d = 2, \quad \sigma = 1 \text{ for } d \geq 3, \quad (18)$$

where  $k, l \in R^d$ ,  $k^2 = l^2 = E$ ,  $s > 0$ ,  $\tau > 0$ ,  $\sin(\varphi_1 - \varphi_2) \neq 0$ .

In order to control error in finding  $f(k, l)$  from  $|\psi^+(x, k)|^2$  at  $x = x_1, x_2$  via formulas (9-15) it is necessary to estimate  $\delta a(x, k) = O(s^{-\sigma})$  in detail. However, detailed estimates for  $\delta a(x, k)$  were not yet given in the literature. For the first time such estimates are given in the present work, see Theorem 1 and Lemmas 1 and 2 of Section 2. These estimates are proved in Sections 3-5.

Finally, we recall that  $2n$ -point version of formulas (9)-(16) with error term estimated as  $O(s^{-n})$ ,  $s \rightarrow +\infty$ , is given in [N5]. Detailed estimates of this  $O(s^{-n})$  generalizing the estimates (20)-(23) (see Section 2) of the present work will be given elsewhere.

## 2 Main results

Let

$$D \subset B_r = \{x \in R^d : |x| \leq r\} \quad (19)$$

for some fixed  $r > 0$ .

**Theorem 1. (A)** Under assumptions (2),(8),(19) for  $d = 3$ , the following estimate holds, for  $|x| \geq 3r$ :

$$|\delta a(x, k)| \leq \rho_3 \frac{\|\psi^+\|_\infty \|v\|_{L_1}}{2\pi|x|} + \left(1 + \frac{2\rho_3}{|x|} + \frac{\rho_3^2}{|x|^2}\right) \frac{\|\psi^+\|_\infty^2 \|v\|_{L_1}^2}{16\pi^2|x|}, \quad (20)$$

$$\rho_3 := r(4.5 + 7.65|k|r + 3.91|k|^2r^2), \quad (21)$$

where  $k, x \in R^3, k^2 = E > 0$ .

**(B)** Under assumptions (2),(8),(19) for  $d = 2$ , the following estimate holds:

$$|\delta a(x, k)| \leq \left(1 + \frac{\rho_2\sqrt{2}}{|x|} + \frac{\rho_2^2}{2|x|^2}\right) \frac{\|\psi^+\|_\infty^2 \|v\|_{L_1}^2}{8\pi|x|^{1/2}|k|} + \rho_2 \frac{\|\psi^+\|_\infty \|v\|_{L_1}}{2\sqrt{\pi}|k|^{1/2}|x|}, \quad (22)$$

$$\rho_2 := r \left( \frac{0.33}{|k|r} + 2.51 + 5.36|k|r + 2.14|k|^2r^2 \right), \quad (23)$$

where  $k, x \in R^2, k^2 = E > 0$ .

In Theorem 1 we use the notation  $\|\psi^+\|_\infty := \|\psi^+(\cdot, k)\|_{L_\infty(D)}$ .

Theorem 1 is proved in Sections 3, 4.

In addition,  $\|\psi^+\|_\infty$  is estimated in Lemmas 1 and 2 given below.

Let

$$Q = C_0(d, s) \sup_{x \in D} |(1 + |x|^2)^s v(x)| |k|^{-1}, \quad (24)$$

where  $C_0$  is the constant of the Agmon estimate (99) (see Section 5).

**Lemma 1. (A)** Under assumptions (2), (19), for  $Q < 1$  and  $d = 3$ , the following estimate holds:

$$\|\psi^+(\cdot, k)\|_{L_\infty(R^3)} \leq 1 + \sqrt{\frac{5}{6}} \frac{\|v\|_{L_\infty(D)} (1 + r^2)^{s/2} r^{1/2}}{(1 - Q)}, \quad (25)$$

where  $s = (d + \varepsilon)/2, \varepsilon > 0, |k| \geq 1$ .

**(B)** Under assumptions (2), (19), for  $Q < 1$  and  $d = 2$ , the following estimate holds:

$$\|\psi^+(\cdot, k)\|_{L_\infty(R^2)} \leq 1 + \frac{\sqrt{\pi} \|v\|_{L_\infty(D)} (1 + r^2)^{s/2} r^{1/2}}{\sqrt{2\varepsilon}|k|(1 - Q)}, \quad (26)$$

where  $s = (d + \varepsilon)/2, \varepsilon > 0, |k| \geq 1$ .

**Lemma 2. (A)** Under assumptions (2), (19), for  $\|v\|_{L_\infty(D)} r^2 \leq 2, d = 3$  the following estimate holds:

$$\|\psi^+\|_{L_\infty(R^3)} \leq \frac{1}{1 - \|v\|_{L_\infty(D)} r^2 / 2}. \quad (27)$$

(B) Under assumptions (2), (19), for  $\sqrt{\frac{2\pi}{|k|} \frac{\|v\|_{L_\infty(D)} r^{5/2}}{5}} < 1, d = 2$  the following estimate holds:

$$\|\psi^+\|_{L_\infty(\mathbb{R}^2)} \leq \frac{1}{1 - \sqrt{\frac{2\pi}{|k|} \frac{\|v\|_{L_\infty(D)} r^{3/2}}{3}}}. \quad (28)$$

Lemmas 1 and 2 are proved in Section 5.

### 3 Proof of Theorem 1 (A)

We have that (see [N2]):

$$|\delta a(x, k)| = |x|^{-\frac{d-1}{2}} |c|^2 |f|^2 + 2|x|^{\frac{d-1}{2}} \operatorname{Re}(\delta\psi^+(x, k) \overline{\psi_1^+(x, k)}) + |x|^{\frac{d-1}{2}} |\delta\psi^+(x, k)|^2, \quad (29)$$

where  $f = f(k, \frac{|k|x}{|x|})$ ,

$$\psi_1^+(x, k) := e^{ikx} + c(d, |k|) \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k, |k| \frac{x}{|x|}), \quad (30)$$

$$\delta\psi^+(x, k) := \psi^+(x, k) - \psi_1^+(x, k). \quad (31)$$

Note that

$$|f| = \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} e^{-ix} v(x) \psi^+(x, k) dx \right| \leq \frac{1}{(2\pi)^d} \|\psi^+\|_\infty \|v\|_{L_1}. \quad (32)$$

Further in this section we always assume that  $d = 3$ .

The following estimate holds:

$$|\delta a(x, k)| \leq |x|^{-1} (2\pi^2)^2 |f|^2 + 2|x| |\delta\psi^+(x, k)| \left(1 + \frac{2\pi^2 |f|}{|x|}\right) + |x| |\delta\psi^+(x, k)|^2. \quad (33)$$

In addition,  $|\delta\psi^+(x, k)|$  is estimated in the following lemma:

**Lemma 3.** *assumptions of Theorem 1A, the following estimate holds, for  $|x| \geq 3r$  :*

$$|\delta\psi^+(x, k)| \leq \frac{\|\psi^+\|_\infty \|v\|_{L_1}}{4\pi |x|^2} \rho_3(r, k). \quad (34)$$

Using estimates (29), (32), (34) we have that:

$$\begin{aligned} |\delta a(x, k)| &\leq |x|^{-1} (2\pi^2)^2 \frac{1}{((2\pi)^3)^2} \|\psi^+\|_\infty^2 \|v\|_{L_1}^2 + \\ &+ 2|x| \frac{1}{4\pi} \|\psi^+\|_\infty \|v\|_{L_1} \frac{\rho_3}{|x|^2} \left(1 + \frac{2\pi^2 \frac{1}{(2\pi)^3} \|\psi^+\|_\infty \|v\|_{L_1}}{|x|}\right) + |x| \left(\frac{1}{4\pi} \|\psi^+\|_\infty \|v\|_{L_1} \frac{\rho_3}{|x|^2}\right)^2 \leq \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|\psi^+\|_\infty^2 \|v\|_{L_1}^2}{16\pi^2|x|} + \|\psi^+\|_\infty \|v\|_{L_1} \frac{\rho_3}{2\pi|x|} \left(1 + \frac{\|\psi^+\|_\infty \|v\|_{L_1}}{4\pi|x|}\right) + \\
&\quad + \|\psi^+\|_\infty^2 \|v\|_{L_1}^2 \frac{\rho_3^2}{16\pi^2|x|^3}, |x| \geq 3r.
\end{aligned} \tag{35}$$

Estimate (20) of Theorem 1A follows from (35). Therefore, in order to prove Theorem 1A it remains to prove Lemma 3.

*Proof of Lemma 3.* Using the Lippmann-Schwinger integral equation (4) and formulas (6), (31) we obtain

$$\begin{aligned}
\delta\psi^+(x, k) &= - \int_{R^3} \frac{e^{i|k||x-y|}}{4\pi|x-y|} v(y)\psi^+(y, k)dy + \\
&\quad + 2\pi^2 \frac{e^{i|k||x|}}{|x|} \frac{1}{(2\pi)^3} \int_{R^3} e^{-i|k|\frac{xy}{|x|}} v(y)\psi^+(y, k)dy = \\
&= \frac{1}{4\pi} \int_{R^3} \left( \frac{e^{i|k||x|-i|k|\frac{xy}{|x|}}}{|x|} - \frac{e^{i|k||x-y|}}{|x-y|} \right) v(y)\psi^+(y, k)dy.
\end{aligned} \tag{36}$$

From (36) we obtain (37):

$$|\delta\psi^+(x, k)| \leq \frac{1}{4\pi} \|\psi^+(\cdot, k)\|_\infty \int_{R^3} \left| \frac{e^{i|k||x|-i|k|\frac{xy}{|x|}}}{|x|} - \frac{e^{i|k||x-y|}}{|x-y|} \right| |v(y)|dy. \tag{37}$$

**Lemma 4.** *Let  $x, y \in R^3$ ,  $|y| \leq r$ ,  $|x| \geq 3r$ . Then:*

$$\left| \frac{e^{i|k||x|-i|k|\frac{xy}{|x|}}}{|x|} - \frac{e^{i|k||x-y|}}{|x-y|} \right| \leq \frac{r}{|x|^2} (4.5 + 7.65|k|r + 3.91|k|^2r^2). \tag{38}$$

Estimate (34) of Lemma 3 follows from estimates (37), (38). Thus, in order to prove Lemma 3, it remains to prove Lemma 4.

*Proof of Lemma 4.* To prove Lemma 4 we use in particular Lemma 5.

**Lemma 5.** *Let  $x, y \in R^3$ ,  $|y| \leq r$ ,  $|x| \geq 3r$ . Then the following estimates hold:*

$$|x-y| = |x| \left( 1 - \frac{xy}{|x|^2} + \frac{|y|^2}{2|x|^2} - \frac{(xy)^2}{2|x|^4} + L_3(x, y) \right), |L_3(x, y)| \leq \frac{4.13r^3}{|x|^3}; \tag{39}$$

$$|x-y| = |x| \left( 1 - \frac{xy}{|x|^2} + L_2(x, y) \right), |L_2(x, y)| \leq \frac{2.38r^2}{|x|^2}. \tag{40}$$

*Proof of Lemma 5.* Recall that

$$(1 + \varepsilon)^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1 - 2n)n!2^4} \varepsilon^n = \sum_{n=0}^{\infty} a_n \varepsilon^n, \forall |\varepsilon| < 1. \quad (41)$$

Note that

$$k_{n+1} := \frac{a_{n+1}}{a_n} = \frac{(-1)(2n+1)(2n+2)(1-2n)}{(n+1)(n+1)4(1-2n-2)} = -\frac{2(2n+1)(2n-1)}{4(2n+1)(n+1)} = -\left(1 - \frac{3}{2n+2}\right) \quad (42)$$

$|k_{n+1}| < 1$ , for  $n \in N \cup \{0\}$ .

Therefore,

$$\begin{aligned} \left| \sum_{n=3}^{\infty} a_n \varepsilon^n \right| &\leq |a_3| \sum_{n=3}^{\infty} |\varepsilon^n| \leq |a_3| \frac{|\varepsilon|^3}{1 - |\varepsilon|}, |\varepsilon| < 1, \\ |a_3| &= \frac{6!}{5 * 3!3!4^3} = \frac{6 * 5 * 4 * 6}{5 * 6 * 6 * 4^3} = \frac{1}{16}. \end{aligned} \quad (43)$$

In the present work we use formulas (41), (43) for

$$\varepsilon = -\frac{2xy}{|x|^2} + \frac{|y|^2}{|x|^2}, \quad x, y \in R^d, |y| \leq r, |x| \geq 3r. \quad (44)$$

From (44) it follows that:

$$|\varepsilon| \leq \frac{7}{9}, |\varepsilon| \leq \frac{7}{3} \frac{r}{|x|}. \quad (45)$$

Using (41),(44) we have that

$$\begin{aligned} |x - y| &= |x| \left| \frac{x}{|x|} - \frac{y}{|x|} \right| = |x| \left( 1 - \frac{2xy}{|x|^2} + \frac{|y|^2}{|x|^2} \right)^{\frac{1}{2}} = \\ &= |x| \left( 1 - \frac{xy}{|x|^2} + \frac{|y|^2}{2|x|^2} - \frac{1}{8} \left( \frac{|y|^2}{|x|^2} - \frac{2xy}{|x|^2} \right)^2 + R_3(x, y) \right), \end{aligned} \quad (46)$$

where

$$|R_3(x, y)| = \left| \sum_{n=3}^{\infty} a_n \varepsilon^n \right| \leq |a_3| \frac{|\varepsilon|^3}{1 - |\varepsilon|} \leq \frac{1}{16} \frac{7^3 r^3}{3^3 |x|^3} \frac{1}{2/9} \leq 3.58 \frac{r^3}{|x|^3}. \quad (47)$$

Using (46), (47) and gathering the terms with equal degrees in  $|x|$ , we obtain:

$$|x - y| = |x| \left( 1 - \frac{xy}{|x|^2} + \frac{|y|^2}{2|x|^2} - \frac{(xy)^2}{2|x|^4} + L_3(x, y) \right), \quad (48)$$



$$|L_3(x, y)| \leq \frac{1}{8} \frac{|y|^4}{|x|^4} + \frac{1}{2} \frac{xy|y|^2}{|x|^4} + R_3(x, y) \leq \frac{r^3}{24|x|^3} + \frac{r^3}{2|x|^3} + 3.58 \frac{r^3}{|x|^3} \leq \frac{4.13r^3}{|x|^3}. \quad (49)$$

Thus, estimate (39) is proved.

In addition to (46), (47) we also need the following formulas:

$$|x - y| = |x| \left( 1 - \frac{xy}{|x|^2} + \frac{|y|^2}{2|x|^2} + R_2(x, y) \right), \quad (50)$$

$$|R_2(x, y)| = \left| \sum_{n=2}^{\infty} a_n \varepsilon^n \right| \leq |a_2 \varepsilon^2| + R_3(x, y) \leq \frac{7^2/3^2}{8} \frac{r^2}{|x|^2} + 3.58 \frac{r^3}{|x|^3} \leq 1.88 \frac{r^2}{|x|^2}. \quad (51)$$

In a similar way with (48), (49) we have

$$|x - y| = |x| \left( 1 - \frac{xy}{|x|^2} + L_2(x, y) \right), \quad (52)$$

$$|L_2(x, y)| \leq \frac{r^2}{2|x|^2} + 1.88 \frac{r^2}{|x|^2} \leq \frac{2.38r^2}{|x|^2}. \quad (53)$$

Thus, estimate (40) is proved. This completes the prove of Lemma 5.  $\square$

Now we are ready to prove estimate (38). We have

$$\begin{aligned} & \left| \frac{e^{i|k||x|-i|k|\frac{xy}{|x|}}}{|x|} - \frac{e^{i|k||x-y|}}{|x-y|} \right| = \\ & = \left| \frac{e^{i|k||x|-i|k|\frac{xy}{|x|}}}{|x|} - \frac{e^{i|k||x|\left(1-\frac{xy}{|x|^2}+\frac{|y|^2}{2|x|^2}-\frac{(xy)^2}{2|x|^4}+L_3(x,y)\right)}}{|x|\left(1-\frac{xy}{|x|^2}+L_2(x,y)\right)} \right| = \\ & = \frac{1}{|x|} \left| 1 - \frac{e^{i|k||x|\left(\frac{|y|^2}{2|x|^2}-\frac{(xy)^2}{2|x|^4}+L_3(x,y)\right)}}{1-\frac{xy}{|x|^2}+L_2(x,y)} \right| =: \frac{1}{|x|} \left| 1 - \frac{e^{iL}}{1-t} \right|. \end{aligned} \quad (54)$$

Note that

$$|e^{iL} - 1 - iL| \leq \frac{L^2}{2}, \text{ for } L \in R, \text{ and } \left| \frac{1}{1-t} - 1 - t \right| \leq t^2 \frac{1}{1-|t|}, |t| < 1, \quad (55)$$

and further

$$\begin{aligned} \left| 1 - \frac{e^{iL}}{1-t} \right| & \leq \left| 1 - (1 + L + L^2/2)(1 + t + t^2/(1 - |t|)) \right| \leq \\ & \leq L + L^2/2 + t + \frac{t^2}{1 - |t|} + (L + L^2/2)(t + t^2/(1 - |t|)). \end{aligned} \quad (56)$$

In our case:

$$|L| = |k||x| \left| \frac{|y|^2}{2|x|^2} - \frac{(xy)^2}{2|x|^4} + L_3(x, y) \right| \leq \frac{|k|}{|x|} \left( \frac{r^2}{2} + \frac{r^2}{2} + \frac{4.13r^3}{|x|} \right) \leq \frac{2.38|k|r^2}{|x|}; \quad (57)$$

$$L^2/2 \leq \frac{2.38^2|k|^2r^4}{2|x|^2};$$

$$|t| = \left| \frac{xy}{|x|^2} - L_2(x, y) \right| \leq \frac{r}{|x|} + \frac{2.38r^2}{|x|^2} \leq \frac{r}{|x|} + \frac{0.8r}{|x|} \leq \frac{1.8r}{|x|};$$

$$\frac{t^2}{1-t} \leq \frac{\frac{1.8^2r^2}{|x|^2}}{1-1.8/3} = \frac{1.8^2r^2}{|x|^2} \frac{1}{0.4} = 8.1 \frac{r^2}{|x|^2}.$$

Finally,

$$\begin{aligned} \left| 1 - \frac{e^{iL}}{1-t} \right| &\leq \frac{2.38|k|r^2}{|x|} + \frac{2.38^2|k|^2r^4}{2|x|^2} + \frac{1.8r}{|x|} + \frac{8.1r^2}{|x|^2} + \\ &\quad + \left( \frac{2.38|k|r^2}{|x|} + \frac{2.38^2|k|^2r^4}{2|x|^2} \right) \left( \frac{1.8r}{|x|} + \frac{8.1r^2}{|x|^2} \right) \leq \\ &\leq \frac{3.06|k|r^2}{|x|} + \frac{3.06^2|k|^2r^4}{2|x|^2} + \frac{1.8r}{|x|} + \frac{8.1r^2}{|x|^2} + 1.5 \left( \frac{3.06|k|r^2}{|x|} + \frac{3.06^2|k|^2r^4}{2|x|^2} \right) \leq \\ &\leq \frac{r}{|x|} (4.5 + 7.65|k|r + 3.91|k|^2r^2). \end{aligned} \quad (58)$$

Estimate (38) follows from (54), (58), that proves Lemma 4.  $\square$ .

This also completes the proof of Theorem 1A.  $\square$ .

## 4 Proof of the Theorem 1(B)

Proceeding from (29)-(31), for  $d = 2$ , we have:

$$|\delta a(x, k)| \leq |x|^{-1/2} |c|^2 |f|^2 + 2|x|^{1/2} |\delta\psi^+(x, k)| \left( 1 + \frac{|c||f|}{|x|^{1/2}} \right) + |x|^{1/2} |\delta\psi^+(x, k)|^2, \quad (59)$$

where

$$c = c(2, |k|) = -\pi i (-2\pi i)^{\frac{2-1}{2}} |k|^{\frac{2-3}{2}} = -(1+i)\pi^{3/2} |k|^{-1/2}. \quad (60)$$

Now  $|f|$  is estimated in (32) for  $d = 2$  and  $|\delta\psi^+(x, k)|$  is estimated in the following lemma:

**Lemma 6.** *Under the assumptions of Theorem 1B, the following estimate holds for  $|x| \geq 3r$ :*

$$|\delta\psi^+(x, k)| \leq \frac{\|v\|_{L_1} \|\psi^+\|_\infty \rho_2(|k|, r)}{4(\pi|k|)^{1/2}|x|^{3/2}}. \quad (61)$$

Using estimates (59), (32), (61) we have that

$$|\delta a(x, k)| \leq |x|^{-1/2}|k|^{-1} \frac{\|\psi^+\|_\infty^2 \|v\|_{L_1}^2}{8\pi} + \quad (62)$$

$$+ |x|^{1/2} \frac{\|v\|_{L_1} \|\psi^+\|_\infty \rho_2}{2\sqrt{\pi}|k|^{1/2}|x|^{3/2}} \left( 1 + \frac{\|\psi^+\|_\infty \|v\|_{L_1}}{2|x|^{1/2}(2\pi)^{1/2}|k|^{1/2}} \right) + |x|^{1/2} \frac{\|v\|_{L_1}^2 \|\psi^+\|_\infty^2 \rho_2^2}{16\pi|k||x|^3} \quad (63)$$

Estimate (22) of Theorem 1B follows from (62). Therefore, in order to prove Theorem 1B it remains to prove Lemma 6.

*Proof of Lemma 6.* Using the Lippmann-Schwinger integral equation (4) and formulas (6), (31), (7) we obtain

$$\begin{aligned} \delta\psi^+(x, k) &= \int_{R^2} -\frac{iH_0^{(1)}(|k||x-y|)}{4} v(y)\psi^+(y, k) dy + \\ &+ (1+i) \frac{\pi^{3/2}}{|k|^{1/2}} \frac{e^{i|k||x|}}{|x|^{1/2}} \frac{1}{(2\pi)^2} \int_{R^2} e^{-i|k|\frac{xy}{|x|}} v(y)\psi^+(y, k) dy = \\ &= \int_{R^2} \left( -\frac{iH_0^{(1)}(|k||x-y|)}{4} + \frac{(1+i)e^{i|k||x|-i|k|\frac{xy}{|x|}}}{4(\pi|k||x|)^{1/2}} \right) v(y)\psi^+(y, k) dy. \end{aligned} \quad (64)$$

From (64) we obtain (65):

$$|\delta\psi^+(x, k)| \leq \|\psi^+(\cdot, k)\|_\infty \int_{R^2} \left| -\frac{iH_0^{(1)}(|k||x-y|)}{4} + \frac{(1+i)e^{i|k||x|-i|k|\frac{xy}{|x|}}}{4(\pi|k||x|)^{1/2}} \right| |v(y)| dy. \quad (65)$$

**Lemma 7.** *Let  $x, y \in R^2$ ,  $|y| \leq r$ ,  $|x| \geq 3r$ . Then*

$$\left| -\frac{iH_0^{(1)}(|k||x-y|)}{4} + \frac{(1+i)e^{i|k||x|-i|k|\frac{xy}{|x|}}}{4(\pi|k||x|)^{1/2}} \right| = M_1 + M_2, \text{ where} \quad (66)$$

$$M_1 \leq \frac{\sqrt{2}}{4(\pi k)^{1/2}|x|^{3/2}} (1.77 + 3.79|k|r + 1.51|k|^2 r^2); \quad (67)$$

$$M_2 \leq \frac{3\sqrt{3}}{64(\pi|k||x|)^{1/2}|k||x|}. \quad (68)$$

Note that

$$M_1 + M_2 \leq \frac{\sqrt{2}r}{4(\pi|k|)^{1/2}|x|^{3/2}} \left( \frac{3\sqrt{3}}{16\sqrt{2}|k|r} + 1.77 + 3.79|k|r + 1.51|k|^2r^2 \right). \quad (69)$$

Estimate (61) of Lemma 6 follows from (65)-(69). Thus, in order to prove Lemma 6 it remains to prove Lemma 7.

*Proof of Lemma 7.* To prove Lemma 7 we use, in particular, estimate (39) of Lemma 5, which remains the same for  $d = 2$ . Besides, we use, in particular, Lemma 8.

**Lemma 8.** *For Hankel function the following representation holds:*

$$H_0^{(1)}(|k||x|) = \left( \frac{2}{\pi|k||x|} \right)^{1/2} e^{i|k||x| - i\pi/4} (1 + h(|k||x|)), \quad |h(|k||x|)| \leq \frac{1}{8|k||x|}. \quad (70)$$

Lemma 8 is proved at the end of this Section.

Using formula (70) we have

$$\begin{aligned} & \left| -\frac{iH_0^{(1)}(|k||x-y|)}{4} + \frac{(1+i)e^{i|k||x| - i|k|\frac{xy}{|x|}}}{4(\pi|k||x|)^{1/2}} \right| \leq \\ & \left| -\frac{i}{4} \left( \frac{2}{\pi|k||x-y|} \right)^{1/2} e^{i|k||x-y| - i\pi/4} (1 + h(|k|(x-y))) + \frac{(1+i)e^{i|k||x| - i|k|\frac{xy}{|x|}}}{4(\pi|k||x|)^{1/2}} \right| \leq \\ & \leq \frac{1}{4(\pi|k|)^{1/2}} \left| -i\sqrt{2} \left( \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \right) \frac{e^{i|k||x-y|}}{|x-y|^{1/2}} + (1+i) \frac{e^{i|k||x| - i|k|\frac{xy}{|x|}}}{|x|^{1/2}} \right| + \\ & + \frac{1}{4} \left( \frac{2}{\pi|k||x-y|} \right)^{1/2} \frac{1}{8|k||x-y|} =: M_1 + M_2. \end{aligned} \quad (71)$$

The term  $M_2$  can be estimated easily:

$$M_2 \leq \frac{3\sqrt{3}}{64(\pi|k||x|)^{1/2}|k||x|}, \quad |y| \leq r, \quad |x| \geq 3r. \quad (72)$$

**Lemma 9.** *The term  $M_1$  can be estimated as follows:*

$$M_1 \leq \frac{\sqrt{2}}{4(\pi|k|)^{1/2}|x|^{3/2}} (1.77 + 3.79|k|r + 1.51|k|^2r^2), \quad |y| \leq r, \quad |x| \geq 3r. \quad (73)$$

*Proof of Lemma 9.* One can see that

$$M_1 = \frac{\sqrt{2}}{4(\pi|k|)^{1/2}} \left| \frac{e^{i|k||x| - i|k|\frac{xy}{|x|}}}{|x|^{1/2}} - \frac{e^{i|k||x-y|}}{|x-y|^{1/2}} \right|. \quad (74)$$

Further, we obtain

$$\begin{aligned}
M_1 &\leq \frac{\sqrt{2}}{4(\pi|k|)^{1/2}} \left| \frac{e^{i|k||x|-i|k|\frac{xy}{|x|}}}{|x|^{1/2}} - \frac{e^{i|k||x|(1-\frac{xy}{|x|^2}+\frac{|y|^2}{2|x|^2}-\frac{(xy)^2}{2|x|^4}+L_3(x,y))}}{|x|^{1/2}(1-\frac{xy}{2|x|^2}+\widetilde{L}_2(x,y))} \right| = \\
&= \frac{\sqrt{2}}{4(\pi|k||x|)^{1/2}} \left| 1 - \frac{e^{i|k||x|(\frac{|y|^2}{2|x|^2}-\frac{(xy)^2}{2|x|^4}+L_3(x,y))}}{(1-\frac{xy}{2|x|^2}+\widetilde{L}_2(x,y))} \right|. \tag{75}
\end{aligned}$$

Here we use the same expansion as in (39) and the following additional expansion:

$$|x-y|^{1/2} = |x|^{1/2} \left( 1 - \frac{2xy}{|x|^2} + \frac{|y|^2}{|x|^2} \right)^{1/4} = |x|^{1/2} \left( 1 - \frac{xy}{2|x|^2} + \widetilde{L}_2(x,y) \right). \tag{76}$$

To complete the proof of Lemma 9 we use, in particular, the estimate for  $L_3$  in (39) and the estimate for  $\widetilde{L}_2$  given in the following Lemma.

**Lemma 10.** *Let  $x, y \in R^2$ ,  $|y| \leq r$ ,  $|x| \geq 3r$ . Then:*

$$|\widetilde{L}_2(x,y)| \leq 1.81 \frac{r^2}{|x|^2}. \tag{77}$$

*Proof of Lemma 10.* Recall that

$$(1+\varepsilon)^{1/4} = \sum_{n=0}^{\infty} b_n \varepsilon^n, \quad |\varepsilon| < 1, \quad \text{where } b_n = \prod_{k=1}^n \frac{1/4 - k + 1}{k}. \tag{78}$$

Note that

$$l_{n+1} = \frac{b_{n+1}}{b_n} = \frac{1/4 - n - 1 + 1}{n+1} = (-1) \left( 1 - \frac{5}{4(n+1)} \right),$$

$|l_{n+1}| < 1$ , for  $n \in N \cup \{0\}$ .

Therefore,

$$\left| \sum_{n=2}^{\infty} b_n \varepsilon^n \right| \leq |b_2| \varepsilon^2 + |b_3| \frac{|\varepsilon|^3}{1-|\varepsilon|}, \tag{79}$$

$$|b_2| = 3/32, \quad |b_3| = 7/128.$$

In the present work we use formulas (78),(79) for

$$\varepsilon = -\frac{2xy}{|x|^2} + \frac{|y|^2}{|x|^2}, \quad x, y \in R^2, \quad |y| \leq r, \quad |x| \geq 3r, \tag{80}$$

$$|\varepsilon| \leq \frac{7}{9}, |\varepsilon| \leq \frac{7}{3|x|}. \quad (81)$$

Using (78), (80) we have that

$$|x - y|^{1/2} = |x|^{1/2} \left( 1 - \frac{2xy}{|x|^2} + \frac{|y|^2}{|x|^2} \right)^{1/4} = |x|^{1/2} \left( 1 - \frac{xy}{2|x|^2} + \frac{|y|^2}{4|x|^2} + \widetilde{R}_2(x, y) \right), \quad (82)$$

where

$$\begin{aligned} |\widetilde{R}_2(x, y)| &= \left| \sum_{n=2}^{\infty} b_n \varepsilon^n \right| \leq |b_2| \varepsilon^2 + |b_3| \frac{\varepsilon^3}{1 - |\varepsilon|} \leq \frac{3}{32} \frac{7^2}{3^2} \frac{r^2}{|x|^2} + \frac{7}{128} \frac{7^3}{3^3} \frac{r^3}{|x|^3} \frac{9}{2} \leq \\ &\leq \left( \frac{7^2}{32 * 3} + \frac{7 * 7^3 * 9}{128 * 3^3 * 3 * 2} \right) \frac{r^2}{|x|^2} \leq 1.56 \frac{r^2}{|x|^2} \end{aligned} \quad (83)$$

Using (82), (83) and gathering the terms with equal degrees in  $|x|$ , we obtain:

$$|\widetilde{L}_2(x, y)| = \left| \frac{|y|^2}{4|x|^2} + \widetilde{R}_2(x, y) \right| \leq \frac{r^2}{4|x|^2} + 1.56 \frac{r^2}{|x|^2} \leq 1.81 \frac{r^2}{|x|^2}. \quad (84)$$

*Lemma 10 is proved.*

Returning to the proof of Lemma 9, we rewrite (75) as:

$$M_1 \leq \frac{\sqrt{2}}{4(\pi|k|x)^{1/2}} \left| 1 - \frac{e^{iL}}{1-t} \right|. \quad (85)$$

We have

$$\begin{aligned} \left| 1 - \frac{e^{iL}}{1-t} \right| &\leq \left| 1 - (1 + L + L^2/2)(1 + t + t^2/(1-t)) \right| \leq \\ &\leq L + L^2/2 + t + \frac{t^2}{1-t} + (L + L^2/2)(t + t^2/(1-t)). \end{aligned} \quad (86)$$

In addition:

$$L \leq |k||x| \left( \frac{r^2}{2|x|^2} + \frac{r^2}{2|x|^2} + \frac{4.13r^3}{|x|^3} \right) \leq \frac{|k|r^2}{|x|} \left( 1 + \frac{4.13}{3} \right) \leq \frac{2.38|k|r^2}{|x|}; \quad (87)$$

$$L^2/2 \leq 2.84 \frac{|k|^2 r^4}{|x|^2}; \quad (88)$$

$$t \leq \frac{r}{2|x|} + 1.81 \frac{r^2}{|x|^2} \leq 1.11 \frac{r}{|x|}; \quad (89)$$

$$\frac{t^2}{1-t} \leq \frac{(1.11 \frac{r}{|x|})^2}{1 - 1.11/3} \leq 1.9558 \frac{r^2}{|x|^2}; \quad (90)$$

$$\begin{aligned}
(L + L^2/2)(t + t^2/(1-t)) &\leq \left(\frac{2.38|k|r^2}{|x|} + \frac{2.84|k|^2r^4}{|x|^2}\right)\left(1.11\frac{r}{|x|} + 1.9558\frac{r^2}{|x|^2}\right) \leq \\
&\leq 0.59\left(\frac{2.38|k|r^2}{|x|} + \frac{2.84|k|^2r^4}{|x|^2}\right). \tag{91}
\end{aligned}$$

Using (86)-(91), we obtain:

$$\begin{aligned}
\left|1 - \frac{e^{iL}}{1-t}\right| &\leq \frac{2.38 * 1.59|k|r^2}{|x|} + \frac{2.84 * 1.59|k|^2r^4}{|x|^2} + 1.77\frac{r}{|x|} \leq \\
&\leq \frac{r}{|x|} (1.77 + 3.79|k|r + 1.51|k|^2r^2). \tag{92}
\end{aligned}$$

The estimate (73) follows from (85) and (92). This completes the proof of Lemma 9.  $\square$

*Proof of Lemma 8.* For  $H_0^1$  the following equality holds, for  $s > 0$ , see in [S] formula (43) for  $\nu = 0, \beta = 0$ :

$$H_0^{(1)}(s) = \left(\frac{2}{\pi s}\right)^{1/2} \frac{e^{is-i\pi/4}}{\Gamma(1/2)} \int_0^\infty e^{-u} u^{-1/2} \left(1 + \frac{iu}{2s}\right)^{-1/2} du. \tag{93}$$

Then we apply the Taylor's expansion with one term and the integral reminder to the function  $g(t) = (1 + \frac{iut}{2s})^{-1/2}, t = 0, 1, u, s \in R$ . We obtain:

$$\left(1 + \frac{iu}{2s}\right)^{-1/2} = 1 + \frac{1/2}{0!} \frac{u}{2is} \int_0^1 \left(1 - \frac{ut}{2is}\right)^{-3/2} dt. \tag{94}$$

So,

$$\left|\left(1 + \frac{iu}{2s}\right)^{-1/2}\right| \leq 1 + \frac{u}{4s} \int_0^1 \left|1 - \frac{ut}{2is}\right|^{-3/2} dt \leq 1 + \frac{u}{4s}. \tag{95}$$

Using (93), (95), we obtain:

$$\left|\frac{H_0^{(1)}(s) - \left(\frac{2}{\pi s}\right)^{1/2} e^{is-i\pi/4}}{\left(\frac{2}{\pi s}\right)^{1/2} e^{is-i\pi/4}}\right| \leq \frac{1}{4s\Gamma(1/2)} \int_0^\infty e^{-u} u^{-1/2} u du \leq \frac{\Gamma(3/2)}{4s\Gamma(1/2)} = \frac{1}{8s}. \tag{96}$$

Estimate (96) implies (70).  $\square$

## 5 Estimates for $\|\psi^+\|_\infty$

The Lippmann-Schwinger integral equation (4) can also be rewritten as:

$$(I - A^+(|k|))\varphi^+(\cdot, k) = \varphi_0^+(\cdot, k), \quad (97)$$

where

$$\begin{aligned} \varphi^+(x, k) &= \Lambda^{-s}\psi^+(x, k), \quad \varphi_0^+(x, k) = \Lambda^{-s}e^{ikx}, \\ A^+(|k|) &= \Lambda^{-s}G^+(|k|)\Lambda^{-s}(\Lambda^{2s}v), \end{aligned} \quad (98)$$

where  $I$  is the identity operator,  $\Lambda$  denotes the multiplication operator by the functions  $(1 + |x|^2)^{1/2}$ ,  $G^+(|k|)$  denotes the integral operator with the Schwartz kernel  $G^+(x - y, k)$  of (4), (5),  $v$  is the multiplication operator by the function  $v(x)$ ,  $s = (d + \varepsilon)/2$ ,  $\varepsilon > 0$ ,  $k \in \mathbb{R}^d \setminus \{0\}$ ,  $x \in \mathbb{R}^d$ .

We recall that the following Agmon estimate holds:

$$\|\Lambda^{-s}G^+(|k|)\Lambda^{-s}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_0(d, s)|k|^{-1}, \quad s > 1/2, |k| \geq 1, \quad (99)$$

see, for example, [E], [N1] and references therein.

Using (99) one can see that

$$\|A^+(|k|)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq Q, \quad s > 1/2, |k| \geq 1, \quad (100)$$

$$Q := C_0(d, s)\|\Lambda^{2s}v\|_\infty|k|^{-1}. \quad (101)$$

As a corollary of (97), (98), (100), we have that if  $Q < 1$ ,  $s > d/2$ ,  $|k| \geq 1$ , then:

$$\|\varphi^+\|_{L_2(\mathbb{R}^d)} \leq \frac{\|\varphi_0^+\|_{L_2(\mathbb{R}^d)}}{1 - Q}, \quad (102)$$

where

$$\|\varphi_0^+\|_{L_2(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \frac{dx}{(1 + |x|^2)^s} \right)^{1/2} =: I_d(s). \quad (103)$$

*Proof of Lemma 1.* Using (97), (98), (102) we obtain that

$$\begin{aligned} |\psi^+(x, k) - e^{ikx}| &\leq \left| \int_{B_r} G^+(x - y, k)v(y) \langle y \rangle^s \varphi^+(y, k) dy \right| \leq \\ &\leq \|\varphi^+(\cdot, k)\|_{L_2(\mathbb{R}^d)} J(x) \leq \frac{\|\varphi_0^+(\cdot, k)\|_{L_2} J(x)}{1 - Q}, \quad x \in \mathbb{R}^d, \end{aligned} \quad (104)$$

$$J(x) = \left( \int_{B_r} |G^+(x - y, k)v(y) \langle y \rangle^s|^2 dy \right)^{1/2}. \quad (105)$$



For  $d = 3$  we have:

$$\begin{aligned}
\|\varphi_0^+\|_{L_2(R^3)}^2 &\leq \int_{R^3} \frac{dy}{(1+|y|^2)^s} \leq 4\pi \int_0^\infty \frac{r^2 dr}{(1+r^2)^s} \leq \\
&\leq 4\pi \left( \int_0^1 \frac{r^2 dr}{(1+r^2)^s} + \int_1^\infty \frac{r^2 dr}{(1+r^2)^s} \right) \leq \\
&\leq 4\pi \left( \frac{1}{3} + \int_1^\infty \frac{r^2 dr}{(r^2)^s} \right) = 4\pi \left( \frac{1}{3} + \frac{1}{(2s-1)} \right) \leq \frac{10}{3}\pi; \tag{106}
\end{aligned}$$

$$J(x) = \left( \int_{B_r} \frac{v^2(y)(1+y^2)^s}{16\pi^2|x-y|^2} dy \right)^{1/2} \leq \frac{\|v\|_{L_\infty(D)}(1+r^2)^{s/2}}{4\pi} \left( \int_{B_r} \frac{1}{|y|^2} dy \right)^{1/2} \tag{107}$$

$$\leq \frac{1}{4\pi} \|v\|_{L_\infty(D)}(1+r^2)^{s/2} \|(4\pi \int_0^r d\rho)^{1/2}\| = \frac{\|v\|_{L_\infty(D)}(1+r^2)^{s/2}r^{1/2}}{2\sqrt{\pi}}, x \in R^3. \tag{108}$$

For  $d = 2$  we have:

$$\begin{aligned}
\|\varphi_0^+\|_{L_2}^2 &= \int_{R^d} |\varphi_0^+(y, k)|^2 dy = \int_{R^2} \frac{dy}{(1+|y|^2)^s} = \\
&= \pi \int_0^\infty \frac{dr^2}{(1+r^2)^s} \leq \pi \int_0^\infty \frac{dz}{(1+z)^s} = \frac{2\pi}{\varepsilon}; \tag{109}
\end{aligned}$$

$$\begin{aligned}
J(x) &= \left( \int_{B_r} \frac{|H_0^{(1)}(|k||x-y)|^2}{16} |v(y)|^2 (1+|y|^2)^s dy \right)^{1/2} \leq \\
&\leq \|v\|_{L_\infty(D)}(1+r^2)^{s/2} \frac{1}{4} \left( \int_{B_r} |H_0^{(1)}(|k||x-y)|^2 dy \right)^{1/2}. \tag{110}
\end{aligned}$$

From (93) we obtain:

$$|H_0^{(1)}(s)| \leq \left( \frac{2}{\pi s} \right)^{1/2} \frac{1}{\Gamma(1/2)} \int_0^\infty e^{-u} u^{-1/2} \left| 1 + \frac{iu}{2s} \right|^{-1/2} du \leq \left( \frac{2}{\pi s} \right)^{1/2}. \tag{111}$$

Hence, we have:

$$\begin{aligned}
&\left( \int_{B_r} |H_0^{(1)}(|k||x-y)|^2 dy \right)^{1/2} \leq \left( \int_{B_r} \frac{2}{\pi|k||x-y|} dy \right)^{1/2} \leq \\
&\leq \left( \int_0^r 2\pi r \frac{2}{\pi|k|r} dr \right)^{1/2} = 2\sqrt{\frac{r}{|k|}}. \tag{112}
\end{aligned}$$

Estimate (25) follows from (104), (106), (107).

Estimate (26) follows from (104), (109), (110), (112).

Lemma 9 is proved.

*Proof of Lemma 2 (A).* Using the Lippmann-Schwinger equation (4) for  $d = 3$ , we obtain that

$$\begin{aligned} \|\psi^+\|_\infty &\leq 1 + \|\psi^+\|_\infty \|v\|_\infty \int_{B_r} \frac{dx}{4\pi|x|} \leq 1 + \|\psi^+\|_\infty \|v\|_\infty \int_0^r \frac{r^2}{r} dr \leq \\ &\leq 1 + \|\psi^+\|_\infty \|v\|_\infty \frac{r^2}{2}. \end{aligned} \quad (113)$$

Estimates (27) follows from (113).

**(B).** Using the Lippmann-Schwinger equation (4) for  $d = 2$ , we obtain that

$$\begin{aligned} \|\psi^+\|_\infty &\leq 1 + \|\psi^+\|_\infty \|v\|_\infty \int_{B_r} \frac{|H_0^1(|x||k|)|dx}{4} \leq 1 + \frac{\|\psi^+\|_\infty \|v\|_\infty}{4} \int_0^r \left(\frac{2}{\pi|k|r}\right)^{1/2} 2\pi r dr \leq \\ &\leq 1 + \|\psi^+\|_\infty \|v\|_\infty \sqrt{\frac{\pi}{2|k|}} \int_0^r r^{1/2} dr \leq 1 + \sqrt{\frac{2\pi}{|k|}} \frac{\|\psi^+\|_\infty \|v\|_\infty r^{3/2}}{3}. \end{aligned} \quad (114)$$

Estimates (28) follows from (114).

Lemma 10 is proved.  $\square$

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