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# FIRST PASSAGE PROBLEMS FOR UPWARDS SKIP-FREE RANDOM WALKS VIA THE $\Phi, W, Z$ PARADIGM

FLORIN AVRAM AND MATIJA VIDMAR

ABSTRACT. We develop the theory of the  $W$  and  $Z$  scale functions for right-continuous (upwards skip-free) discrete-time discrete-space random walks, along the lines of the analogous theory for spectrally negative Lévy processes. Notably, we introduce for the first time in this context the one and two-parameter scale functions  $Z$ , which appear for example in the joint problem of deficit at ruin and time of ruin, and in problems concerning the walk reflected at an upper barrier. Comparisons are made between the various theories of scale functions as one makes time and/or space continuous. The theory is shown to be fruitful by providing a convenient unified framework for studying dividends-capital injection problems under various objectives, for the so-called compound binomial risk model of actuarial science.

**Key words:** skip-free Markovian jump processes; random walks; scale functions; martingales; compound binomial risk model; dividends; capital injections.

*2010 Mathematics Subject Classification:* Primary: 60G50; Secondary: 91B30.

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## 1. INTRODUCTION

First passage theory for random walks is a classic topic, excellently treated for example in the textbooks [Spi13, Fel71, Tak77, Bor12], and this includes the upwards skip-free compound binomial model of the actuarial literature. However, in light of recent developments in the parallel continuous-time theory of spectrally negative/upwards skip-free Lévy and Markov additive processes — see for example [AKP04, Iva11, IP12, Vid13, AIZ16, AGVA17] — it seems worthwhile to revisit this topic.

Indeed, while it is well-known that optimization problems in the discrete setup (which is in many ways more natural than the continuous one) may be tackled numerically by dynamic programming algorithms, it is less known that when restricting to the skip-free case, the solutions of a great variety of first passage problems may be parsimoniously expressed in terms of two families of scale functions, just like in the continuous-time Lévy case.

Recall that in the Lévy case the scale functions  $W^{(q)}$  and  $Z^{(q)}$  have been known since [Sup76] and [AKP04], and that these functions intervene in important optimization problems. For example,  $W^{(q)}$  provides the value function of the classic de Finetti problem of optimizing expected dividends until ruin with discount factor  $q$  [APP07], and  $Z^{(q)}(\cdot, \theta)$  intervenes for instance in the moment generating function (as function of  $\theta$ ) of the capital injections [IP12] and in the combined dividend payout-capital injections problem for a doubly reflected process [APP07, AI17]. These are just two examples from an ever increasing list of problems [Pis05, Kyp14, AGVA17], which can be now tackled by simple lookup in the list and using off-shelf packages computing the functions  $W$  and  $Z$  [Iva11].

It was expected that the first passage theory developed in the world of spectrally negative Lévy processes, which we call the  $\Phi, W, Z$  paradigm, should have parallels for other classes of spectrally negative/skip-free Markov processes. In particular, the three cases listed below, being precisely the processes with stationary independent increments that exhibit non-random overshoots [Vid15] (modulo trivial processes with monotone paths), were expected to be very similar:

- (i) (discrete-time, discrete-space) right-continuous (i.e. skip-free to the right) random walks, also known in insurance as the compound binomial model;
- (ii) (continuous-time, discrete-space) compound Poisson processes that live on a lattice  $h\mathbb{Z}$ ,  $h \in (0, \infty)$ , jumping up only by  $h$  (what were called upwards skip-free Lévy chains in [Vid13]);
- (iii) (continuous-time, continuous-space) spectrally negative Lévy processes.

However, important steps were missing for the fully discrete setup. Notably, the second scale function  $Z_v(\cdot, w)$  was absent from the previous literature, and we provide below for the first time its generating function ( $z$ -transform) (17).

A second contribution of our paper is spelling out the connections between the three types of first passage problems listed above. In particular, we provide in Appendix B a concise table featuring side-by-side some of the salient features of the  $\Phi, W, Z$  theory for the three types of process (i)-(ii)-(iii) delineated above. It may serve as an inexhaustive summary and a quick reference; for the complete exposition, the main body of the text must be consulted.

A third contribution is showing the convenience of using the  $\Phi, W, Z$  theory for solving dividends-capital injections problems – see Sections 6 and 7.

Now, the doubly discrete (in time and space) random walk risk model is defined by [Ger88, Shi89]:

$$X_n = X_0 + cn - \sum_{i=1}^n C_i, \quad n \in \mathbb{N}_0,$$

where  $X_0$ , taking values in  $\mathbb{Z}$ , is the initial capital,  $c \in \mathbb{N}$  is the premium rate and the  $C_i$ ,  $i \in \mathbb{N}$ , take values in  $\mathbb{N}_0$  and are independent, identically distributed random variables with probability mass function  $p_k = P(C_1 = k)$  for  $k \in \mathbb{N}_0$ . One advantage of the discrete setup over the more popular continuous time models is the possibility to replace the Wiener-Hopf factorization by the conceptually simpler factorization of Laurent series (see for example [BF02] and [Xin04]); another advantage is that one has access to Panjer recursions for computing compound distributions.

The results simplify considerably for the upwards skip-free compound binomial model obtained when  $c = 1$  ([Qui04, BPR10, Mar01] [Spi13, *passim*] [CP16, Section 4.1] among others):

$$(1) \quad X_n = X_0 + n - \sum_{i=1}^n C_i, \quad n \in \mathbb{N}_0,$$

that we now consider as having been fixed and to which we specialize all discussion henceforth. We insist throughout that  $p_0 > 0$ .

Notation-wise, we let

$$\tilde{p}(z) := Ez^{C_1} = \sum_{k=0}^{\infty} p_k z^k, \quad z \in (0, 1],$$

denote the probability generating function of the claims. Then, for  $n \in \mathbb{N}$ , (in the obvious notation)  $Ez^{\sum_{i=1}^n C_i} = [\tilde{p}(z)]^n = (p_0 + (1-p_0)\tilde{p}_{C \geq 1}(z))^n$ , which makes it manifest that  $\sum_{i=1}^n C_i$ , the total claims arising from  $n$  time periods, has a compound binomial distribution, explaining the name compound binomial model: at each instant in discrete time, a positive claim either occurs or not, with probability  $1 - p_0$  and  $p_0$ , respectively, independently of the sizes of the positive claims.

**Remark 1.** *By the independence of the claims, we may also write, for  $n \in \mathbb{N}_0$ :*

$$E \left[ z^{\sum_{i=1}^n (C_i - 1)} \right] = \left( \frac{\tilde{p}(z)}{z} \right)^n \implies \sum_{m=0}^{\infty} v^m E \left[ z^{\sum_{i=1}^m (C_i - 1)} \right] = \frac{1}{1 - v\tilde{p}(z)/z}, \quad v \in \left( 0, \frac{z}{\tilde{p}(z)} \right).$$

The last expression, called the “unrestricted generating function” in [BF02, Eq. (8)], identifies already potential singularities as the roots of the Lundberg equation [Lun03]  $\frac{\tilde{p}(z)}{z} = v^{-1}$ . The smallest (positive) root of this equation plays a central role in our story — see next section.

Next, we will denote by

$$(2) \quad \tau_b^- = \inf\{t \geq 0 : X_t \leq b\} \text{ and } \tau_b^+ = \inf\{t \geq 0 : X_t \geq b\},$$

respectively, the first passage times below and above a level  $b$  (with  $\inf \emptyset = \infty$ ).

**Remark 2.** *Note this differs slightly from the usual definition of these quantities for a spectrally negative Lévy process, say  $U$ . There one replaces  $t \geq 0$  by  $t > 0$  and  $\leq b$  ( $\geq b$ ) by  $< b$  ( $> b$ ); and, of course,  $X$  by  $U$ . When considering  $\tau_b^\pm$  for a spectrally negative Lévy process  $U$ , we shall mean these quantities with the latter replacements having been effected.*

Lastly, for convenience, we assume given a family of measures  $(P_x)_{x \in \mathbb{Z}}$  with corresponding expectation operators  $(E_x)_{x \in \mathbb{Z}}$ , for which: (i)  $P_x(X_0 = x) = 1$  for all  $x \in \mathbb{Z}$ ; and (ii) the  $C_i$ ,  $i \in \mathbb{N}$ , have the same law under all the  $P_x$ ,  $x \in \mathbb{Z}$ , as they do under  $P = P_0$ .

**Remark 3.** *The discrete-time discrete-space compound binomial model is embedded into continuous time via subordination (time-change) by an independent homogeneous Poisson process  $N$ . In precise terms, allowing also a scaling of space, we have the following correspondence between the right-continuous random walk  $X$  of (1) and the upwards skip-free Lévy chain of [Vid13, Sec. 2] that we will here denote by  $Y$ :*

$$X \rightsquigarrow Y : Y_t := hX_{N_t}, \quad t \in [0, \infty),$$

where  $h \in (0, \infty)$  is space scaling. In particular, denoting the intensity of  $N$  by  $\gamma$ , the Lévy measure  $\lambda$  of  $Y$  is given by  $\lambda = \gamma \sum_{i \in \mathbb{Z} \setminus \{1\}} p_i \delta_{h(1-i)}$ ; and if we denote the Laplace exponent of  $Y$  by  $\psi$  (so  $\psi(\beta) = \frac{\log E[e^{\beta Y_t}]}{t}$  for  $\beta \in [0, \infty)$ ), then  $\psi(\beta) = \gamma[e^{\beta h} \tilde{p}(e^{-\beta h}) - 1]$ . Note that the mass of the Lévy measure  $\lambda$  is  $\gamma(1 - p_1)$ , which may be strictly less than  $\gamma$ .

**Remark 4.** *In the following, when the  $\tau_b^\pm$  appear in the context of the upwards skip-free Lévy chain  $Y$ , they are to be interpreted in the sense of (2) with  $Y$  replacing  $X$ .*

Here is now a brief guide to the contents. In Sections 2, 3 and 4, we review, respectively (with  $v$  indicating discounting):

- (1) the smooth one-sided first passage problem, which introduces the Lundberg root  $\varphi_v$  (analogue of  $\Phi(q)$  from the Lévy theory);
- (2) the non-smooth one-sided first passage problem, which involves the ruin and survival probabilities  $\Psi_v, \bar{\Psi}_v$ ;
- (3) the smooth two-sided first passage problem, where the fundamental scale function  $W_v$  first appears.

We turn then to new material in Section 5, computing the generating function ( $z$ -transform) of the second hero of first passage theory: the  $Z_v(\cdot, w)$  scale function. This is introduced via the problem of deficit at ruin: we provide the analogue (15) of the following two-sided exit identity for a spectrally negative Lévy processes (in standard notation):

$$E_x[e^{-q\tau_0^- + \theta X(\tau_0^-)}; \tau_0^- < \tau_b^+] = Z^{(q)}(x, \theta) - \frac{W^{(q)}(x)}{W^{(q)}(b)} Z^{(q)}(b, \theta),$$

with its beautiful probabilistic interpretation [IP12, Cor. 3]. We also determine the analogue (21) of the formula [Kyp14, Eq. (8.9)] (again for a spectrally negative Lévy process, in standard notation)

$$E_x[e^{-q\tau_0^-}; \tau_0^- < \infty] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x), \quad q > 0,$$

which is interesting, for example, since it reveals that the two protagonists of the “reflected” and “absorbed” smooth passage problems,  $Z^{(q)}$  and  $W^{(q)}$ , have the same asymptotics at  $\infty$ , up to a constant. A distinguishing element of the scale functions  $W_v$  and  $Z_v(\cdot, w)$ , in the present context, are explicit recursions available for their computation: see (12) and (16), respectively. Section 6 discusses some important applications, like the de Finetti dividends optimization problem, and the optimization of dividends for the doubly reflected process. These are complemented by illustrative numerical examples in Section 7. Finally, note that while our motivation for this investigation comes chiefly from risk models in the insurance context, the results presented are general and hence more widely applicable.

## 2. SMOOTH ONE-SIDED FIRST PASSAGE PROBLEM: THE LUNDBERG EQUATION

The first key observation is that for the first passage upwards, the stationary independent increments and skip-free properties imply a multiplicative structure; thus, for integer  $x \leq b$ , and for  $v \in (0, 1]$ , we have

$$(3) \quad E_x \left[ v^{\tau_b^+}; \tau_b^+ < \infty \right] = \varphi_v^{b-x},$$

where

$$\varphi_v := E \left[ v^{\tau_1^+}; \tau_1^+ < \infty \right] = \sum_{k=1}^{\infty} v^k P[\tau_1^+ = k] \in (0, v).$$

Conditioning at time 1, we obtain

$$\varphi_v = v E \left[ E_{1-C_1} [v^{\tau_1^+}; \tau_1^+ < \infty] \right] = v \sum_{k=0}^{\infty} p_k \varphi_v^k = v \tilde{p}(\varphi_v),$$

which reveals that  $\varphi_v$  appearing in (3) satisfies the Lundberg equation [CGS00, Eq. (3.3)], [GSY10, Eq. (6.8)]:

$$(4) \quad \frac{\varphi_v}{\tilde{p}(\varphi_v)} = v.$$

Alternatively, this relation may be derived by looking for exponential martingales of the form  $(v^t \xi^{-X_t})_{t \in \mathbb{N}_0}$ , for fixed  $v$ , and  $\xi$  from  $(0, 1]$ :  $(v^t \xi^{-X_t})_{t \in \mathbb{N}_0}$  is a martingale iff  $\frac{\xi}{\tilde{p}(\xi)} = v$ ; and then applying optional sampling.

**Remark 5.** *The function  $(0, 1] \ni \xi \mapsto \tilde{p}(\xi)/\xi = E\xi^{C_1-1}$  is strictly convex, equal to 1 at 1, and tending to  $\infty$  at 0. It follows that the equation (in  $\xi \in (0, 1]$ )  $\frac{\tilde{p}(\xi)}{\xi} = v^{-1}$  has as its unique solution  $\varphi_v \in (0, 1)$ , when  $v < 1$  (furthermore, in this case,  $\varphi_v < v$ ), whereas in the case  $v = 1$ , this equation has one or two solutions (one of which is always 1), according as to whether  $EC_1 \leq 1$  or  $EC_1 > 1$ . In the latter case  $X$  drifts to  $-\infty$ , and  $\varphi_1 \in (0, 1)$  is the smallest solution to  $\xi = \tilde{p}(\xi)$  (in  $\xi \in (0, 1]$ ). Altogether, this gives a continuous strictly increasing bijection  $\varphi : (0, 1] \rightarrow (0, \varphi_1]$ .*

**Remark 6.** *If for  $q \in [0, \infty)$ , we let  $\Phi(q)$  be the largest zero of  $\psi - q$ , then we see from Remark 3 that  $\varphi_v = e^{-h\Phi(\gamma(v^{-1}-1))}$  for all  $v \in (0, 1]$ .*

**Remark 7.** *Note that (4) identifies  $\tau_1^+$  as a Lagrangian type distribution [CKF06]. Indeed the distribution of  $\tau_1^+$  may be obtained using the Lagrange inversion formula*

$$\varphi_v = \sum_{n=1}^{\infty} \frac{v^n}{n!} \left[ \left( \frac{d}{dw} \right)^{n-1} \tilde{p}(w)^n \right]_{w=0} = \sum_{n=1}^{\infty} \frac{v^n}{n} p^{n*}(n-1),$$

where for  $n \in \mathbb{N}$ ,  $p^{n*}$  is the  $n$ -fold convolution of the distribution  $p$  with itself. More generally, for  $b \in \mathbb{N}$ ,

$$\varphi_v^b = b \sum_{n=b}^{\infty} \frac{v^n}{n} p^{n*}(n-b),$$

yielding Kemperman's formula [Kem61] for the distribution of  $\tau_b^+$ :

$$P[\tau_b^+ = n] = \frac{b}{n} p^{n*}(n-b) = \frac{b}{n} P[X_n = b], \quad n \in \mathbb{N}_{\geq b}.$$

### 3. NON-SMOOTH ONE-SIDED FIRST PASSAGE PROBLEM: RUIN AND SURVIVAL PROBABILITIES; THE LUNDBERG RECURRENCE

For initial capital  $x \in \mathbb{Z}$ , the finite time and eventual ruin probabilities are defined by:

$$\Psi(n; x) := P_x[\tau_{-1}^- \leq n] \text{ for } n \in \mathbb{N}_0, \quad \Psi(x) := \lim_{n \rightarrow \infty} \Psi(n; x) = P_x[\tau_{-1}^- < \infty];$$

similarly we introduce the finite time and perpetual survival probabilities:

$$\bar{\Psi}(n; x) := P_x[\tau_{-1}^- > n] \text{ for } n \in \mathbb{N}_0, \quad \bar{\Psi}(x) := \lim_{n \rightarrow \infty} \bar{\Psi}(n; x) = P_x[\tau_{-1}^- = \infty].$$

Of course  $\Psi(n; x) + \bar{\Psi}(n; x) = 1$ ,  $\Psi(x) + \bar{\Psi}(x) = 1$ , and one has the recursions, valid for all integer  $x \geq 0$ ,  $n \geq 1$ :

$$(5) \quad \bar{\Psi}(n; x) = \sum_{i=0}^{x+1} p_i \bar{\Psi}(n-1; x+1-i), \quad \bar{\Psi}(0; x) = 1,$$

$$(6) \quad \Psi(n; x) = \sum_{i=0}^{x+1} p_i \Psi(n-1; x+1-i) + \sum_{k=x+2}^{\infty} p_k, \quad \Psi(0; x) = 0.$$

These two recurrences may, for a sequence of functions  $f_n : \mathbb{Z} \rightarrow [0, 1]$ , standing in lieu of  $\Psi(n; \cdot)$ ,  $\bar{\Psi}(n; \cdot)$ , be written symbolically as

$$f_n = K \tilde{p}(K^{-1}) f_{n-1} \text{ on } \mathbb{N}_0,$$

which passes to the limit (as  $n \rightarrow \infty$ )

$$(7) \quad f = K \tilde{p}(K^{-1}) f \text{ on } \mathbb{N}_0,$$

where  $K$  is the translation operator,  $Kg(x) := g(x+1)$ , and  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ . This limiting recurrence (satisfied by the eventual ruin and perpetual survival probabilities  $\Psi$  and  $\bar{\Psi}$ ) may be called the ‘‘Lundberg recurrence’’. It constitutes a linear difference equation for  $f$ , whose characteristic equation is (in  $x \neq 0$ )  $1 = x\tilde{p}(1/x)$ . The latter is (formally) just the Lundberg equation (4) with  $v = 1$  upon substituting  $x^{-1}$  for  $\varphi_1$ . When the distribution  $p$  has a finite support, then from the theory of finite order linear difference equations with constant coefficients, this implies that  $f$ , in particular the ultimate ruin and perpetual survival probabilities, may be expressed as combinations of powers of the roots of the characteristic equation (in  $x \neq 0$ )

$$(8) \quad 1 = x\tilde{p}(1/x).$$

Classical ruin theory proceeds by computing double (generating function) transforms, briefly reviewed in Appendix A. For example, one useful result, similar to the Pollaczek-Khinchine formula for the Cramér-Lundberg model, is [Wil93, Eq. (3.5)]

$$(9) \quad \tilde{\Psi}(z) := \sum_{x=0}^{\infty} z^x \bar{\Psi}(x) = \frac{(1 - E[C_1]) \vee 0}{\tilde{p}(z) - z}, \quad z \in (0, 1).$$

Another is

$$(10) \quad \tilde{\Psi}_v(z) := \sum_{x=0}^{\infty} \sum_{n=0}^{\infty} z^x v^n \Psi(n; x) = \frac{1}{z - v\tilde{p}(z)} \left( \frac{v(z - \tilde{p}(z))}{(1-v)(1-z)} + \frac{\varphi_v}{1 - \varphi_v} \right), \quad v, z \in (0, 1), z \neq \varphi_v.$$

We will follow next an alternate approach, which focuses on the two-sided exit problem from an interval.

#### 4. SMOOTH TWO-SIDED FIRST PASSAGE PROBLEM: THE $W$ SCALE FUNCTIONS

In the context of Lévy processes, the  $W^{(q)}$  scale function is often defined first for  $q = 0$ , in the case when the underlying process drifts to  $\infty$ , by proportionality to the survival probability, and then in the remainder of the cases by an Esscher transform/approximation [Ber97, Sec. VII.2] [Kyp14, Sec. 8.2] [Vid13, Sec. 4.2].

In our setting of the right-continuous random walk  $X$ , we introduce, for  $v \in (0, 1]$ , the discrete-time analogue  $W_v$  of  $W^{(q)}$ , by setting  $W_v(y) := (p_0 E[v^{\tau_y^+}; \tau_y^+ < \tau_{-1}^-])^{-1}$  for  $y \in \mathbb{N}_0$  and  $W_v(y) = 0$  for  $y \in -\mathbb{N}$ . The Markov property at the time  $\tau_x^+$  and the skip-free property (yielding  $X_{\tau_x^+} = x$  on  $\{\tau_x^+ < \infty\}$ ) then imply the ‘‘gambler’s winning’’ relation [Mar01, GLY06], for integer  $x \leq N$ ,  $0 \leq N$ :

$$(11) \quad E_x[v^{\tau_N^+}; \tau_N^+ < \tau_{-1}^-] = \frac{W_v(x)}{W_v(N)}.$$

We call  $W_v$  the  $v$ -scale function and we write simply  $W$  for the 1-scale function  $W_1$ . (The choice of the normalization  $W_v(0) = 1/p_0$  is somewhat arbitrary, though it is guided by obtaining the simplest possible form for the  $z$ -transform of  $W_v$  ((13) below); by comparison to the  $W$  scale function of [Vid13] (see Remark 10 below); and the simplicity of subsequent formulae in which  $W_v$  features.)

**Remark 8.** We use the subscript notation  $W_v$  for the scale functions of  $X$ , reserving the superscript version  $W^{(q)}$  for the corresponding quantities from the Lévy setting. When only  $W$  appears, it will be clear from context which of the two is meant. We will adhere to a similar convention with respect to the scale functions  $Z^{(q)}(\cdot, \theta)$ ,  $Z^{(q)} := Z^{(q)}(\cdot, 0)$  and (hence the notation) their discrete-time analogues  $Z_v(\cdot, w)$ ,  $Z_v$ .

Conditioning on the first jump, (11) implies the harmonic recursion [Mar01, Eq. (3.1)]

$$(12) \quad W_v(x) = v \sum_{y=-1}^x W_v(x-y)p_{y+1}, \quad x \in \mathbb{N}_0.$$

Taking  $z$ -transform yields [Mar01, Eq. (3.2)]

$$(13) \quad \widetilde{W}_v(z) := \sum_{x=0}^{\infty} z^x W_v(x) = \frac{1}{\widetilde{p}(z) - \frac{z}{v}}, \quad z \in (0, \varphi_v).$$

Since the  $z$ -transform (13) of  $W_v$  is known, the computation of the scale function  $W_v$  reduces finally to Taylor coefficient extraction of (13) expanded in a power series.

**Remark 9.** *It is seen from (12), or directly from (11), that one has  $W_v/v = {}^vW$ , where  ${}^vW$  is the 1-scale function of the process  $X$  geometrically killed with probability  $1 - v$ , i.e. of the process which has, ceteris paribus, the sub-probability pmf  $(vp_k)_{k \in \mathbb{N}_0}$  governing the sizes of the  $C_n$ ,  $n \in \mathbb{N}$ .*

**Remark 10.** *For  $X$  embedded into continuous time as an upwards skip-free Lévy chain, i.e. for the process  $Y$  of Remark 3, (12) and (13) become, respectively, [Vid13, Eqs. (4.10) & (4.6)]. This is seen through the identification  $W^{(q)}(mh) = \frac{1}{\gamma h} W_{\frac{\gamma}{\gamma+q}}(m)$  for  $m \in \mathbb{N}_0$ ,  $q \in [0, \infty)$ , where  $W^{(q)}$  is the  $q$ -scale function of [Vid13]. Note also that the normalization  $W_v(0) = p_0^{-1}$  is consistent with  $W^{(q)}(0) = 1/(h\lambda(\{h\})) = 1/(\gamma h p_0)$  of [Vid13, Prop. (4.7)]. On the other hand, in the spectrally negative case, there is no direct analogue of recursion (12), though one can consider the heuristic relation (it is rigorous in the upwards skip-free case [Vid13, Rem. 4.16])  $(L - q)W^{(q)} = 0$  on  $(0, \infty)$  [KKR13, p. 136],  $L$  being the infinitesimal generator of the underlying Lévy process, to be a close relative. (13) has the Laplace transform equivalent [KKR13, Eq. (8.8)] that formally differs from [Vid13, Eq. (4.6)] only by the factor  $(e^{\beta h} - 1)/(\beta h) \rightarrow 1$  as  $h \downarrow 0$  (with  $\beta$  the argument of the Laplace transform).*

**Remark 11.** *An alternative form of recursion (12) is [Vid13, Eq. (4.13)]*

$$W_v(n+1) = W_v(0) + \sum_{k=1}^{n+1} \frac{\frac{1}{v} - \sum_{l=0}^k p_l}{p_0} W_v(n+1-k), \quad n \in \mathbb{N}_0.$$

*In particular we see via induction that for each fixed integer  $x$ , the map  $[1, \infty) \ni \xi \mapsto W_{1/\xi}(x)$  extends to a polynomial function defined on the whole of the complex plane.*

**Remark 12.** *When  $X$  drifts to  $\infty$ , i.e. when  $EC_1 < 1$ , then with  $v = 1$ , (13) coincides up to a multiplicative constant with the perpetual survival transform (9). We conclude that*

$$\overline{\Psi}(x) = (1 - E(C_1))W(x).$$

**Remark 13.** *It follows from (13) that  ${}^vW(x) = \frac{\varphi_v}{v} W_v(x) \varphi_v^x$ , where  ${}^vW$  is the 1-scale function of the Esscher transformed process in which  $C_1$  has the geometrically tilted probability mass function  $\mathbb{N}_0 \ni k \mapsto \frac{v}{\varphi_v} p_k \varphi_v^k$ . Hence by monotone convergence,  $\lim_{x \rightarrow \infty} W_v(x) \varphi_v^{x+1} = v \lim_{x \rightarrow \infty} {}^vW(x) = v \lim_{z \uparrow 1} \sum_{x=0}^{\infty} (1-z) z^x {}^vW(x) = \frac{v}{1-v\overline{p}'(\varphi_v)}$ , where we understand  $1/0 = \infty$ . This confirms [Vid13, Prop. 4.8(i)]. For a more detailed study of the behaviour of  $W_1$  in the case when  $\overline{p}'(1-) = 1$  and  $\varphi_1 = 1$ , i.e. when  $X$  oscillates, see [Vid13, Prop. 4.8(ii)].*

**Remark 14.** *We note the following interesting observation of [Mar01] that the scale function is essentially a determinant. For an arbitrary homogeneous Markov chain  $(V_n)_{n \in \mathbb{N}_0}$  on a countable state space, let  $(V'_n)_{n \in \mathbb{N}_0}$  denote the chain killed outside a finite non-empty set  $M$ , and let  $Q$  denote the corresponding restriction of the transition matrix to  $M$ . For  $v \in (0, 1)$ , denote by  $D_v$  the determinant of the matrix  $I - vQ$ . Then the killed resolvent expresses as*

$$\sum_{n=0}^{\infty} P_i[V'_n = j] v^n = ((I - vQ)^{-1})_{ij} = \frac{N_{ij}(v)}{D_v}, \quad \{i, j\} \subset M,$$

*where  $N_{ij}(v)$  are the entries of the adjoint matrix  $\text{adj}(I - vQ)$  (see for example [Mar01, Cor. 2.2]). Restricting now to the upwards skip-free case (while [Mar01] considers the downwards skip-free case),*

let, for  $v \in (0, 1]$ ,  $D_v(N)$ ,  $N \in \mathbb{N}$ , denote the determinant corresponding (in the above sense) to the restriction of  $X$  to  $\{0, 1, 2, \dots, N-1\}$ , and set  $D_v(0) := 1$ . From [Mar01, Prop. 3.3],

$$E_i[v^{\tau_N^+}, \tau_N^+ < \tau_{-1}^-] = (p_0 v)^{N-i} \frac{D_v(i)}{D_v(N)}, \quad \{i, N\} \subset \mathbb{N}_0, \quad v \in (0, 1).$$

It follows that  $W_v(i) = p_0^{-1} (p_0 v)^{-i} D_v(i)$  for all  $i \in \mathbb{N}_0$ ,  $v \in (0, 1]$ .

**Remark 15.** For  $N \in \mathbb{N}$ , the resolvent of the process  $X$  killed on exiting  $I_N := \{0, \dots, N-1\}$ , denoted  $X'$ , is given by [Mar01, Prop. 3.2]

$$\sum_{n=0}^{\infty} P_i[X'_n = j] v^n = v^{-1} \left( \frac{W_v(N-1-j)W_v(i)}{W_v(N)} - W_v(i-j-1) \right), \quad \{i, j\} \subset I_N, \quad v \in (0, 1).$$

For the analogue of the latter in the spectrally negative case see e.g. [Kyp14, Thm. 8.7].

We conclude this section with the important observation that

**Proposition 16.** For each  $v \in (0, 1]$ ,  $(v^{n \wedge \tau_{-1}^-} W_v(X_{n \wedge \tau_{-1}^-}))_{n \in \mathbb{N}_0}$  is a martingale under each  $P_x$ ,  $x \in \mathbb{Z}$ .

*Proof.* This follows from the harmonic recurrence (12).  $\square$

**Remark 17.** The analogue of Proposition 16 in the setting of upwards skip-free Lévy chains are the martingales, for  $q \in [0, \infty)$ ,  $(e^{-q(t \wedge \tau_{-h}^-)} W^{(q)}(Y_{t \wedge \tau_{-h}^-}))_{t \in [0, \infty)}$  [Vid13, Cor. 4.17]. In the case of a spectrally negative Lévy process  $U$ ,  $(e^{-q(t \wedge \tau_0^-)} W^{(q)}(U_{t \wedge \tau_0^-}))_{t \in [0, \infty)}$  is a local martingale with localizing sequence  $(\tau_n^+)_{n \in \mathbb{N}}$  [Kyp14, Ex. 8.12]. There are no issues with integrability in the discrete space case, because thanks to the skip-free property,  $P_x$ -a.s. for any  $x \in \mathbb{Z}$ , by any deterministic time, the stopped process  $X^{\tau_{-1}^-}$  is automatically bounded /and, for the upwards skip-free Lévy chain  $Y$ , the further subordination by the independent homogeneous Poisson process  $N$  does not ruin this/.

**Corollary 18.** For each  $v \in (0, 1]$  and integer  $x \leq N$ ,  $b \leq N$ ,

$$E_x(W_v(X_{\tau_{b-1}^-}) v^{\tau_{b-1}^-}; \tau_{b-1}^- < \tau_N^+) = W_v(x) - \frac{W_v(x-b)}{W_v(N-b)} W_v(N).$$

In particular,  $E_x(W_v(X_{\tau_{b-1}^-}) v^{\tau_{b-1}^-}; \tau_{b-1}^- < \infty) = W_v(x) - W_v(x-b) \varphi_v^b$ .

*Proof.* For any integer  $x$ , by optional sampling, the skip-free property and spatial homogeneity,  $W_v(x) = E_x[W_v(X(\tau_{b-1}^-)) v^{\tau_{b-1}^-}; \tau_{b-1}^- < \tau_N^+] + E_x[W_v(X(\tau_N^+)) v^{\tau_N^+}; \tau_N^+ < \tau_{b-1}^-] = E_x[W_v(X(\tau_{b-1}^-)) v^{\tau_{b-1}^-}; \tau_{b-1}^- < \tau_N^+] + W_v(N) E_x[v^{\tau_N^+}; \tau_N^+ < \tau_{b-1}^-] = E_x[W_v(X(\tau_{b-1}^-)) v^{\tau_{b-1}^-}; \tau_{b-1}^- < \tau_N^+] + W_v(N) E_{x-b}[v^{\tau_{N-b}^+}; \tau_{N-b}^+ < \tau_{-1}^-]$ . The first identity then follows from (11). In particular, letting  $N \uparrow \infty$  and using Remark 13, we obtain the second identity (for instance first for  $v < 1$  and then taking the limit  $v \uparrow 1$ ).  $\square$

## 5. PROBLEM OF DEFICIT AT RUIN WITH KILLING AT AN UPPER BOUNDARY: THE $Z$ SCALE FUNCTIONS

Let  $v \in (0, 1]$ ,  $w \in (0, 1]$ . For integer  $x \leq b$ ,  $b \geq 0$ , by the Markov property at time  $\tau_b^+$  and the skip-free property (yielding  $X_{\tau_b^+} = b$  on  $\{\tau_b^+ < \infty\}$ ),

$$\begin{aligned} E_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \tau_b^+] &= E_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \infty] - E_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_b^+ < \tau_{-1}^- < \infty] \\ &= E_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \infty] - E_x[v^{\tau_b^+}; \tau_b^+ < \tau_{-1}^-] E_b[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \infty]. \end{aligned}$$

Putting  $\Psi_v(x, w) := E_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \infty]$ , we have then from the preceding and using (11), the neat identity  $E_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \tau_b^+] = \Psi_v(x, w) - \frac{W_v(x)}{W_v(b)} \Psi_v(b, w)$ . We introduce now, for some  $\alpha_v(w) \in [0, \infty)$  that we shall specify in the sequel,

$$(14) \quad Z_v(x, w) := \Psi_v(x, w) + \alpha_v(w) W_v(x),$$

a slightly modified  $\Psi_v(\cdot, w)$ , which also satisfies the identity

$$(15) \quad E_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \tau_b^+] = Z_v(x, w) - \frac{W_v(x)}{W_v(b)} Z_v(b, w)$$

(easy to check). The first motivation for preferring to use  $Z_v(\cdot, w)$  with a suitable choice of  $\alpha_v(w)$  instead of  $\Psi_v(\cdot, w)$  appears below in (17), and then in Section 6; many other formulas where the analogue of  $Z_v(\cdot, w)$  is preferable are known in the literature on spectrally negative Lévy processes – see for example [IP12, AGVA17].

**Remark 19.** Note that  $Z_v(x, w) = \Psi_v(x, w) = w^{-x}$  for all integer  $x \leq -1$ .

We compute now the  $z$ -transform of  $Z$ . Conditioning on the first jump, we obtain from (14) and the definition of  $\Psi_v(\cdot, w)$ , via (12), the recurrence relation

$$(16) \quad Z_v(x, w)/v = \sum_{k=-1}^x p_{k+1} Z_v(x-k, w) + \sum_{k=x+1}^{\infty} w^{k-x} p_{k+1}, \quad x \in \mathbb{N}_0.$$

Hence the generating function  $\tilde{Z}_v(z, w) := \sum_{x=0}^{\infty} z^x Z_v(x, w)$  satisfies, for  $z \in (0, \varphi_v) \setminus \{w\}$ ,

$$\begin{aligned} \tilde{Z}_v(z, w)/v &= p_0 \frac{\tilde{Z}_v(z, w) - Z_v(0, w)}{z} + \sum_{x=0}^{\infty} z^x \sum_{k=0}^x Z_v(x-k, w) p_{k+1} + \sum_{x=0}^{\infty} z^x \sum_{k=x+1}^{\infty} w^{k-x} p_{k+1} \\ &= p_0 \frac{\tilde{Z}_v(z, w) - Z_v(0, w)}{z} + \sum_{k=0}^{\infty} p_{k+1} z^k \sum_{x=k}^{\infty} z^{x-k} Z_v(x-k, w) + \sum_{k=1}^{\infty} p_{k+1} w^k \sum_{x=0}^{k-1} \left(\frac{z}{w}\right)^x \\ &= p_0 \frac{\tilde{Z}_v(z, w) - Z_v(0, w)}{z} + \tilde{Z}_v(z, w) \sum_{k=0}^{\infty} p_{k+1} z^k + \sum_{k=1}^{\infty} p_{k+1} w^k \frac{1 - \left(\frac{z}{w}\right)^k}{1 - \frac{z}{w}} \\ &= p_0 \frac{\tilde{Z}_v(z, w) - Z_v(0, w)}{z} + \tilde{Z}_v(z, w) \frac{\tilde{p}(z) - p_0}{z} + \frac{\frac{\tilde{p}(w) - p_0}{w} - \frac{\tilde{p}(z) - p_0}{z}}{1 - \frac{z}{w}}, \end{aligned}$$

i.e., in view of (13),

$$\tilde{Z}_v(z, w) = -p_0(1 - Z_v(0, w)) \tilde{W}_v(z) + \frac{z\tilde{p}(w) - w\tilde{p}(z)}{(z-w)(\tilde{p}(z) - \frac{z}{v})}.$$

Recall now that in the Lévy case,  $Z^{(q)}(0, \theta)$  is chosen so as to ensure a “smooth fit” [APP15, Def. 5.8] to the boundary condition  $e^{x\theta}$  for  $x \in (-\infty, 0)$ . The analog in the discrete case is to insist on  $Z_v(0, w) = 1$ , which we may do by an appropriate choice of  $\alpha_v(w)$ . Furthermore, this choice (that we assume henceforth) leads to the simple expression

$$(17) \quad \tilde{Z}_v(z, w) = \frac{1}{\tilde{p}(z) - \frac{z}{v}} \frac{z\tilde{p}(w) - w\tilde{p}(z)}{z-w}, \quad z \in (0, \varphi_v), \quad v \in (0, 1], \quad w \in (0, 1]$$

(where the quotient must be understood in the limiting sense when  $z = w$ ).

Extracting the coefficients of the  $z$ -power series yields finally an expression similar to that of the Dickson-Hipp type representation in the Lévy case (see [IP12])

$$(18) \quad Z_v(x, w) = \left(\tilde{p}(w) - \frac{w}{v}\right) \sum_{k=0}^{\infty} w^k W_v(x+k), \quad w \in (0, \varphi_v), \quad v \in (0, 1], \quad x \in \mathbb{N}_0$$

(it is easy to check that this expression has  $z$ -transform (17)).

In the special case  $w = 1$  we set  $Z_v(x) := Z_v(x, 1)$ , (17) simplifies to

$$(19) \quad \tilde{Z}_v(z) := \sum_{x=0}^{\infty} z^x Z_v(x) = \frac{\tilde{p}(z) - z}{(\tilde{p}(z) - \frac{z}{v})(1-z)}, \quad z \in (0, \varphi_v), \quad v \in (0, 1],$$

and we have the representation

$$(20) \quad Z_v(x) = 1 + \left(\frac{1}{v} - 1\right) \sum_{y=0}^{x-1} W_v(y), \quad v \in (0, 1], \quad x \in \mathbb{N}_0.$$

**Remark 20.** Using (10) in the form

$$\tilde{\Psi}_v(z) = \frac{1}{\frac{z}{v} - \tilde{p}(z)} \left( \frac{z - \tilde{p}(z)}{(1-v)(1-z)} + \frac{\varphi_v}{v(1-\varphi_v)} \right), \quad v, z \in (0, 1), \quad z \neq \varphi_v,$$

it follows from (19) and (13) that

$$\Psi_v(x) := \sum_{n=0}^{\infty} v^n \Psi(n; x) = \frac{1}{1-v} Z_v(x) - \frac{\varphi_v}{v(1-\varphi_v)} W_v(x),$$

i.e.

$$(21) \quad E_x[v^{\tau_-^{-1}}; \tau_-^{-1} < \infty] = Z_v(x) - \frac{\varphi_v(1-v)}{v(1-\varphi_v)} W_v(x) = Z_v(x) - \alpha_v W_v(x), \quad x \in \mathbb{N}_0, \quad v \in (0, 1),$$

where we have set  $\alpha_v := \alpha_v(1)$  (recall that we have chosen  $\alpha_v(1)$  so that  $Z_v(0) = 1 = E[v^{\tau_-^{-1}}; \tau_-^{-1} < \infty] + \alpha_v(1)W_v(0)$ ). Passing to the limit  $v \uparrow 1$ , we find that  $P_x(\tau_-^{-1} < \infty) = 1 - W(x)(1 - \tilde{p}'(1-) \wedge 1)$ .

**Remark 21.** It is seen from (20), Remark 10 and [Vid13, Def. 4.9] that one has the identification  $Z^{(q)}(mh) = Z_{\frac{\gamma}{\gamma+q}}(m)$  for  $q \in [0, \infty)$ ,  $m \in \mathbb{Z}$ , where  $Z^{(q)}$  is the  $Z$   $q$ -scale function of [Vid13]. Then (17), (15) and (16), with  $w = 1$ , become [Vid13, Eq. (4.9), Prop. 4.13 and Eq. (4.11)], respectively; (21) becomes [Vid13, Eq. (4.8)]. For an alternative form of (16) (when  $w = 1$ ) see [Vid13, Eq. (4.14)].

**Proposition 22.** For each  $v \in (0, 1]$ ,  $w \in (0, 1]$ , the process  $(v^{n \wedge \tau_-^{-1}} Z_v(X_{n \wedge \tau_-^{-1}}, w))_{n \in \mathbb{N}_0}$  is a martingale.

*Proof.* This follows for instance by linearity, from Proposition 16, and from the definition of  $Z_v(\cdot, w)$  via the Markov property and the terminal time property of  $\tau_-^{-1}$ .  $\square$

**Remark 23.** For the case  $w = 1$ , the analogue of Proposition 22 in the setting of upwards skip-free Lévy chains are the martingales, for  $q \in [0, \infty)$ ,  $(e^{-q(t \wedge \tau_-^{-h})} Z^{(q)}(Y_{t \wedge \tau_-^{-h}}))_{t \in [0, \infty)}$  [Vid13, Cor. 4.17]. In the case of a spectrally negative Lévy processes  $U$ ,  $(e^{-q(t \wedge \tau_0^-)} Z^{(q)}(U_{t \wedge \tau_0^-}))_{t \in [0, \infty)}$  is a local martingale with localizing sequence  $(\tau_n^+)_{n \in \mathbb{N}}$  [Kyp14, Ex. 8.12]. See also [APP15]: There, Gerber-Shiu functions are defined as solutions to martingale problems [APP15, Def. 5.1], and the  $Z^{(q)}(\cdot, \theta)$  function is the Gerber-Shiu function with boundary condition  $e^{x\theta}$  for  $x \in (-\infty, 0)$  [APP15, Def. 5.8].

**Remark 24.** Assume  $EC_1 < \infty$ ; let  $v \in (0, 1]$ ,  $x \in \mathbb{Z}$ . We can obtain the expected undershoot at ruin by differentiating (15) with respect to  $w$  from the left at 1. Putting  $Z_{1,v}(x) := -\frac{\partial Z_v(x,w)}{\partial w} \Big|_{w=1-}$ , we find that for  $b \in \mathbb{N}_0$ ,

$$(22) \quad E_x[X(\tau_-^{-1}) v^{\tau_-^{-1}}; \tau_-^{-1} < \tau_b^+] = Z_{1,v}(x) - \frac{W_v(x)}{W_v(b)} Z_{1,v}(b), \quad x \leq b.$$

The generating function transform of  $Z_{1,v}$  is given by

$$(23) \quad \tilde{Z}_{1,v}(z) := \sum_{k=0}^{\infty} z^k Z_{1,v}(k) = \frac{z}{1-z} \frac{1}{\tilde{p}(z) - z/v} \left( \frac{\tilde{p}(z) - z}{1-z} - (1 - \tilde{p}'(1-)) \right), \quad z \in (0, \varphi_v).$$

Setting for  $f : \mathbb{N}_0 \rightarrow \mathbb{R}$  and  $y \in \mathbb{N}_0$ ,  $\bar{f}(y) := \sum_{z=0}^{y-1} f(z)$  (in particular,  $\bar{f}(0) = 0$ ), and using  $\sum_{k=0}^{\infty} z^k \bar{f}(k) = \frac{z}{1-z} \sum_{k=0}^{\infty} z^k f(k)$  for  $z \in (0, 1]$ , we find that for  $x \in \mathbb{N}_0$ , this coincides with the generating function of  $\mathbb{N}_0 \ni x \mapsto \bar{Z}_v(x) - (1 - \tilde{p}'(1-))\bar{W}_v(x)$ , i.e.

$$(24) \quad Z_{1,v}(x) = \bar{Z}_v(x) - (1 - \tilde{p}'(1-))\bar{W}_v(x), \quad x \in \mathbb{N}_0,$$

Note also that when  $x < 0$ ,  $Z_{1,v}(x) = x$ .  $Z_{1,v}$  will play a central role in the modified de Finetti problem – see Subection 6.3, and in its doubly reflected variant presented in Subection 6.4.

## 6. APPLICATIONS TO THE STUDY OF A COMPANY'S CAPITAL SURPLUS PROCESS

In this section we investigate various forms of the (combined) capital injections-dividend payouts-penalty at ruin problem. One typically has in mind an insurance company, but this need not be the case.

**6.1. The moment generating function of cumulative capital injections.** For the simplest case, we begin by considering a company, whose surplus capital process  $\tilde{X} = (\tilde{X}_k)_{k \in \mathbb{N}_0}$  obeys the following dynamics: for  $k \in \mathbb{N}_0$ , given that at the end of period  $k$ , its capital is  $\tilde{X}_k$ , then in period  $k + 1$  the company receives (the premium) 1, pays out the (claim) amount  $C_{k+1}$ , and, should its net capital at this point be strictly negative, receives a capital injection that just brings its capital back to zero at the end of the  $(k + 1)$ -th period, i.e.  $\tilde{X}_{k+1} = (\tilde{X}_k + 1 - L_{k+1}) \vee 0$ . If the initial capital  $x \in \mathbb{Z}$  of the company is strictly negative, the company receives immediately the capital injection  $-x$ , so that its capital at the end of the zeroth period is nonnegative, i.e.  $\tilde{X}_0 = (-x) \vee 0$ . One says that the surplus process has the dynamics of  $X$  reflected at 0.

Let then  $R_*(n) := (-\inf_{m \leq n} X_m) \vee 0$ ,  $n \in \mathbb{N}_0$ , denote the cumulative capital injections for the process  $X$  reflected at 0, and let, for  $b \in \mathbb{N}_0$ ,  $\tilde{\tau}_b^+$  denote the first entrance time into  $[b, \infty)$  by the reflected process. It was discovered by [IP12] that their joint moment generating function is very simply expressible in terms of the second scale function of two parameters. In our context, their formula becomes

**Proposition 25.** For  $b \in \mathbb{N}_0$ ,

$$(25) \quad B_v^b(x, w) := E_x[v^{\tilde{\tau}_b^+} w^{R_*(\tilde{\tau}_b^+)}; \tilde{\tau}_b^+ < \infty] = \begin{cases} \frac{Z_v(x, w)}{Z_v(b, w)} & x \leq b \\ 1 & x > b \end{cases}, \quad \{v, w\} \subset (0, 1].$$

*Proof.* The case  $x > b$  is trivial; assume  $x \leq b$ . Then this formula is “equivalent” to (15), since by the strong Markov property of  $X$ ,

$$E_x[v^{\tilde{\tau}_b^+} w^{R_*(\tilde{\tau}_b^+)}; \tilde{\tau}_b^+ < \infty] = E_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \tau_b^+] E_0[v^{\tilde{\tau}_b^+} w^{R_*(\tilde{\tau}_b^+)}; \tilde{\tau}_b^+ < \infty] + E_x[v^{\tau_b^+}; \tau_b^+ < \tau_{-1}^-],$$

i.e.

$$(26) \quad B_v^b(x, w) = E_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \tau_b^+] B_v^b(0, w) + W_v(x)W_v(b)^{-1}.$$

Thus, if  $B_v^b(x, w)$  is known from (25), one gets an equation for the deficit at ruin quantities

$$Z_v(x, w)Z_v(b, w)^{-1} = W_v(x)W_v(b)^{-1} + E_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \tau_b^+] Z_v(b, w)^{-1},$$

with solution (15). And if the solution to the deficit at ruin problem is known as (15), one may use (26) to obtain, first with  $x = 0$ ,  $B_v^b(0, w) = Z_v(b, w)^{-1}$ , and then (25).  $\square$

**6.2. The de Finetti dividends optimization problem.** Now the company pays dividends, but does not receive capital injections. Letting for  $k \in \mathbb{N}_0$ ,  $r(k)$  denote the dividend amount (necessarily  $\mathbb{N}_0$ -valued) paid out at the end of period  $k$ , we have the following dynamics for the end-of-period surplus process  $\tilde{X}$ : for  $k \in \mathbb{N}$ , in period  $k$ , the company receives 1, pays out  $C_k$  and then, assuming ruin has not yet occurred, the amount  $r(k)$ , yielding  $\tilde{X}_k = \tilde{X}_{k-1} + 1 - C_k - r(k)$ . Once ruin has occurred, the process is stopped, and no dividends are paid out thereafter. At end of period zero, if the initial capital  $x \in \mathbb{Z}$  is strictly positive, the dividend amount  $r(0)$  is paid out, so that  $\tilde{X}_0 = x - r(0)$ . We insist  $r(k) \leq \tilde{X}_{k-1} + 1 - C_k$  for  $k \in \mathbb{N}$  and  $r(0) \leq x$  (i.e. dividend payouts cannot lead to ruin). The dividend policy process  $(r(k))_{k \in \mathbb{N}_0}$  must be adapted to the natural filtration of  $(C_k)_{k \in \mathbb{N}}$ .

The classic de Finetti problem then consists in computing the optimal discounted dividends until ruin under all dividend policies satisfying the above constraints – see de Finetti [dF57], Miller and Modigliani [MM61] (in a deterministic setup), Miyasawa [Miy61] and Gerber [Ger72]. Here we agree that in the optimization objective,  $r(k)$  is discounted (multiplied) by  $v^k$ , where  $v \in (0, 1]$  is the discount factor. To exclude some degeneracy, we assume throughout this subsection that  $(p_0 + p_1) \wedge v < 1$ .

**Definition 26.** For  $b \in \mathbb{N}_0$ , a dividend policy  $\pi_b$  with barrier  $b$  consists in taking  $r(0) = (x - b)^+$  and  $r(k) = (\tilde{X}_{k-1} + 1 - C_k - b)^+$  for  $k \in \mathbb{N}$ , up to ruin, i.e. (since we are in the upwards skip-free case) in reducing the reserves each time they reach  $b + 1$  (except possibly at time zero, when  $x - b$  may be strictly larger than 1). We will write the expectation operator  $E_x^{[b]}$  and the probability  $P_x^{[b]}$  to indicate this policy and the initial capital  $x$ . One says that under  $E_x^{[b]}$ ,  $\tilde{X}$  follows the dynamics of the process  $X$  reflected at  $b$ . The sets

$$\mathcal{C}^b := [0, b] \text{ and } \mathcal{D}^b := (b, \infty)$$

are called the continuation and dividend taking set, respectively;  $\tilde{\tau}_b^+ := \inf\{k \in \mathbb{N}_0 : \tilde{X}_k \geq b\}$ .

The ruin time, i.e. the first time the surplus process becomes strictly negative, will be denoted by  $\tilde{\tau}_{-1}^-$ . Note that  $r(k) = 0$  for  $k \geq \tilde{\tau}_{-1}^-$ . We also set, for  $k \in \mathbb{N}_0$ ,  $R(k) := \sum_{i=0}^k r(i)$ , the cumulative dividends paid out up to (including) period  $k$ , and interpret  $R(k) = 0$  for  $k < 0$ .

**Proposition 27.** The value function under a barrier dividend distribution policy  $\pi_b$  with barrier  $b \in \mathbb{N}_0$  is given by:

$$(27) \quad V_D^b(x) := E_x^{[b]} \sum_{i=0}^{\tilde{\tau}_{-1}^- - 1} v^i r(i) = \begin{cases} \frac{W_v(x)}{\Delta W_v(b)} & x \leq b \\ x - b + V_D^b(b) = x - b + \frac{W_v(b)}{\Delta W_v(b)} & x > b \end{cases},$$

where for  $f : \mathbb{N}_0 \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N}_0$ ,  $\Delta f(k) := f(k+1) - f(k)$  gives the forward difference operator.

**Remark 28.** It is clear from (11) that under the stipulation  $(p_0 + p_1) \wedge v < 1$ ,  $W_v$  is strictly increasing.

*Proof.* The case  $x > b$  is trivial; assume  $x \leq b$ . Then (27) is “equivalent” to (11), since using the strong Markov property of  $X$ , one has clearly the relation

$$(28) \quad V_D^b(x) = E_x[v^{\tau_{b+1}^+}; \tau_{b+1}^+ < \tau_{-1}^-](1 + V_D^b(b)).$$

Thus if (27) is known, one obtains (11), and vice versa, if (11) is known, then one obtains by setting  $x = b$  in (28), first  $V_D^b(b) = \frac{W_v(b)}{\Delta W_v(b)}$  and then by substituting back, (27).  $\square$

**Remark 29.** The “factorization result”  $\frac{W_v(x)}{\Delta W_v(b)}$  of (27) has been known for a long time [Mor66, Eq. (19)] [GSY10, Sec. 5, Eq. (3.1)], and in the simplest case when  $\Delta W_v$  is “unimodal with minimum at  $b^*$ ”, i.e. when  $\Delta W_v$  is nondecreasing after  $b^*$  and nonincreasing before  $b^*$ , it yields in fact the optimal value function over all dividend distribution policies. The optimal “barrier” policy of taking dividends in  $\mathcal{D}^{b^*} = (b^*, \infty)$  and continuing in  $\mathcal{C}^{b^*} = [0, b^*]$  can then be viewed as a transformation of

the scale function into the value function  $V_D^b$ , which must be concave, by “linearization” of the convex piece of  $W_v$ .<sup>1</sup>

When  $\Delta W_v$  is not unimodal, the optimal policy may be “multi-band”, and requires a complicated recursive construction [Mor66, Sch07, APP15]. We will recall this concept briefly in Definition 40, but the main concern of our applications is optimization among barrier policies, by which we mean optimizing the limit  $b$  of the continuation interval  $[0, b]$ , in the sense of finding

$$V_D(x) := \sup_{b \in \mathbb{N}_0} V_D^b(x).$$

With the objective given by (27), this is related to maximizing the “barrier influence function”  $1/\Delta W_v$ , i.e. minimizing  $\Delta W_v$  (as is customary<sup>2</sup>, we will say  $b \in \mathbb{N}_0$  is optimal for  $V_D(x)$ , if  $V_D(x) = V_D^b(x)$ ):

**Lemma 30.** (I) If  $q := \inf_{b \in \mathbb{N}_0} \Delta W_v(b)$  is attained, letting  $b^*$  be any minimizer of  $\Delta W_v$ , it follows that  $b^*$  is optimal for  $V_D(x)$ , whenever  $x \leq b^*$ . (II) If the infimum defining  $q$  is not attained, then the supremum defining  $V_D(x)$  is not attained either. (III) If for some  $b \in \mathbb{N}_0$ , the function  $\Delta W_v$  is nondecreasing after  $b^*$ , i.e. satisfies  $\Delta W_v(b) \leq \Delta W_v(b')$  for all  $b' > b \geq b^*$ , and nonincreasing before  $b^*$ , i.e. satisfies  $\Delta W_v(b) \geq \Delta W_v(b')$  for all  $b < b' \leq b^*$ , and if furthermore  $b^* < x$ , then  $b^*$  is optimal for  $V_D(x)$ .

**Remark 31.** This dovetails nicely with Remark 29: when  $\Delta W_v$  is unimodal with minimum at  $b^*$ , then  $b^*$  is optimal for  $V_D(x)$ , whether or not  $x \leq b^*$ .

*Proof.* (I) To see this, note that for  $x \leq b$ ,  $V_D^b(x) = \frac{W_v(x)}{\Delta W_v(b)} \leq \frac{W_v(x)}{\Delta W_v(b^*)} = V_D^{b^*}(x)$ . And for  $b < x$ ,  $V_D^b(x) = x - b + \frac{W_v(b)}{\Delta W_v(b)} \leq x - b + \frac{W_v(b)}{\Delta W_v(b^*)} \leq \frac{W_v(x)}{\Delta W_v(b^*)} = V_D^{b^*}(x)$ , where the final inequality follows from (telescopic sum)  $W_v(x) - W_v(b) = \sum_{k=b}^{x-1} \Delta W_v(k) \geq \sum_{k=b}^{x-1} \Delta W_v(b^*) = (x - b)\Delta W_v(b^*)$ . (II) Indeed, there exists a sequence  $(b_n)_{n \in \mathbb{N}}$  in  $\mathbb{N}$ , with  $\Delta W_v(b_n)$  satisfying  $\Delta W_v(b') > \Delta W_v(b_n)$  for all  $b' < b_n$ ,  $n \in \mathbb{N}$ . Let now  $b \in \mathbb{N}_0$ . There is an  $n \in \mathbb{N}$  such that  $b_n \geq x \vee b$ . Then if  $b \geq x$ , clearly  $V_D^b(x) = \frac{W_v(x)}{\Delta W_v(b)} \leq \frac{W_v(x)}{\Delta W_v(b_n)} = V_D^{b_n}(x)$ . And if  $b < x$ , then  $V_D^b(x) = x - b + \frac{W_v(b)}{\Delta W_v(b)} \leq x - b + \frac{W_v(b)}{\Delta W_v(b_n)} \leq \frac{W_v(x)}{\Delta W_v(b_n)} = V_D^{b_n}(x)$ , where the last inequality follows from  $W_v(x) - W_v(b) = \sum_{k=b}^{x-1} \Delta W_v(k) \geq \sum_{k=b}^{x-1} \Delta W_v(b_n) = (x - b)\Delta W_v(b_n)$ . In other words, as  $n \uparrow \infty$ ,  $V_D^{b_n}(x) \uparrow \sup_{b \in \mathbb{N}_0} V_D^b(x) = V_D(x)$ , which however is not attained. We also see that  $q > 0$ , since  $V_D^b(x)$  is bounded by  $(x - b)^+ + \sum_{k=1}^{\infty} v^k$ , as  $b$  ranges over  $\mathbb{N}_0$ . (III) Since for  $y \in \mathbb{N}_0$ ,  $\left(\frac{W_v(y+1)}{\Delta W_v(y+1)} - (y+1)\right) - \left(\frac{W_v(y)}{\Delta W_v(y)} - y\right) = W_v(y) \left((\Delta W_v(y+1))^{-1} - (\Delta W_v(y))^{-1}\right)$ , it follows from the assumption, that the map  $\mathbb{N}_0 \ni b \mapsto \frac{W_v(b)}{\Delta W_v(b)} + (x - b)$  has a maximum at  $b^*$ . Thus if  $b \leq x$ , then it follows at once that  $V_D^b(x) \leq V_D^{b^*}(x)$ . And if  $b > x$ , then  $V_D^b(x) \geq V_D^x(x) = \frac{W_v(x)}{\Delta W_v(x)} \geq \frac{W_v(x)}{\Delta W_v(b)} = V_D^b(x)$ .  $\square$

**Remark 32.** For  $x \leq b$ , by the skip-free property,

$$(29) \quad V_D^b(x) = E_x^{bl} \left[ \sum_{n=1}^{\infty} v^n \mathbb{1}(n < \tilde{\tau}_{-1}^-, \tilde{X}_{n-1} = b, C_n = 0) \right] = E_x^{bl} \left[ \sum_{n=1}^{\tilde{\tau}_{-1}^- \wedge \mathcal{E}_v - 1} \mathbb{1}(\tilde{X}_{n-1} = b, C_n = 0) \right] = E_x^{bl} R(\tilde{\tau}_{-1}^- \wedge \mathcal{E}_v - 1).$$

where  $\mathcal{E}_v$  is an independent random variable with distribution  $\text{geom}_{\mathbb{N}}(1 - v)$ .<sup>3</sup>

<sup>1</sup>By a convex (concave) function  $f : \mathbb{N}_0 \rightarrow \mathbb{R}$  we mean a function whose forward difference  $\Delta f$  is nondecreasing (nonincreasing).

<sup>2</sup>And we will follow an analogous convention with respect to the optimization problems of Subsections 6.3 and 6.4 to follow.

<sup>3</sup>For  $r \in (0, 1]$ , we denote by  $\text{geom}_{\mathbb{N}}(r)$ , resp.  $\text{geom}_{\mathbb{N}_0}(r)$ , the geometric law on  $\mathbb{N}$ , resp.  $\mathbb{N}_0$ , with success parameter  $r$ , i.e. having p.m.f.  $\mathbb{N} \ni k \mapsto r(1 - r)^{k-1}$ , resp.  $\mathbb{N}_0 \ni k \mapsto r(1 - r)^k$ . The degenerate cases  $\text{geom}_{\mathbb{N}}(0)$  and  $\text{geom}_{\mathbb{N}_0}(0)$  are both interpreted as  $\delta_{\infty}$ , the Dirac mass at  $\infty$ .

**Example 33.** For  $b = 0$ , plugging  $W_v(0) = p_0^{-1}$  and  $W_v(1) = p_0^{-2}(v^{-1} - p_1)$  into (27), yields

$$(30) \quad V_D^0(0) = \frac{p_0 v}{1 - p_1 v - p_0 v}.$$

For  $p_1 = 0$ , this reduces to (note that we start with initial capital zero, hence pay no dividends at time zero, and that dividends of 1 are taken all the times strictly prior to ruin)

$$(31) \quad V_D^0(0) = \frac{p_0 v}{1 - p_0 v} = E_0^0 \left[ \sum_{n=1}^{\infty} v^n \mathbb{1}_{\{n < \tilde{\tau}_{-1}^-\}} \right] = E_0^0 [\tilde{\tau}_{-1}^- \wedge \mathcal{E}_v - 1] = E_0^0 R(\tilde{\tau}_{-1}^- \wedge \mathcal{E}_v - 1),$$

where  $\tilde{\tau}_{-1}^- \sim \text{geom}_{\mathbb{N}}(1 - p_0)$  and  $\mathcal{E}_v \sim \text{geom}_{\mathbb{N}}(1 - v)$  and hence  $R(\tilde{\tau}_{-1}^- \wedge \mathcal{E}_v - 1) = \tilde{\tau}_{-1}^- \wedge \mathcal{E}_v - 1 \sim \text{geom}_{\mathbb{N}_0}(1 - p_0 v)$ .

When  $p_1$  is not necessarily equal to 0, one may still decompose  $X$  into the process which records  $X$  only when it changes its value — it does so each time independently according to the law of  $C_1$  conditioned on  $\{C_1 \neq 1\}$  — and into the independent amounts of time that elapse in-between these changes, them being i.i.- $\text{geom}_{\mathbb{N}}(1 - p_1)$ -d.<sup>4</sup> From the perspective of the surplus process, this means that it may be seen as evolving (up to ruin) according to the following probabilistic prescription: for  $k \in \mathbb{N}_0$ , if 0 at end of period  $k$  (i.e. ruin has not yet occurred), then for  $L$  subsequent periods, where  $L \sim \text{geom}_{\mathbb{N}_0}(1 - p_1)$ , the claims are equal to 1, just off-setting the premia, and then during period  $k + L$ , independently, the surplus process goes up by 1 with probability  $p_0/(1 - p_1)$  or down by  $l$  with probability  $p_{l+1}/(1 - p_1)$ ,  $l \in \mathbb{N}$  — if the former, a dividend of one is taken; if the latter, ruin occurs. It follows that in this case the total discounted dividends are equal to

$$V_D^0(0) = E \left[ \sum_{i=1}^{\tau-1} v^{\sum_{j=1}^i Q_j} \right] = E \sum_{i=1}^{\tau-1} \left( \frac{v(1-p_1)}{1-vp_1} \right)^i = \frac{p_0}{1-p_1} \frac{v(1-p_1)}{1-vp_1} = \frac{p_0 v}{1-p_1 v - p_0 v},$$

where  $\tau \sim \text{geom}_{\mathbb{N}}(1 - \frac{p_0}{1-p_1})$  and  $Q_j \sim \text{geom}_{\mathbb{N}}(1 - p_1)$ ,  $j \in \mathbb{N}$ , are independent, confirming again (30). In other words, it is the same as the case  $p_1 = 0$ , except that one has conditioned the claims not to be equal to 1,  $p_0 \rightsquigarrow \frac{p_0}{1-p_1}$ , and changed the discount factor,  $v \rightsquigarrow \frac{v(1-p_1)}{1-vp_1}$ , reflecting the  $\text{geom}_{\mathbb{N}}(1 - p_1)$  distributed “holding periods” during which  $X$  does not move. Thus, for all intents and purposes, the case  $p_1 \neq 0$  is reduced to the case  $p_1 = 0$ . For instance, under  $P_0^0$ , the law of the cumulative paid-out dividends, i.e. of  $R(\tilde{\tau}_{-1}^- - 1)$ , is  $\text{geom}_{\mathbb{N}_0}(1 - \frac{p_0}{1-p_1})$ , and hence

$$(32) \quad R(\tilde{\tau}_{-1}^- \wedge \mathcal{E}_v - 1) \sim \text{geom}_{\mathbb{N}_0} \left( 1 - \frac{p_0 v}{1 - p_1 v} \right)$$

(replacing  $p_0$  and  $p_1$  by  $p_0 v$  and  $p_1 v$ , respectively, has the same effect as independent geometric killing with probability  $1 - v$  (the mass  $(1 - v)(p_0 + p_1)$  may, for instance, be added to  $p_2$ , it matters not)). See Proposition 34 below for a generalization.

Finally, expanding (30) in  $v$ -series, reveals that the probability that dividends are paid in the  $n$ -th step is

$$P_0^0[r(n) = 1] = P_0^0[\tilde{\tau}_{-1}^- > n, C_n = 0] = (p_0 + p_1)^{n-1} p_0, \quad n \in \mathbb{N},$$

which also has a clear interpretation:  $(n - 1)$ -times ruin must not occur, i.e. the claim is zero or one, and then the  $n$ -th claim must be zero. Incidentally, the above is the survival function of a modified geometric r.v.  $\tilde{T}$  with

$$P[\tilde{T} = 1] = 1 - p_0, \quad P[\tilde{T} = k] = p_0(1 - p_0 - p_1)(p_0 + p_1)^{k-2}, \quad k \in \mathbb{N}_{\geq 2}.$$

The next result gives another probabilistic interpretation to the objective  $V_D^b(b) = \frac{W_v(b)}{\Delta W_v(b)} = \frac{W_v(b+1)}{\Delta W_v(b)} - 1 = \left( \frac{\Delta W_v(b)}{W_v(b+1)} \right)^{-1} - 1$ , which is the mean of  $\text{geom}_{\mathbb{N}_0} \left( \frac{\Delta W_v(b)}{W_v(b+1)} \right)$ . Note that much more is

<sup>4</sup>This is analogous to the decomposition of a continuous time Markov chain into its jump chain and its sojourn times.

known in the case of spectrally negative Lévy processes, where  $(V_D^b(b))^{-1} = \frac{(W^{(q)})'(b)}{W^{(q)}(b)}$ , coincides with the rate of “excursions” larger than  $b$  of the Poisson process of heights of downward excursions from a running maximum, in the presence of exponential killing at rate  $q$  – see [Ber98, Sec. VII.8] for  $q = 0$  and [Don05] for  $q > 0$ .

**Proposition 34.** *Let  $b \in \mathbb{N}_0$ . Under a barrier policy  $\pi_b$ , starting from  $x = b$ , the killed cumulative dividends until ruin,  $R(\tilde{\tau}_{-1}^- \wedge \mathcal{E}_v - 1)$ , have the law  $\text{geom}_{\mathbb{N}_0} \left( \frac{\Delta W_v(b)}{W_v(b+1)} \right)$  (recall  $\mathcal{E}_v \sim \text{geom}_{\mathbb{N}}(1-v)$ , independent of  $\tilde{X}$ ). In particular,*

$$(33) \quad E_b^{[z]R(\tilde{\tau}_{-1}^- \wedge \mathcal{E}_v - 1)} = \frac{1 - \frac{W_v(b)}{W_v(b+1)}}{1 - z \frac{W_v(b)}{W_v(b+1)}}, \quad z \in (0, 1].$$

*Proof.* First one assumes  $p_0 + p_1 < 1$  and  $v = 1$ . We have the representation of  $R(\tilde{\tau}_{-1}^- \wedge \mathcal{E}_v - 1)$  as the sum  $\sum_{i=1}^N \tilde{R}_i$ , where  $\tilde{R}_i$  are i.i.d. with the law given in (32), and  $N \sim \text{geom}_{\mathbb{N}}(1-\alpha(b))$  is an independent geometric r.v. with  $\alpha(b)$  yet to be determined. Indeed, the successive  $\tilde{R}_i$  come from the dividends collected during the periods of time that the surplus process either stays at the level  $b$ , or else increases to  $b+1$ , only to be taken down to  $b$  by a paid-out dividend. These amounts have the same law as does the amount of dividends collected until ruin when starting from 0 under  $\pi_0$ . On the other hand,  $\alpha(b)$  is the probability that the surplus process, once it has jumped to a level strictly below  $b$ , then goes on to reach the level  $b$  before ruin occurs, i.e. (the quotients  $\frac{p_k}{1-p_0-p_1}$  come from conditioning to jump strictly below  $b$  from  $b$ )  $\alpha(b) = \sum_{k=2}^{b+1} \frac{p_k}{1-p_0-p_1} P_{b-k+1}[\tau_b^+ < \tau_{-1}^-] = \sum_{k=2}^{b+1} \frac{p_k}{1-p_0-p_1} \frac{W(b-k+1)}{W(b)}$ , which equals, using (12),  $\frac{W(b)-p_1W(b)-p_0W(b+1)}{(1-p_0-p_1)W(b)} = 1 - \frac{p_0}{1-p_0-p_1} \frac{\Delta W(b)}{W(b)}$ . The conclusion of the proposition then follows e.g. by computing the probability generating function of the “geometric sum of geometrics”  $\sum_{i=1}^N \tilde{R}_i$  and recognizing the geometric random variable and its parameter. The general case for  $p_0 + p_1 < 1$  is got by replacing  $p_0, \dots, p_{b+1}$  by  $p_0v, \dots, p_{b+1}v$  (and for instance adding the mass  $(1-v)(p_0 + \dots + p_{b+1})$  to  $p_{b+2}$ , it matters not), using Remark 9. When  $p_0 + p_1 = 1$ , then the result clearly still holds true (one gets, using (12), the law of (32), i.e.  $\text{geom}_{\mathbb{N}_0}(\frac{1-v}{1-p_1v})$ , as one should).  $\square$

The following proposition gives a dividends-deficit at ruin type law for the compound binomial risk processes reflected at  $b$ , in the style of [GSY10, Sec. 4]. See [IP12, Thm. 6], [AGVA17, Lem. 6] for the Lévy analog.

**Proposition 35.** *The joint generating function of the ruin time, deficit at ruin and of the cumulative dividends for a compound binomial risk process reflected at  $b \in \mathbb{N}_0$  is given by, with  $\{v, z, w\} \subset (0, 1]$ ,*

$$(34) \quad DP_v^{[b]}(x, w, z) := E_x^{[b]} \left[ v^{\tilde{\tau}_{-1}^-} w^{-\tilde{X}(\tilde{\tau}_{-1}^-)} z^{R(\tilde{\tau}_{-1}^-)}; \tilde{\tau}_{-1}^- < \infty \right] = \begin{cases} Z_v(x, w) - \frac{Z_v(b+1, w) - z Z_v(b, w)}{W_v(b+1) - z W_v(b)} W_v(x) & x \leq b \\ z^{x-b} DP_v^{[b]}(b, w, z) & x > b \end{cases}.$$

**Remark 36.** *When  $p_0 + p_1 < 1$ , by setting  $v = z = w = 1$ , one obtains (as one should)  $P_x^{[b]}(\tilde{\tau}_{-1}^- < \infty) = 1$  for all  $x \in \mathbb{Z}$ . When  $p_0 + p_1 = 1$ , we have of course  $\tilde{\tau}_{-1}^- = \infty$ ,  $P_x^{[b]}$ -a.s. for all  $x \in \mathbb{N}_0$  (and  $\tilde{\tau}_{-1}^- = 0$ ,  $P_x^{[b]}$ -a.s. for all  $x \in -\mathbb{N}$ ).*

*Proof.* The case  $x > b$  is trivial; let  $x \leq b$ . Using the strong Markov property for  $X$  at the exit time from the interval  $[0, b)$  yields that  $g(x) := DP_v^{[b]}(x, w, z)$  satisfies:

$$\begin{aligned} g(x) &= Z_v(x, w) - \frac{W_v(x)}{W_v(b)} Z_v(b, w) + \frac{W_v(x)}{W_v(b)} g(b) = Z_v(x, w) + W_v(x) \frac{g(b) - Z_v(b, w)}{W_v(b)} \\ &\Rightarrow \frac{g(x) - Z_v(x, w)}{W_v(x)} = \frac{g(b) - Z_v(b, w)}{W_v(b)} =: -H_v^b(z, w). \end{aligned}$$

Now by conditioning on the first jump

$$g(b) = p_0 v z g(b) + p_1 v g(b) + \sum_{k=1}^b p_{k+1} v g(b-k) + v \sum_{k=b+1}^{\infty} p_{k+1} w^{k-b}.$$

Plugging in  $g(x) = Z_v(x, w) - W_v(x) H_v^b(z, w)$  gives us

$$(1 - p_0 v z - p_1 v)(Z_v(b, w) - W_v(b) H_v^b(z, w)) = \sum_{k=1}^b p_{k+1} v (Z_v(b-k, w) - W_v(b-k) H_v^b(z, w)) + v \sum_{k=b+1}^{\infty} p_{k+1} w^{k-b}.$$

Using now (12) and (16) reduces this to

$$(1 - p_0 v z - p_1 v)(Z_v(b, w) - W_v(b) H_v^b(z, w)) = -H_v^b(z, w)(W_v(b) - v p_1 W_v(b) - v p_0 W_v(b+1)) + Z_v(b, w) - v p_1 Z_v(b, w) - v p_0 Z_v(b+1, w),$$

i.e.  $p_0 v (Z_v(b+1, w) - z Z_v(b, w)) = p_0 v H_v^b(z, w)(W_v(b+1) - z W_v(b)).$   $\square$

**Remark 37.** As a check, setting  $v = w = 1$  and  $x = b$  in (34) recovers (33) in the case  $v = 1$ .

Taking  $z = 1$  in (34) yields

**Corollary 38.** For  $\{v, w\} \subset (0, 1]$ , the joint generating function of the (reflected) ruin time and of the deficit at ruin for a compound binomial risk process reflected at  $b \in \mathbb{N}_0$  is given by

$$(35) \quad \Psi_v^b(x, w) := E_x^b \left[ v^{\tilde{\tau}^-} w^{-\tilde{X}(\tilde{\tau}^-)}; \tilde{\tau}^- < \infty \right] = Z_v(x, w) - \frac{\Delta Z_v(b, w)}{\Delta W_v(b)} W_v(x), \quad x \leq b. \quad \square$$

**Remark 39.** This result is similar to identity (15) for the joint generating function of the ruin time and of the deficit at ruin, with absorption at  $b$ ; this is to be expected, since we only replaced the boundary condition  $\Psi_v^b(b, w) := E_b[v^{\tau^-} w^{-X(\tau^-)}; \tau^- < \tau_b^+]$  by  $\Delta \Psi_v^b(b, w) = 0$ .

We recall finally some further background information for the general de Finetti dividends optimization problem with no penalty for the deficit at ruin, when  $\Delta W_v$  is not unimodal. This is useful for the numerics Section 7, to understand the examples where the optimal dividends policy is “multi-band”.

**Definition 40.** A multi-band dividends policy is specified by a partition of  $\mathbb{N}$  into continuation intervals  $\mathcal{C}_1 = [0, b_1]$ ,  $\mathcal{C}_2 = [a_2, b_2]$ ,  $\dots$ , and dividend taking intervals  $\mathcal{D}_1 = (b_1, a_2)$ ,  $\mathcal{D}_2 = (b_2, a_3)$ ,  $\dots$ , intertwined as follows:  $\mathcal{C}_1 < \mathcal{D}_1 < \mathcal{C}_2 < \mathcal{D}_2 < \dots$ . When the capital position is in  $\mathcal{D}_i$ , dividends are taken bringing the process down to the upper boundary  $b_i$  of  $\mathcal{C}_i$ .

When there is only one such pair  $\mathcal{C}_1 = [0, b_1]$ ,  $\mathcal{D}_1 = (b_1, \infty)$ , this is the barrier policy  $\pi_{b_1}$  of Definition 26. Subsequent  $\mathcal{C}_i$  and  $\mathcal{D}_i$ ,  $i \in \mathbb{N}_{\geq 2}$ , appear in the optimal policy when  $\Delta W_v$  is not unimodal and its global minimum  $b_1$  is followed by other local minima. Intuitively, the existence of local minima succeeding the global one offers incitement to postpone bringing the process to  $b_1$  (and thus the eventual ruin below  $-1$ ) – see [Mor66] for more details.<sup>5</sup>

**Remark 41.** The first multi-band example is [Mor66, Ex. 2], and in the Lévy case [AM05]; also, the absence of local minima after the global one is known to be sufficient for the optimality of single barrier policies, and sufficient conditions in terms of the Lévy measure have been provided in [Loe08, Thm. 2]. However, until today, no necessary and sufficient condition in terms of  $W_v$  has been provided.

<sup>5</sup>The barriers  $b_i$ ,  $i \in \mathbb{N}_{\geq 2}$ , may arise then, by “shifting optimally” these local minima. See [APP15] for a recursive algorithm achieving this, which is based on the idea that the process starting in  $\mathcal{C}_i$  will never visit states above  $b_i + 1$ . Since the process at  $x$  only needs to see the bands below  $x$ ,  $b_1$  may be computed as if only barrier policies were allowed, i.e. taken at the global maximum of the barrier influence function. For  $[a_2, b_2]$ , however, we need to take into account that the process may jump down either to ruin, or into  $\mathcal{C}_1 \cup \mathcal{D}_1$ . Now the latter case can be viewed as termination with final payoff given by the value function  $V_D^{b_1}$  over barrier policies, and this allows computing a value function  $V_D^{b_1, a_2, b_2}$ , and so on.

**6.3. Deficit at ruin with reflection at an upper boundary and the modified de Finetti problem.** This problem is masterly dealt with in [GSY10]. It may be useful however to provide an alternative treatment via the  $\Phi, W, Z$  paradigm, as in the parallel Lévy papers [Loe09, LR10, APP15, AGVA17].

Specifically, we assume  $EC_1 < \infty$  in addition to  $v \wedge (p_0 + p_1) < 1$ , and consider the de Finetti problem with dividends and no capital injections of Subsection 6.2, modified by the addition of an extra linear penalty/bailout cost  $ky$  upon ruin ( $y$  being the (positive) deficit at ruin;  $k \in (0, \infty)$ ). Under barrier strategies, this requires the computation of  $Z_{1,v}$  (22) (and of  $Z_v(\cdot, w)$  (17) under exponential “risk-sensitive” bailout costs [BJ15]). In precise terms, we have that, under a barrier strategy  $\pi_b$ ,  $b \in \mathbb{N}_0$ , the additional expected (positive) final bailout is  $kV_B^b(x)$ , where

**Proposition 42.** *For  $b \in \mathbb{N}_0$ ,*

$$(36) \quad V_B^b(x) := E_x^b[v^{\tilde{\tau}_-^{-1}}(-\tilde{X}(\tilde{\tau}_-^{-1}); \tilde{\tau}_-^{-1} < \infty)] = \begin{cases} W_v(x) \frac{\Delta Z_{1,v}(b)}{\Delta W_v(b)} - Z_{1,v}(x) & x \leq b \\ V_B^b(b) & x > b \end{cases}.$$

*Proof.* In the nontrivial case, when  $x \leq b$ , using the strong Markov property for  $X$  at the exit time from the interval  $[0, b]$  yields:

$$(37) \quad V_B^b(x) = E_x[v^{\tau_b^+}; \tau_b^+ < \tau_{-1}^-] V_B^b(b) + E_x[v^{\tau_{-1}^-}(-X(\tau_{-1}^-)); \tau_{-1}^- < \tau_b^+] = \frac{W_v(x)}{W_v(b)} V_B^b(b) - \left( Z_{1,v}(x) - \frac{W_v(x)}{W_v(b)} Z_{1,v}(b) \right),$$

where the second term was computed in Remark 24. Making  $x = 0$  yields  $V_B^b(0) = \frac{W_v(0)}{W_v(b)} V_B^b(b) + \frac{W_v(0)}{W_v(b)} Z_{1,v}(b)$ , and substituting it back in (37) gives us

$$(38) \quad V_B^b(x) + Z_{1,v}(x) = W_v(x) \frac{V_B^b(0)}{W_v(0)}.$$

This formula coincides with (36), up to showing that  $\frac{V_B^b(0)}{W_v(0)} = \frac{\Delta Z_{1,v}(b)}{\Delta W_v(b)}$ . To see this, note that using the strong Markov property for  $X$  at the exit time from the interval  $[0, b]$  yields  $V_B^b(0) = E_0[v^{\tau_{b+1}^+}; \tau_{b+1}^+ < \tau_{-1}^-] V_B^b(b) + E_0[v^{\tau_{-1}^-}(-X(\tau_{-1}^-)); \tau_{-1}^- < \tau_{b+1}^+] = \frac{W_v(0)}{W_v(b+1)} V_B^b(b) + \frac{W_v(0)}{W_v(b+1)} Z_{1,v}(b+1)$ . Plugging into this (38) with  $x = b$ , i.e.  $V_B^b(b) = -Z_{1,v}(b) + W_v(b) \frac{V_B^b(0)}{W_v(0)}$ , we obtain the desired identity.  $\square$

It seems on the basis of numeric examples, that adding a bailout penalty typically makes the optimal policy single barrier. With this in mind and for simplicity, we restrict here to the version of the problem, under which only barrier dividend policies are allowed. Under this proviso, optimizing under barrier policies the combined objective

$$V(x) := \sup_{b \in \mathbb{N}_0} V^b(x), \quad V^b(x) := V_D^b(x) - kV_B^b(x),$$

amounts to optimizing the relevant linear combination of the expressions (27) and (36), viz.  $V^b(x) = (x \vee b) - b + W_v(x \wedge b)H(b) + kZ_{1,v}(x \wedge b)$ , where  $H$ , the “barrier influence function”, is given by

$$H(b) := \frac{1 - k\Delta Z_{1,v}(b)}{\Delta W_v(b)} = \frac{1 - k(Z_v(b) - (1 - \tilde{p}'(1-))W_v(b))}{\Delta W_v(b)},$$

see [AGVA17, Eq. (86)] for the Lévy case. Finding the optimum  $V(x)$  is related to maximizing  $H$  (cf. Lemma 30):

**Lemma 43.** (I) *If  $r := \sup_{b \in \mathbb{N}_0} H(b)$  is attained, letting  $b^*$  be any maximizer of  $H$ , then  $b^* \geq x$  implies that  $b^*$  is optimal for  $V(x)$ .* (II) *If the supremum defining  $r$  is not attained, then the supremum defining  $V(x)$  is not attained either.*

*Proof.* (I) To see this, note that for  $b \geq x$ ,  $V^b(x) = W_v(x)H(b) + kZ_{1,v}(x) \leq W_v(x)H(b^*) + kZ_{1,v}(x) = V^{b^*}(x)$ . And for  $b < x$ ,  $V^b(x) = x - b + W_v(b)H(b) + kZ_{1,v}(b) \leq x - b + W_v(b)H(b^*) + kZ_{1,v}(b) \leq W_v(x)H(b^*) + kZ_{1,v}(x) = V^{b^*}(x)$ , where the final inequality follows from (telescopic sum)  $H(b^*)(W_v(x) - W_v(b)) = H(b^*) \sum_{l=b}^{x-1} \Delta W_v(l) \geq \sum_{l=b}^{x-1} (1 - k\Delta Z_{1,v}(l)) = (x - b) - k(Z_{1,v}(x) - Z_{1,v}(b))$ . (II) Indeed, there exists a sequence  $(b_n)_{n \in \mathbb{N}}$  in  $\mathbb{N}$ , with  $H(b_n)$  satisfying  $H(b') < H(b_n)$  for all  $b' < b_n$ ,  $n \in \mathbb{N}$ . Let now  $b \in \mathbb{N}_0$ . There is an  $n \in \mathbb{N}$  such that  $b_n \geq x \vee b$ . Then if  $b \geq x$ , clearly  $V^b(x) = W_v(x)H(b) + kZ_{1,v}(x) \leq W_v(x)H(b_n) + kZ_{1,v}(x) = V^{b_n}(x)$ . And if  $b < x$ , then  $V^b(x) = x - b + W_v(b)H(b) + kZ_{1,v}(b) \leq x - b + W_v(b)H(b_n) + kZ_{1,v}(b) \leq W_v(x)H(b_n) + kZ_{1,v}(x) = V^{b_n}(x)$ , where the final inequality follows from  $H(b_n)(W_v(x) - W_v(b)) = H(b_n) \sum_{l=b}^{x-1} \Delta W_v(l) \geq \sum_{l=b}^{x-1} (1 - k\Delta Z_{1,v}(l)) = (x - b) - k(Z_{1,v}(x) - Z_{1,v}(b))$ . In other words, as  $n \uparrow \infty$ ,  $V^{b_n}(x) \uparrow \sup_{b \in \mathbb{N}_0} V^b(x) = V(x)$ , which however is not attained.  $\square$

**6.4. Optimizing a combination of dividends and capital injections for a doubly reflected process.** This problem is another very good illustration of the  $\Phi, W, Z$  paradigm and is quite hard analytically. Indeed, the recent paper [WGT11] falls short of reaching an explicit solution, which has been however available in the Lévy literature [APP07] for a while. Since the Lévy solution is a consequence of the Markov and skip-free properties, we may expect that it continues to hold in the discrete setup; and this is indeed the case.

We assume claims have a finite mean,  $EC_1 < \infty$ , and linear capital injection costs  $w(y) = ky$  ( $y$  being the capital injection), where  $k \in (1, \infty)$  is a proportionality parameter. There is also a fixed discount factor  $v \in (0, 1)$  and  $x$  is the initial capital.

The description of the behavior of the surplus process is an amalgamation of those given in Subsections 6.1 and 6.2, so we may be slightly more brief here. Namely, we stipulate that for  $l \in \mathbb{N}$ , during period  $l$ , a premium of 1 is collected and the claim amount  $C_l$  is incurred; then at the end of period  $l$ : (i) capital is injected in the amount  $r_*(l)$ , which is the amount by which the surplus process is negative ( $r_*(l) = 0$  if the surplus process remains nonnegative); (ii) the dividend amount  $r(l)$  is paid out ( $r(l) = 0$  if the surplus process has become nonpositive). At end of period 0 we inject  $r_*(0) = (-x) \vee 0$  and a dividend  $r(0)$  may be paid out, provided  $x > 0$ . One says that the surplus process thus obtained is doubly reflected (at 0 and  $b$ ). The quantities paid out/injected at end of period  $l$  are to be discounted by the factor  $v^l$ ,  $l \in \mathbb{N}_0$ .

Then, using the fact proved in [WGT11] ([APP07] in the spectrally negative case), that barrier policies are optimal, the problem reduces to expressing, in terms of  $W$  and  $Z$ , for a barrier dividend distribution policy  $\pi_b$ ,  $b \in \mathbb{N}_0$ , the values of: (i) the expected discounted dividends,

$$(39) \quad V_D^b(x) := E_x^{[0,b]} \sum_{l=0}^{\infty} v^l r(l) = \begin{cases} \frac{Z_v(x)}{\Delta Z_v(b)} & x \leq b \\ x - b + V_D^b(b) & x > b \end{cases},$$

where  $E_x^{[0,b]}$  indicates expectation with respect to the process doubly reflected at 0 and  $b$ ; and of (ii) the expected discounted bailouts,

$$(40) \quad V_B^b(x) := E_x^{[0,b]} \sum_{l=0}^{\infty} v^l r_*(l) = \begin{cases} Z_v(x) \frac{\Delta Z_{1,v}(b)}{\Delta Z_v(b)} - Z_{1,v}(x) & x \leq b \\ V_B^b(b) & x > b \end{cases}.$$

We give now the derivation of these two formulas. The cases  $x > b$  are trivial, we limit the discussion to  $x \leq b$ .

*Proof of (39), for dividends.* We break the objective in two, following [APP07]: the ‘‘De Finetti part’’ until the first bailout time (27), and the rest (35):

$$V_D^b(x) = E_x^b \left[ \sum_{l=0}^{\tilde{\tau}_{-1}^- - 1} v^l r(l) \right] + E_x^b [v^{\tilde{\tau}_{-1}^-}; \tilde{\tau}_{-1}^- < \infty] V_D^b(0) = \frac{W_v(x)}{\Delta W_v(b)} + V_D^b(0) \left( Z_v(x) - \frac{\Delta Z_v(b)}{\Delta W_v(b)} W_v(x) \right).$$

Making  $x = 0$  yields  $V_D^b(0) = \frac{1}{\Delta Z_v(b)}$  and the result follows.  $\square$

*Proof of (40), for bailouts.* Using the strong Markov property at the exit time from the interval  $[0, b]$  for the process  $X$ , yields an equation with three unknowns,  $V_B^b(x)$ ,  $V_B^b(0)$  and  $V_B^b(b)$ :

$$\begin{aligned} V_B^b(x) &= E_x[v^{\tau_b^+}; \tau_b^+ < \tau_{-1}^-]V_B^b(b) + E_x[v^{\tau_{-1}^-}; \tau_{-1}^- < \tau_b^+]V_B^b(0) + E_x[(-X(\tau_{-1}^-))v^{\tau_{-1}^-}; \tau_{-1}^- < \tau_b^+] \\ &= \frac{W_v(x)}{W_v(b)}V_B^b(b) + \left(Z_v(x) - \frac{W_v(x)}{W_v(b)}Z_v(b)\right)V_B^b(0) - \left(Z_{1,v}(x) - \frac{W_v(x)}{W_v(b)}Z_{1,v}(b)\right), \end{aligned}$$

where, on the event that the first bailout occurs before the level  $b$  is reached, the last term is the expectation of this first bailout, before resetting to 0, computed in Remark 24, the penultimate term gives the expectation of the remaining bailouts and is given by (15), finally the first term follows from (11). Making  $x = 0$ , yields  $0 = V_B^b(b) + Z_{1,v}(b) - Z_v(b)V_B^b(0)$ , and it follows that

$$\frac{V_B^b(x) + Z_{1,v}(x) - V_B^b(0)Z_v(x)}{W_v(x)} = \frac{V_B^b(b) + Z_{1,v}(b) - V_B^b(0)Z_v(b)}{W_v(b)} = 0.$$

It remains to show that  $V_B^b(0) = \frac{\Delta Z_{1,v}(b)}{\Delta Z_v(b)}$ . To this end, using the strong Markov property at the exit time from the interval  $[0, b]$  for the process  $X$ , produces  $V_B^b(0) = \frac{W_v(0)}{W_v(b+1)}V_B^b(b) + \left(1 - \frac{W_v(0)}{W_v(b+1)}Z_v(b+1)\right)V_B^b(0) + \frac{W_v(0)}{W_v(b+1)}Z_{1,v}(b+1)$ . We conclude by plugging in  $V_B^b(b) = Z_v(b)V_B^b(0) - Z_{1,v}(b)$ .  $\square$

**Remark 44.** For  $x < 0$ , by Remark 24, (40) reduces to  $V_B^b(0) - x$ , as it should.

The combined objective is

$$V(x) := \sup_{b \in \mathbb{N}_0} V^b(x), \quad V^b(x) := \left[ V_D^b(x) - kV_B^b(x) \right] = (x \vee b) - b + Z_v(x \wedge b)H(b) + kZ_{1,v}(x \wedge b),$$

with ‘‘barrier influence function’’

$$(41) \quad H(b) := \frac{1 - k\Delta Z_{1,v}(b)}{\Delta Z_v(b)} = \frac{1 - k(Z_v(b) - (1 - \tilde{p}'(1-))W_v(b))}{\left(\frac{1}{v} - 1\right)W_v(b)}.$$

As in the previous subsection, with an analogous justification, finding  $V(x)$  is related to finding the supremum of  $H$ : (I) If  $r := \sup_{b \in \mathbb{N}_0} H(b)$  is attained, letting  $b^*$  be a maximizer of  $H$ , then  $b^* \geq x$  implies that  $b^*$  is optimal for  $V(x)$ . (II) If the supremum defining  $r$  is not attained, then the supremum defining  $V(x)$  is not attained either. Since in this problem there is an optimal barrier strategy that does not depend on the initial reserve [WGT11, Theorem 3.2(B)], it follows, at least when the maximizer of  $H$  is unique, that in (I),  $b^*$  is in fact optimal for all  $V(x)$ ,  $x \in \mathbb{Z}$ . Finally, note that  $H$  differs from  $\overline{H}(b) := \frac{1 - kZ_v(b)}{W_v(b)}$  only up to a positive affine transformation, so finding the supremum of, resp. a maximizer for,  $H$  is equivalent to finding the supremum of, resp. a maximizer for,  $\overline{H}$ .

## 7. EXAMPLES

**7.1. Eventual ruin probabilities and the de Finetti dividends optimization.** The eventual ruin probability is a straightforward application of (9), followed by Taylor series coefficient extraction. Similarly, by using (10)-(4) and generating function inversion, one can obtain the probability mass function of the time to ruin. One may also use the recursions (5)-(6)-(7). Indeed, in the case when the support of the distribution of the claims is finite, the Lundberg recurrence (7) reduces the problem of finding the eventual ruin probability to determining the roots of the characteristic equation (8).

For instance, suppose  $C_1$  takes on the values 0, 1, and 3, with probabilities  $2/3$ ,  $2/9$  and  $1/9$ , respectively. Then  $E[C_1] = 5/9 < 1$  and eventual upwards passage has probability  $\varphi_1 = 1$ . The generating function is  $\tilde{p}(z) = \frac{6}{9} + \frac{2}{9}z + \frac{1}{9}z^3$ ,  $z \in (0, 1]$ ; and Lundberg’s equation is

$$\frac{\varphi_v}{v} = \frac{6}{9} + \frac{2}{9}\varphi_v + \frac{1}{9}\varphi_v^3, \quad v \in (0, 1].$$

The recurrence for the perpetual survival and eventual ruin probabilities writes as (with  $f$  standing in place of  $\Psi$  or  $\bar{\Psi}$ )

$$f(x) = \frac{2}{3}f(x+1) + \frac{2}{9}f(x) + \frac{1}{9}f(x-2)$$

for  $x \in \mathbb{N}_0$ . The characteristic equation (8) for this recurrence is (in  $x$ )

$$\frac{2}{3}x^3 - \frac{7}{9}x^2 + \frac{1}{9} = \frac{2}{3}(x-1)\left(x - \frac{1}{2}\right)\left(x + \frac{1}{3}\right) = 0$$

(coinciding formally with the transformation of Lundberg's equation  $\frac{6}{9} - \frac{7}{9}\varphi_1 + \frac{1}{9}\varphi_1^3 = 0$ , via  $\varphi_1 \rightsquigarrow 1/x$ ). Satisfying the boundary conditions  $\Psi(-1) = \Psi(-2) = 1$ , we arrive at

$$\Psi(x) = \frac{2}{5}\left(\frac{1}{2}\right)^x - \frac{1}{15}\left(-\frac{1}{3}\right)^x \quad \text{and} \quad \bar{\Psi}(x) = 1 - \frac{2}{5}\left(\frac{1}{2}\right)^x + \frac{1}{15}\left(-\frac{1}{3}\right)^x, \quad x \in \mathbb{N}_0.$$

Taking  $z$ -transform yields, for  $z \in (0, 1)$ ,  $\tilde{\Psi}(z) = \frac{z+2}{(z+3)(2-z)}$  and  $\tilde{\bar{\Psi}}(z) = \frac{4}{6-7z+z^3}$ , which confirms (44)-(45). Finally, consider the de Finetti dividends optimization, under a discount factor  $v = 150/169$ . Taylor expanding the scale transform (13) yields (the right-hand side features the consecutive values  $\{W_v(0), W_v(1), \dots\}$ )

$$W_v = \{1.5, 2.035, 2.76082, 3.49551, 4.40307, 5.51337, 6.89721, 8.62338, 10.7802, 13.4755, 16.8446, 21.0558, 26.3198, \dots\},$$

which may be checked to be a convex function with increasing forward difference

$$\Delta W_v = \{0.535, 0.725817, 0.734691, 0.907565, 1.11029, 1.38385, 1.72616, 2.15678, 2.69539, 3.36905, 4.21121, 5.26398, \dots\}.$$

It follows that, irrespective of the initial capital, the optimal dividend policy is bringing the process to the barrier  $b = 0$  by taking dividends whenever possible.

**7.2. Modified geometric claims.** We consider next modified geometric claims, defined by  $p_k = (1 - \alpha)\alpha^{k-I}(1 - p_0 - p_1 - \dots - p_{I-1})$ ,  $k = I, I+1, \dots$ ,  $\alpha \in [0, 1)$ . We restrict to  $I = 2$ , which is equivalent to having two Lundberg roots [SdR07]. We assume  $p_0 + p_1 < 1$ . The probability generating function is

$$\tilde{p}(z) = \frac{p_0 + z(p_1 - \alpha p_0) + z^2[(1 - \alpha)(1 - p_0) - p_1]}{1 - \alpha z}, \quad z \in (0, 1].$$

The mean is  $m := EC_1 = 1 - p_0 + \frac{1-p_0-p_1}{1-\alpha}$ , and the positive profit/subcritical case  $m < 1$  occurs when  $p_0(1 - \alpha) > 1 - p_0 - p_1$ , which we assume henceforth. Fix  $v \in (0, 1]$ . The Lundberg equation (in  $z$ )  $k_v z^2 + z(p_1 - \alpha p_0 - v^{-1}) + p_0 = 0$ , with  $k_v := (1 - \alpha)(1 - p_0) - p_1 + \alpha/v > 0$ , has two (complex) solutions, the smaller one is  $\varphi_v$ , and the larger of the two we will denote by  $R_v$ ; their product is  $\varphi_v R_v = \frac{p_0}{k_v} = \frac{p_0}{(1-\alpha)(1-p_0)-p_1+\alpha/v}$ .

For  $v = 1$ , the roots are  $\varphi_1 = 1$  and  $R := R_1 = \frac{p_0}{1-p_1-(1-\alpha)p_0} > 1$ . The eventual ruin probability is given by

$$(42) \quad \Psi(x) = \Psi(0)R^{-x} = \left(1 - \frac{1 - EC_1}{p_0}\right)R^{-x} = \frac{1 - p_0 - p_1}{p_0(1 - \alpha)}R^{-x}, \quad x \in \mathbb{N}_0.$$

This may be checked using (45). Note the last formula does not hold for  $x = -1$ , except for special constellations of  $p_0, p_1, \alpha$ . Whatever the value of  $v$ ,  $\varphi_v \leq 1 < R_v$ .

Some particular cases are:

- (1) If  $\alpha = 0$ , the claims cannot exceed 2,  $R^{-1} = \frac{p_2}{p_0}$ ,  $\Psi(0) = R^{-1}$ , and the eventual ruin probability is

$$\Psi(x) = \left(\frac{p_2}{p_0}\right)^{x+1}, \quad x \in \mathbb{N}_0 \cup \{-1\},$$

recovering the classic gambler's ruin problem.

- (2) Geometric:  $p_0 = 1 - \alpha$ ,  $p_1 = \alpha(1 - \alpha)$ .
- (3) Geometric shifted by one:  $p_0 = 0$ ,  $p_1 = 1 - \alpha$ .
- (4) Geometric shifted by two:  $p_0 = 0 = p_1$ .

Writing now  $\tilde{p}(z) - z/v = k_v(z - \phi_v)(z - R_v)$ , we find, using (13) & (19), for  $z \in (0, \phi_v)$ :

$$\tilde{W}_v(z) = \frac{1}{k_v(R_v - \phi_v)} \left( \frac{1}{\phi_v - z} - \frac{1}{R_v - z} \right)$$

and

$$\tilde{Z}_v(z) = \frac{1}{1 - z} \left( 1 - \frac{v^{-1} - 1}{k_v(1 - \phi_v)(R_v - 1)} \right) + \frac{v^{-1} - 1}{k_v(R_v - \phi_v)} \left( \frac{(\phi_v^{-1} - 1)^{-1}}{\phi_v - z} + \frac{(1 - R_v^{-1})^{-1}}{R_v - z} \right),$$

so that, for  $x \in \mathbb{N}_0$ ,

$$W_v(x) = \frac{1}{k_v(R_v - \phi_v)} \left( \phi_v^{-x-1} - R_v^{-x-1} \right)$$

and

$$Z_v(x) = 1 - \frac{v^{-1} - 1}{k_v(1 - \phi_v)(R_v - 1)} + \frac{v^{-1} - 1}{k_v(R_v - \phi_v)} \left( (\phi_v^{-1} - 1)^{-1} \phi_v^{-x-1} + (1 - R_v^{-1})^{-1} R_v^{-x-1} \right).$$

As a check,  $\tilde{W}_v(0+) = W_v(0) = p_0^{-1}$  and  $\tilde{Z}_v(0+) = Z_v(0) = 1$ . Given specific values of the parameters  $\alpha, p_1, p_0, v$ , the above expressions for  $W_v$  and  $Z_v$  may be easily used to optimize combinations of expected bailouts/penalties at ruin and dividends.

**Remark 45.** *This model has a long history in branching processes as well [AN72, Mod71]. Its utilization there goes back to Steffensen and Lotka (under the name of linear fractional branching) – see [Ken66], and is still of interest nowadays – see for example [Sag16].*

### 7.3. Multi-band dividend policies and modified de Finetti optimization.

**Example 46.** *Recall Morrill's historic example [Mor66, Ex. 2], with claims taking the values 0 and 3 with probabilities 12/13 and 1/13, respectively ( $\rightarrow E[C_1] = 3/13 < 1$ ), and with discount factor  $v = 65/72$ . Taylor expanding the scale transform (13) yields*

$$W_v = \{1.08333, 1.3, 1.56, 1.78172, 2.02973, 2.30568, 2.61834, 2.97286, 3.3753, 3.83216, 4.35085, \dots\},$$

which may be checked to have a forward difference

$$\Delta W_v = \{0.216667, 0.26, 0.221722, 0.248011, 0.275947, 0.312659, 0.354523, 0.402433, 0.456864, \dots\}$$

with a global minimum at  $b^* = 0$  and another local minimum at 2.

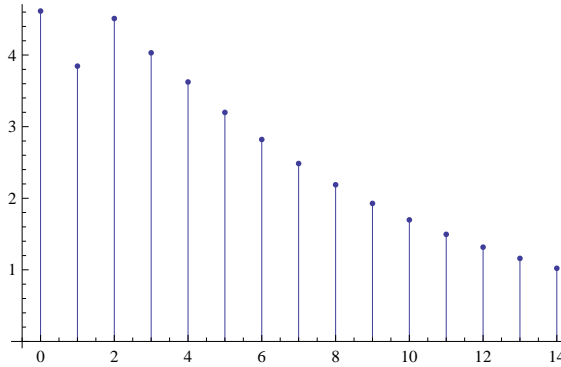


FIGURE 1. The barrier influence function  $1/\Delta W_v(b)$  for Morrill's example. The maximum  $b^* = 0$  is followed by the local maximum 2. The optimal dividend policy is multi-band, with two continuation sets  $\{0\}$  and  $\{2\}$ .

Consider now the modified de Finetti objective of Subsection 6.3. For  $k$  big enough, for example  $k = 3.2$ , the barrier influence function  $(1 - 3.2\Delta Z_{1,v}(b))/\Delta W_v(b)$  is unimodal – see Figure 2.

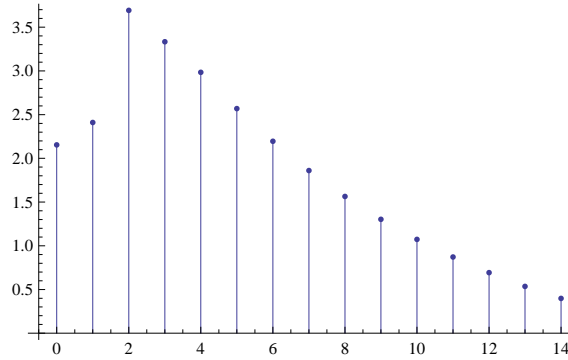


FIGURE 2. The barrier influence function  $(1 - 3.2\Delta Z_{1,v}(b))/\Delta W_v(b)$  for Morrill's example. The maximum  $b^* = 2$  is now the unique local (and hence global) maximum.

**Example 47.** We turn now to the Gerber-Shiu-Yang example [GSY10, Ex. 3], in which the barrier influence function  $1/\Delta W_v(b)$  has three local minima. The claims take the values 0, 1, 2 and 7 with probabilities  $3/4, 1/20, 1/10$  and  $1/10$ , respectively ( $\rightarrow E[C_1] = 19/20 < 1$ ), and the discount factor is  $v = 0.999$ . Now the barrier influence function has a global maximum at  $b^* = 1$  and two further local maxima at 7 and 38 – see Figure 3.

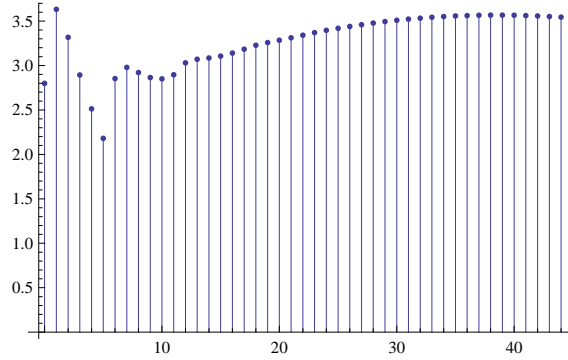


FIGURE 3. The barrier influence function  $1/\Delta W_v(b)$  for the Gerber-Shiu-Yang example. The maximum  $b^* = 1$  is followed by the local maxima 7 and 38. The optimal dividend policy is multi-band.

Adopting a modified de Finetti objective of Subsection 6.3, for example with  $k = 1.2$  — see Figure 4 — shifts the global maximum to  $b^* = 41$ . The barrier influence function is not unimodal. With capital injections however, the barrier influence function is unimodal – see next subsection and Figure 5.

**7.4. Combined dividends and bailouts optimization objective for the doubly reflected process.** We optimize finally in the Gerber-Shiu-Yang example (Example 47), the combined dividends-bailouts objective of Subsection 6.4 for the doubly reflected process with  $k = 1.2$ . Recall that for this optimization problem there is always an optimal barrier policy [WGT11]. We obtain Figure 5. This objective seems to have achieved a “compromise” between the peaks of the pure de Finetti objective.

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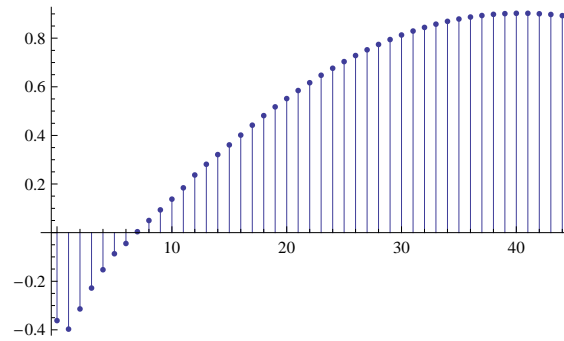


FIGURE 4. The barrier influence function  $(1 - 1.2\Delta Z_{1,v}(b))/\Delta W_v(b)$  for the Gerber-Shiu-Yang example. The unique local and global maximum is  $b^* = 41$ .

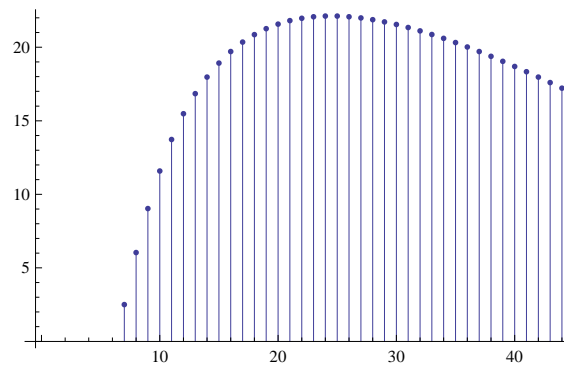


FIGURE 5. The barrier influence function (41) with  $k = 1.2$  for the doubly reflected Gerber-Shiu-Yang example has a unique global maximum at  $b^* = 25$ . The first seven values  $-89.91, -59.1845, -43.5339, -30.8171, -19.8565, -10.3512, -2.10264$  are too negative to be represented. The optimal policy is to take dividends above the level  $b^* = 25$ .

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#### APPENDIX A. DOUBLE (GENERATING FUNCTION) TRANSFORMS OF RUIN PROBABILITIES

Recall the notation of Section 3. From [Wil93, Eqs. (2.7) & (2.13)], one may deduced the double transform

$$(43) \quad \tilde{\Psi}_v(z) := \sum_{n=0}^{\infty} v^n \bar{\Psi}_z(n) := \sum_{n=0}^{\infty} v^n \left( \sum_{x=0}^{\infty} z^x \bar{\Psi}(n; x) \right) = \frac{\frac{z}{1-z} - \frac{\varphi_v}{1-\varphi_v}}{z - v\tilde{p}(z)}, \quad v, z \in (0, 1), \quad z \neq \varphi_v,$$

where  $\varphi_v \in (0, 1)$  is the Lundberg root (4) (note that  $z = \varphi_v$  is a removable singularity). Indeed, from (5), for all  $n \geq 1$ , [Wil93, Eq. (2.3)]

$$z\bar{\Psi}_z(n) = \tilde{p}(z)\bar{\Psi}_z(n-1) - p_0\bar{\Psi}(n-1; 0),$$

and summing over  $n$  after multiplication by  $v^n$  yields [Wil93, Eq. (2.7)]

$$z(\tilde{\Psi}_v(z) - (1-z)^{-1}) = v\tilde{p}(z)\tilde{\Psi}_v(z) - p_0v \sum_{n=0}^{\infty} v^n \bar{\Psi}(n; 0) \Rightarrow (z - v\tilde{p}(z))\tilde{\Psi}_v(z) = \frac{z}{1-z} - p_0v \sum_{n=0}^{\infty} v^n \bar{\Psi}(n; 0),$$

from where (43) is obtained by requiring that the root  $z = \varphi_v$  on the left-hand side annihilates also the right-hand side.

Eq. (43) implies the transform (for  $v, z \in (0, 1)$ ,  $z \neq \varphi_v$ )

$$\tilde{\Psi}_v(z) := \sum_{x=0}^{\infty} \sum_{n=0}^{\infty} z^x v^n \Psi(n; x) = \frac{1}{(1-z)(1-v)} - \tilde{\Psi}_v(z) = \frac{1}{z - v\tilde{p}(z)} \left( \frac{v(z - \tilde{p}(z))}{(1-v)(1-z)} + \frac{\varphi_v}{1-\varphi_v} \right).$$

**Remark 48.** Note the single transforms [Wil93, Eq. (3.5)]

$$(44) \quad \tilde{\bar{\Psi}}(z) := \sum_{x=0}^{\infty} z^x \bar{\Psi}(x) = \lim_{v \uparrow 1} (1-v)\tilde{\Psi}_v(z) = \frac{(1 - E[C_1]) \vee 0}{\tilde{p}(z) - z}, \quad z \in (0, 1),$$

$$(45) \quad \tilde{\Psi}(z) := \sum_{x=0}^{\infty} z^x \Psi(x) = \frac{1}{1-z} - \frac{(1 - E[C_1]) \vee 0}{\tilde{p}(z) - z}, \quad z \in (0, 1),$$

which are similar to the Pollaczek-Khinchine formulas of the Cramér-Lundberg model. One also has [Shi89, 2.14]

$$\bar{\Psi}(0) = \lim_{z \downarrow 0} \tilde{\bar{\Psi}}(z) = \frac{(1 - EC_1) \vee 0}{p_0}.$$

## APPENDIX B. SUMMARY TABLE

Right-continuous random walk	Upwards skip-free Lévy chain	Spectrally negative Lévy process
$\{n, b\} \subset \mathbb{N}_0$ , $x \in \mathbb{Z}$ , $x \leq b$ , $v \in (0, 1]$	$\{q, \beta\} \subset [0, \infty)$ , $\{b, x\} \subset h\mathbb{Z}$ , $x \leq b$ , $b \geq 0$	$\{q, \beta\} \subset [0, \infty)$ , $\{b, x\} \subset \mathbb{R}$ , $x \leq b$ , $b > 0$
$X_n = X_0 + n - \sum_{i=1}^n C_i$	$Y = hX_N$ , $N$ independent homogeneous Poisson process of intensity $\gamma$ , $\{\gamma, h\} \subset (0, \infty)$	Lévy process $U$ having a.s. non-monotone paths and no positive jumps
$C_n$ i.i.d., $\mathbb{N}_0$ -valued; p.m.f. $p$ , $p_0 \in (0, 1)$ ; p.g.f. $\tilde{p}$	Lévy measure $\lambda$ ; $\lambda = \gamma \sum_{i \in \mathbb{Z} \setminus \{1\}} p_i \delta_{h(1-i)}$ ; Laplace exponent $\psi$ ; $\psi(\beta) = \gamma[e^{\beta h} \tilde{p}(e^{-\beta h}) - 1]$	Laplace exponent $\psi$ ; $\delta$ is the drift, when $X$ has bounded variation
$\tau_b^- = \inf\{m \in \mathbb{N}_0 : X_m \leq b\}$ ; $\tau_b^+ = \inf\{m \in \mathbb{N}_0 : X_t \geq b\}$	$\tau_b^- = \inf\{t \in [0, \infty) : Y_t \leq b\}$ ; $\tau_b^+ = \inf\{t \in [0, \infty) : Y_t \geq b\}$	$\tau_b^- = \inf\{t \in (0, \infty) : U_t < b\}$ ; $\tau_b^+ = \inf\{t \in (0, \infty) : U_t > b\}$
$\varphi_v =$ smallest root of $\tilde{p}(\xi)/\xi = v^{-1}$ (in $\xi \in (0, 1]$ )	$\Phi(q) =$ largest root of $\psi(\lambda) - q$ (in $\lambda \in [0, \infty)$ ); $e^{-h\Phi(q)} = \varphi_{\frac{\gamma}{\gamma+q}}$	$\Phi(q) =$ largest root of $\psi(\lambda) - q$ (in $\lambda \in [0, \infty)$ )
$E_x[v^{\tau_b^+}; \tau_b^+ < \infty] = \varphi_v^{b-x}$	$E_x[e^{-q\tau_b^+}; \tau_b^+ < \infty] = e^{-\Phi(q)(b-x)}$	
$\sum_{y=0}^{\infty} z^y W_v(y) = \frac{1}{\tilde{p}(z) - \frac{z}{v}}$ , $z \in (0, \varphi_v)$	$\int_0^{\infty} e^{-\beta y} W^{(q)}(y) dy = \frac{e^{\beta h} - 1}{\beta h(\psi(\beta) - q)}$ , $\beta \in (\Phi(q), \infty)$ ; $W^{(q)}$ càd & constant on each interval $[x, x+h]$ ; $W^{(q)}(x) = \frac{1}{\gamma h} W_{\frac{\gamma}{\gamma+q}}(x/h)$	$\int_0^{\infty} e^{-\beta y} W^{(q)}(y) dy = \frac{1}{\psi(\beta) - q}$ , $\beta \in (\Phi(q), \infty)$ ; $W^{(q)}$ continuous on $[0, \infty)$
$E_x[v^{\tau_b^+}; \tau_b^+ < \tau_{-1}^-] = \frac{W_v(x)}{W_v(b)}$	$E_x[e^{-q\tau_b^+}; \tau_b^+ < \tau_{-h}^-] = \frac{W^{(q)}(x)}{W^{(q)}(b)}$	$E_x[e^{-q\tau_b^+}; \tau_b^+ < \tau_0^-] = \frac{W^{(q)}(x)}{W^{(q)}(b)}$
$P_x(\tau_{-1}^- < \infty) = 1 - W_1(x)(1 - \tilde{p}'(1-) \wedge 1)$	$P_x(\tau_{-h}^- < \infty) = 1 - W^{(0)}(x)(\psi'(0+) \vee 0)$	$P_x(\tau_0^- < \infty) = 1 - W^{(0)}(x)(\psi'(0+) \vee 0)$
$W_v(x) = 0$ , $x < 0$ ; $W_v(0) = 1/p_0$	$W^{(q)}(x) = 0$ , $x < 0$ ; $W^{(q)}(0) = 1/(h\lambda(\{h\}))$	$W^{(q)}(x) = 0$ , $x < 0$ ; $W^{(q)}(0) = 0$ if $U$ has unbounded variation, $= 1/\delta$ o/w
$\lim_{y \rightarrow \infty} \frac{W_v(y) \varphi_v^{y+1}}{1 - v \tilde{p}'(\varphi_v^-)} =$	$\lim_{y \rightarrow \infty} W^{(q)}(y) e^{-\Phi(q)(y+h)} = 1/\psi'(\Phi(q)+)$	$\lim_{y \rightarrow \infty} W^{(q)}(y) e^{-\Phi(q)(y)} = 1/\psi'(\Phi(q)+)$
$Z_v(x) = 1 + (\frac{1}{v} - 1) \sum_{y=0}^{x-1} W_v(y)$ , $x \geq 0$	$Z^{(q)}(x) = 1 + q \sum_{k=0}^{x/h-1} W^{(q)}(kh)$ , $x \geq 0$ ; $Z^{(q)}(x) = Z_{\frac{\gamma}{\gamma+q}}(x/h)$	$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy$ , $x \geq 0$
$Z_v(x) = 1$ , $x \leq 0$	$Z^{(q)}(x) = 1$ , $x \leq 0$	
$\sum_{y=0}^{\infty} z^y Z_v(y) = \frac{\tilde{p}(z) - z}{(1-z)(\tilde{p}(z) - \frac{z}{v})}$ , $z \in (0, \varphi_v)$	$\int_0^{\infty} Z^{(q)}(y) e^{-\beta y} dy = \frac{\psi(\beta)}{\beta(\psi(\beta) - q)}$ , $\beta \in (\Phi(q), \infty)$	
$E_x[v^{\tau_{-1}^-}; \tau_{-1}^- < \tau_b^+] = Z_v(x) - \frac{W_v(x)}{W_v(b)} Z_v(b)$	$E_x[e^{-q\tau_{-h}^-}; \tau_{-h}^- < \tau_b^+] = Z^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(b)} Z^{(q)}(b)$	$E_x[e^{-q\tau_0^-}; \tau_0^- < \tau_b^+] = Z^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(b)} Z^{(q)}(b)$
$E_x[v^{\tau_{-1}^-}; \tau_{-1}^- < \infty] = Z_v(x) - \frac{\varphi_v(1-v)}{v(1-\varphi_v)} W_v(x)$ , $v < 1$	$E_x[e^{-q\tau_{-h}^-}; \tau_{-h}^- < \infty] = Z^{(q)}(x) - \frac{qh}{e^{\Phi(q)h} - 1} W^{(q)}(x)$ , $q > 0$	$E_x[e^{-q\tau_0^-}; \tau_0^- < \infty] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x)$ , $q > 0$
$v^{m \wedge \tau_{-1}^-} W_v(X_{m \wedge \tau_{-1}^-})$ a martingale in $m \in \mathbb{N}_0$	$e^{-q(t \wedge \tau_{-h}^-)} W^{(q)}(X_{t \wedge \tau_{-h}^-})$ a martingale in $t \in [0, \infty)$	$e^{-q(t \wedge \tau_0^- \wedge \tau_b^+)} W^{(q)}(X_{t \wedge \tau_0^- \wedge \tau_b^+})$ a martingale in $t \in [0, \infty)$

**Remark 49.** Every spectrally negative Lévy process may be seen as a (weak) limit of a net  $Y^h$  of upwards skip-free Lévy chains, as  $h \downarrow 0$  [MVJ15]. This means that a great many relations in the

*spectrally negative Lévy setting may be got (at least naively) by simply passing to the limit  $h \downarrow 0$  (formally, one must of course pay attention to whether or not the relevant functional is continuous with respect to such a weak limit).*

**Remark 50.** *One of the important contributions of having a unified  $\Phi, W, Z$  theory developed in all the three settings featuring in the table above, is that whenever a result is available for one of them, it may often be simply “guessed” in the others, by “translating” one set of quantities into the other (though ultimately it still needs to be proved). We have seen this time and again in the results of this paper.*

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