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On the stability of laminar flows between plates

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Abstract

Consider a two-dimensional laminar flow between two plates, so that $(x_1, x_2) \in \mathbb{R} \times [-1, 1]$, given by $\mathbf{v}(x_1, x_2) = (U(x_2), 0)$, where $U \in C^4([-1, 1])$ satisfies $U' \neq 0$ in $[-1, 1]$. We prove that the flow is linearly stable in the large Reynolds number limit, in two different cases:

- $\sup_{x \in [-1, 1]} |U''(x)| + \sup_{x \in [-1, 1]} |U'''(x)| \ll \min_{x \in [-1, 1]} |U'(x)|$
(nearly Couette flows),
- $U'' \neq 0$ in $[-1, 1]$.

We assume either no-slip or fixed traction force conditions on the plates, and an arbitrary large (but much smaller than the Reynolds number) period in the x_1 direction.

1 Introduction

Consider the incompressible Navier-Stokes equations in the two-dimensional pipe $D = \mathbb{R} \times (-1, 1)$

$$\begin{cases} \partial_t \mathbf{v} - \epsilon \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p & \text{in } \mathbb{R}_+ \times D \\ \mathbf{v} = v_b \hat{i}_1 & \text{on } \mathbb{R}_+ \times \partial D, \end{cases} \quad (1.1)$$

where $\hat{i}_1 = (1, 0)$ and $\mathbf{v} = (v_1, v_2)$.

We require periodicity in the x_1 direction, i.e., for all $(t, x_1, x_2) \in \mathbb{R}_+ \times D$ we have

$$\mathbf{v}(t, x_1 + L, x_2) = \mathbf{v}(t, x_1, x_2), \quad (1.2)$$

for some $L > 0$, which may be arbitrary large, but must satisfy $L \ll \epsilon^{-1}$ as $\epsilon \rightarrow 0$. An initial condition on \mathbf{v} at $t = 0$ must be placed as well. The vector $\mathbf{v} = (v_1, v_2)$ denotes the fluid velocity field which belongs, for all $T > 0$, to

$$\mathcal{W}_{T,L} = \{ \mathbf{u} \in L^2(0, T; H_{loc}^2(\overline{D}, \mathbb{R}^2)) \mid \operatorname{div} \mathbf{u} = 0, \mathbf{u}(t, x_1 + L, x_2) = \mathbf{u}(t, x_1, x_2) \},$$

where the divergence is taken in the spatial coordinates. Here $u \in H_{loc}^k(\overline{D})$ for some $k \in \mathbb{N}$ means that for any $\phi \in C_0^\infty(\mathbb{R})$, it holds that $(x_1, x_2) \mapsto \phi(x_1)u(x_1, x_2) \in H^k(D)$. Similarly, $u \in H_{0,loc}^k(\overline{D})$ means that for any $\phi \in C_0^\infty(\mathbb{R})$, $(x_1, x_2) \mapsto \phi(x_1)u(x_1, x_2) \in H_0^k(D)$. Recall that

$$\mathbf{v} \cdot \nabla \mathbf{v} = v_1 \frac{\partial \mathbf{v}}{\partial x_1} + v_2 \frac{\partial \mathbf{v}}{\partial x_2}.$$

The pressure p belongs, for all $T > 0$, to

$$\mathcal{Q}_{T,L} = \{ p \in L^2(0, T; H_{loc}^1(\overline{D})) \mid \nabla p(t, x_1 + L, x_2) = \nabla p(t, x_1, x_2) \}.$$

The trace of \mathbf{v} on the boundary is constant on each connected component of $\mathbb{R}_+ \times \partial D$:

$$\{v_b(-1), v_b(1)\} \in \mathbb{R}^2.$$

The parameter $\epsilon > 0$ denotes the inverse of the flow's Reynolds number $\text{Re} > 0$. Beyond the above no-slip boundary condition we shall also consider a prescribed constant traction force on the boundary, i.e.,

$$\frac{\partial v_1}{\partial x_2} = s_b \quad , \quad v_2 = 0, \quad (1.3)$$

where $s_b(\pm 1) \in \mathbb{R}$ denote the prescribed traction force on $\partial D = \mathbb{R} \times \{\pm 1\}$.

We consider the stability of a stationary pair (\mathbf{v}, p) where the flow \mathbf{v} is of the form, for $(x_1, x_2) \in D$,

$$\mathbf{v}(x_1, x_2) = U(x_2) \hat{i}_1.$$

For such flows, (1.1) is satisfied for (\mathbf{v}, p) if and only if there exists some constant a and p_0 such that, for $(x_1, x_2) \in D$,

$$U''(x_1, x_2) = a \quad ; \quad p(x_1, x_2) = \epsilon a x_1 + p_0. \quad (1.4)$$

Depending on the boundary condition (no-slip or fixed traction force), U should satisfy an inhomogeneous Dirichlet condition $U(\pm 1) = v_b(\pm 1)$ or an inhomogeneous Neumann condition $U'(\pm 1) = s_b(\pm 1)$.

We shall not confine ourselves, however, in the sequel to such unperturbed velocity fields and discuss a more general class of motions (cf. also [12, p. 154]), which can be obtained if we add a non-uniform body force $\mathbf{b}(x_1, x_2) = b(x_2) \hat{i}_2$ to the right-hand-side of (1.1). Such a generalization can be useful if one attempts to examine the stability of a flow in an arbitrary 2D cross-section but uniform in the longitudinal direction. The linearized operator associated with (1.1), at the flow (\mathbf{v}, p) , assumes the form

$$(u, q) \mapsto \mathcal{T}_0(\mathbf{u}, q) := \mathfrak{I}\mathbf{u} - \nabla q \quad (1.5)$$

where

$$\mathfrak{I}\mathbf{u} = -\epsilon \Delta \mathbf{u} + U \frac{\partial \mathbf{u}}{\partial x_1} + u_2 U' \hat{i}_1, \quad \text{in } D, \quad (1.6)$$

where $q \in \mathcal{Q}_L$ given by

$$\mathcal{Q}_L = \{p \in H_{loc}^1(\overline{D}) \mid \nabla p(\cdot + L, \cdot) = \nabla p(\cdot, \cdot) \text{ and } \int_{(0,L) \times (-1,+1)} p(x_1, x_2) dx_1 dx_2 = 0\},$$

and $\mathbf{u} = (u_1, u_2)$ belongs to either

$$\mathcal{W}_{\mathfrak{D}} = \left\{ \mathbf{u} \in \mathcal{W} \mid u_1 \Big|_{\partial D} = 0, u_2 \Big|_{\partial D} = 0 \right\},$$

or to

$$\mathcal{W}_{\mathfrak{S}} = \left\{ \mathbf{u} \in \mathcal{W} \mid \frac{\partial u_1}{\partial x_2} \Big|_{\partial D} = 0, u_2 \Big|_{\partial D} = 0 \right\},$$

where

$$\mathcal{W} = \{ \mathbf{u} \in H_{loc}^2(\overline{D}, \mathbb{R}^2) \mid \text{div } \mathbf{u} = 0; \mathbf{u}(\cdot + L, \cdot) = \mathbf{u}(\cdot, \cdot) \}.$$

Remark 1.1.

- Note that when no-slip boundary conditions, introduced in (1.1), are applied, then $\mathbf{u} \in \mathcal{W}_{\mathfrak{D}}$. Otherwise, if we select (1.3) instead, then $\mathbf{u} \in \mathcal{W}_{\mathfrak{E}}$.
- Note that for any $p \in \mathcal{Q}_L$ there exists $A \in \mathbb{R}$ and \tilde{p} such that $p(x_1, x_2) = Ax_1 + \tilde{p}(x_1, x_2)$ with \tilde{p} satisfying the periodicity condition $\tilde{p}(\cdot + L, \cdot) = \tilde{p}(\cdot, \cdot)$.

We now attempt to define a spectral problem for \mathcal{T}_0 . We seek an estimate for the solution (\mathbf{u}, q) for some $\Lambda \in \mathbb{C}$ and F in a suitable space of the equation

$$\mathcal{T}_0(\mathbf{u}, q) - \Lambda \mathbf{u} = \mathbf{F}. \quad (1.7)$$

To this end we need to define the function space in which the solution should reside, and then to formulate an effective spectral problem involving only \mathbf{u} , so that q is recovered in a final step directly from \mathbf{u} .

The local stability of the flow (1.4) has been addressed mostly by physicists and engineers [12, 22, 36, 32]. Poiseuille flow ($U(x) = 1 - x^2$), which falls outside the scope of this work, and Couette flow ($U(x) = x$), have received some special attention. Thus, for Couette flows it has been established, using a mix of numerical and analytical techniques that Couette flow is always stable [35, 30], and established numerically [28] that Poiseuille flow loses its stability for $\epsilon^{-1} \approx 5772$. In [34] a survey of results published in Russian is presented where the locus of part of the spectrum (but not its left margin) for Couette and Poiseuille flows is approximated in the limit $\epsilon \rightarrow 0$.

It should be emphasized that experimental observations (cf. [11]) conclude that Couette and Poiseuille flows lose their stability for Reynolds numbers that are much lower than $\epsilon^{-1} = 5772$. It is commonly believed that these instabilities arise due to finite, though small, initial conditions. Thus, it has been established in [8] that unbounded Couette flows (in \mathbb{R} instead of $[-1, 1]$ as is presently considered), assuming a period $L = 1$, are finitely stable for $\mathcal{O}(\epsilon^{3/2})$ initial data. From a different perspective, in [14], Sobolev stability of a Prandtl boundary layer is studied, leading to a similar problem in $D = \mathbb{R} \times \mathbb{R}_+$, for sufficiently small period ($L \ll 1$).

To obtain non-linear stability one needs in [8] to use semigroup estimates (and not only the locus of the eigenvalues as in [34]), associated with time dependent equation

$$u_t - \mathcal{T}_0(\mathbf{u}, q) = 0$$

in $\mathcal{W}_{\mathfrak{D}}$ or $\mathcal{W}_{\mathfrak{E}}$.

Unlike the unbounded Couette flow in [8], the semigroup associated with more general laminar flows in $[-1, 1]$ is not explicitly known. In the present contribution we thus consider, in the limit $\epsilon \rightarrow 0$, velocity fields $U \in C^4([-1, 1])$ satisfying $U' \neq 0$ in $[-1, 1]$, of two different types: nearly Couette flows, so that

$$\sup_{x \in [-1, 1]} |U''(x)| + \sup_{x \in [-1, 1]} |U'''(x)| \ll \min_{x \in [-1, 1]} |U'(x)|,$$

and velocity fields for which $U'' \neq 0$ in $[-1, 1]$. We prove that these laminar flows are stable and provide $C(\mathbb{R}_+; \mathcal{L}(L^2))$ estimates for their associated semigroup norm. We

believe that these linear estimates would be useful when considering the nonlinear stability of these flows in a bounded interval.

The rest of this contribution is arranged as follows.

In the next section we formulate the spectral problem by using Hodge decomposition. Since the standard Hodge decomposition [15] is not a direct sum in this periodic setup, we define a space of zero-flux perturbations, and then formulate our main results in this space.

In Section 3 we present the problem in terms of the stream function and its Fourier coefficients and arrive at the Orr-Sommerfeld operator.

In the next section we consider the inviscid problem (where $\epsilon = 0$) in Fourier space. Section 5 includes some resolvent estimates obtained for one-dimensional Schrödinger operators on the entire real line and for their Dirichlet realization in $(-1, 1)$.

In Section 6 we consider the same Schrödinger operator on $(-1, 1)$ and \mathbb{R}_+ but this time in a Sobolev space of functions satisfying certain orthogonality conditions (cf. [34]).

Section 7 provides inverse estimates for the Orr-Sommerfeld operator for the fixed-traction problem, whereas Section 8 provides the same estimates for the no-slip realization.

In Section 9 we prove the main results.

Finally in the appendix we bring some auxiliary estimates obtained for Airy functions and generalized Airy functions.

2 Spectral problem formulation and main results

We now amend the spectral question presented in (1.7) to a more standard spectral problem. To this end we use a variant of Hodge decomposition adapted to our periodic setting (see [15, Theorem 3.4] for the standard case), which allows us to eliminate q .

2.1 Hodge theory

Let

$$\mathcal{H} = \{\mathbf{u} \in L^2_{loc}(\overline{D}, \mathbb{R}^2) \mid \mathbf{u}(\cdot + L, \cdot) = \mathbf{u}(\cdot, \cdot)\}, \quad (2.1)$$

where the scalar product for the Hilbert space \mathcal{H} is given by

$$\mathcal{H} \times \mathcal{H} \ni (\mathbf{u}, \mathbf{v}) \mapsto \int_{(0,L) \times (-1,1)} \bar{\mathbf{u}} \cdot \mathbf{v} \, dx_1 dx_2,$$

and the two closed subspaces of \mathcal{H}

$$\mathcal{H}_{\text{curl}} = \{\mathbf{u} \in L^2_{loc}(\overline{D}, \mathbb{R}^2) \mid \text{curl } \mathbf{u} = 0; \mathbf{u}(\cdot + L, \cdot) = \mathbf{u}(\cdot, \cdot)\},$$

and

$$\mathcal{H}_{\text{div}} = \{\mathbf{u} \in L^2_{loc}(\overline{D}, \mathbb{R}^2) \mid \text{div } \mathbf{u} = 0; \mathbf{u} \cdot \mathbf{n}|_{\partial D} = 0; \mathbf{u}(\cdot + L, \cdot) = \mathbf{u}(\cdot, \cdot)\}.$$

We have

Lemma 2.1.

$$\mathcal{H}_{\text{div}} \cap \mathcal{H}_{\text{curl}} = \text{span}(1, 0).$$

Proof. Let $u \in \mathcal{H}_{\text{div}} \cap \mathcal{H}_{\text{curl}}$, and $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$ denote its partial Fourier transform with respect to x_1 , i.e.,

$$\hat{\mathbf{u}}(n, x_2) = \frac{1}{L} \int_0^L \mathbf{u}(x_1, x_2) e^{-i2\pi n x_1/L} dx_1. \quad (2.2)$$

It can be easily verified that, for any $n \in \mathbb{Z}$,

$$\begin{aligned} i\frac{2\pi}{L}n\hat{u}_2(n, x_2) - \frac{d}{dx_2}\hat{u}_1(n, x_2) &= 0 \\ i\frac{2\pi}{L}n\hat{u}_1(n, x_2) + \frac{d}{dx_2}\hat{u}_2(n, x_2) &= 0 \\ \hat{u}_2(n, -1) = \hat{u}_2(n, +1) &= 0. \end{aligned}$$

From the above we can conclude that $\hat{\mathbf{u}}(n, x_2) = 0$ for $n \neq 0$, $\hat{u}_2(0, x_2) = 0$ and $\hat{u}_1(0, x_2) = \text{Const.}$ ■

With the above in mind, we introduce

$$\mathcal{H}_{\text{div}}^0 := \{\mathbf{u} \in \mathcal{H}_{\text{div}}, \langle \mathbf{u}, (1, 0) \rangle = 0\}. \quad (2.3)$$

The orthogonality condition reads

$$\int_{(0,L) \times (-1,+1)} u_1(x_1, x_2) dx_1 dx_2 = 0. \quad (2.4)$$

We can now prove the following Hodge decomposition:

Lemma 2.2. $\mathcal{H}_{\text{div}}^0$ and $\mathcal{H}_{\text{curl}}$ are orthogonal subspaces of \mathcal{H} and

$$\mathcal{H} = \mathcal{H}_{\text{curl}} \oplus \mathcal{H}_{\text{div}}^0.$$

Proof. Let $\mathbf{u} \in \mathcal{H}$. Let further $\phi_c \in H_{loc}^1(\bar{D})$ and $H_{loc}^1(\bar{D})$ denote the weak solutions of

$$\begin{cases} -\Delta \phi_c = \text{div } \mathbf{u} & \text{in } D \\ \frac{\partial \phi_c}{\partial n} = 0 & \text{on } \partial D \\ \phi_c(\cdot, \cdot) = \phi_c(\cdot + L, \cdot) + AL & \text{in } D, \end{cases} \quad (2.5)$$

and

$$\begin{cases} -\Delta \phi_d = \text{curl } \mathbf{u} & \text{in } D \\ \phi_d = 0 & \text{on } \partial D \\ \phi_d(\cdot, \cdot) = \phi_d(\cdot + L, \cdot) & \text{in } D. \end{cases} \quad (2.6)$$

In the above

$$A = \frac{1}{L} \langle \mathbf{u}, (1, 0) \rangle. \quad (2.7)$$

It can be easily verified, by using the Lax-Milgram Lemma, that there exists a unique solution for (2.6). Similarly, by using (see Remark 1.1) the ansatz

$$\phi_c = Ax_1 + \tilde{\phi}_c, \quad (2.8)$$

there where $\tilde{\phi}_c(\cdot, \cdot) = \tilde{\phi}_c(\cdot + L, \cdot)$, it follows that there exists a unique, solution of (2.5). (Note that the Neumann condition in the second line of (2.5) is satisfied in $H_{loc}^{-1/2}(\partial D)$ sense.) Equivalently, we can say that $\tilde{\phi}_c$ is the unique periodic solution, orthogonal to the constant function, of

$$\int_{(-1,+1) \times (0,L)} (\nabla \tilde{\phi}_c - \mathbf{u}) \cdot \nabla v \, dx_1 dx_2 = 0,$$

for every L -periodic $v \in H^1((-1, +1) \times (0, L))$.

Clearly, $\nabla \phi_c \in \mathcal{H}_{\text{curl}}$, $\nabla_{\perp} \phi_d \in \mathcal{H}_{\text{div}}^0$, $\langle \mathbf{u} - \nabla \phi_c - \nabla_{\perp} \phi_d, (1, 0) \rangle = 0$, and, by the periodicity of ϕ_d and $\nabla \phi_c$, it holds that

$$\langle \nabla \phi_c, \nabla_{\perp} \phi_d \rangle = 0.$$

Finally we set $u = v + \nabla \phi_c + \nabla_{\perp} \phi_d$ to obtain that $v \in \mathcal{H}_{\text{curl}} \cap \mathcal{H}_{\text{div}}^0$ and hence, by Lemma 2.1, $v \equiv 0$. ■

2.2 Zero flux solution

Since $\mathcal{W}_{\#} \not\subset \mathcal{H}_{\text{div}}^0$ for $\# \in \{\mathfrak{D}, \mathfrak{S}\}$, we need to introduce the following spaces, as the domain of the operator in the spectral formulation

$$\mathcal{W}_{\mathfrak{D}}^0 = \left\{ \mathbf{u} \in \mathcal{W}^0 \mid u_1 \Big|_{\partial D} = 0, u_2 \Big|_{\partial D} = 0 \right\}, \quad (2.9a)$$

and

$$\mathcal{W}_{\mathfrak{S}}^0 = \left\{ \mathbf{u} \in \mathcal{W}^0 \mid \frac{\partial u_1}{\partial x_2} \Big|_{\partial D} = 0, u_2 \Big|_{\partial D} = 0 \right\}, \quad (2.9b)$$

where

$$\mathcal{W}^0 = \{ \mathbf{u} \in H_{loc}^2(\overline{D}, \mathbb{R}^2) \mid \text{div } \mathbf{u} = 0; \mathbf{u}(\cdot + L, \cdot) = \mathbf{u}(\cdot, \cdot), \langle \mathbf{u}, (1, 0) \rangle_{\mathcal{H}} = 0 \}.$$

While it seems that the orthogonality requirement (2.4) is an unjustified limitation of the perturbation space, we show in the sequel how one can solve the stability problem in $\mathcal{W}_{\#}$ from its solution in $\mathcal{W}_{\#}^0$. Hence, (2.4) does not result in any loss of generality.

Let then $P : \mathcal{H} \rightarrow \mathcal{H}_{\text{div}}^0$ denote the orthogonal projection on $\mathcal{H}_{\text{div}}^0$, given explicitly for some $u \in \mathcal{H}$ by $\nabla_{\perp} \phi_d$, where ϕ_d is the solution of (2.6). Rewriting (1.7) in the form

$$\mathfrak{T}\mathbf{u} - \Lambda\mathbf{u} - \mathbf{F} = \nabla q,$$

we observe that $\nabla q \in \mathcal{H}_{\text{curl}}$. Then, projecting on $\mathcal{H}_{\text{div}}^0$, we may write, for $\mathbf{u} \in \mathcal{W}_{\#}^0$

$$P((\mathfrak{T} - \Lambda)\mathbf{u} - \mathbf{F}) = 0.$$

With the above in mind, we now define as an unbounded operator on $\mathcal{H}_{\text{div}}^0$ whose domain is $\mathcal{W}_{\#}^0$ (which is clearly dense in $\mathcal{H}_{\text{div}}^0$)

$$\mathfrak{T}_P^{\#} := P\mathfrak{T}. \quad (2.10)$$

By this definition we have

$$(\mathfrak{T}_P^{\#} - \Lambda)\mathbf{u} = P\mathbf{F}, \quad (2.11)$$

which appears to be a proper formulation of the resolvent equation.

Proposition 2.3. \mathfrak{T}_P^\sharp is semi-bounded on $\mathcal{H}_{\text{div}}^0$ and has compact resolvent. Furthermore,

$$\|e^{-t\mathfrak{T}_P^\sharp}\| \leq e^{\frac{1}{2}\|U'\|_\infty t}. \quad (2.12)$$

Proof. Let $u \in \mathcal{W}_\#^0$. As $u \perp \mathcal{H}_{\text{curl}}$ we have

$$\Re\langle u, \mathfrak{T}_P^\sharp u \rangle = \Re\langle u, \mathfrak{T}u \rangle = \epsilon\|\nabla \mathbf{u}\|_2^2 + \Re\langle u_2, U'u_1 \rangle \geq \epsilon\|\nabla \mathbf{u}\|_2^2 - \frac{1}{2}\|U'\|_\infty\|\mathbf{u}\|_2^2, \quad (2.13)$$

verifying, thereby, semi-boundedness. More precisely, the resolvent set of \mathfrak{T}_P^\sharp contains $\{\lambda \in \mathbb{C} \mid \Re\lambda < -\frac{1}{2}\|U'\|_\infty\}$. The semigroup estimate (2.12) is then a consequence of the Hille-Yosida theorem. The compactness of the resolvent is proved by observing that $\mathcal{W}_\#^0$ is compactly embedded in $\mathcal{H}_{\text{div}}^0$. ■

Remark 2.4.

- Suppose that for some $\Lambda_0 \in \mathbb{R}$ and $C > 0$, we have

$$\sup_{\substack{\Lambda \leq \Lambda_0 \\ \Lambda \in \rho(\mathfrak{T}_P^\sharp)}} \|(\mathfrak{T}_P^\sharp - \Lambda)^{-1}\| \leq C.$$

Then, by the compactness of the resolvent the spectrum is discrete, and hence it holds that

$$\sup_{\Re\Lambda \leq \Lambda_0} \|(\mathfrak{T}_P^\sharp - \Lambda)^{-1}\| \leq C.$$

- It results from (2.13) that, for any $\mathbf{F} \in \mathcal{H}$ and $\# \in \{\mathfrak{G}, \mathfrak{D}\}$, it holds that

$$\sup_{\Re\Lambda \leq -\frac{1}{2}\|U'\|_\infty - \epsilon} \|(\mathfrak{T}_P^\sharp - \Lambda)^{-1}\mathbf{F}\|_{1,2} \leq \frac{1}{\epsilon}\|\mathbf{F}\|_2. \quad (2.14)$$

If Λ is in the resolvent set of \mathfrak{T}_P^\sharp , we recover \mathbf{u} by

$$\mathbf{u} = (\mathfrak{T}_P^\sharp - \Lambda)^{-1}P\mathbf{F}. \quad (2.15)$$

Once we have derived \mathbf{u} we can obtain q in the following manner

Proposition 2.5. Let $(\mathbf{u}, \Lambda, \mathbf{F}) \in \mathcal{W}_\#^0 \times \mathbb{C} \times \mathcal{H}$ satisfy (2.15). Then, there exists a unique $q \in \mathcal{Q}_L$ such that (1.7) holds for $(\mathbf{u}, q, \Lambda, \mathbf{F})$.

Proof. From (2.11) it follows that

$$(\mathfrak{T} - \Lambda)\mathbf{u} - \mathbf{F} = \mathbf{G} \in \mathcal{H}_{\text{curl}}.$$

It remains to prove the existence of a unique $q_{\mathbf{G}} \in \mathcal{Q}_L$ satisfying

$$\nabla q_{\mathbf{G}} = \mathbf{G}. \quad (2.16)$$

This, however, easily follows from the proof of Lemma 2.2, i.e., one obtain $q_{\mathbf{G}}$ as the unique solution of (2.5) with \mathbf{G} in the place of \mathbf{u} . ■

Corollary 2.6. *If Λ is in the resolvent set of \mathfrak{T}_P^\sharp , then for any $F \in \mathcal{H}$ there exists a unique pair $(\mathbf{u}_0, q_0) \in \mathcal{W}_\#^0 \times \mathcal{Q}_L$ such that (1.7) holds.*

We use the term “the zero flux solution of (1.7)” for this solution.

Remark 2.7.

- *From the proof of Proposition 2.5 we learn, in addition, that if $\Lambda \in \sigma(\mathfrak{T}_P^\sharp)$ and \mathbf{u}_Λ is a corresponding eigenfunction in $\mathcal{W}_\#^0$ then there exists q_Λ such that $(\mathbf{u}_\Lambda, q_\Lambda)$ satisfies (1.7) with $F = 0$. We cannot exclude, at the moment, the possibility that Λ is not a simple eigenvalue.*
- *The proof shows also that, for any $\Lambda \in \rho(\mathfrak{T}_P^\sharp)$, the map $F \mapsto (\mathbf{u}_0, q_0)$ (as defined in the corollary) is continuous from \mathcal{H} onto $\mathcal{W}_\#^0 \times \mathcal{Q}_L$.*

2.3 General solution

Once the zero flux solution of (1.7) has been found in $\mathcal{W}_\#^0 \times \mathcal{Q}_L$, we attempt to solve the problem more generally in $\mathcal{W}_\# \times \mathcal{Q}_L$.

Proposition 2.8. *Let $\Lambda \in \rho(\mathfrak{T}_P^\sharp)$, then for any F and any $\gamma \in \mathbb{R}$, there exists a unique pair $(\mathbf{u}_\gamma, q_\gamma) \in \mathcal{W}_\# \times \mathcal{Q}_L$ satisfying (1.7) and*

$$\langle \mathbf{u}_\gamma, (1, 0) \rangle = \gamma. \quad (2.17)$$

Moreover, $(\mathbf{u}_\gamma, \nabla q_\gamma)$ continuously depends on γ in

$$H^2((0, L) \times (-1, 1]) \times L^2((0, L) \times (-1, 1)).$$

Proof.

Fixed traction

In this case, we observe that $\gamma \hat{i}_1 \in \mathcal{W}_\mathfrak{E}$. Let $\mathbf{F} \in \mathcal{H}$, and $(\mathbf{u}_0, q_0) \in \mathcal{W}_\mathfrak{E}^0 \times \mathcal{Q}_L$ satisfy (1.7) for some $\Lambda \in \rho(\mathfrak{T}_\mathfrak{E})$. Set then

$$\mathbf{u}_\gamma = \mathbf{u}_0 + \frac{\gamma}{L} \hat{i}_1, \quad q_\gamma = q_0 + \Lambda \frac{\gamma}{L} x_1.$$

It can be easily verified that $(\mathbf{u}_\gamma, q_\gamma, \Lambda, F)$ satisfy both (1.7) and (2.17).

No-slip

Let again $\mathbf{F} \in \mathcal{H}$, and $(\mathbf{u}_0, q_0) \in \mathcal{W}_\mathfrak{D}^0 \times \mathcal{Q}_L$ satisfy (1.7) for some $\Lambda \in \rho(\mathfrak{T}_\mathfrak{D})$. Then, set

$$\tilde{q} = q_0 + \epsilon \gamma x_1 \quad ; \quad \tilde{\mathbf{u}} = \tilde{\mathbf{u}}_0 + \gamma V(x_2) \hat{i}_1,$$

where $V \in \mathcal{W}_\mathfrak{D}$ satisfies

$$V''(x_2) = -\frac{3}{2L}.$$

As $V(x_2) = \frac{3}{4L}(1 - x_2^2)$, we have

$$\int_{-1}^1 V(x_2) dx_2 = \frac{1}{L}.$$

and hence $\langle \tilde{\mathbf{u}}, (1, 0) \rangle = \gamma$.

Clearly,

$$\mathfrak{T}(\tilde{\mathbf{u}}) - \Lambda \tilde{\mathbf{u}} - \nabla \tilde{q} = \mathbf{F} - \Lambda \gamma V \hat{i}_1.$$

To construct $(\mathbf{u}_\gamma, q_\gamma)$ we introduce $(\hat{\mathbf{u}}, \hat{q}) \in \mathcal{W}_\mathfrak{D}^0 \times \mathcal{Q}_L$ satisfying

$$\mathfrak{T}(\hat{\mathbf{u}}) - \Lambda \hat{\mathbf{u}} - \nabla \hat{q} = V \hat{i}_1.$$

The existence of $(\hat{\mathbf{u}}, \hat{q})$ follows from Proposition 2.5. We then set

$$(\mathbf{u}_\gamma, q_\gamma) = (\mathbf{u}_0, q_0) + \Lambda \gamma (\hat{\mathbf{u}}, \hat{q}) + \gamma (\epsilon x_1, V \hat{i}_1). \quad (2.18)$$

Clearly, $(\mathbf{u}_\gamma, q_\gamma)$ is a solution of (1.7) satisfying (2.17). It must also be unique by Proposition 2.5. ■

Note that the above non-uniqueness results from the periodic setup we have chosen. If instead we had selected to work in $L^2(D)$ (an infinite pipe) the solution of (1.7) would have been unique, since the perturbation flux must vanish at infinity, and hence must be zero on each cross section. Nevertheless, as will become evident in the sequel the periodic setting offers significant simplification over the infinite pipe and is not necessarily less effective as an approximation of a more realistic setup.

2.4 Averaging with respect to x_1

We begin by defining the projection $\mathfrak{p} : L^2((0, L) \times (-1, 1)) \rightarrow L^2((0, L) \times (-1, 1))$ by

$$\mathfrak{p}u(x_1, x_2) = \frac{1}{L} \int_0^L u(s, x_2) ds, \quad (2.19)$$

and then extend it to $\Pi : L^2((0, L) \times (-1, 1); \mathbb{R}^2) \rightarrow L^2((0, L) \times (-1, 1); \mathbb{R}^2)$ by writing

$$\Pi \mathbf{u} = (\mathfrak{p}u_1, \mathfrak{p}u_2) \text{ for } \mathbf{u} = (u_1, u_2). \quad (2.20)$$

We first show

Lemma 2.9. Π is a projection on $\mathcal{H}_{\text{div}}^0$.

Moreover for any $\# \in \{\mathfrak{D}, \mathfrak{S}\}$, we have $\Pi \mathcal{W}_\#^0 \subset \mathcal{W}_\#^0$.

Proof. Let $\mathbf{u} \in \mathcal{H}_{\text{div}}^0$. Then $\mathbf{u} = \nabla_\perp \phi_d$ where ϕ_d is a solution of (2.6). We may then write, using the periodicity of ϕ_d ,

$$\Pi \mathbf{u} = \partial_{x_2} (\mathfrak{p} \phi_d) \hat{i}_1. \quad (2.21)$$

Obviously, $\text{div} \Pi \mathbf{u} = 0$, and the orthogonality of $\Pi \mathbf{u}$ to $(1, 0)$ in \mathcal{H} follows from

$$\int_{-1}^1 \partial_{x_2} (\mathfrak{p} \phi_d) dx_2 = 0.$$

Hence $\Pi \mathbf{u} \in \mathcal{H}_{\text{div}}^0$. It can now be easily verified that $\Pi \mathcal{W}_\#^0 \subset \mathcal{W}_\#^0$. ■

Lemma 2.10. P commutes with Π .

Proof. Let $\mathbf{F} \in \mathcal{H}$. Then, by the proof of Lemma 2.2

$$\mathbf{F} = \nabla_{\perp} \phi_d + \nabla \phi_c,$$

where ϕ_d is a solution of (2.6) and ϕ_c is a solution of (2.5). Clearly,

$$\Pi \mathbf{F} = (\partial_{x_2}(\mathbf{p}\phi_d) + A)\hat{i}_1 + \partial_{x_2}(\mathbf{p}\tilde{\phi}_c)\hat{i}_2,$$

where A is given by (2.7) (with \mathbf{F} instead of \mathbf{u}).

Hence

$$\Pi \mathbf{F} = \nabla_{\perp}(\mathbf{p}\phi_d) + A\hat{i}_1 + \nabla(\mathbf{p}\tilde{\phi}_c),$$

and by uniqueness of the Hodge decomposition we obtain

$$P\Pi \mathbf{F} = (\partial_{x_2}\mathbf{p}\phi_d)\hat{i}_1.$$

Next, we compute $\Pi P \mathbf{F}$. Observing that

$$P \mathbf{F} = \nabla_{\perp} \phi_d,$$

we get

$$\Pi P \mathbf{F} = (\partial_{x_2}\mathbf{p}\phi_d)\hat{i}_1 = P\Pi \mathbf{F}. \quad (2.22)$$

■

We can now prove the following commutation result

Lemma 2.11. For any $\# \in \{\mathfrak{S}, \mathfrak{D}\}$, $\mathfrak{T}_P^{\#}$ commutes with Π .

Proof.

We simply observe that for all $\mathbf{u} \in \mathcal{W}_{\#}^0$

$$\mathfrak{T} \Pi \mathbf{u} = \Pi \mathfrak{T} \mathbf{u},$$

and use the commutation of P and Π . ■

An immediate consequence follows

Proposition 2.12. For any $\# \in \{\mathfrak{D}, \mathfrak{S}\}$ it holds that

$$\Pi e^{-t\mathfrak{T}_P^{\#}} = e^{-t\mathfrak{T}_P^{\#}} \Pi. \quad (2.23)$$

2.5 Main results

Throughout this work we assume that:

Assumption 2.13. U' does not vanish in $[-1, +1]$, or

$$\mathbf{m} := \inf_{x \in [-1, 1]} |U'(x)| > 0. \quad (2.24)$$

The statement of the main results below involves the spectral properties of the complex Airy operator on \mathbb{R}_+

$$\mathcal{L}_+ = -\frac{d^2}{dx^2} + ix, \quad (2.25a)$$

defined on

$$D(\mathcal{L}_+) = \{u \in H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+) \mid xu \in L^2(\mathbb{R}_+)\}. \quad (2.25b)$$

We denote its leftmost eigenvalue [3] by ν_1 . We further set

$$J_m(U) = \min(|U'(-1)|, |U'(1)|) \quad (2.25c)$$

We also need below

$$\delta_2(U) := \left\| U'' \right\|_{1, \infty}, \quad (2.26)$$

where $\|u\|_{1, \infty} = \|u\|_\infty + \|u'\|_\infty$.

Finally we define, for any $r > 1$ and $k \geq 2$,

$$\mathcal{S}_r^k = \{v \in C^k([-1, +1]), \inf_{x \in [-1, +1]} |v'(x)| \geq r^{-1} \text{ and } \|v\|_{k, \infty} \leq r\}, \quad (2.27)$$

and then set for convenience of notation

$$\mathcal{S}_r = \mathcal{S}_r^4. \quad (2.28)$$

For $U \in \mathcal{S}_r$, $\epsilon > 0$ and $L > 0$, we recall that

$$\mathfrak{I}_P^\mathfrak{S} := \mathfrak{I}_P^\mathfrak{S}(U, \epsilon, L)$$

is defined in (2.9b) and (2.10) (where ϵ appears in the definition of \mathfrak{I} and L is the x_1 periodicity).

For $\beta > 0$, we introduce

$$\Omega(\beta) := \{(\epsilon, L) \in (0, 1] \times \mathbb{R}_+, (L\epsilon)^{-1} \geq \beta/(2\pi)\}.$$

Theorem 2.14. *The following statements hold for any $r > 1$.*

1. *For all $\check{\delta} > 0$, there exist $\beta_0 > 0$ and $C > 0$ such that, for any $U \in \mathcal{S}_r$ satisfying*

$$\inf_{x \in [-1, 1]} |U''(x)| \geq 1/r, \quad (2.29)$$

any $(\epsilon, L) \in \Omega(\beta_0)$, and any $t > 0$, it holds that

$$\|e^{-t\mathfrak{I}_P^\mathfrak{S}(U, \epsilon, L)}\| \leq C \epsilon^{-2} e^{-ct(\pi^2/4 - \check{\delta})}. \quad (2.30a)$$

Furthermore, for all $\check{\delta} > 0$, there exist $\Upsilon > 0$, $\beta_0 > 0$ and $C > 0$ such that for all $(\epsilon, L) \in \Omega(\beta_0)$, $t > 0$, and any $U \in \mathcal{S}_r$ satisfying (2.29) we have

$$\|e^{-t\mathfrak{I}_P^\mathfrak{S}(U, \epsilon, L)}(I - \Pi)\| \leq C L^{\frac{1}{3} - \check{\delta}} \epsilon^{-\frac{5}{3} - \check{\delta}} e^{-\epsilon\Upsilon[L\epsilon]^{-2/3}t}, \quad (2.30b)$$

where Π is given by (2.19).

2. For all $\hat{\delta} > 0$, there exist $\delta > 0$, $\beta_0 > 0$ and $C > 0$ such that, for any $U \in \mathcal{S}_r$ satisfying $\delta_2(U) < \delta$, (2.30a) holds true for all $(\epsilon, L) \in \Omega(\beta_0)$ and $t > 0$.
 Moreover, for all $\Upsilon < J_m^{2/3} \mathfrak{R}\nu_1$, there exist $\beta_0 > 0$, $\delta > 0$ and $C > 0$ such that, for any $(\epsilon, L) \in \Omega(\beta_0)$, $U \in \mathcal{S}_r$ satisfying $\delta_2(U) < \delta$, and $t > 0$, we have

$$\|e^{-t\mathfrak{T}_P^{\mathfrak{S}}(U, \epsilon, L)}(I - \Pi)\| \leq CL^{2/3}\epsilon^{-\frac{4}{3}}e^{-\epsilon\Upsilon[L\epsilon]^{-2/3}t}. \quad (2.31)$$

For the case $\# = \mathfrak{D}$, we first define, for some $\theta > 0$, the operator \mathcal{L}^θ whose differential operator is given by (2.25a) and its domain by

$$D(\mathcal{L}^\theta) = \{u \in H^2(\mathbb{R}_+) \mid \langle e^{-\theta \cdot}, u \rangle = 0, xu \in L^2(\mathbb{R}_+)\}.$$

We show later (see Proposition 6.4 and Corollary 6.7) that \mathcal{L}^θ is a closed operator and that

$$\hat{\mu}_m := \inf_{\theta \geq 0} (\inf \Re \sigma(\mathcal{L}^\theta) + \frac{\theta^2}{2}). \quad (2.32)$$

is finite and positive. For $U \in \mathcal{S}_r$, $\epsilon > 0$ and $L > 0$, we recall that

$$\mathfrak{T}_P^{\mathfrak{D}} := \mathfrak{T}_P^{\mathfrak{D}}(U, \epsilon, L)$$

is defined in (2.9a) and (2.10).

Theorem 2.15. *For $r > 1$, the following properties hold.*

1. For all $\check{\delta} > 0$, there exist $\beta_0 > 0$ and $C > 0$ such that for any $U \in \mathcal{S}_r$ satisfying (2.29), any $(\epsilon, L) \in \Omega(\beta_0)$ and any $t > 0$ it holds that

$$\|e^{-t\mathfrak{T}_P^{\mathfrak{D}}(U, \epsilon, L)}\| \leq C\epsilon^{-2}e^{-\epsilon t(\pi^2/4 - \check{\delta})}. \quad (2.33a)$$

Furthermore, for all $\check{\delta} > 0$, there exist $\Upsilon > 0$, $\beta_0 > 0$ and $\hat{C} > 0$ such that, for any $U \in \mathcal{S}_r$ satisfying (2.29), $(\epsilon, L) \in \Omega(\beta_0)$ and $t > 0$, we have

$$\|e^{-t\mathfrak{T}_P^{\mathfrak{D}}(U, \epsilon, L)}(I - \Pi)\| \leq \hat{C}L^{\frac{1}{3} - \check{\delta}}\epsilon^{-\frac{5}{3} - \check{\delta}}e^{-\epsilon\Upsilon[L\epsilon]^{-2/3}t}, \quad (2.33b)$$

where Π is given by (2.19).

2. For any $\check{\delta} > 0$, there exist $\delta > 0$, $C > 0$, and $\beta_0 > 0$, such that for any $U \in \mathcal{S}_r$ satisfying $\delta_2(U) \leq \delta$, $(\epsilon, L) \in \Omega(\beta_0)$, and $t > 0$, (2.33a) holds true.
 Moreover, for all $\Upsilon < J_m^{2/3}\hat{\mu}_m$, there exist $\delta_0 > 0$, $\beta_0 > 0$ and $C > 0$, such that for all $U \in \mathcal{S}_r$ satisfying $\delta_2(U) \leq \delta_0$, $(\epsilon, L) \in \Omega(\beta_0)$, and $t > 0$ it holds that

$$\|e^{-t\mathfrak{T}_P^{\mathfrak{D}}(U, \epsilon, L)}(I - \Pi)\| \leq CL^{2/3}\epsilon^{-\frac{4}{3}}e^{-\epsilon\Upsilon[L\epsilon]^{-2/3}t}. \quad (2.34)$$

3 The Orr-Sommerfeld operator

We focus attention in the sequel on $\mathfrak{T}_P^\#$ and its resolvent.

3.1 Stream Function

When considering a two-dimensional incompressible fluid flow, it is customary to introduce a stream function, i.e., to let $\mathbf{u} = \nabla_{\perp}\psi$. Its introduction is again related to Hodge decomposition theory.

Lemma 3.1. *Let $\mathbf{u} \in \mathcal{H}_{\text{div}}^0$. Then, there exists a unique $\psi \in H_{0,\text{loc}}^1(\overline{D}, \mathbb{R}^2)$ such that $\psi(x_1 + L, x_2) = \psi(x_1, x_2)$ and $\mathbf{u} = \nabla_{\perp}\psi$. If in addition, $\mathbf{u} \in \mathcal{W}_{\#}^0$, then $\psi \in H_{\text{loc}}^3(\overline{D}, \mathbb{R}^2)$ and ψ satisfies $\partial_{x_2}\psi = 0$ on ∂D if $\# = \mathfrak{D}$ and $\partial_{x_2}^2\psi = 0$ on ∂D if $\# = \mathfrak{S}$.*

Proof. Existence and uniqueness of ψ follow from the proof of Lemma 2.2. In particular, for any $\mathbf{u} \in \mathcal{H}_{\text{div}}^0$ we have $\psi = \phi_d$ where ϕ_d is a solution of (2.6). The second part of the lemma is immediate. ■

We next substitute $\mathbf{u} = \nabla_{\perp}\psi$ into (1.6) and take the curl of the ensuing equation, which leads to the following equation, in the distributional sense,

$$\mathcal{P}_{\Lambda,\epsilon}^{\#}\psi = \text{curl } \mathbf{F} \quad \text{in } D, \quad (3.1)$$

with $\# \in \{\mathfrak{D}, \mathfrak{S}\}$,

$$\mathcal{P}_{\Lambda,\epsilon}^{\#} := -\epsilon\Delta^2 + U\frac{\partial}{\partial x_1}\Delta - U''\frac{\partial}{\partial x_1} - \Lambda\Delta. \quad (3.2)$$

We treat $\mathcal{P}_{\Lambda,\epsilon}^{\#}$ as an unbounded operator on $L_{\text{per}}^2(D)$, where

$$L_{\text{per}}^2(D) := \{u \in L_{\text{loc}}^2(\overline{D}), u(x_1 + L, x_2) = u(x_1, x_2)\},$$

is equipped with the $L^2([-1, 1] \times [0, L])$ norm. Note that additional regularity is needed while attempting to use results obtained on $L_{\text{per}}^2(D)$ for spectral problem for $\mathfrak{T}_P^{\#}$. We shall obtain the necessary regularity at a later stage. Similarly, we introduce for $k = 1, 2$ and $s \geq 0$

$$H_{\text{per}}^s(D, \mathbb{C}^k) := \{u \in H_{\text{loc}}^s(\overline{D}, \mathbb{C}^k), u(x_1 + L, x_2) = u(x_1, x_2)\},$$

and assign to it the $H^1([-1, 1] \times [0, L])$ norm. In the interest of brevity we write in the sequel $H_{\text{per}}^s(D, \mathbb{C}) = H_{\text{per}}^s(D)$. For no-slip boundary conditions we take

$$D(\mathcal{P}_{\Lambda,\epsilon}^{\mathfrak{D}}) = \{\psi \in H_{\text{per}}^4(D), \psi = 0 \text{ and } \partial_{x_2}\psi = 0 \text{ on } \partial D\}.$$

For fixed traction, the domain of $\mathcal{P}_{\Lambda,\epsilon}^{\mathfrak{S}}$ is given by

$$D(\mathcal{P}_{\Lambda,\epsilon}^{\mathfrak{S}}) = \{\psi \in H_{\text{per}}^4(D), \psi = 0 \text{ and } \partial_{x_2}^2\psi = 0 \text{ on } \partial D\}.$$

We can now make the following statement

Lemma 3.2. *The operator $\mathcal{P}_{\Lambda,\epsilon}^{\#}$ is invertible for each $\# \in \{\mathfrak{S}, \mathfrak{D}\}$ and $\Lambda \in \rho(\mathfrak{T}_P^{\#})$.*

Proof. Let $\psi \in D(P_{\Lambda, \epsilon}^{\#})$ and $f \in L_{per}^2(D)$ satisfy $\mathcal{P}_{\Lambda, \epsilon}^{\#} \psi = f$ for some $\# \in \{\mathfrak{S}, \mathfrak{D}\}$ and $\Lambda \in \rho(\mathfrak{T}_P^{\#})$. Let \mathbf{F} denote the unique vector field in $\mathcal{H}_{\text{div}}^0$ satisfying $\text{curl } \mathbf{F} = f$. As

$$(\mathfrak{T} - \Lambda) \nabla_{\perp} \psi - \mathbf{F} \in \mathcal{H}_{\text{curl}},$$

it follows that

$$\|\nabla \psi\|_2 \leq \|(\mathfrak{T}_P^{\#} - \Lambda)^{-1}\| \|\mathbf{F}\|_2 \leq \|(\mathfrak{T}_P^{\#} - \Lambda)^{-1}\| \|f\|_2.$$

■

Due to the periodicity of our function spaces and the fact that the coefficients of the differential operator $\mathcal{P}_{\Lambda, \epsilon}^{\#}$ and the associated boundary conditions do not depend on x_1 , it is natural to consider the operator in a Fourier space. Hence, we introduce $L_{per}^2(D) \ni \psi \mapsto (\mathcal{F}\psi) \in \ell^2(\frac{2\pi}{L}\mathbb{Z}) \otimes L^2(-1, +1)$ satisfying

$$(\mathcal{F}\psi)(\alpha_n, x_2) = \frac{1}{L} \int_0^L e^{-i\alpha_n x_1} \psi(x_1, x_2) dx_1, \quad (3.3)$$

for $\alpha_n = 2\pi n/L$ ($n \in \mathbb{Z}$).

We then obtain the Hilbertian sum

$$\mathcal{F} \mathcal{P}_{\Lambda, \epsilon}^{\#} \mathcal{F}^{-1} = \epsilon \left(\bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \mathcal{B}_{\lambda, \alpha_n, \beta_n}^{\#} \oplus \mathcal{B}_{\hat{\Lambda}}^{\#} \right),$$

where

$$\mathcal{B}_{\hat{\Lambda}}^{\#} = -\frac{d^4}{dx^4} - \hat{\Lambda} \frac{d^2}{dx^2},$$

in which

$$\hat{\Lambda} = \frac{\Lambda}{\epsilon}. \quad (3.4)$$

The unbounded operator $\mathcal{B}_{\lambda, \alpha, \beta}^{\#}$ on $L^2(-1, +1)$, which is commonly referred to as the Orr-Sommerfeld operator is given by

$$\phi \mapsto \mathcal{B}_{\lambda, \alpha, \beta}^{\#} \phi = (\mathcal{L}_{\beta} - \beta \lambda) \left(\frac{d^2}{dx^2} - \alpha^2 \right) \phi - i\beta U'' \phi, \quad (3.5)$$

in which

$$\beta = \alpha \epsilon^{-1}, \quad (3.6)$$

$$\mathcal{L}_{\beta} = -\frac{d^2}{dx^2} + i\beta U, \quad (3.7)$$

and, for $\beta \neq 0$,

$$\lambda = \beta^{-1} (\hat{\Lambda} - \alpha^2). \quad (3.8)$$

In the sequel, unless stated otherwise, we consider β and α as independent parameters.

We now define two different realizations associated with the differential operator appearing in (3.5). The domain of $\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{S}}$, corresponding to the prescribed traction force boundary condition, is given by

$$D(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{S}}) = \{u \in H^4(-1, 1), u(\pm 1) = 0 \text{ and } u''(\pm 1) = 0\}, \quad (3.9)$$

whereas the operator $\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D}}$, corresponding to the no-slip condition, is defined on

$$D(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D}}) = \{u \in H^4(-1, 1), u(\pm 1) = 0 \text{ and } u'(\pm 1) = 0\}. \quad (3.10)$$

The domain of $\mathcal{B}_{\hat{\Lambda}}^{\#}$ is similarly defined by (3.9) for $\# = \mathfrak{S}$ and (3.10) for $\# = \mathfrak{D}$.

3.2 Inverse estimates

The Orr-Sommerfeld operator given by (3.5) has extensively been studied in the Physics literature [12, 36, 32]. Very few rigorous studies, however, address its spectrum (cf. [30] in the Couette case $U'' = 0$) and none, to the best of our knowledge provide estimates for its inverse norm $\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\#})^{-1}\|$ in $\mathcal{L}(L^2(-1, +1))$. Assuming that $\Lambda \in \rho(\mathcal{T}_P^{\#})$, the inverse of $\mathcal{P}_{\Lambda,\epsilon}$ is bounded. To estimate its norm, one needs a proper uniform bound of $\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\#})^{-1}\|$ for all $\alpha > 0$ and $(\mathcal{B}_{\hat{\Lambda}}^{\#})^{-1}$. Hence we write,

$$\|(\mathcal{P}_{\Lambda,\epsilon}^{\#})^{-1}\| \leq \epsilon^{-1} \max \left(\sup_{n \in \mathbb{Z} \setminus \{0\}} \|(\mathcal{B}_{\lambda_n, \alpha_n, \beta_n}^{\#})^{-1}\|, \|(\mathcal{B}_{\hat{\Lambda}}^{\#})^{-1}\| \right),$$

where, for $n \in \mathbb{Z} \setminus \{0\}$, $\Lambda = \epsilon \hat{\Lambda}$,

$$\lambda_n = \beta_n^{-1}(\hat{\Lambda} - \alpha_n^2) = \epsilon \alpha_n^{-1}(\hat{\Lambda} - \alpha_n^2) = \alpha_n^{-1} \Lambda - \epsilon \alpha_n.$$

Clearly, for any $\epsilon > 0$ and $\hat{\Lambda} \in \mathbb{R}$,

$$\sup_{n \in \mathbb{Z} \setminus \{0\}} \|(\mathcal{B}_{\lambda_n, \alpha_n, \beta_n}^{\#})^{-1}\| \leq \sup_{\substack{\alpha \geq \alpha_1 \\ \lambda = \alpha^{-1} \epsilon (\hat{\Lambda} - \alpha^2)}} \|(\mathcal{B}_{\lambda, \alpha, \alpha \epsilon^{-1}}^{\#})^{-1}\| \leq \sup_{\beta \geq \beta_1} \sup_{\substack{\alpha \geq \alpha_1 \\ \lambda = \beta^{-1}(\hat{\Lambda} - \alpha^2)}} \|(\mathcal{B}_{\lambda, \alpha, \beta}^{\#})^{-1}\|,$$

with

$$\beta_1(\epsilon, L) = \epsilon^{-1} \alpha_1 = 2\pi / (L\epsilon).$$

Consequently, for any $\hat{\Lambda}_0 \in \mathbb{R}$ and $\epsilon > 0$, we have, the following inequality

$$\sup_{\substack{\Re \Lambda \leq \epsilon \hat{\Lambda}_0 \\ \Lambda \in \rho(\mathcal{T}_P^{\#})}} \|(\mathcal{P}_{\Lambda,\epsilon}^{\#})^{-1}\| \leq \epsilon^{-1} \max \left(\sup_{\beta \geq \beta_1} \sup_{\substack{\alpha \geq 0 \\ \Re \lambda \leq \beta^{-1}(\hat{\Lambda}_0 - \alpha^2)}} \|(\mathcal{B}_{\lambda, \alpha, \beta}^{\#})^{-1}\|, \sup_{\Re \hat{\Lambda} \leq \hat{\Lambda}_0} \|(\mathcal{B}_{\hat{\Lambda}}^{\#})^{-1}\| \right). \quad (3.11)$$

Note that $\mathcal{B}_{\lambda,\alpha,\beta}^{\#} = \overline{\mathcal{B}}_{\lambda,-\alpha,\beta}^{\#}$ and hence, it is sufficient to consider $\alpha \geq 0$ in the above. We emphasize that the supremum with respect to λ , β and α of $\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\#})^{-1}\|$ is obtained while ignoring the dependence of β on α . Note also that $\beta_1 = \beta_1(\epsilon)$ tends to $+\infty$ as $L\epsilon \rightarrow 0$. Hence we shall attempt to obtain a bound on $\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\#})^{-1}\|$ for large values of β , since we are interested in the small ϵ limit.

Throughout this work we recall that we always assume Assumption 2.13. Without any loss of generality we can assume that

$$U' > 0 \text{ on } [-1, +1]. \quad (3.12)$$

Indeed, the case $U' < 0$ can similarly be treated after applying the transformation $x \rightarrow -x$.

In view of (3.11), we attempt to obtain, in the large β limit, a bound on $\|\mathcal{B}_{\lambda,\alpha,\beta}^{-1}\|$. To this end, we introduce

$$\mathcal{A}_{\lambda,\alpha} \stackrel{def}{=} (U + i\lambda) \left(-\frac{d^2}{dx^2} + \alpha^2 \right) + U''. \quad (3.13)$$

We define $\mathcal{A}_{\lambda,\alpha}$, for $\Re\lambda \neq 0$ or when $\Re\lambda = 0$ and $\Im\lambda \notin [U(-1), U(1)]$, on

$$D(\mathcal{A}_{\lambda,\alpha}) = H^2(-1, 1) \cap H_0^1(-1, 1). \quad (3.14)$$

In the case $\Re\lambda = 0$ and $\Im\lambda \in [U(-1), U(1)]$ we set

$$D(\mathcal{A}_{i\nu,\alpha}) = H^2((-1, 1); |U - \nu|^2 dx) \cap H_0^1(-1, 1), \quad (3.15)$$

where $\nu \in [U(-1), U(1)]$.

Note that $\mathcal{A}_{\lambda,\alpha} = \mathcal{A}_{\lambda,-\alpha}$ and hence we consider it only in the case $\alpha \geq 0$.

One can formally obtain $\mathcal{A}_{\lambda,\alpha}$ from $\mathcal{B}_{\lambda,\alpha,\beta}$ by dividing it by β and taking the limit $\beta \rightarrow \infty$ which corresponds to the limit $\alpha^{-1}\epsilon \rightarrow 0$. This is why it has been commonly referred to as the ‘‘inviscid operator’’ in [12].

We note that the formal limit of the Orr-Sommerfeld operator as $\beta \rightarrow \infty$ is very different from that of the Schrödinger operator $-\frac{d^2}{dx^2} + i\beta(U - \nu)$ ($\nu = \Im\lambda$). In the latter case, we expect the resolvent to be small away from the set where $U = \nu$. This fact was used in [3, 4], for instance, to obtain resolvent estimates via localization techniques. For $\mathcal{B}_{\lambda,\alpha,\beta}$, the best one can expect is that $v = \mathcal{A}_{\lambda,\alpha}\phi$ would be small outside a close neighborhood of the set where $U = \Im\lambda$. We note that $\mathcal{A}_{\lambda,\alpha}$ raises considerable interest independently of the viscous operator $\mathcal{B}_\lambda^\#$ (cf. [12, Sections 21-24] and [25]).

4 The inviscid operator

We consider here the inviscid operator $\mathcal{A}_{\lambda,\alpha}$ associated with the differential operator (3.13), whose domain of definition is given either by (3.14) where $\Re\lambda \neq 0$ or $\Im\lambda \notin [U(-1), U(1)]$, or by (3.15) in the case $\lambda = i\nu$ for $\nu \in [U(-1), U(1)]$. While the spectrum of $\mathcal{A}_{\lambda,\alpha}$ has been studied in the context of inviscid (Euler) flows and their stability [12, 25], it appears that its inverse norm has not been estimated in the literature. The purpose of this section is therefore to offer a systematic study of $\|\mathcal{A}_{\lambda,\alpha}^{-1}\|$, with emphasis on the limit $\Re\lambda \rightarrow 0$ for $\nu \in [U(-1), U(1)]$.

4.1 The case $\Re\lambda = 0$: preliminaries

We begin by showing that $\mathcal{A}_{\lambda,\alpha}$ is a closed operator on $L^2(-1, 1)$. In the cases where $D(\mathcal{A}_{\lambda,\alpha})$ is given by (3.14) the proof is standard and will therefore be omitted.

Proposition 4.1. *Let U satisfy Assumption 2.13 and $\nu \in [U(-1), U(1)]$. Then $\mathcal{A}_{i\nu, \alpha}$, whose domain is given by (3.15), is closed as an unbounded operator on $L^2(-1, +1)$ and the space*

$$\mathcal{V} := \{u \in C^\infty([-1, +1]), \text{ s.t. } u(-1) = u(1) = 0\}$$

is dense in $D(\mathcal{A}_{i\nu, \alpha})$ under the graph norm.

Proof. Let $\{u_n\}_{n=1}^\infty$ denote a sequence in \mathcal{V} such that

$$u_n \rightarrow \hat{u} \text{ in } L^2(-1, +1) \text{ and } \mathcal{A}u_n \rightarrow \hat{v} \text{ in } L^2(-1, +1) \quad (4.1)$$

To prove the proposition we need to show that $\hat{u} \in D(\mathcal{A}_{i\nu, \alpha})$ and that $\mathcal{A}_{i\nu, \alpha}\hat{u} = \hat{v}$.

Let $U(x_\nu) = \nu$. We use Hardy's inequality for weighted Sobolev spaces associated with the intervals $(-1, x_\nu)$ and $(x_\nu, 1)$ separately.

Set then, for $k \in \{1, 2\}$

$$W_{1, \nu}^k(x_\nu, 1) := \{u \in H^1(x_\nu, 1), (U - \nu)u^{(k)} \in L^2(x_\nu, 1), u(1) = 0\}.$$

Recall the one-dimensional Hardy inequality (see for example [23, Eq. 0.6] or [9, Lemma 2.1]) which holds for any $v \in W_{1, \nu}^2(x_\nu, 1)$,

$$\|v\|_{L^2(x_\nu, 1)} \leq 2 \|(x - x_\nu)v'\|_{L^2(x_\nu, 1)}. \quad (4.2)$$

Note that (4.2) follows by extension from Hardy's inequality in $(0, +\infty)$, given for any $v \in W_{1, 0}^2(0, +\infty)$ by

$$\|v\|_{L^2(0, +\infty)} \leq 2 \|xv'\|_{L^2(0, +\infty)}. \quad (4.3)$$

Hence, for any $u \in W_{1, \nu}^2(x_\nu, 1)$ (and hence for the restriction to $[x_\nu, 1]$ of any $u \in \mathcal{V}$)

$$\|u'\|_{L^2(x_\nu, 1)} \leq 2 \|(x - x_\nu)u''\|_{L^2(x_\nu, 1)} \leq \frac{C}{\mathbf{m}} \|(U - \nu)u''\|_{L^2(x_\nu, 1)}. \quad (4.4)$$

From (4.1) and (4.4) we deduce that u_n is a Cauchy sequence in $H^1(x_\nu, 1)$ and in $H^1(-1, x_\nu)$. Hence there are two corresponding limits $u_+ \in H^1(x_\nu, 1)$ and $u_- \in H^1(-1, x_\nu)$, with $\hat{u}_{/(-1, x_\nu)} = u_-$ and $\hat{u}_{/(x_\nu, 1)} = u_+$. By continuity we have $u_-(-1) = 0$, $u_+(1) = 0$ and $u_-(x_\nu) = u_+(x_\nu)$. This shows that $\hat{u} \in H_0^1(-1, +1)$. Finally it is clear that $\hat{v} = \mathcal{A}_{i\nu, \alpha}\hat{u}$ in $\mathcal{D}'(-1, +1)$ and hence $\hat{u} \in D(\mathcal{A}_{i\nu, \alpha})$.

The density argument is a consequence, after localization, of Proposition 2.1 in [10]. ■

The cases $x_\nu = \pm 1$ are easier, since one can apply Hardy's inequality only once, in $(-1, 1)$.

Consider the case of Couette flow $U = x$ in $(-1, 1)$, which is one of the most popular examples of uniaxial flows ([35, 30, 7]). In this case

$$\mathcal{A}_{i\nu, \alpha}^c \stackrel{\text{def}}{=} (x - \nu) \left(-\frac{d^2}{dx^2} + \alpha^2 \right).$$

We can construct in some cases the explicit solution of the inhomogeneous problem

$$\mathcal{A}_{i\nu,\alpha}^c u = f. \quad (4.5)$$

For example, when $\alpha = 0$, $\nu \in (-1, +1)$ and $f \equiv 1$. It can be easily verified that

$$u_\nu(x) = A_1(x - \nu) + A_2 + (x - \nu) \log |x - \nu|, \quad (4.6a)$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}[(1 - \nu) \log |1 - \nu| + (1 + \nu) \log |1 + \nu|], \\ A_2 &= \frac{1}{2}[(1 - \nu) \log |1 - \nu| - (1 + \nu) \log |1 + \nu|]. \end{aligned} \quad (4.6b)$$

It can be easily verified that u_ν satisfies (4.5).

While it seems at a first glance that $\mathcal{A}_{i\nu,0}^c : D(\mathcal{A}_{i\nu,0}^c) \rightarrow L^2(-1, 1)$ is injective for all $\nu \in \mathbb{R}$, it is wrong for $\nu \in (-1, +1)$. It has indeed a non-trivial solution of the form

$$\varphi_\nu(x) = \begin{cases} \frac{x-\nu}{1-\nu} - 1 & \text{if } \nu \leq x, \\ -\frac{x-\nu}{1+\nu} - 1 & \text{if } x < \nu. \end{cases}$$

While $\varphi_\nu \notin H^2(-1, 1)$ for all $\nu \in (-1, 1)$, it does satisfy (4.8) in the sense of distributions.

We now look at the injectivity of $\mathcal{A}_{i\nu,\alpha}^c$ when $\nu \in (-1, +1)$. We have already explicitly obtained a non trivial element in $\text{Ker}\mathcal{A}_{i\nu,0}^c$. More generally, we prove

Lemma 4.2. *For all $\nu \in (-1, 1)$, $\alpha \geq 0$, it holds that*

$$\text{Dim Ker}\mathcal{A}_{i\nu,\alpha}^c = 1. \quad (4.7)$$

Proof. We observe that if u is in the kernel, we have $u \in H_0^1(-1, +1)$ and

$$\left(-\frac{d^2}{dx^2} + \alpha^2\right)u = c\delta(x - \nu),$$

where c is the jump u' undergoes through $x = \nu$. Since $\delta(x - \nu)$ belongs to $H^{-1}(-1, +1)$, we may use the Lax-Milgram Lemma for the Dirichlet problem to obtain a unique solution $u_1 \in H_0^1(-1, +1)$ for

$$\left(-\frac{d^2}{dx^2} + \alpha^2\right)u_1 = \delta(x - \nu).$$

It follows that $u = cu_1$. ■

Using [10, Proposition 3.1] and the same argument as in the proof of [10, Theorem 3.1] one can show that $\mathcal{A}_{i\nu,\alpha}^c$, which is a bounded operator from the Hilbert space $D(\mathcal{A}_{i\nu,\alpha}^c)$ into $L^2(-1, +1)$, is a Fredholm operator of index 1 for $\nu \in (-1, +1)$. With the aid of Lemma 4.2, we can then conclude the surjectivity of $\mathcal{A}_{i\nu,\alpha}^c$ for any $\nu \in (-1, +1)$ or more precisely the existence of a right inverse. We note that surjectivity of $\mathcal{A}_{i\nu,\alpha}^c$ follows from the surjectivity $\mathcal{A}_{i\nu,\alpha}$ we prove in the sequel.

Remark 4.3. When $f \in C^1([-1, 1])$, we can prove (4.7) in the following alternative manner. Recall the solution of $A_{i\nu, 0}^c u_\nu = 1$ we have obtained in (4.6). If $f(\nu) = 0$, we can solve (4.5) by dividing it by $(x - \nu)$. When $f(\nu) \neq 0$ we write $u = f(\nu)u_\nu + \tilde{u}$, to obtain

$$\mathcal{A}_{i\nu, \alpha}^c \tilde{u} = \tilde{f}, \quad \text{with } \tilde{f}(x) := f(x) - f(\nu) - \alpha^2(x - \nu)u_\nu,$$

which can be easily solved as $\tilde{f}(\nu) = 0$. It is not clear, however, how to extend by density the above solution to any $f \in L^2(-1, +1)$.

In the next subsection, we obtain the surjectivity of $A_{i\nu, \alpha}$ (for any U satisfying (3.12)) via a non-explicit compactness argument.

4.2 Construction of a right inverse of $\mathcal{A}_{i\nu, \alpha}$

We begin by establishing the surjectivity of $\mathcal{A}_{i\nu, \alpha}$

Lemma 4.4. Suppose that (3.12) holds, and that $\lambda = i\nu$ for some $\nu \in [U(-1), U(1)]$. Then, for any $v \in L^2(-1, 1)$ and $\alpha \geq 0$ there exists some $\phi \in D(A_{\lambda, \alpha})$ satisfying $A_{\lambda, \alpha}\phi = v$. Furthermore, there exists $C > 0$, such that for all ν, α and $v \in L^2$, ϕ satisfies

$$\|\phi\|_{1,2} + [1 + \alpha^2]^{-1} \|(U - \nu)\phi''\|_2 \leq C \|v\|_2, \quad (4.8)$$

where $\|\phi\|_{1,2}$ denotes the norm of ϕ in the Sobolev space $H^1(-1, 1)$.

Proof. Let $\kappa > 0$. With (3.13) in mind, we can use the following alternative form of $A_{\lambda, \alpha}\phi = v$, which is valid wherever $U \neq \nu$,

$$-\left((U - \nu)^2 \left(\frac{\phi}{U - \nu}\right)'\right)' + \alpha^2(U - \nu)\phi = v. \quad (4.9)$$

We look first for some $\phi_\kappa \in H^2(-1, +1) \cap H_0^1(-1, +1) \subset D(\mathcal{A}_{i\nu, \alpha})$ satisfying the regularized equation

$$-\left([(U - \nu)^2 + \kappa^2] \left(\frac{\phi_\kappa}{U - \nu + i\kappa}\right)'\right)' + \alpha^2(U - \nu - i\kappa)\phi_\kappa = v. \quad (4.10)$$

We may set $w_\kappa = (U - \nu + i\kappa)^{-1}\phi_\kappa$ to obtain

$$-([(U - \nu)^2 + \kappa^2]w_\kappa')' + \alpha^2[(U - \nu)^2 + \kappa^2]w_\kappa = v$$

Taking the inner product with w_κ then yields

$$\|[(U - \nu)^2 + \kappa^2]^{1/2}w_\kappa'\|_2^2 + \alpha^2\|[(U - \nu)^2 + \kappa^2]^{1/2}w_\kappa\|_2^2 = \langle w_\kappa, v \rangle. \quad (4.11)$$

This immediately implies, by the Lax-Milgram Lemma, that w_κ (and hence also ϕ_κ) uniquely exists in $H^2(-1, 1) \cap H_0^1(-1, 1)$.

Since $\nu \in [U(-1), U(1)]$, there exists $x_\nu \in [-1, 1]$ such that $U(x_\nu) = \nu$. Let $\mathbf{m} > 0$ be defined by (2.24). Clearly, we have

$$(U - \nu)^2 + \kappa^2 \geq \mathbf{m}^2(x - x_\nu)^2.$$

Hence

$$\begin{aligned} \|[(U - \nu)^2 + \kappa^2]^{1/2} w'_\kappa\|_2^2 &\geq \mathbf{m}^2 \|(x - x_\nu) w'_\kappa\|_2^2 \\ &= \mathbf{m}^2 \left[\|(x - x_\nu) w'_\kappa\|_{L^2(-1, x_\nu)}^2 + \|(x - x_\nu) w'_\kappa\|_{L^2(x_\nu, 1)}^2 \right]. \end{aligned}$$

Upon translation we apply Hardy's inequality (4.3), and the fact that $w_\kappa(1) = 0$ (permitting the extension $w_\kappa = 0$ in $(1, +\infty)$ so that $w_\kappa \in H^1(x_\nu, \infty)$), it holds that

$$\|(x - x_\nu) w'_\kappa\|_{L^2(x_\nu, 1)}^2 \geq \frac{1}{4} \|w_\kappa\|_{L^2(x_\nu, 1)}^2.$$

A similar bound can be established on $(-1, x_\nu)$. Hence, by (4.11),

$$\|w_\kappa\|_2 \leq \frac{4}{\mathbf{m}^2} \|v\|_2. \quad (4.12)$$

Substituting once again into (4.11) yields, in addition,

$$\|[(U - \nu)^2 + \kappa^2]^{1/2} w'_\kappa\|_2 \leq \frac{2}{\mathbf{m}} \|v\|_2. \quad (4.13)$$

We note that (4.12) and (4.13) do not imply convergence of w_κ in $L^2(-1, 1)$ or $(U - \nu)^{-1} \phi$ where ϕ is a solution of (3.13) (with $\lambda = i\nu$). As a matter of fact it is expected that, in most cases, $(U - \nu)^{-1} \phi$ will be unbounded in L^2 . We thus use (4.13) and (4.12) to obtain that

$$\|\phi'_\kappa\|_2 = \|([U - \nu + i\kappa] w_\kappa)'\|_2 \leq \|[(U - \nu)^2 + \kappa^2]^{1/2} w'_\kappa\|_2 + \|U' w_\kappa\|_2 \leq C \|v\|_2. \quad (4.14)$$

It follows, either by Poincaré's inequality or by (4.12), that $\{\phi_\kappa\}_{\kappa=1}^\infty$ is bounded in a ball of size $\hat{C} \|v\|_2$ in $H_0^1(-1, 1)$. By weak compactness, there exists a sequence $\kappa_n > 0$ tending to 0 such that ϕ_{κ_n} converges weakly in $H_0^1(-1, 1)$ to a limit $\phi_0 \in H_0^1(-1, 1)$ as $n \rightarrow +\infty$ in the same ball, and hence

$$\|\phi_0\|_{H_0^1(-1, 1)} \leq \hat{C} \|v\|. \quad (4.15)$$

It remains to establish that ϕ_0 is a weak solution of (3.13) for $\lambda = i\nu$. We thus write (4.10) in its weak form for some $\psi \in C_0^\infty(-1, 1)$

$$\langle \psi', (U - \nu - i\kappa) \phi'_\kappa \rangle - \left\langle \psi', \frac{(U - \nu - i\kappa)}{(U - \nu + i\kappa)} U' \phi_\kappa \right\rangle + \alpha^2 \langle \psi, (U - \nu - i\kappa) \phi_\kappa \rangle = \langle \psi, v \rangle. \quad (4.16)$$

Letting $\kappa = \kappa_n$ and then taking the limit $n \rightarrow +\infty$ yields by the above established weak convergence in H^1 that

$$\langle \psi', (U - \nu) \phi'_0 \rangle - \langle \psi', U' \phi_0 \rangle + \alpha^2 \langle \psi, (U - \nu) \phi_0 \rangle = \langle \psi, v \rangle.$$

Consequently, ϕ_0 is a weak solution of (3.13) satisfying (3.12). Finally, since for all $x \neq x_\nu$ we have

$$(U - \nu)[\phi_0'' - \alpha^2 \phi_0] = U'' \phi_0,$$

we easily obtain from (4.15) that

$$\|(U - \nu) \phi_0''\|_2 \leq C(1 + \alpha^2) \|v\|_2.$$

The lemma is proved. ■

Remark 4.5. *As in the case of a Couette flow, one can show that the index of $\mathcal{A}_{i\nu,\alpha}$ as a Fredholm operator from the Hilbert space $D(\mathcal{A}_{i\nu,\alpha})$ (equipped with the graph norm) into $L^2(-1, 1)$ is one for $\nu \in [U(-1), U(1)]$. Since the multiplication by U'' is a compact operator from $D(\mathcal{A}_{i\nu,\alpha})$ into $L^2(-1, 1)$ we may conclude from Fredholm theory that*

$$\tilde{A}_{\nu,\alpha} = (U - \nu) \left(-\frac{d^2}{dx^2} + \alpha^2 \right)$$

and $\mathcal{A}_{i\nu,\alpha}$ have the same index. Then, we observe that the multiplication operator by $(U(x) - \nu)/(x - x_\nu)$ has index 0, and hence $\tilde{A}_{\nu,\alpha}$ and $\mathcal{A}_{i\nu,\alpha}^c$ again have the same index. Consequently, the index of $\mathcal{A}_{i\nu,\alpha}$ equals to one, and by Lemma 4.4 it holds that $\dim \ker \mathcal{A}_{i\nu,\alpha} = 1$. We note that one can construct a direct proof of injectivity with much greater difficulty.

We have proved above, the surjectivity of $\mathcal{A} = \mathcal{A}_{i\nu,\alpha}$. By Fredholm theory, there is a natural right inverse $\mathcal{E}_{i\nu,\alpha}^{\text{right}}$ which associates with v the solution ϕ of $\mathcal{A}\phi = v$ which is orthogonal to $\text{Ker} \mathcal{A}_{i\nu,\alpha}$ in $D(\mathcal{A}_{i\nu,\alpha})$ for the scalar product

$$(\phi, \psi) \mapsto \langle \phi, \psi \rangle_{\nu,\alpha} := \langle \mathcal{A}\phi, \mathcal{A}\psi \rangle_{L^2} + \langle \phi, \psi \rangle_{L^2}.$$

Note that $\langle \phi, \psi \rangle_{\nu,\alpha}$ coincides with the ordinary L^2 product whenever $\psi \in \text{Ker} \mathcal{A}_{i\nu,\alpha}$.

Employing the estimates of Lemma 4.4 we now prove:

Proposition 4.6. *Suppose that (3.12) holds, and that $\lambda = i\nu$ for some $\nu \in [U(-1), U(1)]$. Then, there exists $C > 0$, such that for all ν, α*

$$\|\mathcal{E}_{i\nu,\alpha}^{\text{right}}\|_{\mathcal{L}(L^2, H_0^1)} + \|\mathcal{E}_{i\nu,\alpha}^{\text{right}}\|_{\mathcal{L}(L^2, D(\mathcal{A}_{i\nu,\alpha}))} \leq C. \quad (4.17)$$

Proof. We first observe that the solution constructed in Lemma 4.4 satisfies

$$\|\phi\|_{\alpha,\nu} \leq C \|v\|_2$$

To obtain $\mathcal{E}_{i\nu,\alpha}^{\text{right}} v$ we now need to subtract $\pi_{\alpha,\nu} \phi$, where $\pi_{\alpha,\nu}$ is the orthogonal projector from $D(\mathcal{A}_{i\nu,\alpha})$ onto $\text{Ker} \mathcal{A}_{i\nu,\alpha}$. Obviously,

$$\begin{aligned} \|\mathcal{E}_{i\nu,\alpha}^{\text{right}} v\|_{\alpha,\nu}^2 &= \|\phi - \pi_{\alpha,\nu} \phi\|_{\alpha,\nu}^2 \\ &\leq \|\phi\|_{\alpha,\nu}^2 \\ &\leq C \|v\|_2^2, \end{aligned}$$

establishing, thereby, a bound on the right inverse in $\mathcal{L}(L^2, D(\mathcal{A}_{i\nu,\alpha}))$. To estimate $\mathcal{E}_{i\nu,\alpha}^{\text{right}}$ in $\mathcal{L}(L^2, H_0^1)$ we observe that by (4.4) there exists $C_0 > 0$ such that for any $\alpha \in \mathbb{R}$, $\nu \in [U(-1), U(1)]$, and $\phi \in D(\mathcal{A}_{i\nu,\alpha})$

$$\|\phi\|_{1,2} \leq C_0 \|\phi\|_{\alpha,\nu}.$$

■

4.3 Nearly Couette velocity fields

We now attempt to estimate $\|\mathcal{A}_{\lambda,\alpha}^{-1}\|$ in the case where $\Re\lambda \neq 0$. We shall begin with the case where $\delta_2(U)$, defined in (2.26), is small. To this end we introduce

$$I(\phi, \lambda) = \frac{1}{2}\|\phi'\|_2^2 + \left\langle \frac{U''(U-\nu)\phi}{(U-\nu)^2 + \mu^2}, \phi \right\rangle, \quad (4.18a)$$

and

$$\gamma_m(\lambda, U) = \inf_{\phi \in H_0^1(-1,1) \setminus \{0\}} \frac{I(\phi, \lambda)}{\|\phi\|_2^2}. \quad (4.18b)$$

The following result proves that under suitable assumptions on U , the infimum of $\gamma_m(\lambda)$ over $\mathbb{C} \setminus \mathfrak{J}$, where $\mathfrak{J} = \{\lambda \in \mathbb{C} \mid \Re\lambda = 0, \nu \in [U(-1), U(1)]\}$ is positive.

Lemma 4.7. *For any $r > 1$, there exists $\delta_0 > 0$ and $\gamma_0 > 0$ such that for any $U \in \mathcal{S}_r^3$ satisfying $\delta_2(U) \leq \delta_0$, we have*

$$\inf_{\lambda \in \mathbb{C} \setminus \mathfrak{J}} \gamma_m(\lambda, U) \geq \gamma_0 > 0. \quad (4.19)$$

Proof. Writing $I(\phi, \lambda)$ in the form

$$I(\phi, \lambda) = \frac{1}{2}\|\phi'\|_2^2 + \Re \left\langle \phi, \frac{U''\phi}{U+i\lambda} \right\rangle.$$

we attempt to estimate the second term on the right hand side. For some $\nu_0 > 0$ to be chosen later, we consider two different cases depending on the size of $|\lambda|$.

Case I: $\sqrt{|\mu|^2 + |\nu|^2} < \nu_0$.

Integration by parts yields, accounting for the Dirichlet condition ϕ satisfies at $x = \pm 1$,

$$\left\langle \phi, \frac{U''\phi}{U+i\lambda} \right\rangle = \left\langle \frac{U''}{U'} |\phi|^2, \frac{U'}{U+i\lambda} \right\rangle = - \left\langle \left(\frac{U''|\phi|^2}{U'} \right)', \log(U+i\lambda) \right\rangle.$$

We estimate the right-hand-side as follows

$$\left| \left\langle \left(\frac{U''|\phi|^2}{U'} \right)', \log(U+i\lambda) \right\rangle \right| \leq C(\nu_0) \delta_2(U) [\|\phi'\|_2 \|\phi\|_\infty + \|\phi\|_\infty^2]. \quad (4.20)$$

Sobolev's embedding and Poincaré's inequality then yield that for some $\hat{C}(\nu_0, r) > 0$,

$$\left| \Re \left\langle \phi, \frac{U''\phi}{U+i\lambda} \right\rangle \right| \leq \hat{C}(\nu_0, r) \delta_2(U) \|\phi'\|_2^2. \quad (4.21)$$

Case II: $\sqrt{|\mu|^2 + |\nu|^2} \geq \nu_0$.

As $U \in \mathcal{S}_r$ we conclude that there exists $C(r) > 0$ such that

$$\left| \Re \left\langle \phi, \frac{U''\phi}{U+i\lambda} \right\rangle \right| \leq \left| \left\langle \phi, \frac{U''\phi}{U+i\lambda} \right\rangle \right| \leq \frac{C}{|\nu_0|} \|\phi\|_2^2 \leq \frac{\hat{C}(r)}{\nu_0} \|\phi'\|_2^2. \quad (4.22)$$

Poincaré's inequality was applied to obtain the last estimate. We can now set $\nu_0 = 8\hat{C}$ in (4.22) and $\delta_2(U) \leq \delta_0 = \frac{1}{8\hat{C}(\nu_0, r)}$ in (4.22), to obtain

$$I(\phi, \lambda) \geq \frac{1}{4}\|\phi'\|_2^2.$$

The lemma is proved by using Poincaré's inequality once again. \blacksquare

If $\delta_2(U)$ is not small we may still show

Lemma 4.8. *For any $r > 1$, there exists $\gamma_0 \in \mathbb{R}$ such that for any $U \in \mathcal{S}_r^3$ we have*

$$\inf_{\lambda \in \mathbb{C} \setminus \mathfrak{J}} \gamma_m(\lambda, U) \geq \gamma_0.$$

Proof. Indeed, we obtain by (4.20), Poincaré's inequality and Sobolev's embeddings, that for any $\varepsilon > 0$ and any $\nu_0 > 0$ there exists C_{ε, ν_0} such that, for any $\lambda \in \mathbb{C} \setminus \mathfrak{J}$ and $\phi \in H_0^1(-1, 1)$,

$$I(\phi, \lambda) \geq \frac{1}{2} \|\phi'\|_2^2 - C \|\phi'\|_2^{3/2} \|\phi\|_2^{1/2} \geq \left(\frac{1}{2} - \varepsilon\right) \|\phi'\|_2^2 - C_{\varepsilon, \nu_0} \|\phi\|_2^2.$$

For $|\lambda| > \nu_0$, semiboundedness of I follows immediately from (4.22). ■

We now obtain an estimate for $\|\mathcal{A}_{\lambda, \alpha}^{-1}\|$ which is neither singular as $|\mu| \rightarrow 0$ (unlike (4.49)), nor does it necessitate any assumption on the injectivity of $\mathcal{A}_{i\nu, \alpha}$. Instead, we simply assume (4.19). We also observe that $\mathbf{m} \geq 1/r$ for any $U \in \mathcal{S}_r^2$ where \mathbf{m} is defined in (2.24).

Proposition 4.9. *For any $r > 1$ and $p > 1$, there exists a constant C such that for any $U \in \mathcal{S}_r^2$ satisfying (4.19) we have:*

1. *For all $v \in H^1(-1, 1)$,*

$$\sup_{\substack{|\Re \lambda| \leq (2r)^{-1} \\ 0 \leq \alpha}} \frac{1}{\log(|\Re \lambda|^{-1})} \|\mathcal{A}_{\lambda, \alpha}^{-1} v\|_{1,2} \leq C \|v\|_{\infty}. \quad (4.23)$$

2. *For all $v \in W^{1,p}(-1, 1)$,*

$$\sup_{\substack{|\Re \lambda| \leq (2r)^{-1} \\ 0 \leq \alpha}} \|\mathcal{A}_{\lambda, \alpha}^{-1} v\|_{1,2} \leq C (\|v'\|_p + \|v\|_{\infty}). \quad (4.24)$$

3. *For all $v \in L^p(-1, +1)$,*

$$\sup_{\substack{|\Re \lambda| \leq (2r)^{-1} \\ 0 \leq \alpha}} |\Re \lambda|^{1/p} \|\mathcal{A}_{\lambda, \alpha}^{-1} v\|_{1,2} \leq C \|v\|_p. \quad (4.25)$$

Proof.

Proof of (4.23). Let $v \in H^1(-1, 1)$ and $\phi = \mathcal{A}_{\lambda, \alpha}^{-1} v$ for some $\lambda = \mu + i\nu$ with $0 < |\mu| \leq \mathbf{m}/2$. We begin by observing that

$$\Re \left\langle \phi, \frac{v}{U - \nu + i\mu} \right\rangle = \|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2 + \Re \left\langle \phi, \frac{U'' \phi}{U + i\lambda} \right\rangle, \quad (4.26)$$

and note that by (4.19), which is assumed to hold here,

$$\Re \left\langle \phi, \frac{v}{U - \nu + i\mu} \right\rangle = \frac{1}{2} \|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2 + I(\phi, \lambda) \geq \frac{1}{2} \|\phi'\|_2^2. \quad (4.27)$$

The left hand side of (4.27) can be estimated as follows

$$\left| \Re \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| \leq \|\phi\|_\infty \|v\|_\infty \left\| \frac{1}{U + i\lambda} \right\|_1 \leq C \log(\mathbf{m}|\mu|^{-1}) \|\phi\|_\infty \|v\|_\infty. \quad (4.28)$$

To obtain (4.28), we consider two different cases.

We first apply the following computation, valid whenever $\nu \in [U(-1), U(1)]$, and $0 < |\mu| \leq \mathbf{m}/2$,

$$\begin{aligned} \left\| \frac{1}{U + i\lambda} \right\|_1 &= \int_{-1}^{+1} \frac{1}{\sqrt{(U(x) - \nu)^2 + \mu^2}} dx \\ &\leq \mathbf{m}^{-1} \int_{-1}^1 \frac{1}{\sqrt{|x - x\nu|^2 + |\mu/\mathbf{m}|^2}} dx \\ &\leq \mathbf{m}^{-1} \int_{-\frac{2m}{\mu}}^{\frac{2m}{\mu}} \frac{1}{\sqrt{1 + y^2}} dy \\ &\leq C \mathbf{m}^{-1} \log(\mathbf{m}|\mu|^{-1}). \end{aligned} \quad (4.29)$$

In the case $\nu < U(-1)$ we may write

$$\begin{aligned} \left\| \frac{1}{U + i\lambda} \right\|_1 &= \int_{-1}^{+1} \frac{1}{\sqrt{(U(x) - \nu)^2 + \mu^2}} dx \\ &\leq \int_{-1}^{+1} \frac{1}{\sqrt{(U(x) - U(-1))^2 + \mu^2}} dx \\ &\leq \mathbf{m}^{-1} \int_{-1}^{+1} \frac{1}{\sqrt{(x+1)^2 + (\mu/\mathbf{m})^2}} dx, \end{aligned}$$

and proceed as before. The case $U(1) > \nu$ is similar. Hence (4.28) is proved in both cases.

Sobolev's embedding and Poincaré's inequality then yield

$$\left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| \leq C \log(\mathbf{m}|\mu|^{-1}) \|\phi'\|_2 \|v\|_\infty. \quad (4.30)$$

Together with (4.27), it implies for $0 < |\mu| \leq \mathbf{m}/2$,

$$\|\phi\|_{1,2} \leq C \log(\mathbf{m}|\mu|^{-1}) \|v\|_\infty, \quad (4.31)$$

and hence also (4.23).

Proof of (4.24).

Let $|\nu| > 2 \max(|U(1)|, |U(-1)|)$. Then we have

$$\left| \Re \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| \leq C \|\phi\|_2 \|v\|_2. \quad (4.32)$$

If $|\nu| \leq 2 \max(|U(1)|, |U(-1)|)$ we integrate by parts to obtain

$$\left\langle \phi, \frac{v}{U + i\lambda} \right\rangle = - \left\langle \log(U + i\lambda), \left(\frac{\bar{\phi} v}{U'} \right)' \right\rangle.$$

Use of Hölder's inequality and Sobolev embedding lead, for every $p > 1$, to the conclusion that there exists $C_p > 0$ such that

$$\left| \left\langle \left(\frac{\bar{\phi} v}{U'} \right)', \log(U + i\lambda) \right\rangle \right| \leq C_p (\|\phi\|_\infty \|v'\|_p + \|\phi'\|_p \|v\|_\infty).$$

where we have used the fact that the L^q -norm (q being the Hölder conjugate) of $\log(U + i\lambda)$ can be uniformly bounded for

$$(|\Re\lambda|, |\Im\lambda|) \in [0, \mathbf{m}/2] \times [0, 2 \max(|U(1)|, |U(-1)|)].$$

Combining the above with (4.32), and making use of Sobolev embedding once again yield

$$\left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| \leq C_p (\|v'\|_p \|\phi\|_\infty + \|v\|_\infty \|\phi'\|_p). \quad (4.33)$$

By (4.27) we obtain that

$$\frac{1}{2} \|\phi'\|_2^2 \leq C \|\phi\|_{1,p} \min(\|v'\|_p, \|v\|_\infty), \quad (4.34)$$

from which (4.24) easily follows for $1 < p \leq 2$, and then for $p > 2$ by the inequality $\|v'\|_2 \leq 2^{1-2/p} \|v'\|_p$.

Proof of (4.25). We first observe that

$$\left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| \leq \left\| \frac{1}{U + i\lambda} \right\|_q \|\phi\|_\infty \|v\|_p, \quad (4.35)$$

where $q = p/(p-1)$.

If $\nu \in [U(-1), U(1)]$, we estimate the right-hand-side as follows

$$\begin{aligned} \left\| \frac{1}{U + i\lambda} \right\|_q &\leq \mathbf{m}^{-1} \left[\int_{-1}^1 \frac{dx}{[(x-x_\nu)^2 + |\mu/\mathbf{m}|^{2q/2}]^{1/q}} \right]^{1/q} \\ &\leq \mathbf{m}^{-1} \left[\int_{-\infty}^{\infty} \frac{dx}{[x^2 + |\mu/\mathbf{m}|^{2q/2}]^{1/q}} \right]^{1/q} \\ &= \mathbf{m}^{-1} \left(\frac{|\mu|}{\mathbf{m}} \right)^{\frac{1}{q}-1} \left[\int_{-\infty}^{\infty} \frac{dx}{[x^2+1]^{q/2}} \right]^{1/q} \\ &= C_q \mathbf{m}^{-\frac{1}{p}} |\mu|^{-1/p}. \end{aligned} \quad (4.36)$$

If $\nu < U(-1)$, we write

$$\left\| \frac{1}{U + i\lambda} \right\|_q \leq \mathbf{m}^{-1} \left[\int_{-1}^1 \frac{dx}{[(x+1)^2 + |\mu/\mathbf{m}|^{2q/2}]^{1/q}} \right]^{1/q}$$

and proceed as before. The case $\nu > U(-1)$ is similar. Thus, without any restriction on ν , we get

$$\left\| \frac{1}{U + i\lambda} \right\|_q \leq \hat{C}_q |\mu|^{-1/p}. \quad (4.37)$$

Substituting the above into (4.35) then yields

$$\left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| \leq C |\mu|^{-1/p} \|\phi\|_\infty \|v\|_p. \quad (4.38)$$

Combining the above with (4.27), we obtain

$$\frac{1}{2} \|\phi'\|_2^2 \leq C |\mu|^{-1/p} \|\phi'\|_2 \|v\|_p, \quad (4.39)$$

from which (4.25) easily follows ■

We shall need in Section 8 the following immediate consequence of Proposition 4.9.

Corollary 4.10. *For any $r > 1$ there exists $C > 0$ such that for all $v \in L^2(-1, 1)$, $U \in \mathcal{S}_r$ satisfying (4.19), $\lambda \in \mathbb{C} \setminus \mathfrak{J}$ and $\alpha \geq 0$ it holds that*

$$\|\mathcal{A}_{\lambda, \alpha}^{-1} v\|_{1,2} \leq C \left\| (1 \pm x)^{1/2} \frac{v}{U + i\lambda} \right\|_1. \quad (4.40)$$

Proof. Let $v \in L^2(-1, 1)$ and $\phi = \mathcal{A}_{\lambda, \alpha}^{-1} v$. By (4.27) we have that

$$\frac{1}{2} \|\phi'\|_2^2 \leq \Re \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle. \quad (4.41)$$

As $\phi \in H_0^1(-1, 1)$ we may write

$$\Re \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle = \Re \left\langle \phi - \phi(\pm 1), \frac{v}{U + i\lambda} \right\rangle \leq \|\phi'\|_2 \left\| (1 \pm x)^{1/2} \frac{v}{U + i\lambda} \right\|_1.$$

Combining the above with (4.41) yields (4.40) with the aid of Poincaré's inequality. ■

In the next lemma we address the optimality of (4.25).

Lemma 4.11. *Let $U \in C^3([-1, 1])$ satisfy (3.12). Then, (4.25) is optimal, i.e., there exists a sequence $\{\lambda_k, v_k\}_{k=1}^\infty \in \mathbb{C}^\mathbb{N} \times [L^p(-1, 1)]^\mathbb{N}$ and $\alpha \in \mathbb{R}_+$ such that $\|v_k\|_p = 1$, $\mu_k = \Re \lambda_k \rightarrow 0$, and*

$$\liminf_{k \rightarrow \infty} |\mu_k|^{1/p} \|\mathcal{A}_{\lambda_k, \alpha}^{-1} v_k\|_{1,2} > 0. \quad (4.42)$$

Proof. We prove (4.42) for $\alpha = 0$. Consider then $\mathcal{A}_{\lambda, 0}$ for some $\lambda = \mu + i\nu$ with $\mu \neq 0$ and $\nu \in (U(-1), U(1))$. Let (ϕ, v) satisfy

$$\mathcal{A}_{\lambda, 0} \phi = v, \quad \text{with } v \in L^p(-1, 1) \text{ and } \|v\|_p = 1.$$

We may rewrite this equation in the form

$$\left[(U + i\lambda)^2 \left(\frac{\phi}{U + i\lambda} \right)' \right]' = v.$$

Integrating once yields

$$(U(t) + i\lambda)^2 \left(\frac{\phi}{U + i\lambda} \right)'(t) = A_1 + \int_{-1}^t v(\tau) d\tau.$$

Integrating again leads to

$$\frac{\phi(x)}{U(x) + i\lambda} = A_1 \int_{-1}^x \frac{dt}{(U(t) + i\lambda)^2} + \int_{-1}^x \frac{1}{(U(t) + i\lambda)^2} \left(\int_{-1}^t v(\tau) d\tau \right) dt.$$

The Dirichlet boundary condition at $x = 1$ is then satisfied through the requirement that A_1 satisfies

$$A_1 \int_{-1}^1 \frac{dt}{(U(t) + i\lambda)^2} + \int_{-1}^1 \frac{1}{(U(t) + i\lambda)^2} \left(\int_{-1}^t v(\tau) d\tau \right) dt = 0.$$

Making use of Fubini's Theorem, we finally obtain

$$\phi(x) = (U(x) + i\lambda) \left[\int_{-1}^x v(t) \int_t^x \frac{d\tau}{(U(\tau) + i\lambda)^2} dt - A_1 \int_{-1}^x \frac{d\tau}{(U(\tau) + i\lambda)^2} \right], \quad (4.43a)$$

where

$$A_1 = \frac{\int_{-1}^1 v(t) \int_t^1 \frac{d\tau}{(U(\tau) + i\lambda)^2} dt}{\int_{-1}^1 \frac{d\tau}{(U(\tau) + i\lambda)^2}}. \quad (4.43b)$$

We now write

$$\int_t^x \frac{d\tau}{(U(\tau) + i\lambda)^2} = -\frac{1}{U'(U + i\lambda)} \Big|_t^x - \int_t^x \frac{U''(\tau) d\tau}{|U'(\tau)|^2 (U(\tau) + i\lambda)}. \quad (4.44)$$

It can be verified (as in the proof of (4.29)) that for some positive, independent of μ , constants C_1 and C ,

$$\left| \int_t^x \frac{U''(\tau) d\tau}{|U'(\tau)|^2 (U(\tau) + i\lambda)} \right| \leq C_1 \int_1^1 |U(\tau) + i\lambda|^{-1} d\tau \leq C(1 + |\log |\mu||).$$

Consequently, as $\mu \rightarrow 0$ we have

$$\left| \int_{-1}^x v(t) \int_t^x \frac{d\tau}{(U(\tau) + i\lambda)^2} dt - \int_{-1}^x v(t) \frac{1}{U'(U + i\lambda)} \Big|_t^x dt \right| \leq C |\log |\mu|| \|v\|_1. \quad (4.45)$$

We seek a sequence $\{\lambda_k, v_k\}$ with $v_k \in L^p(-1, 1)$ and $\mu_k > 0$ a decreasing sequence tending to 0 for which (4.42) holds. Let then

$$\lambda_k = \mu_k + i\nu.$$

For sufficiently large k we have that $x_\nu + 2\mu_k < 1$. We then define v_k by

$$v_k(x) = \begin{cases} \mu_k^{-1/p} & x \in [x_\nu, x_\nu + \mu_k] \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\|v_k\|_p = 1$, and $\|v_k\|_1 = \mu_k^{1-1/p}$.

By (4.45) we then have as μ_k tends to 0,

$$\int_{-1}^x v_k(t) \int_t^x \frac{d\tau}{(U(\tau) + i\lambda_k)^2} dt = \int_{-1}^x v_k(t) \left(\frac{1}{U'(U + i\lambda_k)} \Big|_t^x \right) dt + \mathcal{O}(|\mu_k|^{1-1/p} |\log \mu_k|). \quad (4.46)$$

We now write, for $(x_\nu + 1)/2 < x \leq 1$,

$$\begin{aligned} \int_{-1}^x v_k(t) \left(\frac{1}{U'(U + i\lambda_k)} \Big|_t^x \right) dt &= \\ &= \frac{1}{U'(x)(U(x) + i\lambda_k)} \mu_k^{1-1/p} - \mu_k^{-1/p} \int_{x_\nu}^{x_\nu + \mu_k} \frac{1}{U'(t)(U(t) + i\lambda_k)} dt. \end{aligned}$$

As

$$\left| \int_{x_\nu}^{x_\nu + \mu_k} \frac{d\tau}{U'(U + i\lambda_k)} - \int_{x_\nu}^{x_\nu + \mu_k} \frac{d\tau}{J_\nu[J_\nu(\tau - x_\nu) + i\mu_k]} \right| \leq C \mu_k (1 + \log \mu_k^{-1}),$$

where $J_\nu = U'(x_\nu)$, we obtain, for $(x_\nu + 1)/2 < x \leq 1$,

$$\int_{-1}^x v_k(t) \left(\frac{1}{U'(U + i\lambda_k)} \Big|_t^x \right) dt = -\frac{\mu_k^{-1/p}}{J_\nu^2} \log [J_\nu(\tau - x_\nu) + i\mu_k] \Big|_{x_\nu}^{x_\nu + \mu_k} + \mathcal{O}(\mu_k^{1-1/p}).$$

Consequently,

$$\int_{-1}^x v_k(t) \left(\frac{1}{U'(U + i\lambda_k)} \Big|_t^x \right) dt = -\frac{\mu_k^{-1/p}}{J_\nu^2} \log(1 - iJ_\nu) + \mathcal{O}(\mu_k^{1-1/p} |\log \mu_k|). \quad (4.47)$$

Substituting the above into (4.43a) and (4.43b) yields, for all $(x_\nu + 1)/2 < x \leq 1$,

$$\phi_k(x) = -(U(x) + i\lambda_k) \frac{\mu_k^{-1/p}}{J_\nu^2} \log(1 - iJ_\nu) \left[\frac{\int_x^1 (U(\tau) + i\lambda_k)^{-2} d\tau}{\int_{-1}^1 (U(\tau) + i\lambda_k)^{-2} d\tau} \right] + \mathcal{O}(\mu_k^{1-1/p} |\log \mu_k|),$$

where $\phi_k := \mathcal{A}_{\lambda_k, 0}^{-1} v_k$.

Clearly, for all $(x_\nu + 1)/2 < x \leq 1$,

$$\lim_{k \rightarrow \infty} \int_x^1 (U(\tau) + i\lambda_k)^{-2} d\tau = \int_x^1 (U(\tau) - \nu)^{-2} d\tau > 0,$$

the convergence being uniform in x .

We now prove (4.42) by establishing that

$$\liminf_{k \rightarrow \infty} |\mu_k|^{1/p} \|\phi_k\|_{L^2((x_\nu+1)/2, 1]} > 0,$$

which will immediately imply $\liminf_{k \rightarrow \infty} |\mu_k|^{1/p} \|\phi_k\|_{L^2(-1, 1)} > 0$ and consequently (4.42).

To this end we need to prove that

$$\left| \int_{-1}^1 (U(\tau) + i\lambda_k)^{-2} d\tau \right| \leq C. \quad (4.48)$$

We now use (4.44) together with the fact that $U \in C^3([-1, 1])$ to obtain that

$$\int_{-1}^1 \frac{d\tau}{(U(\tau) + i\lambda_k)^2} = \left[-\frac{1}{U'(U + i\lambda_k)} - \frac{U'' \log(U + i\lambda_k)}{|U'|^3} \right]_{-1}^1 + \int_{-1}^1 \left(\frac{U''}{|U'|^3} \right)' \log(U + i\lambda_k) d\tau.$$

Since $U' > 0$ and $|U(\pm 1) + i\lambda_k| \geq C(1 \pm x_\nu)$, for some $C > 0$, we can conclude (4.48) from the above. ■

4.4 Non-vanishing U''

We dedicate this subsection to the case when $U \in C^2([-1, +1])$ and $U'' \neq 0$ which may result from a combination of non-vanishing pressure gradient and relative velocity between the plates at $x_2 = \pm 1$. We begin by establishing the following, rather straightforward, result.

Proposition 4.12. *Suppose that $U'' \neq 0$ on $[-1, +1]$, then, for any $\lambda \in \mathbb{C}$ for which $\Re\lambda \neq 0$ and $\alpha \geq 0$, $\mathcal{A}_{\lambda,\alpha}$ is invertible. Moreover, for any $r > 1$ there exists $C > 0$, such that, for any λ with $\Re\lambda \neq 0$, $\alpha \geq 0$, and $U \in \mathcal{S}_r^2$ satisfying (2.29), it holds that*

$$\|\mathcal{A}_{\lambda,\alpha}^{-1}\| + \left\| \frac{d}{dx} \mathcal{A}_{\lambda,\alpha}^{-1} \right\| \leq C \frac{1 + |\Re\lambda|^{1/2}}{|\Re\lambda|}. \quad (4.49)$$

Proof. For a pair (ϕ, v) such that $v = \mathcal{A}_{\lambda,\alpha}\phi$, with $\lambda = \mu + i\nu$, we write,

$$\Im \left\langle \phi, \frac{v}{U - \nu + i\mu} \right\rangle = -\mu \left\langle \frac{U''\phi}{(U - \nu)^2 + \mu^2}, \phi \right\rangle. \quad (4.50)$$

Consequently, since $U'' \neq 0$ (hence has constant sign) we obtain

$$\left\| \frac{\phi}{U - \nu + i\mu} \right\|_2 \leq \frac{C}{|\mu|} \|v\|_2. \quad (4.51)$$

Let $\nu_0 > 0$ be chosen at a later stage and consider the following two cases.

In the case $\sqrt{|\mu|^2 + |\nu|^2} < \nu_0$, we immediately deduce from (4.51) that there exists $C(\nu_0)$ such that

$$\|\phi\|_2 \leq \frac{C(\nu_0)}{|\mu|} \|v\|_2. \quad (4.52)$$

In the case $\sqrt{|\mu|^2 + |\nu|^2} \geq \nu_0$, as

$$\Re \left\langle \phi, \frac{v}{U - \nu + i\mu} \right\rangle = \|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2 + \left\langle \frac{U''(U - \nu)\phi}{(U - \nu)^2 + \mu^2}, \phi \right\rangle, \quad (4.53)$$

we can use (4.51) once again to obtain that, for $\nu_0 \geq 2 \sup |U|$, there exists $C > 0$ such that

$$\|\phi'\|_2^2 + \left(\alpha^2 - \frac{C}{\nu_0} \right) \|\phi\|_2^2 \leq \frac{C}{|\mu|} \|v\|_2^2.$$

Since $\phi \in H_0^1(-1, 1)$ we can use Poincaré's inequality to obtain for sufficiently large ν_0 that there exists $C > 0$ such that, for any $\alpha \geq 0$ (and $|\lambda| \geq \nu_0$),

$$\|\phi\|_{1,2} \leq \frac{C}{|\mu|^{1/2}} \|v\|_2, \quad (4.54)$$

which, combined with (4.52) yields (4.49).

Once injectivity of $\mathcal{A}_{\lambda,\alpha}$ is established, we may apply Fredholm theory to prove its surjectivity. By the compactness of the multiplication with U'' from $D(\mathcal{A}_{\lambda,\alpha})$ into $L^2(-1, +1)$, we can conclude, as in Remark 4.5, that the index of $\mathcal{A}_{\lambda,\alpha}$ is the same as the index of $(U + i\lambda) \left(-\frac{d^2}{dx^2} + \alpha^2 \right)$. Since for $\mu \neq 0$, $U + i\lambda \neq 0$ on $[-1, +1]$, it follows that the indices of $\mathcal{A}_{\lambda,\alpha}$ and $-\frac{d^2}{dx^2} + \alpha^2$ from $H^2(-1, 1) \cap H_0^1(-1, 1)$ onto $L^2(-1, 1)$ are the same. Consequently, the index of $\mathcal{A}_{\lambda,\alpha}$ is 0 and surjectivity follows from injectivity. ■

It should be noted that (4.9) is unsatisfactory. Clearly, it is significantly inferior to (4.23)-(4.25), where a bound of $\mathcal{O}(|\Re\lambda|^{-1/2})$ for $\|\mathcal{A}_{\lambda,\alpha}^{-1}\|$ is obtained. We seek, therefore, a better estimate for $\|\mathcal{A}_{\lambda,\alpha}^{-1}\|$ that will be applicable in Sections 7 and 8.

Proposition 4.13. *Let $r > 1$ and $p > 1$. There exist $\mu_0 > 0$ and $C > 0$ such that for all $v \in W^{1,p}(-1, 1)$ and $U \in \mathcal{S}_r^3$ satisfying (2.29) we have*

$$\sup_{\substack{|\Re\lambda| \leq \mu_0 \\ 0 \leq \alpha}} \frac{1}{\log(|\Re\lambda|^{-1})} \|\mathcal{A}_{\lambda,\alpha}^{-1} v\|_{1,2} \leq C \|v\|_\infty, \quad (4.55a)$$

$$\sup_{\substack{|\Re\lambda| \leq \mu_0 \\ 0 \leq \alpha}} \|\mathcal{A}_{\lambda,\alpha}^{-1} v\|_{1,2} \leq C (\|v'\|_p + \|v\|_\infty), \quad (4.55b)$$

and

$$\sup_{\substack{|\Re\lambda| \leq \mu_0 \\ 0 \leq \alpha}} |\Re\lambda|^{1/p} \|\mathcal{A}_{\lambda,\alpha}^{-1} v\|_{1,2} \leq C \|v\|_p. \quad (4.55c)$$

Proof. In the case where $\nu \in [U(-1), U(1)]$ we (uniquely) select $x_\nu \in [-1, 1]$ where $U(x_\nu) = \nu$. Otherwise if $\nu > U(1)$ ($\nu < U(-1)$) we set $x_\nu = 1$ ($x_\nu = -1$).

Step 1: For $p > 1$ and $\Re\lambda \neq 0$ define $N_{m,p}^\pm$ by

$$v \mapsto N_{m,p}^\pm(v, \lambda) := \min \left(\left\| (1 \pm \cdot)^{1/2} \frac{v}{U + i\lambda} \right\|_1, \|v\|_{1,p} \right).$$

We prove that there exists $C > 0$ such that, for all $\varepsilon > 0$ and $0 < |\mu| \leq 1$ it holds that

$$|\phi(x_\nu)| \leq C(\varepsilon^{-1/2} N_{m,p}^\pm(v, \lambda) + (|\mu|^{1/2} + \varepsilon^{1/2}) \|\phi'\|_2), \quad (4.56)$$

for all pairs $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,r}(-1, 1)$ satisfying $\mathcal{A}_{\lambda,\alpha}\phi = v$.

As

$$|\phi(x)|^2 \geq \frac{1}{2} |\phi(x_\nu)|^2 - |\phi(x) - \phi(x_\nu)|^2,$$

we may use (4.50) to obtain

$$|\Im \left\langle \phi, \frac{v}{U - \nu + i\mu} \right\rangle| \geq |\mu| \left\langle \frac{|U''|}{(U - \nu)^2 + \mu^2}, \frac{1}{2} |\phi(x_\nu)|^2 - |\phi(x) - \phi(x_\nu)|^2 \right\rangle. \quad (4.57)$$

We note that, for any $p > 1$, there exists $C > 0$ such that

$$\begin{aligned} \left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| &= \left| \left\langle \phi' \bar{v} + \phi \bar{v}', \log(U + i\lambda) \right\rangle \right| \\ &\leq C (\|\phi'\|_2 \|v\|_\infty + \|\phi\|_\infty \|v'\|_p) \\ &\leq C \|\phi'\|_2 \|v\|_{1,p}. \end{aligned}$$

The constant C depends on a uniform bound of $\|\log(U + i\lambda)\|_2$ and $\|\log(U + i\lambda)\|_{p/(p-1)}$. On the other hand, observing that

$$|\phi(x)| = |\phi(x) - \phi(\pm 1)| \leq \|\phi'\|_2 (1 \pm x)^{\frac{1}{2}},$$

we may conclude that

$$\left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| \leq \|\phi'\|_2 \left\| (1 \pm \cdot)^{1/2} \frac{v}{U + i\lambda} \right\|_1$$

and hence, there exists $C > 0$, such that

$$\left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| \leq C \|\phi'\|_2 N_{m,p}^\pm(v, \lambda). \quad (4.58)$$

Substituting the above into (4.57) yields

$$\frac{|\mu|}{2} |\phi(x_\nu)|^2 \left\| \frac{|U''|^{1/2}}{U + i\lambda} \right\|_2^2 \leq |\mu| \sup |U''| \left\| \frac{\phi - \phi(x_\nu)}{U + i\lambda} \right\|_2^2 + C \|\phi'\|_2 N_{m,p}^\pm(v, \lambda).$$

We now observe, as in (4.36) (but with a lower bound in mind), that, for some positive C and \hat{C} (note that $|\mu| \leq 1$), it holds

$$\left\| \frac{|U''|^{1/2}}{U + i\lambda} \right\|_2^2 \geq \frac{1}{C} \left\| \frac{1}{(x - x_\nu)^2 + \mu^2} \right\|_1 \geq \frac{1}{\hat{C}|\mu|}.$$

Hence, for another constant $C > 0$, we get

$$|\phi(x_\nu)|^2 \leq C \left[|\mu| \left\| \frac{\phi - \phi(x_\nu)}{U + i\lambda} \right\|_2^2 + \|\phi'\|_2 N_{m,p}^\pm(v, \lambda) \right]. \quad (4.59)$$

To estimate the first term on the right-hand-side of (4.59) we first observe that for some $C > 0$ we have

$$\left| \frac{1}{U(x) + i\lambda} \right| \leq \frac{C}{|x - x_\nu|}, \quad \forall x \in (-1, +1).$$

Then we notice that for any $w \in H_0^1(\mathbb{R}_+)$ we have by (4.3) and some integration by parts

$$\|(xw)'\|_2^2 = \|xw'\|_2^2 \geq \frac{1}{4} \|w\|_2^2.$$

Using a constant smaller than $1/4$ on its right-hand-side, the above inequality can be extended in an interval (a, b) to the space $\{w \in H^1(a, b), v(a) = 0\}$. We thus apply it to $w = (\phi - \phi(x_\nu))/(x - x_\nu)$ in $(-1, x_\nu)$ and $(x_\nu, 1)$ to obtain

$$\left\| \frac{\phi - \phi(x_\nu)}{U + i\lambda} \right\|_2^2 \leq C \|\phi'\|_2^2,$$

which when substituted into (4.59) readily yields (4.56) via Cauchy's inequality. Note that, for $\nu \notin [U(-1), U(1)]$, (4.56) is trivial as $\phi(x_\nu) = 0$.

Step 2: Let $d = \min(1 - x_\nu, 1 + x_\nu)$. We prove that for any $A > 0$, and $d_1 > 0$, there exists C_{A,d_1} and μ_{A,d_1} such that, for $\alpha^2 \leq A$, $x_\nu \in (-1, 1)$, $|\mu| \leq \mu_{A,d_1}$, and $d \geq d_1$,

$$\|\phi\|_{1,2} \leq C_{A,d_1} N_{m,p}^\pm(v, \lambda). \quad (4.60)$$

holds for any pair (ϕ, v) satisfying $\mathcal{A}_{\lambda,\alpha}\phi = v$.

Let $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ satisfy

$$\chi(x) = \begin{cases} 1 & |x| < 1/2 \\ 0 & |x| > 1. \end{cases}$$

Let $\chi_d(x) = \chi((x - x_\nu)/d)$ and set

$$\phi = \varphi + \phi(x_\nu)\chi_d. \quad (4.61)$$

Note that by the choice of d , φ satisfies also the boundary condition at ± 1 . It can be easily verified that

$$\mathcal{A}_{\lambda, \alpha}\varphi = v + \phi(x_\nu)((U + i\lambda)(\chi_d'' - \alpha^2\chi_d) - U''\chi_d).$$

By construction we have that $w = (U - \nu)^{-1}\varphi \in H^2(-1, 1)$, and hence we can rewrite the above equality (using (3.13) twice) in the form

$$\begin{aligned} & - \left((U - \nu)^2 \left(\frac{\varphi}{U - \nu} \right)' \right)' + \alpha^2(U - \nu)\varphi \\ & = v + \phi(x_\nu)((U - \nu)(\chi_d'' - \alpha^2\chi_d) - U''\chi_d) + i\mu(\phi'' - \alpha^2\phi) \\ & = \frac{(U - \nu)v}{U + i\lambda} + \phi(x_\nu)((U - \nu)(\chi_d'' - \alpha^2\chi_d) - U''\chi_d) + i\mu \frac{U''\phi}{U + i\lambda}. \end{aligned}$$

Taking the scalar product with w and integrating by parts then yield

$$\begin{aligned} \|(U - \nu)w'\|_2^2 + \alpha^2\|\varphi\|_2^2 & = \left\langle \varphi, \frac{v}{U + i\lambda} \right\rangle - \langle w, \phi(x_\nu)U''\chi_d \rangle \\ & \quad + \phi(x_\nu)\langle \varphi, \chi_d'' - \alpha^2\chi_d \rangle + i\mu \left\langle w, \frac{U''\phi}{U + i\lambda} \right\rangle. \end{aligned} \quad (4.62)$$

As in the proof of (4.58), the first term on the right-hand side is estimated as follows

$$\left| \left\langle \varphi, \frac{v}{U + i\lambda} \right\rangle \right| \leq \|\varphi'\|_2 \left\| (1 + \cdot)^{1/2} \frac{v}{U + i\lambda} \right\|_1 \leq C(\|\phi'\|_2 + d^{-1/2}|\phi(x_\nu)|)N_{m,p}^\pm(v, \lambda). \quad (4.63)$$

To estimate the second term on the right-hand-side, we note that by Hardy's inequality, we have

$$\|w\|_2 \leq C\|\varphi'\|_2 \leq \hat{C} \left(\|\phi'\|_2 + \frac{1}{d^{1/2}}|\phi(x_\nu)| \right). \quad (4.64)$$

From (4.64) we get

$$|\langle w, \phi(x_\nu)U''\chi_d \rangle| \leq \check{C} d^{\frac{1}{2}} |\phi(x_\nu)| \left(\|\phi'\|_2 + \frac{1}{d^{1/2}}|\phi(x_\nu)| \right). \quad (4.65)$$

Then, we write for the third term on the right-hand-side of (4.62), using integration by parts and the upper bound on α^2

$$|\langle \varphi, \chi_d'' - \alpha^2\chi_d \rangle| \leq \|\varphi'\|_2 \|\chi_d'\|_2 + C\|\varphi\|_2 \leq \hat{C} \left(\frac{1}{d^{1/2}}\|\varphi'\|_2 + \|\varphi\|_2 \right).$$

Consequently, by (4.61),

$$\begin{aligned} |\phi(x_\nu)| |\langle \varphi, \chi_d'' - \alpha^2 \chi_d \rangle| &\leq \\ &\leq C |\phi(x_\nu)| \left(\frac{1}{d^{1/2}} \left(\|\phi'\|_2 + \frac{1}{d^{1/2}} |\phi(x_\nu)| \right) + (\|\phi\|_2 + d^{1/2} |\phi(x_\nu)|) \right). \end{aligned}$$

Hence, using Poincaré's inequality,

$$|\phi(x_\nu)| |\langle \varphi, \chi_d'' - \alpha^2 \chi_d \rangle| \leq C \frac{1}{d^{1/2}} |\phi(x_\nu)| \left(\|\phi'\|_2 + \frac{1}{d^{1/2}} |\phi(x_\nu)| \right),$$

from which we conclude the existence of C such that for any $\varepsilon_1 \in (0, 1)$, we have

$$|\phi(x_\nu)| |\langle \varphi, \chi_d'' - \alpha^2 \chi_d \rangle| \leq C \left(\varepsilon_1 \|\phi'\|_2^2 + \frac{1}{\varepsilon_1 d} |\phi(x_\nu)|^2 \right). \quad (4.66)$$

To estimate the last term on the right-hand-side of (4.62), we use (4.64), (4.37), and (4.64), to obtain

$$\left| \left\langle w, U'' \frac{\phi}{U + i\lambda} \right\rangle \right| \leq C \|w\|_2 \|\phi\|_\infty \left\| \frac{1}{U + i\lambda} \right\|_2 \leq C |\mu|^{-1/2} \|\phi\|_\infty \left(\|\phi'\|_2 + \frac{1}{d^{1/2}} |\phi(x_\nu)| \right) \quad (4.67)$$

Substituting (4.67) together with (4.63), (4.65), and (4.66) into (4.62) yields that there exists $C > 0$ such that for every $0 < \varepsilon_1 < 1$ it holds that

$$\|(U - \nu)w'\|_2^2 + \alpha^2 \|\varphi\|_2^2 \leq C \left((\varepsilon_1 + |\mu|^{1/2}) \|\phi'\|_2^2 + \frac{1}{\varepsilon_1 d} |\phi(x_\nu)|^2 + \varepsilon_1^{-1} N_{m,p}^\pm(v, \lambda)^2 \right) \quad (4.68)$$

By Hardy's inequality (4.2), Poincaré's inequality, and (4.56) we obtain, for $0 < |\mu| \leq 1$, $\varepsilon \in (0, 1)$, and $\varepsilon_1 \in (0, 1)$, that

$$\|w\|_2 \leq C \left(\left[|\mu|^{1/4} + \varepsilon_1^{1/2} + \frac{\varepsilon^{1/2} + |\mu|^{1/2}}{[\varepsilon_1 d]^{1/2}} \right] \|\phi'\|_2 + [\varepsilon_1 d \varepsilon]^{-1/2} N_{m,p}^\pm(v, \lambda) \right).$$

Selecting $\varepsilon = \varepsilon_1^2$ then yields, for any $|\mu| \leq \mu_{A,d_1}$, and $\varepsilon_1 \in (0, 1)$,

$$\|w\|_2 \leq C \left(|\mu|^{1/4} (1 + |\mu|^{1/4} [\varepsilon_1 d]^{-1/2}) + d^{-1/2} \varepsilon_1^{1/2} \right) \|\phi'\|_2 + C [\varepsilon_1^3 d]^{-1/2} N_{m,p}^\pm(v, \lambda).$$

As $d \geq d_1$ we then write

$$\|w\|_2 \leq C \left(|\mu|^{1/4} (1 + |\mu|^{1/4} [\varepsilon_1 d_1]^{-1/2}) + d_1^{-1/2} \varepsilon_1^{1/2} \right) \|\phi'\|_2 + C [\varepsilon_1^3 d_1]^{-1/2} N_{m,p}^\pm(v, \lambda). \quad (4.69)$$

Recalling the definitions of ϕ , φ and w , we immediately conclude that

$$\|\phi'\|_2 \leq \|(U - \nu)w'\|_2 + \|U'w\|_2 + Cd^{-1/2} |\phi(x_\nu)|.$$

which together with (4.56) gives

$$\|\phi'\|_2 \leq \|(U - \nu)w'\|_2 + \|U'w\|_2 + C \left([d_1^{1/2} \varepsilon_1]^{-1} N_{m,p}^\pm(v, \lambda) + d_1^{-\frac{1}{2}} (|\mu|^{\frac{1}{2}} + \varepsilon_1) \right) \|\phi'\|_2. \quad (4.70)$$

Substituting (4.69) and (4.68) into (4.70) yields

$$\|\phi'\|_2 \leq C(d_1) \left(\varepsilon_1^{-3/2} N_{m,p}^\pm(v, \lambda) + (|\mu|^{\frac{1}{4}} + |\mu|^{\frac{1}{2}} \varepsilon_1^{-\frac{1}{2}} + \varepsilon_1^{1/2}) \|\phi'\|_2 \right)$$

Hence, we can choose first ϵ_1 and then $\mu_{A,d_1} > 0$ such that (4.60) follows for $|\mu| \leq \mu_{A,d_1}$ and $d \geq d_1$.

Step 3: We prove (4.60) under the assumption that

$$\alpha^2 \geq C_U := \frac{1}{4}C_0^4\delta_2(U)^4, \quad (4.71)$$

with

$$C_0 = 2 \sup_{\substack{|\mu| \leq 1 \\ \nu \in [U(-1), U(1)]}} \|\log(U + i\lambda)\|_2. \quad (4.72)$$

We recall (4.26) which reads

$$\Re \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle = \|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2 - \Re \left\langle U''\phi, \frac{\phi}{U + i\lambda} \right\rangle.$$

For the last term we have, using Poincaré's inequality and Sobolev's embeddings

$$\begin{aligned} \left| \left\langle U''\phi, \frac{\phi}{U + i\lambda} \right\rangle \right| &\leq \left| \left\langle \left(\frac{U''}{U'} |\phi|^2 \right)', \log(U + i\lambda) \right\rangle \right| \\ &\leq \|\log(U + i\lambda)\|_2 \|\phi\|_\infty \left(2 \left\| \frac{U''}{U'} \right\|_\infty \|\phi'\|_2 + \left\| \left(\frac{U''}{U'} \right)' \right\|_\infty \|\phi\|_2 \right) \\ &\leq C_0 \delta_2(U) \|\phi'\|_2^{3/2} \|\phi\|_2^{1/2}, \end{aligned}$$

where C_0 is given by (4.72).

Consequently, by (4.58)

$$\|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2 \leq C_0 \delta_2(U) \|\phi'\|_2^{3/2} \|\phi\|_2^{1/2} + C \|\phi'\|_2 N_{m,p}^\pm(v, \lambda).$$

Using Young's inequality we obtain

$$\frac{1}{8} \|\phi'\|_2^2 \leq \left(\frac{1}{4} C_0^4 \delta_2(U)^4 - \alpha^2 \right) \|\phi\|_2^2 + \hat{C} N_{m,p}^\pm(v, \lambda)^2 \quad (4.73)$$

Hence, for $\alpha^2 \geq C_U$ (4.60) follows immediately from the above inequality in conjunction with Poincaré's inequality.

Step 4: We prove that there exist $d_0 > 0$, $\mu_0 > 0$ and $C > 0$ such that, for all $d \leq d_0$, $\nu \in [U(-1), U(1)]$, $\alpha \geq 0$ and $|\mu| \leq \mu_0$,

$$\|\phi\|_{1,2} \leq C N_{m,p}^\pm(v, \lambda). \quad (4.74)$$

holds for any pair (ϕ, v) such that $\mathcal{A}_{\lambda,\alpha}\phi = v$.

Without any loss of generality we can assume that $d = 1 - x_\nu$. As

$$(U - U(1))(-\phi'' + \alpha^2\phi) - U''\phi = v - (U(1) - \nu + i\mu)(-\phi'' + \alpha^2\phi)$$

or equivalently, by (3.13),

$$\begin{aligned} - \left((U - U(1))^2 \left(\frac{\phi}{U - U(1)} \right)' \right)' + \alpha^2 (U - U(1))\phi \\ = \frac{(U - U(1))v}{U + i\lambda} + (U(1) - \nu + i\mu) \frac{U''\phi}{U + i\lambda}. \end{aligned}$$

Taking the scalar product with $(U - U(1))^{-1}\phi$ and integrating by parts then yield

$$\begin{aligned} & \left\| (U - U(1)) \left(\frac{\phi}{U - U(1)} \right)' \right\|_2^2 + \alpha^2 \|\phi\|_2^2 \\ &= \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle + (U(1) + i\lambda) \left\langle \frac{\phi}{U - U(1)}, \frac{U''\phi}{U + i\lambda} \right\rangle. \end{aligned} \quad (4.75)$$

For the first term on the right-hand-side of (4.75) we use (4.58).

Next, we estimate the second inner product on the right-hand-side of (4.75) by splitting the domain of integration in two sub-intervals: $(1 - 2d, 1)$ and $(-1, 1 - 2d)$.

The integral over $(1 - 2d, 1)$. To estimate the integral over $(1 - 2d, 1)$ we use the identity

$$\frac{1}{[U(1) - U](U + i\lambda)} = \frac{1}{U(1) + i\lambda} \left[\frac{1}{U + i\lambda} + \frac{1}{U(1) - U} \right],$$

to obtain that

$$\begin{aligned} (U(1) + i\lambda) \int_{1-2d}^1 \frac{U''|\phi|^2}{[U(1) - U](U + i\lambda)} dx = \\ \int_{1-2d}^1 \frac{U''|\phi|^2}{U + i\lambda} dx + \int_{1-2d}^1 \frac{U''|\phi|^2}{U(1) - U} dx. \end{aligned} \quad (4.76)$$

As

$$\int_{1-2d}^1 \frac{U''|\phi|^2}{U + i\lambda} dx = -\frac{U''}{U'} |\phi|^2 \log(U + i\lambda) \Big|_{1-2d} - \int_{1-2d}^1 \left(\frac{U''}{U'} |\phi|^2 \right)' |\log(U + i\lambda)| dx,$$

we may conclude, having in mind that μ and d are bounded, that

$$\begin{aligned} \left| \int_{1-2d}^1 \frac{U''|\phi|^2}{U + i\lambda} dx \right| &\leq C \left\| \frac{U''}{U'} \right\|_{1,\infty} \left(d \log d \|\phi'\|_2^2 + \int_{1-2d}^1 (|\phi| |\phi'| + |\phi|^2) |\log(U + i\lambda)| dx \right) \\ &\leq \hat{C} d^{1/2} \|\phi'\|_2^2. \end{aligned} \quad (4.77)$$

Furthermore, employing Hardy's inequality and Cauchy-Schwarz inequality yields

$$\left| \int_{1-2d}^1 \frac{U''|\phi|^2}{U - U(1)} dx \right| \leq C d^{1/2} \|\phi'\|_2 \|\phi\|_\infty.$$

Substituting the above inequalities into (4.76) yields

$$\left| (U(1) + i\lambda) \int_{1-2d}^1 \frac{U''|\phi|^2}{[U(1) - U](U + i\lambda)} dx \right| \leq C d^{1/2} \|\phi'\|_2^2. \quad (4.78)$$

The integrals over $(-1, 1 - 2d)$. We now estimate the integrals over $[-1, 1 - 2d]$ for the inner products on the right-hand-side of (4.75). To this end we write, using Hardy's inequality (4.2) and lower bounds of $|U + i\lambda|$,

$$\begin{aligned} \left| \int_{-1}^{1-2d} \frac{U''|\phi|^2}{[U(1) - U](U + i\lambda)} dx \right| \\ \leq C \|\phi'\|_2 \|\phi\|_\infty \left[\int_{-1}^{1-2d} \frac{1}{|U + i\lambda|^2} \right]^{1/2} \\ \leq \hat{C} (\max(d, |\mu|))^{-1/2} \|\phi'\|_2^2. \end{aligned} \quad (4.79)$$

Returning to the estimate of the right hand side of (4.75), we use (4.79), and the fact that $|U(1) - \nu| \leq \hat{C}d$ (following from by definition of d) together with Poincaré's inequality, to obtain

$$|U(1) - \nu| \left| \int_{-1}^{1-2d} \frac{U''|\phi|^2}{[U(1) - U](U + i\lambda)} dx \right| \leq Cd^{1/2} \|\phi'\|_2^2.$$

and

$$|\mu| \left| \int_{-1}^{1-2d} \frac{\bar{\phi}(v + U''\phi)}{[U(1) - U](U + i\lambda)} dx \right| \leq C|\mu|^{1/2} \|\phi'\|_2^2.$$

Combining the above and (4.78) yields

$$\left| (U(1) - \nu + i\mu) \left\langle \frac{\phi}{U - U(1)}, \frac{U''\phi}{U + i\lambda} \right\rangle \right| \leq C[d^{1/2} + |\mu|^{1/2}] \|\phi'\|_2^2. \quad (4.80)$$

Substituting (4.80) together with (4.58), and (4.76) into (4.75), yields

$$\left\| (U - U(1)) \left(\frac{\phi}{U - U(1)} \right)' \right\|_2^2 \leq C ([d^{1/2} + |\mu|^{1/2}] \|\phi'\|_2^2 + \|\phi'\|_2 N_{m,p}^\pm(v, \lambda)).$$

From Hardy's inequality in the form (4.2) we then conclude

$$\left\| \frac{\phi}{U - U(1)} \right\|_2^2 \leq C([d^{1/2} + |\mu|^{1/2}] \|\phi'\|_2^2 + \|\phi'\|_2 N_{m,p}^\pm(v, \lambda)).$$

Combined with the following straightforward observation

$$\|\phi'\|_2 \leq \left\| (U - U(1)) \left(\frac{\phi}{U - U(1)} \right)' \right\|_2 + \left\| U' \frac{\phi}{U - U(1)} \right\|_2, \quad (4.81)$$

this yields,

$$\|\phi'\|_2^2 \leq C([d^{1/2} + |\mu|^{1/2}] \|\phi'\|_2^2 + \|\phi'\|_2 N_{m,p}^+(v, \lambda)).$$

Hence, there exists $d_0 > 0$ and $\mu_0 > 0$ such that (4.74) holds for all $d \leq d_0$ and $|\mu| \leq \mu_0$.

Step 5: We prove that there exist $C > 0$ and $\mu_0 > 0$ such that (4.74) holds for all $\nu \in \mathbb{R} \setminus [U(-1), U(1)]$ and $|\mu| \leq \mu_0$.

Without any loss of generality we assume $\nu > U(1)$. We begin by rewriting $\mathcal{A}_{\lambda,\alpha}\phi = v$ in the form

$$-\left((U - \nu)^2 \left(\frac{\phi}{U - \nu}\right)'\right)' + \alpha^2(U - \nu)\phi = v - i\mu \frac{v + U''\phi}{U + i\lambda} = \frac{(U - \nu)v - i\mu U''\phi}{U + i\lambda}.$$

Taking the inner product with $\phi/(U - \nu)$ on the left yields

$$\left\| (U - \nu) \left(\frac{\phi}{U - \nu}\right)' \right\|_2^2 + \alpha^2 \|\phi\|_2^2 = \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle - i\mu \left\langle \frac{\phi}{U - \nu}, \frac{U''\phi}{U + i\lambda} \right\rangle. \quad (4.82)$$

Let

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & x \in [-1, 1] \\ 0 & |x| > 1. \end{cases}$$

Let

$$\hat{U} = \begin{cases} U(x) & x \in [-1, 1] \\ U(1) + U'(1)(x - 1) & x > 1. \end{cases}$$

Let $\hat{x}_\nu > 1$ denote the unique zero of $\hat{U} - \nu$. We may now use Hardy's inequality to obtain the existence of $C > 0$ such that for all $\nu \in \mathbb{R} \setminus [U(-1), U(+1)]$, all $\phi \in H_0^1(-1, +1)$,

$$\begin{aligned} \left\| (U - \nu) \left(\frac{\phi}{U - \nu}\right)' \right\|_2^2 &= \left\| (\hat{U} - \nu) \left(\frac{\tilde{\phi}}{\hat{U} - \nu}\right)' \right\|_{L^2(-1, \hat{x}_\nu)}^2 \\ &\geq \frac{1}{C} \left\| \frac{\tilde{\phi}}{\hat{U} - \nu} \right\|_{L^2(-1, \hat{x}_\nu)}^2 \\ &= \frac{1}{C} \left\| \frac{\phi}{U - \nu} \right\|_2^2. \end{aligned} \quad (4.83)$$

Next we use the analog of (4.81)

$$\|\phi'\|_2^2 \leq 2 \left\| (U - \nu) \left(\frac{\phi}{U - \nu}\right)' \right\|_2^2 + 2 \left\| U' \frac{\phi}{U - \nu} \right\|_2^2,$$

which leads, together with (4.83), to

$$\|\phi'\|_2 \leq \hat{C} \left\| (U - \nu) \left(\frac{\phi}{U - \nu}\right)' \right\|_2. \quad (4.84)$$

On the other hand we have from (4.82) and (4.37)

$$\begin{aligned} \left\| (U - \nu) \left(\frac{\phi}{U - \nu}\right)' \right\|_2^2 &\leq \|\phi'\|_2 N_{m,p}^\pm(v, \lambda) + |\mu| \left\| \frac{\phi}{U - \nu} \right\|_2 \|\phi\|_\infty \left\| \frac{1}{U + i\lambda} \right\|_2 \\ &\leq C \|\phi'\|_2 \left(N_{m,p}^\pm(v, \lambda) + |\mu|^{1/2} \left\| \frac{\phi}{U - \nu} \right\|_2 \right). \end{aligned}$$

Combining the above with (4.83) yields first

$$\left\| (U - \nu) \left(\frac{\phi}{U - \nu}\right)' \right\|_2 \leq \check{C} \left(N_{m,p}^\pm(v, \lambda) + |\mu|^{1/2} \left\| (U - \nu) \left(\frac{\phi}{U - \nu}\right)' \right\|_2 \right),$$

hence, for sufficiently small $|\mu|$,

$$\left\| (U - \nu) \left(\frac{\phi}{U - \nu}\right)' \right\|_2 \leq 2\check{C} N_{m,p}^\pm(v, \lambda). \quad (4.85)$$

Finally, using (4.84) once again leads to the existence of $C > 0$ and $\mu_0 > 0$ such that if $|\mu| \leq \mu_0$

$$\|\phi'\|_2 \leq C N_{m,p}^\pm(v, \lambda), \quad (4.86)$$

and we obtain (4.74) in this case as well.

Step 6: Prove (4.55).

By (4.74) (established in steps 4 and 5 for $d \leq d_0$) and (4.60) (proved in steps 2 and 3 for $d \geq d_1 = d_0$), there exist $C > 0$ and $\mu_0 > 0$ such that for $|\mu| \leq \mu_0$ we have

$$\|\phi\|_{1,2} \leq C N_{m,p}^\pm(v, \lambda), \quad (4.87)$$

As $N_{m,p}^\pm(v, \lambda) \leq \|v\|_{1,p}$, we can immediately conclude (4.55b). To conclude (4.55a,c) we first observe that

$$N_{m,p}^\pm(v, \lambda) \leq 2 \left\| \frac{v}{U + i\lambda} \right\|_1$$

and then use Hölder inequality

$$\left\| \frac{v}{U + i\lambda} \right\|_1 \leq \|v\|_p \left\| \frac{1}{U + i\lambda} \right\|_q,$$

valid for any $1 < p \leq \infty$, together with (4.37), and (4.29). ■

For later reference we also need the following estimate which can be deduced from the proofs of Propositions 4.12 and 4.13.

Proposition 4.14. *Let $r > 1$. Then, there exists $C > 0$ such that, for all $v \in L^2(-1, 1)$, λ such that $\Re \lambda \neq 0$, and $U \in \mathcal{S}_r^3$ satisfying (2.29) it holds that*

$$\|\mathcal{A}_{\lambda,\alpha}^{-1}v\|_{1,2} \leq C \left\| (1 \pm x)^{1/2} \frac{v}{U + i\lambda} \right\|_1. \quad (4.88)$$

Proof. Let μ_0 be as in the statement of Proposition 4.13. Let $\lambda = \mu + i\nu$, $v \in L^2(-1, 1)$, and $\phi = \mathcal{A}_{\lambda,\alpha}^{-1}v$. For $|\mu| < \mu_0$, (4.88) is an immediate result of (4.87).

Consider then the case $|\mu| \geq \mu_0$. By (4.50) and (4.58), we obtain that

$$|\mu| \min_{x \in [-1, 1]} |U''| \left\langle \frac{\phi}{(U - \nu)^2 + \mu^2}, \phi \right\rangle \leq \left| \left\langle \phi, \frac{v}{U - \nu + i\mu} \right\rangle \right| \leq C \|\phi'\|_2 N_{m,p}^\pm(v, \lambda).$$

Consequently, as $|\mu| > \mu_0$, there exists $C(\mu_0) > 0$ such that

$$\left\| \frac{\phi}{U + i\lambda} \right\|_2^2 \leq C \|\phi'\|_2 N_{m,p}^\pm(v, \lambda). \quad (4.89)$$

Then, we use (4.53) to establish that

$$\|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2 \leq \left| \left\langle \phi, \frac{v}{U - \nu + i\mu} \right\rangle \right| + \left\| \frac{\phi}{U + i\lambda} \right\|_2 \|\phi\|_2,$$

from which we conclude, with the aid of (4.89), that

$$\|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2 \leq C \left([\|\phi'\|_2 N_{m,p}^\pm(v, \lambda)]^{1/2} \|\phi\|_2 + \|\phi'\|_2 N_{m,p}^\pm(v, \lambda) \right).$$

Using Poincaré's inequality we can now establish (4.88). ■

5 Some Schrödinger operators and their resolvents

In this section we derive several refinements of estimates obtained in [21, 4] for the resolvent of $\mathcal{L}_\beta = -\frac{d^2}{dx^2} + i\beta U$, (as in (3.7)) defined over different domains. As in the rest of this contribution, we are assuming (2.24). These estimates will be used in Sections 7 and 8.

5.1 The entire real line

We begin by stating the following result on \mathbb{R} .

Proposition 5.1. *For $\tilde{U} \in C^1(\mathbb{R})$, let $\tilde{\mathcal{L}}_{\beta, \mathbb{R}}$ be given by*

$$\tilde{\mathcal{L}}_{\beta, \mathbb{R}} = -\frac{d^2}{dx^2} + i\beta\tilde{U}, \quad (5.1)$$

with domain

$$D(\tilde{\mathcal{L}}_{\beta, \mathbb{R}}) = \{u \in H^2(\mathbb{R}) \mid xu \in L^2(\mathbb{R})\}. \quad (5.2)$$

Then, for all positive Υ , m , M , C , and ϵ , there exist $\beta_0 > 0$ and \hat{C} , such that for all $\beta \geq \beta_0$ and $\tilde{U} \in C^1(\mathbb{R})$ satisfying

$$\begin{cases} 0 < m \leq \tilde{U}'(x) \leq M \text{ for all } x \in \mathbb{R} \\ |\tilde{U}'(x) - \tilde{U}'(y)| \leq C|x - y|^\epsilon, \text{ for all } x, y \in \mathbb{R} \text{ s.t. } |x - y| \leq 1, \end{cases} \quad (5.3)$$

it holds that

$$\begin{aligned} \sup_{\Re\lambda \leq \Upsilon\beta^{-1/3}} (1 + \beta^{1/3}|\Re\lambda|) \|(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)^{-1}\| + \\ \beta^{-1/3} \sup_{\Re\lambda \leq \Upsilon\beta^{-1/3}} \left\| \frac{d}{dx} (\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)^{-1} \right\| \leq \frac{\hat{C}}{\beta^{2/3}}, \end{aligned} \quad (5.4)$$

and

$$\sup_{\Re\lambda \leq \Upsilon\beta^{-1/3}} \|(\tilde{U} - \Im\lambda)(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)^{-1}\| \leq \frac{\hat{C}}{\beta}. \quad (5.5)$$

Proof.

The estimation of the first term on the left-hand-side of (5.4), can be obtained, for $-1 < \beta^{1/3}\mu < \Upsilon$ (with $\mu = \Re\lambda$) as in the proof of [21, Theorem 1.1 (ii)]. The difference is that the interval is infinite and that we impose here less regularity on the potential. To accommodate $C^{1, \epsilon}$ potential in the proof in [21], we construct a partition of unity composed of intervals of size $\beta^{-\frac{\rho}{2}}$, and select $\rho \in (\frac{2}{3(1+\epsilon)}, \frac{2}{3})$ instead of $\rho \in (\frac{1}{3}, \frac{2}{3})$ (as in p. 16, line 6 in [21]). The remaining details are skipped. We now observe that

$$\Re\langle u, (\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)u \rangle = \|u'\|_2^2 - \beta\mu\|u\|_2^2, \quad (5.6)$$

For $\mu \leq -\beta^{-1/3}$, we deduce

$$\|u\|_2 \leq \frac{C}{\beta|\mu|} \|(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)u\|_2 \leq \frac{2C}{\beta^{2/3}(1 + \beta^{1/3}|\mu|)} \|(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)u\|_2,$$

which gives the estimate of the first term on the left-hand-side of (5.4) for $\beta^{1/3}\mu \leq -1$.

To estimate the second term on the right-hand-side of (5.4), we return to (5.6) to conclude from the bound of $\|(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)^{-1}\|$, we have just obtained, that

$$\|u'\|_2 \leq \frac{C}{\beta^{1/3}} \|(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)u\|_2. \quad (5.7)$$

Finally, to prove (5.5) we use the identity

$$\Im\langle(\tilde{U} - \nu)u, (\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)u\rangle = \beta\|(\tilde{U} - \nu)u\|_2^2 + \Im\langle\tilde{U}'u, u'\rangle,$$

to obtain with the aid of (5.4) that

$$\beta\|(\tilde{U} - \nu)u\|_2^2 \leq \|(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)u\|_2 \|(\tilde{U} - \nu)u\|_2 + \frac{C}{\beta} \|(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)u\|_2^2,$$

from which (5.5) easily follows. ■

5.2 A Dirichlet problem

We now obtain some resolvent estimate for the Dirichlet realization $\mathcal{L}_\beta^{\mathfrak{D}}$ of \mathcal{L}_β in $(-1, 1)$.

Proposition 5.2. *For any $r > 1$ and $\Upsilon < J_m^{2/3} \Re\nu_1$, there exist $C > 0$ and $\beta_0 > 0$ such that for all $U \in \mathcal{S}_r^2$ and $\beta \geq \beta_0$*

$$\sup_{\Re\lambda \leq \Upsilon\beta^{-1/3}} \left[\|(\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)^{-1}\| + \beta^{-1/3} \left\| \frac{d}{dx} (\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)^{-1} \right\| + \beta^{-2/3} \left\| \frac{d^2}{dx^2} (\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)^{-1} \right\| \right] \leq \frac{C}{\beta^{2/3}}, \quad (5.8)$$

and

$$\sup_{\Re\lambda \leq \Upsilon\beta^{-1/3}} \|(U - \Im\lambda)(\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)^{-1}\| \leq \frac{C}{\beta}. \quad (5.9)$$

Furthermore, for every $1 < p < 2$ there exists $C_p > 0$ such that for all $f \in L^2(-1, 1)$

$$\sup_{\Re\lambda \leq \Upsilon\beta^{-1/3}} \left\| \frac{d}{dx} (\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)^{-1} f \right\|_p \leq \frac{C_p}{\beta^{\frac{2+p}{6p}}} \|f\|_2. \quad (5.10)$$

Proof. By [21, Theorem 1.1] we have, under the assumptions of the proposition,

$$\|(\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)^{-1}\| \leq \frac{C}{\beta^{2/3}}. \quad (5.11)$$

We next observe that for any $u \in D(\mathcal{L}_\beta^{\mathfrak{D}})$

$$\Re\langle u, (\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)u \rangle = \|u'\|_2^2 - \beta\mu\|u\|_2^2,$$

which together with (5.11) yields

$$\|u'\|_2 \leq \frac{C}{\beta^{1/3}} \|(\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)u\|_2. \quad (5.12)$$

To complete the proof of (5.8) we write, with $\mu = \Re\lambda$,

$$-\Re\langle u'', (\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)u \rangle = \|u''\|_2^2 - \beta\mu\|u'\|_2^2 - \beta\Im\langle U'u, u' \rangle.$$

From (5.11) and (5.12) we then obtain that

$$\|u''\|_2 \leq C \|(\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)u\|_2,$$

completing, thereby, the proof of (5.8). We establish (5.9) in the same way we have established (5.5).

It remains to prove (5.10). Let $f \in L^2(-1, 1)$ and $u \in D(\mathcal{L}_\beta^\mathfrak{D})$ satisfy

$$(\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)u = f. \quad (5.13)$$

Let η and $\tilde{\eta}$ denote C^∞ functions such that

$$\tilde{\eta}(t) = \begin{cases} 0 & |t| < 1/2 \\ 1 & |t| > 1, \end{cases} \quad \text{and } \eta(t) = \sqrt{1 - \tilde{\eta}(t)^2}. \quad (5.14)$$

Let $x_\nu \in [-1, 1]$ be such that $U(x_\nu) = \nu$ (otherwise, if $U(x) \neq \nu$ for all $x \in [-1, 1]$, we arbitrarily set $x_\nu = -2$).

Let

$$\eta_\nu(x) = \tilde{\eta}(\beta^{1/3}(x - x_\nu))\mathbf{1}_{\mathbb{R}_+}(x - x_\nu). \quad (5.15)$$

As

$$\begin{aligned} & \Re\langle \eta_\nu^2(U - \nu)u, (\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)u \rangle \\ &= \|\eta_\nu|U - \nu|^{1/2}u'\|_2^2 + \Re\langle \eta_\nu(U'\eta_\nu + 2(U - \nu)\eta'_\nu)u, u' \rangle - \mu\beta\|\eta_\nu|U - \nu|^{1/2}u\|_2^2, \end{aligned}$$

we obtain, observing that $|(U - \nu)\eta'_\nu|$ is uniformly bounded,

$$\begin{aligned} \|\eta_\nu|U - \nu|^{1/2}u'\|_2^2 &\leq \|(U - \nu)u\|_2\|f\|_2 + \\ &C(\beta^{2/3}\|\eta_\nu|U - \nu|^{1/2}u\|_2^2 + \|u\|_2\|u'\|_2). \end{aligned} \quad (5.16)$$

Furthermore, since

$$\Im\langle \eta_\nu^2 u, (\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)u \rangle = \beta\|\eta_\nu|U - \nu|^{1/2}u\|_2^2 + 2\Im\langle \eta'_\nu u, \eta_\nu u' \rangle, \quad (5.17)$$

we obtain

$$\|\eta_\nu|U - \nu|^{1/2}u\|_2^2 \leq \frac{1}{\beta}(\|u\|_2\|f\|_2 + \beta^{1/3}\|u\|_2\|u'\|_2),$$

and hence, by (5.8),

$$\|\eta_\nu|U - \nu|^{1/2}u\|_2 \leq \frac{C}{\beta^{5/6}}\|f\|_2.$$

Substituting the above into (5.16) yields, with the aid of (5.8) and (5.9),

$$\|\eta_\nu|U - \nu|^{1/2}u'\|_2 \leq \frac{C}{\beta^{1/2}}\|f\|_2.$$

Setting $\eta_\nu^-(x) = \eta_\nu(-x)$, we obtain in a similar manner

$$\|\eta_\nu^-|U - \nu|^{1/2}u'\|_2 \leq \frac{C}{\beta^{1/2}}\|f\|_2,$$

and hence, with the aid of (5.8), we can conclude that

$$\begin{aligned} \| |U + i\lambda|^{1/2}u'\|_2 &\leq \|\eta_\nu|U - \nu|^{1/2}u'\|_2 + \|\eta_\nu^-|U - \nu|^{1/2}u'\|_2 + C\beta^{-1/6}\|u'\|_2 \\ &\leq \hat{C}\beta^{-1/2}\|f\|_2. \end{aligned}$$

We can now conclude, assuming first $|\mu| \geq \Upsilon\beta^{-1/3}/2$, with the aid (4.36),

$$\|u'\|_p \leq \| |U + i\lambda|^{1/2}u'\|_2 \| |U + i\lambda|^{-1/2}\|_{\frac{2p}{2-p}} \leq C\beta^{-\frac{2+p}{6p}}\|f\|_2. \quad (5.18)$$

Otherwise, if $|\mu| < \Upsilon\beta^{-1/3}/2$, we rewrite (5.13) in the form

$$(\mathcal{L}_\beta^\mathfrak{D} - \beta[\Upsilon\beta^{-1/3} + i\nu])u = f - (\Upsilon\beta^{-\frac{1}{3}} - \mu)\beta u$$

to obtain from (5.18) that

$$\|u'\|_p \leq C\beta^{-\frac{2+p}{6p}}(\|f\|_2 + \beta^{2/3}\|u\|_2).$$

The proof of (5.10) can now be completed using (5.8). \blacksquare

We can now deduce the following corollary

Corollary 5.3. *For any $r > 1$ and $\Upsilon < J_m^{2/3}\mathfrak{R}\nu_1$, there exist $C > 0$ and $\beta_0 > 0$ such that for all $U \in \mathcal{S}_r^2$, $\beta \geq \beta_0$, $2 < p \leq \infty$, and $f \in L^2(-1, 1)$ it holds that*

$$\sup_{\Re\lambda \leq \Upsilon\beta^{-1/3}} \|(\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)^{-1}f\|_p \leq C\beta^{-\frac{3p+2}{6p}}\|f\|_2. \quad (5.19)$$

Proof. Let $f \in L^2(-1, 1)$ and set $v = (\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)^{-1}f$. By (5.8), there exist $C > 0$ and β_0 such that, for $\beta \geq \beta_0$

$$\|v\|_2 \leq \frac{C}{\beta^{2/3}}\|f\|_2, \quad \text{and} \quad \|v'\|_2 \leq C\beta^{-1/3}\|f\|_2. \quad (5.20)$$

Consequently, using the interpolation inequality

$$\|v\|_p \leq \|v\|_2^{\frac{2}{p}} \|v\|_\infty^{1-\frac{2}{p}},$$

and the Sobolev embedding

$$\|v\|_\infty \leq \sqrt{2} \|v\|_2^{\frac{1}{2}} \|v'\|_2^{\frac{1}{2}},$$

we can conclude that

$$\|v\|_p \leq 2^{\frac{p-2}{2p}} \|v\|_2^{\frac{p+2}{2p}} \|v'\|_2^{\frac{p-2}{2p}} \leq C\beta^{-\frac{3p+2}{6p}}\|f\|_2.$$

\blacksquare

For later reference we also need the following refined estimate.

Proposition 5.4. *For any $r > 1$ and $\Upsilon < \Re\nu_1$, , there exist $C > 0$ and $\beta_0 > 0$ such that, for all $U \in \mathcal{S}_r$, $\beta \geq \beta_0$, and $f \in L^\infty(-1, 1)$,*

$$\sup_{\Re\lambda \leq \Upsilon J_m^{2/3} \beta^{-1/3}} \|(\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)^{-1}f\|_2 \leq \frac{C}{\beta^{5/6}} \|f\|_\infty. \quad (5.21)$$

Proof. Let η be given by (5.14) and for any $x \in [-1, 1]$,

$$\eta_x(t) = \eta(\beta^{1/3}(t-x))\mathbf{1}_{[-1,1]}(t).$$

Let further

$$\eta_{x,2}(t) = \eta(\beta^{1/3}(t-x)/2)\mathbf{1}_{[-1,1]}(t),$$

implying that

$$\eta_x \eta_{x,2} = \eta_x.$$

Step 1: We prove that, for any $\nu \in [U(-1), U(1)]$, there exist $C > 0$ and β_0 such that, for all $\beta \geq \beta_0$, $x \in [-1, 1]$, $f \in L^\infty(-1, 1)$, $\Re\lambda \leq \Upsilon\beta^{-\frac{1}{3}}$,

$$\|(U - \nu)\eta_x v\|_2^2 \leq \frac{C}{\beta^2} \|\eta_x f\|_2^2 + C \frac{\beta^{1/3}|x - x_\nu| + 1}{\beta^{2/3}} \|\eta_{x,2} v\|_2^2, \quad (5.22)$$

where $x_\nu \in [-1, +1]$ is chosen so that $U(x_\nu) = \nu$ and

$$v = (\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)^{-1}f. \quad (5.23)$$

Clearly,

$$\Im \langle \eta_x^2 (U - \nu)v, (\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)v \rangle = \beta \|(U - \nu)\eta_x v\|_2^2 + \Im \langle [\eta_x^2 (U - \nu)]'v, v \rangle,$$

which implies

$$\|(U - \nu)\eta_x v\|_2^2 \leq \frac{C}{\beta^2} \|\eta_x f\|_2^2 + \frac{C}{\beta} \left(\sup_t |\eta_x'(t)(U(t) - \nu)| \right) \|\eta_x v'\|_2 \|\eta_{x,2} v\|_2. \quad (5.24)$$

To estimate $\|\eta_x v'\|_2$ we use the identity

$$\Re \langle \eta_x^2 v, (\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)v \rangle = \|(\eta_x v)'\|_2^2 - \|\eta_x' v\|_2^2 - \mu\beta \|\eta_x v\|_2^2, \quad (5.25)$$

from which we easily conclude, using the fact that $\beta^{\frac{1}{3}}\mu$ is bounded by assumption, that

$$\|\eta_x v'\|_2^2 \leq C(\beta^{-2/3} \|\eta_x f\|_2^2 + \beta^{2/3} \|\eta_{x,2} v\|_2^2). \quad (5.26)$$

Finally we note that

$$C^{-1}|x - x_\nu| \leq |U(x) - \nu| \leq C|x - x_\nu|. \quad (5.27)$$

Suppose first that $|x - x_\nu| < 3\beta^{-1/3}$.

Since by (5.27) it holds that $(\sup_t |\eta_x'(t)(U(t) - \nu)|) \leq C$ we may use (5.24) to obtain

$$\|(U - \nu)\eta_x v\|_2^2 \leq \frac{C}{\beta^2} \|\eta_x f\|_2^2 + \frac{C}{\beta} \|\eta_x v'\|_2 \|\eta_{x,2} v\|_2.$$

which together with (5.26) yields (5.22).

Suppose now that $|x - x_\nu| \geq 3\beta^{-1/3}$.

This time we have by (5.27) that $(\sup_t |\eta'_x(t)(U(t) - \nu)|) \leq C\beta^{1/3}|x - x_\nu|$, and hence we get from (5.24)

$$\|(U - \nu)\eta_x v\|_2^2 \leq \frac{C}{\beta^2} \|\eta_x f\|_2^2 + C \frac{|x - x_\nu|}{\beta^{2/3}} \|\eta_{x,2} v\|_2 \|\eta_x v'\|_2. \quad (5.28)$$

To estimate the last term on the right-hand-side of (5.28), we use (5.25) once again and get instead of (5.26)

$$\|\eta_x v'\|_2^2 \leq C \left(\beta^{2/3} \|\eta_{x,2} v\|_2^2 + \|(U - \nu)\eta_x v\|_2 \left\| \eta_x \frac{f}{U - \nu} \right\|_2 \right), \quad (5.29)$$

or, alternatively, for any $\delta > 0$,

$$\|\eta_x v'\|_2 \leq C \left(\beta^{1/3} \|\eta_{x,2} v\|_2 + \delta \frac{\beta^{1/2}}{|x - x_\nu|^{1/2}} \|(U - \nu)\eta_x v\|_2 + \frac{|x - x_\nu|^{1/2}}{\delta \beta^{1/2}} \left\| \eta_x \frac{f}{U - \nu} \right\|_2 \right).$$

Substituting the above into (5.28) then yields

$$\begin{aligned} \|(U - \nu)\eta_x v\|_2^2 &\leq C \left(\frac{1}{\beta^2} \|\eta_x f\|_2^2 + \frac{|x - x_\nu|}{\beta^{1/3}} \|\eta_{x,2} v\|_2^2 + \right. \\ &\quad \left. + \frac{|x - x_\nu|^{3/2}}{\delta \beta^{7/6}} \|\eta_{x,2} v\|_2 \left\| \eta_x \frac{f}{U - \nu} \right\|_2 + \frac{\delta |x - x_\nu|^{1/2}}{\beta^{1/6}} \|\eta_{x,2} v\|_2 \|(U - \nu)\eta_x v\|_2 \right). \end{aligned}$$

Observing that

$$\left\| \eta_x \frac{f}{U - \nu} \right\|_2 \leq C |x - x_\nu|^{-1} \|\eta_x f\|_2,$$

yields (5.22) by choosing a sufficiently small value of δ .

To proceed to the next step, we need to define, yet, two additional β dependent cutoff functions. Let then for $s \geq 2$, η_s and $\tilde{\eta}_s$ in $C^\infty(\mathbb{R}, [0, 1])$ satisfy

$$\eta_s(t) = \begin{cases} 1 & |t - x_\nu| \leq s\beta^{-1/3} \\ 0 & |t - x_\nu| \geq (s+1)\beta^{-1/3} \end{cases} \quad \text{and} \quad \tilde{\eta}_s = \sqrt{1 - \eta_s^2}.$$

We further require that there exists C and β_0 such that for any $s \geq 2$ and $\beta \geq \beta_0$

$$\|\eta'_s\|_\infty \leq C\beta^{1/3} \quad ; \quad \|\eta''_s\|_\infty \leq C\beta^{2/3}.$$

Step 2: We prove that there exist $s_0 > 0$, and $C > 0$ such that, for all $s \geq s_0$, there exists β_s such that $\beta \geq \beta_s$

$$\|\tilde{\eta}_s v\|_2^2 \leq C(\beta^{-5/3} \|f\|_\infty^2 + s^{-1} \|v\|_2^2), \quad (5.30)$$

for any pair (f, v) satisfying (5.23).

By (5.22) we have

$$\|\eta_x v\|_2^2 \leq \frac{C}{|x - x_\nu|^2 \beta^2} \|\eta_x f\|_2^2 + \frac{C}{|x - x_\nu| \beta^{1/3}} \|\eta_{x,2} v\|_2^2. \quad (5.31)$$

We now integrate the above inequality with respect to x over $(-1, x_\nu - s\beta^{-1/3}/2) \cup (x_\nu + s\beta^{-1/3}/2, 1)$. By changing the order of integration we obtain that for all $s > 4$

$$\begin{aligned} & \int_{s\beta^{-1/3}/2 < |x-x_\nu|} \mathbf{1}_{[-1,1]}(x) \|\eta_x v\|_2^2 dx \\ &= \int_{-1,1} |v|^2(t) dt \int_{s\beta^{-1/3}/2 < |x-x_\nu|} \mathbf{1}_{[-1,1]}(x) \eta^2(\beta^{1/3}(x-t)) dx \\ &\geq \int_{-1,1} |\tilde{\eta}_s v|^2(t) dt \int_{s\beta^{-1/3}/2 < |x-x_\nu|} \mathbf{1}_{[-1,1]}(x) \eta^2(\beta^{1/3}(x-t)) dx \\ &\geq \beta^{-1/3} \|\tilde{\eta}_s v\|_2^2. \end{aligned}$$

Note that

$$\int_{s\beta^{-1/3}/2 < |x-x_\nu|} \mathbf{1}_{[-1,1]}(x) \eta^2(\beta^{1/3}(x-t)) dx = \beta^{-\frac{1}{3}} \left(\int_{s/2 < |\tau + \beta^{1/3}(t-x_\nu)|} \eta^2(\tau) d\tau \right) \geq \beta^{-\frac{1}{3}},$$

for any t in the support of $\tilde{\eta}_s$. We rewrite the above in the form

$$\|\tilde{\eta}_s v\|_2^2 \leq \beta^{\frac{1}{3}} \int_{s\beta^{-1/3}/2 < |x-x_\nu|} \mathbf{1}_{[-1,1]}(x) \|\eta_x v\|_2^2 dx. \quad (5.32)$$

As $\|\eta_x f\|_2^2 \leq 2\beta^{-1/3} \|f\|_\infty^2$ we have

$$\begin{aligned} & \int_{s\beta^{-1/3}/2 < |x-x_\nu|} \mathbf{1}_{[-1,1]}(x) \frac{1}{|x-x_\nu|^2 \beta^2} \|\eta_x f\|_2^2 dx \\ &\leq C \beta^{-7/3} \|f\|_\infty^2 \int_{s\beta^{-1/3}/2 < |x-x_\nu|} \mathbf{1}_{[-1,1]}(x) \frac{1}{|x-x_\nu|^2} dx \\ &\leq \hat{C} \beta^{-2} \|f\|_\infty^2. \end{aligned} \quad (5.33)$$

Finally, we have for all $s \geq 16$,

$$\begin{aligned} & \int_{s\beta^{-1/3}/2 < |x-x_\nu|} \mathbf{1}_{[-1,1]}(x) \frac{1}{|x-x_\nu|^{\beta^{1/3}}} \|\eta_{x,2} v\|_2^2 dx \\ &\leq \int_{-1,1} |v|^2(t) dt \int_{s\beta^{-1/3}/2 < |x-x_\nu|} \mathbf{1}_{[-1,1]}(x) \frac{\eta^2(\beta^{1/3}(x-t)/2)}{|x-x_\nu|^{\beta^{1/3}}} dx \\ &\leq \beta^{-1/3} \int_{s\beta^{-1/3}/4 < |t-x_\nu|} \mathbf{1}_{[-1,1]}(t) \log \frac{|t-x_\nu| + 2\beta^{-1/3}}{|t-x_\nu| - 2\beta^{-1/3}} |v(t)|^2 dt \\ &\leq \frac{C\beta^{-1/3}}{s} \|v\|_2^2. \end{aligned}$$

Note that $\eta(\beta^{1/3}(x-t)/2)$ vanishes for $s\beta^{-1/3}/2 < |x-x_\nu|$ (for $r \geq 16$) and $|t-x_\nu| < s\beta^{-1/3}/4$. Note further that

$$0 < \log \frac{|t-x_\nu| + 2\beta^{-1/3}}{|t-x_\nu| - 2\beta^{-1/3}} \leq \log \frac{1 + 8/s}{1 - 8/s} \leq \frac{C}{s}.$$

Combining the above with (5.33), (5.32), and (5.31) easily yields (5.30).

Step 3: We now prove (5.21).

Writing

$$(\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)(\eta_s v) = \eta_s f + 2\eta'_s v' + \eta''_s v,$$

we obtain from (5.8) and the definitions and properties of the cut-off functions η_s and η_x ,

$$\|\eta_s v\|_2^2 \leq C(\beta^{-4/3} \|\eta_s f\|_2^2 + \beta^{-2/3} \|\eta_{x_s} v'\|_2^2 + \|\tilde{\eta}_{s/2} v\|_2^2), \quad (5.34)$$

where $x_s = x_\nu + \frac{s+1}{2}\beta^{-1/3}$.

We now use (5.29) (with $x = x_s$) together with (5.22) to obtain that

$$\begin{aligned} \|\eta_{x_s} v'\|_2^2 &\leq C \left(\beta^{2/3} \|\eta_{x_s,2} v\|_2^2 + \|(U - \nu)\eta_{x_s} v\|_2 \left\| \eta_{x_s} \frac{f}{U - \nu} \right\|_2 \right) \\ &\leq \hat{C} \left(\beta^{2/3} \|\eta_{x_s,2} v\|_2^2 + s^{-1} \beta^{1/3} [\beta^{-1} \|\eta_{x_s} f\|_2 + s\beta^{-1/3} \|\eta_{x_s,2} v\|_2] \|\eta_{x_s} f\|_2 \right), \end{aligned}$$

from which we can conclude, using the inequality $\tilde{\eta}_{s/2} \geq \eta_{x_s,2}$ for $s > 16$, that

$$\|\eta_{x_s} v'\|_2^2 \leq C(\beta^{2/3} \|\tilde{\eta}_{r/2} v\|_2^2 + \beta^{-2/3} \|\eta_{x_s} f\|_2^2).$$

Substituting the above into (5.34), yields that, for some $C > 0$, it holds:

$$\|\eta_s v\|_2^2 \leq C_s(\beta^{-5/3} \|f\|_\infty^2 + \|\tilde{\eta}_{s/2} v\|_2^2).$$

Using (5.30) (used with r replaced by $\frac{s}{2}$ and for s large enough) then yields

$$\|\eta_s v\|_2^2 \leq C(\beta^{-5/3} \|f\|_\infty^2 + s^{-1} \|v\|_2^2).$$

Combining the above with again (5.30) yields (5.21), by choosing a sufficiently large value of s . ■

5.3 L^1 estimates

In this subsection we establish new L^1 estimates for the resolvent of $\tilde{\mathcal{L}}_{\beta,\mathbb{R}}$, defined by (5.1) and (5.2). We first observe that the proof of Proposition 5.4 can be applied to the entire real line case, replacing the estimates of resolvent for the Dirichlet problem $(\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)^{-1}$ by the corresponding estimates of the resolvent $(\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\lambda)^{-1}$ for $\Re\lambda \leq \beta^{-\frac{1}{3}}\Upsilon$. Hence, we may state the following

Lemma 5.5. *For any $r > 1$, any $\Upsilon > 0$ and $a > 0$ there exists $C > 0$ and $\beta_0 > 0$ such that, for all $U \in \mathcal{S}_r$, $g \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\beta \geq \beta_0$,*

$$\sup_{\Re\lambda \leq \Upsilon\beta^{-1/3}} \|(\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\lambda)^{-1} g\|_{L^2(-a,a)} \leq \frac{C}{\beta^{5/6}} \|g\|_\infty. \quad (5.35)$$

We continue this subsection with the following L^1 estimate

Lemma 5.6. *For any $r > 1$ and $\Upsilon > 0$, there exist $C > 0$ and $\beta_0 > 0$ such that, for all $\beta \geq \beta_0$, $U \in \mathcal{S}_r^2$, $\Re\lambda \leq \Upsilon J_m^{2/3} \beta^{-1/3}$, and any $g \in L^\infty(\mathbb{R})$ supported in $[-1, 1]$, we have*

$$\|(\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\lambda)^{-1} g\|_{L^1(-1,1)} \leq C \min(\beta^{-5/6} \|g\|_2, \beta^{-1} \log \beta \|g\|_\infty). \quad (5.36)$$

Proof. Let Υ denote a fixed positive value and $\Re\lambda \leq \Upsilon\beta^{-1/3}$. Let further $g \in L^\infty(\mathbb{R})$, supported on $[-1, 1]$, and $v \in D(\tilde{\mathcal{L}}_{\beta,\mathbb{R}})$ satisfy

$$(\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\lambda)v = g. \quad (5.37)$$

By (5.4)-(5.5) and (4.37), we have

$$\|v\|_{L^1(-1,1)} \leq \|(\tilde{U} - \nu + i\beta^{-1/3})^{-1}\|_{L^2(-1,+1)} \|(\tilde{U} - \nu + i\beta^{-1/3})v\|_2 \leq C\beta^{-5/6} \|g\|_2,$$

which proves the first inequality of (5.36).

Let $x_\nu \in \mathbb{R}$ satisfy $\tilde{U}(x_\nu) = \nu$. Let η and η_ν be given by (5.14) and (5.15) and

$$\zeta_\nu(x) = \eta_\nu(x) \eta(|x - x_\nu|/6).$$

Taking the inner product of (5.37) with $\zeta_\nu^2 v$ yields for the imaginary part (see (5.17) with η_ν instead of ζ_ν)

$$\beta \|\zeta_\nu |\tilde{U} - \nu|^{1/2} v\|_2^2 + 2\Im \langle \zeta'_\nu v, (\zeta_\nu v)' \rangle = \langle \zeta_\nu v, \zeta_\nu g \rangle. \quad (5.38)$$

By (5.35) we have that

$$\|\zeta'_\nu v\|_2 \leq \frac{C}{\beta^{1/2}} \|g\|_\infty. \quad (5.39)$$

To estimate $\|(\zeta_\nu v)'\|_2$ we use the identity

$$\|(\zeta_\nu v)'\|_2^2 - \Re \lambda \beta \|\zeta_\nu v\|_2^2 - \|\zeta'_\nu v\|_2^2 = \Re \langle \zeta_\nu v, \zeta_\nu g \rangle,$$

to obtain from (5.4), (5.35), and (5.39)

$$\|(\zeta_\nu v)'\|_2^2 \leq \frac{C}{\beta} \|g\|_\infty^2 + \|\zeta_\nu v\|_1 \|g\|_\infty.$$

Substituting the above into (5.38) yields

$$\|\zeta_\nu |\tilde{U} - \nu|^{1/2} v\|_2 \leq C \left(\frac{1}{\beta} \|g\|_\infty + \beta^{-1/2} \|\zeta_\nu v\|_1^{1/2} \|g\|_\infty^{1/2} \right). \quad (5.40)$$

As

$$\begin{aligned} \|\zeta_\nu v\|_1 &\leq \|\zeta_\nu |\tilde{U} - \nu|^{1/2} v\|_2 \mathbf{1}_{(x_\nu + \frac{1}{2}\beta^{-1/3}, x_\nu + 6)} |\tilde{U} - \nu|^{-1/2} \|v\|_2 \\ &\leq C (\log \beta)^{\frac{1}{2}} \|\zeta_\nu |\tilde{U} - \nu|^{1/2} v\|_2, \end{aligned} \quad (5.41)$$

we obtain from (5.40) that

$$\|\zeta_\nu |\tilde{U} - \nu|^{1/2} v\|_2 \leq \tilde{C} \frac{(\log \beta)^{\frac{1}{2}}}{\beta} \|g\|_\infty.$$

Then, using (5.41) once again yields

$$\|\zeta_\nu v\|_1 \leq \tilde{C} \frac{\log \beta}{\beta} \|g\|_\infty,$$

from which we easily conclude that

$$\|\mathbf{1}_{(x_\nu, x_\nu + 3)} v\|_1 \leq \|\zeta_\nu v\|_1 + C \beta^{-1/6} \|\mathbf{1}_{(x_\nu, x_\nu + \beta^{-1/3})} v\|_2 \leq \check{C} \frac{\log \beta}{\beta} \|g\|_\infty.$$

In a similar manner we obtain that

$$\|\mathbf{1}_{(x_\nu - 3, x_\nu)} v\|_1 \leq \check{C} \frac{\log \beta}{\beta} \|g\|_\infty.$$

This proves (5.36) in the case where $(-1, +1) \subset (x_\nu - 3, x_\nu + 3)$.

It remains to prove (5.36) in the case $|x_\nu| \geq 2$. By (5.5) and the fact that g is supported on $[-1, 1]$, we have that

$$\|v\|_{L^1(-1,1)} \leq \sqrt{2} \|v\|_{L^2(-1,1)} \leq C \|(\tilde{U} - \nu)v\|_{L^2(-1,1)} \leq \frac{C}{\beta} \|g\|_2 \leq \frac{C}{\beta} \|g\|_\infty.$$

The lemma is proved \blacksquare

A similar statement can be proved in the Dirichlet case.

Lemma 5.7. *For any $r > 1$ and $\Upsilon \in (0, \Re\nu_1)$, there exist $C > 0$ and $\beta_0 > 0$ such that, for all $\beta \geq \beta_0$, $U \in \mathcal{S}_r^2$ and $\Re\lambda \leq J_m^{2/3} \Upsilon \beta^{-1/3}$, and for any $g \in L^\infty(-1, 1)$ we have*

$$\|(\mathcal{L}_\beta^\mathfrak{D} - \beta\lambda)^{-1}g\|_1 \leq C \min(\beta^{-5/6}\|g\|_2, \beta^{-1} \log \beta \|g\|_\infty). \quad (5.42)$$

The proof is similar to the proof of Lemma 5.6 and is therefore skipped.

6 No-slip resolvent estimates

6.1 A no-slip Schrödinger operator

We begin by providing a short explanation of the difficulties arising when the no-slip boundary condition (3.10) is prescribed. Complete details will be given in Section 8.

In the zero-traction case, estimating $\phi \in D(\mathcal{B}_{\lambda,\alpha,\beta}^\mathfrak{E})$ satisfying $\mathcal{B}_{\lambda,\alpha,\beta}^\mathfrak{E} \phi = f$ for some $f \in L^2(-1, 1)$, we may write, by (3.5),

$$\left(-\frac{d^2}{dx^2} + i\beta U - \beta\lambda\right)(-\phi'' + \alpha^2\phi) = i\beta U''\phi + f.$$

Since by (3.9), it holds that $-\phi'' + \alpha^2\phi$ satisfies a Dirichlet condition at ± 1 , one can now use, for instance, (5.8) and (5.21) to obtain

$$\|-\phi'' + \alpha^2\phi\|_2 \leq C\beta^{1/6}\|\phi\|_\infty + \beta^{-2/3}\|f\|_2.$$

Such an estimate is particularly useful in the case $\alpha \gg \beta^{1/6}$, but also in other cases (detailed in Section 8).

Similar estimates can be obtained for $v = \mathcal{A}_{\lambda,\alpha}\phi$ and $\tilde{v} = (U + i\lambda)^{-1}v$.

If we now consider the same problem in the no-slip case the above approach is inapplicable. Thus, for $\phi \in D(\mathcal{B}_{\lambda,\alpha,\beta}^\mathfrak{D})$ satisfying $\mathcal{B}_{\lambda,\alpha,\beta}^\mathfrak{D} \phi = f$, we can no longer use neither (5.8) nor (5.21), as $-\phi'' + \alpha^2\phi$ does not satisfy a Dirichlet condition at $x = \pm 1$. However, integration by parts easily yields that for all $\phi \in D(\mathcal{B}_{\lambda,\alpha,\beta}^\mathfrak{D})$ we have

$$\langle e^{\pm\alpha x}, -\phi'' + \alpha^2\phi \rangle = 0.$$

If we consider v or \tilde{v} instead of $-\phi'' + \alpha^2\phi$ we can still obtain similar orthogonality conditions (see (8.22) and (8.32c)). These conditions read

$$\langle v, \zeta_\pm \rangle_{L^2(-1,+1)} = 0, \quad (6.1)$$

where ζ_- and ζ_+ are linearly independent, β dependent, and belong to $H^1(-1, +1)$.

With (3.5) in mind, we let \mathcal{L}_β^ζ be the differential operator $-d^2/dx^2 + i\beta U$ with domain

$$D(\mathcal{L}_\beta^\zeta) = \{u \in H^2(-1, 1) \mid \langle \zeta_\pm, u \rangle = 0\}. \quad (6.2)$$

For convenience we require that ζ_\pm satisfy

$$\begin{bmatrix} \zeta_+(1) & \zeta_-(-1) \\ \zeta_+(-1) & \zeta_-(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (6.3)$$

Note that $\zeta_{\pm} = e^{\pm\alpha x}$ do not satisfy the above requirement, and we shall therefore need to replace them by a pair of proper linear combinations of them [34] (a more detailed explanation is brought in Section 8). We seek resolvent estimates for $\mathcal{L}_{\beta}^{\zeta}$ in the following. In the absence of a Dirichlet boundary condition, it seems reasonable to approximate the solution of

$$(\mathcal{L}_{\beta}^{\zeta} - \beta\lambda)v = g, \quad (6.4)$$

by a sum of a solution in \mathbb{R} of the inhomogeneous equation and a linear combination of two independent approximate solutions of the homogeneous equation whose coefficients will be determined by the above integral conditions. Using affine approximations of U in $(-1, +1)$ or extensions outside of $(-1, +1)$, the approximate solutions can be described by a pair of dilated and translated Airy functions in $(-1, +\infty)$ and $(-\infty, +1)$.

The solution in \mathbb{R} .

We now explain our construction of an approximate inverse in \mathbb{R} by defining first a natural C^1 -extension \tilde{U} of U outside of $[-1, +1]$, satisfying (5.3), by

$$\tilde{U}(x) = \begin{cases} U(x) & x \in [-1, 1] \\ U(1) + U'(1)(x - 1) & x > 1 \\ U(-1) + U'(-1)(x + 1) & x < -1. \end{cases}$$

We note that \tilde{U} satisfies the conditions of Proposition 5.1. We also extend g by

$$\tilde{g}(x) = \begin{cases} g(x) & x \in [-1, 1] \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$u = \Gamma_{(-1,1)}(\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\lambda)^{-1}\tilde{g}. \quad (6.5)$$

where $\tilde{\mathcal{L}}_{\beta,\mathbb{R}}$ is defined by (5.1) and (5.2) and $\Gamma_{(-1,1)}$ denotes the restriction to $(-1, 1)$.

Boundary terms.

To obtain the boundaries effect, we replace $U(x)$ by its affine approximation at ± 1 and consider the $L^2 \cap L^1$ solutions ψ_{\pm} of the approximate problems

$$\begin{cases} (-d^2/dx^2 + i\beta[U(-1) + J_-(x + 1)] - \beta\lambda)\psi_- = 0 \text{ in } (-1, +\infty) \\ \int_{-1}^{+\infty} \psi_-(x) dx = (J_-\beta)^{-1/3}, \end{cases} \quad (6.6)$$

and

$$\begin{cases} (-d^2/dx^2 + i\beta[U(1) + J_+(x - 1)] - \beta\lambda)\psi_+ = 0 \text{ in } (-\infty, 1) \\ \int_{-\infty}^1 \psi_+(x) dx = (J_+\beta)^{-1/3}, \end{cases} \quad (6.7)$$

with

$$J_{\pm} = U'(\pm 1).$$

Except, perhaps for some particular values of λ , the above solutions are unique, and ψ_{\pm} rapidly decays as $x \rightarrow \mp\infty$, but their existence (due to the additional integral condition) could depend on $(\beta, \lambda, J_{\pm})$ as is clarified below. We express ψ_{\pm}

using Airy functions. Having in mind the definition of the generalized Airy functions [35, eq. (39)] or [30, Lemma 2] (for more details see [12, Appendix] or our short review in Appendix A). These solutions are given, assuming that the denominator does not vanish, by

$$\psi_-(x) = e^{i\pi/6} \frac{\text{Ai}((J_-\beta)^{1/3} e^{i\pi/6} [(1+x) + iJ_-^{-1}(\lambda - iU(-1))])}{A_0(i\beta^{1/3} J_-^{-2/3} [\lambda - iU(-1)])}, \quad (6.8a)$$

and

$$\overline{\psi_+(x)} = -e^{i\pi/6} \frac{\text{Ai}((J_+\beta)^{1/3} e^{i\pi/6} [(1-x) + iJ_+^{-1}(\bar{\lambda} + iU(1))])}{A_0(i\beta^{1/3} J_+^{-2/3} [\bar{\lambda} + iU(1)])}. \quad (6.8b)$$

where A_0 is the holomorphic extension to \mathbb{C} of

$$x \mapsto A_0(x) = e^{i\pi/6} \int_x^{+\infty} \text{Ai}(e^{i\pi/6} t) dt. \quad (6.9)$$

Much of the properties of A_0 are recalled (mainly from Wasow's paper [35]) in Appendix A. It has been established in [35] (see also Appendix A) that The zeroes of $z \mapsto A_0(iz)$ are located in the sector $\arg z \in (\frac{\pi}{6}, \frac{\pi}{2})$. Let

$$\mathcal{S}_\lambda = \{ z \mid A_0(iz) = 0 \},$$

and further define

$$\vartheta_1^r := \inf_{z \in \mathcal{S}_\lambda} \Re z. \quad (6.10)$$

In addition, we prove in Appendix A.2 that $\mathcal{S}_\lambda \neq \emptyset$ and (relying on [35]) that $\vartheta_1^r > 0$. It follows that the denominators in (6.8b) and (6.8a) do not vanish, if

$$\Re \lambda < \vartheta_1^r \beta^{-1/3} J_m^{2/3},$$

where J_m is given by (2.25c).

The functions ψ_\pm are not exact solutions of $(-d^2/dx^2 + i\beta U - \beta\lambda)\psi = 0$ and hence we must introduce a correction term. We thus consider

$$g_\pm = \begin{cases} (-\frac{d^2}{dx^2} + i\beta(U + i\lambda))\psi_\pm & \text{for } x \in (-1, 1) \\ 0 & \text{otherwise,} \end{cases} \quad (6.11)$$

and then introduce

$$\tilde{v}_\pm = \Gamma_{(-1,1)}(\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\lambda)^{-1} g_\pm. \quad (6.12)$$

This correction term can be estimated as follows

Lemma 6.1. *For any $r > 1$ and $\Upsilon < \vartheta_1^r$, there exist C and β_0 such that, for all $U \in \mathcal{S}_r^2$, $\lambda \in \mathbb{C}$ satisfying*

$$\beta^{1/3} \Re \lambda \leq \Upsilon J_m^{2/3}, \quad (6.13)$$

and $\beta \geq \beta_0$, we have

$$\|(U + i\lambda)\tilde{v}_\pm\|_2 + \beta^{-1/3} \|\tilde{v}_\pm\|_2 \leq C\beta^{-5/6}. \quad (6.14)$$

Proof.

A simple computation shows that:

$$g_{\pm} = i\beta [U - U(\pm 1) - J_{\pm}(x \mp 1)] \psi_{\pm} \text{ in } (-1, 1).$$

Let

$$\lambda_{\pm} = \mu - i(U(\pm 1) - \nu). \quad (6.15)$$

We note that

$$\psi_{\pm}(x) = e^{i\frac{\pi}{6}} \Psi_{(J_{\pm}\beta)^{\frac{1}{3}} J_{\pm}^{-1} \lambda_{\pm}}((J_{\pm}\beta)^{\frac{1}{3}}(1 \mp x)) \quad (6.16)$$

where Ψ_{λ} is defined in the appendix (see (A.42)).

Using translation, dilation, and (A.43a), we can conclude that, under the assumptions of Lemma 6.1, it holds, for $k \in [0, 4]$, that

$$\|(1 \mp x)^k \psi_{\pm}\|_2 \leq C [1 + |\lambda_{\pm}| \beta^{1/3}]^{\frac{1-2k}{4}} \beta^{-(1+2k)/6}. \quad (6.17)$$

Hence, as $|g_{\pm}(x)| \leq C\beta(1 \mp x)^2 |\psi_{\pm}(x)|$, we have

$$\|g_{\pm}\|_2 \leq C \beta^{1/6} [1 + |\lambda_{\pm}| \beta^{1/3}]^{-3/4}. \quad (6.18)$$

By (5.4) and (5.5) we have

$$\|(U + i\lambda)\tilde{v}_{\pm}\|_2 + \beta^{-1/3} \|\tilde{v}_{\pm}\|_2 \leq C\beta^{-5/6} [1 + |\lambda_{\pm}| \beta^{1/3}]^{-3/4}, \quad (6.19)$$

establishing thereby (6.14).
■

We are now ready for introducing a solution of (6.4) in the form

$$v = A_+(\psi_+ - \tilde{v}_+) + A_-(\psi_- - \tilde{v}_-) + u. \quad (6.20)$$

We observe that $(\mathcal{L}_{\beta} - \beta\lambda)v = g$ for any pair (A_-, A_+) . Therefore, one can attempt to find two linear forms $g \mapsto A_-(g)$ and $g \mapsto A_+(g)$ such that v belongs to the domain of $\mathcal{L}_{\beta}^{\zeta}$, hence is the solution of (6.4). This is the object of the next lemma.

Lemma 6.2. *Let $\theta > 0$ and $C_{\zeta} > 0$, and suppose that $\zeta_-, \zeta_+ \in H^1(-1, 1)$ satisfy the conditions (6.3),*

$$\|\zeta_{\pm}\|_{\infty} \leq C_{\zeta}, \quad (6.21)$$

and

$$\|\zeta'_{\pm}\|_2 \leq \theta \beta^{1/6}. \quad (6.22)$$

Let further $r > 1$ and $\Upsilon < \vartheta_1^r$. Then, there exist $\beta_0 > 0$ and $\theta_0 > 0$ such that for all $\beta \geq \beta_0$, $0 < \theta \leq \theta_0$, $U \in \mathcal{S}_r^2$ and $\lambda \in \mathbb{C}$ satisfying

$$\beta^{1/3} \Re \lambda \leq J_m^{2/3} \Upsilon, \quad (6.23)$$

(6.4) and (6.20) hold true with $v \in D(\mathcal{L}_{\zeta}^{\beta})$ and $A_{\pm} = A_{\pm}(g)$ denoting a pair of linear forms $A_{\pm}(g) : L^{\infty}(-1, 1) \rightarrow \mathbb{C}$. Furthermore, there exists $C > 0$ such that, for all $\beta \geq \beta_0$, $U \in \mathcal{S}_r^2$ and $g \in L^{\infty}(-1, +1)$, we have

$$|A_{\pm}(g)| \leq C \min(\beta^{-1/2} \|g\|_2, \beta^{-2/3} \ln \beta \|g\|_{\infty}). \quad (6.24)$$

Proof. In view of the discussion preceding the statement of the lemma, it remains to show the existence of $A_{\pm}(g)$ satisfying (6.24).

Taking the inner product of (6.20) in $L^2(-1, +1)$, first by ζ_+ and then by ζ_- while having (6.1) in mind yields the following system

$$\begin{bmatrix} \langle \zeta_+, (\psi_+ - \tilde{v}_+) \rangle & \langle \zeta_+, (\psi_- - \tilde{v}_-) \rangle \\ \langle \zeta_-, (\psi_+ - \tilde{v}_+) \rangle & \langle \zeta_-, (\psi_- - \tilde{v}_-) \rangle \end{bmatrix} \begin{bmatrix} A_+(g) \\ A_-(g) \end{bmatrix} = \begin{bmatrix} \langle \zeta_+, u \rangle \\ \langle \zeta_-, u \rangle \end{bmatrix}. \quad (6.25)$$

We now write

$$\langle \zeta_{\pm}, \psi_{\pm} \rangle = \langle 1, \psi_{\pm} \rangle + \langle \zeta_{\pm} - 1, \psi_{\pm} \rangle. \quad (6.26)$$

For the first term on the right-hand-side we have

$$\langle 1, \psi_- \rangle = (J_- \beta)^{-1/3} - \int_1^{\infty} \psi_-(x) dx.$$

The integral on the other side can be estimated as follows: we first write

$$\left| \int_1^{\infty} \psi_-(x) dx \right| \leq \|(1+x)^3 \psi_-\|_1.$$

Then, using (A.43b), (6.16) and dilation, we obtain for all $s \in [0, 3]$,

$$\|(1 \mp x)^s \psi_{\pm}\|_1 \leq C [1 + |\lambda_{\pm}| \beta^{1/3}]^{-s/2} \beta^{-(s+1)/3}. \quad (6.27)$$

The above estimate for $s = 3$ yields,

$$\left| \int_1^{\infty} \psi_-(x) dx \right| \leq C \beta^{-4/3}.$$

A similar estimate can be obtained for $\langle 1, \psi_+ \rangle$. Consequently we have

$$\langle 1, \psi_{\pm} \rangle = (J_{\pm} \beta)^{-1/3} [1 + \mathcal{O}(\beta^{-1})]. \quad (6.28)$$

For the second term on the right-hand-side of (6.26) we use the fact that, for all $x \in [-1, 1]$, we have by (6.3)

$$|\zeta_{\pm}(x) - 1| \leq [1 \mp x]^{1/2} \|\zeta'_{\pm}\|_2.$$

We obtain, using (6.27) with $s = \frac{1}{2}$ and (6.22),

$$|\langle \zeta_{\pm} - 1, \psi_{\pm} \rangle| \leq \|\zeta'_{\pm}\|_2 \|[1 \mp x]^{1/2} \psi_{\pm}\|_1 \leq C(\Upsilon) \theta_0 \beta^{-1/3}. \quad (6.29)$$

Furthermore, by (4.36) and (6.19), we have

$$\begin{aligned} |\langle \zeta_{\pm}, \tilde{v}_{\pm} \rangle| &\leq 2 \|(U - \nu + i\beta^{-1/3})^{-1}\|_2 \|(U - \nu + i\beta^{-1/3}) \tilde{v}_{\pm}\|_2 \\ &\leq C \beta^{-2/3} [1 + |\lambda_{\pm}| \beta^{1/3}]^{-3/4}. \end{aligned} \quad (6.30)$$

By the above, (6.28), and (6.29), there exists $C > 0$ and β_0 , such that, for any $\beta \geq \beta_0$ and any λ satisfying $\Re \lambda \leq \Upsilon \beta^{-1/3}$, we have

$$|\langle \zeta_{\pm}, (\psi_{\pm} - \tilde{v}_{\pm}) \rangle - (J_{\pm} \beta)^{-1/3}| \leq C(\theta_0 \beta^{-1/3} + \beta^{-2/3}). \quad (6.31)$$

As $\zeta_{\pm}(\mp 1) = 0$, we obtain as in (6.29)

$$|\langle \zeta_{\pm}, \psi_{\mp} \rangle| \leq C \theta_0 \beta^{-1/3}. \quad (6.32)$$

Furthermore, as in (6.30) we obtain that

$$|\langle \zeta_{\pm}, \tilde{v}_{\mp} \rangle| \leq C \beta^{-2/3} [1 + |\lambda_{\pm}| \beta^{1/3}]^{-3/4}. \quad (6.33)$$

Substituting the above, together with (6.31) and (6.32) into (6.25) then yields, for θ_0 small enough, and β large enough, the invertibility of (6.25) together with the estimate

$$|A_{\pm}(g)| \leq |\langle \zeta_{\pm}, u \rangle| (J_{\pm} \beta)^{1/3} [1 + C \theta_0] + C \theta_0 \beta^{-1/3} |\langle \zeta_{\mp}, u \rangle|. \quad (6.34)$$

By (5.36) we obtain that

$$|\langle \zeta_{\pm}, u \rangle| \leq C \min(\beta^{-1/2} \|g\|_2, \beta^{-2/3} \log \beta \|g\|_{\infty}).$$

Substituting the above into (6.34) yields (6.24). ■

Remark 6.3. *We may replace in Lemma 6.2, $\tilde{\mathcal{L}}_{\beta, \mathbb{R}}$ by $\mathcal{L}_{\beta}^{\mathfrak{D}}$ in (6.5) and (6.12), but under the condition $\Upsilon < J_m^{2/3} \mathfrak{R}\nu_1$. In this case, we have by (6.20) (with $\mathcal{L}_{\beta}^{\mathfrak{D}}$ instead of $\tilde{\mathcal{L}}_{\beta, \mathbb{R}}$) and having in mind the Dirichlet condition at ± 1 ,*

$$v(\pm 1) = A_{\pm} \psi_{\pm}(\pm 1) + A_{\mp} \psi_{\mp}(\pm 1).$$

By (A.35) we have

$$|\psi_{\pm}(\pm 1)| \leq C [1 + |\lambda_{\pm}|^{1/2} \beta^{1/6}],$$

and by (A.43c)

$$|\psi_{\mp}(\pm 1)| \leq C \beta^{-4/3} [1 + |\lambda_{\pm}| \beta^{1/3}]^{-2}.$$

Combining the above with (6.24) yields that

$$|v(\pm 1)| \leq C [1 + |\lambda_{\pm}|^{1/2} \beta^{1/6}] \beta^{-1/2} \|g\|_2.$$

6.2 A no-slip Schrödinger in \mathbb{R}_+

In the previous subsection we have considered a space of functions satisfying the orthogonality condition (6.1). We have assumed that the functions spanning the orthogonal space ζ_+ and ζ_- satisfy the bound

$$\|\zeta'_{\pm}\|_2 \leq \theta \beta^{1/6},$$

where $\theta \in (0, \theta_0]$ for some sufficiently small $\theta_0 > 0$.

Of particular interest is the example $\zeta_{\pm} = e^{-\alpha(1 \mp x)}$ (or a proper linear combination of them satisfying (6.3)). In this case, we have

$$\|(e^{-\alpha(1 \mp x)})'\|_2 \cong \sqrt{\alpha/2},$$

for sufficiently large α .

Consequently, as long as $\alpha \ll \beta^{1/3}$, Lemma 6.2 is applicable in this case. We,

however, need to consider also the case where $\alpha \sim \beta^{1/3}$, or even $\alpha \gg \beta^{1/3}$. These cases can, nevertheless, be treated using localization techniques as in [20, 4]. To this end we have to consider a localized version of \mathcal{L}_ζ near $x = \pm 1$. This subsection is devoted therefore to the study of the ensuing linearized operator.

We begin by establishing a proper spectral formulation for the problem.

Proposition 6.4. *Let, for some $\theta > 0$,*

$$\mathcal{L}^\theta = -\frac{d^2}{dx^2} + ix, \quad (6.35a)$$

be defined on

$$D(\mathcal{L}^\theta) = \{ u \in H^2(\mathbb{R}_+) \mid \langle e_\theta, u \rangle = 0, xu \in L^2(\mathbb{R}_+) \}. \quad (6.35b)$$

where

$$e_\theta(x) := e^{-\theta x}.$$

Then, \mathcal{L}^θ is a closed operator with non empty resolvent set and compact resolvent. Moreover \mathcal{L}^θ has index 0.

Before proceeding to the proof of the proposition we establish the following H^1 estimate of any $v \in D(\mathcal{L}^\theta)$ in term of the L^2 -norm of $\mathcal{L}^\theta v$ and v .

Lemma 6.5. *There exists some constant $C(\theta)$ such that, for any $\lambda \in \mathbb{C}$ and any $v \in D(\mathcal{L}^\theta)$, we have*

$$\|v'\|_2 + |v(0)| \leq C(\theta)[(1 + |\Re\lambda|^{1/2} \text{sign } \Re\lambda)\|v\|_2 + \|(\mathcal{L}^\theta - \lambda)v\|_2]. \quad (6.36)$$

Proof. Let $v \in D(\mathcal{L}^\theta)$ and $g \in L^2(\mathbb{R}_+)$ satisfy

$$(\mathcal{L}^\theta - \lambda)v = g. \quad (6.37)$$

Taking the inner product with v yields

$$\|v'\|_2^2 - \lambda\|v\|_2^2 + i\langle xv, v \rangle + v'(0)\bar{v}(0) = \langle v, g \rangle. \quad (6.38)$$

To obtain an estimate for the fourth term on the left-hand-side of (6.38) we need an effective bound on $v'(0)$.

Integration by parts yields, with the aid of the fact that $\langle e_\theta, v \rangle = 0$,

$$-\langle e_\theta, v'' \rangle = v'(0) + \theta v(0).$$

Taking the inner product of (6.37) with e_θ , we then obtain

$$v'(0) + \theta v(0) + i\langle xe_\theta, v \rangle = \langle e_\theta, g \rangle, \quad (6.39)$$

from which we conclude that

$$|\bar{v}(0)v'(0)| \leq \theta |v(0)|^2 + \frac{1}{2\theta^{3/2}} |v(0)| (\|v\|_2 + \theta\sqrt{2}\|g\|_2). \quad (6.40)$$

As

$$|v(0)|^2 \leq \|v'\|_2 \|v\|_2, \quad (6.41)$$

we obtain, using (6.40) that

$$|\bar{v}(0)v'(0)| \leq \frac{\theta}{2} \|v'\|_2 \|v\|_2 + \frac{1}{2\theta^{3/2}} |v(0)| (\|v\|_2 + \theta\sqrt{2}\|g\|_2). \quad (6.42)$$

Combining the real part of (6.38) with (6.42) yields

$$\|v'\|_2^2 - \mu \|v\|_2^2 \leq \frac{\theta}{2} \|v'\|_2 \|v\|_2 + \|v\|_2 \|g\|_2 + \frac{1}{2\theta^{3/2}} |v(0)| (\|v\|_2 + \theta\sqrt{2}\|g\|_2). \quad (6.43)$$

Consequently, we obtain (6.36). ■

Proof of Proposition 6.4.

Step 1: We prove that \mathcal{L}^θ is a closed operator.

Let $\{(v_n, \mathcal{L}^\theta v_n)\}_{n=1}^\infty \in [D(\mathcal{L}^\theta)]^\mathbb{N} \times [L^2(\mathbb{R}_+)]^\mathbb{N}$ converge, as $n \rightarrow \infty$, in $L^2(0, +\infty) \times L^2(0, +\infty)$ to (v_∞, g_∞) . We need to establish that $v_\infty \in D(\mathcal{L}^\theta)$. The orthogonality of v_∞ with e_θ immediately follows from the L^2 convergence. From (6.36) (with $\lambda = 0$) we conclude that v_n is a Cauchy sequence in $H^1(\mathbb{R}_+)$ and hence must converge to v_∞ in the $H^1(\mathbb{R}_+)$ norm.

Let $\chi \in C^\infty(\mathbb{R})$ be supported on $[\frac{1}{2}, +\infty)$ and satisfy $\chi = 1$ for $x > 1$. Clearly,

$$\left(-\frac{d^2}{dx^2} + ix\right)(\chi v_n) = \chi g_n + 2\chi'v_n' + \chi''v_n.$$

Since we can smoothly extend χv_n to $H^2(\mathbb{R})$, it follows from (5.4) and (5.5) (with $\beta = 1$ and $\tilde{U} = x$) that χv_n and $\chi x v_n$ are Cauchy sequences in $H^2(\mathbb{R})$ and in $L^2(\mathbb{R})$ respectively. Hence, its limit χv_∞ satisfies $\chi v_\infty \in H^2(\mathbb{R})$ and $\chi x v_\infty \in L^2(\mathbb{R})$. By the $H^1(\mathbb{R}_+)$ convergence of $\{v_n\}_{n=1}^\infty$ it follows that v_∞ is a weak solution of

$$v_\infty'' = -g_\infty + ixv_\infty.$$

Since the right-hand-side is in $L^2(\mathbb{R}_+)$, it follows that $v_\infty \in H^2(\mathbb{R}_+)$ and hence \mathcal{L}^θ is closed.

Step 2: We prove that $\mathcal{L}^\theta - \lambda$ has index 0.

Let $\tilde{\mathcal{L}} : H^2(\mathbb{R}_+) \cap L^2(\mathbb{R}^+; x^2 dx) \rightarrow L^2(\mathbb{R}^+)$ be associated with the same differential operator as \mathcal{L}^θ . Clearly, $\tilde{\mathcal{L}}$ is a Fredholm operator of index 1. Indeed, it is clearly surjective (we can find a unique solution satisfying a Dirichlet condition at $x = 0$) and it is easy to see that the kernel has dimension 1 ($\text{span}\{Ai(e^{i\pi/6} \cdot)\}$). Consequently, for any $\lambda \in \mathbb{C}$, $\tilde{\mathcal{L}} - \lambda$ has index 1. We now observe that \mathcal{L}_θ is obtained by imposing a single orthogonality condition in the domain. Hence the index of $\mathcal{L}^\theta - \lambda$ equals 0.

Step 3: We show that $\rho(\mathcal{L}_\theta) \neq \emptyset$.

We prove that there exists $\tilde{\mu} < 0$ such that for all $\Re\lambda < \tilde{\mu}$ the operator $\mathcal{L}^\theta - \lambda$ is injective. Combined with the above zero index property, it would yield that the resolvent set contains the half plane $\Re\lambda < \tilde{\mu}$. The injectivity follows from (6.41) and (6.43), by which there exist μ_0 and $C > 0$ such that for all $\Re\lambda < \mu_0$, we have

$$\|v\|_2 + \|v'\|_2 + |v(0)| \leq C \|g\|_2. \quad (6.44)$$

Finally, the compactness of the resolvent follows from the fact that $D(\mathcal{L}^\theta)$ is compactly embedded in $L^2(\mathbb{R}_+)$. ■

The previous proof has also shown to us that \mathcal{L}^θ is semi-bounded. The next proposition provides a [a more explicit](#) lower bound for the spectrum as a function of θ .

Proposition 6.6. *For all $\theta \in \mathbb{R}_+$, we have*

$$\mu_0(\theta) = \inf \Re\sigma(\mathcal{L}^\theta) \geq -\frac{1}{2} \min(\theta^2, \theta^{-2}). \quad (6.45)$$

Proof. Suppose that for some positive $\hat{\theta}_0$ there exist $\lambda_0 \in \sigma(\mathcal{L}^{\hat{\theta}_0})$ and $v_0 \in D(\mathcal{L}^{\hat{\theta}_0})$ such that

$$(\mathcal{L}^{\hat{\theta}_0} - \lambda_0)v_0 = 0. \quad (6.46)$$

Since v_0 is an L^2 solution of the complex Airy equation in \mathbb{R}_+ it is expressible, up to a multiplicative constant, in the form

$$v_0(x) = \text{Ai}(e^{i\pi/6}(x + i\lambda_0)). \quad (6.47a)$$

The orthogonality condition for v_0 reads

$$F(\lambda_0, \hat{\theta}_0) = 0, \quad (6.47b)$$

where

$$F(\lambda, \theta) = \int_{\mathbb{R}_+} e_\theta(x) \text{Ai}(e^{i\pi/6}(x + i\lambda)) dx. \quad (6.47c)$$

Step 1: We prove that the set $\{\theta \in (0, +\infty), \exists \lambda \in \sigma(\mathcal{L}_\theta) \text{ with } \Re\lambda < \Re\nu_1\}$ is open.

We use the implicit function theorem. If indeed $F(\lambda_0, \hat{\theta}_0) = 0$, we get after integration by parts that

$$\frac{\partial F}{\partial \lambda}(\lambda_0, \hat{\theta}_0) = -i \text{Ai}(e^{i2\pi/3}\lambda_0) \neq 0.$$

Hence there exists a neighborhood of $\hat{\theta}_0$ and a C^1 -solution $\lambda(\theta)$ in this neighborhood such that $\lambda(\hat{\theta}_0) = \lambda_0$.

Step 2: Let $\varepsilon < \Re\nu_1$. Consider the set

$$\Sigma(\varepsilon) := \{(\lambda, \theta) \in \mathbb{C} \times (0, +\infty), \Re\lambda < \varepsilon \text{ and } F(\lambda, \theta) = 0\},$$

which can be described as a countable (or finite) union of simple analytic curves denoted by $\{\lambda_k(\theta)\}_{k \in \mathcal{K}}$, each with an interval of definition (θ_k, θ_k^*) . Let further $\mu_k(\theta) = \Re \lambda_k(\theta)$. We prove that for all $k \in \mathcal{K}$ and $\theta \in (\theta_k, \theta_k^*)$

$$\mu_k(\theta) + \theta^2/2 \geq (\mu_k(\theta_k) + \theta_k^2/2) \exp \left\{ - \int_{\theta_k}^{\theta} \frac{\|v_k(\cdot, \tau)\|_2^2}{|v_k(0, \tau)|^2} d\tau \right\}, \quad (6.48)$$

where

$$v_k(x, \theta) := \text{Ai}(e^{i\pi/6}(x + i\lambda_k(\theta))).$$

Let $k \in \mathcal{K}$. For convenience of notation we set

$$\lambda(\theta) = \lambda_k(\theta), \quad \mu(\theta) = \Re \lambda(\theta) \quad \text{and} \quad v(\theta) = v_k(\theta).$$

By (6.47b,c) we have that

$$F(\lambda(\theta), \theta) = 0, \quad \forall \theta \in \Sigma(0). \quad (6.49)$$

Differentiating this identity with respect to θ yields

$$e^{2\pi i/3} \frac{d\lambda}{d\theta}(\theta) I_1(\theta) - I_2(\theta) = 0, \quad (6.50a)$$

where

$$I_1(\theta) = \int_{\mathbb{R}_+} e_{\theta}(x) \text{Ai}'(e^{i\pi/6}(x + i\lambda(\theta))) dx, \quad (6.50b)$$

and

$$I_2(\theta) = \int_{\mathbb{R}_+} x e_{\theta}(x) \text{Ai}(e^{i\pi/6}(x + i\lambda(\theta))) dx. \quad (6.50c)$$

Integration by parts yields, in conjunction with (6.49),

$$I_1(\theta) = -e^{-i\pi/6} \text{Ai}(e^{2\pi i/3} \lambda(\theta)). \quad (6.51)$$

We now write, with the aid of (6.49) and Airy's equation

$$\begin{aligned} I_2(\theta) &= \int_{\mathbb{R}_+} (x + i\lambda(\theta)) e_{\theta}(x) \text{Ai}(e^{i\pi/6}(x + i\lambda(\theta))) dx \\ &= e^{-i\pi/6} \int_{\mathbb{R}_+} e_{\theta}(x) \text{Ai}''(e^{i\pi/6}(x + i\lambda(\theta))) dx. \end{aligned}$$

Integration by parts and (6.51) then yield

$$I_2(\theta) = -e^{-i\pi/3} \text{Ai}'(e^{2\pi i/3} \lambda(\theta)) + i\theta \text{Ai}(e^{2\pi i/3} \lambda(\theta)).$$

Substituting the above, together with (6.51) into (6.50) yields

$$\frac{d\lambda}{d\theta}(\theta) = -\frac{\partial_x v(0, \theta)}{v(0, \theta)} - \theta. \quad (6.52)$$

Taking the inner product of (6.46) with $v(\cdot, \theta)$ we obtain for the real part

$$\|\partial_x v(\cdot, \theta)\|_2^2 - \mu(\theta) \|v(\cdot, \theta)\|_2^2 + \Re\{\bar{v}(0, \theta) \partial_x v(0, \theta)\} = 0,$$

where $\mu(\theta) = \Re\lambda(\theta)$.

Combining the above with (6.52) then yields

$$\frac{d\mu}{d\theta}(\theta) + \theta = \frac{\|\partial_x v(\cdot, \theta)\|_2^2}{|v(0, \theta)|^2} - \mu(\theta) \frac{\|v(\cdot, \theta)\|_2^2}{|v(0, \theta)|^2}.$$

We then have on the branch

$$\frac{d(\mu + \theta^2/2)}{d\theta} + \frac{\|v(\cdot, \theta)\|_2^2}{|v(0, \theta)|^2}(\mu + \theta^2/2) > 0.$$

Solving in $[\hat{\theta}_0, \hat{\theta}_0^*]$ yields (6.48).

Step 3: We prove that along every curve in $\Sigma(0)$

$$\Re\lambda(\theta) = \mu(\theta) \geq -\theta^{-2}/2. \quad (6.53)$$

From (6.43) with $g = 0$, and (6.41) we obtain that

$$\frac{1}{2}\|v'\|_2^2 - \mu(\theta)\|v\|_2^2 \leq \frac{1}{2\theta^{3/2}}\|v'\|_2^{1/2}\|v\|_2^{3/2}.$$

The above, in conjunction with Young's inequality yields $\mu(\theta) \geq -\theta^{-2}/2$ which is precisely (6.53). In particular it implies that

$$\liminf_{\theta \rightarrow +\infty} \inf_{F(\lambda, \theta)=0} \Re\lambda \geq 0. \quad (6.54)$$

Step 4: We prove (6.45).

If $\inf_{\theta} \inf_{F(\lambda, \theta)=0} \Re\lambda \geq 0$ then (6.45) readily follows. Hence, we can assume that there exists $(\lambda_0, \hat{\theta}_0)$ such that $\Re\lambda_0 < 0$ and $F(\lambda_0, \hat{\theta}_0) = 0$.

We then look at $\{\lambda_k(\theta)\}_{k \in \mathcal{K}}$ inside $\Sigma(-\varepsilon)$ where $\Re\lambda_0 < -\varepsilon < 0$. By (6.54), all branches exit $\Sigma(-\varepsilon)$ for sufficiently large θ .

We now observe that

$$F(\lambda(0), 0) = 0 \Rightarrow \mu(0) > 0, .$$

Indeed, as

$$F(\lambda, 0) = e^{-\frac{i\pi}{6}} A_0(i\lambda)$$

we can apply Corollary A.4. Hence, these branches must lie outside $\Sigma(0)$ for $\theta \in [0, \theta_{inf})$ for sufficiently small $\theta_{inf} > 0$. Assume that ε is chosen small enough so that

$$-\varepsilon + \frac{1}{2}\theta_{inf}^2 > 0.$$

Consider any branch $\lambda(\theta)$ in $\Sigma(-\varepsilon)$ with θ in some interval $[\theta(\varepsilon), \theta^*(\varepsilon)] \subset (0, +\infty)$ such that $\Re\lambda(\theta(\varepsilon)) = \Re\lambda(\theta^*(\varepsilon)) = -\varepsilon$

We can then use (6.48) to obtain

$$\begin{aligned} \mu(\theta) + \theta^2/2 &\geq (\mu(\theta(\varepsilon)) + \theta(\varepsilon)^2/2) \exp \left\{ - \int_{\theta(\varepsilon)}^{\theta} \frac{\|v\|_2^2}{|v(0)|^2} d\theta \right\} \\ &= (-\varepsilon + \theta(\varepsilon)^2/2) \exp \left\{ - \int_{\theta(\varepsilon)}^{\theta} \frac{\|v\|_2^2}{|v(0)|^2} d\theta \right\} \\ &\geq (-\varepsilon + \theta_{inf}^2/2) \exp \left\{ - \int_{\theta(\varepsilon)}^{\theta} \frac{\|v\|_2^2}{|v(0)|^2} d\theta \right\} \\ &> 0. \end{aligned} \quad (6.55)$$

This completes the proof of (6.45). ■

Corollary 6.7. *It holds that*

$$\hat{\mu}_m := \inf_{\theta \in \mathbb{R}_+} \inf_{\lambda \in \sigma(\mathcal{L}^\theta)} \left(\Re \lambda + \frac{\theta^2}{2} \right) > 0. \quad (6.56)$$

The proof of (6.56) with the non strict inequality follows immediately from (6.45). Note now that by (6.55), taking into account that $\mu(\theta) + \frac{\theta^2}{2} \xrightarrow{\theta \rightarrow +\infty} +\infty$ and that $\mu(0) > 0$ we can conclude the strict inequality in (6.56).

The adjoint operator

We note that $D(\mathcal{L}^\theta)$ is not dense in $L^2(\mathbb{R}_+)$. We thus introduce $\mathcal{H}_0 = \overline{D(A)}$, and then define

$$D(A^*) = \{v \in \mathcal{H} \text{ s.t. } D(A) \ni u \mapsto \langle v, Au \rangle \text{ extends as a continuous linear map on } \mathcal{H}_0\}, \quad (6.57)$$

and set A^*v to be the unique (by Riesz theorem) $y \in \mathcal{H}_0$ for which

$$\langle v, Au \rangle = \langle y, u \rangle.$$

The standard definition is recovered when $\mathcal{H}_0 = \mathcal{H}$. Note that

$$(A - \lambda I)^* = A^* - \bar{\lambda} \Pi_{\mathcal{H}_0},$$

where $\Pi_{\mathcal{H}_0}$ is the projector on \mathcal{H}_0 . We further note that A^*A is an unbounded operator on \mathcal{H}_0 .

In the particular case $A = \mathcal{L}^\theta$, \mathcal{H}_0 is the orthogonal complement in $L^2(\mathbb{R}_+)$ of e_θ . Hence, $\Pi_{\mathcal{H}_0} = I - P_\theta$, where, for any $u \in L^2(\mathbb{R}_+)$,

$$P_\theta u(x) = \frac{1}{2\theta} \langle e_\theta, u \rangle e_\theta.$$

We next provide a more explicit representation of $(\mathcal{L}^\theta)^*$.

Lemma 6.8. *We have*

$$D((\mathcal{L}^\theta)^*) = H_0^2(\mathbb{R}_+) \cap L^2(\mathbb{R}_+; x^2 dx), \quad (6.58a)$$

and for any $v \in D((\mathcal{L}^\theta)^*)$,

$$(\mathcal{L}^\theta)^* v = (I - P_\theta) \left(-\frac{d^2}{dx^2} - ix \right) v. \quad (6.58b)$$

Proof. The proof is reminiscent of the analysis of selfadjointness for (1D)-problems in [29]. Let $\phi \in C_0^\infty(\mathbb{R}^+)$ and then set $u = (I - P_\theta)\phi \in D(\mathcal{L}_\theta)$. Let $v \in D(\mathcal{L}_\theta^*)$, where $D(\mathcal{L}_\theta^*)$ is defined by (6.57). From the definition we deduce that the distribution

$$C_0^\infty(\mathbb{R}_+) \ni \phi \mapsto \langle v, \mathcal{L}_\theta(I - P_\theta)\phi \rangle_{L^2(\mathbb{R}_+)}$$

should extend as a continuous linear map on $L^2(\mathbb{R}^+)$. We then observe that

$$\langle v, \mathcal{L}_\theta(I - P_\theta)\phi \rangle = \left\langle \left(-\frac{d^2}{dx^2} - ix \right) v, \phi \right\rangle - \left\langle v, \left(-\frac{d^2}{dx^2} + ix \right) e_\theta \right\rangle \langle e_\theta, \phi \rangle.$$

The second term on the right hand side defines a linear form on $L^2(\mathbb{R}_+)$. Hence, from (6.57) we get that $\phi \mapsto \langle (-d^2/dx^2 - ix)v, \phi \rangle$ is a distribution in $L^2(\mathbb{R}_+)$. Hence, it holds that $(-d^2/dx^2 - ix)v \in L^2(\mathbb{R}_+)$. We can thus conclude that $v \in H^2(\mathbb{R}_+)$, $xv \in L^2(\mathbb{R}_+)$.

We now compute $\langle v, \mathcal{L}_\theta u \rangle$ using integration by parts to obtain

$$\left\langle v, \left(-\frac{d^2}{dx^2} + ix \right) u \right\rangle = -u'(0)\bar{v}(0) + u(0)\bar{v}'(0) + \left\langle \left(-\frac{d^2}{dx^2} - ix \right) v, u \right\rangle.$$

To conform with (6.57) $u \mapsto -u'(0)\bar{v}(0) + u(0)\bar{v}'(0)$ must be a continuous map on $(I - P_\theta)L^2$. This, however, is possible only if $v(0) = v'(0) = 0$ (consider the sequence $u_n = \chi_n - P_\theta \chi_n$ with $\chi_n(x) = \sqrt{n}\chi(nx)$), leading thereby to (6.58a). Consequently for any $v \in D(\mathcal{L}_\theta^*)$, we have

$$\left\langle v, \left(-\frac{d^2}{dx^2} + ix \right) u \right\rangle = \left\langle \left(-\frac{d^2}{dx^2} - ix \right) v, u \right\rangle.$$

Having in mind that

$$\langle v, \mathcal{L}_\theta u \rangle = \langle \mathcal{L}_\theta^* v, u \rangle, \quad (6.59)$$

leads to

$$\mathcal{L}_\theta^* v = (I - P_\theta) \left(-\frac{d^2}{dx^2} - ix \right) v.$$

We can then extend (6.59) by density to any $u \in \mathcal{H}_0$. ■

Proposition 6.9. *The eigenfunctions of \mathcal{L}^θ are complete in $(I - P_\theta)L^2(\mathbb{R}_+)$.*

Proof. We take a similar approach to the one in [5].

Step 1: By the semi-boundedness of \mathcal{L}^θ and (6.58a) there exists $c_0 > 0$ and $\hat{\mu}_0 \in \mathbb{R}_-$ such that for all $u \in D(\mathcal{L}^\theta)$

$$c_0 \| -u'' + xu \|_2^2 \leq 2c_0 (\|xu\|_2^2 + \|u\|_{2,2}^2) \leq \| -u'' + (ix - \hat{\mu}_0)u \|_2^2. \quad (6.60)$$

Step 2: We now show that the resolvent of \mathcal{L}^θ is in S_p for every $p > 3/2$, where S_p denotes the Schatten of order p .

By the Max-Min principle the singular values $(\mu_n)_{n \in \mathbb{N}^*}$ of the operator $(\mathcal{L}^\theta - \hat{\mu}_0)^{-1}$ satisfy for $k \in \mathbb{N}$

$$\mu_{k+1}^{-2} = \max_{U_k \in \mathcal{H}_0^k} \min_{u \in U_k^\perp \cap D(\mathcal{L}_\theta)} \frac{\| -u'' + (ix - \hat{\mu}_0)u \|_2^2}{\|u\|_2^2}.$$

Let further

$$\kappa_{k+1}^{-2} = \max_{U_k \in (L^2(\mathbb{R}_+))^k} \min_{u \in U_k^\perp \cap D(\mathcal{L}_\theta)} \frac{\| -u'' + xu \|_2^2}{\|u\|_2^2}.$$

By (6.60) we have, for $n \in \mathbb{N}^*$,

$$\kappa_n^{-2} \leq c_0 \mu_n^{-2}.$$

Finally, let

$$\tilde{\kappa}_{k+1}^{-2} = \max_{U_k \in L^2(\mathbb{R}_+)^k} \min_{u \in U_k^\perp \cap H^2(\mathbb{R}_+) \cap L^2(\mathbb{R}_+; x^2 dx)} \frac{\| -u'' + xu \|_2^2}{\|u\|_2^2}.$$

In view of the additional constraint embedded in $D(\mathcal{L}^\theta)$ we have

$$\kappa_n \leq \tilde{\kappa}_n.$$

By the Max-Min principle the $\tilde{\kappa}_n^{-2}$ are eigenvalues of

$$\mathcal{A}_N := (-d^2/dx^2 + x)^2$$

defined on

$$D(\mathcal{A}_N) = \{u \in H^4(\mathbb{R}_+) \cap L^2(\mathbb{R}_+; x^4 dx) \mid u''(0) = (-u'' + xu)'(0) = 0\}.$$

Let $\lambda = \alpha^2$, where $\alpha > 0$, denote an eigenvalue of \mathcal{A}_N . (Note that $\lambda = 0$ is an eigenvalue.) Let u_α denote the corresponding eigenfunction. As

$$\left(-\frac{d^2}{dx^2} + x\right)^2 - \alpha^2 = \left(-\frac{d^2}{dx^2} + x + \alpha\right) \left(-\frac{d^2}{dx^2} + x - \alpha\right),$$

we easily obtain that, up to a product by an arbitrary constant,

$$u_\alpha = -\frac{1}{2\alpha} \text{Ai}(x + \alpha) + A_1 \text{Ai}(x - \alpha),$$

where A_1 has to be determined from the requirement $u_\alpha \in D(\mathcal{A}_N)$. It can now be easily verified that $\alpha \in \sigma(\mathcal{A}_N)$ if and only if

$$\delta(\alpha) := -\frac{\text{Ai}'(\alpha)}{\text{Ai}(\alpha)} + \frac{\text{Ai}'(-\alpha)}{\text{Ai}(-\alpha)} = 0. \quad (6.61)$$

Let $\{\omega_n\}_{n=1}^\infty \subset \mathbb{R}_-$ denote the zeroes of Airy's function $\text{Ai}(x)$. By computation of its derivative $\delta(\alpha)$ is a monotone increasing for $\alpha \in (-\omega_n, -\omega_{n+1})$ and tends to $\pm\infty$ at the edges. Consequently, there is precisely one solution of (6.61) in $(-\omega_n, -\omega_{n+1})$.

As $-\omega_n \sim n^{2/3}$ we may conclude from the foregoing discussion that $\tilde{\kappa}_n^{-1} \sim n^{2/3}$ as well. Consequently, there exists $C > 0$ such that, for sufficiently large n ,

$$\mu_n \leq \frac{C}{n^{2/3}}.$$

As a result, for all $p > 3/2$ it holds that

$$\sum_{n=1}^{\infty} \mu_n^p < \infty.$$

Step 3: We complete the proof of the proposition.

We take a similar approach to the one in [5]. By (6.41) and (6.43) we have, for sufficiently large $-\Re\lambda$,

$$\|(\mathcal{L}^\theta - \lambda)^{-1}\| \leq \frac{C(\theta)}{|\lambda|}.$$

Let $v = (\mathcal{L}^\theta - \lambda)^{-1}g$ for some $g \in L^2(\mathbb{R}_+)$. From the imaginary part of (6.38) we learn that

$$-\Im\lambda\|v\|_2^2 + \|x^{1/2}v\|_2^2 = \Im\langle v, g \rangle - \Im\{v'(0)\bar{v}(0)\}.$$

With the aid of (6.36) and (6.42) we then obtain that

$$-\Im\lambda\|v\|_2^2 \leq C(\theta)[(1 + |\mu|^{1/2})\|v\|_2^2 + \|g\|_2^2].$$

Hence, there exists \hat{C} such that if $-\Im\lambda \geq \hat{C}(1 + |\mu|^{1/2})$, then

$$\|(\mathcal{L}^\theta - \lambda)^{-1}\|_2 \leq \frac{C(\theta)}{|\lambda|}.$$

From the foregoing discussion we may conclude that every direction where $\pi/2 < \arg \lambda < 2\pi$ is a direction of minimal growth for $(\mathcal{L}^\theta - \lambda)^{-1}$. Following the arguments of the proof of [2, Theorem 16.4] (cf. also [16, Theorem X.3.1] or [13, Corollary XI.9.31]) we can conclude that the eigenspace of \mathcal{L}^θ is given by $D(\mathcal{L}^\theta) = (I - P_\theta)L^2(\mathbb{R}_+)$. ■

Proposition 6.10. *Let $\mu_0(\theta) = \inf_{\lambda \in \sigma(\mathcal{L}^\theta)} \Re\lambda$. Then,*

$$\lim_{\theta \rightarrow +\infty} \mu_0(\theta) = \Re\nu_1, \quad (6.62)$$

where ν_1 the left most eigenvalue of \mathcal{L}_+ the Dirichlet realization of \mathcal{L} in \mathbb{R}_+ .

Proof. We begin the proof by applying Rouché's Theorem, in the large θ limit, to the holomorphic functions $\theta F(\lambda, \theta)$ and $\text{Ai}(e^{2i\pi/3}\lambda)$ inside a disk of radius $r > 0$ centered at ν_1 and containing no other eigenvalue of \mathcal{L}_+ . As $\lambda \mapsto \text{Ai}(e^{2i\pi/3}\lambda)$ has a unique zero in this disk, Rouché's Theorem would show the same for the zeros of $F(\cdot, \theta)$. It is therefore necessary to compare the two functions for $\lambda \in \partial B(\nu_1, r)$. We thus write

$$\theta F(\lambda, \theta) - \text{Ai}(e^{i2\pi/3}\lambda) = \theta \int_{\mathbb{R}_+} e^{-\theta x} [\text{Ai}(e^{i2\pi/3}\lambda + e^{i\pi/6}x) - \text{Ai}(e^{i2\pi/3}\lambda)] dx. \quad (6.63)$$

We bound the right-hand-side in the following manner

$$\begin{aligned} & \theta \int_{\mathbb{R}_+} e^{-\theta x} |\text{Ai}(e^{i2\pi/3}\lambda + e^{i\pi/6}x) - \text{Ai}(e^{i2\pi/3}\lambda)| dx \\ & \leq \theta \int_{\mathbb{R}_+} e^{-\theta x} x^{1/2} \|\text{Ai}'(e^{i\pi/6}(\cdot + i\lambda))\|_{L^2(0,x)} dx \\ & \leq \frac{C}{\theta^{1/2}} \|\text{Ai}'(e^{i\pi/6}(\cdot + i\lambda))\|_2. \end{aligned} \quad (6.64)$$

From this, we obtain the existence of $r_0 > 0$ and $C > 0$ such that, for any $r \in (0, r_0]$ and any $\lambda \in \partial B(\nu_1, r)$, we have

$$|\theta F(\lambda, \theta) - \text{Ai}(e^{2i\pi/3}\lambda)| \leq \frac{C}{r\theta^{1/2}} |\text{Ai}(e^{2i\pi/3}\lambda)|.$$

It follows from Rouché's Theorem that for sufficiently large θ , $F(\lambda, \theta)$ has a unique zero in $B(\nu_1, r)$.

At this stage we have obtained

$$\limsup_{\theta \rightarrow +\infty} \mu_0(\theta) \leq \Re \nu_1.$$

Using the arguments as above and supposing now that $r < |\lambda - \nu_1| < R$ and $\Re \lambda \leq \Re \nu_1$, we can establish that

$$\left| \frac{\theta F(\lambda, \theta)}{\text{Ai}(e^{i2\pi/3}\lambda)} - 1 \right| \leq \frac{C(R, r)}{\theta^{1/2}}. \quad (6.65)$$

Consequently, we obtain that, there exists $\theta_1(R, r)$ such that, for all $\theta > \theta_1(R, r)$ $F(\lambda, \theta)$ does not vanish in $(B(\nu_1, R) \setminus B(\nu_1, r)) \cap \{\Re \lambda \leq \Re \nu_1\}$.

To complete the proof we need yet to establish that there exists $R_0 > 0$, and $\theta_2(R_0) > 0$ such that for all $\theta > \theta_2(R_0)$ we have that

$$\inf_{\substack{\Re \lambda \leq \Re \nu_1 \\ |\lambda - \nu_1| > R_0}} \left| \frac{[\theta + (-\lambda)^{1/2}]F(\lambda, \theta)}{\text{Ai}(e^{i2\pi/3}\lambda)} \right| > 0. \quad (6.66)$$

To this end we set as in (A.6)

$$\frac{F(\lambda, \theta)}{\text{Ai}(e^{i2\pi/3}\lambda)} = \int_{\mathbb{R}_+} e^{-\theta x} e^{(-\lambda)^{1/2}x} dx + \int_{\mathbb{R}_+} e^{-\theta x} w(x) dx.$$

To bound the second term we use (A.14) and (A.17) (with $\mu_0 = \Re \nu_1$) together with Sobolev embeddings to obtain

$$\|w\|_{\infty}^2 \leq \|w\|_2 \|w'\|_2 \leq C|\lambda|^{-3/2}.$$

Hence,

$$\left| \frac{[\theta + (-\lambda)^{1/2}]F(\lambda, \theta)}{\text{Ai}(e^{i2\pi/3}\lambda)} - 1 \right| \leq C|\lambda|^{-1/4}, \quad (6.67)$$

from which (6.66) easily follows. ■

6.3 No-slip operator on $(-1, 1)$ for large α

Consider $\mathcal{L}_{\beta}^{\zeta}$, defined in (6.2), with $\zeta_{\pm} = \mathfrak{z}_{\pm}$, where $\mathfrak{z}_{\pm} \in C^2(-1, 1)$ is the solution of

$$\begin{cases} -\mathfrak{z}_{\pm}'' + \alpha^2 \mathfrak{z}_{\pm} = 0 \text{ for } x \in (-1, 1) \\ \mathfrak{z}_{\pm}(\pm 1) = 1 \text{ and } \mathfrak{z}_{\pm}(\mp 1) = 0. \end{cases} \quad (6.68)$$

An immediate computation gives

$$\mathfrak{z}_+(x) = \frac{\sinh \alpha(1+x)}{\sinh 2\alpha}, \quad (6.69)$$

and a similar formula for \mathfrak{z}_- .

We attempt to obtain a resolvent estimate for \mathcal{L}_β^ζ in the case $\alpha \geq \theta_1 \beta^{1/3}$ where $\theta_1 > 0$. If we try to use the arguments of Subsection 6.1 we would encounter a problem while attempting to use (6.29). It can be verified from (6.69) that

$$\|\mathfrak{z}_\pm - e^{-\alpha(1 \mp x)}\|_\infty \leq C e^{-2\alpha} \leq C e^{-2\theta_1 \beta^{1/3}}. \quad (6.70)$$

Then, one can deduce in the same manner that for some $C > 0$, $C_1 \in (0, 1)$, and sufficiently large β ,

$$\|\mathfrak{z}'_\pm\|_2 \geq \alpha^{1/2}(1 - C e^{-2\alpha}) \geq C_1 \theta_1^{1/2} \beta^{1/6}.$$

Thus, the error introduced by (6.29) is not necessarily small and one needs an alternate route for the estimation of $\|(\mathcal{L}_\beta^\zeta - \lambda)^{-1}\|$.

Since for $\alpha \geq \theta_1 \beta^{1/3}$ we need to consider, in the next section, only the case $\zeta_\pm \approx \mathfrak{z}_\pm$, we focus attention here on the resolvent of \mathcal{L}_β^ζ in that case. Thus, we no longer approximate ζ_\pm near $x = \pm 1$ by 1, as in Subsection 6.1, and use instead the approximation $\zeta_\pm \approx e^{-\alpha(1 \mp x)}$ as observed in (6.70). Note that \mathcal{L}_β^ζ depends on α through the orthogonality conditions appearing in the definition of its domain. Consequently, we need to renormalize ψ_\pm from (6.8b) and (6.8a) in a manner that would suit the approximation used for ζ_\pm .

For some $\theta > 0$, the renormalization factor will be defined by

$$\omega_\pm(\beta, \lambda, \theta) := \frac{F(\tilde{\lambda}_\pm, 0)}{F(\tilde{\lambda}_\pm, \theta J_\pm^{-1/3})}, \quad (6.71)$$

where, (see (6.15) for the definition of λ_\pm),

$$\tilde{\lambda}_\pm = \beta^{1/3} J_\pm^{-2/3} [\lambda - iU(\pm 1)] = \beta^{1/3} J_\pm^{-2/3} \lambda_\pm. \quad (6.72)$$

We now define

$$\psi_{\pm, \theta} = \omega_\pm(\beta, \lambda, \theta) \psi_\pm, \quad (6.73)$$

where ψ_\pm was introduced in (6.8b)- (6.8a).

The above normalization provides the approximation $\langle \zeta_\pm, \psi_{\pm, \theta} \rangle \sim (J_\pm \beta)^{-\frac{1}{3}}$, in the limit $\beta \rightarrow \infty$, as in Subsection 6.1 (see below (6.84)).

We similarly introduce with the notation of (6.11) and (6.12)

$$g_{\pm, \theta} = \omega_\pm(\beta, \lambda, \theta) g_\pm \text{ and } \tilde{v}_{\pm, \theta} = \omega_\pm(\beta, \lambda, \theta) \tilde{v}_\pm. \quad (6.74)$$

We can now state:

Proposition 6.11. *Let $r > 1$, $\theta_1 > 0$ and $\varkappa > 0$. Then, there exist $\beta_0 > 0$ and $C(\varkappa) > 0$ such that, for all $U \in \mathcal{S}_r^2$, $\beta \geq \beta_0$ and $\theta = \alpha \beta^{-1/3} \geq \theta_1$,*

$$\sup_{\Re \lambda \leq (\hat{\mu}_0(\theta) - \varkappa) \beta^{-1/3}} \|(\mathcal{L}_\beta^\zeta - \beta \lambda)^{-1}\| \leq C(\varkappa) \beta^{-2/3}, \quad (6.75)$$

where

$$\hat{\mu}_0(\theta) = \min(J_-^{2/3} \mu_0(J_-^{-1/3} \theta), J_+^{2/3} \mu_0(J_+^{-1/3} \theta)). \quad (6.76)$$

Remark 6.12. In the sequel we apply Proposition 6.11 with $\theta_1 = \theta_0$ where θ_0 is defined in the statement of Lemma 6.2.

Proof.

The proof goes along similar lines to the proof of Lemma 6.2. Let $\lambda \in \mathbb{C}$ satisfy

$$\beta^{1/3} \Re \lambda \leq \Upsilon(\theta, \varkappa) := \hat{\mu}_0(\theta) - \varkappa.$$

Furthermore, let the pair (g, v) in $L^2(-1, 1) \times D(\mathcal{L}_\beta^\zeta)$ satisfy the relation

$$(\mathcal{L}_\beta^\zeta - \beta\lambda)v = g.$$

We assume, as in (6.20),

$$v = A_+(g)(\psi_{+, \theta} - \tilde{v}_{+, \theta}) + A_-(g)(\psi_{-, \theta} - \tilde{v}_{-, \theta}) + u, \quad (6.77)$$

where u is given by (6.5), and then estimate $A_\pm(g)$ in the relevant regime of α values. We first estimate the renormalization factor. We note that

$$\omega_\pm(\beta, \lambda, \theta) = e^{-i\pi/6} \frac{A_0(i\tilde{\lambda}_\pm)}{F(\tilde{\lambda}_\pm, \theta J_\pm^{-1/3})},$$

and from (A.44) which reads,

$$\sup_{\Re \tilde{\lambda} \leq \Upsilon(\hat{\theta})} \left| \frac{A_0(i\tilde{\lambda})}{F(\tilde{\lambda}, \hat{\theta})} \right| \leq C(\varkappa)(1 + \hat{\theta}),$$

we obtain that there exist $C(\varkappa) > 0$ and $\beta_0(\varkappa)$, such that for all $\theta \geq \theta_1$, $\Re \lambda \leq \Upsilon(\theta, \varkappa)\beta^{-1/3}$, and $\beta \geq \beta_0(\varkappa)$ we have

$$|\omega_\pm(\beta, \lambda, \theta)| \leq C\theta. \quad (6.78)$$

We can now use (6.17), (6.18) and (6.19) to obtain that

$$\|\psi_{\pm, \theta}\|_2 \leq C\theta [1 + |\lambda_\pm|^{1/2} \beta^{1/6}]^{1/2} \beta^{-1/6}, \quad (6.79)$$

$$\|g_{\pm, \theta}\|_2 \leq C\theta \frac{\beta^{1/6}}{[1 + |\lambda_\pm|^{1/2} \beta^{1/6}]^{3/2}}, \quad (6.80)$$

and

$$\|(U - \nu + i\beta^{-1/3})\tilde{v}_{\pm, \theta}\|_2 + \beta^{-1/3}\|\tilde{v}_{\pm, \theta}\|_2 \leq C\theta \frac{\beta^{-5/6}}{[1 + |\lambda_\pm|^{1/2} \beta^{1/6}]^{3/2}}. \quad (6.81)$$

We note that (6.25) remain valid in the case $\theta > 0$, i.e.,

$$\begin{bmatrix} \langle \mathfrak{z}_+, (\psi_{+, \theta} - \tilde{v}_{+, \theta}) \rangle & \langle \mathfrak{z}_+, (\psi_{-, \theta} - \tilde{v}_{-, \theta}) \rangle \\ \langle \mathfrak{z}_-, (\psi_{+, \theta} - \tilde{v}_{+, \theta}) \rangle & \langle \mathfrak{z}_-, (\psi_{-, \theta} - \tilde{v}_{-, \theta}) \rangle \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \end{bmatrix} = \begin{bmatrix} \langle \mathfrak{z}_+, u \rangle \\ \langle \mathfrak{z}_-, u \rangle \end{bmatrix}, \quad (6.82)$$

We now write

$$\langle \mathfrak{z}_\pm, \psi_{\pm, \theta} \rangle = \langle e^{-\alpha(1 \pm x)}, \psi_{\pm, \theta} \rangle + \langle \mathfrak{z}_\pm - e^{-\alpha(1 \pm x)}, \psi_{\pm, \theta} \rangle.$$

Since by (6.78) and (A.43b) we have that

$$\|\psi_{\pm, \theta}\|_1 \leq C(\varkappa)\theta\|\psi_{\pm}\|_1 \leq C\theta\beta^{-1/3}, \quad (6.83)$$

we can easily deduce using (6.70) that for sufficiently large β

$$|\mathfrak{z}_{\pm} - e^{-\alpha(1\pm x)}, \psi_{\pm, \theta}\rangle| \leq \hat{C}e^{-2\theta\beta^{1/3}}\|\psi_{\pm, \theta}\|_1 \leq \check{C}\beta^{-\frac{2}{3}}e^{-\theta_1\beta^{1/3}}.$$

Furthermore, as

$$\begin{aligned} & \int_{-1}^{+\infty} e^{-\alpha(1+x)} \text{Ai} \left((J_- \beta)^{1/3} e^{i\pi/6} [(1+x) + iJ_-^{-1}(\lambda - iU(-1))] \right) dx \\ &= (J_- \beta)^{-1/3} \int_{\mathbb{R}_+} e^{-\theta J_-^{1/3} \xi} \text{Ai} \left(e^{i\pi/6} [\xi + iJ_-^{-2/3} \beta^{1/3} (\lambda - iU(-1))] \right) d\xi \\ &= (J_- \beta)^{-1/3} F(\tilde{\lambda}_{\pm}, \theta J_{\pm}^{-1/3}), \end{aligned}$$

we have that

$$\langle e^{-\alpha(1+x)}, \psi_{-, \theta} \rangle = (J_- \beta)^{-1/3} - \langle \mathbf{1}_{1, \infty} e^{-\alpha(1+x)}, \psi_{-, \theta} \rangle.$$

Since by (6.83) we have

$$|\langle \mathbf{1}_{1, \infty} e^{-\alpha(1+x)}, \psi_{-, \theta} \rangle| \leq C e^{-2\theta\beta^{1/3}} \|\psi_{-, \theta}\|_1 \leq \check{C} \beta^{-\frac{1}{3}} e^{-\theta_1 \beta^{2/3}},$$

and hence we obtain

$$\langle e^{-\alpha(1+x)}, \psi_{-, \theta} \rangle = (J_- \beta)^{-1/3} [1 + \mathcal{O}(e^{-\theta_1 \beta^{1/3}})], \quad (6.84)$$

and a similar estimate can be obtained for $\langle e^{-\alpha(1-x)}, \psi_{+, \theta} \rangle$.

We now write

$$|\langle \mathfrak{z}_+, \psi_{-, \theta} \rangle| \leq \|\psi_{-, \theta}\|_{L^1(-1, 0)} e^{-\theta\beta^{1/3}} + \|\psi_{-, \theta}\|_{L^\infty(0, 1)} \|\mathfrak{z}_+\|_{L^1(0, 1)},$$

and then use (6.83), (6.78), and (A.43c) to obtain that

$$|\langle \mathfrak{z}_+, \psi_{-, \theta} \rangle| \leq C\beta^{-5/3}. \quad (6.85)$$

Bounds for $\langle \mathfrak{z}_{\pm}, u \rangle$.

Set,

$$\hat{\lambda} = \begin{cases} \lambda & \text{if } |\Re \lambda| > \max \left(1, \frac{\|U'\|_{\infty}}{\theta} \right) \beta^{-1/3} \\ \max \left(1, \frac{\|U'\|_{\infty}}{\theta} \right) \beta^{-1/3} + i\Im \lambda & \text{if } |\Re \lambda| \leq \max \left(1, \frac{\|U'\|_{\infty}}{\theta} \right) \beta^{-1/3} \end{cases}. \quad (6.86)$$

As $\beta^{\frac{1}{3}} |\Re \hat{\lambda}| \geq 1$ and $|\Re \hat{\lambda}| \geq |\Re \lambda|$, it can be verified that

$$1 + |\Re \lambda|^2 \beta^{2/3} \leq 2 |\Re \hat{\lambda}|^2 \beta^{2/3},$$

and hence

$$1 + \beta^{2/3} |\lambda_{\pm}|^2 \leq 2 \beta^{2/3} |U(\pm 1) + i\hat{\lambda}|^2. \quad (6.87)$$

We now verify that

$$\|\mathfrak{z}_-^2(U + i\hat{\lambda})^{-2}\|_\infty = |U(-1) + i\hat{\lambda}|^{-2} \quad (6.88)$$

Indeed, we have

$$\begin{aligned} [\mathfrak{z}_-^2|U + i\hat{\lambda}|^{-2}]' &= \mathfrak{z}_-^2|U + i\hat{\lambda}|^{-4} \left(-2\theta\beta^{1/3}[(U - \Im\lambda)^2 + |\Re\hat{\lambda}|^2] + 2U'(U - \Im\lambda) \right) \\ &= \mathfrak{z}_-^2|U + i\hat{\lambda}|^{-4} \left(-2\theta\beta^{1/3} \left[(U - \Im\lambda) + \frac{U'}{2\theta\beta^{1/3}} \right]^2 + \frac{|U|^2}{2\theta\beta^{1/3}} - 2\theta\beta^{1/3}|\Re\hat{\lambda}|^2 \right), \end{aligned}$$

which is non positive by (6.86).

Hence the maximum of $\mathfrak{z}_-^2|U + i\hat{\lambda}|^{-2}$ is obtained at $x = -1$. A similar inequality can be established for $\mathfrak{z}_+^2|U + i\hat{\lambda}|^{-2}$. Combining (6.87) and (6.88) yields

$$\begin{aligned} \|\mathfrak{z}_\pm(U + i\hat{\lambda})^{-1/2}\|_1 &\leq \|\mathfrak{z}_\pm^{1/2}\|_1 \|\mathfrak{z}_\pm^{1/2}(U + i\hat{\lambda})^{-1/2}\|_\infty \\ &\leq C \frac{\beta^{-1/6}}{\theta} |\beta^{1/3}(U(\pm 1) + i\hat{\lambda})|^{-1/2} \\ &\leq \hat{C} \frac{\beta^{-1/6}}{\theta[1 + \beta^{1/3}|\lambda_\pm|]^{1/2}}. \quad (6.89) \end{aligned}$$

Note that the replacement of λ by $\hat{\lambda}$ avoids the burden of a vanishing denominator.

We now write

$$|\langle \mathfrak{z}_\pm, u \rangle| \leq \|\mathfrak{z}_\pm(U + i\hat{\lambda})^{-1/2}\|_1 \|(U + i\hat{\lambda})^{1/2}u\|_\infty. \quad (6.90)$$

To estimate $\|(U + i\hat{\lambda})^{1/2}u\|_\infty$ in the right hand side of (6.90), we first obtain a bound for $\| |U - \nu|^{1/2}u \|_\infty$. Thus, integration by parts yields for all $(x, x_0) \in [-1, 1]^2$

$$(U - \nu)|u|^2 \Big|_{x_0}^x = 2\Re \int_{x_0}^x (U - \nu)\bar{u}u' dt - \int_{x_0}^x U'|u|^2 dt.$$

From which we conclude, by integrating the above for $x \in (-1, +1)$ and Cauchy-Schwarz inequalities, that

$$\| |U - \nu|^{1/2}u \|_\infty^2 \leq C(\| |U - \nu| u \|_2 \|u'\|_2 + \|u\|_2^2 + \| |U - \nu|^{1/2} u \|_2^2).$$

By (5.4) and (5.5) we then have

$$\| |U - \nu|^{1/2}u \|_\infty^2 \leq \frac{C}{\beta^{4/3}} \|g\|_2^2.$$

We now write

$$\begin{aligned} \|(U + i\hat{\lambda})^{1/2}u\|_\infty^2 &\leq 2(\| |U - \nu|^{1/2}u \|_\infty^2 + (|\Re\lambda| + 2\beta^{-1/3}) \|u\|_\infty^2) \\ &\leq C(\beta^{-4/3}\|g\|_2^2 + (|\Re\lambda| + 2\beta^{-1/3}) \|u\|_2 \|u'\|_2). \end{aligned}$$

By (5.4) we then have

$$(|\Re\lambda| + 2\beta^{-1/3}) \|u\|_2 \leq \frac{C}{\beta} \|g\|_2,$$

and hence, using(5.4) once again to estimate $\|u'\|_2$, we may conclude that

$$\|(U + i\hat{\lambda})^{1/2}u\|_\infty \leq \frac{C}{\beta^{2/3}} \|g\|_2. \quad (6.91)$$

Combining the above with (6.89) and (6.90) yields

$$|\langle \mathfrak{z}_\pm, u \rangle| \leq \frac{C}{\theta[1 + |\lambda_\pm|\beta^{1/3}]^{1/2}} \beta^{-5/6} \|g\|_2. \quad (6.92)$$

Bounds for $\langle \mathfrak{z}_\pm, \tilde{v}_{\pm,\theta} \rangle$.

The estimation of $\langle \mathfrak{z}_\pm, \tilde{v}_{\pm,\theta} \rangle$ follows a similar path to that of $\langle \mathfrak{z}_\pm, u \rangle$. We begin by writing

$$|\langle \mathfrak{z}_\pm, \tilde{v}_{\pm,\theta} \rangle| \leq \|\mathfrak{z}_\pm(U + i\hat{\lambda})^{-1/2}\|_1 \|(U + i\hat{\lambda})^{1/2}\tilde{v}_{\pm,\theta}\|_\infty.$$

Since $\tilde{v}_{\pm,\theta}$, given by (6.74) satisfies the same problem as u with g replaced by $g_{\pm,\theta}$ we may conclude as in (6.91) that

$$\|(U + i\hat{\lambda})^{1/2}\tilde{v}_{\pm,\theta}\|_\infty \leq \frac{C}{\beta^{2/3}} \|g_{\pm,\theta}\|_2.$$

Consequently, by (6.80) and (6.89)

$$|\langle \mathfrak{z}_\pm, \tilde{v}_{\pm,\theta} \rangle| \leq C \frac{\beta^{-2/3}}{[1 + |\lambda_\pm| \beta^{1/3}]^{5/4}}. \quad (6.93)$$

In a similar manner we can obtain that

$$|\langle \mathfrak{z}_\mp, \tilde{v}_{\pm,\theta} \rangle| \leq C \frac{\beta^{-2/3}}{[1 + |\lambda_\pm| \beta^{1/3}]^{3/4} [1 + |\lambda_\mp| \beta^{1/3}]^{1/2}} \leq C \beta^{-2/3}. \quad (6.94)$$

As in (6.31) we can now write, in view of (6.93) and

$$|\langle \mathfrak{z}_\pm, (\psi_{\pm,\theta} - \tilde{v}_{\pm,\theta}) \rangle - (J_\pm \beta)^{-1/3}| \leq C \beta^{-2/3}.$$

Combining the above with (6.85), (6.93), and (6.94) yields

$$|A_\pm(g)| \leq C \beta^{1/3} |\langle \mathfrak{z}_\pm, u \rangle| + C \beta^{-2/3} |\langle \mathfrak{z}_\mp, u \rangle|.$$

The above, together with (6.92) yields

$$|A_\pm(g)| \leq \frac{C \beta^{-1/2}}{\theta} \left(\frac{1}{[1 + |\lambda_\pm| \beta^{1/3}]^{1/2}} + \beta^{-1} \frac{1}{[1 + |\lambda_\mp| \beta^{1/3}]^{1/2}} \right) \|g\|_2.$$

As $|\lambda_+ - \lambda_-| = |U(1) - U(-1)|$, we obtain for sufficiently large β

$$|A_\pm(g)| \leq \frac{\hat{C} \beta^{-1/2}}{\theta} \frac{1}{[1 + |\lambda_\pm| \beta^{1/3}]^{1/2}} \|g\|_2. \quad (6.95)$$

Combining the above with (6.77), (6.79), and (6.81) yields (6.75). ■

7 Zero traction Orr-Sommerfeld operator

7.1 A short reminder

We recall for the commodity of the reader that $\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{S}}$ is defined by (3.5) and (3.10) i.e.

$$\mathcal{B}_{\lambda,\alpha,\beta} := (\mathcal{L}_\beta - \beta\lambda) \left(\frac{d^2}{dx^2} - \alpha^2 \right) - i\beta U'' ,$$

with domain

$$D(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{S}}) = \{ u \in H^4(-1, 1) \cap H_0^1(-1, 1) \mid u'' \in H_0^1(-1, 1) \}.$$

Here

$$\mathcal{L}_\beta = -\frac{d^2}{dx^2} + i\beta U,$$

and $\mathcal{L}_\beta^{\mathfrak{D}}$ is the Dirichlet realization of \mathcal{L}_β in $(-1, +1)$.

Finally we recall that the inviscid operator $\mathcal{A}_{\lambda,\alpha}$ associated with U is defined by

$$\mathcal{A}_{\lambda,\alpha} = (U + i\lambda) \left(-\frac{d^2}{dx^2} + \alpha^2 \right) + U''$$

with domain $D(\mathcal{A}_{\lambda,\alpha}) = H^2(-1, +1) \cap H_0^1(-1, +1)$.

7.2 The case $U'' \neq 0$.

We now prove

Proposition 7.1. *For all $r > 1$ and $\hat{\delta} > 0$ there exist positive β_0 , Υ , and C such that, for any $\beta \geq \beta_0$ and $U \in \mathcal{S}_r$ satisfying (2.29), it holds that*

$$\sup_{\substack{\Re \lambda \leq \Upsilon \beta^{-1/3} \\ 0 \leq \alpha}} \left\| (\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{S}})^{-1} \right\| + \left\| \frac{d}{dx} \circ (\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{S}})^{-1} \right\| \leq C \beta^{-\left(\frac{1}{2} - \hat{\delta}\right)}. \quad (7.1)$$

Proof. Let $\lambda = \mu + i\nu$. Let further $0 < \Upsilon < \Re \nu_1$ and suppose $\mu \leq \Upsilon \beta^{-1/3}$. Let $\phi \in D(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{S}})$ and $f \in L^2(-1, +1)$ satisfy

$$\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{S}} \phi = f \quad (7.2)$$

and

$$v := \mathcal{A}_{\lambda,\alpha} \phi. \quad (7.3)$$

We note that $v \in D(\mathcal{L}_\beta^{\mathfrak{D}})$ and, defining g by

$$g := (\mathcal{L}_\beta^{\mathfrak{D}} - \lambda\beta)v, \quad (7.4)$$

we have

$$g = -(U + i\lambda)f + 2U'(\phi^{(3)} - \alpha^2\phi') + U''(\phi'' - \alpha^2\phi) - (U''\phi)''. \quad (7.5)$$

By (5.19), there exist $C > 0$ and $\beta_0 > 0$ such that, for $\beta \geq \beta_0$ and $p \in (2, +\infty]$ we have

$$\|v\|_p \leq \hat{C} \beta^{-\frac{3p+2}{6p}} \|g\|_2. \quad (7.6)$$

As

$$\begin{aligned} \Re \langle (U'')^{-1}(\phi'' - \alpha^2\phi), \mathcal{B}_\lambda^{\mathfrak{S}} \phi \rangle &= \|(U'')^{-1/2}(\phi^{(3)} - \alpha^2\phi')\|_2^2 + \\ &+ \Re \langle ((U'')^{-1})'(\phi'' - \alpha^2\phi), \phi^{(3)} - \alpha^2\phi' \rangle - \beta\mu \|\phi'' - \alpha^2\phi\|_2^2, \end{aligned} \quad (7.7)$$

we easily obtain that

$$\|\phi^{(3)} - \alpha^2 \phi'\|_2 \leq C(\|f\|_2 + |\mu|^{1/2} \beta^{1/2} \|\phi'' - \alpha^2 \phi\|_2). \quad (7.8)$$

We now write,

$$(\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)(\phi'' - \alpha^2 \phi) = i\beta U'' \phi + f. \quad (7.9)$$

With the aid of (5.8) and (5.21) we then obtain

$$\|\phi'' - \alpha^2 \phi\|_2 \leq C(\beta^{1/6} \|\phi\|_\infty + \beta^{-2/3} \|f\|_2), \quad (7.10)$$

Substituting the above into (7.8) then yields

$$\|\phi^{(3)} - \alpha^2 \phi'\|_2 \leq C(\|f\|_2 + |\mu|^{1/2} \beta^{1/2} \|\phi'' - \alpha^2 \phi\|_2). \quad (7.11)$$

To bound $\|(U'' \phi)''\|_2$ we first use the fact that

$$\langle \phi'', \phi'' - \alpha^2 \phi \rangle = \|\phi''\|_2^2 + \alpha^2 \|\phi'\|_2^2. \quad (7.12)$$

Then, by (7.10) we obtain that

$$\|\phi''\|_2 \leq C(\beta^{1/6} \|\phi\|_\infty + \beta^{-2/3} \|f\|_2), \quad (7.13)$$

and

$$\|(U'' \phi)''\|_2 \leq C(\beta^{1/6} \|\phi\|_\infty + \beta^{-2/3} \|f\|_2 + \|\phi_{1,2}\|). \quad (7.14)$$

We continue the proof by considering $\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}}$ in a few different regimes of λ values.

Case 1: Bounded $|\lambda|$

Suppose first that

$$|\nu| \leq C_0 \quad ; \quad 0 < |\mu| \leq \Upsilon \beta^{-1/3}, \quad (7.15a,b)$$

where $C_0 \in \mathbb{R}_+$. The value of C_0 to be selected above will be determined in a later stage.

Using (7.10), (7.11), and (7.14), we obtain from (7.5) that

$$\|g\|_2 \leq C(\|f\|_2 + (|\mu|^{1/2} \beta^{2/3} + \beta^{1/6})[\|\phi\|_\infty + \|\phi\|_{1,2}]),$$

or by Sobolev's embedding

$$\|g\|_2 \leq C(\|f\|_2 + (|\mu|^{1/2} \beta^{2/3} + \beta^{1/6}) \|\phi\|_{1,2}). \quad (7.16)$$

(7.3), we may apply Proposition 4.13 to the pair (ϕ, v) to conclude, by (4.24), that for any $q > 1$ there exists $C > 0$ such that

$$\|\phi\|_{1,2} \leq C(\|v'\|_q + \|v\|_\infty). \quad (7.17)$$

Similarly, by (4.25), for any $p > 2$ there exists $C > 0$ such that

$$|\mu|^{1/2} \beta^{2/3} \|\phi\|_{1,2} \leq |\mu|^{1/2-1/p} \beta^{2/3} \|v\|_p. \quad (7.18)$$

Hence,

$$\|g\|_2 \leq C (\|f\|_2 + |\mu|^{1/2-1/p} \beta^{2/3} \|v\|_p + \beta^{1/6} (\|v'\|_q + \|v\|_\infty)).$$

We may now use (7.6) to obtain, for $p > 2$,

$$\begin{aligned} |\mu|^{1/2-1/p} \beta^{2/3} \|v\|_p &\leq C |\mu|^{\frac{1}{2}-\frac{1}{p}} \beta^{\frac{2}{3}-\frac{3p+2}{6p}} \|g\|_2 \\ &\leq \hat{C} \Upsilon^{\frac{1}{2}-\frac{1}{p}} \beta^{\frac{2}{3}-\frac{p-2}{6p}-\frac{3p+2}{6p}} \|g\|_2 \\ &= \hat{C} \Upsilon^{\frac{1}{2}-\frac{1}{p}} \|g\|_2 \end{aligned}$$

Applying (7.6) (which is valid for $p = +\infty$ as well) once again yields

$$\beta^{\frac{1}{6}} \|v\|_\infty \leq C \beta^{\frac{1}{6}-\frac{1}{2}} \|g\|_2 = C \beta^{-\frac{1}{3}} \|g\|_2.$$

Finally, we apply (5.10) (with $q = p$) to the pair (v, g) satisfying (7.4) to conclude, for $1 < q < 2$, that

$$\beta^{1/6} \|v'\|_q \leq \frac{C_p}{\beta^{\frac{1}{3r}}} \|g\|_2.$$

Combining the above we then obtain, for sufficiently small Υ and β^{-1} ,

$$\|g\|_2 \leq C \|f\|_2. \quad (7.19)$$

From (5.10) and (7.19) we now get, for any $p \in (1, 2)$,

$$\|v'\|_p \leq C \beta^{-\frac{2+p}{6p}} \|g\|_2 \leq \hat{C} \beta^{-\frac{2+p}{6p}} \|f\|_2.$$

From which we deduce, for p sufficiently close to 1 and sufficiently small $\delta > 0$, the existence of $C_\delta > 0$ such that

$$\|v'\|_p + \|v\|_\infty \leq C_\delta \beta^{-\left(\frac{1}{2}-\delta\right)} \|f\|_2.$$

We now return to (7.17) to conclude that

$$\|\phi\|_{1,2} \leq \hat{C}_\delta \beta^{-\left(\frac{1}{2}-\delta\right)} \|f\|_2. \quad (7.20)$$

Hence (7.1) is proven in Case 1 for sufficiently small $\Upsilon > 0$.

Case 2: $\Re\lambda$ unbounded negative

Next, consider the case where (7.15a) is kept in place but instead of (7.15b), $\mu < -\Upsilon \beta^{-1/3}$ is assumed. In this case we return to (7.7) and use the positivity of $-\beta\mu$ for the last term on its right hand side. In a similar manner to the one used to derive (7.8) we establish the existence of $\beta(\Upsilon) > 0$ such that for all $\beta \geq \beta(\Upsilon)$,

$$\|\phi'' - \alpha^2 \phi\|_2 \leq \frac{C}{\Upsilon \beta^{2/3}} \|f\|_2.$$

Since $\phi \in H_0^1(-1, 1)$ it can be easily verified that

$$\|\phi\|_{1,2} \leq \frac{4}{\pi} \|\phi'' - \alpha^2 \phi\|_2, \quad (7.21)$$

and hence for any $\Upsilon > 0$, there exist C and β_0 such that, for all $\alpha \geq 0$ and all $\beta \geq \beta_0$

$$\|\phi\|_{1,2} \leq C \beta^{-2/3} \|f\|_2,$$

establishing (7.1) in Case 2.

Remark 7.2.

Case 3: $|\Im\lambda|$ unbounded

Consider next the case, where instead of (7.15) we have

$$|\nu| \geq C_0. \quad (7.22)$$

In this case we write

$$-\Im\langle \phi'' - \alpha^2\phi, \mathcal{B}_\lambda^\Im \phi \rangle = \beta[\langle \phi'' - \alpha^2\phi, (\nu - U)(\phi'' - \alpha^2\phi) \rangle + \langle \phi'' - \alpha^2\phi, U''\phi \rangle]$$

As $\phi \in H_0^1(-1, 1)$ we have that

$$\|\phi\|_2 \leq \frac{4}{\pi^2} \|\phi'' - \alpha^2\phi\|_2,$$

and hence, under (7.22),

$$|\langle \phi'' - \alpha^2\phi, (\nu - U)(\phi'' - \alpha^2\phi) \rangle + \langle \phi'' - \alpha^2\phi, U''\phi \rangle| \geq (C_0 - \|U\|_\infty - 4\pi^{-2}\|U''\|_\infty) \|\phi'' - \alpha^2\phi\|_2^2.$$

Selecting $C_0 > \|U\|_\infty + 4\pi^{-2}\|U''\|_\infty$ we then obtain

$$\|\phi'' - \alpha^2\phi\|_2 \leq \frac{C}{\beta} \|f\|_2.$$

We can now conclude from (7.21) that

$$\|\phi\|_{1,2} \leq \frac{C}{\beta} \|f\|_2.$$

As (4.23)- (4.25) are not valid for $\mu = 0$, we need to establish (7.20) for the case $\mu = 0$. Let

$$\mathcal{B}_{iv,\alpha}^\Im \phi = f.$$

Then, we may write by (7.7) that

$$\|\phi'' + \alpha^2\phi\|_2 \leq C \|f\|_2.$$

Then, as

$$\mathcal{B}_{iv+\beta^{-1},\alpha}^\Im \phi = \phi'' + \alpha^2\phi + f,$$

we may conclude (7.1) from (7.20). ■

7.3 The nearly Couette case

We now proceed to consider the nearly Couette case addressed by both Theorems 2.14 and 2.15. Let $U \in C^4([-1, 1])$ satisfy (2.24) and recall the definition of $\delta_2(U)$ from (2.26),

$$\delta_2(U) := \|U''\|_{1,\infty}.$$

We next recall from (2.28), for some $r > 1$,

$$\mathcal{S}_r = \{v \in C^4([-1, 1]) \mid \inf_{x \in [-1, 1]} v' \geq 1/r \text{ and } \|v\|_{4,\infty} \leq r\}.$$

We shall consider the case where $\delta_2(U)$ is small. Note that in contrast with the assumptions of Proposition 7.1, U'' may change its sign.

Proposition 7.3. For any $r > 1$ and $\Upsilon < \Re\nu_1$, there exist $\delta \in (0, \frac{1}{4})$, and positive β_0 and C , such that, for all $U \in \mathcal{S}_r$ satisfying $0 < \delta_2(U) \leq \delta$ and all $\beta \geq \beta_0$,

$$\sup_{\substack{\beta^{1/3} \Re\lambda \leq J_m^{2/3} \Upsilon \\ 0 \leq \alpha}} \left\| (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1} \right\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1} \right\| \leq \frac{C}{\beta^{5/6}}. \quad (7.23)$$

Proof.

Note that by (2.28)

$$r \geq \mathbf{m} \geq \frac{1}{r}. \quad (7.24)$$

As in the proof of Proposition 7.1 we separately consider different regimes of $\lambda \in \mathbb{C}$.

Case 1: $\Re\nu_1/2 \leq J_m^{-2/3} \beta^{1/3} \mu \leq \Upsilon$ or $\beta^{1/3} \mu \leq -J_m^{2/3} \Re\nu_1/2$

Let $f \in L^2(-1, 1)$. Using the definitions (7.2)-(7.3) we further set

$$\tilde{v} = \frac{v}{U + i\lambda},$$

Clearly

$$(\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)\tilde{v} = f + \left(\frac{U''\phi}{U + i\lambda} \right)'' = h. \quad (7.25)$$

We now write

$$\|v\|_2 \leq \|(U + i\lambda)(\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)^{-1}h\|_2$$

By (5.8) and (5.9) we then have

$$\|v\|_2 \leq \frac{C}{\beta} \|h\|_2. \quad (7.26)$$

Next, we turn to estimate $\|h\|_2$. Clearly, by (7.25),

$$\begin{aligned} \|h\|_2 \leq \|f\|_2 + C(r) & \left(\left\| \frac{\phi''}{|U + i\lambda|} \right\|_2 + \left\| \phi' \left(1 + \frac{1}{|U + i\lambda|^2} \right) \right\|_2 \right. \\ & \left. + \left\| \frac{U''\phi}{|U + i\lambda|^3} \right\|_2 + \left\| \phi \left(1 + \frac{1}{|U + i\lambda|^2} \right) \right\|_2 \right). \end{aligned} \quad (7.27)$$

where we have use that fact that

$$\frac{1}{|U + i\lambda|} \leq \frac{1}{2} \left(1 + \frac{1}{|U + i\lambda|^2} \right).$$

For the third term in the right-hand-side of (7.27) we have

$$\left\| \phi' \left(1 + \frac{1}{|U + i\lambda|^2} \right) \right\|_2 \leq C\beta^{2/3} \|\phi'\|_2. \quad (7.28)$$

We now turn to estimate the second term on the right-hand-side of (7.27). Here we write

$$\left\| \frac{\phi''}{U + i\lambda} \right\|_2 \leq C\beta^{1/3} \|\phi''\|_2.$$

Then, we use (7.13) (which remains valid in the nearly-Couette case) to obtain

$$\left\| \frac{\phi''}{U + i\lambda} \right\|_2 \leq C(r)(\beta^{1/2}\|\phi\|_\infty + \beta^{-1/3}\|f\|_2).$$

Sobolev embedding then yields

$$\left\| \frac{\phi''}{U + i\lambda} \right\|_2 \leq C(r)(\beta^{1/2}\|\phi\|_{1,2} + \beta^{-1/3}\|f\|_2). \quad (7.29)$$

Finally, we turn to estimate the last term on the right-hand-side of (7.27). Suppose first that for some $x_\nu \in [-1, 1]$ we have $U(x_\nu) = \nu$. Clearly, as

$$|U(x) + i\lambda| \geq \left| \mathbf{m}(x - x_\nu) + \frac{1}{2}i J_m^{2/3} \Re \nu_1 \beta^{-1/3} \right| \quad (7.30)$$

it holds by (7.24) that

$$\left\| \frac{U''\phi}{|U + i\lambda|^3} \right\|_2 \leq C\delta \left\| \frac{\phi}{|(x - x_\nu) + i J_m^{2/3} \Re \nu_1 \beta^{-1/3}|^3} \right\|_2.$$

Hence, we obtain

$$\left\| \frac{U''\phi}{|U + i\lambda|^3} \right\|_2 \leq \hat{C}\delta\beta^{5/6}\|\phi\|_\infty. \quad (7.31)$$

Substituting the above together with (7.29) and (7.28) into (7.27) yields with the aid of Sobolev embedding and Poincare inequality

$$\|h\|_2 \leq \|f\|_2 + C(r)(\delta\beta^{5/6} + \beta^{\frac{2}{3}})\|\phi\|_{1,2},$$

which for $\beta \geq \beta_0(\delta)$ gives

$$\|h\|_2 \leq \|f\|_2 + \hat{C}(r)\delta\beta^{5/6}\|\phi\|_{1,2}. \quad (7.32)$$

By (7.26) we then have

$$\|v\|_2 \leq \frac{C}{\beta}\|f\|_2 + C(r)\delta\beta^{-1/6}\|\phi\|_{1,2}. \quad (7.33)$$

The proof can now be completed, for $\Re \nu_1/2 \leq J_m^{-2/3}\beta^{1/3}\mu \leq \Upsilon$ or $\beta^{1/3}\mu \leq -J_m^{2/3}\Re \nu_1/2$, by using (4.25) with $p = 2$. In the case where $U(x) - \nu \neq 0$ on $[-1, +1]$, we may still apply the previous arguments by replacing (7.30) with

$$|U(x) + i\lambda| \geq \left| \mathbf{m}(x \pm 1) + \frac{1}{2}i J_m^{2/3} \Re \nu_1 \beta^{-1/3} \right| \quad (7.34)$$

where \pm denotes $+$ for $\nu < \inf U(x)$ and $-$ for $\nu > \sup U(x)$.

Case 2: $\beta^{1/3}|\mu| < J_m^{2/3}\Re \nu_1/2$

Let

$$s = \beta^{1/3} \left(\frac{J_m^{2/3}\Re \nu_1}{2} - \mu \right),$$

and then write

$$\mathcal{B}_{\lambda+s\beta^{-1/3},\alpha}^{\mathfrak{E}}\phi = f - s\beta^{2/3}(\phi'' - \alpha^2\phi).$$

Since $\Re(\lambda + s\beta^{-1/3}) = \beta^{-1/3}J_m^{2/3}\Re\nu_1/2$, we can use (7.23) to obtain that

$$\|\phi\|_{1,2} \leq C(\beta^{-5/6}\|f\|_2 + \beta^{-1/6}\|\phi'' - \alpha^2\phi\|_2). \quad (7.35)$$

We now recall (7.9)

$$(\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)(\phi'' - \alpha^2\phi) = i\beta U''\phi + f,$$

and then use (5.8) and (5.21) to establish that

$$\|\phi'' - \alpha^2\phi\|_2 \leq C(\beta^{-2/3}\|f\|_2 + \delta\beta^{1/6}\|\phi\|_\infty) \quad (7.36)$$

Substituting the above into (7.35) yields

$$\|\phi\|_{1,2} \leq C(\beta^{-5/6}\|f\|_2 + \delta\|\phi\|_\infty).$$

Using Sobolev embedding yields (7.23), for sufficiently small δ and $\beta^{1/3}\mu \leq \Upsilon$. ■

8 No slip Orr-Sommerfeld operator

In contrast with the prescribed traction condition, the auxiliary function $v = \mathcal{A}_{\lambda,\alpha}\phi$, does not satisfy, if $\phi \in D(B_{\lambda,\alpha,\beta}^{\mathfrak{D}})$, a Dirichlet boundary condition. Consequently, special attention must be given to the behaviour of v near the boundary through (6.20) for instance. Since ϕ satisfies a Dirichlet boundary condition, we expect that the rapidly decaying boundary terms $\psi_\pm - \tilde{v}_\pm$ in (6.20) should have a negligible contribution to ϕ compared with that of u (i.e., $\|\mathcal{A}_{\lambda,\alpha}^{-1}(\psi_\pm - \tilde{v}_\pm)\|_{1,2} \ll \|\mathcal{A}_{\lambda,\alpha}^{-1}u\|_{1,2}$). The next subsection is dedicated to the establishment of such estimates.

8.1 Preliminaries

We recall that $J_\pm = U'(\pm 1)$ and that ϑ_1^r and ψ_\pm are respectively defined by (6.10), (6.6) and (6.7). The next lemma holds true under the assumptions of either Proposition 4.9 or Proposition 4.13.

Lemma 8.1. *For any $r > 1$ and $\Upsilon < \vartheta_1^r$, there exist positive constants C and β_0 such that, for all $\beta \geq \beta_0$, $\lambda \in \mathbb{C}$ for which $\Re\lambda \leq \beta^{-1/3}J_\pm^{2/3}\Upsilon$, and $U \in \mathcal{S}_r$ satisfying either (2.29) or (4.19), it holds that*

$$\|\mathcal{A}_{\lambda,\alpha}^{-1}(U + i\lambda)\psi_\pm\|_{1,2} \leq C[1 + \beta^{1/3}|\lambda_\pm|]^{-1/4}\beta^{-1/2}. \quad (8.1)$$

where

$$\lambda_\pm = \mu + i(\nu - U(\pm 1)). \quad (8.2)$$

Proof.

Let $\phi = \mathcal{A}_{\lambda, \alpha}^{-1}(U + i\lambda)\psi_{\pm}$. We write

$$\left| \left\langle \frac{\phi}{U + i\lambda}, (U + i\lambda)\psi_{\pm} \right\rangle \right| \leq \|(\phi - \phi(\pm 1))\psi_{\pm}\|_1. \quad (8.3)$$

As

$$|\phi(x) - \phi(\pm 1)| \leq \|\phi\|_{1,2} \| (1 \mp x)^{1/2} \|, \quad \forall x \in (-1, 1),$$

we may write

$$\|(\phi - \phi(\pm 1))\psi_{\pm}\|_1 \leq \|\phi\|_{1,2} \|(1 \mp x)^{1/2}\psi_{\pm}\|_1.$$

It follows from (6.27) (with $s = \frac{1}{2}$) that for some positive C

$$\|(1 \mp x)^{1/2}\psi_{\pm}\|_1 \leq C [1 + |\lambda_{\pm}| \beta^{1/3}]^{-1/4} \beta^{-1/2}. \quad (8.4)$$

In the case where (4.19) is satisfied, we may use (4.40) so that

$$\|\phi\|_{1,2} \leq C \|(1 \mp x)^{1/2}\psi_{\pm}\|_1,$$

which, combined with (8.4) yields (8.1).

In the case where (2.29) is satisfied, we may use (4.88) for $v = (U + i\lambda)\psi_{\pm}$ to obtain that

$$\|\mathcal{A}_{\lambda, \alpha}^{-1}(U + i\lambda)\psi_{\pm}\|_{1,2} \leq C \|(1 \mp x)^{1/2}\psi_{\pm}\|_1.$$

Then, we apply (8.4) to obtain (8.1). ■

We shall also need the following

Lemma 8.2. *Let $r > 1$ and $\Upsilon < \vartheta_1^r$. Let further $(\lambda, \check{\lambda}) \in \mathbb{C}^2$ satisfy $\lambda - \check{\lambda} \in \mathbb{R}$, $\Re \lambda$ and $\Re \check{\lambda}$ are in $(-\infty, \beta^{-1/3} J_{\pm}^{2/3} \Upsilon)$ and $|\Re \lambda - \Re \check{\lambda}| \leq 2\vartheta_1^r \beta^{-1/3}$. Then, there exist positive C and β_0 such that, for all $\beta \geq \beta_0$, and all $U \in \mathcal{S}_r$ satisfying either (2.29) or (4.19), it holds that*

$$\|\mathcal{A}_{\check{\lambda}, \alpha}^{-1}(U + i\check{\lambda})\Gamma_{[-1,1]}(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\check{\lambda})^{-1}\tilde{\psi}_{\pm}(\lambda)\|_{1,2} \leq C [1 + |\lambda_{\pm}| \beta^{1/3}]^{-1} \beta^{-7/6}, \quad (8.5)$$

where

$$\tilde{\psi}_{\pm}(x, \lambda) = \begin{cases} \psi_{\pm}(x, \lambda) & x \in [-1, 1] \\ 0 & |x| > 1. \end{cases}$$

Proof.

For later reference we note that the requirement set above that $|\lambda - \check{\lambda}| \leq 2\vartheta_1^r \beta^{-1/3}$ implies the existence of $C > 0$ such that

$$\frac{1}{C} [1 + |\lambda_{\pm}| \beta^{1/3}] \leq [1 + |\check{\lambda}_{\pm}| \beta^{1/3}] \leq C [1 + |\lambda_{\pm}| \beta^{1/3}]$$

and

$$\frac{1}{C} (1 + \beta^{1/3} |\mu|) \leq (1 + \beta^{1/3} |\check{\mu}|) \leq C (1 + \beta^{1/3} |\mu|),$$

where $\check{\mu} = \Re\check{\lambda}$.

Step 1: We prove that there exist positive C , and β_0 such that for all $\beta > \beta_0$ and $k \in \{0, 1, 2\}$

$$\|(1 \mp x)^k (\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\check{\lambda})^{-1} \tilde{\psi}_{\pm}(\lambda)\|_2 \leq C \beta^{-(5/6+k/2)} [1 + \beta^{1/3} |\lambda_{\pm}|]^{-3/4-k/2}. \quad (8.6)$$

For convenience of notation we prove (8.6) only for $(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\check{\lambda})^{-1} \tilde{\psi}_+(\lambda)$. The proof for $(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\check{\lambda})^{-1} \tilde{\psi}_-(\lambda)$ can be obtained in a similar manner.

Let $u_k = (1-x)^k (\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\check{\lambda})^{-1} \tilde{\psi}_+$, for $k \in \{0, 1, 2, 3\}$. For convenience we also set $u_k \equiv 0$ for all $k \leq -1$.

The case $k = 0$.

For $k = 0$, we observe from (5.4) and (6.17), that we have

$$\|u_0\|_2 \leq \frac{C}{\beta^{5/6}(1 + \beta^{1/3} |\mu|)} [1 + \beta^{1/3} |\lambda_+|]^{1/4}. \quad (8.7)$$

We note that (8.6) for $k = 0$ does not follow from (8.7). Some additional estimates for large values $\beta^{1/3} |\nu|$ should be obtained to this end. For $k \in \{1, 2, 3\}$, we now write:

$$(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\check{\lambda})u_k = (1-x)^k \tilde{\psi}_{\pm}(\lambda) - 2ku'_{k-1} + k(k-1)u_{k-2}. \quad (8.8)$$

By (5.4) it holds that

$$\|u_k\|_2 \leq \frac{C}{\beta^{2/3}(1 + \beta^{1/3} |\mu|)} (\|(1-x)^k \tilde{\psi}_{\pm}(\lambda)\|_2 + \|u'_{k-1}\|_2 + (k-1)\|u_{k-2}\|_2). \quad (8.9)$$

As

$$\Re\langle u_{k-1}, (\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\check{\lambda})u_{k-1} \rangle = \|u'_{k-1}\|_2^2 - \beta\check{\mu}\|u_{k-1}\|_2^2, \quad (8.10)$$

we obtain, as $\check{\mu} \leq \beta^{-1/3} J_{\pm}^{2/3} \Upsilon$, that

$$\|u'_{k-1}\|_2 \leq C \left(\beta^{1/3} \|u_{k-1}\|_2 + \beta^{-1/3} (\|(1-x)^{k-1} \tilde{\psi}_{\pm}(\lambda)\|_2 + (k-1)\|u'_{k-2}\|_2 + (k-1)(k-2)\|u_{k-3}\|_2) \right). \quad (8.11)$$

Substituting the above into (8.9) yields, for $k = 1$,

$$\|u_1\|_2 \leq C \beta^{-\frac{2}{3}} (1 + \beta^{\frac{1}{3}} |\mu|)^{-1} \left(\|(1-x) \tilde{\psi}_{\pm}(\lambda)\|_2 + \beta^{\frac{1}{3}} \|u_0\|_2 + \beta^{-\frac{1}{3}} \|\tilde{\psi}_{\pm}(\lambda)\|_2 \right).$$

Using (6.17) for $k = 0$ and $k = 1$, which holds as $\beta^{1/3} \Re\lambda \leq \Upsilon < J_+^{2/3} \vartheta_1^r$, yields

$$\|u_1\|_2 \leq \frac{C}{\beta^{2/3}(1 + \beta^{1/3} |\mu|)} ([1 + \beta^{1/3} |\lambda_+|]^{1/4} \beta^{-1/2} + \beta^{1/3} \|u_0\|_2). \quad (8.12)$$

From (8.7) and (8.12) we then get

$$\|u_1\|_2 \leq C \beta^{-7/6} (1 + \beta^{1/3}|\mu|)^{-1} [1 + \beta^{1/3}|\lambda_+|]^{1/4}. \quad (8.13)$$

Using (8.13) we can now complete the proof of (8.6) for $k = 0$. To this end we write

$$[\beta^{-1/3} + |\lambda_+|]\|u_0\|_2 \leq (\beta^{-1/3} + |\mu|)\|u_0\|_2 + \|[U - U(1)]u_0\|_2 + \|(U - \nu)u_0\|_2.$$

From (5.5) and (6.17), we deduce that

$$\|(U - \nu)u_0\|_2 \leq C\beta^{-1}\|\tilde{\psi}_\pm(\lambda)\|_2 \leq \hat{C}\beta^{-7/6}[1 + \beta^{1/3}|\lambda_+|]^{-3/4}.$$

On the other hand, we have, by (8.13)

$$\begin{aligned} \|[U - U(1)]u_0\|_2 &\leq C\|u_1\|_2 \\ &\leq \hat{C}\beta^{-7/6}(1 + \beta^{1/3}|\mu|)^{-1}[1 + \beta^{1/3}|\lambda_+|]^{1/4} \\ &\leq \hat{C}\beta^{-7/6}[1 + \beta^{1/3}|\lambda_+|]^{1/4}. \end{aligned}$$

Together with (8.7), we obtain

$$\|u_0\|_2 \leq \frac{C}{\beta^{5/6}}[1 + \beta^{1/3}|\lambda_+|]^{-3/4}, \quad (8.14)$$

which proves (6.11) for $k = 0$.

The case $k = 1$.

By (8.10) we may write, instead of (8.11),

$$\|u'_0\|_2 \leq C(\beta^{1/3}[1 + \beta^{1/3}|\lambda_+|]^{1/2}\|u_0\|_2 + \beta^{-1/3}[1 + \beta^{1/3}|\lambda_+|]^{-1/2}\|\tilde{\psi}_\pm(\lambda)\|_2).$$

Hence, by (6.17), it holds that

$$\|u'_0\|_2 \leq \frac{C}{\beta^{1/2}}[1 + \beta^{1/3}|\lambda_+|]^{-1/4}, \quad (8.15)$$

which, when substituted into (8.9) for $k = 1$, yields

$$\|u_1\|_2 \leq \frac{C}{\beta^{7/6}(1 + \beta^{1/3}|\mu|)}[1 + \beta^{1/3}|\lambda_+|]^{-1/4}.$$

Substituting into (8.11) yields, for $k = 2$,

$$\|u'_1\|_2 \leq \frac{C}{\beta^{5/6}}[1 + \beta^{1/3}|\lambda_+|]^{-1/4}, \quad (8.16)$$

and hence, by (8.9) for $k = 2$,

$$\|u_2\|_2 \leq \frac{C}{\beta^{3/2}(1 + \beta^{1/3}|\mu|)}[1 + \beta^{1/3}|\lambda_+|]^{-1/4}. \quad (8.17)$$

As above we now write

$$[\beta^{-1/3} + |\lambda_+|]\|u_1\|_2 \leq (\beta^{-1/3} + |\mu|)\|u_1\|_2 + C\|u_2\|_2 + \|(U - \nu)u_1\|_2, \quad (8.18)$$

to obtain from (8.17), (8.16), (5.5), and (6.17) that

$$\|u_1\|_2 \leq \frac{C}{\beta^{7/6}}[1 + \beta^{1/3}|\lambda_+|]^{-5/4}. \quad (8.19)$$

The case $k = 2$

We briefly repeat the same argument as in the case $k = 1$. By (8.10) with $k = 2$ we may write (instead of using (8.11))

$$\|u'_1\|_2 \leq C(\beta^{1/3}[1+\beta^{1/3}|\lambda_+|]^{1/2}\|u_1\|_2+\beta^{-1/3}[1+\beta^{1/3}|\lambda_+|]^{-1/2}[\|(1-x)\tilde{\psi}_\pm(\lambda)\|_2+\|u'_0\|_2]).$$

From which we obtain, with the aid of (8.19) and (8.15)

$$\|u'_1\|_2 \leq \frac{C}{\beta^{5/6}}[1 + \beta^{1/3}|\lambda_+|]^{-3/4}. \quad (8.20)$$

Substituting (8.20) into (8.9) with $k = 2$ then yields with the aid of (8.14)

$$\|u_2\|_2 \leq \frac{C}{\beta^{3/2}(1 + \beta^{1/3}|\mu|)}[1 + \beta^{1/3}|\lambda_+|]^{-3/4}.$$

Substituting the above, (8.20), and (8.14) into (8.11), with $k = 3$ yields

$$\|u'_2\|_2 \leq \frac{C}{\beta^{7/6}(1 + \beta^{1/3}|\mu|)}[1 + \beta^{1/3}|\lambda_+|]^{-3/4}.$$

From (8.9) with $k = 3$ we then obtain

$$\|u_3\|_2 \leq \frac{C}{\beta^{11/6}(1 + \beta^{1/3}|\mu|)^2}[1 + \beta^{1/3}|\lambda_+|]^{-3/4}.$$

As in (8.18) and (8.19) we can now obtain that

$$\|u_2\|_2 \leq \frac{C}{\beta^{3/2}}[1 + \beta^{1/3}|\lambda_+|]^{-7/4}.$$

Combining the above with (8.19) and (8.14) yields (8.12).

Step 2: We prove (8.5).

We first observe that by interpolation (8.6) holds for any $k \in [0, 2]$. If the conditions of (4.9) are met, we may now obtain (8.5) from (4.19) as in the proof of (8.1).

Otherwise if the assumptions of Proposition 4.13 are met, we may conclude (8.1) from (4.88). ■

8.2 Nearly Couette flows

We begin by considering the case where the flow U is nearly linear, as in Subsection 7.3.

8.2.1 The case $0 \leq \alpha \leq \tilde{\alpha}_0\beta^{1/3}$

Let $\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D}}$ be defined by (3.5). We can now state and prove

Proposition 8.3. *For every $r > 1$ and $\Upsilon < \vartheta_1^r$, there exist positive δ , β_0 , α_0 , and C such that, for all $U \in \mathcal{S}_r$ satisfying $0 < \delta_2(U) \leq \delta$ (where δ_2 is given by (2.26)) and $\beta \geq \beta_0$, it holds that*

$$\sup_{\substack{\Re\lambda \leq \Upsilon J_m^{2/3} \beta^{-1/3} \\ 0 \leq \alpha \leq \alpha_0 \beta^{1/3}}} \left\| (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}})^{-1} \right\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}})^{-1} \right\| \leq C \beta^{-5/6}. \quad (8.21)$$

Proof. Let $\phi \in D(\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}})$ and $f \in L^2(-1, 1)$ satisfy $\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}} \phi = f$. Let $\lambda = \mu + i\nu$. Let further v be given by (7.3), and set $\tilde{v} = (U + i\lambda)^{-1}v$. Without any loss of generality we select

$$\Upsilon > \vartheta_1^r/2.$$

Step 1: We prove (8.21) for $\lambda \in \mathbb{C}$ satisfying either $\vartheta_1^r/2 \leq J_m^{-2/3} \beta^{1/3} \Re\lambda \leq \Upsilon$ or $-3r \leq \Re\lambda \leq -J_m^{2/3} \beta^{-1/3} \vartheta_1^r/2$, and $|\Im\lambda| \leq 3r$.

Let $\zeta_{\pm} \in C^2([-1, 1])$ satisfy the problem

$$\begin{cases} -(U + i\lambda)\zeta_{\pm}'' + (\alpha^2[U + i\lambda] + U'')\zeta_{\pm} = 0 & \text{in } (-1, 1) \\ \zeta_{\pm}(\pm 1) = 1 \text{ and } \zeta_{\pm}(\mp 1) = 0 \end{cases}. \quad (8.22)$$

Note that the differential operator on the left-hand-side is identical with that of $\mathcal{A}_{\lambda, \alpha}$ given by (3.13). We now write

$$\zeta_{\pm}(x) = \zeta_{\pm}^0(x) + \exp\{-\alpha(1 \mp x)\} \tilde{\eta}_{\pm}(x) \quad (8.23)$$

in which $\tilde{\eta}_{\pm}(x) = \eta(1 \mp x)$, where η is given by (5.14), and $\zeta_{\pm}^0 \in D(\mathcal{A}_{\lambda, \alpha})$.

ζ_{\pm}^0 and ζ_{\pm} estimates

We begin by establishing L^∞ and $W^{1,q}$ estimates for ζ_{\pm}^0 . To this end we first write

$$\mathcal{A}_{\lambda, \alpha} \zeta_{\pm}^0 = -(U'' \tilde{\eta}_{\pm} + 2\alpha(U + i\lambda) \tilde{\eta}_{\pm}' + \tilde{\eta}_{\pm}''(U + i\lambda)) \exp\{-\alpha(1 \mp x)\}. \quad (8.24)$$

Note that since the derivatives of $\tilde{\eta}$ are supported on $[-1/2, 1/2]$ we have, since $|\mu|$ is bounded,

$$\|(2\alpha(U + i\lambda) \tilde{\eta}_{\pm}' + \tilde{\eta}_{\pm}''(U + i\lambda)) \exp\{-\alpha(1 \mp \cdot)\}\|_{1, \infty} \leq C(1 + |\nu|)(1 + \alpha^2) e^{-\alpha/2} \leq C(1 + |\nu|).$$

Note further that, for all $1 \leq q \leq 2$, we have

$$\|U'' \tilde{\eta}_{\pm} \exp\{-\alpha(1 \mp \cdot)\}\|_q \leq C(1 + \alpha)^{-1/q}.$$

and

$$\|U'' \tilde{\eta}_{\pm} \exp\{-\alpha(1 \mp \cdot)\}\|_{1, q} \leq C(1 + \alpha)^{1-1/q}.$$

Consequently, for all $1 \leq q \leq 2$, we have

$$\|\mathcal{A}_{\lambda, \alpha} \zeta_{\pm}^0\|_q \leq C(1 + |\nu| + (1 + \alpha)^{-1/q}) \leq \hat{C}(1 + |\nu|). \quad (8.25a)$$

and

$$\|\mathcal{A}_{\lambda,\alpha}\zeta_{\pm}^0\|_{1,q} \leq C(1 + |\nu| + \alpha^{1-1/q}). \quad (8.25b)$$

Observing that

$$\Re\langle \zeta_{\pm}^0, (U + i\lambda)^{-1} \mathcal{A}_{\lambda,\alpha} \zeta_{\pm}^0 \rangle = \frac{1}{2} |(\zeta_{\pm}^0)'|^2 + \alpha^2 |\zeta_{\pm}^0|^2 + I(\zeta_{\pm}^0, \lambda),$$

we deduce from Lemma 4.7 that, for sufficiently small δ there exist $\gamma_0 > 0$ and $C > 0$ such that, for any $\alpha \geq 0$ and any $\lambda \in \mathbb{C} \setminus \mathcal{J}$, (recall that $|\Re\lambda| > J_m^{2/3} \beta^{-1/3} \vartheta_1^r / 2$ in this step)

$$\begin{aligned} \Re\langle \zeta_{\pm}^0, (U + i\lambda)^{-1} \mathcal{A}_{\lambda,\alpha} \zeta_{\pm}^0 \rangle &\geq \\ \frac{1}{2} \|(\zeta_{\pm}^0)'\|_2^2 + (\gamma_0 + \alpha^2) \|\zeta_{\pm}^0\|_2^2 &\geq C(1 + \alpha)^{1/2} \|(\zeta_{\pm}^0)'\|_2^{3/2} \|\zeta_{\pm}^0\|_2^{1/2}. \end{aligned} \quad (8.26)$$

Consequently, by (4.33) (which is applied with $q = p$, $\phi = \zeta_{\pm}^0$ and $v = A_{\lambda,\alpha} \zeta_{\pm}^0$) and (8.25), for any $1 < q < 2$ there exists $C_q > 0$ such that

$$\begin{aligned} \|(\zeta_{\pm}^0)'\|_2^{3/2} \|\zeta_{\pm}^0\|_2^{1/2} &\leq C_q (1 + \alpha)^{-1/2} (\|A_{\lambda,\alpha} \zeta_{\pm}^0\|_q \|\zeta_{\pm}^0\|_{\infty} + \|A_{\lambda,\alpha} \zeta_{\pm}^0\|_{\infty} \|(\zeta_{\pm}^0)'\|_q) \\ &\leq C_q (1 + |\nu|) (1 + \alpha)^{-1/q+1/2} \|(\zeta_{\pm}^0)'\|_2. \end{aligned}$$

(We use the fact that $q \leq 2$ and Poincaré's inequality to obtain the last inequality.) We may thus conclude that

$$\|(\zeta_{\pm}^0)'\|_2^{1/2} \|\zeta_{\pm}^0\|_2^{1/2} \leq C_q (1 + |\nu|) (1 + \alpha)^{-1/q+1/2}.$$

Hence, for $q = \frac{3}{2}$, we get

$$\|\zeta_{\pm}^0\|_{\infty} \leq \|(\zeta_{\pm}^0)'\|_2^{1/2} \|\zeta_{\pm}^0\|_2^{1/2} \leq C_q (1 + |\nu|) (1 + \alpha)^{-1/6},$$

which finally implies

$$\|\zeta_{\pm}^0\|_{\infty} \leq C(1 + |\nu|). \quad (8.27)$$

By (8.26) we also have that

$$\Re\langle \zeta_{\pm}^0, (U + i\lambda)^{-1} \mathcal{A}_{\lambda,\alpha} \zeta_{\pm}^0 \rangle \geq \frac{1}{2} \|(\zeta_{\pm}^0)'\|_2^2,$$

and hence

$$\|(\zeta_{\pm}^0)'\|_2 \leq C_q (1 + |\nu|) (1 + \alpha)^{-1/q+1}.$$

For $q = \frac{3}{2}$ we may thus conclude

$$\|(\zeta_{\pm}^0)'\|_2 \leq C(1 + |\nu|)(1 + \alpha)^{\frac{1}{3}}. \quad (8.28)$$

As (8.27) and (8.28) are unsatisfactory for large values of ν , we use the fact that

$$\begin{aligned} \Re\langle (U - \nu)^2 \zeta_{\pm}^0, (U + i\lambda)^{-1} \mathcal{A}_{\lambda,\alpha} \zeta_{\pm}^0 \rangle &= \|((U - \nu)\zeta_{\pm}^0)'\|_2^2 - \|U'\zeta_{\pm}^0\|_2^2 + \alpha^2 \|(U - \nu)\zeta_{\pm}^0\|_2^2 \\ &\quad - \Re\langle (U - \nu)^2 \zeta_{\pm}^0, (U + i\lambda)^{-1} U'' \zeta_{\pm}^0 \rangle. \end{aligned}$$

Assuming $|\nu| \geq 2|U|_\infty$, we first obtain, for any $\varepsilon > 0$

$$\varepsilon \|(U - \nu)\zeta_\pm^0\|_2^2 + \frac{4}{\varepsilon} \|A_{\lambda, \alpha} \zeta_\pm^0\|_2^2 \geq \|((U - \nu)\zeta_\pm^0)'\|_2^2 - C \|(U - \nu)\zeta_\pm^0\|_2 \|\zeta_\pm^0\|_2.$$

Poincaré's inequality then yields, for sufficiently small $\varepsilon > 0$

$$\nu \|\zeta_\pm^0\|_{1,2} \leq C(\|A_{\lambda, \alpha} \zeta_\pm^0\|_2 + \|\zeta_\pm^0\|_2),$$

which implies, for sufficiently large $|\nu|$

$$\|\zeta_\pm^0\|_{1,2} \leq C. \quad (8.29)$$

We can now use (8.25a), (8.27) and (8.29) to obtain, for any ν ,

$$\|\zeta_\pm^0\|_{1,2} \leq C(1 + \alpha)^{\frac{1}{3}}. \quad (8.30)$$

Combining (8.29) with (8.27) implies, in addition, the following refinement of (8.27) to any ν

$$\|\zeta_\pm^0\|_\infty \leq C. \quad (8.31)$$

Substituting (8.30) and (8.31) into (8.23) respectively yields

$$\|\zeta_\pm\|_{1,2} \leq C(1 + \alpha_0^{1/2} \beta^{1/6}). \quad (8.32a)$$

and

$$\|\zeta_\pm\|_\infty \leq C. \quad (8.32b)$$

Application of Lemma 6.2

It can be easily verified that

$$\langle \zeta_\pm, \tilde{v} \rangle = 0. \quad (8.32c)$$

We can thus consider the problem

$$\left(-\frac{d^2}{dx^2} + i\beta(U + i\lambda) \right) \tilde{v} = h,$$

where \tilde{v} satisfies (8.32c) and h is given by (see (7.25))

$$h = f + \left(\frac{U''\phi}{U + i\lambda} \right)''.$$

By (8.32) we may now use Lemma 6.2 to conclude from (6.20) (applied for $(g, u, v) = (h, \tilde{u}, \tilde{v})$) that \tilde{v} admits the decomposition

$$\tilde{v} = A_+(h)(\psi_+ - \tilde{v}_+) + A_-(h)(\psi_- - \tilde{v}_-) + \tilde{u}. \quad (8.33)$$

where $A_\pm(h)$ and \tilde{v}_\pm respectively satisfy (6.24) and (6.14), and \tilde{u} is given by

$$\tilde{u} = \Gamma_{[-1,1]}(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\lambda)^{-1} \tilde{h}, \quad (8.34)$$

where

$$\tilde{h}(x) = \begin{cases} h(x) & x \in [-1, 1] \\ 0 & \text{otherwise.} \end{cases}$$

By (5.4) we then have

$$\|\tilde{u}\|_2 \leq C\beta^{-2/3} \|h\|_2. \quad (8.35)$$

Estimate of h .

Since (7.27), (7.28), and (7.31) are still valid, we obtain

$$\|h\|_2 \leq \|f\|_2 + C(\delta\beta^{5/6} + \beta^{2/3})\|\phi\|_{1,2} + \left\| \frac{\phi''}{U+i\lambda} \right\|_2.$$

which leads for $\beta \geq \beta_0(\delta)$ to

$$\|h\|_2 \leq \|f\|_2 + \hat{C}\delta\beta^{5/6}\|\phi\|_{1,2} + \left\| \frac{\phi''}{U+i\lambda} \right\|_2. \quad (8.36)$$

To bound the last term on the right-hand-side we write, recalling that $|\mu| \geq J_m^{2/3}\beta^{-1/3}\vartheta_1^r/2$ in this step,

$$\left\| \frac{\phi''}{U+i\lambda} \right\|_2 \leq C\beta^{1/3}\|\phi''\|_2. \quad (8.37)$$

Then, we use the fact that by (7.3) and (3.13)

$$\|\phi'' - \alpha^2\phi\|_2 \leq \|\tilde{v}\|_2 + \|U''(U+i\lambda)^{-1}\phi\|_2.$$

As in the proof of (7.31), we get

$$\|U''(U+i\lambda)^{-1}\phi\|_2 \leq C\delta\beta^{1/6}\|\phi\|_\infty.$$

We now obtain a bound for the norm of \tilde{v} by estimating each term appearing in the right hand side of (8.33). By (6.14) we get for \tilde{v}_\pm ,

$$\|\tilde{v}_\pm\| \leq C\beta^{-\frac{1}{2}},$$

By (6.17) (for $k=0$) we get for ψ_\pm ,

$$\|\psi_\pm\| \leq C\beta^{-\frac{1}{6}}[1 + |\lambda_\pm|\beta^{1/3}]^{\frac{1}{4}}, \quad (8.38)$$

and by (6.24) we get for $A_\pm(h)$

$$|A_\pm(h)| \leq C\beta^{-\frac{1}{2}}\|h\|_2. \quad (8.39)$$

Together with (8.35) for \tilde{u} , we then have

$$\|\tilde{v}\|_2 \leq C\beta^{-2/3}[1 + |\lambda_\pm|\beta^{1/3}]^{\frac{1}{4}}\|h\|_2,$$

where λ_\pm is given by (6.15).

Since (7.12) remains valid for no-slip conditions we may conclude that

$$\|\phi''\|_2 \leq C(\beta^{-2/3}[1 + |\lambda_\pm|\beta^{1/3}]^{\frac{1}{4}}\|h\|_2 + \delta\beta^{1/6}\|\phi\|_\infty).$$

Since $|\lambda_\pm|$ is bounded in this step, we obtain

$$\|\phi''\|_2 \leq C(\beta^{-7/12}\|h\|_2 + \delta\beta^{1/6}\|\phi\|_\infty). \quad (8.40)$$

Combining the above with (8.36) and (8.37) yields

$$\|h\|_2 \leq (1 + C\beta^{-1/4})\|f\|_2 + C\delta\beta^{5/6}\|\phi\|_{1,2}. \quad (8.41)$$

For later reference we note that by (7.31), (8.40) and (8.41) we have

$$\left\| \left(\frac{U''\phi}{U+i\lambda} \right)'' \right\|_2 \leq C(\beta^{-1/4}\|f\|_2 + \delta\beta^{5/6}\|\phi\|_{1,2}). \quad (8.42)$$

Estimates of ϕ

Next we set, with (8.33) in mind,

$$v = v_o + v_+ + v_-,$$

where

$$v_{\pm} = A_{\pm}(h)(U + i\lambda)(\psi_{\pm} - \tilde{v}_{\pm}), \quad (8.43)$$

and

$$v_o = (U + i\lambda)\Gamma_{[-1,1]}\tilde{u} = (U + i\lambda)\Gamma_{[-1,1]}(\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\lambda)^{-1}\tilde{h}. \quad (8.44)$$

Set further

$$\phi_{\pm} = \mathcal{A}_{\lambda,\alpha}^{-1}v_{\pm} \quad ; \quad \phi_o = \mathcal{A}_{\lambda,\alpha}^{-1}v_o.$$

Clearly,

$$\phi = \phi_o + \phi_+ + \phi_-. \quad (8.45)$$

To estimate ϕ_{\pm} we write

$$\phi_{\pm} = A_{\pm}(h)\mathcal{A}_{\lambda,\alpha}^{-1}(U + i\lambda)\psi_{\pm} - A_{\pm}(h)\mathcal{A}_{\lambda,\alpha}^{-1}(U + i\lambda)\tilde{v}_{\pm}. \quad (8.46)$$

For the first term in the right hand side, we use (8.1) to obtain

$$\|A_{\pm}(h)\mathcal{A}_{\lambda,\alpha}^{-1}(U + i\lambda)\psi_{\pm}\|_{1,2} \leq C\beta^{-\frac{1}{2}}|A_{\pm}(h)|.$$

Using (4.25) with $p = 2$ and (6.14), yields for the second term

$$\|A_{\pm}(h)\mathcal{A}_{\lambda,\alpha}^{-1}(U + i\lambda)\tilde{v}_{\pm}\|_{1,2} \leq C\beta^{-1}|A_{\pm}(h)|.$$

Hence using (8.39), we obtain

$$\|\phi_{\pm}\|_{1,2} \leq C(\beta^{-1}\|f\|_2 + \delta\beta^{-1/6}\|\phi\|_{1,2}). \quad (8.47)$$

To estimate ϕ_o we first recall that by (8.34) and (8.44)

$$v_o = (U + i\lambda)\Gamma_{[-1,1]}\tilde{u} = (U + i\lambda)\Gamma_{[-1,1]}(\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\lambda)^{-1}\tilde{h}.$$

By (5.4) and (5.5) we then have

$$\|v_o\|_2 \leq \frac{C}{\beta}\|h\|_2,$$

and hence by (8.41)

$$\|v_o\|_2 \leq C\beta^{-1}\|f\|_2 + C\delta\beta^{-1/6}\|\phi\|_{1,2}. \quad (8.48)$$

By (4.25) (with $p = 2$), and the definition of ϕ_o we obtain

$$\|\phi_o\|_2 \leq C(\beta^{-5/6}\|f\|_2 + \delta\|\phi\|_{1,2}). \quad (8.49)$$

Substituting (8.49) and (8.47) into (8.45) yields, for sufficiently small δ

$$\|\phi\|_{1,2} \leq C\beta^{-5/6}\|f\|_2,$$

which is precisely (8.21) established in this step for all $\lambda \in \mathbb{C}$ such that $|\Im \lambda| \leq 3r$, and either $\vartheta_1^r/2 \leq \beta^{1/3} J_m^{-2/3} \Re \lambda \leq \Upsilon$ or $-3r \leq \beta^{1/3} J_m^{-2/3} \Re \lambda \leq -\vartheta_1^r/2$.

Step 2: We prove (8.21) for sufficiently small α_0 and λ satisfying $\beta^{1/3} J_m^{-2/3} |\Re \lambda| < \vartheta_1^r/2$ and $|\Im \lambda| \leq 3r$.

Let $\mathfrak{z}_\pm \in C^2([-1, 1])$ be given by (6.68). Note that

$$\|\mathfrak{z}_\pm\|_\infty = 1 \quad ; \quad \|\mathfrak{z}_\pm\|_{1,2} \leq C(1 + \alpha^{1/2}) \leq C(1 + \alpha_0^{1/2} \beta^{1/6}). \quad (8.50)$$

By (3.10), (6.68), and two integrations by parts, it holds that

$$\langle \mathfrak{z}_\pm, -\phi'' + \alpha^2 \phi \rangle = 0. \quad (8.51)$$

As (7.9) is still valid, and in view of (8.50) and (8.51), we can apply Lemma 6.2, assuming that α_0 is small enough, with ζ_\pm replaced by \mathfrak{z}_\pm , to obtain for

$$g = f + i\beta U'' \phi \text{ and } v = -\phi'' + \alpha^2 \phi,$$

that

$$-\phi'' + \alpha^2 \phi = B_+(\psi_+ - \tilde{v}_+) + B_-(\psi_- - \tilde{v}_-) + \Gamma_{(-1,1)}(\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\lambda)^{-1} \tilde{g}, \quad (8.52)$$

where

$$B_\pm = A_\pm(f + i\beta U'' \phi),$$

and

$$\tilde{g} := \gamma_{\mathbb{R}}(f + i\beta U'' \phi)(x) = \begin{cases} (f + i\beta U'' \phi)(x) & x \in [-1, 1] \\ 0 & \text{otherwise.} \end{cases}$$

By (6.24) and the fact that $\|U''\|_\infty \leq \delta$ in this nearly Couette case, it holds that

$$|B_\pm| \leq C(\beta^{-1/2} \|f\|_2 + \delta \beta^{1/3} \log \beta \|\phi\|_\infty). \quad (8.53)$$

By (6.14) together with (8.53), we have

$$\|B_\pm \tilde{v}_\pm\|_2 \leq C(\beta^{-1} \|f\|_2 + \delta \beta^{-1/6} \log \beta \|\phi\|_\infty). \quad (8.54)$$

By (5.4) and (5.35) it holds that

$$\|(\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\lambda)^{-1} \gamma_{\mathbb{R}}(f + i\beta U'' \phi)\|_2 \leq C(\beta^{-2/3} \|f\|_2 + \delta \beta^{1/6} \|\phi\|_\infty).$$

Combining the above with (8.52) and (8.54) then yields

$$\|w_0\|_2 \leq C(\beta^{-2/3} \|f\|_2 + \delta \beta^{1/6} \|\phi\|_\infty), \quad (8.55)$$

where

$$w_0 = -\phi'' + \alpha^2 \phi - B_+ \psi_+ - B_- \psi_-.$$

As in the proof of Proposition 7.3, we use the result of the previous step by considering $\lambda + \Upsilon_0\beta^{-1/3}$ for a suitable value of Υ_0 . We choose Υ_0 such that

$$J_m^{2/3} \frac{\vartheta_1^r}{2} \leq \beta^{1/3} \Re \lambda + \Upsilon_0 \leq J_m^{2/3} \Upsilon. \quad (8.56)$$

We now write

$$\mathcal{B}_{\lambda+\Upsilon_0\beta^{-1/3},\alpha}^{\mathfrak{D}} \phi = f - \Upsilon_0 \beta^{2/3} (w_0 + B_+ \psi_+ + B_- \psi_-),$$

and introduce the following decomposition of ϕ

$$\phi = \chi_0 - \Upsilon_0 \beta^{2/3} (B_+ \chi_+ + B_- \chi_-), \quad (8.57)$$

where

$$\chi_0 = (\mathcal{B}_{\lambda+\Upsilon_0\beta^{-1/3},\alpha}^{\mathfrak{D}})^{-1} (f - \Upsilon_0 \beta^{2/3} w_0) \quad \text{and} \quad \chi_{\pm} = (\mathcal{B}_{\lambda+\Upsilon_0\beta^{-1/3},\alpha}^{\mathfrak{D}})^{-1} \psi_{\pm}.$$

For convenience we set

$$\check{\lambda} = \lambda + \Upsilon_0 \beta^{-1/3}.$$

We may now apply (8.21) (with λ replaced by $\check{\lambda}$) to obtain, with the aid of (8.55) that

$$\|\chi_0\|_2 \leq C(\beta^{-5/6} \|f\|_2 + \delta \|\phi\|_{\infty}). \quad (8.58)$$

Estimate of χ_{\pm} .

We seek an estimate of $\|\chi_+\|_{1,2}$. To this end we repeat the same procedure applied in step 1. For convenience of notation we consider only χ_+ in the following. The same estimates for χ_- can be obtained in a similar manner. Let then

$$\check{w}_+ = -\check{\chi}_+'' + \alpha^2 \check{\chi}_+ + \frac{U''}{U + i\check{\lambda}} \check{\chi}_+,$$

and

$$H_+ := \left(-\frac{d^2}{dx^2} + i\beta(U + i\check{\lambda}) \right) \check{w}_+. \quad (8.59)$$

It can be easily verified that

$$H_+ = \psi_+ + \left(\frac{U'' \chi_+}{U + i\check{\lambda}} \right)''.$$

and that $\langle \zeta_{\pm}, \check{w}_+ \rangle = 0$. Consequently, we can use Lemma 6.2 with λ replaced by $\check{\lambda}$. With the notation $\psi_{\pm} = \psi_{\pm}(\lambda)$, $\check{\psi}_+ = \psi_+(\check{\lambda})$, and $\check{v}_{\pm} = \check{v}_{\pm}(\check{\lambda})$ (as defined in (6.12)) Lemma 6.2 yields

$$\check{w}_+ = \hat{B}_+(\check{\psi}_+ - \check{v}_+) + \hat{B}_-(\check{\psi}_- - \check{v}_-) + (\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\check{\lambda})^{-1} \tilde{H}_+,$$

where $\hat{B}_{\pm} = A_{\pm}(H_+)$ and

$$\tilde{H}_+(x) = \begin{cases} H_+(x) & x \in (-1, 1) \\ 0 & |x| \geq 1. \end{cases}$$

Moreover, we have that

$$|\hat{B}_\pm| \leq C\beta^{-\frac{1}{2}}\|H_+\|_2. \quad (8.60)$$

We now follow the arguments of the previous step with $(\chi_+, \check{\psi}_+, H_+, \check{w}_+)$ respectively replacing (ϕ, f, h, \tilde{v}) . We then reach by the equivalent of (8.42)

$$\|H_+\|_2 \leq (1 + C\beta^{-1/4})\|\psi_+\|_2 + C\delta\beta^{5/6}\|\chi_+\|_{1,2},$$

from which, using (6.17), we get

$$\|H_+\|_2 \leq C\beta^{-1/6}[1 + |\lambda_\pm|^{1/4}\beta^{\frac{1}{12}}] + C\delta\beta^{5/6}\|\chi_+\|_{1,2}.$$

Hence, by (8.60) for $k = 0$, we get

$$|\hat{B}_\pm| \leq C(\beta^{-2/3}[1 + |\lambda_\pm|^{1/4}\beta^{\frac{1}{12}}] + \delta\beta^{1/3}\|\chi_+\|_{1,2}). \quad (8.61)$$

As above we set

$$\chi_+ = \hat{\phi}_+ + \hat{\phi}_- + \hat{\phi}_0, \quad (8.62a)$$

where

$$\hat{\phi}_\pm = \hat{B}_\pm \mathcal{A}_{\check{\lambda}, \alpha}^{-1}(U + i\check{\lambda})(\check{\psi}_\pm - \check{v}_\pm) \quad ; \quad \hat{\phi}_0 = \mathcal{A}_{\check{\lambda}, \alpha}^{-1}(U + i\check{\lambda})\Gamma_{[-1,1]}(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\check{\lambda})^{-1}\tilde{H}_+. \quad (8.62b)$$

By (8.1) and (8.61) we have

$$|\hat{B}_\pm| \|\mathcal{A}_{\check{\lambda}, \alpha}^{-1}(U + i\check{\lambda})\check{\psi}_\pm\|_{1,2} \leq C\left(\beta^{-7/6} + \delta\frac{\beta^{-1/6}}{[1 + |\check{\lambda}_\pm|^{1/4}\beta^{\frac{1}{12}}]}\|\chi_+\|_{1,2}\right).$$

Note here that as $|\lambda - \check{\lambda}| = \Upsilon_0\beta^{-1/3}$ we have used that, for some $C > 1$,

$$C^{-1}[1 + |\lambda_\pm|^{1/4}\beta^{\frac{1}{12}}] \leq [1 + |\check{\lambda}_\pm|^{1/4}\beta^{\frac{1}{12}}] \leq C[1 + |\lambda_\pm|^{1/4}\beta^{\frac{1}{12}}].$$

Moreover, by (8.61), (6.14), and (4.25) (with $p = 2$) it holds that

$$|\hat{B}_\pm| \|\mathcal{A}_{\check{\lambda}, \alpha}^{-1}(U + i\check{\lambda})\check{v}_\pm\|_{1,2} \leq C(\beta^{-4/3}[1 + |\check{\lambda}_\pm|^{1/4}\beta^{\frac{1}{12}}] + \delta\beta^{-1/3}\|\chi_+\|_{1,2}).$$

Combining the above yields

$$\|\hat{\phi}_\pm\|_{1,2} \leq C\left(\beta^{-7/6} + \delta\frac{\beta^{-1/6}}{[1 + |\lambda_\pm|^{1/4}\beta^{\frac{1}{12}}]}\|\chi_+\|_{1,2}\right). \quad (8.63)$$

Next, we estimate $\|\hat{\phi}_0\|_{1,2}$. By (8.5) we have

$$\|\mathcal{A}_{\check{\lambda}, \alpha}^{-1}(U + i\check{\lambda})\Gamma_{[-1,1]}(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\check{\lambda})^{-1}\tilde{\psi}_+\|_{1,2} \leq C\frac{\beta^{-7/6}}{1 + |\nu - U(+1)|\beta^{\frac{1}{3}}}. \quad (8.64)$$

Furthermore, by (5.5), (5.4), and (4.25), we have

$$\left\|\mathcal{A}_{\check{\lambda}, \alpha}^{-1}(U + i\check{\lambda})\Gamma_{[-1,1]}(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta\check{\lambda})^{-1}\left(\frac{U''\chi_+}{U + i\check{\lambda}}\right)''\right\|_{1,2} \leq \frac{C}{\beta^{5/6}}\left\|\left(\frac{U''\chi_+}{U + i\check{\lambda}}\right)''\right\|_2 \quad (8.65)$$

By (8.42) with χ_+ instead of ϕ and ψ_+ replacing f , it holds that

$$\left\| \left(\frac{U'' \chi_+}{U + i\tilde{\lambda}} \right)'' \right\|_2 \leq C \left(\beta^{-5/12} [1 + |\lambda_{\pm}|^{1/4} \beta^{1/12}] + \delta \beta^{5/6} \|\chi_+\|_{1,2} \right).$$

Combining the above with (8.65) leads to

$$\left\| \mathcal{A}_{\tilde{\lambda}, \alpha}^{-1} (U + i\tilde{\lambda}) \Gamma_{[-1,1]} (\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta \tilde{\lambda})^{-1} \left(\frac{U'' \chi_+}{U + i\tilde{\lambda}} \right)'' \right\|_{1,2} \leq C \left(\beta^{-5/4} [1 + |\lambda_{\pm}|^{1/4} \beta^{1/12}] + \delta \|\chi_+\|_{1,2} \right),$$

which, together with (8.64), yields

$$\|\hat{\phi}_o\|_{1,2} \leq C \left(\beta^{-7/6} + \delta \|\chi_+\|_{1,2} \right).$$

Combining the above with (8.62) and (8.63) yields, for $\delta > 0$ small enough,

$$\|\chi_+\|_2 \leq C \beta^{-7/6}.$$

In a similar manner we obtain

$$\|\chi_-\|_2 \leq C \beta^{-7/6}.$$

Substituting the above, (8.58), and (8.53) into (8.57) yields that (8.21) holds for $|\Re \lambda| \leq \beta^{-1/3} J_m^{2/3} \vartheta_1^r / 2$.

Step 3: We prove (8.21) for $\lambda \in \mathbb{C}$ satisfying $\Re \lambda \leq -3r$.

As $f = \mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}} \phi$ and

$$-\Re \langle \phi, \mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}} \phi \rangle = \|\phi''\|_2^2 - (\beta \mu - \alpha^2) \|\phi'\|_2^2 - \beta \mu \alpha^2 \|\phi\|_2^2 - \beta \Im \langle U' \phi, \phi' \rangle,$$

we can conclude that

$$\beta (3r \|\phi'\|_2^2 - \|U'\|_{\infty} \|\phi'\|_2 \|\phi\|_2) \leq \|\phi\|_2 \|f\|_2.$$

Since $\phi \in H_0^1(-1, 1)$ it holds that

$$\|\phi'\|_2 \geq \frac{\pi}{2} \|\phi\|_2,$$

and hence (recall that $\|U'\|_{\infty} < r$ and $3r > 3 > 2/\pi$) we can conclude that there exists $C > 0$ such that

$$\|\phi'\|_2 \leq \frac{C}{\beta} \|f\|_2, \tag{8.66}$$

completing, thereby, the proof of (8.21) for $\Re \lambda \leq -3r$.

Step 4: We prove (8.21) for $\lambda \in \mathbb{C}$ satisfying $|\Im \lambda| \geq 3r$.

An integration by parts yields

$$\Im \langle \phi, \mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}} \phi \rangle = \beta \nu (\|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2) - \beta (\langle U \phi', \phi' \rangle + \alpha^2 \langle U \phi, \phi \rangle + \Re \langle U' \phi, \phi' \rangle + \langle U'' \phi, \phi \rangle),$$

and as

$$\Re\langle U'\phi, \phi' \rangle = -\frac{1}{2}\langle U''\phi, \phi \rangle,$$

we obtain that

$$\beta\left(|\nu| - \|U\|_\infty - \frac{1}{2}\|U''\|_\infty\right)\|\phi'\|_2^2 + \alpha^2\beta(|\nu| - \|U\|_\infty)\|\phi\|_2^2 \leq \|\phi\|_2\|f\|_2.$$

As

$$|\nu| - \|U\|_\infty - \frac{1}{2}\|U''\|_\infty \geq |\nu| - \frac{3}{2}r,$$

we establish (8.66) whenever $|\nu| \geq 3r$.

The proposition is proved. ■

8.2.2 The case $\alpha \geq \alpha^\# \beta^{1/6}$

We separately treat the case $\# = \mathfrak{D}$ and the case $\# = \mathfrak{S}$.

Proposition 8.4. *For any $r > 1$ and any $\varkappa > 0$ there exist $\beta_0 > 0$, $\alpha^\mathfrak{D} > 0$ and C such that for all $\beta \geq \beta_0$ and $U \in \mathcal{S}_r$*

$$\sup_{\substack{\Re\lambda \leq \beta^{-1/3}(\hat{\mu}_0(\beta^{-1/3}\alpha) - \varkappa) \\ \alpha^\mathfrak{D}\beta^{1/6} \leq \alpha}} \left((1 + \alpha)\|(\mathcal{B}_{\lambda, \alpha, \beta}^\mathfrak{D})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^\mathfrak{D})^{-1} \right\| \right) \leq C\beta^{-5/6}. \quad (8.67)$$

Proof. Let $\mathfrak{z}_\pm \in C^2([-1, 1])$ be given by (6.68). By (6.75), (7.9), and (8.51), we have

$$\|-\phi'' + \theta^2\beta^{2/3}\phi\|_2 \leq C(\beta^{1/3}\|\phi\|_2 + \beta^{-2/3}\|f\|_2), \quad (8.68)$$

where $\theta = \alpha\beta^{-1/3}$.

Hence,

$$\|\phi'\|_2^2 + \theta^2\beta^{2/3}\|\phi\|_2^2 = \langle -\phi'' + \theta^2\beta^{2/3}\phi, \phi \rangle \leq C(\beta^{1/3}\|\phi\|_2^2 + \beta^{-2/3}\|f\|_2\|\phi\|_2). \quad (8.69)$$

As $\theta \geq \alpha^\mathfrak{D}\beta^{-1/6}$, we obtain that for sufficiently large $\alpha^\mathfrak{D}$ and β ,

$$\|\phi\|_2 \leq \frac{C}{\theta^2\beta^{4/3}}\|f\|_2,$$

which implies

$$\alpha\|\phi\|_2 \leq \frac{C}{\theta\beta}\|f\|_2 \leq \frac{C}{\alpha^\mathfrak{D}\beta^{5/6}}\|f\|_2.$$

Returning to (8.69), we get

$$\|\phi'\|_2^2 \leq \beta^{-2/3}\|f\|_2\|\phi\|_2 \leq \frac{C}{\theta^2\beta^2}\|f\|_2^2,$$

hence

$$\|\phi'\|_2 \leq \frac{C}{\theta\beta}\|f\|_2 \leq \frac{C}{\alpha^\mathfrak{D}\beta^{5/6}}\|f\|_2$$

■

Remark 8.5. Using the definition of $\hat{\mu}_0$ from (6.76) yields that

$$\hat{\mu}_0(\beta^{-1/3}\alpha)\beta^{2/3} + \alpha^2 = \beta^{2/3} \min(J_+^{2/3}\mu_0(J_+^{-1/3}\theta) + \theta^2, J_-^{2/3}\mu_0(J_-^{-1/3}\theta) + \theta^2),$$

where $\theta = \beta^{-1/3}\alpha$. Let $\theta_{\pm} = J_{\pm}^{-1/3}\theta$.

Using the definition of $\hat{\mu}_m$ from (6.56) we then conclude

$$J_{\pm}^{2/3}\mu_0(J_{\pm}^{-1/3}\theta) + \theta^2 = J_{\pm}^{2/3}(\mu_0(\theta_{\pm}) + \theta_{\pm}^2) \geq J_m^{2/3}\hat{\mu}_m.$$

Consequently we obtain that

$$\hat{\mu}_0(\beta^{-1/3}\alpha)\beta^{2/3} + \alpha^2 \geq \beta^{2/3}J_m^{2/3}\hat{\mu}_m.$$

By the foregoing discussion, we may conclude from (8.67) that

$$\sup_{\substack{\Re\lambda \leq \beta^{-1/3}(J_m^{2/3}\hat{\mu}_m - \alpha - \beta^{-2/3}\alpha^2) \\ \alpha^2\beta^{1/6} \leq \alpha}} \left((1 + \alpha) \|(\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}})^{-1} \right\| \right) \leq \frac{C}{\beta^{5/6}}. \quad (8.70)$$

A similar estimate holds true also for $\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}}$.

Proposition 8.6. For any $r > 1$ and $\Upsilon < \Re\nu_1$ there exist positive β_0 , $\alpha^{\mathfrak{S}}$ and C such that for all $\beta \geq \beta_0$ and $U \in \mathcal{S}_r$ such that

$$\sup_{\substack{\Re\lambda \leq \Upsilon J_m^{2/3}\beta^{-1/3} \\ \alpha^{\mathfrak{S}}\beta^{1/6} \leq \alpha}} \left((1 + \alpha) \|(\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1} \right\| \right) \leq C\beta^{-5/6}. \quad (8.71)$$

Proof. To prove (8.71) we note that by (7.9) we have

$$\|\phi'' - \alpha^2\phi\|_2 \leq C(\beta^{1/3}\|\phi\|_2 + \beta^{-2/3}\|f\|_2).$$

Consequently,

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq C(\beta^{1/3}\|\phi\|_2^2 + \beta^{-2/3}\|f\|_2\|\phi\|_2).$$

from which (8.71) follows as in the \mathfrak{D} - case. ■

8.3 Strictly convex/concave flows

8.3.1 Large α or $|\Im\lambda|$

If we assume large α or $|\nu|$ we may obtain resolvent estimates for all $U \in C^4([-1, 1])$, without the necessity to assume any further restrictions on U as in the nearly Couette case or in the case $U'' \neq 0$.

Proposition 8.7. Let $r > 1$, $\Upsilon < \min(\Re\nu_1, \vartheta_1^r)$, and $\# \in \{\mathfrak{S}, \mathfrak{D}\}$. Then, there exist $\tilde{\alpha}_0 > 0$, $\tilde{\alpha}_1 > 0$, $\tilde{\nu} > 0$, $\beta_0 > 0$, and C such that, for all $\beta \geq \beta_0$ and $U \in \mathcal{S}_r$ satisfying (4.23)-(4.25), it holds that

$$\sup_{\substack{\Re\lambda \leq \Upsilon J_m^{2/3}\beta^{-1/3} \\ \tilde{\alpha}_1 \leq \alpha \leq \tilde{\alpha}_0\beta^{1/3}}} \left\| (\mathcal{B}_{\lambda, \alpha, \beta}^{\#})^{-1} \right\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\#})^{-1} \right\| \leq C\beta^{-5/6}, \quad (8.72)$$

and

$$\sup_{\substack{\Re\lambda \leq \Upsilon J_m^{2/3} \beta^{-1/3} \\ d(\Im\lambda, [U(-1), U(1)]) \geq \tilde{\nu} \beta^{-1/3} \\ 0 \leq \alpha \leq \tilde{\alpha}_1}} \|(\mathcal{B}_{\lambda, \alpha, \beta}^\#)^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^\#)^{-1} \right\| \leq C \beta^{-5/6}. \quad (8.73)$$

Proof. Let $f \in L^2(-1, 1)$, $\phi \in D(\mathcal{B}_{\lambda, \alpha, \beta}^\#)$ and h satisfy

$$\mathcal{B}_{\lambda, \alpha, \beta}^\# \phi = f \text{ and } h = f + \left(\frac{U'' \phi}{U + i\lambda} \right)''.$$

Proof of (8.73).

Consider first the case $d(\Im\lambda, [U(-1), U(1)]) \geq \tilde{\nu} \beta^{-1/3}$. Here, following the same derivation as in (7.27)-(7.31) (for $\# = \mathfrak{S}$) and (8.37)-(8.41) (for $\# = \mathfrak{D}$) (the role of δ being replaced by $\frac{1}{\tilde{\nu}}$, which is small too) we obtain

$$\|h\|_2 \leq \|f\|_2 + \frac{C}{\tilde{\nu}} \beta^{5/6} \|\phi\|_{1,2}. \quad (8.74)$$

For sufficiently large $\tilde{\nu}$ we can thus follow the same steps of either the proof of Proposition 7.3 starting from (7.32) now replaced by (8.74), or the proof of Proposition 8.3 starting from (8.41), now replaced by (8.74), to obtain (8.73).

Proof of (8.72).

Case 1: $|\Re\lambda| > J_m^{2/3} \beta^{-1/3} \min(\vartheta_1^r, \Re\nu_1)/2$. To prove (8.72) for $\# = \mathfrak{D}$ we repeat the same procedure used in the proof of Proposition 8.3 to establish (8.48) with $\delta = r$ (note that we always have $\delta_2(U) \leq r$ and recall that v_0 is defined by (8.44))

$$\|v_o\|_2 \leq \frac{C}{\beta} \|f\|_2 + C \beta^{-1/6} \|\phi\|_{1,2}. \quad (8.75)$$

In the case $\# = \mathfrak{S}$ we repeat the same steps as in the proof of Proposition 7.3 to establish (7.33) with $\delta = r$. For convenience we use the notation v_o instead of v (which is defined by (7.3)).

We now estimate $\phi_o = \mathcal{A}_{\lambda, \alpha}^{-1} v_o$. To this end we rewrite (4.27) in the form

$$\Re \left\langle \frac{\phi_o}{U - \nu + i\mu}, v_o \right\rangle \geq \frac{1}{2} \|\phi_o'\|_2^2 + (\alpha^2 + \gamma_m(\lambda, U)) \|\phi_o\|_2^2.$$

The left-hand-side can be bounded by (4.38) (for $p = 2$) to obtain

$$\frac{1}{2} \|\phi_o'\|_2^2 + (\alpha^2 + \gamma_m(\lambda, U)) \|\phi_o\|_2^2 \leq C \beta^{1/6} \|\phi_o\|_\infty \|v_o\|_2.$$

Since by Lemma 4.8, $\gamma_m(\lambda, U) \geq \gamma_0 > -\infty$ we pick $\alpha \in \mathbb{R}$ such that

$$\alpha^2 \geq \sup(1, -2\gamma_0).$$

Then we write

$$\alpha \|\phi_o\|_\infty^2 \leq \alpha \|\phi'_o\|_2 \|\phi_o\|_2 \leq \frac{1}{2} (\|\phi'_o\|_2^2 + \alpha^2 \|\phi_o\|_2^2) \leq C\beta^{1/6} \|\phi_o\|_\infty \|v_o\|_2. \quad (8.76)$$

Consequently,

$$\|\phi_o\|_\infty \leq \frac{C}{\alpha} \beta^{1/6} \|v_o\|_2,$$

and

$$\|\phi_o\|_{1,2}^2 \leq \frac{2C}{\alpha} \beta^{1/3} \|v_o\|_2^2. \quad (8.77)$$

By (8.75) and (8.77), we deduce

$$\|\phi_o\|_{1,2} \leq \frac{\hat{C}}{\sqrt{\alpha}} \beta^{-5/6} \|f\|_2 + \frac{\hat{C}}{\sqrt{\alpha}} \|\phi\|_{1,2}. \quad (8.78)$$

In the case $\# = \mathfrak{S}$ we have $\phi_o = \phi$ and hence (8.72) immediately follows. In the case $\# = \mathfrak{D}$ we continue as in the proof of Proposition 8.3 to establish (8.47) with $\delta = r$, or explicitly,

$$\|\phi_\pm\|_{1,2} \leq C(\beta^{-1} \|f\|_2 + \beta^{-1/6} \|\phi\|_{1,2}).$$

The above, combined with (8.78) and (8.45) yields for $\alpha \geq \tilde{\alpha}_1$ with $\tilde{\alpha}_1$ large enough and $\beta \geq \beta_0$ with β_0 large enough

$$\|\phi\|_{1,2} \leq C \left(\frac{1}{\sqrt{\alpha}} + \beta^{-1/6} \right) \beta^{-5/6} \|f\|_2, \quad (8.79)$$

from which (8.72) readily follows.

Case 2: $|\Re \lambda| \leq \min(\Re \nu_1, \nu_1^r) 2^{-1} J_m^{\frac{2}{3}} \beta^{-\frac{1}{3}}$.

For $\# = \mathfrak{S}$ we use (7.36) for $\delta = r$, i.e.,

$$\|\phi'' - \alpha^2 \phi\|_2 \leq C(\beta^{-2/3} \|f\|_2 + r\beta^{1/6} \|\phi\|_\infty). \quad (8.80)$$

Then, as

$$\mathcal{B}_{\lambda+s\beta^{-1/3}, \alpha, \beta}^{\mathfrak{S}} \phi = f - s\beta^{2/3}(\phi'' - \alpha^2 \phi).$$

we may use (8.78), having in mind that $\phi_o = \phi$, to obtain that

$$\|\phi\|_{1,2} \leq \frac{\hat{C}}{\sqrt{\alpha}} \beta^{-5/6} (\|f\|_2 + \beta^{2/3} \|\phi'' - \alpha^2 \phi\|_2).$$

Substituting (8.80) into the above yields (8.72).

For $\# = \mathfrak{D}$ we first obtain (8.55) with $\delta = r$, which implies

$$\|w_o\|_2 \leq C(\beta^{-2/3} \|f\|_2 + r\beta^{1/6} \|\phi\|_\infty), \quad (8.81)$$

where

$$w_o = -\phi'' + \alpha^2 \phi - B_+ \psi_+ - B_- \psi_- ,$$

and

$$B_{\pm} = A_{\pm}(f + i\beta U''\phi).$$

As

$$\mathcal{B}_{\lambda+\Upsilon_0\beta^{-1/3},\alpha,\beta}^{\mathfrak{D}}\phi = f - \Upsilon_0\beta^{2/3}(w_0 + B_+\psi_+ + B_-\psi_-),$$

where Υ_0 satisfies (8.56), we can write as in (8.57) that

$$\phi = \chi_0 - \Upsilon_0\beta^{2/3}(B_+\chi_+ + B_-\chi_-), \quad (8.82)$$

where

$$\chi_0 = (\mathcal{B}_{\lambda+\Upsilon_0\beta^{-1/3},\alpha,\beta}^{\mathfrak{D}})^{-1}(f - \Upsilon_0\beta^{2/3}w_0) \quad \text{and} \quad \chi_{\pm} = (\mathcal{B}_{\lambda+\Upsilon_0\beta^{-1/3},\alpha}^{\mathfrak{D}})^{-1}\psi_{\pm}.$$

The estimation of χ_0 can be done with the aid of (8.79) and (8.81), yielding

$$\|\chi_0\|_{1,2} \leq C\left(\frac{1}{\sqrt{\alpha}} + \beta^{-1/6}\right)\beta^{-5/6}(\|f\|_2 + \|\phi\|_{\infty}). \quad (8.83)$$

To estimate χ_+ (or χ_-) we write

$$\chi_+ = \chi_+^1 + \chi_+^2,$$

where

$$\chi_+^1 = \mathcal{A}_{\tilde{\lambda},\alpha}^{-1}(U + i\tilde{\lambda})\Gamma_{[-1,1]}(\tilde{\mathcal{L}}_{\beta,\mathbb{R}} - \beta\tilde{\lambda})^{-1}\left(\frac{U''\chi_+}{U + i\tilde{\lambda}}\right)''.$$

As

$$\left\|\left(\frac{U''\chi_+}{U + i\tilde{\lambda}}\right)''\right\|_2 \leq C\left(\beta^{-1/3} + \beta^{5/6}\|\chi_+\|_{1,2}\right),$$

we can conclude from (5.4), (5.5), and (8.77) that

$$\|\chi_+^1\|_{1,2} \leq \frac{C}{\sqrt{\alpha}}(\beta^{-7/6} + \|\chi_+\|_{1,2}).$$

The estimation of χ_+^2 in the proof of Proposition 8.3 does not involve δ at all, but only r and hence we can conclude by (8.63) and (8.64) that

$$\|\chi_+^2\|_{1,2} \leq C(\beta^{-7/6} + \beta^{-1/6}\|\chi_+\|_{1,2}).$$

Consequently we obtain that for sufficiently large α

$$\|\chi_+\|_{1,2} \leq C\beta^{-7/6}.$$

A similar estimate holds for χ_- , and hence we can conclude (8.72) from (8.83), (8.82), and (8.53). ■

8.3.2 The case $U'' \neq 0$

Lemma 8.8. *Let $r > 1$ and $\hat{\delta} \in (0, \frac{1}{2}]$. Then, there exist $\beta_0 > 0$, $\Upsilon > 0$, and $C > 0$ such that for all $\beta \geq \beta_0$ and $U \in \mathcal{S}_r$ satisfying (2.29), it holds that*

$$\sup_{\substack{\Re \lambda \leq \Upsilon J_m^{\frac{2}{3}} \beta^{-1/3} \\ 0 \leq \alpha \leq \tilde{\alpha}_1}} \left\| (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}})^{-1} \right\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}})^{-1} \right\| \leq \frac{C}{\beta^{1/2 - \hat{\delta}}}, \quad (8.84)$$

where $\tilde{\alpha}_1$ is the same as in Proposition 8.7.

Proof. Let $\phi \in D(\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}})$, $\alpha \leq \tilde{\alpha}_1$, $f = \mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}} \phi$ and $v_{\mathfrak{D}} \in H^2(-1, +1)$ defined by

$$v_{\mathfrak{D}} = \mathcal{A}_{\lambda, \alpha} \phi + (U + i\lambda)[\phi''(1)\hat{\psi}_+ + \phi''(-1)\hat{\psi}_-], \quad (8.85)$$

where

$$\hat{\psi}_{\pm} = \psi_{\pm} \Theta_{\pm} / \psi_{\pm}(\pm 1) \quad (8.86)$$

in which ψ_{\pm} is defined in (6.8) and $\Theta_{\pm}(x) = 1 - \tilde{\eta}(1 \mp x)$, with $\tilde{\eta}$ given by (5.14). With $u(x, \lambda)$ given by (A.4), we have

$$\hat{\psi}_{\pm}(x, \lambda) = u((J_{\pm}\beta)^{1/3}(1 \mp x), iJ_{\pm}^{-2/3}\beta^{1/3}\lambda_{\pm}). \quad (8.87)$$

Hence we may thus use all the estimates derived for u in the proof of Proposition A.1.

Note that $v_{\mathfrak{D}} \in H_0^1(-1, 1)$ and hence we may introduce

$$g_{\mathfrak{D}} := (\mathcal{L}_{\beta}^{\mathfrak{D}} - \beta\lambda)v_{\mathfrak{D}}.$$

We have

$$g_{\mathfrak{D}} = (U + i\lambda)(-f + \phi''(1)\hat{g}_+ + \phi''(-1)\hat{g}_-) - (U''\phi)'' - 2U'\tilde{v}'_{\mathfrak{D}} - U''\tilde{v}_{\mathfrak{D}}, \quad (8.88)$$

wherein

$$\hat{g}_{\pm} = \left(-\frac{d^2}{dx^2} + i\beta U - \beta\lambda \right) \hat{\psi}_{\pm},$$

and

$$\tilde{v}_{\mathfrak{D}} = \frac{v_{\mathfrak{D}} - U''\phi}{U + i\lambda} = -\phi'' + \alpha^2\phi + \phi''(1)\hat{\psi}_+ + \phi''(-1)\hat{\psi}_-.$$

We note that

$$(\mathcal{L}_{\beta}^{\mathfrak{D}} - \beta\lambda)\tilde{v}_{\mathfrak{D}} - i\beta U''\phi = -f + \phi''(1)\hat{g}_+ + \phi''(-1)\hat{g}_-. \quad (8.89)$$

As in the proof of Proposition 7.1 (see in particular (7.7)) we can integrate by parts to obtain

$$\begin{aligned} & -\Re\langle (U'')^{-1}\tilde{v}_{\mathfrak{D}}, (\mathcal{L}_{\beta}^{\mathfrak{D}} - \beta\lambda)\tilde{v}_{\mathfrak{D}} - i\beta U''\phi \rangle = \|(U'')^{-1/2}\tilde{v}'_{\mathfrak{D}}\|_2^2 + \\ & + \Re\langle ((U'')^{-1})'\tilde{v}_{\mathfrak{D}}, \tilde{v}'_{\mathfrak{D}} \rangle - \beta\mu \|\tilde{v}_{\mathfrak{D}}\|_2^2 + \beta\Re\langle \phi''(1)\hat{\psi}_+ + \phi''(-1)\hat{\psi}_-, i\phi \rangle. \end{aligned} \quad (8.90)$$

We begin the estimation by obtaining a bound for the last term on the right-hand-side.

Estimate of $\beta \Re \langle \phi''(1) \hat{\psi}_+ + \phi''(-1) \hat{\psi}_-, i\phi \rangle$.

We first write

$$\phi(x) = \int_x^1 (\xi-x)\phi''(\xi) d\xi = \phi''(1) \int_x^1 (\xi-x)\hat{\psi}_+(\xi) d\xi + \int_x^1 (\xi-x)[\phi''(\xi) - \phi''(1)\hat{\psi}_+(\xi)] d\xi.$$

Let

$$w_+(x) = \int_x^1 (\xi-x)\hat{\psi}_+(\xi) d\xi.$$

It can be easily verified that $w'_+(\pm 1) = 0$ and $w_+(1) = 0$. Hence,

$$\Re \langle \phi''(1) \hat{\psi}_+, i\phi''(1)w_+ \rangle = |\phi''(1)|^2 \Re \langle w'_+, iw_+ \rangle = 0.$$

We then obtain for $x \in (-1, +1)$, using the fact that $\hat{\psi}_+\hat{\psi}_- \equiv 0$,

$$\begin{aligned} \left| \int_x^1 (\xi-x)[\phi''(\xi) - \phi''(1)\hat{\psi}_+(\xi)] d\xi \right| &= \left| \int_x^1 (\xi-x)[\tilde{v}'_{\mathfrak{D}}(\xi) + \alpha^2\phi(\xi)] d\xi \right| \\ &\leq C(\|\tilde{v}'_{\mathfrak{D}}\|_2 + \alpha^2\|\phi'\|_2)(1-x)^{5/2}. \end{aligned}$$

We thus get, as $\alpha \leq \tilde{\alpha}_1$,

$$|\Re \langle \phi''(1) \hat{\psi}_+, i\phi \rangle| \leq |\phi''(1)|(\|\tilde{v}'_{\mathfrak{D}}\|_2 + \tilde{\alpha}_1^2\|\phi'\|_2)\|(1-x)^{5/2}\hat{\psi}_+\|_1.$$

Using (8.87), (A.32), translation and dilation (see the proof of (6.15)-(6.16)) we may conclude that

$$\|(1-x)^s \hat{\psi}_+\|_1 \leq C\beta^{-(s+1)/3}[1 + |\lambda_+|^{1/2}\beta^{1/6}]^{-(s+1)}, \quad \forall s \leq 3,$$

and since $\alpha \leq \tilde{\alpha}_1$ we can determine that

$$\beta |\Re \langle \phi''(1) \hat{\psi}_+, i\phi \rangle| \leq C[1 + |\lambda_{\pm}|\beta^{1/3}]^{-7/4} \beta^{-1/6} |\phi''(1)|(\|\tilde{v}'_{\mathfrak{D}}\|_2 + \|\phi'\|_2). \quad (8.91)$$

Estimate of $\|\tilde{v}'_{\mathfrak{D}}\|$.

Next we obtain from (8.90) and (8.91) that

$$\begin{aligned} \|\tilde{v}'_{\mathfrak{D}}\|_2^2 &\leq C\|\tilde{v}_{\mathfrak{D}}\|_2(\|f\|_2 + |\phi''(1)|\|\hat{g}_+\|_2 + |\phi''(-1)|\|\hat{g}_-\|_2) \\ &\quad + C\Upsilon\beta^{2/3}\|\tilde{v}_{\mathfrak{D}}\|_2^2 + C[1 + |\lambda_{\pm}|\beta^{1/3}]^{-2}\beta^{-1/6}|\phi''(1)|(\|v'_{\mathfrak{D}}\|_2 + \|\phi'\|_2). \end{aligned} \quad (8.92)$$

By (8.53), which is valid here with $\delta = r$ and Remark 6.3, we have that

$$|\phi''(\pm 1)| \leq C[1 + |\lambda_{\pm}|^{1/2}\beta^{1/6}](\beta^{-1/6}\|f\|_2 + \beta^{1/3}\log\beta\|\phi\|_{\infty}). \quad (8.93)$$

Furthermore, by (6.18) we have that

$$\|\hat{g}_{\pm}\|_2 \leq C\beta^{-1/6}[1 + |\lambda_{\pm}|^{1/2}\beta^{1/6}]^{-1}, \quad (8.94)$$

which, when substituted into (8.92), yields, with the aid of Sobolev's embeddings,

$$\|\tilde{v}'_{\mathfrak{D}}\|_2 \leq C(\|f\|_2 + \beta^{1/6}\log\beta\|\phi\|_{1,2}) + \Upsilon^{1/2}\beta^{1/3}\|\tilde{v}_{\mathfrak{D}}\|_2. \quad (8.95)$$

Estimate of $\|\tilde{v}_{\mathfrak{D}}\|$.

By (5.21) and (8.89) we have that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C(\beta^{1/6}\|\phi\|_{\infty} + \beta^{-2/3}[\|f\|_2 + |\phi''(1)|\|\hat{g}_+\|_2 + |\phi''(-1)|\|\hat{g}_-\|_2]).$$

Hence, by (8.94) and (8.93) we have that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C(\beta^{1/6}\|\phi\|_{\infty} + \beta^{-2/3}\|f\|_2).$$

Substituting the above into (8.95) yields

$$\|\tilde{v}'_{\mathfrak{D}}\|_2 \leq C(\|f\|_2 + (\Upsilon^{1/2}\beta^{1/2} + \beta^{1/6}\log\beta)\|\phi\|_{1,2}). \quad (8.96)$$

We now combine (8.88), (8.96), (8.95), (8.93), and (8.94) to obtain that

$$\|g_{\mathfrak{D}}\|_2 \leq C(\|f\|_2 + (\Upsilon^{1/2}\beta^{1/2} + \beta^{1/6}\log\beta)\|\phi\|_{1,2}). \quad (8.97)$$

Proof of (8.84)

We continue as in the proof of Proposition 8.3. We first write, in view of (8.85)

$$\phi = \phi_{\mathfrak{D}} + \check{\phi}_+ + \check{\phi}_-, \quad (8.98)$$

where

$$\phi_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}^{-1}v_{\mathfrak{D}} \quad ; \quad \check{\phi}_{\pm} = \mathcal{A}_{\lambda,\alpha}^{-1}([U + i\lambda]\phi''(\pm 1)\hat{\psi}_{\pm}).$$

By (8.1) and (8.93) we have

$$\|\check{\phi}_{\pm}\|_{1,2} \leq C[1 + |\lambda_{\pm}|^{1/2}\beta^{1/6}]^{-1/2}(\beta^{-2/3}\|f\|_2 + \beta^{-1/6}\log\beta\|\phi\|_{1,2}),$$

and hence, by (8.98),

$$\|\check{\phi}_{\pm}\|_{1,2} \leq C[1 + |\lambda_{\pm}|^{1/2}\beta^{1/6}]^{-1/2}(\beta^{-2/3}\|f\|_2 + \beta^{-1/6}\log\beta\|\phi_{\mathfrak{D}}\|_{1,2}). \quad (8.99)$$

Substituting the above into (8.97) yields, with the aid of (8.98)

$$\|g_{\mathfrak{D}}\|_2 \leq C(\|f\|_2 + (\Upsilon^{1/2}\beta^{1/2} + \beta^{1/6}\log\beta)\|\phi_{\mathfrak{D}}\|_{1,2}).$$

By (4.24) and (4.25) we then have for any $q > 1$ and $p > 2$

$$\|g_{\mathfrak{D}}\|_2 \leq C(\|f\|_2 + \Upsilon^{1/2-1/p}\beta^{\frac{3p+2}{6p}}\|v_{\mathfrak{D}}\|_p + \beta^{1/6}\log\beta(\|v'_{\mathfrak{D}}\|_q + \|v_{\mathfrak{D}}\|_{\infty})). \quad (8.100)$$

By (7.6) there exists $C > 0$ such that for all $p > 2$ (including $p = \infty$) it holds that

$$\|v_{\mathfrak{D}}\|_p \leq C\beta^{-\frac{3p+2}{6p}}\|g_{\mathfrak{D}}\|_2. \quad (8.101)$$

Similarly, by (5.10), for all $1 < q < 2$, there exists $C_q > 0$ such that

$$\|v'_{\mathfrak{D}}\|_q \leq C_q\beta^{-\frac{2+q}{6q}}\|g_{\mathfrak{D}}\|_2. \quad (8.102)$$

Substituting (8.101) and (8.102) into (8.100) yields, choosing $\Upsilon > 0$ small enough and β_0 large enough, the existence of $C > 0$ such that for $\beta \geq \beta_0$

$$\|g_{\mathfrak{D}}\|_2 \leq C\|f\|_2.$$

Using (4.24) and (4.25) once again upon (8.101) and (8.102) and the above inequality readily verifies (8.84). ■

We can now conclude.

Proposition 8.9. *Let $r > 1$ and $\hat{\delta} > 0$. Then, there exist $\beta_0 > 0$, $\Upsilon > 0$, and $C > 0$, such that for all $\beta \geq \beta_0$ and $U \in \mathcal{S}_r$ satisfying (2.29), it holds that*

$$\sup_{\substack{\Re\lambda + \beta^{-1}\alpha^2 \leq \Upsilon J_m^{2/3} \beta^{-1/3} \\ 0 < \alpha}} \left\| (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}})^{-1} \right\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{D}})^{-1} \right\| \leq \frac{C}{\beta^{1/2 - \hat{\delta}}}, \quad (8.103)$$

The proof follows immediately by combining (8.70), (8.72), and (8.84) for sufficiently small value of $\Upsilon > 0$.

9 Semigroup estimates

In this section we prove Theorems 2.14 and 2.15.

9.1 Preliminaries

For $\sharp \in \{\mathfrak{S}, \mathfrak{D}\}$, let $\mathbf{F} = (F_1, F_2) \in H_{loc}^1(\bar{D}, \mathbb{R}^2) \cap \mathcal{H}$, $\Lambda \in \mathbb{C}$, and $\mathbf{u} \in \mathcal{W}_{\sharp}^0$ satisfy

$$(\mathfrak{T}_P^{\sharp}(U, \epsilon, L) - \Lambda)\mathbf{u} = P\mathbf{F}. \quad (9.1)$$

Recall that by writing $\mathbf{u} = \nabla_{\perp} \psi$ for some $\psi \in D(P_{\Lambda, \epsilon}^{\sharp})$ we have established in (3.1) that

$$\mathcal{P}_{\Lambda, \epsilon}^{\sharp} \psi = \text{curl } \mathbf{F} \quad \text{in } D. \quad (9.2)$$

This gives, for $\Lambda \in \rho(\mathfrak{T}_P^{\sharp})$,

$$\mathbf{u} = \nabla_{\perp} (\mathcal{P}_{\Lambda, \epsilon}^{\sharp})^{-1} \text{curl } \mathbf{F} = (\mathfrak{T}_P^{\sharp}(U, \epsilon, L) - \Lambda)^{-1} P\mathbf{F}. \quad (9.3)$$

To estimate $(\mathfrak{T}_P^{\sharp}(U, \epsilon, L) - \Lambda)^{-1}$ we seek therefore a bound for $(\mathcal{P}_{\Lambda, \epsilon}^{\sharp})^{-1}$, for Λ in a suitable region of \mathbb{C} . We later derive the properties of the semi-group $e^{-t\mathfrak{T}_P^{\sharp}}$ from these resolvent estimates.

Let $\hat{\Lambda}_0 \in \mathbb{R}$ and $\epsilon > 0$. We seek a bound for the $\mathcal{L}(L_{per}^2, H_{per}^1)$ norm of $(\mathcal{P}_{\Lambda, \epsilon}^{\sharp})^{-1}$ in the domain in $\Re\Lambda \leq \Lambda_0$ with $\Lambda_0 = \epsilon \hat{\Lambda}_0$. Recall that $\mathcal{P}_{\Lambda, \epsilon}^{\sharp}$ depends on $L > 0$ through the periodicity condition appearing in the definition of its domain.

As in Section 3 we can rewrite (3.11), which provides an $\mathcal{L}(L_{per}^2)$ bound for $(\mathcal{P}_{\Lambda, \epsilon}^{\sharp})^{-1}$ to the following $\mathcal{L}(L_{per}^2, H_{per}^1)$ estimate

$$\sup_{\Re\Lambda \leq \Lambda_0} \|(\mathcal{P}_{\Lambda, \epsilon}^{\sharp})^{-1}\|_{\mathcal{L}(L_{per}^2, H_{per}^1)} \leq \epsilon^{-1} \max(B_{*}^{\sharp}(\hat{\Lambda}_0, \epsilon, L), B_0^{\sharp}(\hat{\Lambda}_0)), \quad (9.4)$$

where,

$$B_{*}^{\sharp}(\hat{\Lambda}_0, \epsilon, L) = \sup_{\beta \geq \beta_1(\epsilon, L)} \sup_{\substack{\alpha \geq 0 \\ \Re\lambda \leq \beta^{-1}(\hat{\Lambda}_0 - \alpha^2)}} \left[\|(1 + \alpha)(\mathcal{B}_{\lambda, \alpha, \beta}^{\sharp})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\sharp})^{-1} \right\| \right], \quad (9.5)$$

in which $\beta_1(\epsilon, L) = (2\pi)/(L\epsilon)$, and

$$B_0^{\sharp}(\hat{\Lambda}_0) = \sup_{\Re\hat{\Lambda} \leq \hat{\Lambda}_0} \left[\|(\mathcal{B}_{\hat{\Lambda}}^{\sharp})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\hat{\Lambda}}^{\sharp})^{-1} \right\| \right]. \quad (9.6)$$

9.2 Proof of Theorem 2.14

9.2.1 $(\mathcal{P}_{\hat{\Lambda}, \epsilon}^{\mathfrak{S}})^{-1}$ estimates

Estimation of $B_*^{\mathfrak{S}}(\hat{\Lambda}_0, \epsilon, L)$.

We begin by showing that for all $\hat{\delta} \in (0, 1/3)$, $\hat{\Lambda}_0 > 0$ and $\alpha_M > 0$, there exist $\check{\beta}_0 > 0$ and $C > 0$, such that, for all $\beta \geq \check{\beta}_0$

$$\sup_{\substack{0 \leq \alpha \leq \alpha_M \beta^{1/6} \\ \Re \lambda \leq \beta^{-1} \hat{\Lambda}_0}} \left[\|(1 + \alpha)(\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1} \right\| \right] \leq C \beta^{-1/3+\hat{\delta}}. \quad (9.7)$$

Indeed, let

$$S_1(\beta, \hat{\Lambda}_0) = \{(\Re \lambda, \alpha) \in \mathbb{R}^2 \mid \Re \lambda \leq \beta^{-1} \hat{\Lambda}_0; 0 \leq \alpha \leq \alpha_M \beta^{1/6}\},$$

and

$$S_2(\beta, \Upsilon) = \{(\Re \lambda, \alpha) \in \mathbb{R}^2 \mid \Re \lambda \leq \beta^{-1/3} \Upsilon; 0 \leq \alpha \leq \alpha_M \beta^{1/6}\}.$$

Then there exists $\check{\beta}_0(\Upsilon)$ such that for all $\beta \geq \check{\beta}_0(\Upsilon)$ we have $S_1(\beta, \hat{\Lambda}_0) \subset S_2(\beta, \Upsilon)$.

By (7.1) we have that for sufficiently small $\Upsilon > 0$ there exists $\beta'_0(\Upsilon) > 0$ and $C > 0$ such that

$$\sup_{(\Re \lambda, \alpha) \in S_2(\beta, \Upsilon)} \left[\|(1 + \alpha)(\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1} \right\| \right] \leq C \beta^{-1/3+\hat{\delta}}, \quad (9.8)$$

for all $\beta \geq \beta'_0(\Upsilon)$. For such sufficiently small Υ we may then write, for all $\beta \geq \check{\beta}_0 := \max(\beta_0(\Upsilon), \beta'_0(\Upsilon))$, that

$$\sup_{(\Re \lambda, \alpha) \in S_1(\beta, \hat{\Lambda}_0)} \left[\|(1 + \alpha)(\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1} \right\| \right] \leq C \beta^{-1/3+\hat{\delta}},$$

which is precisely (9.7). Consequently, for all $L > 0$, $\epsilon > 0$ such that $L\epsilon \leq \frac{2\pi}{\check{\beta}_0}$ we have

$$\sup_{\beta \geq \beta_1(\epsilon, L)} \sup_{\substack{0 \leq \alpha \leq \alpha_M \beta^{1/6} \\ \Re \lambda \leq \beta^{-1} \hat{\Lambda}_0}} \left[\|(1 + \alpha)(\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1} \right\| \right] \leq C \beta_1(\epsilon, L)^{-1/3+\hat{\delta}}. \quad (9.9)$$

On the other hand, it follows from (8.71) in a similar manner that for any $\hat{\Lambda}_0 > 0$ there exist $\alpha^{\mathfrak{S}} > 0$, $\beta''_0 > 0$, and $C > 0$, such that for all $\beta \geq \beta''_0$ we have

$$\sup_{\substack{\alpha^{\mathfrak{S}} \beta^{1/6} \leq \alpha \\ \Re \lambda \leq \beta^{-1} \hat{\Lambda}_0}} \left[\|(1 + \alpha)(\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1} \right\| \right] \leq C \beta^{-5/6}. \quad (9.10)$$

As a result, for all $L > 0$ and $\epsilon > 0$ such that $L\epsilon \leq \epsilon_0 := 2\pi/(\max(\beta''_0, \check{\beta}_0))$ (implying that $\beta_1(\epsilon, L) \geq \max(\beta''_0, \check{\beta}_0)$) it holds that

$$\sup_{\beta \geq \beta_1(\epsilon, L)} \sup_{\substack{\alpha^{\mathfrak{S}} \beta^{1/6} \leq \alpha \\ \Re \lambda \leq \beta^{-1} \hat{\Lambda}_0}} \left[\|(1 + \alpha)(\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1} \right\| \right] \leq C \beta_1(\epsilon, L)^{-5/6}.$$

Combining the above with (9.7) for $\alpha_M = \alpha^\mathfrak{S}$ yields that for any $\hat{\delta} \in (0, \frac{1}{3})$ and $\hat{\Lambda}_0 > 0$ there exists $C > 0$ and $\epsilon_0 > 0$ such that for all $L > 0$ and $\epsilon > 0$ satisfying $L\epsilon \leq \epsilon_0$ we have

$$\sup_{\beta \geq \beta_1(\epsilon, L)} \sup_{\substack{0 \leq \alpha \\ \Re \lambda \leq \beta^{-1} \hat{\Lambda}_0}} \left[\|(1 + \alpha)(\mathcal{B}_{\lambda, \alpha, \beta}^\mathfrak{S})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^\mathfrak{S})^{-1} \right\| \right] \leq C \beta_1(\epsilon, L)^{-1/3+\hat{\delta}}.$$

We now observe that

$$B_*^\mathfrak{S}(\hat{\Lambda}_0, \epsilon, L) \leq \sup_{\beta \geq \beta_1(\epsilon, L)} \sup_{\substack{0 \leq \alpha \\ \Re \lambda \leq \beta^{-1} \hat{\Lambda}_0}} \left[\|(1 + \alpha)(\mathcal{B}_{\lambda, \alpha, \beta}^\mathfrak{S})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^\mathfrak{S})^{-1} \right\| \right].$$

Hence we obtain that for any $\hat{\Lambda}_0$ and $\hat{\delta} \in (0, \frac{1}{3})$ there exists $\epsilon_0 > 0$ and $C > 0$ such that for all $L > 0$, $\epsilon > 0$ satisfying $0 < L\epsilon \leq \epsilon_0$ it holds that

$$B_*^\mathfrak{S}(\hat{\Lambda}_0, \epsilon, L) \leq C \beta_1(\epsilon, L)^{-1/3+\hat{\delta}}. \quad (9.11)$$

$B_0^\mathfrak{S}(\hat{\Lambda}_0)$ estimation.

Recall that

$$\mathcal{B}_{\hat{\Lambda}}^\mathfrak{S} = -\frac{d^4}{dx^4} - \hat{\Lambda} \frac{d^2}{dx^2}.$$

As for any $\phi \in D(\mathcal{B}_{\hat{\Lambda}}^\mathfrak{S})$ we have $\phi'' \in H_0^1(-1, 1)$, we easily obtain that

$$\left\| \frac{d^2}{dx^2} (\mathcal{B}_{\hat{\Lambda}}^\mathfrak{S})^{-1} \right\| \leq \frac{1}{\pi^2/4 - \Re \hat{\Lambda}}.$$

Since $D(\mathcal{B}_{\hat{\Lambda}}^\mathfrak{S}) \subset H_0^1(-1, 1)$ we obtain that for all $0 < \hat{\Lambda}_0 < \pi^2/4$

$$B_0^\mathfrak{S}(\hat{\Lambda}_0) = \sup_{\Re \hat{\Lambda} \leq \hat{\Lambda}_0} \left[\|(\mathcal{B}_{\hat{\Lambda}}^\mathfrak{S})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\hat{\Lambda}}^\mathfrak{S})^{-1} \right\| \right] \leq \left(\frac{2}{\pi} + \frac{4}{\pi^2} \right) \frac{1}{(\pi^2/4 - \hat{\Lambda}_0)}. \quad (9.12)$$

In conclusion, we have established the following

Proposition 9.1. *For any $0 < \hat{\Lambda}_0 < \pi^2/4$ and $C > 2/\pi + 4/\pi^2$ there exists $\epsilon_0 > 0$, such that for all positive L and ϵ for which $0 < L\epsilon \leq \epsilon_0$ and $U \in \mathcal{S}_r$ satisfying (2.29), it holds that*

$$\sup_{\Re \Lambda \leq \epsilon \hat{\Lambda}_0} \|(\mathcal{P}_{\Lambda, \epsilon}^\mathfrak{S})^{-1}\|_{\mathcal{L}(L^2_{per}, H^1_{per})} \leq \frac{C}{\epsilon(\pi^2/4 - \hat{\Lambda}_0)}. \quad (9.13)$$

9.2.2 Estimation of $e^{-t \mathfrak{T}_P^\mathfrak{S}(U, \epsilon, L)}$.

Proof of (2.30a)

Let $\check{\delta} > 0$ and set $\hat{\Lambda}_0 = \pi^2/4 - \check{\delta}$. We can now combine (9.2), (9.3), and (9.13) to obtain,

$$\sup_{\Re \Lambda \leq \epsilon(\pi^2/4 - \check{\delta})} \|\mathbf{u}\|_2 \leq \frac{C}{\epsilon} \|\text{curl } \mathbf{F}\|_2 \leq \frac{C}{\epsilon} \|\mathbf{F}\|_{1,2}. \quad (9.14)$$

It has been established in (2.14) that

$$\sup_{\Re\Lambda \leq -\|U'\|_\infty/2 - \epsilon} \|\mathbf{u}\|_{1,2} \leq \frac{1}{\epsilon} \|\mathbf{F}\|_2, \quad (9.15)$$

which already implies, having in mind the definition of \mathcal{H} in (2.1),

$$\sup_{\Re\Lambda < -\|U'\|_\infty/2 - \epsilon} \|(\mathfrak{T}_P^\mathfrak{S} - \Lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{\epsilon}. \quad (9.16)$$

To estimate the resolvent for $\Lambda \in \mathbb{C}$ satisfying $-\|U'\|_\infty/2 - \epsilon \leq \Re\Lambda \leq \epsilon(\pi^2/4 - \check{\delta})$, we introduce $\omega = -\|U'\|_\infty/2 - \epsilon$ and use the resolvent identity

$$(\mathfrak{T}_P^\mathfrak{S} - \Lambda)^{-1} = (\mathfrak{T}_P^\mathfrak{S} + \omega + i\mathfrak{S}\Lambda)^{-1} + (\omega + \Re\Lambda)(\mathfrak{T}_P^\mathfrak{S} - \Lambda)^{-1}(\mathfrak{T}_P^\mathfrak{S} + \omega + i\mathfrak{S}\Lambda)^{-1}, \quad (9.17)$$

to obtain that

$$\begin{aligned} & \|(\mathfrak{T}_P^\mathfrak{S} - \Lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \\ & \leq \|(\mathfrak{T}_P^\mathfrak{S} + \omega + i\mathfrak{S}\Lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \\ & \quad + |\omega + \Re\Lambda| \|(\mathfrak{T}_P^\mathfrak{S} - \Lambda)^{-1}\|_{\mathcal{L}(H_{per}^1(D, \mathbb{C}^2), \mathcal{H})} \|(\mathfrak{T}_P^\mathfrak{S} + \omega + i\mathfrak{S}\Lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}, H_{per}^1(D, \mathbb{C}^2))}. \end{aligned}$$

With the aid of (9.14) and (2.14) we then obtain that for each $\check{\delta} > 0$ there exist $C > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, 1]$, $L > 0$ satisfying $0 < \epsilon L \leq \epsilon_0$

$$\sup_{-\|U'\|_\infty/2 - \epsilon \leq \Re\Lambda \leq \epsilon(\pi^2/4 - \check{\delta})} \|(\mathfrak{T}_P^\mathfrak{S}(U, \epsilon, L) - \Lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{\epsilon^2}. \quad (9.18)$$

We can now use [19, Proposition 2.1] (or [18, Proposition 13.31]), which we repeat here for the benefit of the reader

Proposition 9.2.

Let $S(t)$ be a strongly continuous semigroup, defined on a Hilbert space H , which satisfies for some $\hat{M} \geq 1$ and $\hat{\omega} \in \mathbb{R}$,

$$\|S(t)\| \leq \hat{M}e^{-\hat{\omega}t}. \quad (9.19)$$

Let $-A : D(A) \rightarrow H$ denote the generator of $S(t)$. Suppose further that for some $\omega > \hat{\omega}$ there exists $r(\omega) > 0$ such that

$$\sup_{\Re z \leq \omega} \|(A - z)^{-1}\| \leq \frac{1}{r(\omega)}. \quad (9.20)$$

Then,

$$\|S(t)\| \leq \widehat{M} \left(1 + \frac{2\widehat{M}(\omega - \hat{\omega})}{r(\omega)}\right) e^{-\omega t}. \quad (9.21)$$

We now apply (9.21) to $S(t) = e^{-t\mathfrak{T}_P^\mathfrak{S}(U, \epsilon, L)}$. By (2.12) it follows that (9.19) holds for $\hat{M} = 1$ and $\hat{\omega} = -\|U'\|_\infty/2$. By (9.16) and (9.18) we have that (9.20) holds for $\omega = \epsilon(\pi^2/4 - \check{\delta})$ and $r(\omega) = \epsilon^2/C$.

Consequently by (9.21), we obtain that for any $\check{\delta}$, there exist $C > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, 1]$ and $L > 0$ satisfying $0 < L\epsilon \leq \epsilon_0$ we have

$$\|e^{-t\mathfrak{T}_P^\mathfrak{S}(U, \epsilon, L)}\| \leq \frac{C}{\epsilon^2} e^{-\epsilon t(\pi^2/4 - \check{\delta})}.$$

This completes the proof of (2.30a).

Proof of (2.30b).

Let Π be defined by (2.19). We have (see (2.21)) that

$$\Pi \mathbf{u} = \Pi \nabla_{\perp} \psi = \nabla_{\perp} \mathbf{p} \psi.$$

We also have from (9.2)

$$\mathcal{P}_{\Lambda, \epsilon}^{\mathfrak{S}} (I - \mathbf{p}) \psi = (I - \mathbf{p}) \operatorname{curl} \mathbf{F} \quad \text{in } D. \quad (9.22)$$

and from (9.3)

$$(I - \Pi) \mathbf{u} = \nabla_{\perp} (I - \mathbf{p}) (\mathcal{P}_{\Lambda, \epsilon}^{\mathfrak{S}})^{-1} (I - \mathbf{p}) \operatorname{curl} \mathbf{F}. \quad (9.23)$$

Hence

$$\begin{aligned} \|(I - \Pi) \mathbf{u}\|_2 &= \|\nabla_{\perp} (I - \mathbf{p}) \psi\|_2 \\ &\leq \epsilon^{-1} \sup_{n \in \mathbb{N}^*} [\|(1 + \alpha_n) (\mathcal{B}_{\lambda_n, \alpha_n, \beta_n}^{\mathfrak{S}})^{-1}\| + \|\frac{d}{dx} (\mathcal{B}_{\lambda_n, \alpha_n, \beta_n}^{\mathfrak{S}})^{-1}\|] \|(I - \mathbf{p}) \operatorname{curl} \mathbf{F}\|_2. \end{aligned}$$

We can thus conclude, as in the proof of (2.30a) but with $\hat{\Lambda}_0$ being replaced by $\Upsilon \beta_1(\epsilon, L)^{\frac{1}{3}}$, that

$$\sup_{\Re \Lambda \leq \epsilon \Upsilon \beta_1(\epsilon, L)^{2/3}} \|(I - \Pi) \mathbf{u}\|_2 \leq \frac{1}{\epsilon} B_*^{\mathfrak{S}}(\Upsilon \beta_1(\epsilon, L)^{2/3}, \epsilon, L) \|(I - \mathbf{p}) \operatorname{curl} \mathbf{F}\|_2.$$

Consequently, we obtain, using the fact that $\beta_1^{\frac{2}{3}} \beta^{-1} \leq \beta^{-\frac{1}{3}}$ for $\beta \geq \beta_1$,

$$\begin{aligned} \sup_{\Re \Lambda \leq \epsilon \Upsilon \beta_1^{2/3}} \|(I - \Pi) \mathbf{u}\|_2 &\leq \frac{1}{\epsilon} \sup_{\beta \geq \beta_1} \sup_{\substack{\alpha \geq 0 \\ \Re \lambda \leq \Upsilon \beta^{-1/3}}} [\|(1 + \alpha) (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1}\| + \\ &\quad + \|\frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1}\|] \|(I - \mathbf{p}) \operatorname{curl} \mathbf{F}\|_2. \end{aligned}$$

We also note that

$$(I - \mathbf{p}) \operatorname{curl} \mathbf{F} = \operatorname{curl} (I - \Pi) \mathbf{F}.$$

As in the proof of (2.30a) we can use (9.8) and (9.10) to establish that for all $\hat{\delta} \in (0, 1/3)$ there exist Υ , $\epsilon_0 > 0$ and C such that for all positive L and ϵ for which $0 < \epsilon L \leq \epsilon_0$

$$\sup_{\Re \Lambda \leq \epsilon \Upsilon \beta_1(\epsilon, L)^{2/3}} \|(I - \Pi) \mathbf{u}\|_2 \leq C \epsilon^{-1} \beta_1(\epsilon, L)^{-1/3 + \hat{\delta}} \|\operatorname{curl} (I - \Pi) \mathbf{F}\|_2. \quad (9.24)$$

As in the proof of (2.30b) we need yet to establish a bound on $\|\nabla(I - \Pi) \mathbf{u}\|_2$ for $\Lambda < -\frac{1}{2} \|U'\|_{\infty} - \epsilon$. To this end we first recall Lemma 2.11 to obtain

$$(\mathfrak{T}_P^{\mathfrak{S}}(U, \epsilon, L) - \Lambda)(I - \Pi) \mathbf{u} = (I - \Pi) \mathbf{F},$$

Using (2.13) with \mathbf{u} replaced by $(I - \Pi) \mathbf{u}$, we get

$$\begin{aligned} \epsilon \|\nabla(I - \Pi) \mathbf{u}\|_2^2 + \langle (I - \mathbf{p}) u_1, U'(I - \mathbf{p}) u_2 \rangle - \Lambda \|(I - \Pi) \mathbf{u}\|_2^2 \\ = \langle (I - \Pi) \mathbf{u}, (I - \Pi) \mathbf{F} \rangle, \quad (9.25) \end{aligned}$$

which implies

$$\epsilon \|\nabla(I - \Pi)\mathbf{u}\|_2^2 \leq (|\Lambda| + \frac{1}{2}\|U'\|_\infty)\|(I - \Pi)\mathbf{u}\|_2^2 + \|(I - \Pi)\mathbf{u}\| \|(I - \Pi)\mathbf{F}\|, \quad (9.26)$$

and therefore leads to

$$\sup_{\Re\Lambda \leq -\frac{1}{2}\|U'\|_\infty - \epsilon} \|(I - \Pi)\mathbf{u}\|_{1,2} \leq \frac{1}{\epsilon} \|(I - \Pi)\mathbf{F}\|_2. \quad (9.27)$$

Consequently, as above, we may apply $(I - \Pi)$ to the resolvent identity (9.17) to establish that, for $\epsilon \in (0, 1]$, $L > 0$ such that $L\epsilon \leq \epsilon_0$,

$$\sup_{\Re\Lambda \leq \epsilon \Upsilon \beta_1(\epsilon, L)^{2/3}} \|(I - \Pi)(\mathfrak{I}_P^\mathfrak{S}(U, \epsilon, L) - \Lambda)^{-1}\| \leq CL^{1/3-\hat{\delta}}\epsilon^{-5/3-\hat{\delta}}. \quad (9.28)$$

As by (2.12)

$$\|e^{-t\mathfrak{I}_P^\mathfrak{S}(U, \epsilon, L)}(I - \Pi)\| \leq e^{t\|U'\|_\infty/2},$$

We may apply, as above, [19, Proposition 2.1] to $S(t) = e^{-t\mathfrak{I}_P^\mathfrak{S}}(I - \Pi)$ to establish (2.30b).

Proof of Part 2.

The proof of part 2 is obtained in the same manner.

For the first statement leading to (2.30a), we replace Proposition 9.1 by the following result

Proposition 9.3. *For any $0 < \hat{\Lambda}_0 < \pi^2/4$ and $C > 2/\pi + 4/\pi^2$ there exists $\epsilon_0 > 0$ and $\delta > 0$, such that for all positive ϵ and L for which $0 < L\epsilon \leq \epsilon_0$ and $U \in \mathcal{S}_r$ satisfying $\delta_2(U) < \delta$, it holds that*

$$\sup_{\Re\Lambda \leq \epsilon \hat{\Lambda}_0} \|(\mathcal{P}_{\Lambda, \epsilon}^\mathfrak{S})^{-1}\|_{\mathcal{L}(L^2_{per}, H^1_{per})} \leq \frac{C}{\epsilon(\frac{\pi^2}{4} - \hat{\Lambda}_0)}. \quad (9.29)$$

Proof. As the estimate (9.12) of $B_0^\mathfrak{S}(\hat{\Lambda}_0)$ is unaffected by the change of U in this part, we have only to prove that (9.11) still holds under the new assumptions. Recall that (9.11) follows from (9.10), which remains valid in this part, and (9.9) which needs to be remanded to fit this case. To this end we use (7.23) which gives the improved estimate

$$\sup_{\beta \geq \beta_1(\epsilon, L)} \sup_{\substack{0 \leq \alpha \leq \alpha_M \beta^{1/6} \\ \Re\lambda \leq \beta^{-1}\hat{\Lambda}_0}} \left[\|(1+\alpha)(\mathcal{B}_{\lambda, \alpha, \beta}^\mathfrak{S})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^\mathfrak{S})^{-1} \right\| \right] \leq C \beta_1(\epsilon, L)^{-2/3}. \quad (9.30)$$

Combining the above with (9.10) yields

$$B_*^\mathfrak{S}(\hat{\Lambda}_0, \epsilon, L) \leq C \beta_1(\epsilon, L)^{-2/3},$$

which together with (9.4) yields (9.36). ■

The proof of (2.30a) proceeds from here in the same manner as in Part 1.

We may conclude (2.30b) by following, once again, the same steps as those used in the proof of part 1 of the theorem. We use again (7.23) to establish, for $\Upsilon < J_m^{\frac{2}{3}} \mathfrak{R}\nu_1$,

$$\sup_{\beta \geq \beta_1(\epsilon, L)} \sup_{\substack{0 \leq \alpha \leq \alpha_M \beta^{1/6} \\ \mathfrak{R}\lambda \leq \Upsilon \beta^{-1/3}}} \left[\|(1+\alpha)(\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1}\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda, \alpha, \beta}^{\mathfrak{S}})^{-1} \right\| \right] \leq C \beta_1(\epsilon, L)^{-2/3}, \quad (9.31)$$

Together with (8.71), (9.31) yields, as in Part 1,

$$\sup_{\mathfrak{R}\lambda \leq \epsilon \Upsilon \beta_1(\epsilon, L)^{2/3}} \|(I - \Pi)(\mathfrak{T}_P^{\mathfrak{S}}(U, \epsilon, L) - \Lambda)^{-1}\| \leq CL^{2/3} \epsilon^{-\frac{4}{3}}. \quad (9.32)$$

from which (2.30b) is verified by following the same route used in Part 1.

9.3 Proof of Theorem 2.15

Since the proof is similar to the proof of Theorem 2.14 we address here only its main ingredients.

Proof of Part 1 For the first part of the Theorem we use (8.70) and (8.103) to establish that for any $\hat{\Lambda}_0$ and $\hat{\delta} \in (0, \frac{1}{3})$ there exists $\epsilon_0 > 0$ and $C > 0$ such that for all positive L and ϵ satisfying $0 < L\epsilon \leq \epsilon_0$ we have

$$B_*^{\mathfrak{D}}(\hat{\Lambda}_0, \epsilon, L) \leq C \beta_1(\epsilon, L)^{-1/3+\hat{\delta}}, \quad (9.33)$$

which is similar to (9.11) in the case $\# = \mathfrak{S}$.

To bound $B_0^{\mathfrak{D}}(\hat{\Lambda}_0)$ we use integration by parts to establish for any $\phi \in D(\mathcal{B}_{\hat{\Lambda}}^{\mathfrak{D}})$

$$-\Re \langle \mathcal{B}_{\hat{\Lambda}}^{\mathfrak{D}} \phi, \phi \rangle = \|\phi''\|^2 - \Re \hat{\Lambda} \|\phi'\|^2.$$

As $\phi' \in H_0^1(-1, 1)$ we obtain, for $\Re \hat{\Lambda} \leq \frac{\pi^2}{4}$,

$$-\Re \langle \mathcal{B}_{\hat{\Lambda}}^{\mathfrak{D}} \phi, \phi \rangle \geq \left(\frac{\pi^2}{4} - \Re \hat{\Lambda} \right) \|\phi'\|^2.$$

Upon the above inequality we use the fact that $\phi \in H_0^1$ to obtain the equivalent of (9.12) for all $0 < \hat{\Lambda}_0 < \pi^2/4$,

$$B_0^{\mathfrak{D}}(\hat{\Lambda}_0) = \sup_{\mathfrak{R}\hat{\Lambda} \leq \hat{\Lambda}_0} \left\| (\mathcal{B}_{\hat{\Lambda}}^{\mathfrak{D}})^{-1} \right\| + \left\| \frac{d}{dx} (\mathcal{B}_{\hat{\Lambda}}^{\mathfrak{D}})^{-1} \right\| \leq \left(\frac{2}{\pi} + \frac{4}{\pi^2} \right) \frac{1}{(\pi^2/4 - \hat{\Lambda}_0)}. \quad (9.34)$$

Combining the above with (9.33) and (9.4) we may conclude that

Proposition 9.4. *For any $0 < \hat{\Lambda}_0 < \pi^2/4$ and $C > 2/\pi + 4/\pi^2$ there exists $\epsilon_0 > 0$, such that for all positive L and $\epsilon > 0$ for which $0 < L\epsilon \leq \epsilon_0$ and $U \in \mathcal{S}_r$ satisfying (2.29), it holds that*

$$\sup_{\mathfrak{R}\lambda \leq \epsilon \hat{\Lambda}_0} \|(\mathcal{P}_{\lambda, \epsilon}^{\mathfrak{D}})^{-1}\|_{\mathcal{L}(L_{per}^2, H_{per}^1)} \leq \frac{C}{\epsilon(\pi^2/4 - \hat{\Lambda}_0)}.$$

We may now proceed as in the proof of (2.30a) to establish (2.33a).

To prove (2.33b) we use (8.70) together with (8.103) once again to obtain

$$B_*^{\mathfrak{D}}(\Upsilon\beta_1(\epsilon, L)^{2/3}, \epsilon, L) \leq C \beta_1(\epsilon, L)^{-1/3+\delta}. \quad (9.35)$$

From the above, we can obtain as in the proof of (2.30b) that

$$\sup_{\Re\Lambda \leq \epsilon \Upsilon\beta_1(\epsilon, L)^{2/3}} \|(I - \Pi)(\mathfrak{F}_P^{\mathfrak{D}}(U, \epsilon, L) - \Lambda)^{-1}\| \leq CL^{1/3-\delta} \epsilon^{-5/3-\delta}.$$

We can now establish (2.33b) in the same manner as in the proof of (2.30b).

Proof of Part 2 As in the proof of Part 2 of Theorem 2.14 we establish from (9.34), (8.21), and (8.70) that

Proposition 9.5. *For any $0 < \hat{\Lambda}_0 < \pi^2/4$ and $C > 2/\pi + 4/\pi^2$ there exists $\epsilon_0 > 0$ and $\delta > 0$, such that for all $\epsilon \in (0, 1]$ and $L > 0$ for which $0 < L\epsilon \leq \epsilon_0$ and $U \in \mathcal{S}_r$ satisfying $\delta_2(U) < \delta$, it holds that*

$$\sup_{\Re\Lambda \leq \epsilon \hat{\Lambda}_0} \|(\mathcal{P}_{\Lambda, \epsilon}^{\mathfrak{D}})^{-1}\|_{\mathcal{L}(L_{per}^2, H_{per}^1)} \leq \frac{C}{\epsilon(\frac{\pi^2}{4} - \hat{\Lambda}_0)}. \quad (9.36)$$

We then continue as in the proof of (2.33a).

To establish (2.33b) in Part 2 we use (8.21) and (8.70) to show that

$$B_*^{\mathfrak{D}}(\Upsilon\beta_1(\epsilon, L)^{2/3}, \epsilon, L) \leq C \beta_1(\epsilon, L)^{-2/3}.$$

Then we can continue in the same manner as in the proof of (2.33b) in the first part of the theorem.

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A Basic properties of the Airy function and Wasow's results on A_0

A.1 Airy function properties

In this subsection, we summarize some of the basic properties of Airy function $\text{Ai}(z)$, and the generalized Airy function $A_0(z)$, that are being used throughout this work (see [1] for details) and establish some new inequalities satisfied by these functions. We recall that Airy function is the unique solution of

$$(D_x^2 + x)u = 0,$$

on the line such that $u(x)$ tends to 0 as $x \rightarrow +\infty$ and $\text{Ai}(0) = 1/\left(3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)\right)$. Standard ODE theory shows that Airy function is entire and strictly decreasing on \mathbb{R}_+ , but has an infinite number of zeros in \mathbb{R}_- .

Airy function satisfies different asymptotic expansions as $|z| \rightarrow \infty$ depending on $\arg z$. We bring two of them here

$$\text{Ai}(z) = \frac{1}{2}\pi^{-\frac{1}{2}}z^{-1/4} \exp\left(-\frac{2}{3}z^{3/2}\right) \left(1 + \mathcal{O}(|z|^{-\frac{3}{2}})\right), \quad \text{for } |\arg z| < \pi, \quad (\text{A.1a})$$

$$\text{Ai}(-z) = \pi^{-\frac{1}{2}}z^{-1/4} \sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) \left(1 + \mathcal{O}(|z|^{-\frac{3}{2}})\right), \quad \text{for } |\arg z| < \frac{2\pi}{3}. \quad (\text{A.1b})$$

Moreover the $\mathcal{O}(|z|^{-3/2})$ estimate is, for any $\hat{\delta} > 0$, uniform when $|\arg z| \leq \pi - \hat{\delta}$ in (A.1a) or $|\arg z| \leq 2\pi/3 - \hat{\delta}$ in (A.1b). In particular, $\text{Ai}(z)$ is rapidly decreasing at ∞ if z belongs to a sector $|\arg z| \leq \frac{\pi}{3} - \hat{\delta}$, with $\hat{\delta} > 0$.

The following moment estimates are needed in Subsection 6.1.

Proposition A.1. *Let $\langle \lambda \rangle := \sqrt{1 + |\lambda|^2}$. For every $\mu_0 > 0$ there exists $C > 0$ such that, for any λ with $\Re \lambda \leq \mu_0$, we have,*

$$\int_0^\infty x^{2k} |\text{Ai}(e^{i\pi/6}[x + i\lambda])|^2 dx \leq C \langle \lambda \rangle^{-k-1} e^{-\frac{4}{3}\Re\{(e^{i2\pi/3}\lambda)^{3/2}\}}, \quad \text{for } 0 \leq k \leq 4, \quad (\text{A.2})$$

and

$$\int_0^\infty x^s |\text{Ai}(e^{i\pi/6}[x + i\lambda])| dx \leq C \langle \lambda \rangle^{-\frac{3+2s}{4}} e^{-\frac{2}{3}\Re\{(e^{i2\pi/3}\lambda)^{3/2}\}} \quad \text{for } 0 \leq s \leq 3. \quad (\text{A.3})$$

Proof.

If $|\lambda| \leq 3\mu_0$ then, by (A.1a), all the estimates of the proposition are satisfied for some $C = C(\mu_0)$. Hence, we can consider from now on the case where $|\lambda| > 3\mu_0$. Note that by interpolation, it is sufficient to consider $k \in \mathbb{Z}$.

Proof of (A.2) for $k = 0$.

Let

$$u(x) = \frac{\text{Ai}(e^{i\pi/6}[x + i\lambda])}{\text{Ai}(e^{i2\pi/3}\lambda)}, \quad (\text{A.4})$$

which is well-defined for

$$\mathcal{V}(\mu_0) := \{\Re \lambda \leq \mu_0\} \cap \{|\lambda| > 3\mu_0\},$$

since the denominator can vanish only when $\arg \lambda = \pi/3$.

It can be easily verified that

$$\begin{cases} -u'' + (ix - \lambda)u = 0 & \text{for } x > 0 \\ u(0) = 1. \end{cases} \quad (\text{A.5})$$

Let further

$$w = u - e_\lambda(x) \quad \text{with } e_\lambda(x) := e^{(-\lambda)^{1/2}x}, \quad (\text{A.6})$$

where $(-\lambda)^{\frac{1}{2}}$ is well-defined for $\lambda \in \mathcal{V}(\mu_0)$ as

$$-\pi + \arccos\left(\frac{1}{3}\right) < \arg(-\lambda) < \pi - \arccos\left(\frac{1}{3}\right)$$

Note that for $\lambda \in \mathcal{V}(\mu_0)$

$$\Re(-\lambda)^{\frac{1}{2}} \geq |\lambda|^{\frac{1}{2}} \sin\left(\frac{1}{2} \arccos\left(\frac{1}{3}\right)\right) \geq |3\mu_0|^{\frac{1}{2}} \sin\left(\frac{1}{2} \arccos\left(\frac{1}{3}\right)\right) > 0, \quad (\text{A.7})$$

implying, for any $\mu_0 > 0$, the existence of $C(\mu_0) > 0$, such that, for all $\lambda \in \mathcal{V}(\mu_0)$, it holds that

$$\|e_\lambda\|_2 \leq C(\mu_0)|\lambda|^{-\frac{1}{4}}. \quad (\text{A.8})$$

Substituting into (A.5) yields

$$(\mathcal{L}_+ - \lambda)w = -ixe_\lambda(x), \quad (\text{A.9})$$

where \mathcal{L}_+ is associated with the differential operator $-d^2/dx^2 + ix$ and is defined on the domain

$$D(\mathcal{L}_+) = \{u \in H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+) \mid xu \in L^2(\mathbb{R}_+)\}.$$

It has been established in [17, §5] that, for any $\hat{\mu} \in \mathbb{R}^+$, there exist $\nu_0(\hat{\mu}) > 0$ and $C(\hat{\mu})$ such that

$$\|(\mathcal{L}_+ - \lambda)^{-1}\| \leq C(\hat{\mu}), \quad \text{for } |\Im\lambda| \geq \nu_0(\hat{\mu}) \text{ and } |\Re\lambda| \leq \hat{\mu}. \quad (\text{A.10})$$

In addition, we have

$$\|(\mathcal{L}_+ - \lambda)^{-1}\| \leq C, \quad \text{for } \Re\lambda \leq 0. \quad (\text{A.11})$$

Denote by $\{\nu_n\}_{n=1}^\infty$ the eigenvalues of \mathcal{L}_+ , and recall that they are located on the ray $\arg \lambda = \pi/3$ (see [3, §2.2]). Observing that $\mathcal{V}(\mu_0)$ does not contain any eigenvalue (a consequence of the fact that $\arccos 1/3 > \pi/3$) and combining the above with (A.10)-(A.11) yield the existence of $C(\mu_0) > 0$ such that

$$\sup_{\lambda \in \mathcal{V}(\mu_0)} \|(\mathcal{L}_+ - \lambda)^{-1}\| \leq C(\mu_0). \quad (\text{A.12})$$

Hence, for $\lambda \in \mathcal{V}(\mu_0)$,

$$\|w\|_2 \leq C(\mu_0)\|xe_\lambda\|_2.$$

Using (A.7), we obtain that

$$\|xe_\lambda\|_2 \leq \hat{C}(\mu_0)|\lambda|^{-3/4}, \quad (\text{A.13})$$

and hence

$$\|w\|_2 \leq \check{C}(\mu_0)|\lambda|^{-3/4}. \quad (\text{A.14})$$

By the above, (A.6), and (A.8), we thus have

$$\|u\|_2 \leq C(\mu_0)|\lambda|^{-1/4}. \quad (\text{A.15})$$

To prove (A.2) for $k = 0$, we need to establish yet an upper bound for $\text{Ai}(e^{2i\frac{\pi}{3}}\lambda)$, for $\lambda \in \mathcal{V}(\mu_0)$. This is an immediate consequence of (A.1a). We observe indeed that

for any $\epsilon > 0$, $\arg \lambda \notin (-\frac{\pi}{2} + \epsilon, \frac{\pi}{2} - \epsilon)$ as $|\lambda| \rightarrow +\infty$ with $\Re \lambda \leq \mu_0$. This implies that, for any $\epsilon > 0$, there exists $\lambda_0(\epsilon, \mu_0) > 0$ such that for $|\lambda| \geq \lambda_0(\epsilon)$ and $\Re \lambda \leq \mu_0$, $\arg \left(\lambda \exp \frac{2i\pi}{3} \right) \in (-\frac{5\pi}{6} - \epsilon, \frac{\pi}{6} + \epsilon)$.

Consequently, for any $\mu_0 > 0$, there exists a constant $C(\mu_0)$, such that for all $\lambda \in \mathcal{V}(\mu_0)$

$$|\text{Ai}(e^{2i\frac{\pi}{3}} \lambda)| \leq C(\mu_0) < \lambda >^{-\frac{1}{4}} e^{-\frac{2}{3}\Re\{(e^{i2\pi/3}\lambda)^{3/2}\}}. \quad (\text{A.16})$$

Together with (A.15), (A.16) yields (A.2) for $k = 0$ and $\lambda \in \mathcal{V}(\mu_0)$.

Proof of (A.2) for $k = 1$.

We begin by deriving an estimate of xw . To this end we observe that

$$(D_x^2 + ix - \lambda)(xw) = -ix^2e_\lambda - 2iw'.$$

Following the same procedure applied in the previous proof, we need an estimate of w' . To achieve this end, we first observe that

$$\Re \langle (D_x^2 + ix - \lambda)w, w \rangle = \|w'\|^2 - \Re \lambda \|w\|_2^2,$$

which leads to the estimate

$$\|w'\|^2 \leq \mu_0 \|w\|_2^2 + \|xe_\lambda\| \|w\|_2.$$

Implementing (A.13) and (A.14) leads to

$$\|w'\|_2 \leq C(\mu_0) |\lambda|^{-\frac{3}{4}}. \quad (\text{A.17})$$

We observe in addition that

$$\|x^2e_\lambda\|_2 \leq C(\mu_0) |\lambda|^{-\frac{5}{4}}.$$

Proceeding as in the proof of (A.2) for $k = 0$, we obtain (A.2) for $k = 1$.

Proof of (A.2) for $k = 2$.

In this case the approximation of u by e_λ is unsatisfactory. To improve it, in light of (A.9), we solve in $H^2(\mathbb{R}^+)$ the problem

$$\begin{cases} \left(-\frac{d^2}{dx^2} - \lambda \right) f_\lambda = -ixe_\lambda & \text{for } x > 0, \\ f_\lambda(0) = 0. \end{cases} \quad (\text{A.18})$$

We look for f_λ in the form $f_\lambda = p_\lambda e_\lambda$ which means that p_λ must satisfy

$$p_\lambda''(x) - 2(-\lambda)^{\frac{1}{2}} p_\lambda'(x) = ix, \quad p_\lambda(0) = 0.$$

We search for a polynomial solution. A simple computation leads to

$$f_\lambda = p_\lambda e_\lambda \text{ with } p_\lambda(x) = \frac{i}{4} \left((-\lambda)^{-1/2} x^2 - \lambda^{-1} x \right). \quad (\text{A.19})$$

We now write

$$u = e_\lambda + w = e_\lambda + f_\lambda + w_1, \quad (\text{A.20})$$

to obtain

$$(\mathcal{L}_+ - \lambda)w_1 = -ixf_\lambda = -ixp_\lambda e_\lambda. \quad (\text{A.21})$$

In order to prove (A.2) ($k = 2$) we observe first that

$$\sum_{\ell=0}^3 \langle \lambda \rangle^{\frac{\ell}{2}} \|x^\ell f_\lambda\| \leq C \langle \lambda \rangle^{-\frac{7}{4}}. \quad (\text{A.22})$$

Hence, it remains necessary to obtain an estimate of $x^2 w_1$ in L^2 . Here we use the fact that (A.9) is similar to (A.21), the only difference being that the right-hand-side is given by $-ixf_\lambda$ instead of $-ixe_\lambda$. Note that $\|xf_\lambda\|_2$ is much smaller than $\|xe_\lambda\|_2$ as $|\lambda| \rightarrow \infty$.

By (A.12), (A.21) and (A.22) (for $\ell = 1$), we then have

$$\|w_1\|_2 \leq \frac{C}{|\lambda|^{9/4}}, \quad (\text{A.23})$$

which is significantly smaller than the bound provided by (A.14) for w . We continue as in the case $k = 1$. We first use the identity

$$\Re\langle w_1, (\mathcal{L}_+ - \lambda)w_1 \rangle = \|w_1'\|_2^2 - \mu \|w_1\|_2^2,$$

to conclude with the aid of (A.21), (A.22), (A.23), and recalling that $\mu \leq \mu_0$,

$$\|w_1'\|_2 \leq \frac{C}{|\lambda|^{9/4}}. \quad (\text{A.24})$$

Then we write

$$(\mathcal{L}_+ - \lambda)(xw_1) = x(\mathcal{L}_+ - \lambda)w_1 - 2w_1' = -ix^2 f_\lambda - 2w_1'. \quad (\text{A.25})$$

Combining (A.24), (A.21), and (A.12) yields

$$\|xw_1\|_2 \leq \frac{C}{|\lambda|^{9/4}}. \quad (\text{A.26})$$

From (A.26) and (A.23) we get in addition

$$\|x^{1/2}w_1\|_2 \leq \|xw_1\|_2^{1/2} \|w_1\|_2^{1/2} \leq \frac{C}{|\lambda|^{9/4}}. \quad (\text{A.27})$$

Next, we write

$$\Re\langle xw_1, (\mathcal{L}_+ - \lambda)(xw_1) \rangle = \|(xw_1)'\|_2^2 - \mu \|xw_1\|_2^2,$$

to conclude from (A.25) and (A.26) that

$$\|(xw_1)'\|_2 \leq \frac{C}{|\lambda|^{9/4}}. \quad (\text{A.28})$$

Upon writing

$$(\mathcal{L}_+ - \lambda)(x^2 w_1) = x^2(\mathcal{L}_+ - \lambda)w_1 - 2(xw_1)' = -ix^3 f_\lambda - 2(xw_1)',$$

we use (A.28), (A.21), and (A.12) to obtain

$$\|x^2 w_1\|_2 \leq \frac{C}{|\lambda|^{9/4}}, \quad (\text{A.29})$$

which can be used, together with (A.20) and (A.6) to obtain (A.2) for $k = 2$.

In a similar manner we obtain the estimates for $k \in \{3, 4\}$. We note in particular that

$$\|x^4 w_1\|_2 + \|(x^4 w_1)'\|_2 \leq \frac{C}{|\lambda|^{9/4}},$$

and hence, by Sobolev embeddings,

$$\|x^4 w_1\|_\infty \leq \frac{C}{|\lambda|^{9/4}}.$$

Combining the above with (A.6) and (A.20) yields, for $k \in [0, 4]$,

$$\|x^k u\|_\infty \leq \frac{C}{|\lambda|^{k/2}}. \quad (\text{A.30})$$

Weighted L^1 estimates.

Recall that in deriving (A.2) we needed to establish that, for $k = 0, \dots, 4$, it holds that

$$\|x^k u\|_{L^2} \leq C \langle \lambda \rangle^{-\frac{2k+1}{4}}. \quad (\text{A.31})$$

By interpolation it is enough to treat the case when s is an integer. Then we use Hölder inequality and (A.31) for $k = s$ and $k = s + 1$ to obtain, for $s \leq 3$,

$$\begin{aligned} \|x^s u\|_{L^1} &\leq \|[x + i \langle \lambda \rangle^{-\frac{1}{2}}] x^s u\|_2 \|[x + i \langle \lambda \rangle^{-\frac{1}{2}}]^{-1}\|_2 \\ &\leq \left(\|x^{s+1} u\|_2 + \langle \lambda \rangle^{-\frac{1}{2}} \|x^s u\|_2 \right) \|[x + i \langle \lambda \rangle^{-\frac{1}{2}}]^{-1}\|_2 \\ &\leq C \langle \lambda \rangle^{-\frac{s+1}{2}}. \end{aligned} \quad (\text{A.32})$$

We then recover (A.3) by using (A.16). ■

A.2 Definition of A_0 and the locus of its zeroes.

Let $A_0 : \mathbb{C} \rightarrow \mathbb{C}$ be given (see (6.9)) by the holomorphic extension to \mathbb{C} of

$$\mathbb{R} \ni z \mapsto A_0(z) = e^{i\pi/6} \int_z^{+\infty} \text{Ai}(e^{i\pi/6} t) dt.$$

To use the results of Wasow [35] (see [35, Eq. (39)]) and justify this holomorphic extension we observe the following relation

Lemma A.2.

$$A_0(z) = -\psi(-e^{i\pi/6} z). \quad (\text{A.33})$$

where ψ is the holomorphic extension of the real function

$$\mathbb{R} \ni x \mapsto \psi(x) := \int_{-\infty}^x \text{Ai}(-t) dt = \frac{1}{3} + \int_0^x \text{Ai}(-t) dt. \quad (\text{A.34})$$

It has been proved by Wasow in [35, Section 3] that the zeroes of ψ are all located in the sector $|\arg z| < \frac{\pi}{6}$.

Proposition A.3. *The zeroes of ψ belongs to $-\pi/6 < \arg z < \pi/6$. Moreover ψ has no real zeroes.*

Sketch of the proof. To establish that result is a combination of the argument principle and Rouché's. The change of $\arg \psi$ is estimated along the path $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where, for some $R > 0$

$$\Gamma_1 = \{z = re^{i\pi/6} \mid 0 < r \leq R\} \quad ; \quad \Gamma_2 = \{z = Re^{i\theta} \mid \pi/6 < \theta < \pi\} \quad ; \quad \Gamma_3 = [-R, 0].$$

Since $\text{Ai}(z)$ is real and positive for $z \in \mathbb{R}_+$ it follows that $\arg \psi$ does not change along Γ_3 . Along Γ_2 one uses (A.1), with a bound on the remainder. Finally, to estimate $\Delta \arg \psi$ along Γ_1 one uses the power series of $\text{Ai}(t)$, for $0 < |t| < 9$, and (A.1) for $|t| > 9$. The tails of the ensuing power series of $\Re \psi$ and $\Im \psi$ are Leibniz series with terms of alternating sign and decreasing moduli. Thus, one may truncate the series into finite sums, and the remainders can be easily estimated. Once the above procedure is applied, one can establish that $|\Delta \arg \psi| < 2\pi$ and hence that $\Delta \arg \psi = 0$ along Γ . Since $\psi(z) = 0 \Rightarrow \psi(\bar{z}) = 0$, the first statement of the proposition follows.

The second statement is proved in [35, p. 199]. ■

Corollary A.4. *Let $A_0(iz) = 0$. Then, $\pi/6 < \arg z < \pi/2$.*

We continue with the following result stated in [35] (Eq. (35)) which allows us to obtain additional information on the location of the zeroes of A_0 . It also serves as a useful tool in some of the proofs in Subsection 6.1.

Lemma A.5. *Let $\tilde{\psi}(z) = \psi(-z)$. For any $0 < \hat{\delta} < \pi$ there exists $C_{\hat{\delta}} > 0$ and $r_0(\hat{\delta}) > 0$ such that for all $|z| > r_0(\hat{\delta})$ in the sector $|\arg z| < \pi - \hat{\delta}$ it holds that*

$$\left| \tilde{\psi}(z) - \frac{1}{2} \pi^{-\frac{1}{2}} z^{-3/4} \exp\left(-\frac{2}{3} z^{3/2}\right) \right| \leq \frac{C_{\hat{\delta}}}{|z|^{7/4}} \left| \exp\left(-\frac{2}{3} z^{3/2}\right) \right|. \quad (\text{A.35})$$

The proof is a rather standard application of the method of steepest descent method [26, Chapter 4] and is therefore being skipped.

A.3 Asymptotic of the zeroes

In Subsection 6.1 we also need to establish the following lemma about the asymptotic behavior of the zeroes of A_0 . A similar statement for ψ is made in [35, § 3] without a clear proof.

Proposition A.6. *Let \mathcal{S}_0 denote the set of points $\lambda \in \mathbb{C}$ satisfying $A_0(i\lambda) = 0$. Then, for any $R > 0$, $\mathcal{S}_0 \cap B(0, R)$ is finite and its cardinality tends to $+\infty$ as R tends to $+\infty$. In particular \mathcal{S}_0 is non empty. Moreover, for any $\epsilon > 0$, there exists R such that, for $\lambda \in \mathcal{S}_0 \cap B(0, R)^c$,*

$$|\arg \lambda - \pi/3| \leq \epsilon. \quad (\text{A.36})$$

Proof.

We apply Jensen's formula [24, Theorem 1.7] to $\tilde{\psi}(z) = \psi(-z)$.

$$\log \tilde{\psi}(0) + N_{\tilde{\psi}}(R, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |\tilde{\psi}(Re^{i\theta})| d\theta. \quad (\text{A.37})$$

In the above $N_{\tilde{\psi}}(R, 0)$ is the Nevanlinna counting function,

$$N_{\tilde{\psi}}(R, 0) = \sum_{\substack{|a| < R \\ \tilde{\psi}(a)=0}} \log \left| \frac{R}{a} \right|,$$

where we have used the fact that all zeroes of $\tilde{\psi}$ are simple. We recall from [35] that none of them is real, and all the zeroes of $\tilde{\psi}' = -\text{Ai}$ lie on the negative real axis. Note that if $\mathcal{S}_0 = \emptyset$ then $N_{\tilde{\psi}}(R, 0) \equiv 0$ for all $R > 0$. Recall from the definition that $\tilde{\psi}(0) = 1/3$ and hence if we show that the first term on the right-hand-side of (A.37) is unbounded as $R \rightarrow \infty$, we may conclude the first statement of the proposition.

It follows from (A.35) that for any $C_0 \in (0, \frac{8}{9})$ there exist $\hat{\delta} > 0$ and $r_0 > 0$ such that for all $R > r_0$ we have

$$\int_{-\pi+\hat{\delta}}^{\pi-\hat{\delta}} \log |\tilde{\psi}(Re^{i\theta})| d\theta \geq C_0 R^{3/2}. \quad (\text{A.38})$$

Indeed, using Lemma A.5, we establish the existence for any $\hat{\delta} > 0$ of $r_0(\hat{\delta})$ and C such that, for all $R \geq r_0(\hat{\delta})$,

$$\begin{aligned} \int_{-\pi+\hat{\delta}}^{\pi-\hat{\delta}} \log |\tilde{\psi}(Re^{i\theta})| d\theta &\geq \frac{2}{3} R^{3/2} (1 - CR^{-1}) \int_{-\pi+\hat{\delta}}^{\pi-\hat{\delta}} -\cos\left(\frac{3\theta}{2}\right) d\theta \\ &\geq \frac{8}{9} R^{3/2} (1 - CR^{-1}) \cos(3\hat{\delta}/2). \end{aligned}$$

To estimate the integral for $|\arg z| \in (\pi - \hat{\delta}, \pi)$ we write, owing to the concavity of $\mathbb{R}^+ \ni x \mapsto \log x$,

$$\int_{|\theta| > \pi - \hat{\delta}} \log |\tilde{\psi}(Re^{i\theta})| d\theta = 2 \int_{\pi - \hat{\delta}}^{\pi} \log |\tilde{\psi}(Re^{i\theta})| d\theta \leq 2\hat{\delta} \log \left(\frac{1}{\hat{\delta}} \int_{\pi - \hat{\delta}}^{\pi} |\tilde{\psi}(Re^{i\theta})| d\theta \right). \quad (\text{A.39})$$

Since by (A.34) we have, for any $0 < \hat{\delta} < \pi/3$ and $R > R_0$,

$$\int_{\pi - \hat{\delta}}^{\pi} |\tilde{\psi}(Re^{i\theta})| d\theta \leq \int_0^{R_0} \int_{\pi - \hat{\delta}}^{\pi} |\text{Ai}(se^{i\theta})| d\theta ds + \int_{R_0}^R \int_{\pi - \hat{\delta}}^{\pi} |\text{Ai}(se^{i\theta})| d\theta ds,$$

where R_0 is fixed, but sufficiently large so that $\text{Ai}(se^{i\theta})$ obeys (A.1b) for all $s > R_0$. For the first term on the right-hand side there exists $C(R_0) > 0$ such that

$$\int_0^{R_0} \int_{\pi - \hat{\delta}}^{\pi} |\text{Ai}(se^{i\theta})| d\theta ds \leq C(R_0) \hat{\delta}.$$

For the second term we use (A.1b) to obtain the rather crude estimate for $R > R_0$

$$\int_{R_0}^R \int_{\pi - \hat{\delta}}^{\pi} |\text{Ai}(se^{i\theta})| d\theta ds \leq C\hat{\delta}(R - R_0) e^{-\frac{2}{3}R^{3/2} \cos(3(\pi - \hat{\delta})/2)}.$$

Consequently, for every $C_1 > 1$ there exists R_1 , such that, for all $R > R_1$,

$$\log \left(\frac{1}{\hat{\delta}} \int_{\pi-\hat{\delta}}^{\pi} |\tilde{\psi}(Re^{i\theta})| d\theta \right) \leq \frac{2}{3} C_1 R^{3/2} |\cos(3(\pi - \hat{\delta})/2)| \leq C_1 R^{3/2} \hat{\delta}. \quad (\text{A.40})$$

This implies, by (A.39), the existence, for any $0 < \hat{\delta} < \pi/3$, of $R_2 > 0$, such that for $R \geq R_2$

$$\int_{|\theta| > \pi - \hat{\delta}} \log |\tilde{\psi}(Re^{i\theta})| d\theta \leq 2\hat{\delta}^2 C_1 R^{3/2}.$$

Combining the above with (A.38) yields, by fixing $\hat{\delta}$ small enough, the existence of R_3 and $\hat{C}_0 > 0$ such that for all $R > R_3$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |\tilde{\psi}(Re^{i\theta})| d\theta \geq \hat{C}_0 R^{3/2}.$$

The proof shows that the above bound holds true for any $\hat{C}_0 \in (0, \frac{8}{9})$ for $R_3(\hat{C}_0)$ large enough. Hence, we get a lower bound for $N_{\tilde{\psi}}(R, 0)$ which implies the first statement of the proposition.

To prove (A.36) we notice that for any $\hat{\delta} > 0$, it holds by (A.35) that there exists $r_0 > 0$ such that $\tilde{\psi}$ cannot have any zeroes, for $|z| > r_0$ in the sector $|\arg z| < \pi - \hat{\delta}$. We then observe that $A_0(i\lambda) = -\tilde{\psi}(e^{2i\pi/3}z)$. ■

We can now immediately draw the following conclusion.

Corollary A.7.

$$\inf \Re \mathcal{S}_0 = \vartheta_1^r > 0. \quad (\text{A.41})$$

A.4 Normalized Airy functions

We complete the appendix with some corollaries of (A.2)-(A.3) and (A.35) needed in Subsection 6.1 and with other estimates needed in Subsection 6.3.

Proposition A.8. *Let $\Psi_\lambda \in L^2(\mathbb{R}_+)$ be defined by*

$$\Psi_\lambda(x) = \frac{\text{Ai}(e^{i\pi/6}(x + i\lambda))}{A_0(i\lambda)}. \quad (\text{A.42})$$

Then, for any $\hat{\delta}_1 > 0$ there exists $C > 0$ such that for all $\Re \lambda \leq \vartheta_1^r - \hat{\delta}_1$

$$\|x^k \Psi_\lambda\|_2 \leq C \langle \lambda \rangle^{\frac{1-2k}{4}} \text{ for } k \in [0, 4], \quad (\text{A.43a})$$

$$\|x^s \Psi_\lambda\|_1 \leq C \langle \lambda \rangle^{-\frac{s}{2}} \text{ for } s \in [0, 3], \quad (\text{A.43b})$$

$$\|x^s \Psi_\lambda\|_\infty \leq C \langle \lambda \rangle^{-\frac{s-1}{2}} \text{ for } s \in [0, 4]. \quad (\text{A.43c})$$

Proof. Let $|\lambda| > 3\vartheta_1^r$ (the proof for $|\lambda| \leq 3\vartheta_1^r$ follows by continuity as the denominator is bounded away from zero). As $A_0(i\lambda) = -\tilde{\psi}(e^{i2\pi/3}\lambda)$ we obtain from (A.35) that for any $\hat{\delta} > 0$ and $|\arg \lambda + 2\pi/3| < \pi$

$$|A_0(i\lambda)| \geq \frac{C}{|\lambda|^{3/4}} \exp \left\{ -\frac{2}{3} \Re \{ (e^{i2\pi/3}\lambda)^{3/2} \} \right\},$$

which combined with (A.2) and (A.3) yields (A.43a,b). The proof of (A.43c) follows from (A.30). ■

We also need, in Subsection 6.3 the following estimate

Lemma A.9. *Let $\varkappa > 0$ and, for $\theta > 0$, $\Upsilon(\theta) = \mu_0(\theta) - \varkappa$ where*

$$\mu_0(\theta) := \inf \Re \sigma(\mathcal{L}^\theta).$$

Let $F(\lambda, \theta)$ be defined by (6.47c). There exists $C(\varkappa) > 0$ such that, for all $\theta > 0$,

$$\sup_{\Re \lambda \leq \Upsilon(\theta)} \left| \frac{A_0(i\lambda)}{F(\lambda, \theta)} \right| \leq C(\varkappa)(1 + \theta). \quad (\text{A.44})$$

Proof. Let $R > 0$ be determined later.

The case $|\lambda| < R$.

Let $\theta_1(\varkappa)$ satisfy

$$\sup_{\theta > \theta_1(\varkappa)} \mu_0(\theta) \leq \Re \nu_1 + \frac{\varkappa}{2}$$

By the continuity of $(\lambda, \theta) \mapsto A_0(i\lambda)/F(\lambda, \theta)$ on $B(0, R) \times [0, \theta_1(\varkappa)]$, we have

$$\sup_{\{\Re \lambda \leq \Upsilon(\theta)\} \cap B(0, R), 0 \leq \theta \leq \theta_1} \left| \frac{A_0(i\lambda)}{F(\lambda, \theta)} \right| \leq C(\varkappa, R). \quad (\text{A.45})$$

In the case $\theta > \theta_1(\varkappa)$ we write, as in Subsection 6.3 (see (6.63)),

$$\theta F(\lambda, \theta) - \text{Ai}(e^{i2\pi/3}\lambda) = \theta \int_{\mathbb{R}_+} e^{-\theta x} [\text{Ai}(e^{i2\pi/3}\lambda + e^{i\pi/6}x) - \text{Ai}(e^{i2\pi/3}\lambda)] dx,$$

and deduce (see (6.64)) that, for $|\lambda| \leq R$, we have

$$|\theta F(\lambda, \theta) - \text{Ai}(e^{i2\pi/3}\lambda)| \leq C(R)\theta^{-\frac{1}{2}}.$$

We may now use the fact that $\Upsilon(\theta) < \Re \nu_1 - \varkappa/2$ to get the existence of $\theta(\varkappa, R) \geq \theta_1(\varkappa)$ and $C(\varkappa, R) > 0$ such that, for $\theta \geq \theta(\varkappa, R)$, $|\lambda| \leq R$ and $\Re \lambda \leq \Upsilon(\theta)$ we have

$$|F(\lambda, \theta)| \geq \frac{C(\varkappa, R)}{1 + \theta}.$$

For $\theta \in [\theta_1(\varkappa), \theta(\varkappa, R)]$, we use the continuity of $|F(\lambda, \theta)|$ to get a uniform lower bound for it. Consequently, we obtain that

$$\sup_{\{\Re \lambda \leq \Upsilon(\theta)\} \cap |\lambda| \leq R} \left| \frac{A_0(i\lambda)}{F(\lambda, \theta)} \right| \leq C(\varkappa, R)(1 + \theta). \quad (\text{A.46})$$

The case $|\lambda| \geq R$.

If $|\lambda| \geq R$ we may use (6.67) which reads

$$\left| \frac{[\theta + (-\lambda)^{1/2}]F(\lambda, \theta)}{\text{Ai}(e^{i2\pi/3}\lambda)} - 1 \right| \leq C |\lambda|^{-1/4},$$

to obtain for R large enough with the aid of (A.1), (A.33), and (A.35) that,

$$\left| \frac{A_0(i\lambda)}{F(\lambda, \theta)} \right| \leq \left| \frac{A_0(i\lambda)}{\text{Ai}(e^{i2\pi/3}\lambda)} \right| \left| \frac{\text{Ai}(e^{i2\pi/3}\lambda)}{F(\lambda, \theta)} \right| \leq C \frac{\theta + |\lambda|^{1/2}}{|\lambda|^{1/2}}.$$

Combining the above with (A.45) and (A.46) yields (A.44). ■

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