



Perspectives on design research: the case of didactical engineering

Michèle Artigue

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Abstract: In what is often called the “French didactical culture”, design has always played an essential role in research. This is attested by the introduction and institutionalization of a specific concept, that of *didactical engineering*, already in the early eighties and by the way didactical engineering has accompanied the development of didactical research, both in its fundamental and applied dimensions. In this chapter, I present this vision of design and its characteristics as a research methodology, coming back to its historical origin in close connection with the development of the theory of didactical situations, tracing its evolution along the last three decades, and illustrating this methodology by some particular examples. I also consider current developments within this design culture, especially those linked to the integration of a design dimension into the anthropological theory of didactics and also to the idea of didactical engineering of second generation introduced for addressing more efficiently the development dimension of didactical engineering.

I. Introduction

Design has always played a substantial role in mathematics education up to the point that some researchers consider this field as a design science (see for instance (Wittmann, 1998) or (Cobb, 2007)). But the conception of design and the exact role it is given in research strongly depend on educational cultures. In this chapter we consider the case of what is often called the “French didactical culture” in which design has always played a fundamental role. This importance of design is attested by the introduction and institutionalization of a specific concept, that of *Didactical Engineering* (DE in the following) already in the early eighties. Since that time DE, which developed in close connection with the theory of didactical situations initiated by Brousseau (cf. (Warfield, 2006) for an introduction and (Brousseau, 1997) for a more detailed vision), has accompanied the development of didactical research, both in its fundamental and applied dimensions. This chapter is structured into four main sections. In the first section I briefly review the development of DE from its emergence in the early eighties until now, and clarify its links with the theory of didactical situations (see also (Bessot, 2011)). In the second section I present its characteristics as a research methodology. In the third section I illustrate this methodology with examples taken at different levels of schooling. In the fourth section I consider two recent evolutions of DE. The first one is that conveyed by the anthropological theory of didactics in terms of course of study and research that considers very open forms of design; the second one is that of “didactical engineering of second generation” introduced by Perrin-Glorian for addressing dissemination and up-scaling issues (2011). Beyond the many examples of realizations, the writing of this chapter has been especially inspired by some foundational texts such as (Chevallard, 1982), (Artigue, 1989, 2002, 2008), and by the extensive reflection on didactical engineering carried out at the XV^e Summer School of Didactics of Mathematics in 2009 (Margolinas & al., 2011).

II. Didactical engineering: an historical review

The emergence, consolidation and evolution of didactical engineering can be traced in the successive summer schools of didactics of mathematics organized every two years in France since 1980. In this brief historical review, I focus on three of these (1982, 1989, 2009) for

which DE was a specific theme of study. Already, at the second summer school in 1982, DE was one of the themes addressed. Chevallard prepared a specific manuscript note for supporting the work of the summer school collective (Chevallard, 1982), Brousseau gave a course and practical sessions were organized around this theme. Accessible documents regarding this summer school show the shared conviction that didactical research should give a more central role to the construction and study of classroom realizations. French researchers express concerns about the observed tendency to privilege methodologies borrowed from established fields such as psychology (clinical interviews, questionnaires, pre-test/post-test comparisons...) for ensuring the scientific legitimacy of research in mathematics education. They point out that the didactics of mathematics is a genuine scientific field whose methodologies should be in line with its specific purpose: the study of intentional dissemination of mathematical knowledge through didactical systems, and the associated interaction between teaching and learning processes. As explained in Chevallard's note, the need for developing specific methodologies based on classroom realizations is justified by both theoretical and practical reasons. On the theoretical side, such methodologies are necessary for this essential part of scientific activity which is the production of phenomena (in this case, didactical phenomena), what Bachelard (1938) called *phenomenotechnique*. On the practical side, such methodologies are necessary for establishing productive relationships between research and practice, as they permit researchers to consider didactical systems in their concrete functioning, and to pay the necessary attention to the different constraints and forces acting on these, what could be neglected otherwise.

Didactical engineering thus emerged as a research and development methodology based on classroom realizations in form of sequences of lessons, informed by theory and putting to the test theoretical ideas. At that time, what was predominant in the French didactical community was the theory of didactical situations that had emerged in the late sixties. This theory became thus the natural support of DE. Its systemic perspective, constructions and values shaped DE which progressively became the research methodology privileged within this community. In fact, it would be more adequate to say that theoretical constructions and DE jointly developed along the eighties.

In 1989, for the second time, didactical engineering was a specific theme of the summer school and I was asked to give a course on this methodology. This course (Artigue, 1989) contributed to the institutionalization of DE as a research methodology, making explicit its characteristics and its foundational links with the theory of didactical situations. It also pointed out that its privileged links with the theory of didactical situations did not prevent researchers using this methodology from relying on other theoretical approaches. For instance, several examples mentioned in the course or worked out in the practical sessions associated to it relied on the tool-object dialectics due to Douady (1986). Many contributions to the summer school indeed combined its specific constructs (through the attention paid in design to the dialectics to be organized between the tool and object dimensions of mathematics concepts and to the learning potential offered by moves between mathematical settings, numbers and geometry for instance) with those offered by the theory of didactical situations. In this course too, I pointed out that if DE had consolidated as a research methodology, the problem of establishing productive links between research and practice had not been solved. DE produced by research was disseminating through articles, educational resources and teacher education, but there was some evidence that along this dissemination process, it tended to lose its essence and value.

In fact, in coherence with the theory of didactical situations, in DE design, particular efforts had been made to create situations in which:

- the mathematical knowledge aimed at was an optimal solution to the problem to solve (what is captured in the theory by the idea of *fundamental situation*);
- students as a collective were able to reach this optimal solution through their interaction with the *milieu*¹ of the situation, without significant help from their teacher (what is captured in the theory by the idea of *a-didactical situation*).

The teacher's role, for its part, had been mainly approached in terms of the dual processes of *devolution* and *institutionalization*, coherently with the vision of learning as a combination of *adaptation* and *acculturation* processes underlying the theory. Through the devolution process, the teacher tries to make her students accept the mathematical responsibility of solving the problem at stake. She tries to make thus possible the a-didactic interaction with the milieu required for learning through adaptation. If the devolution process is successful, the students agree to forget for a while the didactical intention of the teacher; to concentrate on the search for mathematical solutions instead of trying to decipher the teacher's expectations. Through the dual process of institutionalization, the teacher connects the knowledge built by students through a-didactic interaction with the milieu to the scholarly and decontextualized forms of knowledge aimed at by the institution, making it possible the acculturation dimension of learning.

In 1989, even if research DEs had been able to approach in many cases this ideal-type, their functioning out of the control of research seemed difficult. Moreover, in the DE produced, high visibility had been given to the situations, often very innovative, designed for introducing new mathematical ideas or overcoming *epistemological obstacles*, and much less to the more standard situations used for consolidating mathematical knowledge and techniques. This situation created a distorted vision of DE products that certainly had negative impact on the quality of their dissemination.

In 2009, twenty years later, DE was once again a theme for the summer school and more, its unique theme. Since 1989, the didactic field had substantially evolved. The anthropological theory of didactics that was just emerging in the late eighties had matured and gained in influence. Moreover, in the last decade, it had created its own design approach in terms of activities of research and study and then programs of study and research (Chevallard, 2006, to appear). A new theoretical framework had also emerged from the theory of didactical situations and the anthropological theory of didactics: the theory of joint action between teachers and students, proposing a renewed vision of the role of the teacher and of students-teacher interactions (Sensevy, 2011, 2012). More generally, teachers' practices and professional development had become a focus of research, and this research had developed its own methodologies involving naturalistic and participative observations of classrooms. DE was still an important research methodology, especially each time the didactical systems one wanted to study could not be observed in natural conditions (as is for instance often the case in research about technology), but was no longer the privileged methodology (Artigue, 2002, 2008). Didactical engineering had also migrated outside its original habitat. It has been extended to teacher education and to the study of innovative pedagogical practices, including informal education; didacticians from other disciplines, for instance physical sciences or sports, had used it (Terrisse, 2002); researchers educated in other countries and cultures, and having different theoretical backgrounds, had used it, for instance researchers referring to the socio-epistemological framework in mathematics education (Farfán, 1997), (Cantoral &

¹ In the theory, the milieu of a situation is defined as the system with which the student interacts, and which provides objective feedback to her. The milieu may comprise material and symbolic elements: artifacts, informative texts, data, results already obtained..., and also other students who collaborate or compete with the learner.

Farfán, 2003) or to semiotic approaches (Maschietto, 2002), (Falcade, 2006). Moreover, design-based research perspectives had emerged and grown in other contexts, independently of it (Burkhardt & Schoenfeld, 2003), (Design-based Research Collective, 2003). These conditions created the need for a thorough reflection about the concept of DE and this was the exact purpose of the 2009 summer school. I have integrated some of the results of this reflection in the next section describing the characteristics of DE as a research methodology, and some others will be dealt with in the fourth section. Nevertheless the size of this chapter does not allow to pay full justice to the work carried out at this summer school and those interested are invited to read (Margolinas & al., 2011).

III. Didactical engineering as a research methodology

In this section, I present the characteristics of DE as a research methodology, using for that purpose its most standard form: the conception, realization, observation, analysis and evaluation of classroom realizations aiming at the learning of a specific content. However, it should be clear that, while obeying fixed principles, this research methodology might take a diversity of forms in practice, according to the nature of the questions addressed by the researchers, and to the contexts involved. I will end this section by pointing out some similarities and differences with design-based research perspectives more and more influential in mathematics education.

One essential characteristic of DE as a research methodology is that, contrary to the traditional use of classroom realizations in educational research, it does not obey the validation paradigm based on the comparison of control and experimental groups. Its validation is internal and based on the comparison between the *a priori* and *a posteriori* analyses of the didactic situations involved. This methodological choice can be easily understood considering the educational culture in which DE has emerged. In this culture, as explained above, research in mathematics education (didactics of mathematics) is seen as a scientific field of its own whose ambition is the study of the intentional dissemination of mathematical knowledge through didactical systems. What is to be understood is the functioning of such didactical systems, and associated didactical phenomena, which requires entering into the intimacy of their functioning. Validating the hypotheses engaged in the conception phase of a DE cannot be thus a matter of comparison between experimental and control groups.

As a methodology, DE is structured into different phases. These are the following: preliminary analyses, conception and *a priori* analysis, realization, observation and data collection, *a posteriori* analysis and validation.

III.1. Preliminary analyses:

Preliminary analyses set the background for the conception phase of the process. They cover different dimensions, and especially the three following:

- An epistemological analysis of the content at stake, often including an historical part. This analysis helps researchers to fix the precise goals of the DE and to identify possible epistemological obstacles to be faced. It also supports the search for mathematical situations representative of the knowledge aimed at, what the theory of didactical situations calls *fundamental situations*. These are problematic situations for the solving of which this knowledge is necessary or in some sense optimal. The epistemological analysis helps the researchers to take the necessary reflective position and distance with respect to the educational world they are embedded in, and to build a reference.

- An institutional analysis whose aim is to identify the characteristics of the context in which the DE takes place, the conditions and constraints it faces. These conditions and constraints may be situated at different levels of what is called the *hierarchy of levels of determination* (Chevallard, 2002) in the anthropological theory of didactics. They may be attached to curricular choices regarding the content at stake and associated teaching practices, to more general curricular characteristics regarding the teaching of the discipline, the (technological) resources accessible, the evaluation practices and the school organization. They can also be linked to the characteristics of the students and teachers involved, to the way the school is connected with its environment... Depending on the precise goals and context of the research, the importance attached to these different levels may of course vary.
- A didactical analysis whose aim is to survey what research has to offer regarding the teaching and learning of the content at stake, and is likely to guide the design.

The three dimensions organizing the phase of preliminary analyses reflect the systemic perspective underlying DE as a research methodology. Each of them has its methodological specificities and needs. The epistemological analysis often involves the use of historical sources and not just secondary sources; the institutional analysis also generally includes an historical dimension. As made clear by the theory of didactical transposition (Bosch & Gascón, 2005), curricular organizations and choices are the result of a long-term historical process; they cannot be understood just by analyzing current curricula, official documents and textbooks. Such understanding is needed for making clear the strength of the constraints faced and the way some of these can be moved in the design. The didactical analysis has generally a substantial cognitive dimension, but this cognitive dimension is only one part of the global picture even if what is aimed at is the development of a didactical strategy allowing students to learn better some part of mathematics.

It must also be pointed out that, according to the precise goals of the research, what is exactly investigated in each of these dimensions, and the respective importance attached to them may vary substantially.

III.2. Conception and a priori analysis:

Conception and a priori analysis is a crucial phase of the methodology. It relies on the preliminary analyses carried out, and is the place where research hypotheses are made explicit and engaged in the conception of didactical situations, where theoretical constructs are put to the test. Conception requires a number of choices and these situate at different levels. Some pilot the global project and in that case it is usual to speak of *macro-choices*; some situate at the level of a particular situation, and in that case it is usual to speak of *micro-choices*. These choices determine *didactical variables*², so we have both *macro-didactic and micro-didactic variables*. These variables condition the milieu, thus the interactions between students and knowledge, the interactions between students and between students and teachers, thus the exact opportunities that students have to learn, how and what they can learn. In line with the theoretical foundations of DE, in these choices particular attention must be paid to the epistemological pertinence of the problems posed and to the mathematical responsibility given to the students.

² Among the many variables influencing the possible dynamics of a situation and its learning outcomes, didactical variables are those under the control of the teacher. In a situation of enlargement such as the well-known “Puzzle situation” by Brousseau, the number of pieces of the puzzle, their shapes and dimensions, the ratio of enlargement are didactical variables; the fact that students work in group, each student being asked to enlarge one piece of the puzzle is also a didactical variable.

The *a priori* analysis makes clear the different choices made and the way they relate to the research hypotheses and preliminary analyses. For each situation, it identifies the main didactical variables, that is to say those which affect the efficiency and cost of the possible strategies developed by students, and their possible dynamics. These variables can be attached to the characteristics of the tasks proposed to students, but they can also be linked to the resources provided to the students for solving these tasks (what in the theory corresponds to the *material milieu* of the situation) and to the way the students' interaction with the *milieu* is socially organized. From these characteristics, conjectures are made regarding the possible development of the situation, students' interaction with the *milieu*, students' strategies and their evolution, and the possible sharing of mathematical responsibilities between teacher and students. It is important to stress that such conjectures regard not individuals but a *generic and epistemic student* who enters the situation with some supposed knowledge background and is ready to play the role that the situation proposes her to play. Of course, the realization will involve students with their personal specificities and history, but the goal of the *a priori* analysis is not to anticipate how each particular student will behave and benefit from the situation, but what the situation *a priori* can offer in terms of learning in the context at stake. It creates a reference with which classroom realizations will be contrasted.

III.3. Realization, observation and data collection

During the realization, data are collected for the analysis *a posteriori*. The nature of the data collected depends on the precise goals of the DE, on the hypotheses put to the test in it and on the conjectures made in the *a priori* analysis. However, particular attention is paid to the collection of data allowing the researcher to understand students' interaction with the milieu, and up to what point this interaction support their autonomous move from initial strategies to the strategies aimed at, and to analyze devolution and institutionalization processes. Generally data collected include the students' productions including computer files when technology is used, field notes from observers, audios and, more and more, videos covering group work and collective phases. The data, collected during the realization are generally complemented by additional data (questionnaires, interviews with students and teacher, tests) allowing a better evaluation of the outcomes of the DE. During the realization, researchers are in the position of observers. It is important to point out that the realization often leads to make some adaptation of the design during the realization, especially when the DE is of substantial size. In that case, adaptations are of course documented together with the rationale for them and taken into account when the *a posteriori* analysis is carried out.

III.4. A posteriori analysis and validation

A posteriori analysis is organized in terms of contrast with the *a priori* analysis. Up to what point do the data collected during the realization support the *a priori* analysis? What are the significant convergences and divergences and how can they be interpreted? What happened that was not anticipated and how can it be interpreted? Through this connection between *a priori* and *a posteriori* analyses, the hypotheses underlying the design are put to the test. It is important to be aware that there are always differences between the reference provided by the *a priori* analysis and the contingency analyzed in the *a posteriori* analysis. As observed above, the *a priori* analysis deals with generic and epistemic students, which it not the case for the contingency of the realization. Thus, the validation of the hypotheses underlying the design does not impose perfect match between the two analyses.

The analyses carried out are qualitative in nature and local, even when the researchers use statistical tools such as for instance implicative analysis for identifying dependences. In accordance with the theoretical foundation of DE, what the researcher looks at is the dynamic of a complex system, and he does so through the comparison of the observed dynamics with

the reference provided by the a priori analysis, trying to make sense of similarities and differences. The precise tools used for that purpose depend on the research questions at stake and the data collected. There is no doubt however that these tools have evolved along the years, influenced by the global evolution of the field and also by the technological evolution. In general, researchers combine and triangulate different scales of analyses. They more and more include microscopic analyses taking into account the multimodality of the semiotic resources used by students and teachers that technology makes today accessible. To this should be added that, as mentioned above, the validation of the research hypotheses generally combines the analysis of data collected during the classroom sessions themselves and of complementary data.

It must be stressed that the results obtained are local, contextualized, and generally in form of existence theorems in their positive forms. For instance, in the research I developed about the teaching of differential equations in the mid eighties, I used DE methodology to investigate the possibility of combining qualitative, algebraic and numerical approaches to the solving of ordinary differential equations in a university mathematics course for first year students. The research showed the possibility of organizing such a course in the French context at that time with the support of technological tools; it made clear what could be expected from such a course in terms of learning outcomes and why. Beyond that, one important result was that a condition for the viability of the course was the acceptance by the didactical system of proofs based on specific graphical arguments, which violated the usual didactical contract regarding proofs in Analysis at university. The difficulty of ensuring this acceptance out of experimental contexts and research control at that time hindered a large-scale dissemination of the didactical strategy developed, despite the fact that its robustness had been attested by experimentations carried out with different categories of students.

Looking back at decades of DE research, what is evident is that the results of DE research are far from being limited to the production and validation of didactical designs. DE research has also been a highly productive tool for understanding the functioning and ecology of didactical systems, and for identifying didactical phenomena. For instance, this was the case for the obsolescence phenomenon that helped researchers question their vision of reproducibility for didactical situations (Brousseau, 1997), (Artigue, 1986). DE research has been for decades an essential tool for the development of theoretical constructs paying justice to the complexity of the systems involved in the teaching and learning of mathematics.

What I have described here are the characteristics of the main form of DE: a research methodology based on the conception, experimentation and evaluation of a succession of classroom sessions having a precise mathematical aim. As already mentioned, this methodology has been extended to other contexts such as teacher education, to more open activities such as project work or modeling, and even to mathematical activities carried out in informal settings such as summer camps which obey a different form of contract, what Pelay defines as the *didactical and ludic contract* (Pelay, 2011). These extensions influence the content of preliminary analyses, but also what the design ambitions to control in terms of learning trajectories. The reference provided by the a priori analysis cannot exactly have the same nature, and this impact the ways a priori and a posteriori analyses are contrasted.

I will end this section by situating didactical engineering with respect to design-based research, using the definition of it provided in the Encyclopedia of mathematics education (Swan, 2013):

Design-based research is a formative approach to research, in which a product or process (or ‘tool’) is envisaged, designed, developed and refined through cycles of enactment, observation, analysis and redesign, with systematic feedback from end-users. In education, such tools might, for example,

include innovative teaching methods, materials, professional development programmes and/or assessment tasks. Educational theory is used to inform the design and refinement of the tools, and is itself refined during the research process. Its goals are to create innovative tools for others to use, to describe and explain how these tools function, account for the range of implementations that occur and develop principles and theories that may guide future designs. Ultimately, the goal is *transformative*; we seek to create new teaching and learning possibilities and study their impact on teachers, children and other end-users.

This definition makes clear that design-based research and DE have some common methodological characteristics. Both methodologies are organized around the design of some educational tool; this design is informed by educational theory, but also contributes to its development. Moreover, both methodologies reject standardized validation processes based on the comparison of experimental and control groups through a pre-test / post-test system. However, differences are visible. The global vision underlying design-based research is that of mathematics education as a design science whose aim is the controlled production of educational tools (Wittman, 1995), (Collins, 1992); the global vision underlying DE is of didactics of mathematics as a fundamental science, whose aim is the understanding of didactical systems and didactical phenomena, and which has also an applied dimension. This fundamental difference reflects in methodological characteristics. Design-based research is interventionist and iterative in nature, and the cyclic nature of its process is essential. Along the successive cycles, the design is refined but also experimented in wider contexts for studying how it functions with different categories of users, not involved in the design process, and what adaptations may be necessary for its large-scale use. Didactical engineering as a research methodology does not obey the same pattern. It is more a “phenomeno-technique” with the meaning given to this term by Bachelard (1934, 1936), a tool for answering didactical questions, for identifying, analyzing and producing didactical phenomena through the controlled organization of teaching experiments. This is the reason why the preliminary analyses with their different dimensions, the a priori analysis are a central part of the research work, and are given so much importance in the articles referring to this methodology. Of course, this does not mean that a DE used in research is built from scratch, but previous constructions when they exist are used to inform the a priori analysis; the process is not theorized as a cyclic process. Moreover, what concerns robustness and up-scaling is considered as a matter of development. I will come back to this point in the last section of the chapter, but will first illustrate the ideas developed up to now with two examples.

IV. Two particular examples

IV.1. A paradigmatic example: the extension of number field by G. and N. Brousseau

The first example I will consider is the paradigmatic example of the didactical engineering developed by N. and G. Brousseau for extending the field of whole numbers towards rational and decimals (Brousseau & Brousseau, 1988), (Brousseau, 1997). This engineering which ranges over 65 classroom sessions is a very big object when compared with usual constructions whose size is much more limited. We cannot enter into its very details but would like to show how this construction is characteristic of a DE piloted by the theory of didactical situations. This construction evidences the importance attached to the preliminaries analyses, and especially to the epistemological and didactical dimensions of it, the realizations taking place in the COREM³ where the institutional pressure was reduced. These analyses led

³ COREM was the Center for observation and research in mathematics education created by Brousseau in Bordeaux in 1976. An experimental elementary school was attached to this center, with very advanced means for

Brousseau to question the usual educational strategy for extending the field of whole numbers. In this extension, the first step was the introduction of decimal numbers in connection with changes in units in the metric system, and fractions played a more marginal role. Emphasis was put on the continuity between the two systems of numbers, especially regarding the techniques for arithmetic operations, and the resistant cognitive difficulties that these strategies generated or reinforced were more and more evidenced by research. Brousseau made the hypothesis that, in their last years at elementary school, students were able to learn much more about rational and decimal numbers, for instance to differentiate the dense order of rational and decimal numbers from the discrete order of whole numbers, to appreciate the computational interest of decimal numbers and the possibility that this system offers for approaching rational numbers with arbitrary levels of precision. The didactical engineering developed aimed at testing the validity of this hypothesis with ordinary students. The epistemological analysis carried out inspired the first macro-choice made, in clear rupture with established practices: to extend first the field of numbers towards rational numbers, and then to particularize decimal numbers among these for the facilities they offer in terms of comparison, estimation and calculation. Regarding the introduction of rational numbers, another macro-choice was made linked to the identification of two different conceptions for rational numbers: a conception in terms of partition of the unit ($1/n$ is then associated with the partition of one unit into n equal parts and the rational m/n represents m such pieces of the unit) and a conception in terms of commensurability, which corresponds to the search of a common multiple to two different magnitudes for instance two lengths (the ratio of two magnitudes is expressed by the rational m/n if m times the second one equals n times the first one). Generally didactical strategies privilege the first conception in the context of pizza parts or other equivalent contexts. This constitutes an easy entrance in the world of fractions but Brousseau hypothesized that it could contribute to the observed cognitive difficulties. This led him to explore the potential offered by an entry in terms of commensurability, and to search for a fundamental situation attached to this conception: a situation that would oblige to consider multiples of magnitudes to compare them.

The problem posed to the grade 4 students was the following: how to compare the thickness of different sheets of paper? There is no doubt that this problem answers the condition just mentioned. The thickness of a sheet of paper cannot be directly measured with usual instruments but taking a sufficient number of such sheets one obtain something measurable. This problem being fixed, different choices must be done for defining a situation. Evident didactic variables are the number of types of paper to compare and their respective thickness. Anticipating that a basic strategy for students is to use their senses (sight and touch) for ordering the different types, it is important to have papers of close thickness invalidating perceptive strategies. Another choices concern, as mentioned above, the organization of the material *milieu* and the students' interaction with this *milieu*, the social organization of the classroom. In the organization adopted in this DE, the material *milieu* was made of piles of sheets of different thickness often very close and students worked in groups. First, they had to find a way of comparing the thickness of the sheets provided to their group, then in a second phase, after selecting one type of paper, to write a message allowing another group of students having the same types of paper to find the paper they had selected. These messages became then themselves an object of study: did the messages produced by the different groups solved the particular problem each group had to address, and beyond that did they provide a technique for solving the problem of comparison in a general way? We can see here a construction which takes into account the distinction made in the theory of didactical

situations between three different functionalities of mathematical knowledge: for acting, for formulating, for proving. Their development obeys different dialectics and thus supposes different types of situations: *situations of action* (here in the first phase), *situations of formulation* when what is the key for the success is the quality of the specific language developed (in the second phase), and *situations of validation* when what is at stake is the validity of assertions (in the third phase).

In an implicit way, the winning strategy in this situation uses the fact that the thickness of a pile is proportional to the number of sheets, which constitutes a reasonable model under certain limits, of course. In fact, couples attached to the same paper obtained through manipulations are not exactly proportional, which shows the distance which separates the real world from mathematical models. In observed realizations, this strategy systematically emerged through a didactic interaction with the *milieu*, this emergence being certainly fostered by the presence of piles of paper in the material *milieu*. In the *a priori* analysis, it is expected that each type of paper will be eventually characterized by one or several couples of whole numbers nearly proportional, in reference to the manipulations carried out by the students, for instance 1mm for 27 sheets in one case, 2mm for 40 sheets in another case. Once such couples are obtained, as they do not necessarily correspond to the same number of millimeters or to the same number of sheets, if students are not allowed more manipulations, the success of the comparison relies on proportional reasoning and thus, for a good functioning of the interaction with the *milieu*, it is necessary that some knowledge about proportional reasoning is part of the mathematical knowledge shared by students (in the *a priori* analysis, this knowledge is supposed from the generic student). For instance, if the task is to compare the types of paper corresponding to the two couples mentioned above, one can develop the following reasoning: for the first paper, 2mm should correspond to 54mm, and 54 is more than 40, thus the second paper is thicker. For close thicknesses, comparison may be more delicate for the reasons mentioned above, and several exchanges of messages needed.

What is mathematically at stake in the solving of this problem is the ordered structure of rational numbers seen as couples of whole numbers or more appropriately families of such couples, and the conception attached is clearly the commensurability conception. Let us point out that if substantial work can be developed in this context about equality and order of rational numbers, if students can progressively discover a good number of properties in a didactic interaction with the successive milieux organized for them, validate them pragmatically using piles of paper, and then use piles of paper more metaphorically for supporting computations and reasoning, the mathematical knowledge built remains attached to this specific context. There is no reason that the notations introduced by students and progressively refined for reasons of economy and efficiency are the conventional notations. This is the responsibility of the teacher to decide when to connect these classroom notations to the usual ones expected by the institution, and also to organize the decontextualization of knowledge through appropriate situations. Of course, in the DE, these steps are also carefully designed.

In this DE, the same context is then used for extending addition to these new numbers but it does not allow to extend in a similar way multiplication to rational numbers. For this extension, the choice is made of privileging a conception of multiplication as an external operation in terms of linear application for which the well-known situation of the puzzle is the associated fundamental situation. With this new situation, it is also expected to make students face the epistemological obstacle of the additive model.

I cannot enter into more details in this DE structured in four main phases and invite the reader interested to consult the references mentioned above or the retrospective analysis provided in (Brousseau & Brousseau, 2006). In the description above, I have focused on the essential

phases of design and a priori analysis of the methodology, trying to show how they were informed by the preliminary analyses and guided by the theory of didactical situations. The experimentations took place in the experimental school attached to the COREM, the sessions being observed by researchers according to specific guidelines and systematically video-recorded. The comparison of the a priori and a posteriori analyses, the complementary tests taken by the students, validated the hypotheses underlying the DE. It must be also stressed that this DE was in fact used for approaching a diversity of research questions, and for instance for investigating dependences between conceptions (Ratsimba-Rajohn, 1982). In his doctoral thesis, indeed, Ratsimba-Rajohn, starting from the two strategies for associating a rational measure to a magnitude mentioned above (commensurability and partition of the unit), precisely differentiated these in terms of situations of effectiveness and mathematical knowledge engaged. This analysis led to the identification of a set of 9 variables conditioning the effectiveness and cost of each strategy, depending on the type of task (game in the terminology used by the author, in line with the use of game theory in the theory of didactical situations). The author used this tool for investigating how students introduced to rational measures through the commensurability strategy, as was the case in the DE, could enrich their strategies by incorporating the partitioning strategy, a priori more intuitive and socially used. For that purpose, a sequence of three situations was designed as part of the DE. In the first situation, the commensurability strategy was extended to other magnitudes (length, weight, capacity); in the second situation, the tasks proposed were out of the domain of effectiveness of the commensuration strategy and adapted to the partition strategy. The goal of the third situation was to initiate the validation of equivalence of the two models when both strategies were effective. The corresponding lessons were implemented two consecutive years. Students' strategies and their evolution were carefully documented. Different dynamics were identified, but the most striking result was the difficulty that these students had at moving from commensuration strategies to partition strategies, even when commensuration was ineffective. These difficulties were confirmed by the evolution of students' answers at a test taken by the students before and after the teaching sequence in the first year of experimentation. All students significantly progressed in their answers to questions favoring the commensuration strategy or neutral, but only one student progressed on questions blocking the commensuration strategy. Difficulties met at using commensuration and efforts made for overcoming these difficulties in fact tended to reinforce this strategy and the associated conception of rational numbers; more was needed for integrating an alternative conception in terms of partition, despite the fact that it seemed a priori much more accessible than the commensuration conception.

This DE was used year after year in the experimental school attached to the COREM. More than 750 students were exposed to it and its robustness was confirmed. However, as often stressed by Brousseau himself, it was never considered that this DE could be easily implemented in ordinary schools and was likely to become a standard teaching strategy. Moreover, the comparison of the successive dynamics attracted Brousseau's attention to the fact that the reproduction of the same situations, year after year, by a teacher generated what he called a phenomenon of obsolescence affecting the internal reproducibility of the DE. This phenomenon more globally raised the issue of the reproducibility of didactical situations that was theorized in further work (Artigue, 1986).

IV.2. An example of DI combining TDS with another theoretical frame

The second example I will consider is substantially different. It corresponds to a didactical engineering developed by Maschietto in her doctoral thesis (Maschietto, 2002) on the transition between algebra and Analysis. The goal of this DE is to explore the possibility of

introducing students very early to the local/global game on functional objects fundamental in Calculus and Analysis, through the introduction of the derivative in terms of local linear approximation. The main hypothesis made is that through an appropriate use of the potential offered by symbolic calculators, this local/global game can be initiated in high school and that the idea of derivative and associated computations can be built by the students as mathematization of a perceptive phenomenon. Another aim of this DE whose theoretical framework combines the theory of didactical situations and the theory of semiotic mediations (Bartolini & Mariotti, 2008) is to analyze how gestures and metaphors (Arzarello & Edwards, 2005), (Lakoff & Nuñez, 2000) contribute to the mathematization process and the cognitive development of students. As summarized in (Maschietto, 2008):

« The research hypothesis is that the transformations of the graphical representation of a function through the use of zoom-controls and the experience of the perceptive phenomena of “micro-straightness” that these transformations provoke, can give rise to the formulation of some specific language, the construction of metaphors and the production of gestures and specific signs by the students. Our hypothesis is also that adequately exploited by the teacher, these germs can lead to an entrance in the local/global game, fundamental in Calculus and Analysis hardly observed at high-school level. »

We find in this DE interesting variations from the standard case which illustrate how, while maintaining the foundational values of this methodology, researchers can adapt it to their theoretical culture and needs. In its presentation, I will try to make clear how this theoretical combination affects the methodological work.

In this DE, we observe still the same attention paid to preliminary analyses. Maschietto develops a detailed analysis of the different perspectives that can be attached to a function: punctual, local, global, of the idea of local straightness, and of thinking modes in Analysis. Her epistemological analyses also aims at understanding how, before the official introduction of the concept of limit, the language of infinitesimals can support the transition from algebra towards Calculus, fostering the identification of rules for computations which take into consideration the respective order of magnitudes of the quantities involved. From an institutional perspective, the DE is strongly constrained. Realizations can only be organized at the end of the school year in grade 11 in the Italian context, and in usual practices very few sessions are devoted to the topic. Moreover the use of calculators is usually limited in ordinary classrooms and that of symbolic calculators nearly inexistent. What is proposed is thus at distance of usual practices and would be impossible to observe in naturalistic conditions. In fact, Maschietto worked with a teacher used to collaborate with researchers, but the institutional constraints limited the realization to a few sessions, two consecutive years. Six were initially planned, but the thesis only analyses the three first ones implemented in the three experimentations carried out.

Didactic analysis classically reviews research carried out in that area which is substantial from the seminal work by Tall (1989). What this review shows nevertheless is that even when the property of local straightness is put to the fore and the visualization potential of technology used for making students aware of it, the responsibility of the mathematization process is hardly devolved to them. Moreover, with few exceptions (see for instance (Defouad, 2000)), the distance between what is seen on the screen of calculators or computers, or the equations provided by the calculator for tangent lines and the ideal mathematical objects is not necessarily questioned; thus the mathematization process is not fully developed. Research also shows that when students enter Calculus, the idea of tangent is not new to them; they have coherent conceptions, geometric and algebraic, coming from the experience gained working with circles. These conceptions lead to characterize the tangent to a curve as a line

having a unique point of intersection with the curve and staying on the same side of it, not in terms of proximity (Castela, 1995). This conception has to be questioned and as research also shows, usual teaching does not pay much attention to the reconstruction needed. It must be pointed out that, in Italy, these conceptions are reinforced through the teaching of conics in grade 10. Preliminary analyses also reviews research developed on gestures and embodiment, as well as the metaphorical vision of mathematics developed by Lakoff and Nuñez (2000).

The conception phase of the DE relies on these preliminary analyses. In the first situation, students are asked to consider six different functions and after entering them in the calculator and getting their graphical representation in the standard window, to make successive zooms around particular points and explore what happens.

They are also asked to sketch the initial representation and those obtained after two zooms and at the end of the exploration (when they have the feeling that the graphical representation is more or less stable), before moving to another function. The number and characteristics of the functions proposed and the points selected are evidently micro-didactical variables for this task. In the DE, the value of these are chosen so that students first meet differentiable functions, then face a function not differentiable at a point but having left and right derivatives (the function defined by $f(x) = -x^3 - 2|x| + 4$), a linear function and a function with a more complex behavior (the function defined by $f(x) = 4 + x \cdot \sin(1/x)$ for $x \neq 0$ and $f(0) = 4$). It is thus hypothesized that the first examples will lead students to perceptively identify the local straightness phenomenon and to expect its emergence for further examples. The examples of non-differentiable functions will then oblige them to realize that there exists exceptions to this apparently common behavior and that these exceptions may present different characteristics. It is also expected that the dynamic process of zooming will make emerge discourses and metaphors able to support the further mathematization of the perceptive phenomena of local straightness. The drawings asked to the students are expected to be a useful support for this emergence, and for the substantial collective discussion ending the session. They are also data used for the *a posteriori* analysis. Moreover, for each function two different points are considered for insisting on the local nature of the observed phenomenon. Students work by pairs with one calculator for each pair and one common graphical production to deliver. This is a classical organization in DE for fostering verbal exchanges and making these accessible to researchers.

The aim of the second situation is the mathematization of this perceptive phenomenon. A differentiable function is selected different from those already envisaged and a particular point of its graphical representation. Students are asked to check its local behavior around this point, then to find the equation of the line they have got on the screen. It is hypothesized that the different groups will manage the zooming process in different ways and that using the Trace command or numerical values from Table for getting coordinates of a second point of the line, they will get different equations. At this stage, it is planned that the teacher will collect and write on the blackboard all these equations and will launch a collective discussion. It is hypothesized that the view of the equations, close but different will lead students to consider all these lines as approximations of one ideal object: the tangent to the curve, whose equation they can conjecture from the equations written on the blackboard. The validation of this conjecture is not supposed to result from mere a-didactical interaction with the *milieu*. In the scenario for this session, it is planned that the teacher will ask students to find a common way of expressing the different computations made and that, if this is not spontaneously proposed by them, she will introduce the idea of giving account of the commonalities between these different calculations through the use of a letter h representing the different small increments chosen by the students. From this point a collective computation is expected to lead to an equation for the line depending on h , but which becomes the ideal equation when h

is made equal to 0 (in some sense when infinite zooming is performed). This should allow the teacher both to institutionalize the definition of the tangent to a curve at a given point in terms of linear approximation, and the specific type of computation which allows to find its equation. For this second situation, the characteristics of the function and of the point are the main micro-didactical variables of the task. In the DE, two different choices have been successively made: a polynomial function of degree 2 and then one of degree 3, with simple coefficients and of a point whose coordinates are such that the ideal equation can be easily conjectured. Choosing a polynomial function makes that, in the symbolic computation using the letter h , the equation of the line is a polynomial in h (after simplification by h), which makes the reasoning easier. Choosing a polynomial of degree 3 makes that the algebraic strategy known from these students for finding tangents to conics does no longer work. Once again students work in pairs. In the third situation, it is planned to begin to consolidate the form of computation that has been introduced and also to connect this conception of the tangent in terms of approximation with that, geometric and algebraic, developed in grade 10.

As mentioned above, it is hypothesized that during the three sessions, the students will combine gestures with the use of language and different semiotic representations for making sense of the situations and exchange with other students and the teacher. However, the exact forms these combinations will take, and the language that students are likely to introduce for qualifying local straightness is not anticipated. From that point of view, the DE has more an exploratory dimension. The semiotic perspective also impacts the collection of data (those in charge of video-recording for instance try to capture students and teacher gestures as much as possible) and the a posteriori analysis of the sessions.

Each session whose duration is 90mn combines a phase of autonomous work by the students and a phase of collective discussion. Its a priori analysis is structured in the thesis around the following dimensions:

- the preparation of students' worksheets and analysis of these in terms of mathematical content, pre-requisites, didactical variables ;
- the analysis of the role to be played by graphic and symbolic calculators in each phase of the session ;
- the analysis of the work expected from the students, the anticipation of possible strategies and difficulties;
- the analysis of the work expected from the teacher in each phase of the session, and of the distribution of responsibility expected between students and teacher.

The data collected consist of students' worksheets and productions, videos of one particular group and of collective phases, observation notes for different groups (two or three depending on the experimentation) according to guidelines defined from the analysis a priori. To this adds a test taken by students two weeks after the teaching experiment and a questionnaire filled by them regarding their participation to this experience.

The a posteriori analysis of each session combines two levels. The first level presents a global analysis of the session in its relation to the a priori analysis (regarding the scenario of the session, the distribution between group work and collective discussions, the strategies developed by the students and the main characteristics of their work, the difficulties observed, the teacher's role...). The second level is a fine-grained analysis of the protocols elucidating the conceptualization processes at stake and their characteristics, through the role of the calculator, of metaphors, of discourse and gestures, of interactions between students during group work and between students and teacher.

We illustrate this methodological work by a few examples taken from the a posteriori analysis of the first session. For this session, the global analysis is structured around four dimensions: the scenario, the localization of the perspective, the emergence of the invariant and the role of the teacher. Regarding the localization of the perspective for instance, the main elements taken into account in this global approach are the characteristics of the graphical representations drawn by the different groups. A specific list of codes has been developed in the a posteriori analysis of the first experimentation, starting from students' productions. It is used again in the a posteriori analysis of the second and third experimentation. These codes show the expected evolution of representations along the zoom process, but they also make evident the strength of the usual didactic contract regarding graphical representations of functions and the difficulty most students thus face when the zooming process makes the axes disappear.

The analysis of protocols for the groups observed and for the collective discussion combines then different semiotic elements for clarifying the conceptualization processes at stake and the characteristics of the situation that foster these conceptualizations (characteristics of the task, of the milieu and of social interactions). In particular, discourse, inscriptions and gestures are tightly connected in the analysis.

In the a posteriori analysis, the different levels of analysis for one particular session are then combined for testing the conjectures made in the a priori analysis regarding this particular session. The same type of a posteriori analysis is made for the three sessions, then the different results obtained are synthetized and triangulated with those resulting from the analysis of the final test and questionnaire.

The following two quotations from (Maschietto, 2008) in which the author gives a synthetic vision of her research work, illustrate the form that these analyses can take. The first quotation (pp. 215-216) regards the emergence of the linear invariant and an interesting phenomenon accompanying this emergence. This phenomenon was non-anticipated in the a priori analysis but it had a positive effect on the dynamics of the situation.

Excerpt 1 DAL-DF-MA group (Exp_A)

15. DF: "Forward zoom" (he carries out the 3rd ZOOMIN)

16. DF: "Again" (he carries out the 4th ZOOMIN)

17. DF: "It becomes straighter and straighter"

18. DF: "The drawing is the same as before. Even if the result is the same, we'll write it down".

After getting the representation in the standard window, DF does 2 ZOOMINS

20. DF: "I want the other piece of function. It's still a line! Draw at least one axis" (addressed to MA. DF carries out the 3rd ZOOMIN)

21. DF: "We'll stop here because it stays the same".

In the pencil-and-paper environment (Fig. 1b), the linearity is emphasised by the use of a ruler to draw the graphical representation that appears on the calculator display on the third sheet (end of the exploration).

In other protocols (Exp_B and Exp_C), the students try to explain the end-point of their exploration, for example: “REASON WHY WE STOPPED CARRYING OUT THE ZOOMS ? The more we used the ZoomIn, the more the curve sector considered tended to become a line”. We observe here a dynamic language, that draws on the infinite approximation process.

In the protocols, there are two distinct phenomena, linked to the local point of view. The first regards the strength of the “straight” nature at a perceptive level. The second regards the interference of the global point of view with the local one. As far as the first phenomenon is concerned, the comments (for example, Excerpt 2) on the exploration of the corner (function y3) highlight that at this stage the students have, in general, clearly identified the graphic phenomenon “it becomes straight using the zoom”.

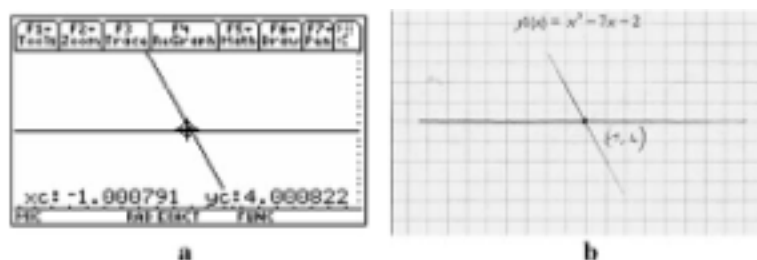


Fig. 1 Window at the end of the exploration process (Exp_A)

Excerpt 2. DAL-DF-MA group (Exp_A)

In all these cases the functions, even with the second zoom, are similar to a line with a gradient C0 but:

- y4 is similar to a line only after the 4th zoom [Note: at $x = 1/\pi$]
- y3 is similar to two lines (one with $m > 0$ and the other with $m < 0$)

However, this recognition does not allow them to distinguish the situation of the function that is differentiable at the given point and that of the function having two different half derivatives and leading to a corner. In fact, these situations, mathematically different, are unified by their common “straightness” recognized at a perceptive level (Excerpt 2). The second function does not therefore represent a counter-example, unlike what is hypothesised in the a-priori analysis. Their distinction will only occur during the mathematization process of the linear invariant. The real counter-example is provided by the y4 function, the graphical representation of which, after subsequent zooms, is perceptively different. In this case there is no move from the “curve” category to the “straight” category, as happens for all the other functions.

The second quotation (pp. 217-218) shows the importance attached to gestures in the a posteriori analysis:

In accordance with the a-priori analysis, the activity presented to the students shows its potential for the production of gestures and metaphors. These appeared both during the communication inside the groups and during the collective discussions. The analysis of the students’ protocols and the discussions show that the conceptualisation of the zoom- controls,

that supports the localisation of the view, appears through gestures that accompany the explanation of the exploration strategies and linguistic expressions that can be analysed in terms of metaphors.

A particularly representative example is the analysis of the gestures of one student, PM (Exp_A), while he is explaining the exploration of a graphical representation. The ZOOMIN control is used in order to see some of the characteristics of the curve in a detailed way and is associated with a downward movement meaning an “entrance into the curve”, that corresponds with moving into the curve (ZoomIn gesture, Fig. 3a). The ZOOMOUT control, which is used to obtain a bigger curve and to study its characteristics better, is associated with an upward movement meaning an “exit from the curve” (ZoomOut gesture, Fig. 3b), which also corresponds with moving away from the curve. PM’s gestures lead the details of the curve to be interpreted as downwards and the overall curve as upwards. PM also creates a space in front of him for controlling these processes (the standard window of the calculator becomes a little rectangle that is constructed by his fingers, Fig. 3c).

The reference to the ZOOMOUT control identifies the space under his eyes, while the palm of one hand is associated with the flat part that is obtained from the ZOOMIN. In this way, PM has created his own space, which is suggested by the activity with the calculator, where the two different transformations of the curve can co-exist and be controlled.

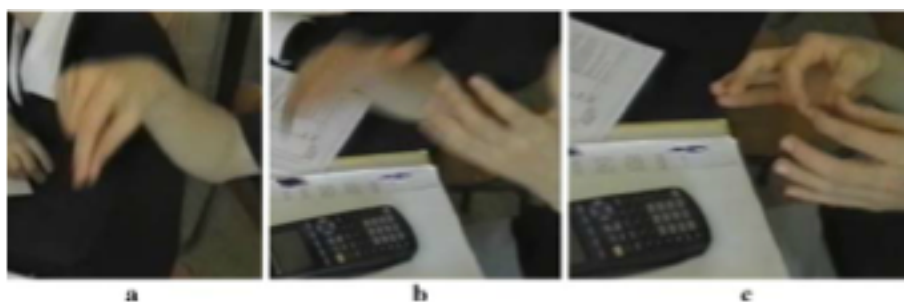


Fig 3: PM’s gestures

The realization took place in three different classes as mentioned above, with some minor adjustments and evident regularities were observed. Globally the hypotheses mentioned above were confirmed despite the fact that it was not possible to cover all what had been planned and that, due to their previous experience with conics, some groups conjectured very early that the line was the tangent and privileged an algebraic strategy for finding its equation, persisting in that strategy with the polynomial of degree 3 in the second and third experiments. Some interesting and non-anticipated phenomena also occurred but they did not necessarily invalidate the *a priori* analysis. For instance, it appeared that most students considered that straight lines and curves were objects belonging to different categories. This conception in fact helped them to consider that the linear representations they obtained by zooming were not exactly linear but just very close to linear objects, and that linearity could only be reached through an infinite succession of zooms. This helped them to make sense of the notion of tangent as approximation and of the computations carried out for finding its equation. This conception nevertheless led also them to think that the function admitting only left and right derivatives at a given point was not very different from the regular ones. This question was considered again later on once the derivative was properly defined. As expected, gestures accompanied students’ verbalizations and work, and the language and metaphors used by students showed evident embodiment. They introduced their own expressions for

qualifying the phenomenon of local straightness saying for instance that the functions were “zoomata lineare” at a particular point and these were accepted and used by the teacher. Validation of the DE did not just use the comparison of the a priori and a posteriori of the sessions, but also the data from a questionnaire taken by the students two weeks after the completion of the process.

We cannot enter into more details here. The interested reader can find these in the references mentioned above. But we would like to stress a few points. According to her author, this methodological construction is a DE and I fully agree with this position, recognizing in it the fundamental features of DE presented above. This is nevertheless a construction sensibly different from that described in the first example. For instance, it is difficult to model the first situation as a game that students enter with basic strategies that they must make evolve towards winning strategies. Students are asked to stop their exploration when they have got the feeling that the graphical representations will no longer substantially evolve, which is a condition rather fuzzy. Moreover, if the situations are designed in order to ensure productive a-didactical interaction with the milieu, in the construction of the situations an important role is given to collective discussions piloted by the teacher and to her mediations. These collective discussions are not just institutionalization phases. As evidenced by the a posteriori analysis, they play an essential role in the progression of knowledge beyond what has been achieved by each pair of students in the phase of autonomous work. In some sense, they play the role given in the theory of didactical situations to situations of formulation and of validation but they do not obey a similar organization; they are not supported by the same theoretical constructs. We can see here the effect of a combination of the theory of didactical situations and the theory of semiotic mediation. It shows us that, as a research methodology, DE can productively combine several theoretical approaches. Another close example is provided by the thesis by Falcade (2006) also combining the theory of didactical situations and the theory of semiotic mediation in an approach to functions using Cabri-Géomètre (see also (Falcade, Laborde & Mariotti, 2007).

After considering these two examples, in the last part of the chapter, we enter into some recent developments of didactical engineering, referring more precisely on the work carried out at the 2009 summer school.

V. Some recent developments of didactical engineering

V.1. Didactical engineering and the Anthropological Theory of Didactics

As mentioned earlier, the anthropological theory of didactics has developed in the last decade a design perspective based on the idea of Programme of Study and Research (PSR in the following). At the 2009 summer school, Chevallard proposed to refund didactical engineering around this idea (Chevallard, 2011). I will not follow him up to this point but would like to situate Chevallard’s perspective with respect to the vision of DE which has been presented in the first sections of this chapter, and briefly explore some possible complementarities between these.

Through PSR, Chevallard ambitions to build a new epistemology opposing what he calls the “monumentalistic” doctrine pervading contemporary school epistemology (Chevallard, 2006, to appear). As explained in (Chevallard, 2006):

“For every praxeology⁴ or praxeological ingredient chosen to be taught, the new epistemology should in the first place make clear that this ingredient is in no way given, or a

⁴ The notion of praxeology is central in the anthropological theory of didactics that considers that knowledge emerges from human practices and is shaped by the institutions where these practices develop. Praxeologies, which model human practices, at the most elemental level (punctual praxeologies), are defined as 4-uplets made

pure echo of something out there, but a purposeful human construct. And it should consequently bring to the fore what its *raison d'être* are, that is, what its reasons are to be here, in front of us, waiting to be studied, mastered, and rightly utilised for the purpose it was created to serve.” (p. 26)

In coherence with this vision, a PSR starts from the will to bring an answer to some generating question. In fact, at the 2009 summer school, Chevallard distinguished between different forms of PSR, and especially between finalized and open PSR. In finalized PSR, the main praxeologies aimed at are known. They correspond for instance to praxeologies aimed at by a given curriculum. The designer must find a question or a succession of questions able to generate the meeting of the corresponding types of tasks and the development of techniques and technological discourse constituting these praxeologies, through a combination of study of existing works and inquiry processes. In open PSR, the situation is quite different. There is a generating question but the praxeological equipment needed for answering it is not *a priori* known; neither is it necessarily limited to mathematical praxeologies. This is for instance often the case in project work, and modeling activities.

Even in the case of finalized PSR, the vision proposed is however at some distance of the forms of DE mentioned above, especially in what concerns the *milieu* and its evolution. This is notably due to the place given to cultural answers to the question at stake in PSR. In the didactical schema that Chevallard proposes (what he calls the Herbartian schema in reference to the famous German pedagogue), a role is given cultural answers or pieces of information accessible to the learners in the media and especially on the Internet. It is supposed that such cultural answers or pieces of information can enter the *milieu* on the initiative of teacher or students and that, duly studied and criticized, they should contribute to the elaboration of the expected answer to the question at stake. In the theory, this is encapsulated in the idea of *media-milieu dialectics*.

Differences also concern more globally what the researcher ambitions to optimize and control in the design phase and consequently they affect the *a priori* analysis. This is especially the case for open PSR. For that case Chevallard denies the possibility of an *a priori* analysis. He thus introduces the idea of *analysis in vivo*, fully integrated into the inquiry work. This position can be questioned all the more as the publications of researchers working within this perspective show that they develop some form of *a priori* analysis to select questions having a strong generating power under the institutional conditions and constraints at stake. What is clear, however, is that, for such open PSR, in the *a priori* analysis researchers are more interested in investigating the didactical potential of the question selected, trying to make clear how its study can develop and generate new and interesting questions, motivate the study and progressive structuration of important praxeologies, than in the optimization of students' learning trajectories. In fact, the *a priori* analysis becomes an on-going process which develops and adjusts along the implementation phase of the DE. The doctoral thesis by Barquero (2009), (Barquero, Bosch & Gascón, 2008) analyzing the possible ecology of a PSR devoted to the modeling of population dynamics with undergraduate students provides a good example of such functioning.

There is no doubt that, from a DE perspective, the notion of open PSR makes it possible to address research issues attached to the ecology of didactical forms more open than those usually addressed by existing DE such as project work and modeling activities. These didactical forms still have a marginal position in educational systems but they are also more

of a type of task, a technique for solving this type of task, a discourse explaining and justifying the technique (technology), and a theory legitimating the technology itself.

and more encouraged as evidenced for instance by the number of European projects currently funded around inquiry-based education in mathematics and science⁵. As a research methodology, DE certainly needs some accommodation in order to cope efficiently with the research issues that emerge from this evolution, and also for taking into account the dramatic changes in access to information of the digital era. From this point of view, the design perspective offered by the anthropological theory of didactics seems promising.

V.2. Research and development: Didactical engineering of second generation

The second evolution I would like to mention is that introduced by Perrin-Glorian (2011) who distinguishes between DE of first and second generation. In this chapter, we have considered DE from a research perspective focusing on its characteristics as a research methodology. We cannot forget nevertheless that from its emergence DE had the ambition to contribute both to research and development. In the historical review we mentioned the difficulties met at converting DE developed for research aims into useful educational resources. This problem is still not solved but the increase in our knowledge of teachers' representations and practices, and of possible dynamics for their evolution makes us better understand the difficulty of the enterprise. The distinction introduced by Perrin-Glorian directly addresses this issue and we consider it because it can also affect the vision of DE as a research methodology. Contrasting RDE and DDE (research didactical engineering and development didactical engineering), she compares the levels of theoretical controls these two forms of DE engage. She thus points out that even if in both cases the analysis of the mathematical knowledge at stake and of the students' knowledge, the definition of the situations and associated *milieux* are under theoretical control, for DDE much more flexibility is needed for preparing the adaptation to a diversity of contexts. The loss of control is even greater in what regards the role of the teacher while institutional constraints cannot be partly removed as is often the case in RDE. These considerations lead her to postulate that before trying to implement a DE product coming from research in ordinary classes, it is necessary to plan at least two different levels of DE, each one having specific aims: This is the whole process that she names DE of second generation.

At the first level, the goal is the theoretical validation of the situations of the DE (i.e. their capacity at producing the knowledge aimed at) and the identification of the fundamental choices of the DE, separating what is essential from what is linked to the particular context and could be changed, adapted. The associated realization takes place in a rather protected environment and under the control of researchers as is the case for RDE.

At the second level, the goal is the study of the adaptability of such validated situations to ordinary classrooms and teachers through the negotiation of the DE with teachers who have not been involved in the first phase. These negotiations and the transformations introduced by the teachers involved in this second phase are taken as objects of study together with their impact on the DE itself and its outcomes. It is expected that the results obtained allow researchers to determine what concessions can be made in such negotiations, what should be preserved and why, and to identify what forms of control can be maintained.

As Perrin-Glorian points out, envisaging this second level modifies in fact the first level because it obliges researchers to move from a top-down conception of transmission of research results to an idea of adaptation much more dialectical. As she adds: "the problem is no longer to control and disseminate engineering products coming from research but to determine the key variables, in terms of knowledge involved, piloting the didactical engineering that one wants to make a resource for ordinary teaching, and to study the

⁵ See the portal www.scientix.eu for information about these projects.

conditions of their dissemination.” (p. 69, our translation). She then illustrates this vision by an example regarding the teaching of axial symmetry at the transition between elementary school and junior high school.

This reflection in fact points out that the transition from research to development needs specific forms of research, extending our view of the ways didactical engineering and educational research can be connected.

VI. Conclusion

In this chapter, I have tried to present didactical engineering, focusing on its dimension of research methodology. To help readers make sense of this methodology, I have reviewed its history from its emergence in the early eighties until now. I have tried to clarify its main characteristics and to show that this methodology, even if it has been shaped by the values and constructs of the theory of didactical situations, is a methodology that can be productively used beyond the frontiers of this theory, and is enriched by the different uses made of it. I have also tried to show that, as many other constructs in educational research, didactical engineering is a living and dynamic concept which adapts to the evolution of the field, to the advances of educational knowledge, and to the evolution of the social and cultural contexts of mathematics education. I also hope to have made clear that this methodology, although flexible, imposes a systemic view of the field, a view of the classroom as a social organization, of learning as a combination of adaptation and acculturation processes and a particular sensitivity to the discipline and its epistemology.

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