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Nonlinear elliptic equations with measure valued absorption potentials

Nicolas Saintier* Laurent Véron†

Abstract We study the semilinear elliptic equation $-\Delta u + g(u)\sigma = \mu$ with Dirichlet boundary conditions in a smooth bounded domain where σ is a nonnegative Radon measure, μ a Radon measure and g is an absorbing nonlinearity. We show that the problem is well posed if we assume that σ belongs to some Morrey class. Under this condition we give a general existence result for any bounded measure provided g satisfies a subcritical integral assumption. We study also the supercritical case when $g(r) = |r|^{q-1}r$, with $q > 1$ and μ satisfies an absolute continuity condition expressed in terms of some capacities involving σ .

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Key words: Radon measures; Morrey class; capacities; potential estimates; θ -regular measures .

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2 boundary, σ a nonnegative Radon measure in Ω and $g : \mathbb{R} \rightarrow \mathbb{R}$ a continuous function satisfying, for some $r_0 \geq 0$,

$$rg(r) \geq 0 \quad \text{for all } r \in (-\infty, -r_0] \cup [r_0, \infty). \tag{1.1}$$

In this article we consider the following problem

$$\begin{aligned} -\Delta u + g(u)\sigma &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega, \end{aligned} \tag{1.2}$$

where μ is a Radon measure defined in Ω . By a solution we mean a function $u \in L^1(\Omega)$ such that $\rho g(u) \in L^1_\sigma(\Omega)$, where $\rho(x) = \text{dist}(x, \partial\Omega)$ and $L^1_\sigma(\Omega)$ is the Lebesgue space of functions integrable with respect to σ , satisfying

$$-\int_\Omega u \Delta \zeta dx + \int_\Omega g(u) \zeta d\sigma = \int_\Omega \zeta d\mu, \tag{1.3}$$

for all $\zeta \in W_0^{1,\infty}(\Omega)$ such that $\Delta \zeta \in L^\infty(\Omega)$. In the sequel, such a solution is called a *very weak solution*. A measure μ such that the problem admits a solution is called a *good measure*. We emphasize on the particular cases where $g(r) = |r|^{q-1}r$ with $q > 0$, or $g(r) = e^{\alpha r} - 1$ with $\alpha > 0$ and $N = 2$.

When σ is a measure with constant positive density with respect to the Lebesgue measure in \mathbb{R}^N , this problem has been initiated by Brezis and Benilan [4], [5] who gave a general existence result for any bounded measure μ under an integrability condition of g at infinity; their proof is based upon an a priori estimate of approximate solutions u_n in Lorentz spaces $L^{q,\infty}(\Omega)$, yielding the uniform integrability of $g(u_n)$ and hence the pre-compactness in $L^1(\Omega)$. If $g(r) = |r|^{q-1}r$, integrability condition is fulfilled if and only if $0 < q < \frac{N}{N-2}$ (any $q > 0$ if $N = 2$). In the 2-dim case the integrability condition have been replaced by the exponential order of growth of g in [27]. When $g(u) = |u|^{q-1}u$ with $q \geq \frac{N}{N-2}$ not any bounded measure is eligible for solving (1.2). In fact Baras and Pierre [3] proved that when $N > 2$ and $q > 1$, a bounded Radon measure μ is eligible if and only if it vanishes on Borel sets with $c_{2,q'}$ -capacity zero, where $q' = \frac{q}{q-1}$ is the conjugate exponent of q . Contrary to the previous subcritical case, the method for proving the necessity of this condition is based upon a duality-convexity argument, while the sufficiency uses the fact that any positive Radon measure absolutely continuous with respect to the $c_{2,q}$ -capacity can be approximated from below by a non-decreasing sequence of positive measures in $W^{-2,q}(\Omega)$ (see [13]). Furthermore they also

gave a necessary and sufficient condition for a compact subset $K \subset \Omega$ to be removable for equation

$$-\Delta u + |u|^{q-1} u = 0 \quad \text{in } \Omega \setminus K, \quad (1.4)$$

namely that $c_{2,q'}(K) = 0$.

The aim of this paper is to extend the previous constructions of Benilan-Brezis, Baras-Pierre and Vazquez to the case where σ is a general measure. In order to be able to deal with the convergence of approximate solutions we assume that σ belongs to the Morrey class $\mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ for some $\theta \in [0, N]$ which means

$$|B_r(x)|_\sigma := \int_{B_r(x)} d\sigma \leq cr^\theta \quad \text{for all } (x, r) \in \Omega \times (0, \infty), \quad (1.5)$$

for some $c > 0$. Note that we extend σ by 0 in $\mathbb{R}^N \setminus \Omega$ and slightly abuse notation putting $\frac{N}{N-\theta} = \infty$ when $\theta = N$.

Our first result is the following:

Theorem A *Assume $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ for some $\theta \in (N-2, N]$ and that g satisfies (1.1). Then, for any $\mu \in L_\rho^1(\Omega)$, there exists a very weak solution u of problem (1.3). If we assume moreover that g is nondecreasing and if u' is a very weak solution of (1.3) with right-hand side $\mu' \in L_\rho^1(\Omega)$, then the following estimates hold*

$$-\int_\Omega |u - u'| \Delta \zeta dx + \int_\Omega |g(u) - g(u')| \zeta d\sigma \leq \int_\Omega |\mu - \mu'| dx, \quad (1.6)$$

and

$$-\int_\Omega (u - u')_+ \Delta \zeta dx + \int_\Omega (g(u) - g(u'))_+ \zeta d\sigma \leq \int_\Omega (\mu - \mu')_+ dx \quad (1.7)$$

for all $\zeta \in W_0^{1,\infty}(\Omega)$ such that $\Delta \zeta \in L^\infty(\Omega)$ and $\zeta \geq 0$.

Note that (1.6) implies the uniqueness of the solution of (1.3), that we denote by u_μ , and (1.7) the monotonicity of the mapping $\mu \mapsto u_\mu$.

The next result extends Benilan-Brezis unconditional existence result for measures.

Theorem B *Let $N > 2$ and $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ with $N \geq \theta > N - \frac{N}{N-1}$. Assume that g satisfies (1.1) and $|g(r)| \leq \tilde{g}(|r|)$ for all $|r| \geq r_0$ where \tilde{g} is a continuous nondecreasing function on $[r_0, \infty)$ verifying*

$$\int_{r_0}^\infty \tilde{g}(t) t^{-1-\frac{\theta}{N-2}} dt < \infty. \quad (1.8)$$

Then, for any bounded Radon measure μ , there exists a very weak solution u of problem (1.3) which moreover belongs to $L^1_\sigma(\Omega)$. Moreover, if we assume that g is nondecreasing then the solution is unique.

Note that we recover Benilan-Brezis result when σ is the Lebesgue measure (so that $\theta = N$). Note also that when $g(r) = |r|^{q-1}r$, the integrability condition (1.8) is fulfilled if and only if $0 < q < \frac{\theta}{N-2}$.

In the 2-dimensional case the condition on θ is $2 \geq \theta > 0$ but (1.8) has to be modified. If $f : \mathbb{R} \mapsto \mathbb{R}_+$ is nondecreasing we define its exponential order of growth at ∞ (see [27]) by

$$a_\infty(f) = \inf \left\{ \alpha \geq 0 : \int_0^\infty f(s)e^{-\alpha s} ds < \infty \right\}. \quad (1.9)$$

Similarly, if $h : \mathbb{R} \mapsto \mathbb{R}_-$ is nondecreasing its exponential order of growth at $-\infty$ is

$$a_{-\infty}(h) = \sup \left\{ \alpha \leq 0 : \int_{-\infty}^0 h(s)e^{\alpha s} ds > -\infty \right\}. \quad (1.10)$$

If $g : \mathbb{R} \mapsto \mathbb{R}$ satisfies (1.1) but is not necessarily nondecreasing, we define the monotone nondecreasing hull g^* of g by

$$g^*(r) = \begin{cases} \sup\{g(s) : s \leq r\} & \text{for all } r \geq r_0 \\ 0 & \text{for all } r \in (-r_0, r_0) \\ \inf\{g(s) : s \geq r\} & \text{for all } r \leq -r_0. \end{cases} \quad (1.11)$$

We set

$$a_\infty(g) = a_\infty(g^*_+) \quad \text{and} \quad a_{-\infty}(g) = a_{-\infty}(g^*_-). \quad (1.12)$$

Theorem C Let $\sigma \in \mathcal{M}^+_{\frac{2}{2-\theta}}(\Omega)$ with $2 \geq \theta > 0$ and $g : \mathbb{R} \mapsto \mathbb{R}$ satisfies (1.1).

(I) If $a_\infty(g) = 0 = a_{-\infty}(g)$, then for any $\mu \in \mathfrak{M}_b(\Omega)$, problem (1.3) admits a very weak solution.

(II) If $0 < a_\infty(g) < \infty$ and $-\infty < a_{-\infty}(g) < 0$ there exists $\delta > 0$ such that if $\mu \in \mathfrak{M}_b(\Omega)$ satisfies $\|\mu\|_{\mathfrak{M}_b} \leq \delta$ problem (1.3) admits a very weak solution.

In the *supercritical case*, that is when (1.8) is not satisfied, all the measures are not eligible for solving (1.3). Following [16], [28, Th 4.2] we can give a sufficient existence condition involving the Green function of the Laplacian. Let $G(\cdot, \cdot)$ be the Green kernel defined in $\Omega \times \Omega$ and $\mathbb{G}[\cdot]$ the corresponding potential operator acting on bounded measures ν namely $\mathbb{G}[\nu](x) = \int_\Omega G(x, y) d\nu(y)$. We have the following result:

Theorem D Let $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ with $N \geq \theta > N - \frac{N}{N-1}$ and assume that g is nondecreasing and vanishes at 0.

(I) If $\mu \in \mathfrak{M}_b(\Omega)$ satisfies

$$\rho g(\mathbb{G}[|\mu|]) \in L_\sigma^1(\Omega), \quad (1.13)$$

then problem (1.3) admits a unique very weak solution.

(II) Let $\mu = \mu_r + \mu_s$ where μ_r is absolutely continuous with respect to the Lebesgue measure and μ_s is singular. Assume that g satisfies the Δ_2 condition, namely that

$$|g(r + r')| \leq a(|g(r)| + |g(r')|) + b \quad \text{for all } r, r' \in \mathbb{R}, \quad (1.14)$$

for some $a > 1$ and $b \geq 0$. Then the previous assertion holds if (1.13) is replaced by

$$\rho g(\mathbb{G}[|\mu_s|]) \in L_\sigma^1(\Omega). \quad (1.15)$$

Notice that (1.13) holds if either (i) σ and μ have disjoint support, or (ii) $\mu \in \mathcal{M}_p(\Omega)$ for some $p > \frac{N}{2}$. Indeed if (i) holds then $\mathbb{G}[|\mu|]$ is bounded pointwise on the support of σ , and if (ii) holds then by Lemma 2.2 $\mathbb{G}[|\mu|]$ is bounded pointwise in Ω . Obviously the same comment holds in the setting of II.

In order to make more explicit conditions (1.13), (1.15), we introduce the following growth assumption on g :

$$|g(r)| \leq c(1 + |r|^q) \quad \text{for all } r \in \mathbb{R}, \quad (1.16)$$

for some $q > 1$. Notice that $\tilde{g}(r) = 1 + r^q$ satisfies (1.8) if and only if $q < \frac{\theta}{N-2}$. When σ is the Lebesgue measure and $g(r) = |r|^{q-1}r$, Baras and Pierre [3] gave a necessary and sufficient condition for the existence of a solution to (1.2) involving certain capacities associated to the Bessel potential spaces $H^{s,p}(\mathbb{R}^N)$ where $s \in \mathbb{R}$ and $p \in [1, \infty]$. Let us recall that

$$H^{s,p}(\mathbb{R}^N) = \{f : f = \mathbf{G}_s * h, h \in L^p(\mathbb{R}^N)\}, \quad (1.17)$$

where \mathbf{G}_s is the Bessel kernel of order s . By extension $\mathbf{G}_0 = \delta_0$, hence $H^{s,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$. When s is a positive integer, it is proved by Calderón [2, Theorem 1.2.3] that $H^{s,p}(\mathbb{R}^N)$ is the standard Sobolev space $W^{s,p}(\mathbb{R}^N)$. If $s > 0$, we denote by $c_{s,p}$ the associated capacity, called the Bessel capacity. It is defined for any compact set $K \subset \mathbb{R}^N$ by

$$c_{s,p}(K) = \inf \{ \|\phi\|_{H^{s,p}}^p : \phi \in \mathcal{S}(\mathbb{R}^N), \phi \geq 1 \text{ on } K \}. \quad (1.18)$$

The definition of $c_{s,p}$ is then extended first to open sets and then to arbitrary sets. We refer to [2] for general properties of Bessel spaces and their associated capacities

$c_{s,p}$. We say that a measure $\mu \in \mathfrak{M}_b(\Omega)$ is absolutely continuous with respect to the $c_{s,p}$ -capacity if for any Borel subset $E \subset \mathbb{R}^N$,

$$c_{s,p}(E) = 0 \implies |\mu|(E) = 0.$$

Baras and Pierre's result states that equation (1.2), with σ standing for the Lebesgue measure and $g(r) = |r|^{q-1}r$, has a solution if and only if μ is absolutely continuous with respect to the $c_{2,q'}$ -capacity. The next result generalizes the "if" part to the case where σ belongs to some Morrey space.

Theorem E *Let $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ with $N \geq \theta > N - \frac{N}{N-1}$ and assume that g is nondecreasing and satisfies (1.1) and (1.16). Let $p > 1$ and $s \geq 0$ such that $N > sp > N - \theta$ and $\frac{\theta p}{N-sp} \geq q$. If $\mu \in \mathfrak{M}_b(\Omega)$ is absolutely continuous with respect to the $c_{2-s,p'}$ -capacity, then (1.2) admits a unique very weak solution.*

As a particular case, we take $p = q$ and obtain that if μ is absolutely continuous with respect to the $c_{2-\frac{N-\theta}{q},q'}$ -capacity, then (1.3) admits a unique solution. We thus recover Baras-Pierre's sufficient condition [3] when $\theta = N$.

We give an explicit condition on the measure μ in terms of Morrey spaces implying that it satisfies the conditions of Theorem E.

Proposition 1.1 *Under the assumptions on σ and g of Theorem E, if $\mu \in \mathcal{M}_{\frac{N}{N-\theta^*}}(\Omega)$ for some $\theta^* > \frac{(N-2)q-\theta}{q-1}$, then (1.3) admits a unique very weak solution.*

Notice that the condition on μ given in Proposition 1.1 is weaker than the one given after Theorem D.

When $g(r) = |r|^{q-1}r$ with $q > 1$, one can find a necessary conditions for the existence of a solution of (1.3) in the supercritical case under additional regularity assumptions on σ . By [2, Def 2.3.3, Prop. 2.3.5], the following expression

$$c_q^\sigma(E) = \inf \left\{ \int_{\Omega} |v|^{q'} d\sigma : v \in L_{\sigma}^{q'}(\Omega), v \geq 0, \mathbb{G}[v\sigma] \geq 1 \text{ on } E \right\}, \quad (1.19)$$

where E is any subset of Ω defines an outer capacity. The measure is called θ -regular if

$$\frac{1}{c} r^\theta \leq \int_{B_r(x)} d\sigma \leq c r^\theta \quad \text{for all } (x, r) \in \Omega \times (0, 1],$$

The next result gives a necessary condition for a measure to be a good measure.

Theorem F *Let $q > 1$ and $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ be θ -regular with $N \geq \theta > N - 2$. If $\mu \in \mathfrak{M}_b^+(\Omega)$ is such that problem (1.3) with $g(r) = |r|^{q-1}r$ admits a very weak solution, then μ vanishes on any Borel set E such that $c_q^\sigma(E) = 0$.*

Furthermore the c_q^σ - capacity admits the following representation in terms of Besov capacities. If $\Gamma \subset \Omega$ is the support of σ , we denote by $B_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(\Omega)$ the closed subspace of distributions $\zeta \in B_{q',\infty}^{2-\frac{N-\theta}{q}}(\Omega)$ such that the support of the distribution $\Delta\zeta$ is a subset of Γ . Then

$$c_q^\sigma(K) \sim c_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(K) := \inf \left\{ \|\zeta\|_{B_{q',\infty}^{2-\frac{N-\theta}{q}}}^{q'} : \zeta \in B_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(\Omega), \zeta \geq \chi_K \right\}, \quad (1.20)$$

for all compact subset $K \subset \Omega$.

Finally a complete characterization of removable sets can be obtained under a much stronger assumption on σ , namely that $d\sigma = w dx$ with $\omega := w^{-\frac{1}{q-1}} \in L_{loc}^1(\Omega)$. If $K \subset \Omega$ is compact, we set

$$c_q^\omega(K) = \inf \left\{ \int_{\Omega} |\Delta\zeta|^{q'} \omega dx : \zeta \in C_0^\infty(\Omega), 0 \leq \zeta \leq 1, \zeta = 1 \text{ in a neighborhood of } K \right\}. \quad (1.21)$$

This defines a capacity on Borel sets of Ω .

Theorem G. *Assume $q > 1$ and there exists a nonnegative Borel function w in Ω in the Muckenhoupt class $A_q(\Omega)$ such that $d\sigma = w dx$. If $K \subset \Omega$ is compact, a function $u \in L_{loc}^1(\Omega \setminus K)$ such that $|u|^q w \in L_{loc}^1(\Omega \setminus K)$ which satisfies*

$$-\Delta u + w |u|^{q-1} u = 0, \quad (1.22)$$

in the sense of distributions in $\Omega \setminus K$ can be extended as a solution of the same equation in the whole Ω if and only if $c_{q,w}(K) = 0$.

The assumption $w \in A_q(\Omega)$ can be weakened and replaced by $\omega = w^{\frac{1}{1-q}}$ is q' -admissible in the sense of [15, Chap 1], a condition which implies in particular the validity of the Gagliardo-Nirenberg and the Poincaré inequalities.

2 Preliminaries

In the whole paper c denotes a generic positive constant whose value can change from one occurrence to another even within a single string of estimates. Sometimes, in order to avoid ambiguity, we are led to introduce other notations for constant, for example c' .

We denote by $\mathfrak{M}_b(\Omega)$ the space of outer regular bounded Borel measures on Ω equipped with the total variation norm, and by $\mathfrak{M}_b^+(\Omega)$ its positive cone. Since Ω is

bounded we can identify bounded Radon measures in Ω with measures μ in $\overline{\Omega}$ such that $|\mu|(\partial\Omega) = 0$. All the measures are extended by 0 in $\mathbb{R}^N \setminus \Omega$.

Let $G(.,.)$ be the Green kernel defined in $\Omega \times \Omega$ and $\mathbb{G}[\cdot]$ the corresponding potential operator acting on bounded measures ν namely $\mathbb{G}[\nu](x) = \int_{\Omega} G(x, y) d\nu(y)$. We denote $L^{p,\infty}(\Omega)$ the usual weak L^p space. The next result is classical and valid in a much more general setting (see e.g. [6], [11]).

Lemma 2.1 *Let $\mu \in \mathfrak{M}_b(\Omega)$ and $v = \mathbb{G}[\mu]$ be the (very weak) solution of*

$$\begin{aligned} -\Delta v &= \mu && \text{in } \Omega \\ v &= 0 && \text{in } \partial\Omega. \end{aligned} \quad (2.1)$$

I- If $N \geq 2$, then $v \in L^{\frac{N}{N-2},\infty}(\Omega)$, $\nabla v \in L^{\frac{N}{N-1},\infty}(\Omega)$ and

$$\|v\|_{L^{\frac{N}{N-2},\infty}} + \|\nabla v\|_{L^{\frac{N}{N-1},\infty}} \leq c \|\mu\|_{\mathfrak{M}_b}. \quad (2.2)$$

II- If $N = 2$, then $v \in BMO(\Omega)$, $\nabla v \in L^{2,\infty}(\Omega)$ and

$$\|v\|_{BMO} + \|\nabla v\|_{L^{2,\infty}} \leq c \|\mu\|_{\mathfrak{M}_b}. \quad (2.3)$$

This result can be refined when more information is available on the degree of concentration of μ . This leads to the definition of Morrey spaces of measures.

2.1 Morrey spaces of measures

If $1 \leq p \leq \infty$ we define the Morrey space $\mathcal{M}_p(\Omega)$ as the set of bounded outer regular Borel measures μ defined in Ω and extended by 0 in Ω^c , satisfying

$$|B_r(x)|_{\mu} := \int_{B_r(x)} d|\mu| \leq cr^{N(1-\frac{1}{p})} \quad \text{for all } (x, r) \in \Omega \times \mathbb{R}_+, \quad (2.4)$$

for some $c > 0$. In particular $\mu \in \mathcal{M}_{\frac{N}{N-\theta}}(\Omega)$, $\theta \in [0, N]$, if

$$\int_{B_r(x)} d|\mu| \leq cr^{\theta} \quad \text{for all } (x, r) \in \Omega \times \mathbb{R}_+.$$

We refer to [19] for a detailed study of $\mathcal{M}_p(\Omega)$ and full proofs of the various results we will recall now. Endowed with the norm

$$\|\mu\|_{\mathcal{M}_p} = \sup_{(x,r) \in \Omega \times \mathbb{R}_+} r^{N(\frac{1}{p}-1)} |B_r(x)|_{\mu}, \quad (2.5)$$

$\mathcal{M}_p(\Omega)$ is a Banach space and $\mathcal{M}_p^+(\Omega) = \mathcal{M}_p(\Omega) \cap \mathfrak{M}_b^+(\Omega)$ is its positive cone. We also set $M_p(\Omega) = \mathcal{M}_p(\Omega) \cap L_{loc}^1(\Omega)$; it is a closed subspace of $\mathcal{M}_p(\Omega)$ and, if $1 < p < \infty$, the following imbedding holds

$$L^p(\Omega) \hookrightarrow L^{p,\infty}(\Omega) \hookrightarrow M_p(\Omega). \quad (2.6)$$

Note that since Ω is bounded and any measure in Ω is extended to \mathbb{R}^N by 0, it is easily seen that if $1 \leq q \leq p \leq \infty$ we have a continuous embedding $\mathcal{M}_p(\Omega) \hookrightarrow \mathcal{M}_q(\Omega)$ with

$$\|v\|_{\mathcal{M}_q} \leq (\text{diam}(\Omega))^{\frac{N}{q} - \frac{N}{p}} \|v\|_{\mathcal{M}_p} \quad \text{for all } v \in \mathcal{M}_p(\Omega). \quad (2.7)$$

Indeed for any $x \in \Omega$ the ball centered at x with radius $\text{diam}(\Omega)$ contains Ω so that it is enough to consider $r \leq \text{diam}(\Omega)$. We have

$$r^{-N(1-1/q)} |B_r(x)|_\mu \leq r^{-N(1-1/q)} \|\mu\|_{\mathcal{M}_p} r^{N(1-1/p)} \leq (\text{diam}(\Omega))^{\frac{N}{q} - \frac{N}{p}} \|\mu\|_{\mathcal{M}_p}.$$

The following imbedding inequalities holds.

Lemma 2.2 *Let $\mu \in \mathcal{M}_p(\Omega)$ and v be the solution of (2.1).*

I- If $1 < p < \frac{N}{2}$, then $v \in M_q(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{2}{N}$ and there holds

$$\|v\|_{\mathcal{M}_q} \leq c \|\mu\|_{\mathcal{M}_p}. \quad (2.8)$$

II- If $p > \frac{N}{2}$, then v is bounded pointwise and

$$\begin{aligned} (i) \quad & v(x) \leq c \|\mu\|_{\mathcal{M}_p} \quad \text{for all } x \in \Omega, \\ (ii) \quad & \sup_{x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\alpha} \leq c \|\mu\|_{\mathcal{M}_p} \quad \text{with } \alpha = 2 - \frac{N}{p} \quad \text{if } N > p > \frac{N}{2}, \\ (iii) \quad & \sup_{x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\alpha} \leq c \|\mu\|_{\mathcal{M}_p} \quad \text{with } \alpha \in (0, 1) \quad \text{if } N = p, \\ (iv) \quad & \sup_x |\nabla v(x)| \leq c \|\mu\|_{\mathcal{M}_p} \quad \text{if } N < p. \end{aligned} \quad (2.9)$$

Remark. The previous regularity results are proved in [19, Prop. 3.1, 3.5] when $v = I_\alpha * \mu$ where I_α is the Riesz potential. However it is easily seen that the proof in [19] can be adapted to our setting. In particular for (2.8) we need that $G(x, y) \leq c|x - y|^{2-N}$, for (i) we use (2.7).

Remark. If we assume that $\mu \in \mathfrak{M}_\rho(\Omega) \cap \mathcal{M}_{p,loc}(\Omega)$, the previous estimates acquire a local aspect and remain valid provided the supremum in the norms on the left-hand sides are taken on compact subsets of Ω .

2.2 Trace embeddings

Some applications of Morrey spaces to imbedding theorems (also called trace inequalities) can be found in Adams-Hedberg's book [2]. For the sake of completeness, we quote here the main result therein we will use in the sequel. If $0 < \alpha < N$ we recall that I_α (resp. G_α) is the Riesz potential (resp. the Bessel potential) of order α in \mathbb{R}^N . The next result is [2, Th 7.2.2, 7.3.2] (recall that the $c_{I_\alpha, p}$ -Riesz capacity of a ball $B_r(x)$ is proportional to $r^{N-\alpha p}$ - see [2, Prop. 5.1.2].)

Proposition 2.3 *Let σ be a nonnegative Radon measure in \mathbb{R}^N , $N > \alpha p$ and $1 < p < q < \frac{Np}{N-\alpha p}$.*

(I)- *The following assertions are equivalent:*

$$\|I_\alpha * f\|_{L_\sigma^q(\mathbb{R}^N)} \leq c_1 \|f\|_{L^p(\mathbb{R}^N)} \quad \text{for all } f \in L^p(\mathbb{R}^N), \quad (2.10)$$

for some $c_1 = c_1(N, \alpha, p, q) > 0$, and

$$\sigma \in \mathcal{M}_r(\mathbb{R}^N) \quad \text{with} \quad \frac{1}{r} = q \left(\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{N} \right). \quad (2.11)$$

(II)- *The mapping $f \mapsto G_\alpha * f$ is continuous from $L^p(\mathbb{R}^N)$ to $L_\sigma^q(\mathbb{R}^N)$ if and only if*

$$\sigma(K)^{\frac{1}{q}} \leq c_2 (c_{\alpha, p}(K))^{\frac{1}{p}} \quad \text{for all } K \subset \mathbb{R}^N, \quad (2.12)$$

where $c_{\alpha, p}$ denotes the Bessel capacity of order α defined in (1.18). In fact this holds if and only if

$$\sigma(B_r(x)) \leq c_3 (c_{\alpha, p}(B_r(x)))^{q/p} \quad \text{for all } x \in \mathbb{R}^N, 0 < r \leq 1. \quad (2.13)$$

(III)- *A necessary and sufficient condition in order the mapping $f \mapsto G_\alpha * f$ be compact from $L^p(\mathbb{R}^N)$ to $L_\sigma^q(\mathbb{R}^N)$ is*

$$\begin{aligned} (i) \quad & \lim_{\delta \rightarrow 0} \sup_{x \in \mathbb{R}^N, r \leq \delta} \frac{\sigma(B_r(x))}{(c_{\alpha, p}(B_r(x)))^{\frac{q}{p}}} = 0 \\ (ii) \quad & \lim_{|x| \rightarrow \infty} \sup_{r \leq 1} \frac{\sigma(B_r(x))}{(c_{\alpha, p}(B_r(x)))^{\frac{q}{p}}} = 0. \end{aligned} \quad (2.14)$$

If \mathbb{R}^N is replaced by a smooth bounded set Ω , we extend any bounded Radon measure in Ω by zero in Ω^c . In view of [2, 5.6.1] the $c_{I_\alpha, p}$ -Riesz capacity and $c_{\alpha, p}$ -Bessel capacity of balls $B_r(x)$ with $x \in \Omega$ and $r \leq 1$ are then equivalent. It follows that $c_{\alpha, p}(B_r(x)) \simeq r^{N-\alpha p}$. Then, it follows from II and III above, the definition of $H^{\alpha, p}(\mathbb{R}^N)$ and the existence of an extension operator $H^{\alpha, p}(\Omega) \hookrightarrow H^{\alpha, p}(\mathbb{R}^N)$ that the following holds,

Proposition 2.4 Under the assumptions of Proposition 2.3, the embedding $H^{\alpha,p}(\Omega) \hookrightarrow L^q_\sigma(\Omega)$ is:

(I)- continuous if and only if $(\sigma(K))^{\frac{1}{q}} \leq c_2 (c_{\alpha,p}(K))^{\frac{1}{p}}$ for all $K \subset \mathbb{R}^N$, i.e. if and only if $\sigma \in \mathcal{M}_r^+(\mathbb{R}^N)$ with $\frac{1}{r} = q \left(\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{N} \right)$.

(II)- compact if and only if

$$\limsup_{r \rightarrow 0} \sup_{x \in \Omega} \frac{\sigma(B_r(x))}{r^{\frac{(N-\alpha p)q}{p}}} = 0. \quad (2.15)$$

As an immediate corollary,

Proposition 2.5 Let $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$, i.e. $\sigma(B_r(x)) \leq cr^\theta$, $N > \alpha p$ and $1 < p < q < \frac{Np}{N-\alpha p}$. Then the embedding

$$H^{\alpha,p}(\Omega) \hookrightarrow L^q_\sigma(\Omega), \quad (2.16)$$

is continuous iff $\sigma(K) \leq c_1 (c_{\alpha,p}(K))^{\frac{q}{p}}$ for all $K \subset \mathbb{R}^N$ which holds iff $q \leq \frac{\theta p}{N-\alpha p}$. And the embedding (2.16) is compact iff $q < \frac{\theta p}{N-\alpha p}$.

Other trace inequalities can be found in [21]. In the case $N = \alpha p$ the following estimate holds, see e.g. [1], [20, Corollary 8.6.2], [31].

Proposition 2.6 Let σ be a nonnegative Radon measure in \mathbb{R}^N with compact support and $N = \alpha p$, $p > 1$. Then there exists a constant $b = b(N, \alpha, p) > 0$ such that

$$\sup_{\|f\|_{L^p} \leq 1} \int_{\mathbb{R}^N} \exp\left(b |G_\alpha * f|^{p'}\right) d\sigma < \infty \quad (2.17)$$

if and only if $\sigma \in \mathcal{M}_\tau^+(\mathbb{R}^N)$ for some $\tau \in (1, \infty)$.

When $p = 1$ the next result is proved in [20, Sec 1.4.3]

Proposition 2.7 Let σ be a nonnegative bounded Radon measure in \mathbb{R}^N , α be an integer such that $1 \leq \alpha \leq N$ and $q \geq 1$. Then the following estimate holds

$$\|f\|_{L^q_\sigma} \leq c_2 \sum_{|\beta|=\alpha} \|D^\beta f\|_1 \quad \text{for all } f \in C_0^\infty(\mathbb{R}^N), \quad (2.18)$$

for some $c_2 = c_2(N, p, q, \alpha) > 0$ if and only if $\sigma \in \mathcal{M}_{\frac{N}{N-q(N-\alpha)}}^+(\mathbb{R}^N)$.

3 The subcritical case

3.1 The variational construction

We prove in this section that if $\mu \in W^{-1,2}(\Omega)$ then, under some assumptions on g and σ , equation (1.2) has a variational solution.

We assume that $g \in C(\mathbb{R})$ satisfies (1.1), and set $G(r) := \int_0^r g(s)ds$. We will find a solution to (1.2) minimizing the functional

$$J(v) := \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx + \int_{\Omega} G(v) d\sigma - \langle \mu, v \rangle, \quad (3.1)$$

over the set

$$X_G(\Omega) := \{v \in W_0^{1,2}(\Omega) : G(v) \in L^1_{\sigma}(\Omega)\}. \quad (3.2)$$

The next proposition is a variant of a result in [8].

Proposition 3.1 *Assume $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > \frac{N}{2} - 1$. If $\mu \in W^{-1,2}(\Omega)$ there exists $u \in X_G(\Omega)$ which minimizes J in $X_G(\Omega)$. Furthermore u is a weak solution of (1.2) in the sense that*

$$\int_{\Omega} \nabla u \cdot \nabla \zeta dx + \int_{\Omega} g(u) \zeta d\sigma = \langle \mu, \zeta \rangle \quad \text{for all } \zeta \in C_0^{\infty}(\Omega). \quad (3.3)$$

If g is nondecreasing this solution is unique and denoted by u_{μ} , and the mapping $\mu \mapsto u_{\mu}$ is nonincreasing.

Proof. Step 1: Existence of a minimizer. If $N > 2$ we apply (2.16) with $\alpha = 1$ and $p = 2$, recalling that by Fourier transform $H^{1,2}(\Omega) = W^{1,2}(\Omega)$ (it is a special case of Calderón's theorem), to obtain that

$$W_0^{1,2}(\Omega) \hookrightarrow L^{\frac{2\theta}{N-2}}_{\sigma}(\Omega). \quad (3.4)$$

If $N = 2$ with $p = 2$ we take any $\alpha < 1$ and obtain

$$\|f\|_{L^{\frac{\theta}{1-\alpha}}_{\sigma}} \leq c_1 \|f\|_{W^{\alpha,2}} \leq c'_1 \|f\|_{W^{1,2}}. \quad (3.5)$$

According to Proposition 2.5 the imbedding of $W_0^{1,2}(\Omega)$ into $L^p_{\sigma}(\Omega)$ is compact for any $p \in [1, \frac{2\theta}{N-2})$ if $N > 2$ and $1 \leq p < \infty$ if $N = 2$.

Let us first assume that g is bounded. Then $|G(v)| \leq m|v|$. Since g is continuous, $G(v) \in L^1_{\sigma}(\Omega)$ for any $v \in W_0^{1,2}(\Omega)$ and the functional J is well defined and is of class C^1 in $W_0^{1,2}(\Omega)$. Furthermore

$$\lim_{\|v\|_{W^{1,2}} \rightarrow \infty} J(v) = +\infty. \quad (3.6)$$

Let $\{u_n\}$ be a minimizing sequence. By (3.6), $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and thus relatively compact in $L_\sigma^1(\Omega)$ and in $L^2(\Omega)$. Hence there exist $u \in L^2(\Omega)$ and $v \in L_\sigma^1(\Omega)$ such that, up to a subsequence, $u_n \rightarrow v$ in $L_\sigma^1(\Omega)$, and $u_n \rightarrow u$ strongly in $L^2(\Omega)$ and weakly in $W_0^{1,2}(\Omega)$. We can also assume that $u_n \rightarrow u$ $c_{1,2}$ -quasi almost everywhere in the sense that there exists $E \subset \Omega$ with $c_{1,2}(E) = 0$ such that $u_n(x) \rightarrow u(x)$ for any $x \in \Omega \setminus E$. According to Proposition 2.5, σ is absolutely continuous with respect to the $c_{1,2}$ -capacity. It follows that $\sigma(E) = 0$ so that $u_n \rightarrow u$ σ -almost everywhere and thus $u = v$ σ -almost everywhere. Thus we have that $u_n \rightarrow u$ in $L^2(\Omega)$, in $L_\sigma^1(\Omega)$, σ -almost everywhere and weakly in $W_0^{1,2}(\Omega)$. Then we have that $\langle \mu, u_n \rangle \rightarrow \langle \mu, u \rangle$. By the dominated convergence theorem we have also that $G(u_n) \rightarrow G(u)$ in $L_\sigma^1(\Omega)$. Therefore

$$J(u) \leq \liminf_{n \rightarrow \infty} J(u_n), \quad (3.7)$$

which implies that u is a minimizer of J in $W_0^{1,2}(\Omega)$.

If g is unbounded, we write $g = g_1 + g_2$ where $g_1 = g\chi_{(-r_0, r_0)}$, $g_2 = g\chi_{(-\infty - r_0] \cup [r_0, \infty)}$, where r_0 is defined in (1.1). Hence $G(r) = G_1(r) + G_2(r)$ where $|G_1(r)| \leq m|r|$ and $G_2(r)$ is nonnegative. Using again (2.14) we obtain that (3.6) holds. A minimizing sequence $\{u_n\}$ inherits the same property as above, hence $u_n \rightarrow u$ σ -almost everywhere in Ω and in $L_\sigma^1(\Omega)$, this implies that $G_1(u_n) \rightarrow G_1(u)$ in $L_\sigma^1(\Omega)$ and $G_2(u)$ is σ -measurable. By Fatou's lemma

$$\int G_2(u) d\sigma \leq \liminf_{n \rightarrow \infty} \int G_2(u_n) d\sigma,$$

which implies that (3.7) holds. Notice that, among the consequences, X_G is closed subset of $W_0^{1,2}(\Omega)$. Hence u is a minimizer of J in $X_G(\Omega)$.

Uniqueness holds if g is nondecreasing since it implies that J is strictly convex and actually X_G is a closed convex set.

Step 2: The minimizer is a weak solution. For $k > r_0$ we define g_k by

$$g_k(r) = \begin{cases} g(r) & \text{if } |r| \leq k \\ g(k) & \text{if } r > k \\ g(-k) & \text{if } r < -k \end{cases}$$

Then g_k is continuous and bounded and the minimizer $u_k \in W_0^{1,2}(\Omega)$ of

$$J_k(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx + \int_\Omega G_k(v) d\sigma - \langle \mu, v \rangle \quad \text{where } G_k(r) = \int_0^r g_k(s) ds,$$

is a weak solution (i.e. in the sense given by (3.3)) of

$$\begin{aligned} -\Delta u + g_k(u)\sigma &= \mu & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (3.8)$$

The following energy estimate holds

$$\int_{\Omega} |\nabla u_k|^2 dx + \int_{\Omega} u_k g_k(u_k) d\sigma = \langle \mu, u_k \rangle \leq \|\mu\|_{W^{-1,2}} \|u_k\|_{W^{1,2}}, \quad (3.9)$$

and it implies

$$\int_{\Omega} |\nabla u_k|^2 dx + \int_{\Omega} |u_k g_k(u_k)| d\sigma \leq \|\mu\|_{W^{-1,2}}^2 + m\sigma(\Omega) = M, \quad (3.10)$$

for some $m = m(r_0) > 0$. Up to a subsequence, $\{u_k\}_k$ converges to some u as $k \rightarrow \infty$, weakly in $W_0^{1,2}(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω . By Proposition 2.4 the imbedding of $W^{1,2}(\Omega)$ in $L_{\sigma}^q(\Omega)$ is compact for any $q < \frac{2\theta}{N-2}$. Hence the subsequence can be taken such that $u_k \rightarrow u$, σ -almost everywhere as $k \rightarrow \infty$, and consequently $g_k(u_k) \rightarrow g(u)$ σ -almost everywhere. Let $E \subset \Omega$ be a Borel set, then for any $\lambda > r_0$,

$$\begin{aligned} M &\geq \int_E |g_k(u_k)u_k| d\sigma \\ &= \int_{E \cap \{|u_k| > \lambda\}} |g_k(u_k)u_k| d\sigma + \int_{E \cap \{|u_k| \leq \lambda\}} |g_k(u_k)u_k| d\sigma \\ &\geq \lambda \int_{E \cap \{|u_k| > \lambda\}} |g_k(u_k)| d\sigma + \int_{E \cap \{|u_k| \leq \lambda\}} |g_k(u_k)u_k| d\sigma. \end{aligned}$$

Therefore

$$\begin{aligned} \int_E |g_k(u_k)| d\sigma &= \int_{E \cap \{|u_k| > \lambda\}} |g_k(u_k)| d\sigma + \int_{E \cap \{|u_k| \leq \lambda\}} |g_k(u_k)| d\sigma \\ &\leq \frac{M}{\lambda} + \max\{|g(r)| : |r| \leq \lambda\} \sigma(E) \end{aligned}$$

For $\epsilon > 0$ we first choose λ such that $\frac{M}{\lambda} \leq \frac{\epsilon}{2}$ and then $\sigma(E) \leq \frac{\epsilon}{1 + 2 \max\{|g(r)| \leq \lambda\}}$. This implies the uniform integrability of $\{g_k(u_k)\}_k$ in $L_{\sigma}^1(\Omega)$. Hence $g_k(u_k) \rightarrow g(u)$ in $L_{\sigma}^1(\Omega)$ by Vitali's convergence theorem. Since u_k is a weak solution of (3.8), there holds for any $\zeta \in C_0^{\infty}(\Omega)$,

$$\int_{\Omega} \nabla u_k \cdot \nabla \zeta dx + \int_{\Omega} g_k(u_k) \zeta d\sigma = \langle \mu, \zeta \rangle. \quad (3.11)$$

Letting $k \rightarrow \infty$ we obtain, using the above convergence results,

$$-\int_{\Omega} \nabla u \cdot \nabla \zeta dx + \int_{\Omega} g(u) \zeta d\sigma = \langle \mu, \zeta \rangle. \quad (3.12)$$

Hence u is a weak solution. If g is monotone, uniqueness is also a consequence of the weak formulation. Furthermore if μ, μ' belong to $W^{-1,2}(\Omega)$ are such that $\mu - \mu'$ is a nonnegative measure, then $\langle \mu' - \mu, (u'_{\mu} - u_{\mu})_{+} \rangle \leq 0$. Taking $(u'_{\mu} - u_{\mu})_{+}$ for test function in the weak formulation yields $(u'_{\mu} - u_{\mu})_{+} = 0$. \square

3.2 The L^1 case

In the sequel we set

$$\mathbb{X}(\Omega) = \{\zeta \in C^1(\overline{\Omega}), \zeta = 0 \text{ on } \partial\Omega \text{ and } \Delta\zeta \in L^\infty(\Omega)\}, \quad (3.13)$$

and $\mathbb{X}_+(\Omega) = \mathbb{X}(\Omega) \cap \{\zeta \in C^1(\overline{\Omega}) : \zeta \geq 0 \text{ in } \overline{\Omega}\}$. We recall (see e.g. [29]) that if $f \in L^1_\rho(\Omega)$ and $u \in L^1(\Omega)$ is a very weak solution of

$$-\Delta u = f \quad \text{in } \Omega, \quad (3.14)$$

there holds

$$-\int_\Omega |u| \Delta\zeta dx \leq \int_\Omega f \text{sign}(u)\zeta dx \quad \text{for all } \zeta \in \mathbb{X}_+(\Omega), \quad (3.15)$$

and

$$-\int_\Omega u^+ \Delta\zeta dx \leq \int_\Omega f \text{sign}_+(u)\zeta dx \quad \text{for all } \zeta \in \mathbb{X}_+(\Omega). \quad (3.16)$$

Proposition 3.2 *Assume $N \geq 2$, $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N - 2$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nondecreasing function vanishing at 0. If $\mu \in L^1_\rho(\Omega)$ there exists a unique $u := u_\mu \in L^1(\Omega)$ very weak solution of (1.2). Furthermore, if $u_\mu, u_{\mu'} \in L^1(\Omega)$ are the very weak solutions of (1.2) with right-hand sides $\mu, \mu' \in L^1_\rho(\Omega)$, then*

$$-\int_\Omega |u_\mu - u_{\mu'}| \Delta\zeta dx + \int_\Omega |g(u_\mu) - g(u_{\mu'})| \zeta d\sigma \leq \int_\Omega (\mu - \mu') \text{sign}(u_\mu - u_{\mu'}) \zeta dx, \quad (3.17)$$

and

$$-\int_\Omega (u_\mu - u_{\mu'})_+ \Delta\zeta dx + \int_\Omega (g(u_\mu) - g(u_{\mu'}))_+ \zeta d\sigma \leq \int_\Omega (\mu - \mu') \text{sign}_+(u_\mu - u_{\mu'}) \zeta dx \quad (3.18)$$

for any $\zeta \in \mathbb{X}_+(\Omega)$. In particular the mapping $\mu \rightarrow u_\mu$ is nondecreasing.

The following result will be used several time in the sequel. Its proof is standard but we present it for the sake of completeness.

Lemma 3.3 *Assume $N > q \geq 1$ and $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}$ with $N \geq \theta > N - q$. Then σ vanishes on any Borel set with $c_{1,q}$ -capacity zero.*

Proof. It suffices to prove the result when E is compact. We define the Λ_θ Hausdorff measure of a set E by

$$\Lambda_\theta(E) = \lim_{\kappa \rightarrow 0} \Lambda_\theta^\kappa(E) := \lim_{\kappa \rightarrow 0} \inf \left\{ \sum_{j=1}^{\infty} r_j^\theta : 0 < r_j \leq \kappa \leq \infty, E \subset \bigcup_{j=1}^{\infty} B_{r_j}(a_j) \right\}. \quad (3.19)$$

Note that $\Lambda_\theta^\infty(E)$ is the Hausdorff content of E and it is smaller than $(\text{diam}(E))^\theta$. For any covering of E by balls $B_{r_j}(a_j)$, $j \geq 1$, we have

$$\sigma(E) \leq \sum_{j=1}^{\infty} \sigma(B_{r_j}(a_j)) \leq \|\sigma\|_{\frac{N}{N-\theta}} \sum_{j=1}^{\infty} r_j^\theta.$$

It follows that

$$\sigma(E) \leq \|\sigma\|_{\frac{N}{N-\theta}} \Lambda_\theta(E).$$

Next, if $c_{1,q}(E) = 0$ then $\Lambda_\theta(E) = 0$ according to [2, Th. 5.1.13], and thus $\sigma(E) = 0$ by the previous inequality. \square

We introduce the flow coordinates near $\partial\Omega$ defined by

$$\Pi(x) = (\rho(x), \tau(x)) \in [0, \epsilon_0] \times \partial\Omega \quad \text{where } \tau(x) = \text{proj}_{\partial\Omega}(x).$$

It is well-known that for ϵ_0 small enough, Π is a C^1 -diffeomorphism from $\Omega_{\epsilon_0} := \{x \in \overline{\Omega} : \rho(x) \leq \epsilon_0\}$ to $[0, \epsilon_0] \times \partial\Omega$. With this diffeomorphism we can assimilate the surface measure dS_ϵ on $\Sigma_\epsilon = \{x \in \Omega : \rho(x) = \epsilon\}$ with the surface measure dS on $\Sigma_0 = \partial\Omega$ by setting

$$\int_{\Sigma_\epsilon} v(x) dS_\epsilon(x) = \int_{\Sigma_0} v(\epsilon, \tau) dS(\tau).$$

Lemma 3.4 *Assume $N \geq 2$ and $\mu \in \mathfrak{M}(\Omega)$ satisfies*

$$\int_{\Omega} \rho d|\mu| < \infty. \tag{3.20}$$

Then $u = \mathbb{G}[\mu]$ satisfies

$$\lim_{\epsilon \rightarrow 0} \int_{\Sigma_0} |u|(\epsilon, \tau) dS(\tau) = 0. \tag{3.21}$$

Proof. If $u = \mathbb{G}[\mu]$, it is the unique weak solution of $-\Delta u = \mu$ in Ω , $u = 0$ on $\partial\Omega$. Hence $u = u_1 - u_2$ where $u_1 = \mathbb{G}[\mu^+]$ and $u_2 = \mathbb{G}[\mu^-]$. Since μ_+ and μ_- satisfy the integrability condition (3.20) both u_1 and u_2 have a zero measure boundary trace (M -boundary trace in the sense of [18, Sec 1.3]). Hence, taking for test function the function $\zeta = 1$,

$$\lim_{\epsilon \rightarrow 0} \int_{\Sigma_0} u_j(\epsilon, \tau) dS(\tau) = 0, \tag{3.22}$$

which implies (3.20). \square

This result allows us to obtain the uniqueness of the solution even if the right-hand side is a measure.

Lemma 3.5 *Assume $N \geq 2$, $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ with $N \geq \theta > N - 2$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nondecreasing function. If $\mu \in \mathfrak{M}(\Omega)$ there exists at most one very weak solution of (1.2).*

Proof. By Lemma 3.3 with $\alpha = 1$, $p = 2$, σ is absolutely continuous with respect to the $c_{1,2}$ capacity (it is diffuse in the terminology of [9]), and if $h \in L^1_\sigma(\Omega)$ the measure $h_+\sigma$, which is the increasing limit of $\inf\{n, h_+\}\sigma$ is also diffuse. Similarly $h_-\sigma$ is diffuse and so is $h\sigma$. Next we assume that u and u' are two very weak solutions of (1.2) and set $w = u - u'$. Hence

$$-\Delta w + (g(u) - g(u'))\sigma = 0.$$

Since $\rho(g(u) - g(u')) \in L^1_\sigma(\Omega)$, it follows from Lemma 3.4 that

$$\lim_{\epsilon \rightarrow 0} \int_{\Sigma_\epsilon} |w|(\epsilon, \tau) dS(\tau) = 0$$

We use Kato inequality for measures as in [10, Th 1.1]: Since $w \in L^1(\Omega)$, Δw^+ is a diffuse measure and

$$\Delta w^+ \geq \chi_{\{w \geq 0\}} \Delta w = \chi_{\{w \geq 0\}} (g(u) - g(u'))\sigma \geq 0 \text{ in } \Omega$$

Since w^+ has a M-boundary trace by Lemma 3.4, we can apply [18, Lemmma 1.5.8] with $\mu = -\chi_{\{w \geq 0\}} (g(u) - g(u'))\sigma$ which is a measure in $\mathfrak{M}_\rho(\Omega) := \{\nu \in \mathfrak{M}(\Omega) : \rho\nu \in \mathfrak{M}_b(\Omega)\}$. Then there exists $\tau \in \mathfrak{M}_\rho^+(\Omega)$ such that

$$-\Delta w^+ = \mu - \tau.$$

Equivalently

$$-\Delta w^+ + \chi_{\{w \geq 0\}} (g(u) - g(u'))\sigma = -\tau.$$

Since the M-boundary trace of w^+ is zero, it follows that $w^+ = -\mathbb{G}[\chi_{\{w \geq 0\}} (g(u) - g(u'))\sigma + \tau]$. Hence $w^+ = 0$ and $u \leq u'$. Similarly $u' \leq u$. \square

The following variant will be useful in the sequel.

Lemma 3.6 *Assume $N \geq 2$, $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ with $N \geq \theta > N - 2$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nondecreasing function. If $u, u' \in L^1(\Omega)$ are such that $\rho g(u)$ and $\rho g(u')$ belong to $L^1_\sigma(\Omega)$ and satisfy*

$$-\int_\Omega (u - u') \Delta \zeta dx + \int_\Omega (g(u) - g(u')) \zeta d\sigma = \int_\Omega \zeta d\nu \quad \text{for all } \zeta \in \mathbb{X}_+(\Omega) \quad (3.23)$$

for some $\nu \in \mathfrak{M}_+(\Omega)$ diffuse with respect to the $c_{1,2}$ -capacity, then $u \geq u'$ $c_{1,2}$ -quasi everywhere in Ω .

Proof. We use Kato's inequality, Lemma 3.4 and [18, Lemma 1.5.8] in the same way as in the proof of Lemma 3.5 since the measures $(g(u) - g(u'))d\sigma$ and ν are diffuse, $\Delta(u' - u)$ is diffuse, hence

$$\Delta(u' - u)_+ \geq \chi_{\{u' \geq u\}} \Delta(u' - u) = (g'(u) - g(u))\chi_{\{u' \geq u\}} + \chi_{\{u' \geq u\}}\nu \geq 0$$

Since $u' - u \in W_0^{1,q}(\Omega)$ for any $1 < q < \frac{N}{N-1}$, we conclude that $(u' - u)_+ = 0$ almost everywhere and $c_{1,2}$ -quasi everywhere by [2, Th 6.1.4]. \square

The next result and the corollary which follows are the key-stone for the proof of Proposition 3.2.

Lemma 3.7 *Let $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ with $N \geq \theta > N - 2$, $h \in L^\infty(\Omega)$, $f \in L^s(\Omega)$ with $s > \frac{N}{2}$ and $w \in L^1(\Omega)$ be the very weak solution of*

$$\begin{aligned} -\Delta w + h\sigma &= f && \text{in } \Omega \\ w &= 0 && \text{in } \partial\Omega. \end{aligned} \quad (3.24)$$

Then w is continuous in $\bar{\Omega}$ and for any nondecreasing bounded function $\gamma \in C^2(\mathbb{R})$ vanishing at 0, there holds

$$-\int_{\Omega} j(w)\Delta\zeta dx + \int_{\Omega} \gamma(w)h\zeta d\sigma \leq \int_{\Omega} \gamma(w)\zeta f dx \quad \text{for all } \zeta \in \mathbb{X}_+(\Omega), \quad (3.25)$$

where $j(r) = \int_0^r \gamma(s)ds$.

Proof. The solution is unique and expressed by $w = \mathbb{G}[f - h\sigma]$. Since $\frac{N}{N-\theta} > \frac{N}{2}$, $w \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0, 1)$ by Lemma 2.2. Hence $\gamma(w)$ is continuous and therefore measurable. We extend σ by zero in Ω^c and denote $\sigma_n = \sigma * \eta_n$ where $\{\eta_n\}$ is a sequence of mollifiers. Then $\sigma_n \rightarrow \sigma$ in the narrow topology of Ω . For $n \in \mathbb{N}^*$, let w_n be the solution of

$$\begin{aligned} -\Delta w_n + h\sigma_n &= T_n(f) && \text{in } \Omega \\ w_n &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (3.26)$$

where $T_n(f) = \min\{|f|, n\}\text{sgn}(f)$. Then $w_n \in W^{2,s}(\Omega) \cap W_0^{1,\infty}(\Omega)$ for all $1 < s < \infty$. By Green's formula

$$-\int_{\Omega} j(w_n)\Delta\zeta dx + \int_{\Omega} \gamma(w_n)h\zeta d\sigma \leq \int_{\Omega} \gamma(w_n)\zeta f dx \quad \text{for all } \zeta \in \mathbb{X}_+(\Omega). \quad (3.27)$$

Since $w_n \rightarrow w$ uniformly in $\bar{\Omega}$, (3.25) follows. \square

Corollary 3.8 *Under the assumptions of Lemma 3.7, there holds*

$$-\int_{\Omega} |w| \Delta \zeta dx + \int_{\Omega} \text{sign}_0(w) h \zeta d\sigma \leq \int_{\Omega} \text{sign}_0(w) \zeta f dx, \quad (3.28)$$

and

$$-\int_{\Omega} w_+ \Delta \zeta dx + \int_{\Omega} \text{sign}_+(w) \zeta h d\sigma \leq \int_{\Omega} \text{sign}_+(w) \zeta f dx, \quad (3.29)$$

for any $\zeta \in \mathbb{X}_+(\Omega)$. Moreover there exists a constant $C > 0$ depending only on Ω such that

$$\int_{\Omega} \text{sign}_0(w) h d\sigma \leq C \int_{\Omega} |f| dx. \quad (3.30)$$

Proof. For proving (3.28) we consider a sequence $\{\gamma_k\}$ of odd nondecreasing functions such that

$$\gamma_k(r) = \begin{cases} 1 & \text{if } r \geq 2k^{-1} \\ 0 & \text{if } -k^{-1} \leq r \leq k^{-1} \\ -1 & \text{if } r \leq -2k^{-1} \end{cases}$$

and such that $\{r\gamma_k(r)\}$ is nondecreasing for any r . Using γ_k in place of γ in (3.25) we obtain

$$-\int_{\Omega} j_k(w) \Delta \zeta dx + \int_{\Omega} \gamma_k(w) \zeta h d\sigma \leq \int_{\Omega} \gamma_k(w) \zeta f dx \quad \text{for all } \zeta \in \mathbb{X}_+(\Omega), \quad (3.31)$$

where $j_k(r) = \int_0^r \gamma_k(s) ds$. Since $\gamma_k(w) \uparrow w$ on $\Omega_+ := \{x \in \Omega : w(x) > 0\}$, there holds by the monotone convergence theorem,

$$\int_{\Omega_+} \gamma_k(w) \zeta |h| d\sigma \uparrow \int_{\Omega_+} w \zeta |h| d\sigma \quad \text{as } k \rightarrow \infty.$$

Since

$$\left| \int_{\Omega_+} (w - \gamma_k(w)) \zeta h d\sigma \right| \leq \int_{\Omega_+} |(w - \gamma_k(w)) \zeta h| d\sigma = \int_{\Omega_+} (w - \gamma_k(w)) \zeta |h| d\sigma,$$

we obtain

$$\int_{\Omega_+} \gamma_k(w) h \zeta d\sigma \rightarrow \int_{\Omega_+} w h \zeta d\sigma \quad \text{as } k \rightarrow \infty.$$

Similarly, $\gamma_k(w) \downarrow w$ on $\Omega_- := \{x \in \Omega : w(x) < 0\}$ so that

$$\int_{\Omega_-} \gamma_k(w) h \zeta d\sigma \rightarrow \int_{\Omega_-} w h \zeta d\sigma \quad \text{as } k \rightarrow \infty.$$

Combining these two results yields

$$\int_{\Omega} \gamma_k(w) \zeta h d\sigma \rightarrow \int_{\Omega_+} w \zeta h d\sigma - \int_{\Omega_-} w \zeta h d\sigma = \int_{\Omega} \text{sign}_0(w) \zeta h d\sigma.$$

Using dominated convergence theorem there holds

$$\int_{\Omega} \gamma_k(w) \Delta \zeta dx \rightarrow \int_{\Omega} \text{sign}_0(w) \Delta \zeta dx,$$

and

$$\int_{\Omega} \gamma_k(w) \zeta f dx \rightarrow \int_{\Omega} \text{sign}_0(w) \zeta f dx.$$

This implies (3.28). The proof of (3.17) is similar.

Eventually we prove (3.30). Let η_1 be the solution of

$$\begin{aligned} -\Delta \eta_1 &= 1 && \text{in } \Omega \\ \eta_1 &= 0 && \text{in } \partial\Omega. \end{aligned} \quad (3.32)$$

Then $\eta_1 = \mathbb{G}[1] \in \mathbb{X}_+(\Omega)$ and there exists $c, c' > 0$ depending only on Ω such that $c\rho \leq \eta_1 \leq c'\rho$. Given $\alpha \in (0, 1]$, let $j_\epsilon(r) = (r + \epsilon)^\alpha - \epsilon^\alpha$, $r \geq 0$, and $\zeta = j_\epsilon(\eta_1)$. Note that $\zeta \in C^2(\bar{\Omega})$, $0 \leq \zeta \leq \eta_1^\alpha$, $\zeta = 0$ on $\partial\Omega$, $j'_\epsilon > 0$, $j''_\epsilon < 0$, so that $-\Delta \zeta = j'_\epsilon(\eta_1) - j''_\epsilon(\eta_1) |\nabla \eta_1|^2 \geq 0$. We deduce from (3.28) that

$$\int_{\Omega} \text{sign}_0(w) (\eta + \epsilon)^\alpha h d\sigma \leq \int_{\Omega} \text{sign}_0(w) \eta^\alpha |f| dx + \epsilon^\alpha \int_{\Omega} \text{sign}_0(w) h d\sigma.$$

We obtain

$$\int_{\Omega} \text{sign}_0(w) \rho^\alpha h d\sigma \leq C \int_{\Omega} \rho^\alpha |f| dx + \epsilon^\alpha |\tilde{\sigma}(\Omega)|$$

Letting $\epsilon \rightarrow 0$ and then $\alpha \rightarrow 0$ we infer the result by dominated convergence. \square

We are now in position to prove Proposition 3.2.

Proof of Proposition 3.2. We divide the proof into several steps.

Step 1: We assume that $\mu \in L^\infty(\Omega)$. Let $\{\eta_n\}$ be a sequence of mollifiers and $\sigma_n = \sigma * \eta_n$. If $\mu \in L^\infty(\Omega)$, the solution $u_n = u_{n,\mu}$ of

$$\begin{aligned} -\Delta u_n + g(u_n) \sigma_n &= \mu && \text{in } \Omega \\ u_n &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (3.33)$$

is continuous in $\bar{\Omega}$. Since

$$-\mathbb{G}[\mu^-] \leq -u_n^- \leq 0 \leq u_n^+ \leq \mathbb{G}[\mu^+] \quad (3.34)$$

by the maximum principle, the sequence $\{u_n\}$ is uniformly bounded. Recalling that g is nondecreasing we have that the sequence $\{g(u_n)\}$ is also uniformly bounded in Ω , hence $g(u_n)\sigma_n$ is bounded in $\mathcal{M}_{\frac{N}{N-\theta}}(\Omega)$ independently of n , and from (2.9) it follows that u_n is bounded in $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1]$ independently of n . Up to some subsequence, $\{u_n\}$, and thus also $\{g(u_n)\}$, are then uniformly convergent in $\overline{\Omega}$ with limit $u = u_\mu$ and $g(u) = g(u_\mu)$. Because $\sigma * \eta_n$ converges to σ in the narrow topology, u_μ is a very weak solution of (1.2). Notice that being continuous, $g(u)$ is measurable for the measure σ . By Lemma 3.5, u_μ is the unique solution of (1.2), hence the whole sequence $\{u_{\mu_n}\}$ converges to u_μ . Applying Corollary 3.8 with $w = u$, $\tilde{\sigma} = \sigma$ and $\zeta = \eta_1$ yields

$$\int_{\Omega} |u| dx + \int_{\Omega} |g(u)| \eta_1 d\sigma \leq \int_{\Omega} |\mu| \eta_1 dx, \quad (3.35)$$

and (3.29) with $\zeta = \eta_1$ gives

$$\int_{\Omega} (u - u')_+ dx + \int_{\Omega} (g(u) - g(u'))_+ \eta_1 d\sigma \leq \int_{\Omega} \eta_1 \text{sign}_+(u - u') (\mu - \mu')_+ dx. \quad (3.36)$$

which implies the monotonicity of the mapping $\mu \mapsto u_\mu$.

Step 2: We assume that $\mu \in L^1(\Omega)$ is bounded from below. Set $\ell = \text{ess inf } \mu$. For $k > 0$ set $\mu_k = \min\{k, \mu\}$ and $u_k := u_{\mu_k} \in L^\infty(\Omega)$. The sequence $\{\mu_k\}$ is nondecreasing, hence according to Step 1, the sequence $\{u_k\}$ is a nondecreasing sequence of continuous functions in $\overline{\Omega}$ bounded from below by $\ell\eta_1$, where η_1 is defined in (3.32). Its pointwise limit, denoted by u , is thus lower semicontinuous. Moreover $g(u_k) \rightarrow g(u)$ pointwise, hence $g(u)$ is lower semicontinuous and thus σ -measurable. Relation (3.35) applied to μ_k and u_k gives

$$\int_{\Omega} |u_k| dx + \int_{\Omega} |g(u_k)| \eta_1 d\sigma \leq \int_{\Omega} |\mu_k| \eta_1 dx.$$

Passing to the limit using Fatou's lemma in the left-hand side and the dominated convergence theorem in the right-hand side yields

$$\int_{\Omega} |u| dx + \int_{\Omega} |g(u)| \eta_1 d\sigma \leq \int_{\Omega} |\mu| \eta_1 dx. \quad (3.37)$$

We deduce that $u \in L^1(\Omega)$ and $\rho g(u) \in L^1_\sigma(\Omega)$. We have indeed a more precise result. Since g vanishes at 0 $g(u_k) = g(u_k^+) + g(-u_k^-)$. Hence $\rho g(u_k^+) \rightarrow \rho g(u^+)$ in $L^1_\sigma(\Omega)$ by the monotone convergence theorem. Furthermore $g(-u_k^-) \leq g(-u_k^-) \leq 0$, which implies that $\rho g(-u_k^-) \rightarrow \rho g(-u^-)$ in $L^1_\sigma(\Omega)$ by the dominated convergence theorem which finally implies that $\rho g(u_k) \rightarrow \rho g(u)$ in $L^1_\sigma(\Omega)$. Using $\zeta \in \mathbb{X}_+(\Omega)$ as a

test function in the very weak formulation of the equation satisfied by u_k gives

$$-\int_{\Omega} u_k \Delta \zeta dx + \int_{\Omega} g(u_k) \zeta d\sigma = \int_{\Omega} \zeta \mu_k dx.$$

Since $u_k \rightarrow u$ almost everywhere and $-l\eta_1 \leq u_k \leq u$ with $u \in L^1(\Omega)$, we can pass to the limit in the first term to obtain $\int_{\Omega} u_k \Delta \zeta dx \rightarrow \int_{\Omega} u \Delta \zeta dx$. Because $|\mu_k| \leq |\mu| \in L^1(\Omega)$ and $\mu_k \rightarrow \mu$ almost everywhere, we can also pass to the limit in the last term: $\int_{\Omega} \zeta \mu_k dx \rightarrow \int_{\Omega} \zeta \mu dx$. It remains to pass to the limit in the nonlinearity. Because $u_k \uparrow u$ and g is nondecreasing, we have $g(u_k) \uparrow g(u)$. Thus by the monotone convergence theorem,

$$-\int_{\Omega} u \Delta \zeta dx + \int_{\Omega} g(u) \zeta d\sigma = \int_{\Omega} \zeta \mu dx,$$

and u is very weak solution of (1.2).

Step 3: We assume that $\mu \in L^1(\Omega)$. For $\ell \in \mathbb{R}$, we set $\mu^\ell = \sup\{\mu, \ell\}$ and denote by u^ℓ the solution of (1.2) with right-hand side μ^ℓ . Note that the sequence $\{\mu^\ell\}_\ell$ is increasing, bounded from above by μ^+ so that $u^\ell \leq u_{\mu^+}$, where u_{μ^+} is the solution of (1.2) with right-hand side μ^+ which exists according to the previous step, and the sequence $\{u^\ell\}_\ell$ is monotone nondecreasing with ℓ with pointwise limit u when $\ell \rightarrow -\infty$. Hence $u \leq u^\ell \leq u_{\mu^+}$ for any $\ell \leq 0$. The sequence $\{g(u^\ell)\}_\ell$ is monotone nondecreasing with limit $g(u)$ when $\ell \rightarrow -\infty$, and there holds $g(u) \leq g(u^\ell) \leq g(u_{\mu^+})$ for any $\ell \leq 0$. Since $g(u^\ell)$ is lower semicontinuous and σ -measurable, $g(u)$ shares the same properties.

Applying (3.37) to $\mu = \mu^\ell$ and $u = u^\ell$ gives

$$\int_{\Omega} |u^\ell| dx + \int_{\Omega} |g(u^\ell)| \eta_1 d\sigma \leq \int_{\Omega} |\mu^\ell| \eta_1 dx.$$

Passing to the limit in the left-hand side using Fatou's lemma we obtain

$$\int_{\Omega} |u| dx + \int_{\Omega} |g(u)| \eta_1 d\sigma \leq \int_{\Omega} |\mu| \eta_1 dx.$$

We deduce that $u \in L^1(\Omega)$ and $\rho g(u) \in L^1_\sigma(\Omega)$. We conclude as in Step 2 that u is solution of (1.2).

Step 4: Proof of (3.17) and (3.18).

For $\ell < 0 < k$ we set $\mu_k^\ell = \sup\{\ell, \inf\{k, \mu\}\}$ and $(\mu')_k^\ell = \sup\{\ell, \inf\{k, \mu'\}\}$, and denote by u_k^ℓ and $(u')_k^\ell$ the solution of (1.2) with right-hand side μ_k^ℓ and $(\mu')_k^\ell$. Then, by Corollary 3.8, for any $\zeta \in \mathbb{X}(\Omega)$ there holds

$$-\int_{\Omega} |u_k^\ell - (u')_k^\ell| \Delta \zeta dx + \int_{\Omega} |g(u_k^\ell) - g((u')_k^\ell)| \zeta d\sigma \leq \int_{\Omega} \text{sign}_0(u_k^\ell - (u')_k^\ell) (\mu_k^\ell - (\mu')_k^\ell) \zeta dx.$$

Using the previous convergence theorem when $k \rightarrow \infty$ and then $\ell \rightarrow -\infty$, we derive (3.17). The proof of (3.18) is similar. \square

Remark. If it is not assumed that g is nondecreasing, the above proof by monotonicity does not work. However the existence will follow from Theorem B if it is assumed that the extra assumptions in this theorem are satisfied: $\theta > N - q$ for some $q \in (1, \frac{N}{N-1})$ and the growth assumptions of Theorem B.

3.3 Diffuse case

We recall that a measure μ is said to be diffuse with respect to the $c_{s,p}$ -capacity defined in (1.18) if $|\mu|$ vanishes on all sets with zero $c_{s,p}$ -capacity. An important result due to Feyel and de la Pradelle [13] is the following:

Proposition 3.9 *Let $\alpha > 0$ and $1 < p < \infty$. If $\lambda \in \mathfrak{M}_b^+(\Omega)$ does not charge sets with zero $c_{\alpha,p}$ -capacity, there exists an increasing sequence $\{\lambda_n\} \subset H^{-\alpha,p'}(\Omega) \cap \mathfrak{M}_b^+(\Omega)$, λ_n with compact support in Ω , which converges to λ .*

Proposition 3.10 *Assume $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+$ with $N \geq \theta > N - 2$, and that $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nondecreasing function vanishing at 0. Then for any $\mu \in \mathfrak{M}_b^+(\Omega)$ diffuse with respect to the $c_{1,2}$ -capacity there exists a unique very weak solution u to (1.2).*

Proof. According to Proposition 3.9, there exists an increasing sequence of nonnegative measures $\{\mu_n\}$ belonging to $W^{-1,2}(\Omega)$ and converging to μ and by Proposition 3.1, $\{u_{\mu_n}\}$ is a nondecreasing sequence of weak solutions of (1.2) with $\mu = \mu_n$. We claim that $u_{\mu_n} \uparrow u_\mu$ which is a very weak solution of (1.2). There holds,

$$\int_{\Omega} u_{\mu_n} dx + \int_{\Omega} g(u_{\mu_n}) \eta_1 d\sigma = \int_{\Omega} \eta_1 d\mu_n \leq \int_{\Omega} \eta_1 d\mu,$$

where η_1 is defined in (3.32). Since $u_{\mu_n} \geq 0$, $u_{\mu_n} \uparrow u$ and $g(u_{\mu_n}) \uparrow g(u)$. Since u_{μ_n} is σ -measurable by Proposition 3.1, u is also σ -measurable. Hence $g(u)$ shares this measurability property since g is continuous. Hence, by the monotone convergence theorem

$$\int_{\Omega} u dx + \int_{\Omega} g(u) \eta_1 d\sigma = \int_{\Omega} \eta_1 d\mu. \quad (3.38)$$

Furthermore $u_{\mu_n} \rightarrow u$ in $L^1(\Omega)$. Indeed it suffices to show that $\{u_{\mu_n}\}$ is uniformly equiintegrable which follows from $0 \leq \int_{\omega} u_{\mu_n} dx \leq \int_{\omega} u dx$ and the fact that $u \in L^1(\Omega)$. We show in the same way that $\rho g(u_{\mu_n}) \rightarrow \rho g(u)$ in $L^1_{\sigma}(\Omega)$. This implies that $u = u_\mu$ is the very weak solution of (1.2). \square

3.4 Subcritical nonlinearities: proof of Theorem B.

Lemma 3.11 *Assume $N > 2$ and $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ with $N \geq \theta > N-2$. If $\mu \in \mathfrak{M}_b(\Omega)$ and $\lambda \geq 0$, we set $E_\lambda[\mu] := \{x \in \Omega : \mathbb{G}[|\mu|](x) > \lambda\}$. Then*

$$e_\lambda^\sigma(\mu) := \int_{E_\lambda[\mu]} d\sigma \leq c \|\mu\|_{\mathfrak{M}_b}^{\frac{\theta}{N-2}} \lambda^{-\frac{\theta}{N-2}} \quad \text{for all } \lambda > 0. \quad (3.39)$$

Proof. It suffices to prove the result if $\mu \geq 0$. Indeed since $\mathbb{G}[|\mu|] = \mathbb{G}[\mu^+] + \mathbb{G}[\mu^-]$, we have $E_\lambda[\mu] \subset E_{\lambda/2}[\mu^+] \cup E_{\lambda/2}[\mu^-]$ and thus $e_\lambda^\sigma(\mu) \leq e_{\lambda/2}^\sigma(\mu^+) + e_{\lambda/2}^\sigma(\mu^-)$. If the result holds for nonnegative measure, in particular for μ^\pm , then

$$\begin{aligned} \lambda^{\frac{\theta}{N-2}} e_\lambda^\sigma(\mu) &\leq c(\mu^+(\Omega)^{\frac{\theta}{N-2}} + \mu^-(\Omega)^{\frac{\theta}{N-2}}) \leq c(\mu^+(\Omega) + \mu^-(\Omega))^{\frac{\theta}{N-2}} \\ &= c \|\mu\|_{\mathfrak{M}_b}^{\frac{\theta}{N-2}}. \end{aligned}$$

Thus, we assume from now on that μ is nonnegative.

If $\mu = \delta_a$ for some $a \in \Omega$, then $\mathbb{G}[\delta_a](x) \leq c_N |x - a|^{2-N}$ so that $E_\lambda[\delta_a] \subset B_{(\frac{c_N}{\lambda})^{\frac{1}{N-2}}}(a)$. Since $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ it follows that

$$e_\lambda^\sigma(\delta_a) \leq c \lambda^{-\frac{\theta}{N-2}}. \quad (3.40)$$

Let $E \subset \Omega$ be a Borel set. For any given $t > 0$ there holds

$$\int_E \mathbb{G}[\delta_a] d\sigma = \int_{E \cap E_t[\delta_a]} \mathbb{G}[\delta_a] d\sigma + \int_{E \cap E_t^c[\delta_a]} \mathbb{G}[\delta_a] d\sigma.$$

Clearly $\int_{E \cap E_t^c[\delta_a]} \mathbb{G}[\delta_a] d\sigma \leq t\sigma(E)$ and

$$\int_{E \cap E_t[\delta_a]} \mathbb{G}[\delta_a] d\sigma \leq \int_{E_t[\delta_a]} \mathbb{G}[\delta_a] d\sigma \leq - \int_t^\infty s de_s^\sigma(\delta_a) \leq c \frac{\theta t^{1-\frac{\theta}{N-2}}}{\theta + 2 - N},$$

where the last inequality follows by integration by parts and the help of (3.40). Then

$$\int_E \mathbb{G}[\delta_a] d\sigma \leq t\sigma(E) + c \frac{\theta t^{1-\frac{\theta}{N-2}}}{\theta + 2 - N}.$$

Minimizing the right-hand side with respect to t , we infer

$$\int_E \mathbb{G}[\delta_a] d\sigma \leq c\sigma(E)^{1-\frac{N-2}{\theta}}. \quad (3.41)$$

We first suppose that $\mu = \sum_{j=1}^{\infty} \alpha_j \delta_{a_j}$ for some $\alpha_j > 0$ and $a_j \in \Omega$. In particular $\sum_{j=1}^{\infty} \alpha_j = \|\mu\|_{\mathfrak{M}^b}$. Using Fubini's theorem and (3.41) we see that for any Borel set $E \subset \Omega$,

$$\int_E \mathbb{G}[\mu](x) d\sigma(x) = \sum_{j=1}^{\infty} \alpha_j \int_E \mathbb{G}[\delta_{a_j}](x) d\sigma(x) \leq c\sigma(E)^{1-\frac{N-2}{\theta}} \|\mu\|_{\mathfrak{M}^b}. \quad (3.42)$$

Taking in particular $E = E_\lambda[\mu]$ we obtain

$$\lambda e_\lambda^\sigma(\mu) \leq \int_{E_\lambda[\mu]} \mathbb{G}[\mu](x) d\sigma(x) \leq c(e_\lambda^\sigma(\mu))^{1-\frac{N-2}{\theta}} \|\mu\|_{\mathfrak{M}^b},$$

which implies the claim. Notice that the constant c in the right-hand side depends only on N and $\|\sigma\|_{\mathcal{M}_{\frac{N}{N-\theta}}}$.

For a general nonnegative measure $\mu \in \mathfrak{M}_b(\Omega)$, we consider a sequence of nonnegative measures $\{\mu_n\} \subset \mathfrak{M}_b(\Omega)$ where each μ_n is a sum of Dirac masses as before and such that $\mu_n \rightarrow \mu$ weakly as $n \rightarrow \infty$. Then we have

$$e_\lambda^\sigma(\mu_n) := \int_{E_\lambda[\mu_n]} d\sigma \leq c \|\mu_n\|_{\mathfrak{M}_b}^{\frac{\theta}{N-2}} \lambda^{-\frac{\theta}{N-2}},$$

with $\|\mu\|_{\mathfrak{M}_b} \leq \liminf_{n \rightarrow \infty} \|\mu_n\|_{\mathfrak{M}_b}$. We thus need to prove that

$$\liminf \int_{E_\lambda[\mu_n]} d\sigma \geq \int_{E_\lambda[\mu]} d\sigma. \quad (3.43)$$

We first observe that for any $t > 0$ and $x \in \Omega$ the set $\{y \in \Omega : \mathbb{G}(x, y) > t\}$ is open (with $\mathbb{G}(x, x) = +\infty$). It follows from [7][Thm 2.1] that $\liminf_{n \rightarrow \infty} \mu_n(\{\mathbb{G}(x, \cdot) > t\}) \geq \mu(\{\mathbb{G}(x, \cdot) > t\})$. We can take the lim inf using Fatou's lemma in

$$\int_\Omega \mathbb{G}(x, y) d\mu_n(y) = \int_0^{+\infty} \mu_n(\{\mathbb{G}(x, \cdot) > t\}) dt,$$

to derive

$$\liminf_{n \rightarrow \infty} \mathbb{G}[\mu_n](x) \geq \int_0^{+\infty} \mu(\{\mathbb{G}(x, \cdot) > t\}) dt = \int_\Omega \mathbb{G}(x, y) d\mu(y) = \mathbb{G}[\mu](x).$$

We infer that for any $x \in \Omega$ such that $\chi_{E_\lambda(\mu)}(x) = 1$ we have $\liminf_{n \rightarrow \infty} \mathbb{G}[\mu_n](x) > \lambda$, hence $\mathbb{G}[\mu_n](x) > \lambda$ for n large enough. Thus $\chi_{E_\lambda(\mu_n)}(x) = 1$ eventually, and then

$$\liminf_{n \rightarrow \infty} \chi_{E_\lambda[\mu_n]}(x) \geq \chi_{E_\lambda[\mu]}(x) \quad \text{for all } x \in \Omega.$$

The claim (3.43) follows by Fatou's lemma. \square

We are now in position to prove Theorem B.

Proof of Theorem B. We note that if g is nondecreasing, uniqueness follows from estimate Lemma 3.5. Let $\{\eta_n\}$ be a sequence of mollifiers, $\mu_n = \mu * \eta_n$ and $u_n \in W_0^{1,2}(\Omega)$ a minimizing weak solution of

$$\begin{aligned} -\Delta u_n + g(u_n)\sigma &= \mu_n && \text{in } \Omega, \\ u_n &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (3.44)$$

given by Proposition 3.1. We write $g(r) = g_1(r) + g_2(r)$ with $g_1 = g\chi_{(-r_0, r_0)}$, $g_2 = g\chi_{(-\infty, -r_0] \cup [r_0, \infty)}$, and set $m = \sup\{g(r) : -r_0 \leq r \leq r_0\} \geq 0$ and $m' = \inf\{g(r) : -r_0 \leq r \leq r_0\} \leq 0$. Then

$$-\mathbb{G}[\mu_n^-] - m\mathbb{G}[\sigma] \leq u_n \leq \mathbb{G}[\mu_n^+] - m'\mathbb{G}[\sigma].$$

Since $\sigma \in \mathcal{M}_p^+(\Omega)$ for some $p > N/2$, $\mathbb{G}[\sigma] \in C^{0,\alpha}(\bar{\Omega})$ by Lemma 2.2. Moreover $\mathbb{G}[|\mu_n|] \in C(\bar{\Omega})$ since $|\mu_n| \in C(\bar{\Omega})$. It follows that

$$|u_n| \leq \mathbb{G}[|\mu_n|] + M \leq c_n, \quad (3.45)$$

where $M, c_n \geq 0$.

Since $u_n \in W_0^{1,2}(\Omega)$, its precise representative (that we identify with u_n) is defined $c_{1,2}$ -quasi-everywhere, is $c_{1,2}$ -continuous and

$$u_n(x) = \lim_{r \rightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} u_n(y) dy$$

for any $y \in \Omega \setminus E_n$ with $c_{1,2}(E_n) = 0$ (see [2]). It follows that $|u_n| \leq c_n$ in $E := \cup E_n$. Note that $c_{1,2}(E) = 0$ so that $\sigma(E) = 0$ by Lemma 3.3. Hence $|u_n| \leq c_n$ σ -almost everywhere, $g(u_n) \in L_\sigma^\infty(\Omega)$, and therefore $g(u_n)\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$. We can then apply Corollary 3.8 to obtain, for any $\zeta \in \mathbb{X}_+(\Omega)$, that

$$-\int_\Omega |u_n| \Delta \zeta dx + \int_\Omega \text{sign}_0(u_n) g(u_n) \zeta d\sigma \leq \int_\Omega \text{sign}_0(u_n) \zeta \mu_n dx,$$

which implies

$$-\int_\Omega |u_n| \Delta \zeta dx + \int_\Omega |g_2(u_n)| \zeta d\sigma \leq \int_\Omega \text{sign}_0(u_n) \zeta \mu_n dx + c \int_\Omega \zeta d\sigma. \quad (3.46)$$

We take $\zeta = \eta_1$ and obtain

$$\begin{aligned} \int_\Omega |u_n| dx + \int_\Omega |g_2(u_n)| \eta_1 d\sigma &\leq \int_\Omega |\mu_n| \eta_1 dx + c \\ &\leq \int_\Omega \eta_1 d|\mu| + c = c', \end{aligned} \quad (3.47)$$

so that $\{u_n\}$ is bounded in $L^1(\Omega)$. We also have from Corollary 3.8 that

$$\int_{\Omega} \text{sign}_0(u_n)g(u_n)d\sigma \leq C \int_{\Omega} |\mu_n|\rho dx$$

and so

$$\int_{\Omega} |g_2(u_n)|d\sigma \leq C \int_{\Omega} |\mu_n|dx + \int_{\Omega} |g_1(u_n)|d\sigma \leq C \quad (3.48)$$

with C independent of n . We deduce that the sequence of measures $\{g(u_n)\}$ is bounded.

By the standard regularity estimates, the sequence $\{u_n\}$ is bounded in $W^{1,q}(\Omega)$, $q < \frac{N}{N-1}$. Then there exists $u \in W^{1,q}(\Omega)$, $q < \frac{N}{N-1}$, such that, up to a subsequence, $u_n \rightarrow u$ in $L^1(\Omega)$ and also pointwise in $\Omega \setminus E$ where $c_{1,q}(E) = 0$. We fix $q \in \left(1, \frac{N}{N-1}\right)$ such that $\theta > N - q$. In view of Lemma 3.3, $\sigma(E) = 0$ so that $g(u_n) \rightarrow g(u)$ σ -almost everywhere. Applying Fatou's lemma in (3.48) gives that $g(u) \in L^1_{\sigma}(\Omega)$.

In order to prove the uniform integrability of $\{g(u_n)\}$ for the measure σ we can assume that $|g_2| \leq \tilde{g}$ with a function satisfying (1.8) still denoted by \tilde{g} and let $E \subset \Omega$ be a Borel set. Then

$$\begin{aligned} \int_E |g_2(u_n)| d\sigma &\leq \int_{E \cap \{|u_n| \leq t\}} |g_2(u_n)| d\sigma + \int_{E \cap \{|u_n| > t\}} |g_2(u_n)| d\sigma \\ &\leq \tilde{g}(t) \int_E d\sigma + \int_{\{|u_n| > t\}} \tilde{g}(|u_n|) d\sigma. \end{aligned}$$

Then we estimate the second integral in the right-hand side: for $\lambda > M$ we set

$$S_n(\lambda) = \{x \in \Omega : |u_n(x)| > \lambda\} \quad \text{and} \quad b_n^{\sigma}(\lambda) = \int_{S_n(\lambda)} d\sigma.$$

In view of (3.45) we have $|u_n| \leq \mathbb{G}(|\mu_n|) + M$ so that $S_n(\lambda) \subset E_{\lambda-M}[\mu_n]$. Hence $b_n^{\sigma}(\lambda) \leq e_{\lambda-M}^{\sigma}(|\mu_n|)$. This implies

$$\begin{aligned} \int_{\{|u_n| > t\}} \tilde{g}(|u_n|) d\sigma &= - \int_t^{\infty} \tilde{g}(\lambda) db_n^{\sigma}(\lambda) \\ &\leq \int_t^{\infty} b_n^{\sigma}(\lambda) d\tilde{g}(\lambda) \\ &\leq \int_t^{\infty} e_{\lambda-M}^{\sigma}(|\mu_n|) d\tilde{g}(\lambda). \end{aligned}$$

Using (3.39) we obtain

$$\begin{aligned} \int_{\{|u_n| > t\}} \tilde{g}(|u_n|) d\sigma &\leq c \|\mu\|_{\mathfrak{M}^b}^{\frac{\theta}{N-2}} \int_t^{\infty} (\lambda - M)^{-\frac{\theta}{N-2}} d\tilde{g}(\lambda) \\ &\leq \frac{c\theta}{N-2} \int_t^{\infty} (\lambda - M)^{-\frac{\theta}{N-2}-1} \tilde{g}(\lambda) d\lambda. \end{aligned}$$

In view of assumption (1.8), given $\epsilon > 0$ we fix $t > M$ such that

$$\frac{c\theta}{N-2} \int_t^\infty (\lambda - M)^{-\frac{\theta}{N-2}-1} \tilde{g}(\lambda) d\lambda \leq \frac{\epsilon}{2}.$$

Then, setting $\delta = \frac{\epsilon}{2\tilde{g}(t)}$, we deduce

$$\int_E d\sigma \leq \delta \implies \int_E |g_2(u_n)| d\sigma \leq \epsilon.$$

Since g_1 is bounded, this implies that $\{g(u_n)\}$ is uniformly integrable in $L^1_\sigma(\Omega)$. Since we already know that $g(u_n) \rightarrow g(u)$ σ -almost everywhere, it follows by Vitali's convergence theorem that $g(u_n) \rightarrow g(u)$ in $L^1_\sigma(\Omega)$. Taking $\zeta \in \mathbb{X}(\Omega)$ and letting $n \rightarrow \infty$ in the equality

$$-\int_\Omega u_n \Delta \zeta dx + \int_\Omega g(u_n) \zeta d\sigma = \int_\Omega \zeta d\mu_n$$

yields the result. \square

4 The 2-D case

In this section Ω is a bounded C^2 planar domain. The next result is the 2-D version of Lemma 3.11.

Lemma 4.1 *Assume $N = 2$ and $\sigma \in \mathcal{M}^+_{\frac{2}{2-\theta}}(\Omega)$ with $2 \geq \theta > 0$. If $\mu \in \mathfrak{M}^b(\Omega)$ and $\lambda \geq 0$, we set $E_\lambda[\mu] := \{x \in \Omega : \mathbb{G}[|\mu|](x) > \lambda\}$. Then*

$$e_\lambda^\sigma(\mu) := \int_{E_\lambda[\mu]} d\sigma \leq |\Omega|_\sigma e^{1 - \frac{\lambda}{\gamma \|\mu\|_{\mathfrak{M}^b}}} \quad \text{for all } \lambda > 0, \quad (4.1)$$

for some $\gamma = \gamma(\theta, \text{diam}(\Omega)) > 0$

Proof. If $\mu = \delta_a$ for some $a \in \Omega$, one has $0 \leq \mathbb{G}[\delta_a](x) \leq \frac{1}{2\pi} \ln \left(\frac{d_\Omega}{|x-a|} \right)$ where $d_\Omega = \text{diam}(\Omega)$. Hence

$$E_\lambda[\delta_a] \subset B_{d_\Omega e^{-2\pi\lambda}} \implies e_\lambda^\sigma(\delta_a) = \int_{E_\lambda[\delta_a]} d\sigma \leq cd_\Omega^\theta e^{-2\theta\pi\lambda}.$$

Let $E \subset \Omega$ be a Borel set, $\int_E d\sigma = |E|_\sigma$ and $t > 0$, then, as in Lemma 3.11,

$$\begin{aligned} \int_E \mathbb{G}[\delta_a] d\sigma &\leq t \int_E d\sigma - \int_t^\infty s de_s^\sigma(\delta_a) \\ &\leq t |E|_\sigma + cd_\Omega^\theta \left(t + \frac{1}{2\pi\theta} \right) e^{-2\theta\pi t}. \end{aligned}$$

If we choose $e^{-2\theta\pi t} = \frac{|E|_\sigma}{|\Omega|_\sigma}$ we infer

$$\int_E \mathbb{G}[\delta_a] d\sigma \leq \gamma |E|_\sigma \left(\ln \left(\frac{|\Omega|_\sigma}{|E|_\sigma} \right) + 1 \right). \quad (4.2)$$

For proving (3.39) we can assume that $\mu \geq 0$. Then there exists $\alpha_j > 0$ and $a_j \in \Omega$ such that

$$\mu = \sum_{j=1}^{\infty} \alpha_j \delta_{a_j} \implies \sum_{j=1}^{\infty} \alpha_j = \|\mu\|_{\mathfrak{M}^b}.$$

Hence, for any Borel set $E \subset \Omega$,

$$\int_E \mathbb{G}[\mu](x) d\sigma(x) = \sum_{j=1}^{\infty} \alpha_j \int_E \mathbb{G}[\delta_{a_j}(x)] d\sigma(x) \leq \gamma |E|_\sigma \left(\ln \left(\frac{|\Omega|_\sigma}{|E|_\sigma} \right) + 1 \right) \|\mu\|_{\mathfrak{M}^b}. \quad (4.3)$$

If $E = E_\lambda[\mu]$ we infer

$$\lambda e_\lambda^\sigma(\mu) \leq \gamma e_\lambda^\sigma(\mu) \left(\ln \left(\frac{|\Omega|_\sigma}{e_\lambda^\sigma(\mu)} \right) + 1 \right) \|\mu\|_{\mathfrak{M}^b},$$

which implies the claim. \square

Theorem 4.2 *Assume $N = 2$, $\sigma \in \mathcal{M}_{\frac{2}{2-\theta}}^+(\Omega)$ with $2 \geq \theta > 0$ and $g : \mathbb{R} \mapsto \mathbb{R}$ a continuous function satisfying (1.1). If $a_\infty(g) = a_{-\infty}(g) = 0$, for any $\mu \in \mathfrak{M}_b(\Omega)$ problem (1.2) admits a very weak solution.*

Proof. Let g^* be the monotone nondecreasing hull of g defined by (1.11). If $m = \sup\{g(r) : -r_0 \leq r \leq r_0\}$ and $m' = \inf\{g(r) : -r_0 \leq r \leq r_0\}$ then $g \leq g^* + m$ on \mathbb{R}_+ and $g^* + m' \leq g$ on \mathbb{R}_- . If $\{\eta_n\}$ is a sequence of mollifiers and $\mu = \mu^+ - \mu^-$, we set $\mu_n^+ = \mu^+ * \eta_n$, $\mu_n^- = \mu^- * \eta_n$, $\mu_n = \mu_n^+ = -\mu_n^-$ and denote by u_n the very weak solution of

$$\begin{aligned} -\Delta u_n + g(u_n)\sigma &= \mu_n && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (4.4)$$

Since $\|\mu_n\|_{L^1} \leq \|\mu\|_{\mathfrak{M}_b}$, there holds by Proposition 3.2,

$$\|u_n\|_{L^1} + \|\rho g(u_n)\|_{L_\sigma^1} \leq c \|\mu\|_{\mathfrak{M}_b} + M, \quad (4.5)$$

and by Lemma 2.1,

$$\|u_n\|_{BMO} + \|\nabla u_n\|_{L^{2,\infty}} \leq c \left(\|\mu\|_{\mathfrak{M}_b} + \|\rho g(u_n)\|_{L_\sigma^1} \right) \leq c' \|\mu\|_{\mathfrak{M}_b}. \quad (4.6)$$

Again, there exists a set E with $c_{1,q}(E) = 0$ for any $q \leq 2 - \theta$ such that $u_n(x) \rightarrow u(x)$ for all $x \in \Omega \setminus E$, hence $u_n(x) \rightarrow u(x)$ and $g(u_n(x)) \rightarrow g(u(x))$ $d\sigma$ -almost everywhere

in Ω . This implies that $g(u)$ is σ -measurable. In order to conclude we have to prove that $g(u_n) \rightarrow g(u)$ in $L^1_\sigma(\Omega)$. Estimate (4.1) is valid, hence, for any $t > 0$,

$$\tau_n(t) = \int_{\{|u_n(x)| > t\}} d\sigma \leq e_{t-M}^\sigma[\mu_n^+] + e_{t-M'}^\sigma[\mu_n^-] \leq ce^{-\frac{t}{\gamma\|\mu\|_{\mathfrak{M}}}},$$

by Lemma 4.1. Since

$$|g(u_n)| \leq (g_+^*(u_n) - g_-^*(u_n)) + m - m',$$

we have that

$$\begin{aligned} \int_E |g(u_n)| d\sigma &\leq \int_E g_+^*(u_n) d\sigma - \int_E g_-^*(u_n) d\sigma + (m - m') |E|_\sigma \\ &\leq - \int_t^\infty g_+^*(s) d|\{u_n > s\}|_\sigma + \int_{-\infty}^{-t} g_-^*(s) d|\{u_n < s\}|_\sigma + (m - m') |E|_\sigma \\ &\leq - \int_t^\infty (g_+^*(s) - g_-^*(-s)) d\tau_n(s) + (g_+^*(t) - g_-^*(-t) + m - m') |E|_\sigma. \end{aligned}$$

By integration by parts,

$$\begin{aligned} - \int_t^\infty (g_+^*(s) - g_-^*(-s)) d\tau_n(s) &= (g_+^*(t) - g_-^*(-t)) \tau_n(t) + \int_t^\infty \tau_n(s) d(g_+^*(s) - g_-^*(-s)) \\ &\leq (g_+^*(t) - g_-^*(-t)) \left(\tau_n(t) - ce^{-\frac{t}{\gamma\|\mu\|_{\mathfrak{M}}^b}} \right) \\ &\quad + \frac{c}{\gamma\|\mu\|_{\mathfrak{M}}^b} \int_t^\infty e^{-\frac{s}{\gamma\|\mu\|_{\mathfrak{M}}^b}} (g_+^*(s) - g_-^*(-s)) ds \\ &\leq \frac{c}{\gamma\|\mu\|_{\mathfrak{M}}^b} \int_t^\infty e^{-\frac{s}{\gamma\|\mu\|_{\mathfrak{M}}^b}} (g_+^*(s) - g_-^*(-s)) ds. \end{aligned} \tag{4.7}$$

By assumption the integral on the right-hand side is convergent. We end the proof as in Theorem B, first by fixing t large enough and then $|E|_\sigma$ small enough, and we derive the uniform integrability of $\{g(u_n)\}$. \square

A similar result holds when g has nonzero order of growth at infinity.

Theorem 4.3 *Assume $N = 2$, $\sigma \in \mathcal{M}_{\frac{2}{2-\theta}}^+(\Omega)$ with $2 \geq \theta > 0$ and $g : \mathbb{R} \mapsto \mathbb{R}$ a continuous function satisfying (1.1). If $0 < a_\infty(g) < \infty$ and $-\infty < a_\infty(g) < 0$, there exists $\delta > 0$ such that for any $\mu \in \mathfrak{M}_b(\Omega)$ satisfying $\|\mu\|_{\mathfrak{M}_b} \leq \delta$ problem (1.2) admits a very weak solution.*

Proof. The proof is a straightforward adaptation of the previous one. The choice of δ is such that

$$\|\mu\|_{\mathfrak{M}_b} \leq \delta < \frac{1}{\gamma} \sup \left\{ \frac{1}{a_\infty(g)}, -\frac{1}{a_\infty(g)} \right\} \tag{4.8}$$

and the conclusion follows from (4.7). \square

5 The supercritical case

5.1 Proof of Theorem D

Proof of assertion I. For $k > 0$ set $g_k(r) = \max\{g(-k), \min\{g(k), g(r)\}\}$ and denote by u_k the very weak solution of

$$\begin{aligned} -\Delta u + g_k(u)\sigma &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{5.1}$$

which exists by Theorem B. It follows from the proof of Theorem B (see (3.48) with $g = g_2$ and $g_1 = 0$) that

$$\int_{\Omega} |g_k(u_k)| d\sigma \leq C, \tag{5.2}$$

where the constant C depends only on Ω and $|\mu|(\Omega)$. Thus the sequence of measures $\{g_k(u_k)\sigma\}$ is bounded. This implies that $\{u_k\}$ is bounded in $W^{1,q}(\Omega)$, $q < \frac{N}{N-1}$, and thus that, up to a subsequence, it converges in $L^1(\Omega)$ to some $u \in W^{1,q}(\Omega)$, $q < \frac{N}{N-1}$. We can also assume that the convergence holds pointwise except on a set E with zero $c_{1,q}$ -capacity, which in turn is σ -negligible by Lemma 3.3 if we fix $q \in \left(1, \frac{N}{N-1}\right)$ such that $\theta > N - q$. We also have that u is finite but on a set with zero $c_{1,q}$ -capacity hence σ -negligible, therefore

$$g_k(u_k) \rightarrow g(u) \quad \sigma\text{-almost everywhere.}$$

Applying Fatou's lemma in (5.2) yields $g(u) \in L^1_{\sigma}(\Omega)$.

By the maximum principle

$$-\mathbb{G}[|\mu|] \leq u_k \leq \mathbb{G}[|\mu|], \tag{5.3}$$

hence

$$g(-\mathbb{G}[|\mu|]) \leq g_k(u_k) \leq g(\mathbb{G}[|\mu|]), \tag{5.4}$$

since g is nondecreasing.

Because of assumption (1.13) and in view of (5.4), we infer from Lebesgue dominated convergence that $\rho g_k(u_k) \rightarrow \rho g(u)$ in $L^1_{\sigma}(\Omega)$. Thus we can pass to the limit in weak formulation of (5.1) with any $\zeta \in \mathbb{X}(\Omega)$.

Proof of assertion II. We first notice that if g is nondecreasing, vanishes at 0 and satisfies (1.14), then the function g_k defined above also satisfies (1.14) with the same constants a and b . We assume first that $\mu = \mu_r + \mu_s$ is nonnegative and we set $\mu_r^n = \mu_r * \eta_n$ where $\{\eta_n\}$ is a sequence of mollifiers. Let u_k^n be the solution of (5.1) with right-hand side $\mu_r^n + \mu_s$ and v_k^n the one of (5.1) with right-hand side μ_r^n (in both

cases existence and uniqueness follows from Theorem B). Then $0 \leq u_k^n \leq v_k^n + \mathbb{G}[\mu_s]$, $v_k^n \geq 0$ and $\mathbb{G}[\mu_s] \geq 0$. Since g is non-decreasing, we deduce with (1.14) that

$$0 \leq g_k(u_k^n) \leq g_k(v_k^n + \mathbb{G}[\mu_s]) \leq a(g_k(v_k^n) + g_k(\mathbb{G}[\mu_s])) + b. \quad (5.5)$$

Since

$$\|v_k^n\|_{L^1} + \|\rho g_k(v_k^n)\|_{L^1_\sigma} \leq c \|\mu_r^n\|_{\mathfrak{M}_b} \leq c \|\mu\|_{\mathfrak{M}_b}, \quad (5.6)$$

up to subsequences, the sequences $\{v_k^n\}$ and $\{u_k^n\}$ converge in $L^1(\Omega)$ to some $v^n \in L^1(\Omega)$ and u^n such that $\nabla v^n, \nabla u^n \in L^q(\Omega)$ for any $q < \frac{N}{N-1}$ when $k \rightarrow \infty$. As in I, $\{g_k(v_k^n)\}$ and $\{g_k(u_k^n)\}$ converge in $L^1_\sigma(\Omega)$ to $\{g(v^n)\}$ and $\{g(u^n)\}$ respectively. Furthermore v^n and u^n satisfy

$$\begin{aligned} -\Delta v^n + g(v^n)\sigma &= \mu_r^n && \text{in } \Omega \\ v^n &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (5.7)$$

and

$$\begin{aligned} -\Delta u^n + g(u^n)\sigma &= \mu_s + \mu_r^n && \text{in } \Omega \\ u^n &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (5.8)$$

respectively and $0 \leq u^n \leq v^n + \mathbb{G}[\mu_s]$. As in the proof of Proposition 3.2, $v^n \rightarrow v$ in $L^1(\Omega)$ and $\rho g(v^n) \rightarrow \rho g(v)$ in $L^1_\sigma(\Omega)$ as $n \rightarrow \infty$, and v is a very weak solution of

$$\begin{aligned} -\Delta v + g(v)\sigma &= \mu_r && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (5.9)$$

As above $\{u^n\}$ converge in $L^1(\Omega)$ to some $u \in L^1(\Omega)$ (always up to some subsequence), there holds $u \leq v + \mathbb{G}[\mu_s]$ and $g(u^n) \rightarrow g(u)$ σ -almost everywhere in Ω since the uniform bound on $\|\nabla u_n\|_{L^{\frac{N}{N-1}, \infty}}$ holds. Furthermore

$$0 \leq g(u^n) \leq a(g(v^n) + g(\mathbb{G}[\mu_s])) + b \implies 0 \leq g(u) \leq a(g(v) + g(\mathbb{G}[\mu_s])) + b, \quad (5.10)$$

and since $g(v^n) \rightarrow g(v)$ in $L^1_\sigma(\Omega)$, the sequence $\{g(u^n)\}$ is uniformly integrable in $L^1_\sigma(\Omega)$. Again this implies that $g(u^n) \rightarrow g(u)$ in $L^1_\sigma(\Omega)$ and u is a very weak solution of (1.2). If μ is signed measure, we construct successively the solutions u_k^n, \bar{u}_k^n and \underline{u}_k^n of (5.1) with right-hand side $\mu_r^n + \mu_s, |\mu_r^n| + |\mu_s|$ and $-|\mu_r^n| - |\mu_s|$ respectively, and the solutions \bar{v}_k^n and \underline{v}_k^n of (5.1) with right-hand side $|\mu_r^n|$ and $-|\mu_r^n|$ respectively. Then $\underline{v}_k^n - \mathbb{G}[\mu_s] \leq u_k^n \leq \bar{v}_k^n + \mathbb{G}[\mu_s]$ which implies by (1.15)

$$a(g_k(\underline{v}_k^n) + g_k(-\mathbb{G}[\mu_s])) + b \leq g_k(u_k^n) \leq a(g_k(\bar{v}_k^n) + g_k(\mathbb{G}[\mu_s])) + b. \quad (5.11)$$

Using the same estimates as above we conclude that $\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} u_k^n = u$ exists in $L^1(\Omega)$, that $\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} g_k(u_k^n) = g(u)$ holds σ almost everywhere in Ω and in $L^1_\sigma(\Omega)$, which ends the proof. \square

5.2 Reduced measures

We adapt here some of the results in [9] which turn out to be useful tools in our framework.

Lemma 5.1 *Let $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ with $N \geq \theta > N - \frac{N}{N-1}$ and g be nondecreasing satisfying (1.1). Assume $\{\mu_n\} \subset \mathfrak{M}_b^+(\Omega)$ is an increasing sequence of good measures for problem (1.2) converging to $\mu \in \mathfrak{M}_b^+(\Omega)$. Then μ is a good measure.*

Proof. Let u_{μ_n} be the solutions of (1.2) with right-hand side μ_n then for any $n, k \in \mathbb{N}$, $k \geq n$, we have since $u_0 \in C^\alpha(\bar{\Omega})$,

$$-m \leq u_0 \leq u_{\mu_n} \leq u_{\mu_k}$$

for some $m \geq 0$ and then

$$g(-m) \leq g(u_0) \leq g(u_{\mu_n}) \leq g(u_{\mu_k}).$$

We use $\zeta := (\eta_1 + \epsilon)^\alpha - \epsilon^\alpha$ as a test-function in the very weak formulation of the equation satisfied by $u_{\mu_n} - u_0$ as in the proof of (3.30); then, recalling that $-\Delta\zeta \geq 0$, we obtain that

$$\int_{\Omega} (g(u_{\mu_n}) - g(u_0))((\eta_1 + \epsilon)^\alpha - \epsilon^\alpha) d\sigma \leq \int_{\Omega} (\eta_1 + \epsilon)^\alpha d\mu_n \leq C\mu_n(\Omega) \leq C\mu(\Omega),$$

where C is independent of n . Letting successively $\epsilon \rightarrow 0$ and $\alpha \rightarrow 0$ we obtain

$$0 \leq \int_{\Omega} (g(u_{\mu_n}) - g(u_0)) d\sigma \leq C.$$

Hence $\{u_{\mu_n}\}$ is bounded in $W_0^{1,q}(\Omega)$ for any $q < \frac{N}{N-1}$. Thus there exists $u \in W_0^{1,q}(\Omega)$, $q < \frac{N}{N-1}$, such that $u_{\mu_n} \uparrow u$ in $L^1(\Omega)$ and pointwise but for a set E with zero $c_{1,q}$ -capacity. Since $\theta > N - \frac{N}{N-1}$ we can find some $q < \frac{N}{N-1}$ such that $\theta > N - q$. It then follows from Lemma 3.3 that $\sigma(E) = 0$. Thus $g(u_{\mu_n}) \uparrow g(u)$ σ -almost everywhere. Fatou's lemma yields $\int_{\Omega} (g(u) - g(u_0)) d\sigma \leq C$, thus $g(u) \in L_\sigma^1(\Omega)$. By the dominated convergence theorem, $g(u_{\mu_n}) \rightarrow g(u)$ in L_σ^1 . We can then pass to the limit in the equation satisfied by u_{μ_n} to obtain that $u = u_\mu$. \square

Proposition 5.2 *Assume σ and g satisfy the assumptions of Lemma 5.1. Consider the set*

$$Z = \left\{ x \in \Omega : \int_{\Omega} \mathbb{G}(x, y)^q \rho(y) d\sigma(y) = \infty \right\}.$$

If $\mu \in \mathfrak{M}_b^+(\Omega)$ is such that $\mu(Z) = 0$ then μ is good.

Proof. We adapt to our case the proof of [30][Thm 3.10]. Consider the sets

$$C_n = \{x \in \Omega : \int_{\Omega} \mathbb{G}(x, y)^q \rho(y) d\sigma(y) \leq n\}, \quad n = 1, 2, \dots$$

Since the function $x \rightarrow \int_{\Omega} \mathbb{G}(x, y)^q \rho(y) d\sigma(y)$ is lsc (by Fatou's lemma) the sets C_n are closed. Moreover $C_n \subset C_{n+1}$ and $\bigcup_n C_n = \Omega \setminus Z$. Define $\mu_n := 1_{C_n} \mu$ i.e. μ_n is the measure μ restricted to C_n . Then each μ_n satisfies (1.13). Indeed

$$\begin{aligned} \int_{\Omega} \mathbb{G}[[\mu_n]]^q \rho d\sigma &\leq \mu_n(\Omega)^{q-1} \int_{\Omega} \int_{\Omega} \mathbb{G}(x, y)^{q-1} d\mu_n(x) d\sigma(y) \\ &\leq \mu(\Omega)^{q-1} \int_{C_n} \left(\int_{\Omega} \mathbb{G}(x, y)^{q-1} d\sigma(y) \right) d\mu(x) \\ &\leq n \mu(\Omega)^q. \end{aligned}$$

It follows from Theorem D that μ_n is good. Since $0 \leq \mu_n \uparrow \mu$ we deduce from Lemma 5.1 that μ is good. \square

Lemma 5.3 *Assume σ and g satisfy the assumptions of Lemma 5.1.*

I- *If $\mu \in \mathfrak{M}_b^+(\Omega)$ is a good measure, any $\nu \in \mathfrak{M}_b^+(\Omega)$ such that $\nu \leq \mu$ is a good measure.*

II- *Let $\mu, \mu' \in \mathfrak{M}_b^+(\Omega)$. If μ and $-\mu'$ are good measures, any $\nu \in \mathfrak{M}_b(\Omega)$ such that $-\mu' \leq \nu \leq \mu$ is a good measure.*

Proof. Step 1. Assume $\mu \in \mathfrak{M}_b^+(\Omega)$ is a good measure. For $k > 0$ define g_k by $g_k(r) = \max\{g(-k), \min\{g(k), g(r)\}\}$, and denote by $u_{k,\mu}$ the solution of (5.1), which exists by Theorem B, and by u_μ the solutions of (1.2). Then $-m \leq u_0 \leq \min\{u_\mu, u_{k,\mu}\}$. If $k > m$, then $g_k(u_{k,\mu}) = \min\{g(k), g(u_{k,\mu})\} \leq g(u_{k,\mu})$. Hence

$$-\Delta(u_\mu - u_{k,\mu}) + (g_k(u_\mu) - g_k(u_{k,\mu})) \sigma \leq 0.$$

Then $u_\mu \leq u_{k,\mu}$ by Lemma 3.6. Similarly $u_{k',\mu} \leq u_{k,\mu}$ for $k' \geq k > m$. Using η_1 as test-function we obtain

$$\int_{\Omega} (u_{k,\mu} - u_\mu) dx + \int_{\Omega} (g_k(u_{k,\mu}) - g_k(u_\mu)) \eta_1 d\sigma = \int_{\Omega} (g(u_\mu) - g_k(u_\mu)) \eta_1 d\sigma. \quad (5.12)$$

Since $g_k(r) \rightarrow g(r)$ for any $r \in \mathbb{R}$ and $|g_k(u_\mu)| \leq |g(u_\mu)|$ with $\rho|g(u_\mu)| \in L_\sigma^1(\Omega)$, the right-hand side converges to 0 as $k \rightarrow \infty$ and the second term on the left-hand side is nonnegative. Hence $u_{k,\mu} \rightarrow u_\mu$ in $L^1(\Omega)$ as $k \rightarrow \infty$, thus $\rho(g_k(u_{k,\mu}) - g_k(u_\mu)) \rightarrow 0$ in $L_\sigma^1(\Omega)$ which in turn yields $\rho g_k(u_{k,\mu}) \rightarrow \rho g(u_\mu)$ in $L_\sigma^1(\Omega)$.

Step 2: proof of I. Denote by $u_{k,\nu}$ the solution of

$$\begin{aligned} -\Delta u + g_k(u) &= \nu & \text{in } \Omega \\ u &= 0 & \text{in } \partial\Omega. \end{aligned} \quad (5.13)$$

Then $-m \leq u_{k,\nu} \leq u_{k,\mu}$, $u_{k',\mu} \leq u_{k,\mu}$ for $k' \geq k > m$ by Lemma 3.6 and $g_k(u_{k,\nu}) \leq g_k(u_{k,\mu})$. Furthermore $\{u_{k,\nu}\}$ is bounded in $W_0^{1,q}(\Omega)$ for $1 < q < \frac{N}{N-1}$ and thus relatively compact in $L^1(\Omega)$. Therefore there exists $u \in W_0^{1,q}(\Omega)$ such that $u_{k,\nu} \downarrow u$ in $L^1(\Omega)$ and also pointwise up to a set with zero $c_{1,q}$ -capacity which is therefore a σ -negligible set. By Step 1, the set $\{\rho g_k(u_{k,\nu})\}$ is uniformly integrable in $L_\sigma^1(\Omega)$, this implies that $u = u_\nu$.

Step 3: Proof of II. Because $-\mu' \leq \nu \leq \mu$ there holds $u_{k,-\mu'} \leq u_{k,\nu} \leq u_{k,\mu}$ and $g_k(u_{k,-\mu'}) \leq g_k(u_{k,\nu}) \leq g_k(u_{k,\mu})$. Since the sets $\{u_{k,-\mu'}\}$, $\{u_{k,\nu}\}$ and $\{u_{k,\mu}\}$ are relatively compact in $L^1(\Omega)$ and bounded in $W_0^{1,q}(\Omega)$ for $1 < q < \frac{N}{N-1}$ and the sets $\{g_k(u_{k,-\mu'})\}$ and $\{g_k(u_{k,\mu})\}$ are uniformly integrable in $L_\sigma^1(\Omega)$, then, up to a subsequence, $u_{k,\nu} \rightarrow u$ in $L^1(\Omega)$ and σ -almost everywhere as $k \rightarrow \infty$. This implies that $g(u) \in L_\sigma^1(\Omega)$ and $\rho g_k(u_{k,\nu}) \rightarrow \rho g(u)$ in $L_\sigma^1(\Omega)$. Hence $u = u_\nu$. \square

The proof of the next result, based upon Zorn's lemma, is a variant of the one of [9, Th 4.1] which uses inverse maximum principle [9, Corollary 4.8].

Lemma 5.4 *Assume σ and g satisfy the assumptions of Lemma 5.1. If $\mu \in \mathfrak{M}_b^+(\Omega)$ there exists a largest good measure smaller than μ , and it is nonnegative.*

Proof. Let \mathcal{Z}_μ be the subset of all bounded nonnegative good measures smaller than μ . Notice first that \mathcal{Z}_μ is non-empty since it contains the regular part μ_r of μ with respect to the N -dimensional Hausdorff measure. We now show that \mathcal{Z}_μ is inductive. Let $\mathcal{C}_I := \{\mu_i\}_{i \in I}$ be a totally ordered subset of \mathcal{Z}_μ . For $\zeta \in C_0(\bar{\Omega})$, $\zeta \geq 0$, the set of nonnegative real numbers

$$\mathcal{C}_I(\zeta) := \left\{ \int_\Omega \zeta d\mu_i \right\}$$

is bounded from above by $\int_\Omega \zeta d\mu$. Note that can we extend μ as a positive linear form on $C_0(\bar{\Omega})$ since it is a Radon measure and $\mu(\partial\Omega) = 0$. Hence $\mathcal{C}_I(\zeta)$ admits an upper bound $L(\zeta)$ and there exists a sequence $\{i_k\} \subset I$ such that

$$\int_\Omega \zeta d\mu_{i_k} \uparrow L(\zeta) \leq \int_\Omega \zeta d\mu \quad \text{as } k \rightarrow \infty.$$

By the Stone-Weierstrass theorem there exists a dense subset $\{\zeta_n\}$ of the set of nonnegative elements in $C_0(\bar{\Omega})$. By Cantor diagonal process there exists a subsequence

$\{i_{n_k}\} \subset I$ such that

$$\int_{\Omega} \zeta_n d\mu_{i_{n_k}} \uparrow L(\zeta_n) \leq \int_{\Omega} \zeta_n d\mu \quad \text{as } k \rightarrow \infty.$$

Clearly the map $\zeta_n \mapsto L(\zeta_n)$ is additive, positively homogeneous of order one and satisfies

$$L(\zeta) \leq \int_{\Omega} \zeta d\mu \quad \text{for all } \zeta \in C_0(\overline{\Omega}), \zeta \geq 0.$$

Hence L extends as a positive linear functional on $C_0(\overline{\Omega})$, dominated by μ denoted by $\mu_{\mathcal{C}_I}$. Since μ is a Radon measure in Ω , $\mu_{\mathcal{C}_I}(\partial\Omega) = 0$, hence it is a Radon measure. Furthermore it is a good measure by Lemma 5.1. It follows that $\mu_{\mathcal{C}_I} \in \mathcal{Z}_{\mu}$. Moreover since $L(\zeta)$ is an upper bound of $\mathcal{C}_I(\zeta)$ for any nonnegative $\zeta \in C_0(\overline{\Omega})$, we have $\mu_{\mathcal{C}_I} \geq \mu_i$ for any $i \in I$. Hence the set \mathcal{Z}_{μ} is inductive.

As a consequence of Zorn's lemma, \mathcal{Z}_{μ} admits at least one maximal element that we denote μ^* . If ν is any nonnegative good measure smaller than μ it belongs to \mathcal{Z}_{μ} and hence it cannot dominate μ^* . It remains to prove that $\nu \leq \mu^*$. Set $\lambda = \sup\{\nu, \mu^*\}$ and let λ^* be a maximal element of \mathcal{Z}_{λ} . Since ν and μ^* are good measures, we have $\nu^* = \nu$ and $(\mu^*)^* = \mu^*$. It follows that $\lambda^* \geq \nu^* = \nu$ and $\lambda^* \geq (\mu^*)^* = \mu^*$ so that $\lambda^* \geq \sup\{\nu, \mu^*\} = \lambda$. This implies that $\lambda^* = \lambda \geq \mu^*$. On the other hand, since $\nu, \mu^* \leq \mu$, we have $\lambda \leq \mu$ and thus $\lambda^* \leq \mu$. By definition of a maximal element it implies that $\lambda^* = \lambda = \mu^*$, and finally $\mu^* = \sup\{\nu, \mu^*\}$. We infer $\nu \leq \mu^*$ and then μ^* is the maximum of \mathcal{Z}_{μ} . \square

Corollary 5.5 *Assume σ and g satisfy the assumptions of Lemma 5.1. If $\mu, \nu \in \mathfrak{M}_b^+(\Omega)$ are good measures, then $\sup\{\mu, \nu\}$ is a good measure.*

Proof. Set $\lambda = \sup\{\mu, \nu\}$. Then

$$\lambda \geq \lambda^* = (\sup\{\mu, \nu\})^* \geq \sup\{\mu^*, \nu^*\} = \sup\{\mu, \nu\} = \lambda. \quad (5.14)$$

This implies $\lambda = \lambda^*$, hence λ is a good measure. \square

5.3 The capacity framework

We start with the following regularity estimate for the Poisson problem

Lemma 5.6 *For any $s \geq 0$ and $1 < p < \infty$, the mapping $\mu \mapsto \mathbb{G}[\mu]$ is continuous from $\mathfrak{M}_b(\Omega) \cap H^{s-2,p}(\Omega)$ to $H^{s,p}(\Omega)$.*

Proof. It is classical that the mapping $G_D : \lambda \mapsto u = G_D(\lambda)$ solution of $-\Delta u = \lambda$ in Ω and $u = 0$ on $\partial\Omega$ is continuous from $H^{s-2,p}(\Omega)$ to $H^{s,p}(\Omega)$ for $1 < p < \infty$

and $s > \frac{1}{p}$ (see e.g. [14, Example 3.15 p. 314]). Thus we are left with the case $0 \leq s \leq \frac{1}{p}$. If $\lambda \in \mathfrak{M}_b(\Omega)$, then $G_D(\lambda) = \mathbb{G}[\lambda]$ is a very weak solution, hence, since $\mathbb{X}(\Omega) \subset C_c^1(\overline{\Omega}) \cap \left(\bigcap_{1 < r < \infty} H^{2,r}(\Omega) \right)$,

$$-\int_{\Omega} G_D(\lambda) \Delta \zeta dx = \int_{\Omega} \zeta d\lambda \leq \|\zeta\|_{H^{2-s,p'}} \|\lambda\|_{H^{s-2,p}} \quad \text{for all } \zeta \in \mathbb{X}(\Omega).$$

In particular, if $\zeta = \mathbb{G}[v]$, then $\|\zeta\|_{H^{2-s,p'}} \leq c \|v\|_{H^{-s,p'}}$ since $-s > -2 + 1/p'$, and

$$\int_{\Omega} G_D(\lambda) v dx \leq c \|v\|_{H^{-s,p'}} \|\lambda\|_{H^{s-2,p}} \quad \text{for all } v \in \Delta(\mathbb{X}(\Omega)).$$

In particular this inequality holds if $v \in C_c(\overline{\Omega})$ which is dense in $H^{-s,p'}(\Omega)$. Finally this inequality means that the mapping $v \mapsto \int_{\Omega} G_D(\lambda) v dx$ is a continuous linear form over $H^{-s,p'}(\Omega)$, it thus belongs to $H^{s,p}(\Omega)$. \square

Proposition 5.7 *Let σ and g satisfy the assumptions in Theorem E. If $\mu \in \mathfrak{M}_b(\Omega)$ is such that $|\mu| \in H^{s-2,p}(\Omega)$ for some $p > 1$ and $s > 0$ such that $N - \theta < sp < N$ and $\frac{\theta p}{N-sp} \geq q$, then (1.3) admits a unique very weak solution.*

Proof. By Lemma 5.6, if $|\mu| \in H^{s-2,p}(\Omega)$ then $\mathbb{G}[|\mu|] \in H^{s,p}(\Omega)$. By Proposition 2.4

$$\|\mathbb{G}[|\mu|]\|_{L^q} \leq c \|\mathbb{G}[|\mu|]\|_{H^{s,p}}$$

if and only if $\sigma \in \mathcal{M}_r^+(\Omega)$ with $\frac{1}{r} = q \left(\frac{1}{q} - \frac{1}{p} + \frac{s}{N} \right) = \frac{N-\theta'}{N}$. Then $q = \frac{\theta' p}{N-sp}$. Hence, if $\frac{\theta p}{N-sp} \geq q$ we get $\theta \geq \theta'$ and then $\mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega) \subset \mathcal{M}_{\frac{N}{N-\theta'}}^+(\Omega)$ by [2.7]. We conclude by Theorem D. \square

Remark. This result covers the case $q = p$, in which any bounded measure such that $|\mu| \in H^{\frac{N-\theta}{q}-2,q}(\mathbb{R}^N)$ is eligible for solving problem (1.2).

Proof of Theorem E. If μ is absolutely continuous with respect to the $c_{2-s,p'}$ -capacity, so are μ^+ and $-\mu^-$. By [13] there exists an increasing sequence of positive bounded Radon measures $\mu_j \in H^{s-2,p}(\Omega)$ converging to μ^+ . By Proposition 5.7 μ_j is a good measure, hence by Lemma 5.1 μ^+ is a good measure. In the same way $-\mu^-$ is a good measure. Since $-\mu^- \leq \mu \leq \mu^+$, it follows from Lemma 5.3-II that μ is a good measure. \square

Proof of Proposition 1.1. Notice first that if $\mu \in \mathcal{M}_{\frac{N}{N-\theta^*}}(\Omega)$ with $\theta^* > N - sp$, then for any compact $K \subset \Omega$,

$$|\mu|(K) \leq c' (c_{s,p}(K))^{\frac{1}{p}}. \quad (5.15)$$

In particular μ is absolutely continuous w.r.t $c_{s,p}$ -capacity. Indeed under the assumption on θ^* we have $H^{s,p}(\Omega) \hookrightarrow L^1_{|\mu|}(\Omega)$. It follows that for any $v \in H^{s,p}(\Omega)$, $v \geq 1$ on K , we have

$$|\mu|(K) \leq \int_K v d|\mu| \leq \|v\|_{L^1_{|\mu|}} \leq C \|v\|_{H^{s,p}}.$$

We deduce (5.15) taking the infimum over v . To apply Theorem E we need μ to be $c_{2-\frac{N-\theta}{q},q}$ -diffuse. It thus suffices to take $\theta^* > N - sp$ with $s = 2 - \frac{N-\theta}{q}$ and $p = q'$. We obtain exactly the condition on θ^* stated in Proposition 1.1. \square

5.4 The case $g(u) = |u|^{q-1} u$.

In the sequel we consider the following equation

$$\begin{aligned} -\Delta u + |u|^{q-1} u \sigma &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (5.16)$$

where $q > 1$. A measure for which there exists a solution, necessarily unique by Lemma 3.5, is called q -good. Assume that $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}$ with $N \geq \theta > N - \frac{N}{N-1}$. Then the critical exponent q from the point of view of (1.8) in Theorem B is

$$q_\theta := \frac{\theta}{N-2}, \quad (5.17)$$

which is larger than 1 if $N > 2$.

Let $q > 1$ and $\sigma \in \mathfrak{M}_b^+(\Omega)$. Recall that the Green function G of the Dirichlet Laplacian in Ω is defined on $\overline{\Omega} \times \overline{\Omega}$ with values in $[0, +\infty]$ with $G(x, x) = +\infty$, $x \in \Omega$, and $G(x, y) = 0$ if $x \in \partial\Omega$ or $y \in \partial\Omega$. We extend G to $\mathbb{R}^N \times \overline{\Omega}$ by setting $G(x, y) = 0$ if $(x, y) \in \overline{\Omega}^c \times \overline{\Omega}$. Hence $x \mapsto G(x, y)$ is lower semicontinuous in \mathbb{R}^N and $y \mapsto G(x, y)$ is lower semicontinuous in Ω , and thus is σ -measurable. Following [2, Sec. 2.3] we then consider the following set function with values in $[0, +\infty]$,

$$c_q^\sigma(E) = \inf \left\{ \int_\Omega |v|^{q'} d\sigma : v \in L^{q'}_\sigma(\Omega), \mathbb{G}[v\sigma](x) \geq 1 \text{ for all } x \in E \right\}, \quad (5.18)$$

for any $E \subset \Omega$. According to the general theory developed in [2, Sec. 2.3] c_q^σ is a regular capacity in the sense of Choquet. Using the lower semicontinuity of $y \mapsto \mathbb{G}[v\sigma](y)$ (see [2, Prop 2.3.2]) it is easy to verify that for any compact set $K \subset \Omega$, there holds

$$c_q^\sigma(K) = \inf \left\{ \int_\Omega |v|^{q'} d\sigma : v \in L^\infty_\sigma(\Omega), \mathbb{G}[v\sigma](x) \geq 1 \text{ for all } x \in K \right\}. \quad (5.19)$$

The dual formulation of the capacity is the following (see [2, Th 2.5.1]),

$$(c_q^\sigma(K))^{\frac{1}{q'}} = \sup \{ \lambda(K) : \lambda \in \mathfrak{M}_b^+(K), \|\mathbb{G}[\lambda]\|_{L_\sigma^q} \leq 1 \} \quad (5.20)$$

for $K \subset \Omega$, K compact. Existence of extremal measures satisfying equality in (5.20) is proved in [2, Th 2.5.3].

Remark. Note that the \geq inequality in (5.20) follows directly from the following one

$$\nu(K) \leq (c_q^\sigma(K))^{\frac{1}{q'}} \|\mathbb{G}\nu\|_{L_\sigma^q}, \quad (5.21)$$

which holds for any $\nu \in \mathfrak{M}_b^+(\Omega)$ such that $\mathbb{G}[\nu] \in L_\sigma^q$ and any $K \subset \Omega$ compact.

We now give some sufficient conditions for a bounded measure to be absolutely continuous with respect to the capacity c_q^σ . First in view of (5.21) and the dual expression of the capacity it is clear that there holds:

Lemma 5.8 *If $\nu \in \mathfrak{M}_b(\Omega)$ is such that $\mathbb{G}[|\nu|] \in L_\sigma^q(\Omega)$, then ν is absolutely continuous with respect to the capacity c_q^σ . This holds in particular if $\nu \in \mathfrak{M}_b(\Omega)$ is such that $|\nu| \in H^{s-2,p}(\Omega)$ for some $p > 1$ and $s > 0$ verifying $N - \theta < sp < N$ and $\frac{\theta p}{N-sp} \geq q$.*

As a direct consequence we have

Lemma 5.9 *If $\nu \in \mathfrak{M}_b(\Omega)$ is $c_{2-s,p}$ -diffuse where s and p are as in Lemma 5.8, then ν is absolutely continuous with respect to the capacity c_q^σ .*

Proof. If $\nu \geq 0$ there exists a sequence of nonnegative measures $\{\nu_n\} \subset H^{s-2,p}(\Omega)$ such that $\nu_n \uparrow \nu$. If K is a compact such that $c_q^\sigma(K) = 0$ then $\nu_n(K) = 0$ by Lemma 5.8 and thus $\nu(K) = 0$. When ν is a signed measure, we apply the above to its positive and negative part ν^\pm . \square

The following particular case will be useful:

Lemma 5.10 *If $\nu \in \mathcal{M}_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N-2$, then ν is absolutely continuous with respect to the capacity c_q^σ .*

Proof. We have $|\nu| \in \mathcal{M}_p(\Omega)$ for some $p > \frac{N}{2}$. We then obtain from (2.9) that $\mathbb{G}[|\nu|]$ is bounded so that $\mathbb{G}[|\nu|] \in L_\sigma^q(\Omega)$. The conclusion follows from the previous lemma. \square

Remark. It is noticeable that if the support of a nonnegative measure μ does not intersect the support of σ , then μ is always q -good. This is due to the fact that $\mathbb{G}[\mu]$ is bounded on the support of σ , hence $\mathbb{G}[\mu] \in L_\sigma^q(\Omega)$ for any $q < \infty$ and the result

follows from Theorem D. Hence, a more accurate necessary condition must involve a notion of density of σ on its support, a property which has been developed by Triebel [26] in connection with fractal measures.

We recall that the θ -dimensional Hausdorff measure H^θ , $0 \leq \theta \leq N$, is defined on subsets E of \mathbb{R}^N by

$$H^\theta(E) = \lim_{\delta \rightarrow 0} \left(\inf \left\{ \sum_{j=1}^{\infty} (\text{diam } U_j)^\theta : E \subset \bigcup_{j=1}^{\infty} U_j, \text{diam } U_j \leq \delta \right\} \right). \quad (5.22)$$

Definition 5.11 *A nonnegative Radon measure σ on $\overline{\Omega}$ with support Γ is θ -regular with $0 \leq \theta \leq N$ if there exists $c > 0$ such that*

$$\frac{1}{c} r^\theta \leq |B_r(x)|_\sigma \leq c r^\theta \quad \text{for all } x \in \Gamma, \text{ for all } r > 0. \quad (5.23)$$

The support Γ of σ is called a θ -set.

By [26, Th 3.4] σ is equivalent in $\overline{\Omega}$ to the restriction $H^\theta|_\Gamma$ of H^θ to Γ in the sense that there exists $c' > 0$ such that

$$\frac{1}{c'} H^\theta(E \cap \Gamma) \leq \sigma(E) \leq c' H^\theta(E \cap \Gamma) \quad \text{for all } E \subset \overline{\Omega}, E \text{ Borel.} \quad (5.24)$$

The description of $L_\sigma^p(\Gamma)$ necessitates to introduce the scale of Besov spaces and their *trace* on Γ . For $0 < s < 1$, $1 \leq p, q \leq \infty$, we denote by $B_{p,q}^s(\Omega)$ the space obtained by the real interpolation method by

$$B_{p,q}^s(\Omega) = [W^{1,p}(\Omega), L^p(\Omega)]_{s,q}. \quad (5.25)$$

Details can be found in [23]. Its norm is equivalent to

$$\|\phi\|_{B_{p,q}^s} = \|v\|_{L^p} + \left(\int_0^\infty \frac{(\omega_p(t; v))^q dt}{t^{sq}} \frac{1}{t} \right)^{\frac{1}{q}}, \quad (5.26)$$

if $q < \infty$ and

$$\|\phi\|_{B_{p,\infty}^s} = \|v\|_{L^p} + \sup_{t>0} \frac{\omega_p(t; v)}{t^s}, \quad (5.27)$$

where

$$\omega_p(t; \phi) = \sup_{|h|<t} \|v(\cdot + h) - v(\cdot)\|_{L^p}.$$

For $k \in \mathbb{N}_*$, $B_{p,q}^{k+s}(\Omega) = \{v \in W^{k,p}(\Omega) : D^\alpha v \in B_{p,q}^s(\Omega), \text{ for all } \alpha \in \mathbb{N}^N, |\alpha| = k\}$ with norm

$$\|v\|_{B_{p,q}^{k+s}} = \|v\|_{W^{k-1,p}} + \sum_{|\alpha|=k} \|D^\alpha v\|_{B_{p,q}^s}.$$

If $\Gamma \subset \mathbb{R}^N$ is a closed set with zero Lebesgue measure, we consider the set

$$B_{p,q}^{s,\Gamma}(\mathbb{R}^N) = \{v \in B_{p,q}^s(\mathbb{R}^N) : \langle v, \phi \rangle = 0 \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^N) \text{ s.t. } \phi|_{\Gamma} = 0\}, \quad (5.28)$$

endowed with the $B_{p,q}^s(\mathbb{R}^N)$ norm, where $\langle v, \phi \rangle$ is the pairing between $\mathcal{S}'(\mathbb{R}^N)$ and $\mathcal{S}(\mathbb{R}^N)$. If $v \in L_{\sigma}^p(\Omega)$ and σ has support $\Gamma \subset \bar{\Omega}$, the linear map

$$\phi \mapsto T_v^{\sigma}(\phi) = \int_{\Gamma} \phi v d\sigma \quad (5.29)$$

defined on $\mathcal{S}(\mathbb{R}^N)$ is a tempered distribution in \mathbb{R}^N . The following results are proved in [26, Th 18.2, 18.6].

Proposition 5.12 *Assume σ is θ -regular, $0 < \theta < N$, with support $\Gamma \subset \mathbb{R}^N$, and let $v \in L_{\sigma}^q(\Omega)$ with $1 < p \leq +\infty$. There holds*

$$|T_v^{\sigma}(\phi)| \leq c \|v\|_{L_{\sigma}^p} \|\phi\|_{B_{p',1}^{-\frac{N-\theta}{p'}}} \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^N). \quad (5.30)$$

It follows that $T_v^{\sigma} \in B_{p,\infty}^{-\frac{N-\theta}{p'},\Gamma}$ with $\|T_v^{\sigma}\|_{B_{p,\infty}^{-\frac{N-\theta}{p'},\Gamma}} \leq c \|v\|_{L_{\sigma}^p}$.

Moreover the map $v \in L_{\sigma}^p(\Gamma) \rightarrow T_v^{\sigma} \in B_{p,\infty}^{-\frac{N-\theta}{p'},\Gamma}$ is a linear isomorphism. We can thus denote $L_{\sigma}^p(\Gamma) \sim \left(B_{p',1}^{-\frac{N-\theta}{p'},\Gamma} \right)' = B_{p,\infty}^{-\frac{N-\theta}{p'},\Gamma}$.

Proposition 5.13 *Assume σ is θ -regular, $0 < \theta < N$ with support $\Gamma \subset \mathbb{R}^N$. Then for any $1 < p \leq \infty$ the restriction operation from $\mathcal{S}(\mathbb{R}^N)$ to $C(\Gamma)$, $\phi \mapsto \phi|_{\Gamma}$ can be extended as a continuous linear operator from $B_{p,1}^{-\frac{N-\theta}{p}}(\mathbb{R}^N)$ to $L_{\sigma}^p(\Gamma)$ that we denote Tr_{Γ} . Furthermore this operator is onto.*

Definition 5.14 *If $\sigma \in \mathfrak{M}_b^+(\Omega)$ is θ -regular, $N \geq \theta > N - 2$ with support $\Gamma \subset \Omega$ and $m, q > 1$, we set*

$$c_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma} K = \inf \left\{ \|\zeta\|_{B_{q',\infty}^{2-\frac{N-\theta}{q}}}^{q'} : \zeta \in B_{q',\infty}^{2-\frac{N-\theta}{q}}(\Omega) \text{ s.t. } \zeta \geq \chi_K \right\}, \quad (5.31)$$

where

$$B_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(\Omega) = \left\{ \zeta \in B_{q',\infty}^{2-\frac{N-\theta}{q}}(\Omega) \text{ s.t. } \Delta \zeta \in B_{q',\infty}^{-\frac{N-\theta}{q},\Gamma}(\Omega) \right\}. \quad (5.32)$$

Notice that $B_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(\Omega)$ is a closed subspace of $B_{q',\infty}^{2-\frac{N-\theta}{q}}(\Omega)$.

Proposition 5.15 *Assume $\sigma \in \mathfrak{M}_b^+(\Omega)$ is θ -regular, $N \geq \theta > N - 2$ with support $\Gamma \subset \Omega$ and $q > 1$. Then there exists a positive constant $M > 0$ such that*

$$\frac{1}{M} c_q^\sigma(K) \leq c_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(K) \leq M c_q^\sigma(K), \quad (5.33)$$

for all compact set $K \subset \Omega$.

Proof. By standard elliptic equations and interpolation theory (see [23], [24]), for any $\psi \in B_{q',\infty}^{-\frac{N-\theta}{q},\Gamma}(\Omega)$, $\mathbb{G}[\psi\sigma] \in B_{q',\infty}^{2-\frac{N-\theta}{q}}(\Omega)$ and there holds

$$\frac{1}{c} \|\mathbb{G}[\psi\sigma]\|_{B_{q',\infty}^{2-\frac{N-\theta}{q}}} \leq \|\psi\|_{B_{q',\infty}^{-\frac{N-\theta}{q},\Gamma}} \leq c \|\mathbb{G}[\psi\sigma]\|_{B_{q',\infty}^{2-\frac{N-\theta}{q}}}. \quad (5.34)$$

By Proposition 5.12 we can replace $\|\psi\|_{B_{q',\infty}^{-\frac{N-\theta}{q},\Gamma}}$ by $\|\psi\|_{L_\sigma^{q'}}$ in the above inequality, up to a change of constants c . Let $\{v_k\} \subset L_\sigma^\infty(\Omega)$ be such that $v_k \geq 0$, $\zeta_k := \mathbb{G}[v_k\sigma] \geq 0$ on K and $\|v_k\|_{L_\sigma^{q'}} \downarrow (c_q^\sigma(K))^{\frac{1}{q'}}$. Since (5.32) is equivalent to

$$\frac{1}{c} \|\zeta_k\|_{B_{q',\infty}^{2-\frac{N-\theta}{q}}} \leq \|v_k\|_{L_\sigma^{q'}} \leq c \|\zeta_k\|_{B_{q',\infty}^{2-\frac{N-\theta}{q}}},$$

we derive $c_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(K) \geq \frac{1}{c^{q'}} c_q^\sigma(K)$. Similarly $c_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(K) \leq c^{q'} c_q^\sigma(K)$. \square

Proof of Theorem F. By Lemma 5.10 the measure u^q vanishes on Borel sets with zero c_q^σ -capacity. Since $u \in L_\sigma^q(\Omega)$ the mapping

$$\phi \mapsto \int_\Gamma u \phi d\sigma = \langle u, \phi \rangle$$

is a tempered distribution that we denote by T_u^σ , hence

$$|\langle \Delta u, \phi \rangle| = |\langle u, \Delta \phi \rangle| = \left| \int_\Omega u \Delta \phi d\sigma \right| \leq \|u\|_{L_\sigma^q} \|\Delta \phi\|_{L_\sigma^{q'}}.$$

Using Proposition 5.12

$$\|\Delta \phi\|_{L_\sigma^{q'}} \leq c \|\Delta \phi\|_{B_{q',\infty}^{-\frac{N-\theta}{q},\Gamma}} \leq c' \|\phi\|_{B_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}}.$$

Therefore the nonnegative measure T_u^σ is a continuous linear form on $B_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(\Omega)$.

Therefore it vanishes on Borel sets with zero $c_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}$ -capacity, which actually coincide with Borel sets with zero zero c_q^σ -capacity. \square

5.5 Removable singularities

It is easy to prove that for any compact set $K \subset \Omega$, there exists $\mu_K \in \mathfrak{M}_b^+(K)$ such that $\int_{\Omega} (\mathbb{G}[\mu_K])^q d\sigma = 1$ and $c_q^\sigma(K) = \mu_K(K)$ (see [2][Th 2.5.3]). Since μ_K is an admissible measure, it follows from Theorem D that (1.3) is solvable with $\mu = \mu_K$, hence K is not removable. Although it could be conjectured that a compact set with zero c_q^σ -capacity is removable we can prove this assertion only for sigma-moderate solutions.

Definition 5.16 *Let $q > 1$, $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}^+(\Omega)$ where $N \geq \theta > N - 2$ and $K \subset \Omega$ a compact set. A nonnegative function $u \in L_{loc}^1(\overline{\Omega} \setminus K) \cap L_{\sigma,loc}^q(\overline{\Omega} \setminus K)$ is a sigma-moderate solution of*

$$\begin{aligned} -\Delta u + |u|^{q-1} u \sigma &= 0 && \text{in } \Omega \setminus K \\ u &= 0 && \text{in } \partial\Omega, \end{aligned} \quad (5.35)$$

if there exists an increasing sequence $\{\mu_n\} \subset \mathfrak{M}_b^+(K)$ of q -good measures such that $u_{\mu_n} \rightarrow u$ in $L_{loc}^1(\overline{\Omega} \setminus K) \cap L_{\sigma,loc}^q(\overline{\Omega} \setminus K)$.

Theorem 5.17 *Under the assumptions on q, σ and K of Definition 5.16, if $c_q^\sigma(K) = 0$ then the only sigma-moderate solution of (5.35) is the trivial one.*

Proof. Since $c_q^\sigma(K) = 0$ the set of nonnegative q -good measures with support in K is reduced to the zero function by Theorem F. This implies the claim. \square

Remark. We conjecture that for any compact set $K \subset \Omega$, any nonnegative local solution of (5.12) is sigma-moderate. This would imply that a necessary and sufficient condition for a local nonnegative solution of (5.12) to be a solution in Ω is $c_q^\sigma(K) = 0$. However this type of result is usually difficult to prove, see [22], [17], [12] in the framework of semilinear equations with measure boundary data.

In order to find necessary and sufficient conditions for the removability of a compact set $K \subset \Omega$, we assume that σ is a positive measure in Ω absolutely continuous with respect to the Lebesgue measure, with a nonnegative density w . For proving our results we will assume that the function $\omega = w^{-\frac{1}{q-1}}$ is q' -admissible in the sense of [15, Chap 1]. One sufficient condition is that w belongs to the Muckenhoupt class A_q , that is

$$\sup_B \left(\frac{1}{|B|} \int_B w dx \right) \left(\frac{1}{|B|} \int_B w^{-\frac{1}{q-1}} dx \right)^{\frac{1}{p-1}} = m_{w,q} < \infty \quad (5.36)$$

for all ball $B \subset \mathbb{R}^N$.

If $K \subset \Omega$ is compact, we set

$$c_q^\omega(K) = \inf \left\{ \int_{\Omega} |\Delta \zeta|^{q'} \omega dx : \zeta \in C_0^\infty(\Omega), \zeta \geq 1 \text{ in a neighborhood of } K \right\}. \quad (5.37)$$

This defines a capacity on Borel subsets of Ω . Since ω is q' -admissible, it satisfies Poincaré inequality, hence a set with zero c_q^ω -capacity is ω -negligible. Furthermore, following the proof of [2, Th 3.3.3], c_q^ω is equivalent to \check{c}_q^ω defined by

$$\check{c}_q^\omega(K) = \inf \left\{ \|\zeta\|_{W_\omega^{2,q'}}^{q'} : \zeta \in C_0^\infty(\Omega), 0 \leq \zeta \leq 1, \zeta \geq 1 \text{ in a neighborhood of } K \right\}. \quad (5.38)$$

The dual definition is (see [2, Th 2.5.1])

$$(c_q^\omega(K))^{\frac{1}{q'}} = \sup \{ \lambda(K) : \lambda \in \mathfrak{M}_b^+(K), \|\mathbb{G}[\lambda]\|_{L_\omega^q} \leq 1 \}. \quad (5.39)$$

Proof of Theorem G. Step 1: The condition is sufficient. We assume first that $L_{w,loc}^q(\Omega \setminus K) \cap u \in L^1(\Omega \setminus K)$ is a nonnegative subsolution of (1.22) in the sense of distributions in $\Omega \setminus K$ where $K \subset \Omega$ is a compact subset with c_q^ω -capacity zero. There exists a sequence of functions $\{\zeta_k\} \subset C_0^\infty(\Omega)$ with value in $[0, 1]$, value 1 in a neighborhood of K and such that $\|\Delta \zeta_k\|_{L_\omega^{q'}} \rightarrow 0$ when $k \rightarrow \infty$. Let $\rho \in C_0^\infty(\Omega)$, $0 \leq \rho \leq 1$, such that $\rho = 1$ in a neighborhood of K containing the support of the ζ_k . Using $\phi_k := (1 - \zeta_k)^\alpha \rho^\alpha$, with $\alpha > 1$, in the very weak formulation of equation (1.22) we obtain,

$$\begin{aligned} \int_{\Omega} u^q \phi_k w dx &\leq \int_{\Omega} u \Delta \phi_k dx \\ &\leq \alpha \int_{\Omega} u (1 - \zeta_k)^\alpha \rho^{\alpha-1} \Delta \rho dx - 2\alpha \int_{\Omega} u (1 - \zeta_k)^{\alpha-1} \nabla \zeta_k \cdot \nabla \rho^\alpha dx \\ &\quad - \alpha \int_{\Omega} u (1 - \zeta_k)^{\alpha-1} \rho^\alpha \Delta \zeta_k dx + \alpha(\alpha - 1) \int_{\Omega} u (1 - \zeta_k)^{\alpha-2} \rho^\alpha |\nabla \zeta_k|^2 dx \\ &\quad + \alpha(\alpha - 1) \int_{\Omega} u (1 - \zeta_k)^\alpha \rho^{\alpha-2} |\nabla \rho|^2 dx. \end{aligned} \quad (5.40)$$

Notice that the second integral in the right-hand side vanishes since $\nabla \zeta_k \cdot \nabla \rho^\alpha = 0$ by the assumption on their support. If we choose $\alpha = 2q'$, we can bound the remaining

integrals as follows:

$$\begin{aligned}
\left| \int_{\Omega} u(1 - \zeta_k)^{2q'-1} \rho^{2q'} \Delta \zeta_k dx \right| &\leq \left(\int_{\Omega} u^q \phi_k w dx \right)^{\frac{1}{q}} \left(\int_{\Omega} |\Delta \zeta_k|^{q'} (1 - \zeta_k)^{q'} \rho^{2q'} \omega dx \right)^{\frac{1}{q'}} \\
&\leq \left(\int_{\Omega} u^q \phi_k w dx \right)^{\frac{1}{q}} \left(\int_{\Omega} |\Delta \zeta_k|^{q'} \omega dx \right)^{\frac{1}{q'}}, \\
\left| \int_{\Omega} u(1 - \zeta_k)^{2q'} \rho^{2q'-1} \Delta \rho dx \right| &\leq \left(\int_{\Omega} u^q \phi_k w dx \right)^{\frac{1}{q}} \left(\int_{\Omega} |\Delta \rho|^{q'} (1 - \zeta_k)^{2q'} \rho^{q'} \omega dx \right)^{\frac{1}{q'}} \\
&\leq \left(\int_{\Omega} u^q \phi_k w dx \right)^{\frac{1}{q}} \left(\int_{\Omega} |\Delta \rho|^{q'} \omega dx \right)^{\frac{1}{q'}}, \\
\left| \int_{\Omega} u(1 - \zeta_k)^{2q'-2} \rho^{2q'} |\nabla \zeta_k|^2 dx \right| &\leq \left(\int_{\Omega} u^q \phi_k w dx \right)^{\frac{1}{q}} \left(\int_{\Omega} |\nabla \zeta_k|^{2q'} \rho^{2q'} \omega dx \right)^{\frac{1}{q'}} \\
&\leq \left(\int_{\Omega} u^q \phi_k w dx \right)^{\frac{1}{q}} \left(\int_{\Omega} |\nabla \zeta_k|^{2q'} \omega dx \right)^{\frac{1}{q'}},
\end{aligned}$$

and finally

$$\begin{aligned}
\left| \int_{\Omega} u(1 - \zeta_k)^{2q'} \rho^{2q'-2} |\nabla \rho|^2 dx \right| &\leq \left(\int_{\Omega} u^q \phi_k w dx \right)^{\frac{1}{q}} \left(\int_{\Omega} |\nabla \rho|^{2q'} (1 - \zeta_k)^{2q'} \omega dx \right)^{\frac{1}{q'}} \\
&\leq \left(\int_{\Omega} u^q \phi_k w dx \right)^{\frac{1}{q}} \left(\int_{\Omega} |\nabla \rho|^{2q'} \omega dx \right)^{\frac{1}{q'}}.
\end{aligned}$$

Since the Gagliardo-Nirenberg inequality holds with the q' -admissible weight ω , we have for some $\tau \in (0, 1)$ and some $c = c(q, N) > 0$,

$$\begin{aligned}
\left(\int_{\Omega} |\nabla \zeta_k|^{2q'} \omega dx \right)^{\frac{1}{2q'}} &\leq c \left(\int_{\Omega} |\Delta \zeta_k|^{q'} \omega dx \right)^{\frac{\tau}{q'}} \|\zeta_k\|_{L^\infty}^{1-\tau} \\
&\leq c' \left(\int_{\Omega} |\Delta \zeta_k|^{q'} \omega dx \right)^{\frac{\tau}{q'}}.
\end{aligned} \tag{5.41}$$

Therefore, if we set

$$X_k = \left(\int_{\Omega} u^q \phi_k w dx \right)^{\frac{1}{q}} \quad \text{and} \quad Z_k = \left(\int_{\Omega} |\Delta \zeta_k|^{q'} \omega dx \right)^{\frac{1}{q'}},$$

we obtain the inequation

$$X_k^q \leq c_1 X_k Z_k + c_2 X_k + c_3 X_k Z_k^\tau, \tag{5.42}$$

for some positive constants c_1, c_2, c_3 depending on q, N and ρ . By definition of ζ_k we have $Z_k \rightarrow 0$. We thus deduce that $X_k^q \leq cX_k$ with $q > 1$ and then that the sequence $\{X_k\}$ is bounded. Since $\zeta_k \rightarrow 0$ almost everywhere, we have $\phi_k \rightarrow \rho^{2q'}$ almost everywhere. It then follows by Fatou's lemma that

$$\int_{\Omega} u^q \rho^{2q'} w dx \leq c. \quad (5.43)$$

We deduce that $u \in L_{w,loc}^q(\Omega)$. Since $\omega^{-\frac{q'}{q}} \in L_{loc}^1(\Omega)$, we obtain that $L_{loc}^1(\Omega)$ by Hölder's inequality. If $u \in L_{w,loc}^q(\Omega \setminus K) \cap u \in L^1(\Omega \setminus K)$ is a distributional solution of (1.22) in $\Omega \setminus K$, then $|u|$ is a nonnegative subsolution with the same integrability constraints and we derive $u \in L_{w,loc}^q(\Omega) \cap L_{loc}^1(\Omega)$.

If $\phi \in C_0^\infty(\Omega)$, we take $\phi(1-\zeta_k)^{2q'}$ for test function of equation (1.22) in $\mathcal{D}'(\Omega \setminus K)$,

$$-\int_{\Omega} u \Delta(\phi(1-\zeta_k)^{2q'}) dx + \int_{\Omega} |u|^{q-1} u \phi(1-\zeta_k)^{2q'} w dx = 0.$$

Since $u \in L_{w,loc}^q(\Omega)$, ϕ has compact support, and $\zeta_k \rightarrow 0$ almost everywhere, we can pass to the limit as $k \rightarrow +\infty$ in the second integral using Lebesgue convergence theorem and obtain

$$\int_{\Omega} |u|^{q-1} u \phi(1-\zeta_k)^{2q'} w dx \rightarrow \int_{\Omega} |u|^{q-1} u \phi w dx.$$

Moreover we can pass to the limit in the first integral expanding the laplacian. Using that $u \in L_{loc}^1(\Omega)$ and that $\Delta \zeta_k \rightarrow 0$ in $L_{loc}^{q'}$, it is easy to prove from the previous computation that

$$\int_{\Omega} u(1-\zeta_k)^{q'} \Delta \phi dx \rightarrow \int_{\Omega} u \Delta \phi dx \quad \text{as } k \rightarrow \infty,$$

and

$$\lim_{k \rightarrow \infty} \int_{\Omega} u(1-\zeta_k)^{2q'-1} \nabla \zeta_k \cdot \nabla \phi dx = 0 = \lim_{k \rightarrow \infty} \int_{\Omega} u(1-\zeta_k)^{2q'-1} \phi \Delta \zeta_k dx.$$

Hence

$$-\int_{\Omega} u \Delta \phi dx + \int_{\Omega} u^q \phi w dx = 0 \quad (5.44)$$

Step 2: The condition is necessary. Let K be a compact set with positive c_q^ω -capacity. According to [2][Th 2.5.3] there exists an extremal $\mu_k \in \mathfrak{M}_b^+(K)$ in the dual formulation (5.39) of the capacity. According to Theorem D, problem (5.16)

with $\mu = \mu_K$ admits a positive solution which is therefore a positive solution of (5.35). \square

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