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# COLLUSIONS AND QUOTIENTS: GENERALIZING EQUIVALENCE RELATIONS AND DEFINITIONS BY ABSTRACTION

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**§1. Introduction.** During the XIXth century, the increasing diffusion in mathematical practice of what the Peano school coined “definitions by abstraction”<sup>1</sup> (i.e. Definitions which “allow us to talk about certain entities by defining what their equality means even if we are not in a position to provide an explicit definition of the terms flanking the equality”, as P. Mancosu says) is attested through various instances (see [1]). Among numerous examples, one may cite *cardinal numbers* abstracted from bijectability between sets, a.k.a. *Hume’s principle*<sup>2</sup>; *directions* abstracted from parallelism relation between lines; *shapes* abstracted from topological homologies; von Helmholtz’s *weights, brightness, pitch of tones* etc.

## 1.1. Historical overview on the theory of definitions by abstraction.

The Peano’s School (Peano, Burali-Forti, Vailati, Padoa, Pieri and others) seems to have provided the first systematic focus on the *theory* of such definitions (within a programme targeting a typology of mathematical definitions), including their characterization as statements of the form:  $f_R(x) = f_R(y) \Leftrightarrow xRy$  where  $f_R$  is a newly introduced operator and  $xRy$  is a binary predicate satisfying the properties of what has been progressively called *equivalence relations*. Their debates were partly definitional (determination of the relevant properties of the relation  $R$ , discussions about the possible codomains for  $f_R$ ), epistemological (discussions about the status for such statements *as definitions*, compared in particular to nominal, stipulatory definitions and to implicit, axiomatic definitions) and ontological (how to relate the status for the *abstracta* to the status of the objects with respect to which the later are induced).

In sensibly the same period, notably from his *Grundlagen der Arithmetik* (see

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<sup>1</sup> Following P. Mancosu, the term *definitions by abstraction* appeared in print in 1894, but it was used earlier by members of the Peano school at least from 1888.

<sup>2</sup> In §63 of his *Grundlagen*, Frege cites Hume as an ancestor of this idea (meanwhile developed by Cantor). The terminology *Hume’s principle* comes from [3].

[17], section 6), Frege also investigated this kind of “definitions”, focusing specifically on higher order abstraction, as tools for his logicist programme.

Early, some problematic epistemological aspects of “definitions by abstraction” *as definitions* have been pointed out: statements of the form presented above do not guarantee unicity nor existence for the definiendum  $f_R$ , and they are generally not eliminable. From 1902, invoking those defects, Russell pleads for abandoning the terminology “definitions by abstraction” when describing statements of the form presented above<sup>3</sup>. He instead qualifies them as “Abstraction principles”, bringing out the role of the introduced operators  $f_R$  as classifiers mapping any element of the domain of  $R$  to its *equivalence class*, and advocating for limiting abstraction principles to that interpretation<sup>4</sup>. The treatment proposed by Russell in those years is centered on the nowadays standard remark that an equivalence relation  $R$  over a set  $a$  induces a *classification of all the elements  $a$  into non empty disjoint subsets of  $a$*  (or, as we would say nowadays, that the set of equivalence classes induced by  $R$  forms a *partition* of  $a$ )<sup>5</sup>.

Even if Russell’s treatment reducing *abstracta* to *equivalence classes* clarified the epistemological status of definitions by abstraction as well as the status of *abstracta* henceforth set theoretically conceived (to such an extent that, later on, Russell himself will concede that Abstraction principles are Occam’s razors which “dispense with abstraction”, see [4], p. 326), this did not lead the debate at its end.

Let us mention a first line, developed in particular by P. Lorenzen (see [10], [11] and [12]), pointing out that the subtraction which accompanies abstraction (entities “disappear” through the equalization induced by the quotientation) propagates to predicates initially definable over the vanishing individuals: some of those predicates cease afterwards to be distinguishable, in other terms they also are equalized, and it is with respect to the “surviving” ones that the abstraction need to be justified. In particular, inasmuch equivalence classes are grasped through particular representatives, the means required to define those ones may be submitted to constructivity demands.

A second line, in particular in J. Vuillemin, stresses that, as the definiens of equivalence classes includes *definite descriptions* (*abstraction* binders) not belonging to the original primitive language, their eliminability has in any case to be justified (cf. J. Vuillemin [7] and [8]), especially as equivalence classes cannot always be defined *by separation* (indeed, in bad cases, they are not sets, but proper classes defined *by comprehension*, as in the case of the bijectability relation).

<sup>3</sup> Cf. [5], §.110, p.114-115 and [19].

<sup>4</sup> Following F. Consuegra, p.132 of [6], that view was already sensible or explicit in the Peano School, with a priority accorded to Pieri by Peano himself. For Mancosu [1], the first explicit reading of equivalence classes as being the defined “entities” – rather than what we would call today a specific *representant of the class* or maybe a proper, independent entity – is also made early by Frege, influenced in that matter by observations of Grassmann from 1844.

<sup>5</sup> Following Mancosu, even if a systematic and complete exposition of the partitioning/quotienting discipline seems to occur only in the late twenties (maybe for the first time in van der Waerdens *Abstrakte Algebra* [9], in the § 5, entitled “Klasseneinteilung. Äquivalenzrelation”), “the technical details were already clear in the 1910s” (see [1], p. 20)

A third line, has been extensively studied by the so called neo-fregean trend initiated by C. Wright in [13], properly giving a second life to the investigations on *Abstraction principles* (see [14], [16]). Those works focus on the fact that higher order principles (typically when the domain of the operator  $f_R$  is the powerset of the co-domain of  $f_R$ ) may induce an injection from the domain to the codomain, hence contradicting Cantor’s theorem. The canonical example of this problem is Frege’s Basic Law V,  $Ext(x) = Ext(y) \Leftrightarrow \forall z (z \in x \leftrightarrow z \in y)$  (the second abstraction principle introduced in the Grungesetze with fatal consequences, beside to Hume’s principle), which assigns objects to concepts and is based on the equi-membership relation. If the domain of concepts is the power-set of the first-order codomain, then, as the operator  $Ext$  (for Extension) is then an injection of concepts into objects, this enters in contradiction with Cantor’s theorem<sup>6</sup>.

**1.2. Aim and structure of the article.** The aim of our paper is to introduce and investigate a generalization of definitions by abstraction<sup>7</sup>. In the § 1, we start introducing a generalization of equivalence relations – *collusion relations* – able to *characterize* the partitioning discipline. Indeed, to be an equivalence relation is a sufficient but *not necessary* condition with that respect (the fact that one can go from whatever equivalence relation to the partition it induces, and then back from that partition to the original equivalence relation, sometimes misleads into believing that the properties defining equivalence relations form a condition not only sufficient but also *necessary* for partitioning – which is not the case). The § 2 is devoted to show that the quotients induced by collusions (collusional quotients) may be recovered as standard equi-quotients induced by equivalence relations defined in terms of the original collusions. The § 3 focuses on the less familiar collusions, namely the non equivalence ones, and investigates in particular whether any partition may be induced by a non equivalence collusion. The § 4, concentrates on how to extend to collusions the theory of abstraction proposed by the Peano’s school for equivalence relations. This requires a double shift: a mathematical one and a philosophical one. *Mathematically*, a wider version of “Abstraction principles” has to be formulated: indeed, under penalty of inconsistency, introducing a classifier  $f_R$  such that  $f_R(x) = f_R(y) \Leftrightarrow xRy$  requires that  $R$  is an equivalence relation, so that one cannot just replace *equivalence* by *collusion* in the formulation of “Abstraction Principles”. *Philosophically*, the “sameness interpretation” of equivalence relations (a commonplace from Peano’s school to Frege-Russell and further) has to be complementarily replaced by another interpretation relevant for general (non equivalence) collusions. The point is not so much to renounce to the conception that a definition by abstraction captures some form of similarity (by abstracting, subtracting differences) and favours “a change of scale” (the classes are higher order notions), than to abandon the idea that the relation  $R$  itself expresses that sameness. To replace the “sameness interpretation”, we will progressively present, along the various sections, an atypic

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<sup>6</sup> A short and good introduction to those problematics can be found at [www.bbk.ac.uk/philosophy/our-research/ppp/ConferenceAntonelli.pdf](http://www.bbk.ac.uk/philosophy/our-research/ppp/ConferenceAntonelli.pdf)

<sup>7</sup> In particular, we will leave for further works an evaluation of the impact of that generalization of “Abstraction principles” on the debates of the so called neo-fregean trend initiated by [13], which gave a second life to the investigations on *Abstraction principles* (see [14], [16]).

interpretation of quotientation and classes in terms of “coalitions”: *the agonal interpretation*.

## §2. Collusions and equivalences: partitioning.

NOTATIONS, TERMINOLOGY, CONVENTIONS. By default,  $R$  will always denote a binary relation on a non empty set  $a$ . When one has  $xRy$  and  $x'Ry$  (for some  $x, x', y \in a$ ), one says that  $x$  and  $x'$  *do converge (on  $y$ )*. Whenever  $x \in a$ , the subset of  $a$ , noted  $[x]_R$  and defined by  $[x]_R =_{\text{def}} \{y \in a ; xRy\}$  is called *the class of  $x$  (for  $R$ )*. But, in the special case  $R$  is an equivalence relation,  $[x]_R$  will be sometimes called the *equivalence class of  $x$  for  $R$* , as usual. If  $b \subseteq a$  is such that  $b = [x]_R$ , for some  $x \in a$ , one says that  $b$  is *a class (for  $R$ )*. Let us recall that a set  $P$  of subsets of  $a$  forms a *partition* of  $a$ , whenever the three following conditions are satisfied:

- for any  $b, c \in P$ , ( $b \neq c \Rightarrow b \cap c = \emptyset$ ) (pairwise disjointness)
- for any  $b \in P$ ,  $b \neq \emptyset$  (non emptiness)
- $a \subseteq \bigcup P$  (exhaustivity).

Reformulated with the notations and terminology just introduced, the standard observation recalled in our introduction and upon which Peano’s, Frege’s and Russell’s approach of abstraction is based, is thus that in case  $R$  is an equivalence relation, the set  $\{[x]_R\}_{x \in a}$  of classes form a partition of  $a$ .

We will now see that though sufficient to get a partition, this condition is however not necessary.

DEFINITION 2.1. The relation  $R$  over  $a$  is:

- *total*, if  $\forall x \in a \exists y \in a xRy$
- *surjective*, if  $\forall y \in a \exists x \in a xRy$
- *collusive*, if  $\forall x \in a \forall x' \in a (\exists y \in a (xRy \wedge x'Ry) \Rightarrow \forall y \in a (xRy \Rightarrow x'Ry))$
- *a collusion relation*, if it is total, surjective and collusive

*Remark 1.* The meaning of the *collusivity* property is more salient (“convergence somewhere entails convergence everywhere”) when one expresses it in the following redundant form:  $\forall x \in a \forall x' \in a (\exists y \in a (xRy \wedge x'Ry) \Rightarrow \forall y \in a (xRy \Leftrightarrow x'Ry))$ .

THEOREM 1. *Let  $R$  be a binary relation on  $a$ .*

$$R \text{ is an equivalence relation} \Leftrightarrow R \text{ is collusive and reflexive}$$

PROOF.

- $\Rightarrow$  Let us show that any symmetrical and transitive relation is collusive. Let  $x, x' \in a$ . Let us assume that  $xRy$  and  $x'Ry$  for some  $y \in a$  and that  $xRy'$  for some  $y' \in a$ . If  $R$  is symmetrical, as  $xRy$ , one has  $yRx$ . So if  $R$  is transitive, as  $x'Ry$ , one has  $x'Ry'$ , and as  $xRy'$ , one has  $x'Ry'$ .
- $\Leftarrow$  • Let us first show that any collusive and reflexive relation, is symmetrical. Let  $x, y \in a$ , such that  $xRy$ . If  $R$  is reflexive,  $yRy$ , so that  $x$  and  $y$  do converge (on  $y$ ). If  $R$  is collusive,  $x$  and  $y$  must have exactly the same images by  $R$ . As  $xRx$  ( $R$  reflexive being assumed), one thus must also have  $yRx$ .

- Let us now show that any collusive and reflexive relation, is transitive. Let  $x, y, z \in a$ , such that  $xRy$  and  $yRz$ . If  $R$  is reflexive,  $yRy$ . So that  $x$  and  $y$  do converge (on  $y$ ). Hence, if  $R$  is collusive, they must have exactly the same images by  $R$ . So, as  $yRz$ , one must have  $xRz$ .

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*Remark 2.* As any reflexive relation is evidently total and surjective, any equivalence relation is a collusion. But in general, collusions are not equivalence relations (actually, some non reflexive collusions are not symmetrical or not transitive; some of them have none of these properties).

We now will see (see the three lemmas below) that collusivity, surjectivity and totality (which, by the way, are completely independent properties) correspond exactly and respectively to the three conditions which define partitions: pairwise disjointness, exhaustivity, non emptiness. Consequently, the conjunction of those three properties (which defines collusions) is a sufficient and necessary condition to produce a partition of the set on which the relation is defined.

**THEOREM 2.**  $\{[x]_R\}_{x \in a}$  is a partition of  $a \Leftrightarrow R$  is a collusion

**PROOF.** By the lemmas 3, 4 and 5 which follow.

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**LEMMA 3.**  $\emptyset \notin \{[x]_R\}_{x \in a} \Leftrightarrow R$  is total

**PROOF.**

- $\Rightarrow$  If  $R$  is not total, then some  $x \in a$  has no image for  $R$ . But then  $[x]_R = \emptyset$ .
- $\Leftarrow$  Let  $b \in \{[x]_R\}_{x \in a}$ . Then  $b = [x]_R$  for some  $x \in a$ . If  $R$  is total, there exists  $y \in a$ , such that  $xRy$ . Thus  $y \in [x]_R$ , hence  $[x]_R \neq \emptyset$

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**LEMMA 4.**  $a \subseteq \bigcup_{x \in a} [x]_R \Leftrightarrow R$  is surjective

**PROOF.**

- $\Rightarrow$  If  $R$  is not surjective, then some  $y \in a$  has no antecedent for  $R$ . But then  $y$  belongs to no  $b$  in  $\{[x]_R\}_{x \in a}$ . So again  $\{[x]_R\}_{x \in a}$  is not a partition of  $a$ .
- $\Leftarrow$  Let  $y \in a$ . As  $R$  is surjective,  $xRy$  for some  $x \in a$ , so that  $y \in [x]_R$ , i.e.  $y$  belongs to at least one class belonging to  $a/R$ .

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**LEMMA 5.**  $(\forall b, b' \in \{[x]_R\}_{x \in a} (b \cap b' \neq \emptyset \Rightarrow b = b')) \Leftrightarrow R$  is collusive

**PROOF.**

- $\Rightarrow$  If  $R$  is not collusive, there exists  $x, x', y, z \in a$  such that:
  1.  $x$  and  $x'$  do converge on  $y$ , which implies that  $[x]_R \cap [x']_R \neq \emptyset$
  2.  $xRz$  but  $x'Rz$ , which implies that  $[x]_R \neq [x']_R$
- $\Leftarrow$  Let  $b, b' \in \{[x]_R\}_{x \in a}$  such that  $b \cap b' \neq \emptyset$ . Let  $y \in b \cap b'$ . By definition of  $\{[x]_R\}_{x \in a}$ , we know that  $b = [x]_R$  for some  $x \in a$  and that  $b' = [x']_R$  for some  $x' \in a$ . As  $y \in [x]_R$ , one has  $xRy$  and as  $y \in [x']_R$ , one has  $x'Ry$ . So that  $x$  and  $x'$  do converge on  $y$ . Now,  $R$  being collusive, we know that  $x$  and  $x'$  do converge on the same elements, so that  $[x]_R = [x']_R$ , i.e.  $b = b'$ .

⊣

TERMINOLOGY AND NOTATION Because of theorem 2, in case  $R$  is a collusion, we will continue to use the standard terminology (*quotient*) and notation  $(a/R)$  for calling and noting  $\{[x]_R\}_{x \in a}$ . However, in situations where we wish to underline that the collusion generating such or such quotient under consideration is an equivalence relation, we will call it an *equi-quotient*.

**§3. Collusional quotients as equi-quotients.** As they introduce an *equality*, “Abstraction principles” underpin the traditional philosophical interpretation of *equivalence* relations in terms of *sameness*: following such a reading, the fact that two elements are in such a relation suggests their similarity *from a point of view* expressed by  $R$ , a similarity which becomes salient when one forms the quotient, i.e. once one forgets all other points of view with respect to which those elements elsewhere differ. Clearly, as some collusions are not equivalence (and in particular not reflexive), such a reading cannot apply anymore ; and to substitute to the *sameness* interpretation an adequate interpretation would be welcome.

A rather intuitive and actually fruitful candidate is what we call the *agonal* interpretation (from *agôn* i.e. *fight* in ancient greek), an interpretation which will be clearer once the results ending the current section will be presented. To give meanwhile a hint, let us stress that the word “collusion” has been chosen in the line of that interpretation. If one reads  $xRy$  as *x targets y* (or as *y is targeted by x* as well) or maybe, using a gorier but mnemonically efficient declination of the same idea, as *x is a predator for y* (or *y is a prey for x* as well), the *collusivity* property corresponds to situations where any two individuals having a common target did collude: from then on, all the targets of one of them are targets of the other one. As we will see, which such an interpretation in mind, the status of “abstracta” induced by collusions, i.e. the *classes* belonging to the quotients induced by them, receive a clear meaning: they are *coalitions* of individuals having the same targets.

To start with, let us prove the following crucial property of collusions.

LEMMA 6. *Collusions are closed by converse:  $R$  collusion  $\Rightarrow R^{-1}$  collusion.*

PROOF.

- $R$  collusive  $\Rightarrow R^{-1}$  collusive:  
Let  $x, x' \in a$  converging on some  $y \in a$  for  $R^{-1}$ . Let now  $z \in a$  such that  $xR^{-1}z$ . As  $xR^{-1}y$  and  $xR^{-1}z$ , one has  $yRx$  and  $zRx$ , so that  $y$  and  $z$  do converge (on  $x$ ). So, if  $R$  is collusive,  $y$  and  $z$  have exactly the same images; so, knowing that  $yRx'$ , one must also have  $zRx'$ . Hence  $x'R^{-1}z$ . Hence  $R^{-1}$  is collusive.
- $R$  total and surjective  $\Rightarrow R^{-1}$  surjective and total:  
As the switching from  $R$  to  $R^{-1}$  inverts the role of images and antecedents, it maps surjectivity on totality and conversely.

⊣

Collusions being closed by converse, any collusion  $R$  over  $a$  actually induces *two* partitions:  $a/R$  and  $a/R^{-1}$ . They will be called the (two) *collusional quotients* generated by  $R$ .

We will now see how the two collusional quotients generated by a collusion  $R$  can be recovered as equi-quotients generated from two equivalence relations *defined in terms of  $R$* .

**DEFINITION 3.1.** (equi-targeting relations) Let  $R$  any binary relation on  $a$ . The two binary relations  $\overset{\text{td}}{\sim}_R$  and  $\overset{\text{tr}}{\sim}_R$  on  $a$  are defined as follows:

- $x \overset{\text{tr}}{\sim}_R x' \Leftrightarrow_{\text{def}} \forall y \in a (xRy \Leftrightarrow x'Ry)$   
 When  $x \overset{\text{tr}}{\sim}_R x'$ , one says that  $x, x'$  are *equi-targeters* in  $R$  (or, when stressing the agonal interpretation, that they are *coalized for attack*);
- $x \overset{\text{td}}{\sim}_R x' \Leftrightarrow_{\text{def}} \forall y \in a (yRx \Leftrightarrow yRx')$   
 When  $x \overset{\text{td}}{\sim}_R x'$ , one says that  $x, x'$  are *equi-targeted* in  $R$  (or that they are *coalized for defence*).

*Remark 3.*  $\overset{\text{td}}{\sim}_R$  and  $\overset{\text{tr}}{\sim}_R$  are (evidently) equivalence relations. By the way, they could have been defined in terms of equality of classes as  $x \overset{\text{tr}}{\sim}_R x' \Leftrightarrow [x]_R = [x']_R$  and  $x \overset{\text{td}}{\sim}_R x' \Leftrightarrow [x]_{R^{-1}} = [x']_{R^{-1}}$ .

**LEMMA 7.**  $\overset{\text{td}}{\sim}_R = \overset{\text{tr}}{\sim}_{R^{-1}}$     and     $\overset{\text{tr}}{\sim}_R = \overset{\text{td}}{\sim}_{R^{-1}}$ .

**PROOF.** For the first statement, the converse operation on binary relations switches the first notion to the second one. The second one then follows from the first, as the converse operation is involutive.  $\dashv$

**LEMMA 8.** *Let  $R$  be a collusive relation on  $a$ . If  $xRy$ , then:*

1.  $[x]_R = [y]_{\overset{\text{td}}{\sim}_R}$
2.  $[y]_{R^{-1}} = [x]_{\overset{\text{tr}}{\sim}_R}$

**PROOF.** Let us assume that  $xRy$ .

1.  $[x]_R = [y]_{\overset{\text{td}}{\sim}_R}$  :
  - Let  $y' \in [x]_R$ . This entails that  $xRy'$ . As we have also  $xRy$ , this means that  $y$  and  $y'$  do converge on  $x$  for  $R^{-1}$ . Now, if  $R$  is collusive, so is  $R^{-1}$  (by lemma 6). The fact that  $y$  and  $y'$  do converge thus implies that  $y \overset{\text{tr}}{\sim}_{R^{-1}} y'$  or, equivalently, that  $y \overset{\text{td}}{\sim}_R y'$ . So that  $y' \in [y]_{\overset{\text{td}}{\sim}_R}$ . We thus proved that  $[x]_R \subseteq [y]_{\overset{\text{td}}{\sim}_R}$ .
  - Let  $y' \in [y]_{\overset{\text{td}}{\sim}_R}$ . One thus has  $y \overset{\text{td}}{\sim}_R y'$  and, as  $xRy$ , one must have  $xRy'$ , so that  $y' \in [x]_R$ . We thus proved that  $[y]_{\overset{\text{td}}{\sim}_R} \subseteq [x]_R$  (an inclusion by the way valid even when  $R$  is not a collusion).



2.  $[y]_{R^{-1}} = [x]_{\sim_R^{\text{tr}}}$ . As  $xRy$ , one has  $yR^{-1}x$ . As  $R$  is collusive, so is  $R^{-1}$  (by lemma 6), so that item 1. applies, which means  $[y]_{R^{-1}} = [x]_{\sim_{R^{-1}}^{\text{td}}}$ . So  $[y]_{R^{-1}} = [x]_{\sim_R^{\text{tr}}}$

⊣

PROPOSITION 9. *Let  $R$  be a relation over  $a$ .*

$$\{[x]_R\}_{x \in a} = a/\sim_R^{\text{td}} \quad \Leftrightarrow \quad R \text{ is a collusion}$$

PROOF.

⇒ As  $\sim_R^{\text{td}}$  is an equivalence,  $a/\sim_R^{\text{td}}$  is a partition of  $a$ . So, if  $a/\sim_R^{\text{td}} = \{[x]_R\}_{x \in a}$ , then  $\{[x]_R\}_{x \in a}$  is also a partition of  $a$ . By theorem 2, this implies that  $R$  is a collusion.

⇐  $a/\sim_R^{\text{td}} \subseteq a/R$ .

Let  $b \in a/\sim_R^{\text{td}}$ . By definition of  $a/\sim_R^{\text{td}}$ ,  $b = [y]_{\sim_R^{\text{td}}}$  for some  $y \in a$ . If  $R$  is a collusion, it is surjective and there exists  $x \in a$  such that  $xRy$ . Then by lemma 8, one has  $[y]_{\sim_R^{\text{td}}} = [x]_R$ . So  $b \in a/R$ .

–  $a/R \subseteq a/\sim_R^{\text{td}}$ .

Let  $b \in a/R$ . By definition of  $a/R$ , one has  $b = [x]_R$  for some  $x \in a$ . If  $R$  is a collusion, it is total and one has  $xRy$  for some  $y \in a$ . Then by lemma 8, one has  $[x]_R = [y]_{\sim_R^{\text{td}}}$ . So  $b \in a/\sim_R^{\text{td}}$ .

⊣

Note that the “⇐” direction of this proposition gives us a second proof for the “⇐” direction of our theorem 2, which states that  $\{[x]_R\}_{x \in a}$  is a partition of  $a$ , inasmuch the relation  $R$  on  $a$  is a collusion.

PROPOSITION 10. *Let  $R$  be a relation over  $a$ .*

$$\{[x]_{R^{-1}}\}_{x \in a} = a/\sim_R^{\text{tr}} \quad \Leftrightarrow \quad R \text{ is a collusion}$$

PROOF.

$R$  is a collusion ⇐  $R^{-1}$  is a collusion (by lemma 6)

$$\Leftrightarrow \{[x]_{R^{-1}}\}_{x \in a} = a/\sim_{R^{-1}}^{\text{td}} \text{ (by proposition 9).}$$

$$\Leftrightarrow \{[x]_{R^{-1}}\}_{x \in a} = a/\sim_R^{\text{tr}} \text{ (by lemma 7).}$$

⊣

THEOREM 11.  $\{[x]_R\}_{x \in a} = a/\sim_R^{\text{td}} \Leftrightarrow R \text{ is a collusion} \Leftrightarrow \{[x]_{R^{-1}}\}_{x \in a} = a/\sim_R^{\text{tr}}$

PROOF. By propositions 9 and 10.

⊣

To make this result more salient, we could rephrase it by abusively using the generalized quotientation notation (*a priori* potentially inappropriate for the first ⇒ and the second ⇐, as the notation  $a/R$  is well defined only for collusions – even if it appears *a posteriori* non problematic):

THEOREM 11. (*reformulated*):

$$a/R = a/\sim_R^{\text{td}} \quad \Leftrightarrow \quad R \text{ is a collusion} \quad \Leftrightarrow \quad a/R^{-1} = a/\sim_R^{\text{tr}}$$

To conclude this section, let us go back to the agonal interpretation and introduce a bit of terminology: when a subset  $b$  of  $a$  is made of individuals pairwise colluded

for Defence (resp. for Attack), we will say that  $b$  is a D-coalition (resp. an A-coalition).

In the same manner an equivalence class is made of pairwise equivalent elements and is maximal for that property w.r.t. inclusion (a remark whose justification is not given here, for sake of brevity), the status to the classes of the collusional quotients can be described through the agonal interpretation in simple terms: they are maximal coalitions.

**§4. Focusing on non equivalence collusions.** Among collusions, the non reflexive ones (which thus are not equivalence relations) are likely unfamiliar to most readers, and this section is devoted to make some observations on them. We will in particular focus on irreflexive collusions<sup>8</sup>, i.e. the ones such that  $\forall x \in a \ x \not R x$ .

If  $R$  is an irreflexive collusion, the picture for classes belonging to the induced collusional quotient is orthogonal to the one for classes induced by equivalence relations. Indeed, as shows the second lemma below, in the irreflexive case, no one is related in  $R$  with someone inside a class (in agonal terms: the coalitions are peaceful – nobody targets a member of its own coalition).

LEMMA 12. *Let  $R$  over  $a$  collusive,  $b$  a class for  $R$  and  $x \in b$ .*

$$\exists x' \in b \ x R x' \Rightarrow \forall x' \in b \ x \not R x'$$

PROOF. Let  $y \in a$  such that  $b = [y]_R$  and let  $x' \in b$ . One thus has  $y R x'$ . So, if  $x R x'$ , then  $x$  and  $y$  do converge (on  $x'$ ). So, as  $R$  is collusive,  $x$  has exactly the same targets/images for  $R$  than  $y$ . Hence  $\forall x' \in b \ x \not R x'$ .  $\dashv$

LEMMA 13. *Let  $R$  be collusive relation on  $a$ . If  $R$  is irreflexive, then for any targeting class  $b$  for  $R$  and any  $x, x' \in b$ , one has  $x \not R x'$ .*

PROOF. One gets the contraposed implication by particularizing to  $x$  the universally quantified  $x'$  in lemma 12.  $\dashv$

In the subsections below, we will now examine some peculiarities of non reflexive collusions with respect to the partitioning discipline. It is well known that for any partition  $P$  of  $a$ , there exists an equivalence relation  $R$  on  $a$  and a *unique* one, such that  $P = a/R$ . As we will see now, the picture is very different (for existence and for unicity as well) when one considers non reflexive collusions, instead of equivalence relations: (a) some (actually rare and special) partitions cease to be “reachable”, namely the *degenerated ones*, i.e. the ones with a unique class (actually, these ones are the only non reachable ones); (b) all other partitions are reachable, but a reachable partition appears to be reachable from *different* collusions.

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<sup>8</sup> Through the agonal reading, there is nothing odd in considering auto-targeting elements, i.e. elements  $x$  such that  $x R x$ : after all, logic knows about paradoxes (and life about suicide). The important point is to observe that such situations generate special effects which do have an agonal meaning: when the relation is a collusion, the two maximal coalitions (for Attack and for Defence) to which such an auto-targeting element belongs are necessarily *degenerated coalitions*, i.e. coalitions in which everybody targets everybody.

#### 4.1. Incompleteness of non reflexive collusions w.r.t. partitions.

LEMMA 14. *Let  $R$  a collusion. If  $a/R = \{a\}$ , then  $R$  is reflexive.*

PROOF. Let us assume that  $a/R = \{a\}$ . As  $a \neq \emptyset$  (by a convention set in section 2), there is some  $x \in a$ . And as  $R$  is total, there exists  $x' \in a$  such that  $xRx'$ . Applying the lemma 12 to (the class)  $a$ , we deduce that  $\forall x, x' \in a$   $xRx'$ . In particular,  $\forall x \in a$   $xRx$ .  $\dashv$

So, when they are non reflexive, and a fortiori irreflexive, collusions are so to speak “incomplete w.r.t. partitions”: they cannot reach the degenerated one<sup>9</sup>. We will now show that the degenerated partition is however the only missed one.

#### 4.2. Irreflexive collusions are complete for non degenerated partitions.

In this subsection, whenever  $P$  is a partition of  $a$  and  $x \in a$ , we will note  $[x]_P$  (by a slight notational abuse) the unique  $b \in P$  such that  $x \in b$ .

DEFINITION 4.1. Let  $P$  a partition of  $a$  and  $\sigma$  an application from  $P$  to  $P$ . The binary relation  $R_\sigma$  over  $a$  is defined by:

$$xR_\sigma x' \Leftrightarrow_{\text{def}} \sigma([x]_P) = [x']_P$$

LEMMA 15. *Let  $P$  a partition of  $a$  and  $\sigma$  a permutation of  $a$  (i.e. a bijection from  $P$  to  $P$ ). Then  $R_\sigma$  is a collusion on  $a$ . Moreover, if  $\sigma$  has no fix-points (i.e. is such that for any  $b \in P$ ,  $\sigma(b) \neq b$ ), the collusion  $R_\sigma$  is irreflexive.*

PROOF.

- $R_\sigma$  is total and surjective. Indeed, let  $x \in a$ . As  $\sigma$  is total (resp. surjective), there exists  $b \in P$  such that  $\sigma([x]_P) = b$  (resp  $\sigma(b) = [x]_P$ ). As  $P$  is a partition,  $b$  is non empty, i.e.  $b = [x']_P$  for some  $x' \in b$ . As  $\sigma([x]_P) = [x']_P$  (resp  $\sigma([x']_P) = [x]_P$ ), by  $\sigma$ 's definition, we have that  $xR_\sigma x'$  (resp.  $x'R_\sigma x$ ). Hence,  $R_\sigma$  is total (resp. surjective).
- $R_\sigma$  is collusive. Indeed, let  $x, y, x', y' \in a$ , such that  $xR_\sigma x'$ ,  $yR_\sigma y'$  and  $xR_\sigma y'$ . By  $R_\sigma$ 's definition, one thus has  $\sigma([x]_P) = [x']_P$ ,  $\sigma([y]_P) = [y']_P$  and  $\sigma([x]_P) = [y']_P$ . So  $[x]_P = [y]_P$  (as  $\sigma$  is injective) and  $[x']_P = [y']_P$  (as  $\sigma$  is functional). So, as  $\sigma([x]_P) = [x']_P$ , one has  $\sigma([y]_P) = [y']_P$ . Thus  $yR_\sigma y'$ .
- $R_\sigma$  is irreflexive. Indeed, one never has  $xR_\sigma x$ , since for any  $b \in P$ ,  $\sigma(b) \neq b$   $\dashv$

LEMMA 16. *Let  $\sigma$  any application from  $P$  (a partition of  $a$ ) to itself. For any  $y \in \sigma([x]_P)$ , one has  $[x]_{R_\sigma} = [y]_P$ .*

PROOF. Let  $y \in \sigma([x]_P)$ . As  $[x]_{R_\sigma} = \{z \in a; xR_\sigma z\} = \{z \in a; \sigma([x]_P) = [z]_P\}$ , it amounts to show that  $\{z \in a; \sigma([x]_P) = [z]_P\} = [y]_P$

- For the  $\subseteq$  direction, let  $z \in a$  such that  $\sigma([x]_P) = [z]_P$ . As we assumed that  $y \in \sigma([x]_P)$ , we have that  $y \in [z]_P$ . Hence  $[z]_P = [y]_P$ . So  $z \in [y]_P$ .

<sup>9</sup> As a corollary, observe that if  $a$  is a singleton, no irreflexive collusion exists over  $a$ . In terms of the targeting interpretation: in a non suicidal world, one needs to be at least two to fight.

- For the  $\supseteq$  direction, let  $z \in [y]_P$ . This entails  $[y]_P = [z]_P$ . But as we assumed that  $y \in \sigma([x]_P)$ , one has  $\sigma([x]_P) = [y]_P$ . Hence  $\sigma([x]_P) = [z]_P$ .

⊖

LEMMA 17. *For any  $b, b' \in P$ :*

$$\sigma(b) = b' \Leftrightarrow \forall x \in b \forall x' \in b' \ x R_\sigma x'$$

PROOF.

⇒ Let us assume that  $\sigma(b) = b'$ , and let any  $x \in b$  and any  $x' \in b'$ . As  $b = [x]_P$  and  $b' = [x']_P$ , one has  $\sigma([x]_P) = [x']_P$ , hence  $x R_\sigma x'$ .

⇐ Let us assume that  $\forall x \in b \forall x' \in b' \ x R_\sigma x'$ , i.e.  $\forall x \in b \forall x' \in b' \ \sigma([x]_P) = [x']_P$ . As  $b, b' \neq \emptyset$ , one has for some  $x, x' \in a$  that  $[x]_P = b$  and  $[x']_P = b'$ . We thus have  $\sigma([x]_P) = [x']_P$ , i.e.  $\sigma(b) = b'$

⊖

LEMMA 18. *Let  $P$  a partition of  $a$  and  $\sigma$  an application from  $P$  to  $P$ . Then  $\sigma([x]_P) = [x]_{R_\sigma}$*

PROOF.

- $\sigma([x]_P) \subseteq [x]_{R_\sigma}$ . Indeed, let  $y \in \sigma([x]_P)$ . This implies that  $\sigma([x]_P) = [y]_P$ . By lemma 17, one has  $x R_\sigma y$ . So  $y \in [x]_{R_\sigma}$ .
- $[x]_{R_\sigma} \subseteq \sigma([x]_P)$ . Indeed, let  $y \in [x]_{R_\sigma}$ . By definition of classes, one has  $x R_\sigma y$ . By definition of  $R_\sigma$ , one has  $\sigma([x]_P) = [z]_P$ . Hence  $z \in \sigma([x]_P)$ .

⊖

PROPOSITION 19. *Let  $P$  a partition of  $a$  and  $\sigma$  a bijection from  $P$  to  $P$  with no fix-points. Then  $a/R_\sigma = P$ .*

PROOF.

1.  $a/R_\sigma \subseteq P$ . Indeed, let  $b \in a/R_\sigma$ . This means that, for some  $x \in a$ , one has  $b = [x]_{R_\sigma}$ . As  $b$  belongs to  $a/R_\sigma$  (which, by lemma 15 and theorem 2, forms a partition of  $a$ ), it is non empty. Let then  $y \in b$ . As  $y \in [x]_{R_\sigma}$ , by definition of targeting classes, one has  $x R_\sigma y$ , which, by definition of  $R_\sigma$  means that  $\sigma([x]_P) = [y]_P$ . One thus has  $y \in \sigma([x]_P)$ . Hence, by the lemma 16,  $[x]_{R_\sigma} = [y]_P$ . Hence  $b \in P$ .
2.  $P \subseteq a/R_\sigma$ . Indeed, let  $b' \in P$ . Classes in a partition being non empty, there exists  $x' \in a$  such that  $b' = [x']_P$ . As  $\sigma$  is a surjective, there exists  $b \in P$  such that  $\sigma(b) = b'$ . Again, there exists  $x \in a$  such that  $b = [x]_P$ . Finally, one has  $b' = \sigma(b) = \sigma([x]_P) = [x]_{R_\sigma}$  (by lemma 18). Finally, one thus has  $b \in a/R_\sigma$ .

⊖

THEOREM 20. *Let  $P$  a non degenerated partition of  $a$ , i.e. with  $P \neq \{a\}$ . There exists an irreflexive collusion such that the collusional quotient it induces coincides with  $P$ <sup>10</sup>.*

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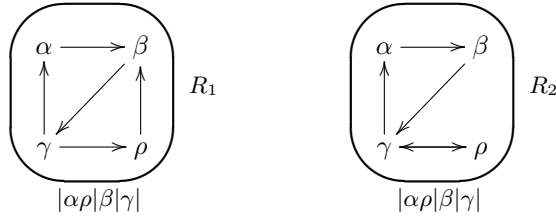
<sup>10</sup> A similar proof has been given independently by Paulo Veloso – private communication, unpublished.

PROOF. Because of proposition 19, we just need to find a permutation  $\sigma$  of  $P$  without fix points. While  $P$  is finite or denumerable, any “*successor like*” relation with no endpoints (see below) would do the job. However, this being not suitable for the non denumerable case, we treat the infinite case through another strategy.

- in case  $P$  is finite, to get a permutation of  $P$  without fix points, we choose for  $\sigma$  any circular permutation with a unique cycle including all the elements of  $a$ , in other terms such that any element of  $P$  has any element of  $P$  in its orbit (note that the *no fix points condition* is satisfiable if and only if the cardinal of partition  $P$  is strictly greater than 1, i.e. when the partition is non degenerated, as assumed);
- in case  $P$  is denumerable,  $\sigma$  could be the binary relation (on  $P$ ) which is the image of the successor relation on  $\mathbb{Z}$  through any bijection from  $\mathbb{Z}$  to  $P$ .
- in case  $P$  is not denumerable (but that treatment applies also for a denumerable  $P$ ), let us consider any  $P_1, P_2 \subseteq P$  such that  $\{P_1, P_2\}$  is a partition of  $P$  and  $\sharp(P_1) = \sharp(P_2)$ . The existence of such a partition of  $P$  is for instance proved in [18] (p. 184 and 287-288; Axiom of Choice used). As  $\sharp(P_1) = \sharp(P_2)$ , there exists a bijection  $\beta$  from  $P_1$  to  $P_2$ . We then define the binary relation  $\sigma$  on  $P$  as  $\beta \cup \beta^{-1}$ . As  $\{P_1, P_2\}$  is a partition of  $P$ , it is immediate, by construction, that  $\sigma$  is a permutation of  $P$  with no fix points. ◊

Let us close this subsection by stressing an intentional refinement introduced by non reflexive collusions.

*Remark 4.* Extensionally, there exists a unique equivalence relation generating a given partition  $P$  (i.e.,  $P$  being given, there is a *unique* reflexive collusion, i.e. a unique equivalence relation,  $R$  on  $a$  such that  $P = a/R$ ). Such an “extensional collapse” does not happen when one also considers non reflexive collusions. Then, indeed, a given partition  $P$  of  $a$  may generally well be generated through *different* collusions on  $a$  (see the relations  $R_1$  and  $R_2$  over  $a = \{\alpha, \beta, \gamma, \rho\}$  pictured below as sagittal diagrams, where the partition induced by  $R_i$  is noted under each diagram).



**§5. Generalizing Abstraction principles.** In this final section, we come back to “Abstraction Principles”:

$$f_R(x) = f_R(y) \Leftrightarrow xRy$$

As introducing such a statement *requires* that  $R$  is an equivalence, one needs to modify their formulation, inasmuch one wants to cover the (more general case of) collusion relations.

Let  $R$  be *any* binary relation over a set  $a$ . We consider the two following “Generalized Abstraction Principles”, where  $R$  is a collusion relation:

1.  $d$ -Generalized Abstraction Principle:  $f_R(x) = f_R(y) \Leftrightarrow x \overset{\text{td}}{\sim}_R y$
2.  $a$ -Generalized-Abstraction principle:  $f_R(x) = f_R(y) \Leftrightarrow x \overset{\text{tr}}{\sim}_R y$

Let us note that the word “generalization” may seem odd in the current situation. Indeed, as for any relation  $R$ , the relations  $\overset{\text{td}}{\sim}_R$  and  $\overset{\text{tr}}{\sim}_R$  are equivalence relations, one could prefer to say that “Regular Abstraction principles” can be *narrowed* to the particular case where the equivalence relation is of the form indicated in these “Generalized” abstraction principles ( $\overset{\text{td}}{\sim}_R$  or  $\overset{\text{tr}}{\sim}_R$ ). Because of theorem 11, the word *generalization* is nevertheless relevant: inasmuch  $R$  is a collusion, one can *extend* the solution proposed by Russell for determining the co-domain of  $f_R$  (i.e.  $f_R(x)$  then denotes the class of  $x$  in the corresponding collusional quotient). Let us finally observe that, because of the proposition below, in the particular case  $R$  is an equivalence relation, both those Generalized Abstraction Principles collapse to the usual, regular “Abstraction principles”, i.e. they both are equivalent to:

$$f_R(x) = f_R(y) \Leftrightarrow xRy$$

PROPOSITION 21.

$$R \text{ is an equivalence relation on } a \Leftrightarrow R = \overset{\text{tr}}{\sim}_R \Leftrightarrow R = \overset{\text{td}}{\sim}_R$$

PROOF. The  $\Rightarrow$  directions amount to say that, when  $R$  is an equivalence relation,  $xRy \Leftrightarrow [x]_R = [y]_R$  (a standard observation, already made by Burali-Forti) and  $xRy \Leftrightarrow [x]_{R^{-1}} = [y]_{R^{-1}}$ . The  $\Leftarrow$  directions are immediate as  $\overset{\text{td}}{\sim}_R$  and  $\overset{\text{tr}}{\sim}_R$  are equivalence relations, whatever the relation  $R$ .  $\dashv$

**§6. Conclusion and Future work.** The generalization of equivalence relations that we introduced under the name *collusions*, characterizes the condition needed for the quotientation of a set from a binary relation. It can thus be seen as generalizing one of the standard tool for the theory of classification. The renewed theory of classification which is at hand, appears no more based on a *primitive* notion of similarity (as in the case of equivalence relations), but on a (non primitive) notion of similarity *induced* by the primitive notion of interaction with the context (agonality)<sup>11</sup>.

As pointed out by Thomas Seiller (currently collaborating with me on those topics), the crucial fact that collusions are closed by converse invites to reformulate the resulting classification theory in terms of *orthogonality* (or equivalently *duality*), this notion being here used with the precise meaning it has in the

<sup>11</sup>As an illustration of the usefulness of the agonal reading of collusions, let us observe that it suggests for instance that, a given irreflexive collusion  $R$  being given, one could extend  $R$  toward a larger collusion, in such a way that a given maximal coalition for  $R$  (i.e. a class in the collusional quotient  $a/R$ ) so to speak absorbs the enemies of the enemies of its members (under the condition, of course, that the later ones are not themselves enemies of members of the initial coalition) – a processus rather frequent in the political life.

*proofs-as-programs* paradigm, where it is employed to define the notion of *type* (i.e. sets of processes closed by bi-orthogonal, for a dynamic dependent notion of orthogonality between processes). In that framework, the words “reaction to the context” receive a proper sense, linked with the dynamics of computation. But as Seiller advocates, this notion of orthogonality may well be studied at an abstract, time-independent level, considering directly axiomatic properties of the relation with respect to which the orthogonality is defined. Independently of this general promising research programme, it would be useful to uncover and investigate “concrete” models of non equivalence collusions. The typical kind of applications we have in mind is algorithms separation<sup>12</sup>. A last (but not least) kind of investigations for the future would be to reinvest, through the notion of collusion, the philosophical and mathematical debates about definitions by abstraction, in particular those based on higher order abstraction principles.

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<sup>12</sup> Far from this computational direction, let us mention that Paulo Veloso presented some concrete arithmetical models of collusions during the third occurrence of *Jornadas de Mirantão* in August 2018.

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