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THIN DOMAIN LIMIT AND COUNTEREXAMPLES TO STRONG DIAMAGNETISM

BERNARD HELFFER AND AYMAN KACHMAR

ABSTRACT. We study the magnetic Laplacian and the Ginzburg-Landau functional in a thin planar, smooth, tubular domain and with a uniform applied magnetic field. We provide counterexamples to strong diamagnetism, and as a consequence, we prove that the transition from the superconducting to the normal state is non-monotone. In some non-linear regime, we determine the structure of the order parameter and compute the super-current along the boundary of the sample. Our results are in agreement with what was observed in the Little-Parks experiment, for a thin cylindrical sample.

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1. INTRODUCTION

1.1. The Ginzburg-Landau model in a non-simply connected domain.

Let ω and Ω be two simply connected bounded open sets in \mathbb{R}^2 such that $\bar{\omega} \subset \Omega$. We assume also that the boundary of Ω , $\partial\Omega$, is smooth of class C^2 . The domain $\Omega \setminus \bar{\omega}$ is then a non-simply connected domain with the single hole ω .

The main question addressed in this paper is the inspection of the Ginzburg-Landau (GL) functional

$$\mathcal{E}_\omega(\psi, \mathbf{A}) = \int_{\Omega \setminus \bar{\omega}} \left(|(\nabla - iH\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right) dx + H^2 \int_{\Omega} |\operatorname{curl}(\mathbf{A} - \mathbf{F})|^2 dx, \quad (1.1)$$

where $\kappa > 0$ is the GL parameter, $H > 0$ the intensity of the applied magnetic field, and

$$(\psi, \mathbf{A}) \in \mathcal{H}_\omega = H^1(\Omega \setminus \bar{\omega}; \mathbb{C}) \times H_{\operatorname{div}}^1(\Omega; \mathbb{R}^2). \quad (1.2)$$

The space $H_{\operatorname{div}}^1(\Omega; \mathbb{R}^2)$ consists of all vector fields in $H^1(\Omega; \mathbb{R}^2)$ satisfying $\operatorname{div}\mathbf{A} = 0$ in Ω and $\nu \cdot \mathbf{A} = 0$ on $\partial\Omega$, where ν is the interior normal vector field of $\partial\Omega$. The vector field \mathbf{F} is the unique vector field satisfying

$$\operatorname{curl}\mathbf{F} = 1 \quad \text{and} \quad \mathbf{F} \in H_{\operatorname{div}}^1(\Omega; \mathbb{R}^2). \quad (1.3)$$

A configuration $(\psi, \mathbf{A}) \in \mathcal{H}_\omega$ is said to be a critical point of the GL functional if it is a weak solution¹ of the corresponding Euler-Lagrange equations, named GL equations in this context, and read as follows

$$\begin{cases} -(\nabla - iH\mathbf{A})^2\psi = \kappa^2(1 - |\psi|^2)\psi & \text{in } \Omega \setminus \bar{\omega}, \\ -\nabla^\perp(\operatorname{curl}(\mathbf{A} - \mathbf{F})) = \frac{1}{H}\operatorname{Im}(\bar{\psi}(\nabla - iH\mathbf{A})\psi) & \text{in } \Omega \setminus \bar{\omega}, \\ \nu \cdot (\nabla - iH\mathbf{A})\psi = 0 & \text{on } \partial\Omega \cup \partial\omega, \\ \operatorname{curl}(\mathbf{A} - \mathbf{F}) = 0 & \text{on } \omega, \end{cases} \quad (1.4)$$

where the operator $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ is the Hodge gradient.

1.2. Normal states.

A critical point (ψ, \mathbf{A}) is said to be trivial (or a normal state) if $\psi \equiv 0$. It is said to be a minimizer if it minimizes the functional in the variational space \mathcal{H}_ω . So one introduces the critical field

$$H_{c,\omega} := \sup\{H > 0 : \exists (\psi, \mathbf{A}), \psi \not\equiv 0, (\psi, \mathbf{A}) \text{ satisfies (1.4)}\}. \quad (1.5)$$

A result by Giorgi-Phillips ensures that this critical field is indeed finite. The question of estimating the critical field is closely related to the spectral analysis of the magnetic Laplacian in $L^2(\Omega \setminus \bar{\omega})$,

$$\mathcal{L}_\omega^b = -(\nabla - ib\mathbf{F})^2 \quad (1.6)$$

with (magnetic) Neumann boundary condition on $\partial\Omega \cup \partial\omega$. Here $b \in \mathbb{R}_+$ is a parameter measuring the strength of the magnetic field. The operator \mathcal{L}_ω^b is actually defined via the closed quadratic form

$$H^1(\Omega \setminus \bar{\omega}) \ni u \mapsto \mathcal{Q}_{\omega,b}(u) = \int_{\Omega \setminus \bar{\omega}} |(\nabla - ib\mathbf{F})u|^2 dx. \quad (1.7)$$

We denote by $\lambda(\omega, b)$ the lowest eigenvalue of the operator \mathcal{L}_ω^b , which is given by the min-max principle as follows

$$\lambda(\omega, b) = \inf_{u \in H^1(\Omega \setminus \bar{\omega}) \setminus \{0\}} \frac{\mathcal{Q}_{\omega,b}(u)}{\|u\|_{L^2(\Omega \setminus \bar{\omega})}^2}. \quad (1.8)$$

The relation between the eigenvalue $\lambda(\omega, b)$ and the critical field is displayed via the following well known result:

¹The weak formulation of (1.4) is precisely given in [5, (10.9a)-(10.9b)].

Proposition 1.1. *For all $\kappa, H > 0$, if $\lambda(\omega, H) < \kappa^2$, then every minimizer of the GL functional is non-trivial. Consequently the GL equations in (1.4) admit a non-trivial solution.*

The proof of Proposition 1.1 simply follows by computing the GL energy $\mathcal{E}_\omega(tu, \mathbf{F})$ with $t > 0$ and u a ground state of the operator \mathcal{L}_ω^H . The parameter t can be selected sufficiently small to ensure that $\mathcal{E}_\omega(tu, \mathbf{F}) < 0 = \mathcal{E}_\omega(0, \mathbf{F})$, which in turn guarantees the existence of a non-trivial minimizer of the GL energy in (1.1).

1.3. The thin domain.

In the sequel, we will introduce a small parameter $\varepsilon > 0$, and choose the hole ω in the following manner

$$\omega := \omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}. \quad (1.9)$$

We will refer to the parameter ε as the ‘thickness’ of our thin domain, Ω_ε , defined as follows

$$\Omega_\varepsilon := \Omega \setminus \overline{\omega_\varepsilon} = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}. \quad (1.10)$$

We define the eigenvalue $\lambda(\omega, b)$ and the GL energy \mathcal{E}_ω as follows

$$\lambda(\varepsilon, b) := \lambda(\omega_\varepsilon, b) \quad \text{and} \quad \mathcal{E}_\varepsilon(\cdot, \cdot) = \mathcal{E}_{\omega_\varepsilon}(\cdot, \cdot). \quad (1.11)$$

Also, we shorten the notation for the critical field, introduced in (1.5), and write,

$$H_c(\varepsilon) := H_{c, \omega_\varepsilon}. \quad (1.12)$$

A critical point, solving (1.4) for $\omega = \omega_\varepsilon$, will be denoted by $(\psi, \mathbf{A})_{\kappa, H, \varepsilon}$, to emphasize the dependence on the parameters κ, H and ε .

We can sharpen the statement in Proposition 1.1 when the thickness parameter ε is ‘small’.

Theorem 1.2. *Given $\kappa > 0$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$ and $H \geq 0$, the following two statements are equivalent.*

- (A) *There exists a non-trivial critical point $(\psi, \mathbf{A})_{\kappa, H, \varepsilon}$.*
- (B) *H satisfies $\lambda(\varepsilon, H) < \kappa^2$.*

1.4. The magnetic Laplacian.

Armed with Theorem 1.2, when estimating the critical field $H_c(\varepsilon)$ in the small ‘thickness’ limit, $\varepsilon \rightarrow 0_+$, we are led to estimating the eigenvalue $\lambda(\varepsilon, b)$, of the magnetic Laplacian $\mathcal{L}_{\omega_\varepsilon}^b$. After doing that, we will find that $H_c(\varepsilon)$ is asymptotically inversely proportional to ε .

For later use, we introduce

$$L = \frac{|\partial\Omega|}{2}, \quad (1.13)$$

where $|\partial\Omega|$ denotes the length of the boundary $\partial\Omega$.

Since the domain Ω_ε is non-simply connected, it is no surprise that the eigenvalue $\lambda(\varepsilon, b)$ depends on the circulation of the magnetic field around the hole of the domain. So we introduce the following quantity,

$$\gamma_0 = \frac{1}{|\partial\Omega|} \int_\Omega \text{curl } \mathbf{F} \, dx = \frac{|\Omega|}{|\partial\Omega|}. \quad (1.14)$$

1.5. Main results.

Our main results, Theorems 1.3 and 1.4 below, display the dependence of the eigenvalue $\lambda(\varepsilon, b)$ on the circulation γ_0 , in the ‘thin domain limit’, $\varepsilon \rightarrow 0_+$.

Theorem 1.3. *For every $N > 0$, there exist positive constants $\varepsilon_0, d_0, \delta_0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, the following holds*

- (A) *If $\varepsilon b \geq d_0$, then $\lambda(\varepsilon, b) \geq N$;*
- (B) *If $0 \leq \varepsilon b \leq \delta_0$, then $\left| \lambda(\varepsilon, b) - \left(\frac{\pi}{L} \right)^2 \inf_{n \in \mathbb{Z}} \left| n + \frac{bL\gamma_0}{\pi} \right|^2 \right| \leq \frac{1}{N}$.*

In light of Theorem 1.3, we see that

$$\lambda(\varepsilon, b) \underset{\substack{\varepsilon b \rightarrow +\infty \\ \varepsilon \rightarrow 0}}{\longrightarrow} +\infty \quad (1.15)$$

and

$$\lambda(\varepsilon, b) \underset{\substack{\varepsilon b \rightarrow 0 \\ \varepsilon \rightarrow 0}}{\sim} \left(\frac{\pi}{L}\right)^2 \inf_{n \in \mathbb{Z}} \left| n + \frac{bL\gamma_0}{\pi} \right|^2 \in \left[0, \frac{\pi^2}{4L^2}\right]. \quad (1.16)$$

So it remains to analyze the regime where $\varepsilon b \propto 1$, thereby bridging the two regimes appearing in Theorem 1.3 above.

Of particular interest is the behavior of the eigenvalue $\lambda(\varepsilon, b_\varepsilon)$, where

$$b_\varepsilon = \frac{a}{\varepsilon} + c \quad (1.17)$$

with $c \in \mathbb{R}$ and $a > 0$. The constant c will have an ‘oscillatory’ effect that will be discussed in Subsection 4.4 below.

In the regime (1.17), a central role is played by the following quantities

$$\beta_n(c, a, \varepsilon) = \left| n + \frac{L}{\pi} \left(\gamma_0 \left(\frac{a}{\varepsilon} + c \right) + \frac{a}{2} \right) \right| \text{ for } n \in \mathbb{Z}, \quad (1.18)$$

and their infimum over \mathbb{Z} :

$$\mathfrak{i}_0(c, a, \varepsilon) := \inf_{n \in \mathbb{Z}} \beta_n(c, a, \varepsilon) \in \left[0, \frac{1}{2}\right]. \quad (1.19)$$

The infimum is attained for one or two minimizers in \mathbb{Z} . The minimizer is unique when $\frac{L}{\pi} \left(\gamma_0 \left(\frac{a}{\varepsilon} + c \right) + \frac{a}{2} \right) \notin \frac{1}{2}\mathbb{Z}$ and denoted by $n_0 = n_0(c, a, \varepsilon)$. If $\frac{L}{\pi} \left(\gamma_0 \left(\frac{a}{\varepsilon} + c \right) + \frac{a}{2} \right) \in \frac{1}{2}\mathbb{Z}$, we have two minimizers, n_0 and $n_0 + 1$.

Theorem 1.4. *If b_ε is defined by (1.17) for some given $c \in \mathbb{R}$ and $a > 0$, then*

$$\lambda(\varepsilon, b_\varepsilon) = \frac{a^2}{12} + \left(\frac{\pi \mathfrak{i}_0(c, a, \varepsilon)}{L} \right)^2 + \mathcal{O}(\varepsilon) \text{ as } \varepsilon \rightarrow 0_+,$$

where $\mathfrak{i}_0(c, a, \varepsilon)$ was introduced in (1.19).

1.6. Remarks.

(1) The conclusion in Theorem 1.4 is formally consistent with the one in Theorem 1.3. Actually, for $a = 0$ and $c = b$, we recover the regime (B) in Theorem 1.3, while regime (A) corresponds to $a = +\infty$. Results on the multiplicity of the eigenvalue $\lambda(\varepsilon, b_\varepsilon)$ are discussed in Subsection 4.1.

(2) **Comparison with the large κ regime.**

In the interesting paper [7], Fournais and Persson-Sundqvist prove that for the disc geometry, $\Omega = D(0, R)$, there exists a thickness ε_0 and a value κ_0 for the GL parameter such that the transition to the normal state is not monotone. Our contribution goes beyond that, since for any geometry Ω and for any value of the GL parameter, we will prove that the transition to the normal state is not monotone for a certain thickness $\tilde{\varepsilon}$ constructed in Sec. 4.4 (see Proposition 4.9 and Remark 4.10).

(3) **Oscillations for bounded fields.**

The interesting contributions by Berger-Rubinstein [2] and Rubinstein-Schatzman [9] establish oscillations for bounded fields H and particular values of the GL parameter. They study the convergence of the GL functional \mathcal{E}_ε to an effective one-dimensional functional. Their results continue to hold for $H \ll \frac{1}{\varepsilon}$. That can be easily checked by the arguments used in this paper. One significant difference of our results is that they hold in the regime of large applied magnetic field and yield an estimate of the critical magnetic field. Also, our arguments differ from those in [9] and are connected to the spectral theory of the magnetic Laplacian in a thin domain.

(4) **Three dimensional rings.**

Shieh and Sternberg [10] study the GL functional in a three dimensional ring (i.e. a domain of the form $\{x \in \mathbb{R}^3, \text{dist}(x, \mathcal{C}) < \varepsilon\}$ where \mathcal{C} is a simple closed and smooth curve) and for an applied magnetic field inversely proportional to ε . They identify a one dimensional limiting problem in the frame work of the Γ -convergence and their limiting problem shows oscillations interpreted in terms of the critical temperature. Our contribution holds in a simpler geometry but it displays the oscillations for the full GL model and not only in the limit problem.

1.7. **Concentration of the GL minimizers.**

It is natural to study the minimization of the GL energy, \mathcal{E}_ε , for $H = \frac{a}{\varepsilon} + c$. We define the ground state energy

$$E_{\text{gs}}(\kappa, H, \varepsilon) = \inf\{\mathcal{G}_\varepsilon(\psi, \mathbf{A}) : (\psi, \mathbf{A}) \in \mathcal{H}_{\omega_\varepsilon}\}, \quad (1.20)$$

where the space $\mathcal{H}_{\omega_\varepsilon}$ was introduced in (1.2).

Theorem 1.5. *Given $\kappa, a > 0$ and $c \in \mathbb{R}$, then, for $H = \frac{a}{\varepsilon} + c$, as $\varepsilon > 0$ tends to 0,*

$$E_{\text{gs}}(\kappa, H, \varepsilon) = -\frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon))_+^2}{2\kappa^2} |\Omega_\varepsilon| + \mathcal{O}(\varepsilon^2), \quad (1.21)$$

where

$$\mathfrak{e}_0(c, a, \varepsilon) = \frac{a^2}{12} + \left(\frac{\pi}{L} \mathfrak{i}_0(c, a, \varepsilon)\right)^2 \quad (1.22)$$

and $\mathfrak{i}_0(c, a, \varepsilon)$ is introduced in (1.19).

Moreover, if $(\psi, \mathbf{A})_{\varepsilon, H, \kappa}$ is a minimizer of the GL functional, then

$$\int_{\Omega_\varepsilon} \left(\kappa |\psi|^2 - \frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon))_+}{\kappa} \right)^2 dx = \mathcal{O}(\varepsilon^2). \quad (1.23)$$

We can estimate the *circulation* of the supercurrent of a minimizing configuration provided for some $\delta \in (0, \frac{1}{2})$, the following two separation conditions hold

$$(\text{SC})_\delta : \quad \text{dist} \left(\frac{L}{\pi} \left(\gamma_0 \left(\frac{a}{\varepsilon} + c \right) + \frac{a}{2} \right), \frac{1}{2} \mathbb{Z} \right) \geq \delta, \quad (1.24)$$

and

$$(\text{SC})'_\delta : \quad \kappa^2 - \mathfrak{e}_0(c, a, \varepsilon) \geq \delta. \quad (1.25)$$

Note that, by Theorem 1.4, the condition $(\text{SC})'_\delta$ in (1.25) yields that $\lambda(\varepsilon, H) < \kappa^2$, for $H = \frac{a}{\varepsilon} + c$. Consequently, Proposition 1.1 yields that the minimizing configurations of the GL functional are non-trivial, thereby confirming the presence of the superconducting phase. The condition $(\text{SC})_\delta$ yields that (1.18) has a unique minimizer n_0 which satisfies $n_0 = \mathcal{O}(\varepsilon^{-1})$. Note finally that, if the constants κ and a satisfy the relation

$$\delta_0(a, \kappa) := \kappa^2 - \frac{a^2}{12} - \left(\frac{\pi}{2L}\right)^2 > 0,$$

then $(\text{SC})'_\delta$ holds for all $\delta \in (0, \delta_0)$.

For a vector field \mathbf{u} , we introduce the circulation along $\partial\Omega$ as follows

$$\oint_{\partial\Omega} \mathbf{u} \cdot dx := \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{t} ds,$$

where ds indicates the arc-length measure along the boundary $\partial\Omega$, and \mathbf{t} is the unit tangent vector along $\partial\Omega$ oriented in the counter clock-wise direction.

Theorem 1.6. *Given $\kappa, a > 0$, $\delta \in (0, \frac{1}{2})$ and $c \in \mathbb{R}$, there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$ satisfying $(\text{SC})_\delta$ and $(\text{SC})'_\delta$, $H = \frac{a}{\varepsilon} + c$ and $(\psi, \mathbf{A})_{\varepsilon, H}$ minimizing the GL functional,*

$$\oint_{\partial\Omega} \mathbf{j} \cdot dx = \frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon))_+}{\kappa^2} \frac{4\pi n_0}{|\partial\Omega|} + o(\varepsilon^{-1}). \quad (1.26)$$

Here $n_0 \in \mathbb{Z}$ is the minimizer of (1.18) and $\mathbf{j} := \text{Re}(i\psi \overline{(\nabla - iH\mathbf{A})\psi})$ is the super-current.

The proof of Theorem 1.6 is given in Section 5, where we establish an estimate compatible with the following expected behavior of the minimizing order parameter (up to a gauge transformation)

$$\psi(x) \sim \frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon))_+}{\kappa^2} \exp\left(i \frac{2\pi n_0 s}{|\partial\Omega|}\right), \quad (1.27)$$

where s is the tangential arc-length variable of x on $\partial\Omega$. The convergence in (1.27) will be made precise in Section 5 later (see (5.4) and (5.5) in Proposition 5.1).

Interestingly, this is reminiscent of the surface superconductivity regime in type II superconductors (see [3] and the references therein).

Notation. Given $p \in [1, +\infty]$ and an open set $U \subset \mathbb{R}^2$, we denote by $\|\cdot\|_{p,U}$ the usual norm in the space $L^p(U)$.

2. PROOF OF THEOREMS 1.3 & 1.4

For the considerations in Theorems 1.3 and 1.4, we assume that $b = b_\varepsilon$ is a function of ε . We will deal with the three regimes:

$$\varepsilon b_\varepsilon \ll 1, \quad \varepsilon b_\varepsilon \approx 1, \quad \varepsilon b_\varepsilon \gg 1.$$

2.1. Boundary coordinates.

Recall the definition of the geometric constants L and γ_0 in (1.13) and (1.14) respectively. Let $M : [-L, L] \rightarrow \partial\Omega$ be the arc-length parameterization of the boundary so that $\mathbf{t} := M'(s)$ is the unit tangent vector of $\partial\Omega$ oriented counter-clockwise. Choose $\varepsilon_0 \in (0, 1)$ sufficiently small such that the transformation

$$\Phi_0 : (s, t) \in [-L, L] \times (0, \varepsilon_0] \mapsto M(s) + t\nu(s) \in \Omega_{\varepsilon_0} \quad (2.1)$$

is bijective, where $\nu(s)$ is the unit interior normal vector of $\partial\Omega$ at the point $M(s)$.

In the sequel, suppose that $\varepsilon \in (0, \varepsilon_0]$. We denote by

$$f(s, t) = -\gamma_0 - t + \frac{t^2}{2}k(s), \quad (2.2)$$

where $k(s)$ is the curvature of $\partial\Omega$ at $M(s)$, and γ_0 is the circulation of the applied magnetic field, introduced in (1.14).

We have (see [5, Lem. F.1.1]):

$$\int_{\Omega_\varepsilon} |(\nabla - ib_\varepsilon \mathbf{F})u|^2 dx = Q_{b_\varepsilon}^{L, \varepsilon}(v) := \int_{-L}^L \int_0^\varepsilon \left(|\partial_t v|^2 + \mathbf{a}^{-2} |(\partial_s - ib_\varepsilon f)v|^2 \right) \mathbf{a} dt ds, \quad (2.3)$$

where

$$\mathbf{a}(s, t) = 1 - tk(s), \quad (2.4)$$

$$v(s, t) = e^{ib_\varepsilon \varphi_0(s, t)} u(\Phi_0(s, t)), \quad (2.5)$$

and $\varphi_0(s, t)$ is a smooth function, $2L$ -periodic with respect to the s -variable, and depends only on the vector field \mathbf{F} and the geometry of the domain Ω . Hence it is independent from ε and the choice of the function u . In fact we can take (see [5, Eq. (F.11)])

$$\varphi_0(s, t) = \int_0^t \tilde{\mathbf{F}}_2(s, t') dt' + \int_0^s \tilde{\mathbf{F}}_1(s', 0) ds' - s\gamma_0,$$

where (see [5, Eq. (F.2)])

$$\tilde{\mathbf{F}}_1(s, t) = \mathbf{a}(s, t)\mathbf{F}(\Phi_0(s, t)) \cdot M'(s) \quad \text{and} \quad \tilde{\mathbf{F}}_2(s, t) = \mathbf{F}(\Phi_0(s, t)) \cdot \nu(s) = 0,$$

since $\mathbf{F} \in H_{\text{div}}^1(\Omega)$.

Moreover, we can express the L^2 -norm of u in the following manner:

$$\int_{\Omega_\varepsilon} |u(x)|^2 dx = \int_{-L}^L \int_0^\varepsilon |v(s, t)|^2 \mathbf{a}(s, t) dt ds. \quad (2.6)$$

2.2. Reduction of the operator.

Let us assume now that $\varepsilon b_\varepsilon \leq M_0$, for some constant $M_0 > 0$. This hypothesis will be valid when for example (1.17) holds, or when we consider the conclusion (B) in Theorem 1.3.

We can estimate the quadratic form and the L^2 norm of v as follows. There exist two constants $K > 0$ and $\tilde{\varepsilon}_0 \in (0, 1)$, depending on the domain Ω only, such that, for all $\varepsilon \in (0, \tilde{\varepsilon}_0]$,

$$(1 - K\varepsilon)q_{b_\varepsilon}^{L, \varepsilon}(v) \leq Q_{b_\varepsilon}^{L, \varepsilon}(v) \leq (1 + K\varepsilon)q_{b_\varepsilon}^{L, \varepsilon}(v) \quad (2.7)$$

and

$$(1 - K\varepsilon) \int_{-L}^L \int_0^\varepsilon |v|^2 dt ds \leq \|u\|_{2, \Omega_\varepsilon}^2 \leq (1 + K\varepsilon) \int_{-L}^L \int_0^\varepsilon |v|^2 dt ds, \quad (2.8)$$

where

$$q_{b_\varepsilon}^{L, \varepsilon}(v) = \int_{-L}^L \int_0^\varepsilon \left(|\partial_t v|^2 + |(\partial_s - ib_\varepsilon f_0)v|^2 \right) dt ds, \quad (2.9)$$

and

$$f_0(t) = -\gamma_0 - t. \quad (2.10)$$

Actually, this follows from the following two estimates:

$$|\mathbf{a}(s, t) - 1| \leq \|\kappa\|_\infty \varepsilon \quad \text{and} \quad |f(t) - f_0(t)| \leq \frac{1}{2} \|\kappa\|_\infty^2 \varepsilon^2.$$

Let us introduce the eigenvalue $\hat{\lambda}(\varepsilon, b_\varepsilon)$ as follows

$$\hat{\lambda}(\varepsilon, b_\varepsilon) = \inf_{v \in H^1([-L, L] \times (0, \varepsilon)) \setminus \{0\}} \frac{q_{b_\varepsilon}(v)}{\|v\|_{L^2([-L, L] \times (0, \varepsilon))}^2}. \quad (2.11)$$

By the min-max principle, we deduce the existence of \tilde{K} and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$,

$$|\lambda(\varepsilon, b_\varepsilon) - \hat{\lambda}(\varepsilon, b_\varepsilon)| \leq \tilde{K} \varepsilon \hat{\lambda}(\varepsilon, b_\varepsilon). \quad (2.12)$$

2.3. Spectral analysis of the reduced operator.

2.3.1. *Fourier modes.* We decompose in Fourier modes to obtain the family of quadratic forms

$$q_{n, b_\varepsilon}^\varepsilon(v_n) = \int_0^\varepsilon \left(|\partial_t v_n|^2 + |(nL^{-1}\pi - b_\varepsilon f_0)v_n|^2 \right) dt. \quad (2.13)$$

So we introduce for $\eta, b > 0$,

$$\tilde{q}_{\eta, b}^\varepsilon(w) = \int_0^\varepsilon \left(|\partial_t w|^2 + |(\eta + bt)w|^2 \right) dt,$$

along with the corresponding eigenvalue

$$\lambda(\eta, \varepsilon, b_\varepsilon) = \inf_{\|w\|_{2, (0, \varepsilon)}^2 = 1} \tilde{q}_{\eta, b}^\varepsilon(w). \quad (2.14)$$

Note that

$$q_{n, b_\varepsilon}^\varepsilon(v_n) = \tilde{q}_{\eta, b_\varepsilon}^\varepsilon(v_n)$$

for

$$\eta = \eta(n, b_\varepsilon) = nL^{-1}\pi + b_\varepsilon \gamma_0. \quad (2.15)$$

The eigenvalue in (2.11) can be expressed using the eigenvalues of the fiber operators as follows,

$$\hat{\lambda}(\varepsilon, b_\varepsilon) = \inf_{n \in \mathbb{Z}} \lambda(\eta(n, b_\varepsilon), \varepsilon, b_\varepsilon). \quad (2.16)$$

2.3.2. Scaling. Now, we assume that b_ε satisfies (1.17) for some constants $a > 0$ and $c \in \mathbb{R}$. We do the change of variable $\tau = a\varepsilon^{-1}t$ and get

$$\lambda(\eta, \varepsilon, b_\varepsilon) = a^2 \varepsilon^{-2} \mu(\alpha, \delta_\varepsilon, a, \zeta_\varepsilon), \quad (2.17)$$

where $\mu(\alpha, \delta, a, \zeta)$ is the lowest eigenvalue in $L^2(0, a)$ of the operator defined via the closed quadratic form, with $\delta \geq 0$ and $\zeta \in \mathbb{R}$,

$$H^1(0, a) \ni u \mapsto h_\delta^{\alpha, a, \zeta}(u) = \int_0^a \left(|\partial_\tau u|^2 + \delta |(\alpha + \tau + \zeta\tau)u|^2 \right) d\tau. \quad (2.18)$$

The formula in (2.17) is valid for η defined by (2.15), $\alpha = \alpha_n$, $\delta = \delta_\varepsilon$ and $\zeta = \zeta_\varepsilon$, where

$$\alpha_n = \frac{n\pi}{L} + \left(\frac{a}{\varepsilon} + c \right) \gamma_0, \quad (2.19)$$

$$\delta_\varepsilon = a^{-2} \varepsilon^2 \quad \text{and} \quad \zeta_\varepsilon = \frac{c\varepsilon}{a}. \quad (2.20)$$

2.3.3. The non-trivial regime.

Comparison with the 1D-Neumann Laplacian.

Let \mathcal{L}_0 be the 1D Neumann Laplace operator defined in $L^2(0, a)$ as follows

$$\mathcal{D}_0 \ni u \mapsto \mathcal{L}_0 u = -\frac{d^2}{d\tau^2} \quad \text{where } \mathcal{D}_0 = \{u \in H^2(0, a) : u'(0) = u'(a) = 0\}. \quad (2.21)$$

The min-max principle yields the following comparison for the eigenvalues defined via the quadratic form in (2.18) and those of the operator \mathcal{L}_0 :

$$\mu_n(\alpha, \delta, a, \zeta) \geq \mu_n(\mathcal{L}_0). \quad (2.22)$$

It is easy to check that

$$\forall n \in \mathbb{N} \setminus \{0\}, \quad \mu_n(\mathcal{L}_0) = \left(\frac{(n-1)\pi}{a} \right)^2, \quad (2.23)$$

hence the comparison in (2.22) is not effective for the first eigenvalue $\mu(\alpha, \delta, a, \zeta)$, since $\mu_1(\mathcal{L}_0) = 0$, however, for the second eigenvalue $\mu_2(\alpha, \delta, a, \zeta)$ we obtain

$$\mu_2(\alpha, \delta, a, \zeta) \geq \left(\frac{\pi}{a} \right)^2. \quad (2.24)$$

Behavior of $\mu(\alpha, \delta, a, \zeta_\varepsilon)$ as $\delta \rightarrow 0$.

We recall from (2.20) that $\lim_{\varepsilon \rightarrow 0} \zeta_\varepsilon = 0$. Fix positive constants ζ_0 and A . We will first write an estimate of the eigenvalue $\mu(\alpha, \delta, a, \zeta)$ that holds uniformly with respect to $\alpha \in [-A, A]$ and $\zeta \in [-\zeta_0, \zeta_0]$. A standard argument of perturbation in δ allows us to expand the eigenvalue $\mu(\alpha, \delta, a, \zeta)$ as follows

$$\mu(\alpha, \delta, a, \zeta) \underset{\delta \rightarrow 0}{=} \mu_0 + \delta \mu_1 + \mathcal{O}(\delta^2).$$

We recall the proof for the commodity of the reader. We introduce a quasi-mode of the form

$$u := u_0 + \delta u_1,$$

so that

$$\left(-\frac{d^2}{d\tau^2} + \delta(\tau + \alpha + \zeta\tau)^2 \right) (u_0 + \delta u_1) = (\mu_0 + \delta \mu_1)(u_0 + \delta u_1) + \mathcal{O}(\delta^2). \quad (2.25)$$

Then the natural choice of μ_0, μ_1, u_0, u_1 (depending smoothly on ζ) would be

$$\begin{aligned} -\frac{d^2}{d\tau^2}u_0 &= \mu_0 u_0, \\ \left(-\frac{d^2}{d\tau^2} - \mu_0\right)u_1 + \left((\tau + \alpha + \zeta\tau)^2 - \mu_1(\zeta)\right)u_0 &= 0. \end{aligned}$$

We choose $\mu_0 = \mu_1(\mathcal{L}_0) = 0$ and $u_0 = 1$, in accordance with (2.23). In order to solve the equation for $u_1(\cdot, \zeta)$, we choose $\mu_1 = \mu_1(\zeta)$ so that $\left((\tau + \alpha + \zeta\tau)^2 - \mu_1\right)u_0$ is orthogonal to u_0 in $L^2(0, a)$, thereby obtaining the Feynman-Hellman formula,

$$\mu_1(\zeta) = \frac{1}{a} \int_0^a (\tau + \alpha + \zeta\tau)^2 d\tau = \alpha^2 + \alpha(1 + \zeta)a + \frac{1}{3}a^2(1 + \zeta)^2. \quad (2.26)$$

We note for later use that

$$\mu_1(\zeta) \leq ((1 + \zeta)a + |\alpha|)^2. \quad (2.27)$$

With this choice, we solve the differential equation satisfied by u_1 , with the boundary conditions $u_1'(0) = u_1'(a) = 0$, and get, imposing that u_1 is orthogonal to u_0 , a unique solution $u_1(\cdot, \zeta)$.

Now, the quasi-mode $u(\cdot, \zeta) = u_0(\cdot, \zeta) + \delta u_1(\cdot, \zeta)$ satisfies

$$u'(0, \zeta) = u'(a, \zeta) = 0,$$

and

$$\left\| \left(-\frac{d^2}{d\tau^2} + \delta(\alpha + \tau + \zeta\tau)^2 - \delta\mu_1(\zeta) \right) u(\cdot, \zeta) \right\|_{L^2(0, a)} \leq C_{A, a, \zeta_0} \delta^2 \|u\|_{L^2(0, a)}^2,$$

which is valid for $\delta \in (0, \delta_{A, a, \zeta_0})$, where δ_{A, a, ζ_0} and C_{A, a, ζ_0} are constants that depend only on A , a and ζ_0 .

Taking $\zeta = \zeta_\varepsilon$, the spectral theorem and the lower bound in (2.24) then yield that there exist $\varepsilon_{A, a}$ and $\widehat{C}_{A, a}$ such that

$$\left| \mu(\alpha, \delta, a, \zeta_\varepsilon) - \delta\mu_1(0) \right| \leq \widehat{C}_{M, a} (\delta + |\zeta_\varepsilon|) \delta. \quad (2.28)$$

This motivates us to introduce the following quantity

$$m(\alpha, a) := \mu_1(0) = \frac{a^2}{12} + \left(\alpha + \frac{a}{2} \right)^2. \quad (2.29)$$

Remark 2.1. Combining (2.24) and (2.28), we see that, if $|\alpha| \leq A$, there exists $\delta_0 > 0$ such that, for $\delta, \varepsilon \in (0, \delta_0)$, the eigenvalue $\mu(\alpha, \delta, a, \zeta_\varepsilon)$ is simple.

Minimization of $\mu(\alpha, \delta, a, \zeta_\varepsilon)$.

We are interested in estimating the quantity

$$\mu_0(\delta, a, \zeta_\varepsilon) := \inf_{\alpha \in \mathcal{J}_\varepsilon} \mu(\alpha, \delta, a, \zeta_\varepsilon) \quad (2.30)$$

where

$$\mathcal{J}_\varepsilon = \left\{ \alpha_n = \frac{n\pi}{L} + \gamma_0 \frac{a}{\varepsilon} + \gamma_0 c : n \in \mathbb{Z} \right\}. \quad (2.31)$$

Choose $n_0 = n_0(\varepsilon) \in \mathbb{Z}$ so that $|\alpha_{n_0}| = \inf_{n \in \mathbb{Z}} |\alpha_n|$. Clearly,

$$\alpha_{n_0} \in \left[-\frac{\pi}{2L}, \frac{\pi}{2L} \right].$$

Using the constant function $u \equiv 1$ as a test function in the quadratic form in (2.18), we get the existence of $\varepsilon_0 > 0$ such that, for all $\delta > 0$ and $\varepsilon \in (0, \varepsilon_0]$,

$$\mu(\alpha_{n_0}, \delta, a, \zeta_\varepsilon) \leq \delta\mu_1(\zeta_\varepsilon) \leq 2\delta \left(a + \frac{\pi}{2L} \right)^2. \quad (2.32)$$

Here we have used for the last inequality that ζ_ε tends to 0 and $\alpha = \alpha_{n_0}$ in (2.27).

Noticing that, for $|\alpha| \geq 10(a + \frac{\pi}{2L})$,

$$\inf_{0 \leq \tau \leq a} (\alpha + \tau + \zeta_\varepsilon \tau)^2 \geq \left(9\left(a + \frac{\pi}{2L}\right) - |\zeta_\varepsilon|a\right)^2 \geq 4\left(a + \frac{\pi}{2L}\right)^2,$$

for ε sufficiently small, we get by the min-max principle

$$\mu(\alpha, \delta, a, \zeta_\varepsilon) \geq 4\left(a + \frac{\pi}{2L}\right)^2 \delta > \mu(\alpha_{n_0}, \delta, a, \zeta_\varepsilon) \geq \mu_0(\delta, a, \zeta_\varepsilon). \quad (2.33)$$

This proves that the minimization in (2.30) can be restricted to $\alpha \in [-A, A] \cap \mathcal{J}_\varepsilon$ with $A = 10(a + \frac{\pi}{2L})$. In light of (2.28), it is enough to minimize the function in (2.29) with respect to α . Therefore, there exist $\delta_0 = \delta_0(a, c, \gamma_0, L) > 0$ and $C_0 = C_0(a, c, \gamma_0, L) > 0$, such that, for all $\delta, \varepsilon \in (0, \delta_0)$,

$$\left| \mu_0(\delta, a, \zeta_\varepsilon) - \left(\frac{a^2}{12} + \inf_{n \in \mathbb{Z}} \left| \frac{n\pi}{L} + \frac{a}{\varepsilon} \gamma_0 + c\gamma_0 + \frac{a}{2} \right|^2 \right) \delta \right| \leq C_0(\delta + \varepsilon)\delta. \quad (2.34)$$

2.4. End of the proofs.

2.4.1. The regime $b_\varepsilon \propto \frac{1}{\varepsilon}$.

Collecting (2.12), (2.16), (2.17) and (2.34) (with $\delta = \delta_\varepsilon$ defined in (2.20)), we get, as ε tends to 0, with $b_\varepsilon = \frac{a}{\varepsilon} + c$, the asymptotics stated in Theorem 1.4.

2.4.2. The regime $\varepsilon b_\varepsilon \ll 1$.

In this case, we restart from Subsections 2.2 and 2.3. We choose $n_0(\varepsilon) \in \mathbb{Z}$ so that

$$\left| n_0(\varepsilon) + \frac{b_\varepsilon L \gamma_0}{\pi} \right| = \inf_{n \in \mathbb{Z}} \left| n + \frac{b_\varepsilon L \gamma_0}{\pi} \right|,$$

and set

$$\beta_{n, \varepsilon} := n + \frac{b_\varepsilon L \gamma_0}{\pi}.$$

Clearly,

$$\beta_{n_0(\varepsilon), \varepsilon} \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Using the function $u(s, t) = e^{in_0(\varepsilon)\pi s/L}$ as a test function, we get by a straightforward computation

$$\hat{\lambda}(\varepsilon, b_\varepsilon) \leq \left(\frac{\pi}{L}\right)^2 \beta_{n_0(\varepsilon), \varepsilon}^2 + \mathcal{O}(\varepsilon b_\varepsilon). \quad (2.35)$$

For the reverse inequality, we decompose in Fourier modes and do the rescaling $\tau = \varepsilon^{-1}t$, to get the following quadratic form,

$$\varepsilon^{-2} \int_0^1 \left(|\partial_\tau u|^2 + \varepsilon^2 \left| \left(\frac{\pi}{L} \beta_{n, \varepsilon} + \varepsilon b_\varepsilon \tau \right) u \right|^2 \right) d\tau \geq \varepsilon^{-2} \int_0^1 \varepsilon^2 \left((1 - \varepsilon b_\varepsilon) \left| \frac{\pi}{L} \beta_{n, \varepsilon} \right|^2 - \varepsilon b_\varepsilon \right) |u|^2 d\tau.$$

So, we get by the min-max principle that

$$\hat{\lambda}(\varepsilon, b_\varepsilon) \geq \inf_{n \in \mathbb{Z}} \left((1 - \varepsilon b_\varepsilon) \left| \frac{\pi}{L} \beta_{n, \varepsilon} \right|^2 - \varepsilon b_\varepsilon \right) = \left(\frac{\pi}{L}\right)^2 \beta_{n_0(\varepsilon), \varepsilon}^2 + \mathcal{O}(\varepsilon b_\varepsilon).$$

Finally, we use (2.12) to conclude the estimate for $\lambda(\varepsilon, b_\varepsilon)$ (Statement (B) in Theorem 1.3).

2.4.3. *The regime $\varepsilon b_\varepsilon \gg 1$.*

In this situation, we can not use the estimate in (2.7), since replacing $b_\varepsilon f$ by $b_\varepsilon f_0$ produces a large error (see (2.2) and (2.10)).

We rescale the variables as follows, $t = \varepsilon \tau$ and $s = \varepsilon^{-2} b_\varepsilon^{-1} \sigma$. We obtain two constants $k > 0$ and $\varepsilon_0 \in (0, 1)$ such that, for all $\varepsilon \in (0, \varepsilon_0]$,

$$\lambda(\varepsilon, b_\varepsilon) \geq (1 - k\varepsilon) \tilde{\lambda}(\varepsilon, b_\varepsilon),$$

where

$$\tilde{\lambda}(\varepsilon, b_\varepsilon) = \inf_{\|v\|_{L^2(\mathbb{T}_\varepsilon)}=1} \tilde{Q}_\varepsilon(v).$$

Here,

$$\mathbb{T}_\varepsilon = [-L_\varepsilon, L_\varepsilon] \times (0, 1),$$

$$L_\varepsilon = \varepsilon^{-2} b_\varepsilon^{-1} L,$$

$$\tilde{Q}_\varepsilon(v) = \int_{\mathbb{T}_\varepsilon} \left(\varepsilon^{-2} |\partial_\tau v|^2 + \varepsilon^2 b_\varepsilon^2 |(\partial_\sigma - i f_\varepsilon) v|^2 \right) d\sigma d\tau,$$

and

$$f_\varepsilon(\sigma, \tau) = \varepsilon^{-1} \gamma_0 - \tau + \frac{\varepsilon \tau^2}{2} \kappa(\varepsilon^{-2} b_\varepsilon^{-1} \sigma).$$

We now prove that $\tilde{\lambda}(\varepsilon, b_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} +\infty$.

Note that

$$\tilde{Q}_\varepsilon(v) \geq \min(\varepsilon^{-2}, \varepsilon^2 b_\varepsilon^2) \int_{\mathbb{T}_\varepsilon} \left(|\partial_\tau v|^2 + |(\partial_\sigma - i f_\varepsilon) v|^2 \right) d\sigma d\tau,$$

and

$$|f_\varepsilon(\sigma, \tau) - f_\varepsilon^0(\tau)| \leq 2\|\kappa\|_\infty \varepsilon \quad \text{where } f_\varepsilon^0(\tau) = \varepsilon^{-1} \gamma_0 - \tau.$$

Consequently,

$$\begin{aligned} \int_{\mathbb{T}_\varepsilon} \left(|\partial_\tau v|^2 + |(\partial_\sigma - i f_\varepsilon) v|^2 \right) d\sigma d\tau &\geq \int_{\mathbb{T}_\varepsilon} \left(|\partial_\tau v|^2 + \frac{1}{2} |(\partial_\sigma - i f_\varepsilon^0) v|^2 - 8\|\kappa\|_\infty^2 \varepsilon^2 |v|^2 \right) d\sigma d\tau \\ &\geq \left(\frac{1}{2} e(\varepsilon) - 8\|\kappa\|_\infty^2 \varepsilon^2 \right) \int_{\mathbb{T}_\varepsilon} |v|^2 d\sigma d\tau, \end{aligned}$$

where

$$e(\varepsilon) = \inf_{\|v\|_{L^2(\mathbb{T}_\varepsilon)}=1} \int_{\mathbb{T}_\varepsilon} |\partial_\tau v|^2 + |(\partial_\sigma - i f_\varepsilon^0(\tau)) v|^2 d\sigma d\tau.$$

The min-max principle now yields

$$\tilde{\lambda}(\varepsilon, b_\varepsilon) \geq \frac{1}{2} \left(\min(\varepsilon^{-2}, \varepsilon^2 b_\varepsilon^2) \right) \left(e(\varepsilon) - 16\|\kappa\|_\infty^2 \varepsilon^2 \right).$$

By decomposition into Fourier modes, we may show that

$$e(\varepsilon) \geq \inf_{\alpha \in \mathbb{R}} \mu(\alpha, 1, 1, 0),$$

where $\mu(\alpha, \delta, a, 0)$ is the eigenvalue defined via the quadratic form in (2.18), for $\delta = 1$, $a = 1$ and $\zeta = 0$.

Using the min-max principle, it is easy to check that the function $\alpha \mapsto \mu(\alpha, 1, 1, 0)$ is continuous, positive-valued, and tends to $+\infty$ as $|\alpha| \rightarrow +\infty$. Consequently, it attains its minimum, i.e. there exists $\alpha_0 \in \mathbb{R}$ such that

$$\inf_{\alpha \in \mathbb{R}} \mu(\alpha, 1, 1, 0) = \mu(\alpha_0, 1, 1, 0) > 0.$$

This proves that $\liminf_{\varepsilon \rightarrow 0_+} e(\varepsilon) > 0$ and finishes the proof of $\tilde{\lambda}(\varepsilon, b_\varepsilon) \rightarrow +\infty$ in this regime (Statement (A) in Theorem 1.3).

3. PROOF OF THEOREMS 1.2 AND 1.5

3.1. **A priori estimates.**

There exists $\varepsilon_0 \in (0, 1)$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, $\kappa, H > 0$, every critical point (ψ, \mathbf{A}) satisfies [5, Ch. 10]

$$\begin{aligned} \|\psi\|_{\infty, \Omega_\varepsilon} &\leq 1, \\ \|(\nabla - iH\mathbf{A})\psi\|_{2, \Omega_\varepsilon} &\leq \kappa \|\psi\|_{2, \Omega_\varepsilon}, \\ \|\nabla(\operatorname{curl} \mathbf{A} - \mathbf{F})\|_{2, \Omega} &\leq \frac{1}{H} \|(\nabla - iH\mathbf{A})\psi\|_{2, \Omega_\varepsilon} \|\psi\|_{2, \Omega_\varepsilon}. \end{aligned} \quad (3.1)$$

Noting that the first eigenvalue of the one dimensional Dirichlet Laplacian $-\frac{d^2}{dt^2}$ in $L^2(0, \varepsilon)$ equals $(\frac{\pi}{\varepsilon})^2$, we get from (3.1), observing that $\operatorname{curl}(\mathbf{A} - \mathbf{F})$ satisfies the Dirichlet condition on $\partial\Omega_\varepsilon$,

$$\begin{aligned} \|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{2, \Omega} &= \|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{2, \Omega_\varepsilon} \\ &\leq \frac{C\varepsilon}{\pi} \|\nabla \operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{2, \Omega_\varepsilon} \\ &\leq \frac{C\varepsilon}{\pi H} \|(\nabla - iH\mathbf{A})\psi\|_{2, \Omega_\varepsilon} \|\psi\|_{2, \Omega_\varepsilon} \\ &\leq \frac{C\kappa\varepsilon}{\pi H} \|\psi\|_{2, \Omega_\varepsilon}^2 \\ &= \mathcal{O}(\varepsilon H^{-1}) \|\psi\|_{2, \Omega_\varepsilon}^2. \end{aligned} \quad (3.2)$$

Consequently, by the div-curl inequality in Ω

$$\|\mathbf{A} - \mathbf{F}\|_{H^1(\Omega)} \leq \tilde{C} \|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{2, \Omega} = \mathcal{O}(\varepsilon H^{-1}) \|\psi\|_{2, \Omega_\varepsilon}^2. \quad (3.3)$$

By the embedding of $H^1(\Omega)$ in $L^p(\Omega)$ for $p \in [2, +\infty)$, we find, using the first line of (3.1),

$$\|\mathbf{A} - \mathbf{F}\|_{p, \Omega} = \mathcal{O}(\varepsilon H^{-1}) \|\psi\|_{2, \Omega_\varepsilon}^2 = \mathcal{O}(\varepsilon^2 H^{-1}). \quad (3.4)$$

We write by Cauchy's inequality,

$$\|(\nabla - iH\mathbf{A})\psi\|_{2, \Omega_\varepsilon}^2 \geq (1 - \eta) \|(\nabla - iH\mathbf{F})\psi\|_{2, \Omega_\varepsilon}^2 - \eta^{-1} H^2 \|(\mathbf{A} - \mathbf{F})\psi\|_{2, \Omega_\varepsilon}^2, \quad (3.5)$$

where $\eta \in (0, 1)$ and $(\psi, \mathbf{A})_{\kappa, H, \varepsilon}$ is a critical configuration.

We estimate the term $\|(\mathbf{A} - \mathbf{F})\psi\|_{2, \Omega_\varepsilon}^2$ using Hölder's inequality and (3.4) as follows

$$\|(\mathbf{A} - \mathbf{F})\psi\|_{2, \Omega_\varepsilon}^2 \leq \|\mathbf{A} - \mathbf{F}\|_{4, \Omega_\varepsilon}^2 \|\psi\|_{4, \Omega_\varepsilon}^2 = \mathcal{O}(\varepsilon^2 H^{-2}) \|\psi\|_{2, \Omega_\varepsilon}^4 \|\psi\|_{4, \Omega_\varepsilon}^2. \quad (3.6)$$

Again, Hölder's inequality yields

$$\|\psi\|_{2, \Omega_\varepsilon}^2 \leq |\Omega_\varepsilon|^{1/2} \|\psi\|_{4, \Omega_\varepsilon}^2 = \mathcal{O}(\varepsilon^{1/2}) \|\psi\|_{4, \Omega_\varepsilon}^2.$$

Thus, from (3.5) and (3.6), we get the following lower bound,

$$\|(\nabla - iH\mathbf{A})\psi\|_{2, \Omega_\varepsilon}^2 \geq (1 - \eta) \|(\nabla - iH\mathbf{F})\psi\|_{2, \Omega_\varepsilon}^2 - C\eta^{-1} \varepsilon^3 \|\psi\|_{4, \Omega_\varepsilon}^6, \quad (3.7)$$

where $C > 0$ is a constant independent from η and H .

Using this estimate, we can bound the GL functional from below as follows:

$$0 \geq \mathcal{E}_\varepsilon(\psi, \mathbf{A}) \geq (1 - \eta) \mathcal{E}_\varepsilon(\psi, \mathbf{F}) - \eta \kappa^2 \|\psi\|_{2, \Omega_\varepsilon}^2 - C\eta^{-1} \varepsilon^3 \|\psi\|_{4, \Omega_\varepsilon}^6, \quad (3.8)$$

and this is true for any critical configuration $(\psi, \mathbf{A})_{\kappa, H, \varepsilon}$.

3.2. **Proof of Theorem 1.2.**

Having Proposition 1.1 in mind, we have only to prove that (A) implies (B).

Step 1: First restriction.

Using the constant function as a quasi-mode, we get that, for all $\varepsilon, H > 0$,

$$\lambda(\varepsilon, H) \leq H^2 \|\mathbf{F}\|_\infty^2, \quad (3.9)$$

where we take the L^∞ -norm on Ω in order to get the uniformity in ε .

Thus, if $H < \sigma_0(\kappa)$ with

$$\sigma_0(\kappa) := \kappa / \|\mathbf{F}\|_\infty,$$

we have $\lambda(\varepsilon, H) < \kappa^2$ and (B) is satisfied.

From now on, we consider $H \geq \sigma_0(\kappa)$ and prove that (A) implies (B) under this additional condition.

Step 2: Second restriction.

We assume that (A) holds. Since $\lambda(\varepsilon, H) \rightarrow +\infty$ as $\varepsilon H \rightarrow +\infty$ and $\varepsilon \rightarrow 0$ (see (1.15)), we find Λ_0 and $\varepsilon_0 > 0$ such that, for $\varepsilon H \geq \Lambda_0$ and $0 < \varepsilon \leq \varepsilon_0$, $\lambda(\varepsilon, H) > 2\kappa^2$.

The lower bound in (3.8) used with $\eta = \varepsilon$, and the min-max principle, yield that,

$$\begin{aligned} 0 &\geq (1 - \varepsilon)(\lambda(\varepsilon, H) - \kappa^2) \|\psi\|_{2, \Omega_\varepsilon}^2 - \varepsilon \kappa^2 \|\psi\|_{2, \Omega_\varepsilon}^2 - C\varepsilon^2 \|\psi\|_{4, \Omega_\varepsilon}^6 \\ &\geq (1 - 2\varepsilon)\kappa^2 \|\psi\|_{2, \Omega_\varepsilon}^2 - C\varepsilon^2 \|\psi\|_{4, \Omega_\varepsilon}^6. \end{aligned}$$

Noting that, because $|\psi| \leq 1$,

$$\|\psi\|_{4, \Omega_\varepsilon}^6 = \left(\int_{\Omega_\varepsilon} |\psi|^4 dx \right)^{\frac{3}{2}} \leq \|\psi\|_{2, \Omega_\varepsilon}^3 \leq |\Omega_\varepsilon|^{\frac{1}{2}} \|\psi\|_{2, \Omega_\varepsilon}^2,$$

we get, for some positive constants C_κ and $\varepsilon_0(\kappa)$,

$$0 \geq (1 - 2\varepsilon - C_\kappa \varepsilon^2) \kappa^2 \|\psi\|_{2, \Omega_\varepsilon}^2,$$

for $\varepsilon \in (0, \varepsilon_0(\kappa)]$ and any ψ corresponding to a critical configuration.

This proves the existence of a positive $\varepsilon_1(\kappa)$ such that $\psi \equiv 0$ when $\varepsilon \in (0, \varepsilon_1(\kappa)]$ in contradiction with (A).

Hence at this stage, we have proven the existence of Λ_0 and ε_1 such that if (A) holds then $H \leq \Lambda_0 \varepsilon^{-1}$ for $\varepsilon \in (0, \varepsilon_1]$.

Step 3: Proof in the remaining case.

We assume that (A) holds and that $0 < \sigma_0(\kappa) \leq H \leq \Lambda_0 \varepsilon^{-1}$. There exist ε_0 and Λ such that, for all $\varepsilon \in (0, \varepsilon_0]$,

$$\lambda(\varepsilon, H) \leq \Lambda. \quad (3.10)$$

This simply follows after combining (2.12) and (2.35).

We introduce

$$\Delta = \kappa^2 \|\psi\|_{2, \Omega_\varepsilon}^2 - \|(\nabla - iH\mathbf{A})\psi\|_{2, \Omega_\varepsilon}^2 = \kappa^2 \|\psi\|_{4, \Omega_\varepsilon}^4. \quad (3.11)$$

The hypothesis on the non-triviality of ψ ensures that $\Delta > 0$. Also, as a consequence of the first inequality in (3.1), we get

$$0 < \Delta \leq \kappa^2 |\Omega_\varepsilon| = \mathcal{O}(\varepsilon). \quad (3.12)$$

Notice that the Hölder inequality yields that

$$\kappa^2 \|\psi\|_{2, \Omega_\varepsilon}^2 \leq \kappa^2 |\Omega_\varepsilon|^{\frac{1}{2}} \|\psi\|_{4, \Omega_\varepsilon}^2 \leq C\sqrt{\varepsilon} \Delta^{1/2}. \quad (3.13)$$

By (3.7) and the min-max principle, we write, for any $\eta \in (0, 1)$,

$$\|(\nabla - iH\mathbf{A})\psi\|_{2, \Omega_\varepsilon}^2 \geq (1 - \eta)\lambda(\varepsilon, H) \|\psi\|_{2, \Omega_\varepsilon}^2 - C\eta^{-1}\varepsilon^3 \|\psi\|_{4, \Omega_\varepsilon}^6, \quad (3.14)$$

and we infer the following lower bound,

$$-\Delta \geq (\lambda(\varepsilon, H) - \kappa^2) \|\psi\|_{2, \Omega_\varepsilon}^2 - \eta\lambda(\varepsilon, H) \|\psi\|_{2, \Omega_\varepsilon}^2 - C\eta^{-1}\varepsilon^3 \|\psi\|_{4, \Omega_\varepsilon}^6. \quad (3.15)$$

Using (3.10), (3.11), (3.12), and (3.13), we get, from (3.15) with $\eta = \varepsilon\Delta^{1/2}$ (note that $\eta \in (0, 1)$ by (3.12) for ε small enough),

$$-(1 - \hat{C}\varepsilon^{\frac{3}{2}})\Delta \geq (\lambda(\varepsilon, H) - \kappa^2)\|\psi\|_{2, \Omega_\varepsilon}^2.$$

But $\Delta > 0$, by our hypothesis, hence this yields for ε small enough that

$$(\lambda(\varepsilon, H) - \kappa^2)\|\psi\|_{2, \Omega_\varepsilon}^2 < 0,$$

which implies (B) after observing that $\|\psi\|_{2, \Omega_\varepsilon} \neq 0$.

3.3. Proof of Theorem 1.5.

Let $(\psi, \mathbf{A})_{\kappa, H, \varepsilon}$ be a minimizing configuration for $H = \frac{a}{\varepsilon} + c$. We start with the inequality in (3.8) with $\eta = \varepsilon$. Since $|\psi| \leq 1$ everywhere, (3.8) yields, for some constant $C > 0$,

$$\mathcal{E}_\varepsilon(\psi, \mathbf{A}) \geq (1 - \varepsilon)\mathcal{E}_\varepsilon(\psi, \mathbf{F}) - C\varepsilon^2. \quad (3.16)$$

The quadratic form part in $\mathcal{E}_\varepsilon(\psi, \mathbf{F})$ can be bounded from below by the min-max principle and Theorem 1.4, so that

$$\begin{aligned} \mathcal{E}_\varepsilon(\psi, \mathbf{F}) &\geq (\mathfrak{e}_0(c, a, \varepsilon) - \kappa^2 + \mathcal{O}(\varepsilon))\|\psi\|_{2, \Omega_\varepsilon}^2 + \frac{\kappa^2}{2}\|\psi\|_{4, \Omega_\varepsilon}^4 \\ &\geq -(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon) - \mathcal{O}(\varepsilon))_+ \|\psi\|_{2, \Omega_\varepsilon}^2 + \frac{\kappa^2}{2}\|\psi\|_{4, \Omega_\varepsilon}^4 =: \mathcal{R}. \end{aligned} \quad (3.17)$$

We rewrite \mathcal{R} in the form

$$\mathcal{R} = \frac{1}{2} \int_{\Omega_\varepsilon} \left(\kappa|\psi|^2 - \frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon) - \mathcal{O}(\varepsilon))_+}{\kappa} \right)^2 dx - \frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon) - \mathcal{O}(\varepsilon))_+^2}{2\kappa^2} |\Omega_\varepsilon|,$$

and get

$$\mathcal{E}_\varepsilon(\psi, \mathbf{F}) \geq \mathcal{R} \geq -\frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon))_+^2}{2\kappa^2} |\Omega_\varepsilon| - \mathcal{O}(\varepsilon) |\Omega_\varepsilon|. \quad (3.18)$$

After inserting this lower bound into (3.16), we get the lower bound part in Theorem 1.5.

To obtain the matching upper bound, we write

$$E_{\text{gs}}(\kappa, H, \varepsilon) \leq \mathcal{E}_\varepsilon(u, \mathbf{F}),$$

and choose as function $u(x) = \tilde{u}(s(x), t(x))$, which is defined in the (s, t) coordinates by

$$\tilde{u}(s, t) = v(s) \exp\left(-iH\varphi_0(s, t)\right), \quad v(s) = \frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon))_+^{1/2}}{\kappa} \exp\left(i\frac{n_0\pi s}{L}\right).$$

Here φ_0 is the smooth function introduced in (2.5) and $n_0 \in \mathbb{Z}$ is defined just after (1.19). Collecting (2.3), (2.7) and (2.8), with the choice $b_\varepsilon = H$, we get

$$\begin{aligned} \mathcal{E}_\varepsilon(u, \mathbf{F}) &\leq (1 + \mathcal{O}(\varepsilon)) \int_{-L}^L \int_0^\varepsilon \left(|(\partial_s - iHf_0)v|^2 - \kappa^2|v|^2 + \frac{\kappa^2}{2}|v|^4 \right) dx + \mathcal{O}(\varepsilon)\kappa^2 \int_{-L}^L \int_0^\varepsilon |v|^2 dt ds \\ &\leq (1 + \mathcal{O}(\varepsilon)) (2L\varepsilon) \frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon))_+^2}{2\kappa^2} + \mathcal{O}(\varepsilon^2). \end{aligned}$$

The last statement in Theorem 1.5 follows immediately of the upper bound, and the more accurate lower bound of $\mathcal{E}_\varepsilon(\psi, \mathbf{F})$:

$$\begin{aligned} \mathcal{E}_\varepsilon(\psi, \mathbf{F}) + \frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon))_+^2}{2\kappa^2} |\Omega_\varepsilon| &\geq \mathcal{R} + \frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon))_+^2}{2\kappa^2} |\Omega_\varepsilon| \\ &\geq \frac{1}{2} \int_{\Omega_\varepsilon} \left(\kappa|\psi|^2 - \frac{(\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon))_+}{\kappa} \right)^2 dx - C\varepsilon |\Omega_\varepsilon|, \end{aligned} \quad (3.19)$$

together with (3.16).

4. ANALYSIS OF GROUND STATES AND STRONG DIAMAGNETISM – APPLICATIONS

We discuss in this section some consequences that we obtain from the statement of Theorem 1.4 or along its proof.

4.1. On the multiplicity of the eigenvalue $\lambda(\varepsilon, b_\varepsilon)$.

Along the proof of Theorem 1.4, we get some information regarding the multiplicity of the eigenvalue $\lambda(\varepsilon, b_\varepsilon)$ when (1.17) holds. Interestingly, we get that $\lambda(\varepsilon, b_\varepsilon)$ is simple when the ‘separation’ condition $(\text{SC})_\delta$ is satisfied.

Proposition 4.1. *For any $\delta \in (0, \frac{1}{2})$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$ satisfying the separation condition $(\text{SC})_\delta$ (see (1.24)) the eigenvalue $\lambda(\varepsilon, b_\varepsilon)$ is simple (where b_ε is given by (1.17)).*

Proposition 4.1 can not be used for the sequence $\left(\varepsilon_n = a \left(\frac{1}{2\gamma_0} \left(\frac{\pi n}{L} - a\right) - c\right)^{-1}\right)_{n \geq 1}$ since for any $\delta > 0$ the values ε_n violate the separation condition $(\text{SC})_\delta$ for n large enough. Proposition 4.2 addresses this degenerate situation, but unfortunately, it does not provide the exact value of the multiplicity.

Proposition 4.2. *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, the multiplicity of $\lambda(\varepsilon, b_\varepsilon)$ is ≤ 2 .*

Proof of Proposition 4.1.

From Theorem 1.4, we can choose $\varepsilon_0, M > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, the eigenvalue $\lambda(\varepsilon, b_\varepsilon)$ satisfies

$$\lambda(\varepsilon, b_\varepsilon) \leq \frac{a^2}{12} + \left(\frac{\pi}{L} \mathfrak{i}_0(c, a, \varepsilon)\right)^2 + M\varepsilon, \quad (4.1)$$

where $b_\varepsilon = \frac{a}{\varepsilon} + c$.

Let us denote by $\mathfrak{H}_{n,\varepsilon}$ the self-adjoint operator defined by the quadratic form in (2.18) for $\alpha = \alpha_n$, $\delta = \delta_\varepsilon$ and $\zeta = \zeta_\varepsilon$ given in (2.19) and (2.20). We also denote by $(\mu_k(\mathfrak{H}_{n,\varepsilon}))_{k \geq 1}$ the non decreasing sequence of eigenvalues of $\mathfrak{H}_{n,\varepsilon}$ counting multiplicities. Note that, for all $k \geq 1$, the eigenvalue $\mu_k(\mathfrak{H}_{n,\varepsilon})$ is simple, and by (2.24),

$$\mu_2(\mathfrak{H}_{n,\varepsilon}) \geq \left(\frac{\pi}{a}\right)^2. \quad (4.2)$$

Now, using (2.7)-(2.9), the min-max principle and the decomposition into Fourier modes (see (2.13), (2.17) and (2.18)), we get that,

$$\lambda_k(\mathcal{L}_{\omega_\varepsilon}^{b_\varepsilon}) \geq \delta_\varepsilon^{-1}(1 - \tilde{K}\varepsilon)\lambda_k\left(\bigoplus_{n \in \mathbb{Z}} \mathfrak{H}_{n,\varepsilon}\right), \quad (4.3)$$

where $\tilde{K} > 0$ is a constant, and for an operator \mathfrak{P} , $\lambda_k(\mathfrak{P})$ denotes the k 'th min-max eigenvalue of \mathfrak{P} .

As a consequence of (4.3),

$$\begin{aligned} & N\left(\mathcal{L}_{\omega_\varepsilon}^{b_\varepsilon}, \frac{a^2}{12} + \left(\frac{\pi}{L} \mathfrak{i}_0(c, a, \varepsilon)\right)^2 + M\varepsilon\right) \\ & \leq \text{Card}\left(\left\{(n, k) \in \mathbb{Z} \times \mathbb{N}^* : \mu_k(\mathfrak{H}_{n,\varepsilon}) \leq \delta_\varepsilon(1 - \tilde{K}\varepsilon)^{-1}\left(\frac{a^2}{12} + \left(\frac{\pi}{L} \mathfrak{i}_0(c, a, \varepsilon)\right)^2 + M\varepsilon\right)\right\}\right), \end{aligned} \quad (4.4)$$

where $N(\mathcal{L}_{\omega_\varepsilon}^{b_\varepsilon}, \lambda)$ denotes the number of eigenvalues of the operator $\mathcal{L}_{\omega_\varepsilon}^{b_\varepsilon}$ below λ , counting multiplicities.

For $0 < \tilde{K}\varepsilon < 1$, we have $(1 - \tilde{K}\varepsilon)^{-1} \leq 1 + 2\tilde{K}\varepsilon$ and consequently,

$$\delta_\varepsilon(1 - \tilde{K}\varepsilon)^{-1} \left(\frac{a^2}{12} + \left(\frac{\pi}{L} \mathbf{i}_0(c, a, \varepsilon) \right)^2 + M\varepsilon \right) \leq \delta_\varepsilon(1 + 2\tilde{K}\varepsilon) \left(\frac{a^2}{12} + \left(\frac{\pi}{L} \mathbf{i}_0(c, a, \varepsilon) \right)^2 + M\varepsilon \right).$$

Thus, there exists $K_1 > 0$ such that, for all $\varepsilon \in (0, 1/\tilde{K})$,

$$\begin{aligned} \delta_\varepsilon(1 - \tilde{K}\varepsilon)^{-1} \left(\frac{a^2}{12} + \left(\frac{\pi}{L} \mathbf{i}_0(c, a, \varepsilon) \right)^2 + M\varepsilon \right) &\leq \delta_\varepsilon \left(\frac{a^2}{12} + \left(\frac{\pi}{L} \mathbf{i}_0(c, a, \varepsilon) \right)^2 + K_1\varepsilon \right) \\ &= \delta_\varepsilon \left(\inf_{\ell \in \mathbb{Z}} m(\alpha_\ell, a) + K_1\varepsilon \right), \end{aligned}$$

where $m(\alpha_\ell, a)$ is introduced in (2.29).

Furthermore, by (4.2), for all $k \geq 2$,

$$\delta_\varepsilon \left(\inf_{\ell \in \mathbb{Z}} m(\alpha_\ell, a) + K_1\varepsilon \right) < \mu_k(\mathfrak{H}_{n, \varepsilon}).$$

Thus, we infer from (4.4),

$$\begin{aligned} N \left(\mathcal{L}_{\omega_\varepsilon}^{b_\varepsilon}, \frac{a^2}{12} + \left(\frac{\pi}{L} \mathbf{i}_0(c, a, \varepsilon) \right)^2 + M\varepsilon \right) \\ \leq \text{Card} \left(\left\{ n \in \mathbb{Z} : \mu_1(\mathfrak{H}_{n, \varepsilon}) \leq \delta_\varepsilon \left(\inf_{\ell \in \mathbb{Z}} m(\alpha_\ell, a) + K_1\varepsilon \right) \right\} \right). \end{aligned} \quad (4.5)$$

The condition of separation ensures that there exist a unique $n_0 \in \mathbb{Z}$ minimizing the problem in (1.18) and $d_0 > 0$ such that, for all $n \in \mathbb{Z} \setminus \{n_0\}$ and $\varepsilon \in (0, \varepsilon_0]$ satisfying (SC) $_\delta$,

$$m(\alpha_n, a) \geq m(\alpha_{n_0}, a) + d_0. \quad (4.6)$$

Using (2.33), we can restrict to counting the set of $n \in \mathbb{Z}$ satisfying the conditions

$$|\alpha_n| \leq 10(a + \frac{\pi}{2L}) \text{ and } \mu_1(\mathfrak{H}_{n, \varepsilon}) \leq \delta_\varepsilon(m(\alpha_{n_0}, a) + K_1\varepsilon).$$

For $n \in \mathbb{Z} \setminus \{n_0\}$ and $|\alpha_n| \leq 10(a + \frac{\pi}{2L})$, we know, thanks to (2.28), that

$$\mu_1(\mathfrak{H}_{n, \varepsilon}) = \delta_\varepsilon m(\alpha_n, a) + o(\delta_\varepsilon).$$

We infer from the condition in (4.6), that, for ε sufficiently small,

$$\mu_1(\mathfrak{H}_{n, \varepsilon}) \geq \delta_\varepsilon \left(m(\alpha_{n_0}, a) + \frac{d_0}{2} \right).$$

Consequently, for ε sufficiently small,

$$N \left(\mathcal{L}_{\omega_\varepsilon}^{b_\varepsilon}, \frac{a^2}{12} + \left(\frac{\pi}{L} \mathbf{i}_0(c, a, \varepsilon) \right)^2 + M\varepsilon \right) \leq 1,$$

which, when combined with (4.1), yields the simplicity of the eigenvalue $\lambda(\varepsilon, b_\varepsilon)$. \square

Remark 4.3. Collecting (4.3) and (4.6), we get under the assumptions of Proposition 4.1 that the spectral gap between the first and second eigenvalues of $\mathcal{L}_{\omega_\varepsilon}^{b_\varepsilon}$ satisfies for ε sufficiently small,

$$\lambda_2(\varepsilon, b_\varepsilon) - \lambda(\varepsilon, b_\varepsilon) \geq \frac{d_0}{2}. \quad (4.7)$$

Proof of Proposition 4.2.

The problem in (1.18) may have at most two minimizers. Let n_0 be the *smallest* minimizer of (1.18). There exist $d' > 0$ and $\varepsilon_0 > 0$, such that for $n \in \mathbb{Z} \setminus \{n_0, n_0 + 1\}$ and $\varepsilon \in (0, \varepsilon_0]$, we have

$$m(\alpha_n, a) \geq m(\alpha_{n_0}, a) + d'. \quad (4.8)$$

Consequently, (4.5) yields the existence of $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$,

$$N \left(\mathcal{L}_{\omega_\varepsilon}^{b_\varepsilon}, \frac{a^2}{12} + \left(\frac{\pi}{L} \mathbf{i}_0(c, a, \varepsilon) \right)^2 + M\varepsilon \right) \leq 2,$$

where M is the constant in (4.1). \square

Remark 4.4. Assume that $0 < \varepsilon \leq \varepsilon_0$ and the problem (1.18) has two minimizers n_0 and $m_0 = n_0 + 1$. Then, there exists $M' > 0$ and a possibly smaller $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$, the second min-max eigenvalue satisfies,

$$\lambda_2(\varepsilon, b_\varepsilon) \leq \frac{a^2}{12} + \left(\frac{\pi}{L} \mathbf{i}_0(c, a, \varepsilon) \right)^2 + M' \varepsilon. \quad (4.9)$$

This can be achieved by using the min-max formula with the two dimensional eigenspace $V_0 := \text{span}(v_{n_0}, v_{m_0})$, where, for every integer n , the function v_n is defined as follows,

$$v_n(s, t) = \exp\left(-i \frac{n\pi}{L} s\right) u_n(b_\varepsilon^{-1} t), \quad (4.10)$$

with u_n the normalized ground state of the effective operator $\mathfrak{H}_{n, \varepsilon}$.

This case covers the sequence $(\varepsilon_n)_{n \geq 1}$ with $\varepsilon_n = a \left(\frac{1}{2\gamma_0} \left(\frac{\pi n}{L} - a \right) - c \right)^{-1}$, where, by Theorem 1.4,

$$\lambda(\varepsilon_n, b_{\varepsilon_n}) \underset{n \rightarrow +\infty}{\sim} \frac{a^2}{12} + \frac{\pi^2}{4L^2}.$$

An interesting question would be to determine the gap $\lambda_2(\varepsilon_n, b_{\varepsilon_n}) - \lambda(\varepsilon_n, b_{\varepsilon_n})$.

4.2. Structure of ground states. When the separation condition $(\text{SC})_\delta$ holds, the eigenvalue $\lambda(\varepsilon, b_\varepsilon)$ is simple. We can prove that the ground states of the operator $\mathcal{L}_{\omega_\varepsilon}^{b_\varepsilon}$ have a simple structure. We denote by Π_ε the orthogonal projection on the space of ground states of $\mathcal{L}_{\omega_\varepsilon}^{b_\varepsilon}$ and will have:

Proposition 4.5. *For any $\delta \in (0, \frac{1}{2})$, there exists $\varepsilon_0, M_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$ satisfying condition $(\text{SC})_\delta$, we have*

$$\|u_0 - \Pi_\varepsilon u_0\|_{2, \Omega_\varepsilon} \leq M_0 \varepsilon,$$

where

- b_ε is given by (1.17),
- $u_0(x) = \exp\left(-ib_\varepsilon \varphi_0(s, t)\right) \exp\left(\frac{in_0 \pi s}{L}\right)$, ($x = \Phi_0(s, t)$),
- $n_0 \in \mathbb{Z}$ is the minimizer of (1.19),
- φ_0 is the function in (2.5), and Φ_0 is the diffeomorphism introduced in (2.1).

Before the proof we recall an abstract lemma in Hilbertian analysis which reads in our application as follows:

Lemma 4.6. *Assume that $\varepsilon \in (0, 1)$, $\mathcal{K} > 0$ and $v \in H^1(\Omega_\varepsilon)$ satisfy*

$$Q_{\varepsilon, b_\varepsilon}(v) := \int_{\Omega_\varepsilon} |(\nabla - ib_\varepsilon \mathbf{F})v|^2 dx \leq \lambda(\varepsilon, b_\varepsilon) \|v\|_{2, \Omega_\varepsilon}^2 + \mathcal{K}. \quad (4.11)$$

Then

$$Q_{\varepsilon, b_\varepsilon}(v - \Pi_\varepsilon v) \leq \lambda(\varepsilon, b_\varepsilon) \|v - \Pi_\varepsilon v\|_{2, \Omega_\varepsilon}^2 + \mathcal{K}, \quad (4.12)$$

and

$$\left(\lambda_2(\varepsilon, b_\varepsilon) - \lambda(\varepsilon, b_\varepsilon) \right) \|v - \Pi_\varepsilon v\|_{2, \Omega_\varepsilon}^2 \leq \mathcal{K}. \quad (4.13)$$

Here b_ε is given in (1.17).

We will use Lemma 4.6 in the proof of Proposition 4.5 and also later in Section 5. For the convenience of the reader, we recall its standard proof.

Proof of Lemma 4.6.

We start by observing the following two identities

$$\|v\|_{2,\Omega_\varepsilon}^2 = \|v - \Pi_\varepsilon v\|_{2,\Omega_\varepsilon}^2 + \|\Pi_\varepsilon v\|_{2,\Omega_\varepsilon}^2,$$

and

$$\begin{aligned} Q_{\varepsilon,b_\varepsilon}(v) &= Q_{\varepsilon,b_\varepsilon}(v - \Pi_\varepsilon v) + Q_{\varepsilon,b_\varepsilon}(\Pi_\varepsilon v) \\ &= Q_{\varepsilon,b_\varepsilon}(v - \Pi_\varepsilon v) + \lambda(\varepsilon, b_\varepsilon)\|\Pi_\varepsilon v\|_{2,\Omega_\varepsilon}^2. \end{aligned}$$

This implies through (4.11) the inequality (4.12).

Now, we write by the min-max principle,

$$Q_{\varepsilon,b_\varepsilon}(v - \Pi_\varepsilon v) \geq \lambda_2(\varepsilon, b_\varepsilon)\|v - \Pi_\varepsilon v\|_{2,\Omega_\varepsilon}^2.$$

Collecting the foregoing estimates and (4.11), we get

$$\lambda_2(\varepsilon, b_\varepsilon)\|v - \Pi_\varepsilon v\|_{2,\Omega_\varepsilon}^2 + \lambda(\varepsilon, b_\varepsilon)\|\Pi_\varepsilon v\|_{2,\Omega_\varepsilon}^2 \leq \lambda(\varepsilon, b_\varepsilon)\left(\|v - \Pi_\varepsilon v\|_{2,\Omega_\varepsilon}^2 + \|\Pi_\varepsilon v\|_{2,\Omega_\varepsilon}^2\right) + \mathcal{K},$$

which gives (4.13) and finishes the proof of Lemma 4.6. \square

Proof of Proposition 4.5.

Let $\mathfrak{e}_0(c, a, \varepsilon)$ be the quantity introduced in (1.22). It is easy to check that

$$\int_{\Omega_\varepsilon} |u_0|^2 dx = 2L\varepsilon + \mathcal{O}(\varepsilon^2),$$

and

$$Q_{\varepsilon,b_\varepsilon}(u_0) := \int_{\Omega_\varepsilon} |(\nabla - ib_\varepsilon \mathbf{F})u_0|^2 dx \leq 2L\varepsilon \mathfrak{e}_0(c, a, \varepsilon) + \mathcal{O}(\varepsilon^2).$$

Now, using Theorem 1.4, we may write

$$Q_{\varepsilon,b_\varepsilon}(u_0) \leq \lambda(\varepsilon, b_\varepsilon)\|u_0\|_{2,\Omega_\varepsilon}^2 + \mathcal{O}(\varepsilon^2). \quad (4.14)$$

By Lemma 4.6, we deduce that

$$\left(\lambda_2(\varepsilon, b_\varepsilon) - \lambda(\varepsilon, b_\varepsilon)\right)\|u_0 - \Pi_\varepsilon u_0\|_{2,\Omega_\varepsilon}^2 = \mathcal{O}(\varepsilon^2).$$

To finish the proof, we use the lower bound of the spectral gap given in Remark 4.3. \square

Remark 4.7. Proposition 4.5 yields the existence of \tilde{M}_0 such that, for all $\varepsilon \in (0, \varepsilon_0]$ and $u \in L^2(\Omega_\varepsilon)$,

$$\left\| \Pi_\varepsilon u - \frac{1}{|\Omega_\varepsilon|} \langle u, u_0 \rangle u_0 \right\|_{2,\Omega_\varepsilon} \leq \tilde{M}_0 \varepsilon^{1/2} \|u\|_{2,\Omega_\varepsilon}.$$

Indeed, since the eigenvalue $\lambda(\varepsilon, b_\varepsilon)$ is simple, the corresponding eigenspace is spanned by the following normalized ground state

$$u_\varepsilon = \frac{1}{\|\Pi_\varepsilon u_0\|_{2,\Omega_\varepsilon}} \Pi_\varepsilon u_0,$$

and

$$\Pi_\varepsilon u = \langle u, u_\varepsilon \rangle u_\varepsilon.$$

4.3. Breakdown of superconductivity. A celebrated result by Giorgi-Phillips [4] establishes the breakdown of superconductivity when the parameter measuring the strength of the magnetic field is sufficiently large. One consequence of the main results of this paper is the following ‘quantitative’ version of the breakdown of superconductivity.

Proposition 4.8. *Given $\kappa > 0$ and $a > 2\sqrt{3}\kappa$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, all $H \geq \frac{a}{\varepsilon}$, every critical point $(\psi, \mathbf{A})_{\kappa, H, \varepsilon}$ is trivial.*

Proof. We first see from Theorem 1.3 (assertion (A), with $N > \kappa^2$) together with Theorem 1.2, that this is true for $H \geq \frac{d_0}{\varepsilon}$.

Assuming now that $H \leq \frac{d_0}{\varepsilon}$, we prove it by contradiction. If there were sequences $(H_n)_{n \geq 1}$ and $(\varepsilon_n)_{n \geq 1}$ such that $\varepsilon_n \rightarrow 0$, $H_n \varepsilon_n \rightarrow \alpha$ for some $\alpha \in [a, +\infty)$, and a non trivial minimizer, then an easy adjustment of the proof of Theorem 1.4 yields that

$$\lambda(\varepsilon_n, H_n) \sim \frac{\alpha^2}{12} + \left(\frac{\pi}{L} i_0(0, \alpha, \varepsilon_n) \right)^2 .$$

Consequently, we get $\lambda(\varepsilon_n, H_n) \geq \kappa^2$ for n large enough, because $\frac{\alpha^2}{12} > \kappa^2$. Theorem 1.2 leads to a contradiction. \square

4.4. Lack of strong diamagnetism and oscillations in the Little-Parks framework.

The behavior of the eigenvalue in Theorem 1.4 shows a pleasant connection to the oscillatory behavior of the Little-Parks experiment. The following statement displays counterexamples to strong diamagnetism.

Proposition 4.9. *There exists a sequence $(\varepsilon_N)_{N \geq 1} \subset \mathbb{R}_+$ which converges to 0 such that, for all $N \geq 1$, the function $H \mapsto \lambda(\varepsilon_N, H)$ is not monotone increasing.*

Proof. Choose $a > 0$ so that

$$\frac{a^2}{12} < \kappa^2 < \frac{a^2}{12} + \frac{1}{4} \left(\frac{\pi}{L} \right)^2 .$$

Let us define the following sequence

$$\tilde{\varepsilon}_N = \frac{aL\gamma_0}{\pi} \left(N - \frac{a}{2} \right)^{-1} \quad \text{for } N \in \mathbb{N} \cap \left(\frac{a}{2}, +\infty \right) .$$

Define $H_N^{(1)} < H_N^{(2)} < H_N^{(3)}$ by

$$H_N^{(1)} = \frac{a}{\tilde{\varepsilon}_N} - \frac{\pi}{2\gamma_0 L}, \quad H_N^{(2)} = \frac{a}{\tilde{\varepsilon}_N}, \quad H_N^{(3)} = \frac{a}{\tilde{\varepsilon}_N} + \frac{\pi}{2\gamma_0 L} .$$

Then, we notice that, as $N \rightarrow +\infty$,

$$\lambda(\tilde{\varepsilon}_N, H_N^{(1)}) \sim \frac{a^2}{12} + \frac{1}{4} \left(\frac{\pi}{L} \right)^2 > \kappa^2, \quad \lambda(\tilde{\varepsilon}_N, H_N^{(2)}) \sim \frac{a^2}{12} < \kappa^2, \quad \lambda(\tilde{\varepsilon}_N, H_N^{(3)}) \sim \frac{a^2}{12} + \frac{1}{4} \left(\frac{\pi}{L} \right)^2 > \kappa^2 .$$

Hence we find N_0 such that the statement of the proposition holds for $\varepsilon_N = \tilde{\varepsilon}_{N+N_0}$. \square

Remark 4.10. Along the proof of Proposition 4.9, we obtain the two remarkable observations:

- For N sufficiently large $H_N^{(1)} < H_N^{(2)}$ while $\lambda(\tilde{\varepsilon}_N, H_N^{(1)}) > \lambda(\tilde{\varepsilon}_N, H_N^{(2)})$.
- By Theorem 1.2, for large N , the minimizers $(\psi, \mathbf{A})_{\kappa, H_N^{(i)}, \tilde{\varepsilon}_N}$, $i = 1, 3$, are non-trivial, while any critical point $(\psi, \mathbf{A})_{\kappa, H_N^{(2)}, \tilde{\varepsilon}_N}$ is trivial.

Thus, the transition from the superconducting to the normal state is not monotone, which is in agreement with the Little-Parks experiment.

5. STRUCTURE OF THE ORDER PARAMETER AND CIRCULATION OF THE SUPER-CURRENT

5.1. Hypotheses. Throughout this section, we work under the following hypothesis on the parameter H :

$$H = b_\varepsilon := \frac{a}{\varepsilon} + c \quad (5.1)$$

where $a > 0$ and $c \in \mathbb{R}$ are fixed constants.

The results of this section will concern an arbitrary minimizer $(\psi, \mathbf{A})_{\varepsilon, H}$ of the GL functional, provided H satisfies (5.1), and ε satisfies the ‘separation’ conditions $(\text{SC})_\delta$ and $(\text{SC})'_\delta$ introduced in (1.24)-(1.25).

5.2. Approximation of the order parameter.

In light of Theorem 1.5, we introduce the following quantity

$$\Lambda_\kappa^\varepsilon := \frac{\kappa^2 - \mathfrak{e}_0(c, a, \varepsilon)}{\kappa^2}, \quad (5.2)$$

where $\mathfrak{e}_0(c, a, \varepsilon)$ is introduced in (1.22). Note that, under the hypotheses in Subsection 5.1, there exists a constant $c_0 > 0$ such that, for all ε sufficiently small,

$$c_0 \leq \Lambda_\kappa^\varepsilon \leq 1. \quad (5.3)$$

Let u_0 be the function introduced in Proposition 4.5. We will prove that, up to multiplication by $\sqrt{\Lambda_\kappa^\varepsilon}$ and a complex phase, the function u_0 provides us with a good approximation of the GL order parameter ψ .

Proposition 5.1. *There exist constants $C, \varepsilon_0 > 0$ such that, if*

- $\varepsilon \in (0, \varepsilon_0]$ satisfies the separation conditions $(\text{SC})_\delta$ and $(\text{SC})'_\delta$;
- H satisfies (5.1);
- $(\psi, \mathbf{A})_{\varepsilon, H}$ is a minimizer of the GL functional in (1.1);

then, there exists $\alpha_\varepsilon \in \mathbb{C}$ such that $|\alpha_\varepsilon| = 1$, ψ satisfies

$$\|\psi - \alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0\|_{2, \Omega_\varepsilon} \leq C \varepsilon, \quad (5.4)$$

and its trace on $\partial\Omega$ satisfies

$$\|\psi|_{\partial\Omega} - \alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} (u_0)|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq C \varepsilon^{1/2}. \quad (5.5)$$

Proof.

Proof of (5.4). Collecting (3.16) and (3.19), we infer from Theorem 1.5,

$$\mathcal{E}_\varepsilon(\psi, \mathbf{F}) = -\frac{1}{2}(\Lambda_\kappa^\varepsilon)^2 |\Omega_\varepsilon| + \mathcal{O}(\varepsilon^2). \quad (5.6)$$

Furthermore, it results from Theorem 1.5 (see (1.23)) together with the definition of $\Lambda_\kappa^\varepsilon$ in (5.2) that

$$\left\| |\psi|^2 - \Lambda_\kappa^\varepsilon \right\|_{2, \Omega_\varepsilon} = \mathcal{O}(\varepsilon) \quad \text{and} \quad \left\| |\psi| - \sqrt{\Lambda_\kappa^\varepsilon} \right\|_{2, \Omega_\varepsilon} = \mathcal{O}(\varepsilon). \quad (5.7)$$

Consequently,

$$\|\psi\|_{4, \Omega_\varepsilon}^4 = (\Lambda_\kappa^\varepsilon)^2 |\Omega_\varepsilon| + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad \|\psi\|_{2, \Omega_\varepsilon}^2 = \Lambda_\kappa^\varepsilon |\Omega_\varepsilon| + \mathcal{O}(\varepsilon^2). \quad (5.8)$$

Note that (5.2) yields that $\kappa^2(-(\Lambda_\kappa^\varepsilon)^2 + \Lambda_\kappa^\varepsilon) = \Lambda_\kappa^\varepsilon \mathfrak{e}_0(c, a, \varepsilon)$, which in turn yields the following identity,

$$\begin{aligned} -\frac{\kappa^2}{2}(\Lambda_\kappa^\varepsilon)^2 |\Omega_\varepsilon| + \kappa^2 \|\psi\|_{2, \Omega_\varepsilon}^2 - \frac{\kappa^2}{2} \|\psi\|_{4, \Omega_\varepsilon}^4 &= |\Omega_\varepsilon| \Lambda_\kappa^\varepsilon \mathfrak{e}_0(c, a, \varepsilon) + \mathcal{O}(\varepsilon^2) \\ &= \mathfrak{e}_0(c, a, \varepsilon) \|\psi\|_{2, \Omega_\varepsilon}^2 + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Now we insert this identity into (5.6) to get (see (1.1) and (1.11)):

$$\int_{\Omega_\varepsilon} |(\nabla - iH\mathbf{F})\psi|^2 dx = \mathfrak{e}_0(c, a, \varepsilon) \|\psi\|_{2, \Omega_\varepsilon}^2 + \mathcal{O}(\varepsilon^2).$$

Recall that $H = b_\varepsilon$ with b_ε given in (5.1). Using Theorem 1.4 and the definition of $\mathbf{e}_0(c, a, \varepsilon)$ in (1.22), we get further

$$\begin{aligned} \int_{\Omega_\varepsilon} |(\nabla - ib_\varepsilon \mathbf{F})\psi|^2 dx &= (\lambda(b_\varepsilon, \varepsilon) + \mathcal{O}(\varepsilon)) \|\psi\|_{2, \Omega_\varepsilon}^2 + \mathcal{O}(\varepsilon^2) \\ &\leq \lambda(b_\varepsilon, \varepsilon) \|\psi\|_{2, \Omega_\varepsilon}^2 + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Now, we can apply Lemma 4.6 (with $v = \psi$). Using the estimate in (4.7), we get

$$\|\psi - \Pi_\varepsilon \psi\|_{2, \Omega_\varepsilon} = \mathcal{O}(\varepsilon), \quad (5.9)$$

and

$$\|(\nabla - iH\mathbf{F})(\psi - \Pi_\varepsilon \psi)\|_{2, \Omega_\varepsilon} = \mathcal{O}(\varepsilon). \quad (5.10)$$

Let u_0 be the function introduced in Proposition 4.5. By Remark 4.7, we know that

$$\left\| \Pi_\varepsilon \psi - \frac{1}{|\Omega_\varepsilon|} \langle \psi, u_0 \rangle u_0 \right\|_{2, \Omega_\varepsilon} = \mathcal{O}(\varepsilon^{1/2}) \|\psi\|_{2, \Omega_\varepsilon} = \mathcal{O}(\varepsilon). \quad (5.11)$$

We can estimate $\langle \psi, u_0 \rangle$ as follows.

On one hand we have

$$\begin{aligned} \langle \Pi_\varepsilon \psi, \psi \rangle &= \|\Pi_\varepsilon \psi\|_{2, \Omega_\varepsilon}^2 \\ &= \|\psi\|_{2, \Omega_\varepsilon}^2 - \|\psi - \Pi_\varepsilon \psi\|_{2, \Omega_\varepsilon}^2 = \Lambda_\kappa^\varepsilon |\Omega_\varepsilon| + \mathcal{O}(\varepsilon^2), \end{aligned}$$

by (5.8) and (5.9).

On the other hand, using (5.9) and (5.11), we have

$$\langle \Pi_\varepsilon \psi, \psi \rangle = \frac{1}{|\Omega_\varepsilon|} |\langle \psi, u_0 \rangle|^2 + \mathcal{O}(\varepsilon) \|\psi\|_{2, \Omega_\varepsilon} = \frac{1}{|\Omega_\varepsilon|} |\langle \psi, u_0 \rangle|^2 + \mathcal{O}(\varepsilon^{3/2}),$$

thereby obtaining that

$$|\langle \psi, u_0 \rangle|^2 = |\Omega_\varepsilon| \left(\Lambda_\kappa^\varepsilon |\Omega_\varepsilon| + \mathcal{O}(\varepsilon^{3/2}) \right) = |\Omega_\varepsilon|^2 \Lambda_\kappa^\varepsilon + \mathcal{O}(\varepsilon^{5/2}).$$

Now, we set

$$\alpha_\varepsilon := \frac{\langle \psi, u_0 \rangle}{|\langle \psi, u_0 \rangle|}, \quad (5.12)$$

We observe that

$$\left| \alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} - \frac{1}{|\Omega_\varepsilon|} \langle \psi, u_0 \rangle \right| = \mathcal{O}(\varepsilon^{1/2}), \quad (5.13)$$

and, after collecting (5.9) and (5.11),

$$\|\psi - \alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0\|_{2, \Omega_\varepsilon} = \mathcal{O}(\varepsilon). \quad (5.14)$$

□

Proof of (5.5). We first compute,

$$\begin{aligned} \|(\nabla - iH\mathbf{F})(\Pi_\varepsilon \psi - \alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0)\|_{2, \Omega_\varepsilon}^2 &= \|(\nabla - iH\mathbf{F})\Pi_\varepsilon \psi\|_{2, \Omega_\varepsilon}^2 + \|(\nabla - iH\mathbf{F})\alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0\|_{2, \Omega_\varepsilon}^2 \\ &\quad + 2\operatorname{Re}\langle (\nabla - iH\mathbf{F})\Pi_\varepsilon \psi, (\nabla - iH\mathbf{F})\alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0 \rangle. \end{aligned}$$

We perform an integration by parts to rewrite the last term of above in the form

$$\begin{aligned} \langle (\nabla - iH\mathbf{F})\Pi_\varepsilon \psi, (\nabla - iH\mathbf{F})\alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0 \rangle &= \langle -(\nabla - iH\mathbf{F})^2 \Pi_\varepsilon \psi, \alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0 \rangle \\ &= \lambda(\varepsilon, H) \langle \Pi_\varepsilon \psi, \alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0 \rangle. \end{aligned}$$

We then insert (4.14) and get,

$$\begin{aligned} \|(\nabla - iH\mathbf{F})(\Pi_\varepsilon \psi - \alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0)\|_{2, \Omega_\varepsilon}^2 &\leq \lambda(\varepsilon, H) \|\Pi_\varepsilon \psi - \alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0\|_{2, \Omega_\varepsilon}^2 + \mathcal{O}(\varepsilon^2) \\ &= \mathcal{O}(\varepsilon^2), \end{aligned}$$

where we used (5.11) and (5.13) for the last statement above. Combining this estimate and (5.10), we get

$$\|(\nabla - iH\mathbf{F})(\psi - \alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0)\|_{2, \Omega_\varepsilon} = \mathcal{O}(\varepsilon). \quad (5.15)$$

Let us introduce the function

$$w = \left| \psi - \alpha_\varepsilon \sqrt{\Lambda_\kappa^\varepsilon} u_0 \right|.$$

By the diamagnetic inequality, we infer from (5.15),

$$\|\nabla w\|_{2, \Omega_\varepsilon} = \mathcal{O}(\varepsilon). \quad (5.16)$$

Define now the re-scaled function

$$[-L, L] \times (0, \varepsilon_0) \ni (s, \tau) \mapsto \tilde{w}_\varepsilon(s, \tau) = \tilde{w}(s, \varepsilon_0^{-1} \varepsilon \tau),$$

where $\tilde{w} = w \circ \Phi_0$, Φ_0 is the transformation introduced in (2.1), and $\varepsilon_0 \in (0, 1)$ is a sufficiently small constant so that the transformation $\Phi_0 : [-L, L] \times (0, \varepsilon_0) \rightarrow \Omega_{\varepsilon_0}$ is bijective.

We can define a function $w_\varepsilon \in H^1(\Omega_{\varepsilon_0})$ by means of the function \tilde{w}_ε as follows

$$w_\varepsilon(x) = \tilde{w}_\varepsilon(s, \tau) \quad \text{for } x = \Phi_0(s, \tau).$$

Consequently, we obtain from (5.14) and (5.16),

$$\|w_\varepsilon\|_{H^1(\Omega_{\varepsilon_0})} = \mathcal{O}(\varepsilon^{1/2}).$$

By the trace theorem, we deduce that

$$\|w_\varepsilon\|_{L^2(\partial\Omega)} = \mathcal{O}(\varepsilon^{1/2}).$$

□

Having proved (5.4) and (5.5), we have achieved the proof of Proposition 5.1. □

5.3. More a priori estimates.

Using the curl-div estimate, we can write,

$$\|\mathbf{A} - \mathbf{F}\|_{H^1(\Omega)} \leq C \|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon^3), \quad (5.17)$$

where we used (3.1) and (3.2) to get the estimate $\mathcal{O}(\varepsilon^3)$.

Also, the following estimate holds (see [1, Lem. B.1])

$$\|\mathbf{A} - \mathbf{F}\|_{H^2(\Omega)} \leq C \|\nabla \operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon^2), \quad (5.18)$$

where we used (3.1) to get the estimate $\mathcal{O}(\varepsilon^2)$.

Consequently, the Sobolev embedding theorem yields, for every $\alpha \in (0, 1)$,

$$\|\mathbf{A} - \mathbf{F}\|_{C^{0, \alpha}(\bar{\Omega})} = \mathcal{O}(\varepsilon^2). \quad (5.19)$$

5.4. Proof of Theorem 1.6.

With the following notation

$$(a, b) = \operatorname{Re}(a\bar{b}) \quad (a, b \in \mathbb{C}),$$

we may express the super-current as follows

$$\mathbf{j} = (i\psi, (\nabla - iH\mathbf{A})\psi).$$

We will prove (see (1.26)) that

$$\frac{1}{|\partial\Omega|} \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{j} ds = \Lambda_\kappa^\varepsilon \frac{4\pi n_0}{|\partial\Omega|} + o(\varepsilon^{-1}) \quad (\varepsilon \rightarrow 0_+), \quad (5.20)$$

where \mathbf{t} is the unit tangent vector of $\partial\Omega$ oriented in the counter-clockwise direction, and $n_0 \in \mathbb{Z}$ is the minimizer of (1.18). Note that n_0 depends on ε and is $\mathcal{O}(\varepsilon^{-1})$, as $\varepsilon \rightarrow 0_+$.

Lemma 5.2.

$$\int_{\partial\Omega} \mathbf{t} \cdot \mathbf{j} \, ds = -H\Lambda_\kappa^\varepsilon |\Omega| + \int_{\partial\Omega} \mathbf{t} \cdot (i\psi, \nabla\psi) \, ds + o(\varepsilon^{-1}) \quad (\varepsilon \rightarrow 0_+).$$

Proof. We perform the simple decomposition

$$\mathbf{j} = -H\mathbf{A}|\psi|^2 + (i\psi, \nabla\psi).$$

In light of (5.5) and (5.19), we write,

$$\int_{\partial\Omega} \mathbf{t} \cdot \mathbf{A} |\psi|^2 \, ds = \Lambda_\kappa^\varepsilon \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{F} \, ds + o(1) \quad (\varepsilon \rightarrow 0_+).$$

By the Stokes formula,

$$\int_{\partial\Omega} \mathbf{t} \cdot \mathbf{F} \, ds = \int_{\Omega} \operatorname{curl} \mathbf{F} \, dx = |\Omega|.$$

□

Let Φ_0 be the transformation introduced in (2.1). Denote by $\tilde{\psi} = \psi \circ \Phi_0^{-1}$ and define the function $u = \tilde{u} \circ \Phi_0$ as follows

$$[-L, L] \times (0, \varepsilon) \ni (s, t) \mapsto \tilde{u}(s, t) := (\Lambda_\kappa^\varepsilon)^{-\frac{1}{2}} \alpha_\varepsilon^{-1} e^{iH\varphi_0(s,t)} e^{-in_0\pi s/L} \tilde{\psi}(s, t), \quad (5.21)$$

where φ_0 is introduced in (2.5) and α_ε is the unit complex number defined in (5.12). Thanks to (5.3), \tilde{u} is well defined by (5.21) and satisfies

$$|\tilde{u}(s, t)| \leq \frac{1}{\sqrt{c_0}} |\tilde{\psi}(s, t)| \leq \frac{1}{\sqrt{c_0}}. \quad (5.22)$$

Furthermore, it results from (5.5) that $\tilde{u}|_{t=0}$ converges to 1 in $L^2([-L, L])$.

Lemma 5.3.

$$\int_{-L}^L \int_0^\varepsilon \left(|\partial_t \tilde{u}|^2 + |\partial_s \tilde{u}|^2 \right) dt ds = \mathcal{O}(\varepsilon).$$

Proof. By a computation analgous with the one in (2.3),

$$\mathcal{E}_\varepsilon(\psi, \mathbf{F}) = \Lambda_\kappa^\varepsilon (1 + \mathcal{O}(\varepsilon)) \int_{-L}^L \int_0^\varepsilon \left(|\partial_t \tilde{u}|^2 + |(\partial_s + iV_\varepsilon)\tilde{u}|^2 \right) dt ds - \kappa^2 \|\psi\|_{2, \Omega_\varepsilon}^2 + \frac{\kappa^2}{2} \|\psi\|_{4, \Omega_\varepsilon}^4,$$

where

$$V_\varepsilon(s, t) = \frac{\pi}{L} n_0 + Hf(s, t),$$

and f is introduced in (2.2).

Consequently, we infer from (5.6) and (5.8),

$$\int_{-L}^L \int_0^\varepsilon |\partial_t \tilde{u}|^2 dt ds = \mathcal{O}(\varepsilon) \quad \text{and} \quad \int_{-L}^L \int_0^\varepsilon |(\partial_s - iV_\varepsilon)\tilde{u}|^2 dt ds = \mathcal{O}(\varepsilon).$$

To finish the proof, it remains to to prove that $\int_{-L}^L \int_0^\varepsilon |\partial_s \tilde{u}|^2 dt ds = \mathcal{O}(\varepsilon)$. To that end, it is enough to prove that $\int_{-L}^L \int_0^\varepsilon |V_\varepsilon \tilde{u}|^2 dt ds = \mathcal{O}(\varepsilon)$.

Since n_0 minimizes (1.18),

$$\left| n_0 + \frac{L}{\pi} \left(\frac{a}{\varepsilon} + c \right) \gamma_0 \right| \leq \beta_{n_0}(c, a, \varepsilon) + \frac{a}{2} \leq \frac{1}{2} + \frac{a}{2}. \quad (5.23)$$

Since $H = \frac{a}{\varepsilon} + c$ and $t \in (0, \varepsilon)$,

$$\begin{aligned} V_\varepsilon(s, t) &= \frac{\pi}{L} \left(n_0 + \frac{\gamma_0 H L}{\pi} + \frac{H t L}{\pi} + \frac{H t^2 L}{2\pi} k(s) \right) \\ &= \frac{\pi}{L} \left(n_0 + \frac{L}{\pi} \left(\frac{a}{\varepsilon} + c \right) \gamma_0 \right) + \mathcal{O}(Ht) + \mathcal{O}(Ht^2) \\ &= \mathcal{O}(1). \end{aligned}$$

Now, the foregoing estimate and (5.22) yield,

$$\int_{-L}^L \int_0^\varepsilon |V_\varepsilon \tilde{u}|^2 dt ds = \mathcal{O}(1) \int_{-L}^L \int_0^\varepsilon |\tilde{u}|^2 dt ds = \mathcal{O}(\varepsilon).$$

□

Lemma 5.4.

$$\frac{1}{|\partial\Omega|} \int_{\partial\Omega} \mathbf{t} \cdot (i\psi, \nabla\psi) ds = \Lambda_\kappa^\varepsilon \frac{4\pi n_0}{|\partial\Omega|} + o(\varepsilon^{-1}).$$

Proof. Notice that

$$\int_{\partial\Omega} \mathbf{t} \cdot (i\psi, \nabla\psi) ds = \int_{-L}^L (i\tilde{\psi}, \partial_s \tilde{\psi})|_{t=0} ds,$$

where (see (5.21))

$$(i\tilde{\psi}, \partial_s \tilde{\psi})|_{t=0} = \Lambda_\kappa^\varepsilon \left(-H \partial_s \varphi_0(s, 0) + n_0 \frac{\pi}{L} \right) |\tilde{u}(s, 0)|^2 + \Lambda_\kappa^\varepsilon (i\tilde{u}, \partial_s \tilde{u})|_{t=0}.$$

Consequently,

$$\begin{aligned} \int_{-L}^L (i\tilde{\psi}, \partial_s \tilde{\psi})|_{t=0} ds &= \Lambda_\kappa^\varepsilon \frac{\pi n_0}{L} \int_{-L}^L |\tilde{u}(s, 0)|^2 ds - \Lambda_\kappa^\varepsilon H \int_{-L}^L \partial_s \varphi_0(s, 0) \tilde{u}(s, 0) ds \\ &\quad + \Lambda_\kappa^\varepsilon \int_{-L}^L (i\tilde{u}, \partial_s \tilde{u})|_{t=0} ds, \end{aligned} \tag{5.24}$$

with

$$\int_{-L}^L |\tilde{u}(s, 0)|^2 ds = 2L + o(1) = |\partial\Omega| + o(1), \tag{5.25}$$

since $\tilde{u} \rightarrow 1$ in $L^2([-L, L])$, as $\varepsilon \rightarrow 0_+$.

We estimate the integral of $\partial_s \varphi_0(s, 0) |\tilde{u}(s, 0)|^2$. By the periodicity of the function φ_0 ,

$$\int_{-L}^L \partial_s \varphi_0(s, 0) ds = \varphi_0(L, 0) - \varphi_0(-L, 0) = 0.$$

Using (5.22),

$$\left| \int_{-L}^L \partial_s \varphi_0(s, 0) (|\tilde{u}(s, 0)|^2 - 1) ds \right| \leq \left(1 + \frac{1}{\sqrt{c_0}} \right) \|\partial_s \varphi_0\|_\infty \int_{-L}^L ||\tilde{u}(s, 0)| - 1| ds = o(1),$$

since $\tilde{u} \rightarrow 1$ in $L^2([-L, L])$. Thus,

$$\int_{-L}^L \partial_s \varphi_0(s, 0) |\tilde{u}(s, 0)|^2 ds = \int_{-L}^L \partial_s \varphi_0(s, 0) ds + \int_{-L}^L \partial_s \varphi_0(s, 0) (|\tilde{u}(s, 0)|^2 - 1) ds = o(1). \tag{5.26}$$

It remains to estimate the integral of $(i\tilde{u}, \partial_s \tilde{u})|_{t=0}$. In fact,

$$\int_{-L}^L (i\tilde{u}, \partial_s \tilde{u})|_{t=0} ds = \int_{-L}^L \int_0^\varepsilon \partial_t (\chi_\varepsilon (i\tilde{u}, \partial_s \tilde{u})) dt ds \tag{5.27}$$

where $\chi_\varepsilon(t)$ is a cut-off function in $C_c^\infty([0, +\infty))$ satisfying $\chi_\varepsilon = 1$ in $[0, \frac{\varepsilon}{2})$, $\text{supp} \chi_\varepsilon \subset [0, \varepsilon)$, $0 \leq \chi_\varepsilon \leq 1$ and $|\nabla \chi_\varepsilon| = \mathcal{O}(\varepsilon^{-1})$ in $[0, +\infty)$.

Note that

$$\partial_t \left(\chi_\varepsilon (i\tilde{u}, \partial_s \tilde{u}) \right) = (\partial_t \chi_\varepsilon) (i\tilde{u}, \partial_s \tilde{u}) + \chi_\varepsilon (i\partial_t \tilde{u}, \partial_s \tilde{u}) + (i\chi_\varepsilon \tilde{u}, \partial_t \partial_s \tilde{u}).$$

Using Lemma 5.3 and the Cauchy-Schwarz inequality, we get

$$\int_{-L}^L \int_0^\varepsilon |\partial_t \chi_\varepsilon| |(i\tilde{u}, \partial_s \tilde{u})| dt ds \leq \|\partial_t \chi_\varepsilon\|_{\infty, \Omega_\varepsilon} \|\tilde{u}\|_{2, \Omega_\varepsilon} \|\partial_s \tilde{u}\|_{2, \Omega_\varepsilon} = \mathcal{O}(1),$$

and

$$\int_{-L}^L \int_0^\varepsilon |\chi_\varepsilon (i\partial_t \tilde{u}, \partial_s \tilde{u})| dt ds \leq \|\chi_\varepsilon\|_{\infty, \Omega_\varepsilon} \|\partial_t \tilde{u}\|_{2, \Omega_\varepsilon} \|\partial_s \tilde{u}\|_{2, \Omega_\varepsilon} = \mathcal{O}(\varepsilon).$$

As for the term $(i\chi_\varepsilon \tilde{u}, \partial_t \partial_s \tilde{u})$, we do an integration by parts in the s -variable and use the periodicity with respect to s to get

$$\int_{-L}^L \int_0^\varepsilon (i\chi_\varepsilon \tilde{u}, \partial_t \partial_s \tilde{u}) dt ds = - \int_{-L}^L \int_0^\varepsilon (i\chi_\varepsilon \partial_s \tilde{u}, \partial_t \tilde{u}) dt ds.$$

Now, by Lemma 5.3 and the Cauchy-Schwarz inequality,

$$\left| \int_{-L}^L \int_0^\varepsilon (i\chi_\varepsilon \tilde{u}, \partial_t \partial_s \tilde{u}) dt ds \right| \leq \|\chi_\varepsilon\|_{\infty, \Omega_\varepsilon} \|\partial_t \tilde{u}\|_{2, \Omega_\varepsilon} \|\partial_s \tilde{u}\|_{2, \Omega_\varepsilon} = \mathcal{O}(\varepsilon).$$

Collecting the foregoing estimates, we infer from (5.27),

$$\int_{-L}^L (i\tilde{u}, \partial_s \tilde{u})|_{t=0} ds = \mathcal{O}(1). \quad (5.28)$$

Inserting (5.25), (5.26) and (5.28) into (5.24), we finish the proof of Lemma 5.2. \square

Proof of (5.20). By collecting the formulas in Lemmas 5.2 and 5.4, we obtain

$$\frac{1}{|\partial\Omega|} \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{j} ds = \Lambda_\kappa^\varepsilon \left(\frac{2\pi n_0}{|\partial\Omega|} - H\gamma_0 \right) + o(\varepsilon^{-1}).$$

This formula yields (5.20) since $n_0 = -\frac{\pi}{L} H\gamma_0 + \mathcal{O}(1)$, by (5.23). Having proved (5.20), we have finished the proof of Theorem 1.6. \square

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